Expectile Correlation Coefficient

:A new measure for tail correlation

Ju Yeon Kim

Sejong University

jgys014@naver.com

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Presentation Overview

Introduction

Pearson correlation coefficient

- Pearson correlation coefficient between two random variables X and Y are defined as

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}. \tag{1}$$

- It is commonly used to measure the dependence between two variables.
- However, it does not give the information about the tail correlation.

Introduction

Literature Review

- There were several literatures that studied the tail correlation, for example:
  - [Adrian and Brunnermeier, 2016] and [Girardi and Ergün, 2013] proposed co-value at risk (CoVaR);
  - The copula also has been considered in, e.g., [Joe et al., 2010, Nikoloulopoulos et al., 2012].
- Measures which summarize a strength of tail dependence are also studied:
  - [Han et al., 2016, Hill, 2009, Hill, 2011a, Hill, 2011b] considered the cross-quantilogram $corr(I(X < q_\tau(X)), I(Y < q_\tau(Y)))$.
  - [Li et al., 2015] suggested another measure $corr(I(X < q_\tau(X)), Y)$.

Introduction

Literature Review

- [Choi and Shin, 2022] proposed a new measure called quantile correlation coefficient, which is defined as a geometric mean of quantile regression coefficients.
- The goal of this study is to extend this quantile correlation coefficient to the analogue of expectile and investigate its properties.

Quantile Correlation Coefficient

Pearson correlation coefficient

- Let σ_{XX} = Var(X) , σ_{YY} = Var(Y) , σ_{XY} = $COV(X, Y)$
- $\beta_{2.1} = \frac{\sigma_{XY}}{\sigma_{XX}}$ is the second element of (α, β) minimizing the expected squared error loss $E[(Y - \alpha - X\beta)^2]$ and $\beta_{1.2} = \frac{\sigma_{XY}}{\sigma_{YY}}$ is the second element of (α, β) minimizing $E[(X - \alpha - Y\beta)^2]$

$$\rho = \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}\sigma_{YY}}} = \mathrm{sign}(\beta_{2.1})\sqrt{\beta_{2.1}\beta_{1.2}}$$

Quantile Correlation Coefficient

Definitions & Examples

- τ-quantile regression coefficients of Y on X and X on Y are defined by minimizing

$$L_\tau^{X,Y}(\alpha, \beta) = \mathrm{E}\left[l_\tau(Y - \alpha - \beta X)\right]$$
$$L_\tau^{Y,X}(\alpha, \beta) = \mathrm{E}\left[l_\tau(X - \alpha - \beta Y)\right], \ \tau \in (0, 1)$$

respectively, where

$$l_\tau(e) = e(\tau - \mathrm{I}(e < 0))$$

is the check loss function of the τ-quantile regression and I is the indicator function.

Quantile Correlation Coefficient

Definitions & Examples

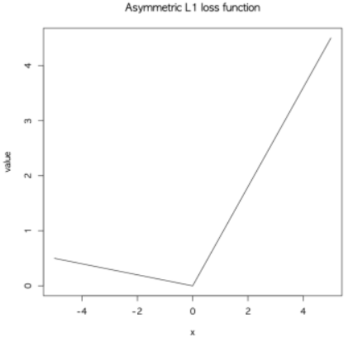


Figure: Asymmetric L1 loss function

- Then for quantile regression coefficients

$$\big(a_{2.1}(\tau), b_{2.1}(\tau)\big) = \arg\min_{\alpha,\beta} L_\tau^{X,Y}(\alpha,\beta)$$
$$\big(a_{1.2}(\tau), b_{1.2}(\tau)\big) = \arg\min_{\alpha,\beta} L_\tau^{Y,X}(\alpha,\beta),$$

a quantile correlation coefficient at level τ is defined as

$$\rho_\tau(X,Y) = \begin{cases} \text{sign}(b_{2.1}(\tau))\sqrt{b_{2.1}(\tau)b_{1.2}(\tau)}, & \textit{if } b_{2.1}(\tau)b_{1.2}(\tau) \geq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Quantile Correlation Coefficient

Properties

1. Quantile correlation coefficient has similar properties as Pearson correlation:
   - if X,Y is independent, $\rho_\tau(X, Y) = 0$
   - $\rho_\tau(aX + b, cY + d) = sign(ac)\rho_\tau(X, Y)$
   - If (X, Y) are bivariate normally distributed, $\rho_\tau(X, Y)$ is the same as $corr(X, Y)$ for any $\tau \in (0, 1)$.

2. However, a quantile correlation coefficient may exceed 1, while Pearson correlation is always not larger than 1.

Quantile Correlation Coefficient

Properties

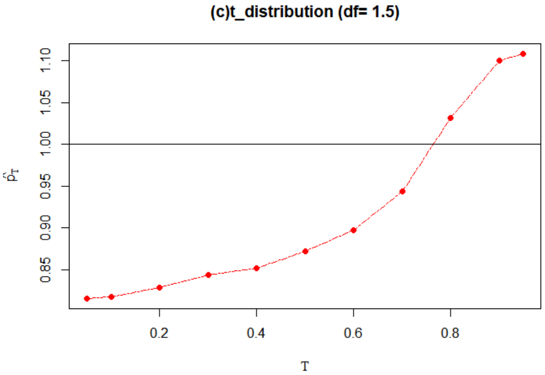
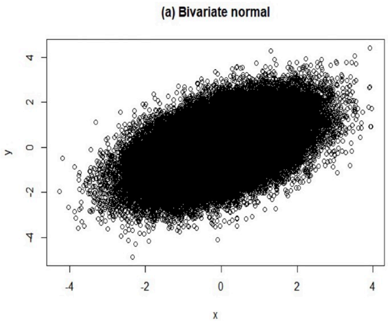


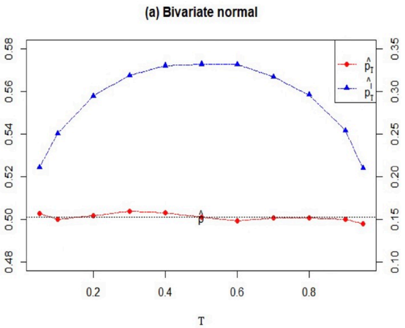
Figure: $|\rho_\tau| \geq 1$

Example

Bivariate normal distribution

We generated 1000 samples from bivariate normal distribution $N_2\left(0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right)$ and obtained a quantile correlation coefficient (red line) and a cross-quantilogram (blue line).





Example

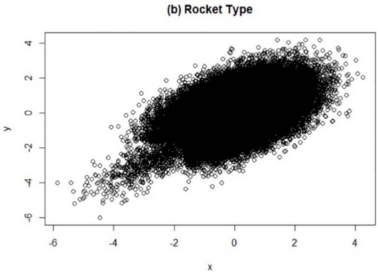
‘Rocket-type’ distribution

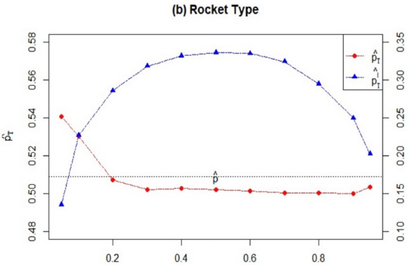
We generated 1000 samples from the distribution of (X, Y), where

$$X = \tilde{X} + eI(\tilde{X} \leq c, \tilde{Y} \leq c), Y = \tilde{Y} + eI(\tilde{Y} \leq c, \tilde{Y} \leq c),$$

$$(\tilde{X}, \tilde{Y}) \sim N_2\left(0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right), \; e \sim N(0,1)$$

and obtained a quantile correlation coefficient (red line) and a cross-quantilogram (blue line).





Inference

Quantile Correlation Coefficient

- [Choi and Shin, 2022] derived the asymptotic distribution of their proposed estimator, which can be used for the inference.
- However, the standard error depends on the conditional density functions, and hence the procedure includes density estimation.

Expectiles

Expectile and Expectile Regression

- For the convenience of calculation and the ease of covariance calculation, expectile regression analysis was introduced (Newey and Powell, 1987).
- Expectile can be estimated by minimizing the asymmetric sum of squares of the residuals.

$$l_{\tau,2}(e) = e^2(\tau - \mathrm{I}(e < 0))$$

Expectile Correlation Coefficient

Definitions & Examples

- Let (X, Y) be a bivariate random vector with $\mathrm{E}|X|^2 < \infty$ and $\mathrm{E}|Y|^2 < \infty$.
- Also define

$$l_{\tau,2}(e) = e^2(\tau \ - \mathrm{I}(e < 0))$$

and

$$S_\tau^{X,Y}(\alpha, \beta) = \mathrm{E}\left[l_{\tau,2}(Y - \alpha - \beta X)\right]$$
$$S_\tau^{Y,X}(\alpha, \beta) = \mathrm{E}\left[l_{\tau,2}(X - \alpha - \beta Y)\right]$$

- Also define expectile regression coefficients as

$$\left(\alpha_{YX}(\tau), \beta_{YX}(\tau)\right) = \arg\min_{\alpha,\beta} S_\tau^{X,Y}(\alpha, \beta)$$
$$\left(\alpha_{XY}(\tau), \beta_{XY}(\tau)\right) = \arg\min_{\alpha,\beta} S_\tau^{Y,X}(\alpha, \beta)$$

- Then a strength of tail correlation at τ-th expectile can be measured as

$$\rho_{\tau,2} = \begin{cases} \text{sign}(\beta_{YX}(\tau))\sqrt{\beta_{YX}(\tau)\beta_{XY}(\tau)}, & \textit{if } \beta_{YX}(\tau)\beta_{XY}(\tau) \geq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Expectile Correlation Coefficient

Quantile vs. Expectile

$$l_\tau(e) = e(\tau - \mathrm{I}(e < 0))$$
$$l_{\tau,2}(e) = e^2(\tau - \mathrm{I}(e < 0))$$

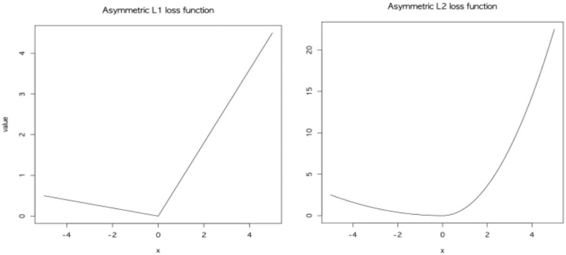
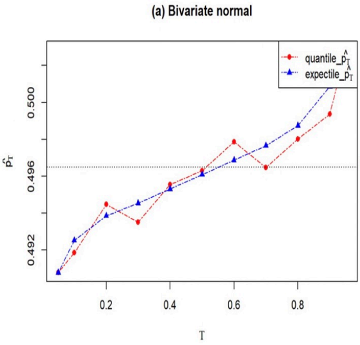


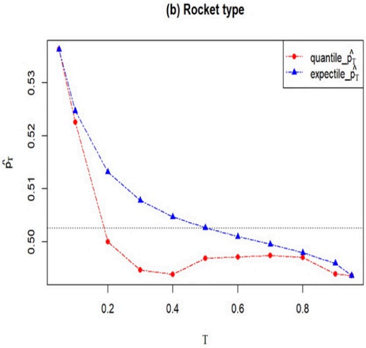
Figure: Quantile VS Expectile

Example

Quantile vs. Expectile

We also obtained expectile correlation coefficients (blue line) from the previous examples and displayed them with quantile correlation coefficients (red line).





Inference

Expectile Correlation Coefficient

- Unlike quantile correlation, standard error of expectile correlation does not rely on conditional densites.
- Furthermore, estimation of expectile regression is expected to be more stable than quantile regression.

Conclusion

- In this study we propose an expectile correlation coefficient, obtained as a straightforward extension from quantile correlation coefficient, to measure the tail dependence.
- Its computation is more stable and does not require the density estimation for the inference.
- We plan to study about the expected properties of the expectile correlation coefficient and simulation about various distributions according to tau value.

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The End

Questions? Comments?