CPSC 331 — Solutions for Tutorial Exercise #1

In this exercise you were asked to use mathematical induction to prove two results.

Solutions for Problems

1. ***Claim:***

$$\prod_{i=2}^{n} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{n}$$

   for every integer n such that $n \geq 2$.

   Proof. This will be proved by induction on n. The standard form of mathematical induction will be used (so that the case that $n = 2$ will be considered in the basis).

   Basis: Suppose that $n = 2$. Then

$$\begin{aligned}\prod_{i=2}^{n} \left(1 - \tfrac{1}{i}\right) &= \prod_{i=2}^{2} \left(1 - \tfrac{1}{i}\right) \\ &= 1 - \tfrac{1}{2} \\ &= \tfrac{1}{2} = \tfrac{1}{n},\end{aligned}$$

   as required.

   Inductive Step: Let k be an integer such that $k \geq 2$. It is necessary to use the following

   <u>Inductive Hypothesis:</u> $\displaystyle\prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k}.$

   to prove the following

   <u>Inductive Claim:</u> $\displaystyle\prod_{i=2}^{k+1} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k+1}.$

Now

$$
\begin{aligned}
\prod_{i=1}^{k+1} \left(1 - \tfrac{1}{i}\right) &= \left(\prod_{i=1}^{k} \left(1 - \tfrac{1}{i}\right)\right) \cdot \left(1 - \tfrac{1}{k+1}\right) \\
&= \left(\prod_{i=1}^{k} \left(1 - \tfrac{1}{i}\right)\right) \cdot \tfrac{k}{k+1} \\
&= \tfrac{1}{k} \cdot \tfrac{k}{k+1} && \text{(by the inductive hypothesis, since } k \geq 2\text{)} \\
&= \tfrac{1}{k+1}
\end{aligned}
$$

establishing the Inductive Claim. This completes the Inductive Step, and establishes the claim. □

2. Recall that the ***Dumbledore numbers*** are a sequence $D_0, D_1, D_2, D_3, \ldots$ of numbers that are defined as follows: For every integer n such that $n \geq 0$,

$$
D_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 3D_{n-1} - 3D_{n-2} + D_{n-3} & \text{if } n \geq 3. \end{cases}
$$

Claim: $D_n = n^2$ for every integer n such that $n \geq 0$.

Proof. This will be proved by induction on n. The strong form of mathematical induction will be used, and the cases $n = 0$, $n = 1$ and $n = 2$ will be considered in the basis.

Basis: Suppose first that $n = 0$. Then

$$
\begin{aligned}
D_n &= D_0 \\
&= 0 && \text{(by the above definition of } D_n\text{)} \\
&= 0^2 = n^2,
\end{aligned}
$$

establishing the claim in this case.

Suppose, next, that $n = 1$. Then

$$
\begin{aligned}
D_n &= D_1 \\
&= 1 && \text{(by the above definition of } D_n\text{)} \\
&= 1^2 = n^2,
\end{aligned}
$$

establishing the claim in this case as well.

Finally, suppose that $n = 2$. Then

$$\begin{aligned} D_n &= D_2 \\ &= 4 && \text{(by the above definition of } D_n\text{)} \\ &= 2^2 = n^2, \end{aligned}$$

establishing the claim in this case too.

Inductive Step: Let k be an integer such that $k \geq 2$. It is necessary and sufficient to use the following

> Inductive Hypothesis: $D_h = h^2$ for every integer h such that $0 \leq h \leq k$.

to prove the following

> Inductive Claim: $D_{k+1} = (k+1)^2$.

Let $n = k + 1$. Since $k \geq 2$, $0 \leq k-2, k-1, k \leq k$, and it follows by the Inductive Hypothesis that $D_{k-2} = (k-2)^2$, $D_{k-1} = (k-1)^2$, and $D_k = k^2$. Now, since $k+1 \geq 3$,

$$\begin{aligned} D_n &= D_{k+1} \\ &= 3D_k - 3D_{k-1} + D_{k-2} && \text{(by the above definition of } D_n\text{)} \\ &= 3k^2 - 3(k-1)^2 + (k-2)^2 && \text{(by the Inductive Hypothesis, as noted above)} \\ &= 3k^2 - 3(k^2 - 2k + 1) + (k^2 - 4k + 4) \\ &= (3 - 3 + 1)k^2 + (0 + 6 - 4)k + (0 - 3 + 4) && \text{(reordering terms)} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

establishing the Inductive Claim. This completes the Inductive Step and establishes the claim. ☐

Common Problems

Students often do not write enough. This seems to be especially true of students whose academic performance is below average.

It is *always* a good idea to begin with a precise statement of the claim that you are trying to prove.

It is also, generally, a good idea to begin a proof with the description of the proof strategy. In the above proofs this included identifying the proof technique as "mathematical induction", saying what value (or function of values) you are going to induct *on*, saying which form of mathematical induction will be used and identifying the case (or, for strong induction, cases) to be considered in the basis.

When mathematical induction is being used, the inductive step should clearly introduce a new value (in the above cases, k) to be considered and include clear statements of the *Inductive Hypothesis* (which can be assumed) and the *Inductive Claim* (which must be proved).