CPSC 331 — Solutions for Tutorial Exercise #7

Asymptotic Notation

1. Consider the functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 4n^2 + 2n + 1$ and $g(n) = n^2$ for all $n \in \mathbb{N}$.

   (a) You were asked to use the definitions of "$O(g)$" and of "$\Omega(g)$" ,and one or more results stated during Lecture #7, to prove that $f \in \Theta(g)$.

   Solution:

   Claim 1. *$4n^2 + 2n + 1 \in O(n^2)$.*

   Proof. It follows by the definition of "$O(n^2)$" that it is sufficient to show *that there* exist constants $c > 0$ and $N_0 \geq 0$ such that $n^2 + 2n + 1 \leq c \cdot n^2$ for *all* $n \in \mathbb{N}$ such that $n \geq N_0$.
   Let $c = 7$ and $N_0 \in \mathbb{N}$.
   With that noted, let $n \in \mathbb{N}$ such that $n \geq N_0 = 1$. Then $n^2 \geq n$ and $n^2 \geq 1$ as well, so that

   $$\begin{aligned} 4n^2 + 2n + 1 &\leq 4n^2 + 2n^2 + n^2 \\ &= 7n^2 = c \cdot n^2. \end{aligned}$$

   Since n was arbitrarily chosen from $\mathbb{N}$ such that $n \geq N_0$, it follows that $4n^2+2n+1 \leq c \cdot n^2$ for *all* $n \in \mathbb{N}$ such that $n \geq N_0$.
   Thus *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $4n^2 + 2n + 1 \leq c \cdot n^2$ for *all* $n \in \mathbb{N}$ such that $n \geq N_0$.
   It follows by the definition of "$O(n^2)$" that $4n^2 + 2n + 1 \in O(n^2)$. $\square$

   Claim 2. *$4n^2 + 2n + 1 \in \Omega(n^2)$.*

   Proof. It follows by the definition of "$\Omega(n^2)$" that it is sufficient to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $n^2 + 2n + 1 \geq c \cdot n^2$ *for all* $n \in \mathbb{N}$ such that $n \geq N_0$.
   Let $c = 4$ and $N_0 = 0$.

With that noted, let $n \in \mathbb{N}$ such that $n \geq N_0 = 0$. Then

$$\begin{aligned} 4n^2 + 2n + 1 &\geq 4n^2 + 1 && \text{(since } 2 > 0 \text{ and } n \geq 0\text{)} \\ &\geq 4n^2 \\ &= c \cdot n^2. \end{aligned}$$

Since n was arbitrarily chosen from $\mathbb{N}$ such that $n \geq N_0$, it follows that $4n^2+2n+1 \geq n^2$ *for all* $n \in \mathbb{N}$ such that $n \geq N_0$.

Thus *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $4n^2 + 2n + 1 \geq c \cdot n^2$ *for all* $n \in \mathbb{N}$ such that $n \geq N_0$.

It follows by the definition of "$\Omega(n^2)$" that $4n^2 + 2n + 1 \in \Omega(n^2)$. $\square$

Since $n^2+4n+1 \in O(n^2)$ and $n^2+4n+1 \in \Omega(n^2)$ by Claims 1 and 2, respectively, it follows by the "alternative definition of $\Theta(g)$" from Lecture #6 that $n^2 + 4n + 1 \in \Theta(n^2)$ as well.

(b) You were asked to use one or more limit tests to prove that $f \in \Theta(g)$, instead.

Solution:

$$\begin{aligned} \lim_{n \to +\infty} \frac{4n^2 + 2n + 1}{n^2} &= \lim_{n \to +\infty} \left(4 + \frac{2}{n} + \frac{1}{n^2} \right) \\ &= 4. \end{aligned}$$

Since this is a positive constant, that is not equal to $+\infty$, it follows by the "Limit Test for Big-Oh" that $4n^2 + 2n + 1 \in O(n^2)$. On the other hand, since this is a *positive* constant — strictly greater than zero — it follows by the "Limit Test for Big-Omega" that $4n^2 + 2n + 1 \in \Omega(n^2)$ as well. It now follows by the "alternative definition of $\Theta(g)$" that $4n^2 + 2n + \in \Theta(n^2)$, as claimed.

2. Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ are asymptotically positive functions. You were asked to prove that if $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.

Solution:

Claim 3. *Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ are asymptotically positive functions. If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.*

Proof. Suppose, as above, that $f \in O(g)$. Then it follows by the definition of "$O(g)$" that *there exist* constants $\widehat{c} > 0$ and $\widehat{N}_0 \geq 0$ such that $f(x) \leq \widehat{c} \cdot g(x)$ for *all* $x \in \mathbb{R}$ such that $x \geq \widehat{N}_0$.

Suppose, as well, that $g \in O(h)$. Then it follows by the definition of "$O(h)$" that *there exist* constants $\widetilde{c} > 0$ and $\widetilde{N}_0 \geq 0$ such that $g(x) \leq \widetilde{c} \cdot h(x)$ *for all* $x \in \mathbb{R}$ such that $x \geq \widetilde{N}_0$.

In order to show that $f \in O(h)$, it suffices to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $f(x) \leq c \cdot h(x)$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

With that noted, let $c = \widehat{c} \cdot \widetilde{c}$ and let $N_0 = \max(\widehat{N}_0, \widetilde{N}_0)$. Then c and N_0 are constants, since $\widehat{c}$, $\widetilde{c}$, $\widehat{N}_0$ and $\widetilde{N}_0$ are. Furthermore, $c > 0$ since $\widehat{c} > 0$ and $\widetilde{c} > 0$, and $N_0 \geq 0$ since $\widehat{N}_0 \geq 0$ and $N_0 \geq \widehat{N}_0$.

Now let $x \in \mathbb{R}$ such that $x \geq N_0$. Then

$$
\begin{aligned}
f(x) &\leq \widehat{c} \cdot g(x) && \text{(since } x \geq N_0 \geq \widehat{N}_0\text{)} \\
&\leq \widehat{c} \cdot (\widetilde{c} \cdot h(x)) && \text{(since } \widehat{c} > 0 \text{ and } x \geq N_0 \geq \widetilde{N}_0\text{)} \\
&= (\widehat{c} \cdot \widetilde{c}) \cdot h(x) \\
&= c \cdot h(x).
\end{aligned}
$$

Since x was arbitrarily chosen from $\mathbb{R}$ such that $x \geq N_0$, it follows that $f(x) \leq c \cdot h(x)$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

It now follows by the definition of "$O(h)$" that $f \in O(h)$. □