Expectile correlation: A measure for tail dependence based on asymmetric least squares

Juyeon Kim · Joonpyo Kim

(Department of Statistics, Sejong University)

Introduction

• This poster introduces a novel measure of tail dependence using asymmetric least squares.

• Pearson correlation coefficient fails to capture tail dependence. To address this limitation, several studies have proposed measures of tail correlation, such as the quantile correlation coefficient suggested by Choi and Shin (2022).

• We extend this quantile correlation coefficient to its analogue in terms of expectiles and investigate its properties.

Expectile correlation coefficient

Quantile correlation coefficient

• τ - quantile regression coefficients of Y on X and X on Y are defined by minimizing

$$L_\tau^{X,Y}(\alpha,\beta) = \mathsf{E}\left[l_\tau(Y-\alpha-\beta X)\right]$$
$$L_\tau^{Y,X}(\alpha,\beta) = \mathsf{E}\left[l_\tau(X-\alpha-\beta Y)\right]$$

respectively, where $l_\tau(e) = e(\tau - I(e<0))$ is the check loss function.

• Then for quantile regression coefficients

$$(a_{2.1}(\tau), b_{2.1}(\tau)) = \arg\min_{\alpha,\beta} L_\tau^{X,Y}(\alpha,\beta)$$
$$(a_{1.2}(\tau), b_{1.2}(\tau)) = \arg\min_{\alpha,\beta} L_\tau^{Y,X}(\alpha,\beta),$$

Choi and Shin (2022) proposed a quantile correlation coefficient at level τ defined as

$$\rho_\tau(X,Y) = \begin{cases} \mathrm{sign}(b_{2.1}(\tau))\sqrt{b_{2.1}(\tau)b_{1.2}(\tau)}, & \text{if } b_{2.1}(\tau)b_{1.2}(\tau) \ge 0 \\ 0, & \text{otherwise}. \end{cases}$$

• Quantile correlation coefficient shares similar properties as Pearson correlation coefficient.

Expectile correlation coefficient

• Expectile can be estimated by minimizing the asymmetric sum of squares of the residuals

$$S_\tau^{X,Y}(\alpha,\beta) = \mathrm{E}[I_{\tau,2}(Y-\alpha-\beta X)]$$
$$S_\tau^{Y,X}(\alpha,\beta) = \mathrm{E}[I_{\tau,2}(X-\alpha-\beta Y)]$$

respectively, where $I_{\tau,2}(e) = e^2(\tau - I(e<0))$ is an asymmetric squared loss.

•Then for expectile regreesion coefficients

$$(\alpha_{YX}(\tau), \beta_{YX}(\tau)) = \arg\min_{\alpha,\beta} S_\tau^{X,Y}(\alpha,\beta)$$
$$(\alpha_{XY}(\tau), \beta_{XY}(\tau)) = \arg\min_{\alpha,\beta} S_\tau^{Y,X}(\alpha,\beta)$$

we propose to measure a strength of tail correlation at τ -th expectile as

$$\rho_{\tau,2} = \begin{cases} \mathrm{sign}(\beta_{YX}(\tau))\sqrt{\beta_{YX}(\tau)\beta_{XY}(\tau)}, & \text{if } \beta_{YX}(\tau)\beta_{XY}(\tau) \ge 0, \\ 0, & \text{otherwise}. \end{cases}$$

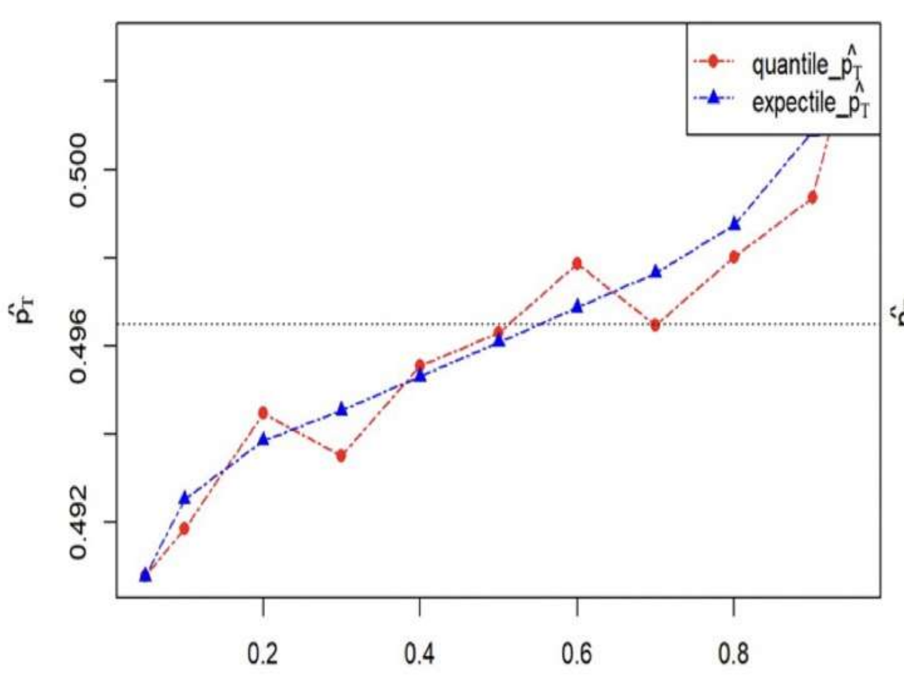
Example

We generated 1000 samples from bivariate distributions and estimated quantile and expectile correlation coefficients for each level τ:

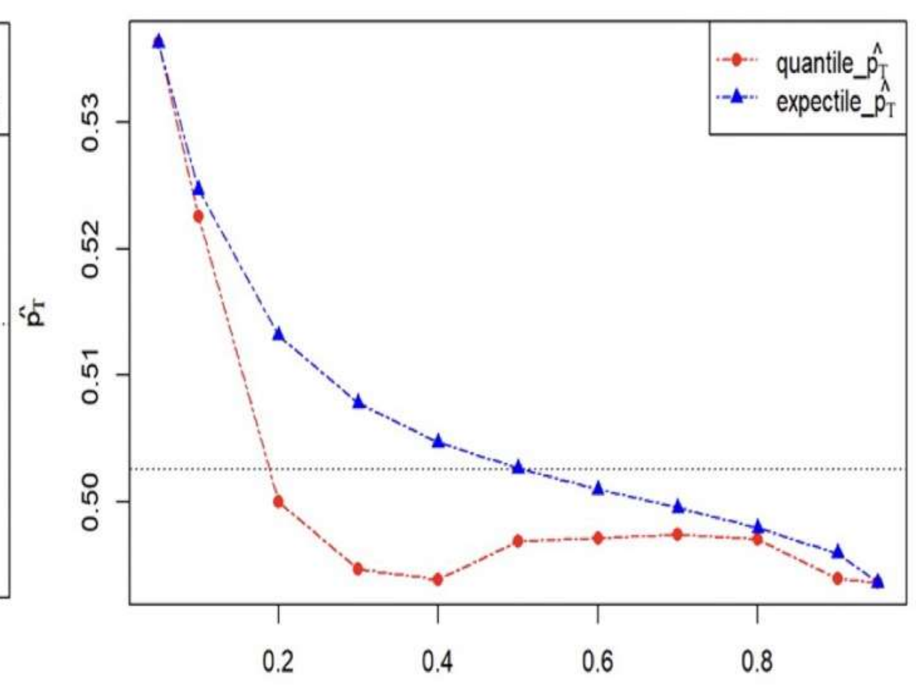
$$N_2\left(0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right)$$

$$X = \tilde{X} + e\mathrm{I}(\tilde{X} \le c, \tilde{Y} \le c), Y = \tilde{Y} + e\mathrm{I}(\tilde{X} \le c, \tilde{Y} \le c)$$
$$(\tilde{X},\tilde{Y}) \sim N_2\left(0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right), e \sim N(0,1)$$

(a) Bivariate normal



(b) Rocket type



• (a) Bivariate normal distribution: Both correlation coefficients closely approximate the original correlation coefficient of 0.5, indicating a similar trend.

• (b) When a stronger dependence is imposed to the left tail, both correlation coefficients successfully capture this strong correlation for lower levels.

• The quantile correlation coefficients (represented by red lines) show larger values compared to other quantiles when the τ values are below 0.1.

• Expectile correlation coefficients exhibit smoother trends compared to quantiles.

• We can verify that the proposed expectile correlation coefficient also reflects tail dependence and shares favorable properties with the quantile correlation coefficient.

Inference

Estimation and Confidence Interval

• Expectile correlation coefficient can be estimated via estimating expectile regression coefficients.

• Confidence interval can be obtained using the asymptotic distribution

$$\sqrt{n}(\hat{\rho}_{\tau,2} - \tau_{\tau,2}) \xrightarrow[n\to\infty]{d} N(0,\sigma_\tau^2),$$

where σ_τ^2 is a function of regression coefficients and residuals, and does not rely on the density terms.

• In contrast, asymptotic standard error of the quantile correlation coefficient derived by Choi and Shin (2022) depends on the conditional density functions, and hence the procedure requires density estimation.

Simulation Results

Data Generating Processes

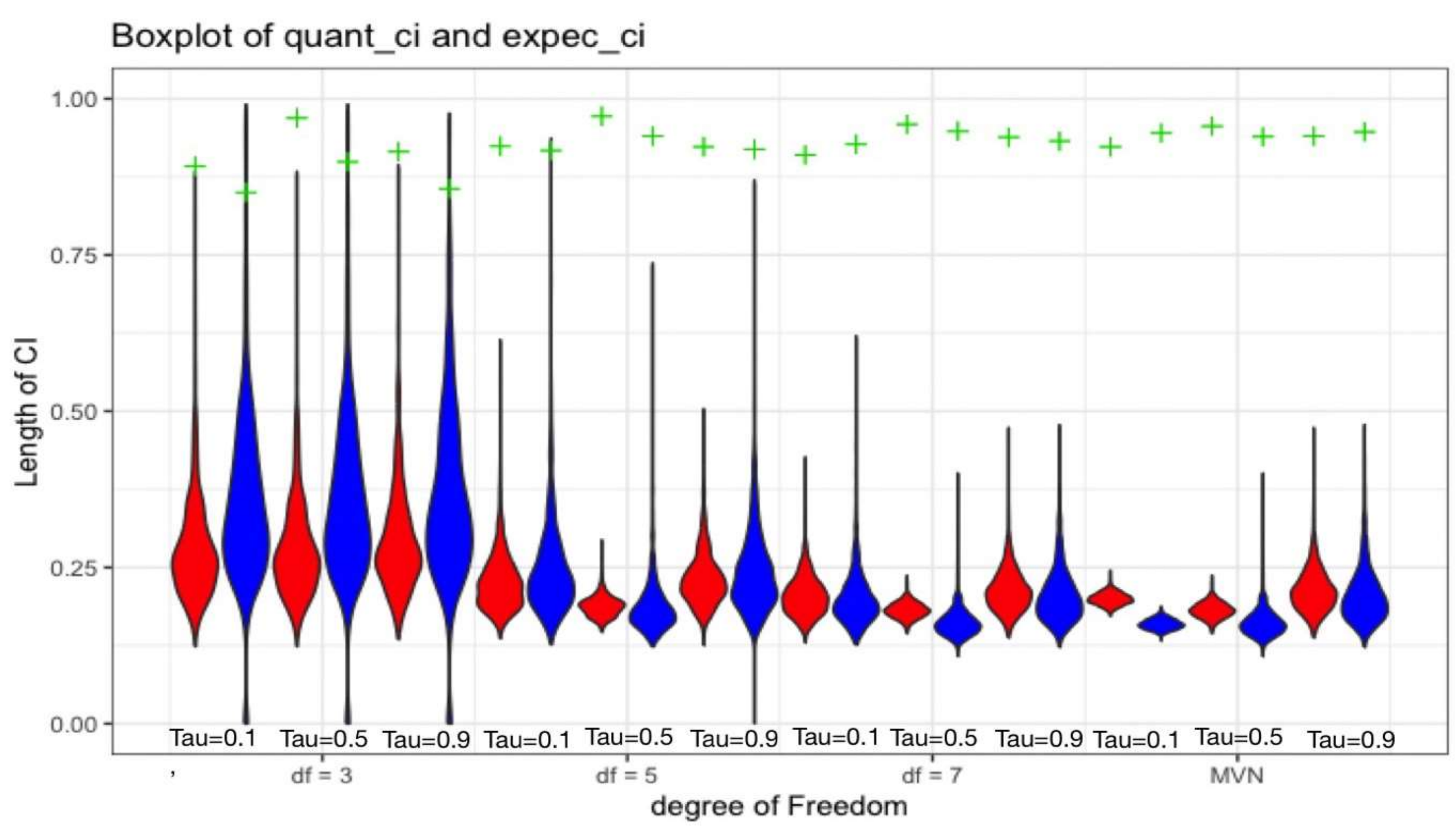
• We generate $n = 500$ samples from the bivariate t-distribution with v degrees of freedom and the scale matrix $\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$.

• We evaluate the estimated correlation coefficients and the resulting 95% confidence intervals through 1,000 repetitions and obtained the empirical coverage probabilities.

Results

• Following plot summarizes the simulation result with $v = 5, 7$ and ∞. Here the t-distribution with $v = \infty$ is regarded as multivariate normal distribution.

• For each v, vase plots are drawn to display the lengths of confidence intervals for quantile correlation coefficients (in red) and expectile correlation coefficients (in blue) at level $\tau = 0.1, 0.5,$ and 0.9. The green points represent the empirical coverage probabilities.



• We can verify that especially when v is large, the expectile correlation coefficients have shorter confidence intervals compared to quantiles. It indicates the stability of expectiles.

• However, if the underlying data generating process a t-distribution with 3 degrees of freedom, the confidence interval for expectiles become slightly wider. This may be due to the moment condition required when obtaining the standard

Conclusion

• In this study we propose an expectile correlation coefficient, obtained as a straightforward extension from quantile correlation coefficient, to measure the tail dependence.

•Its computation is more stable and does not require the density estimation for the inference.

References

[1] Choi, J.-E. and Shin, D. W. (2022).Quantile correlation coefficient: a new tail dependence measure. Statistical Papers, 63(4):1075–1104.