Constrained Optimization

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A 2D Example

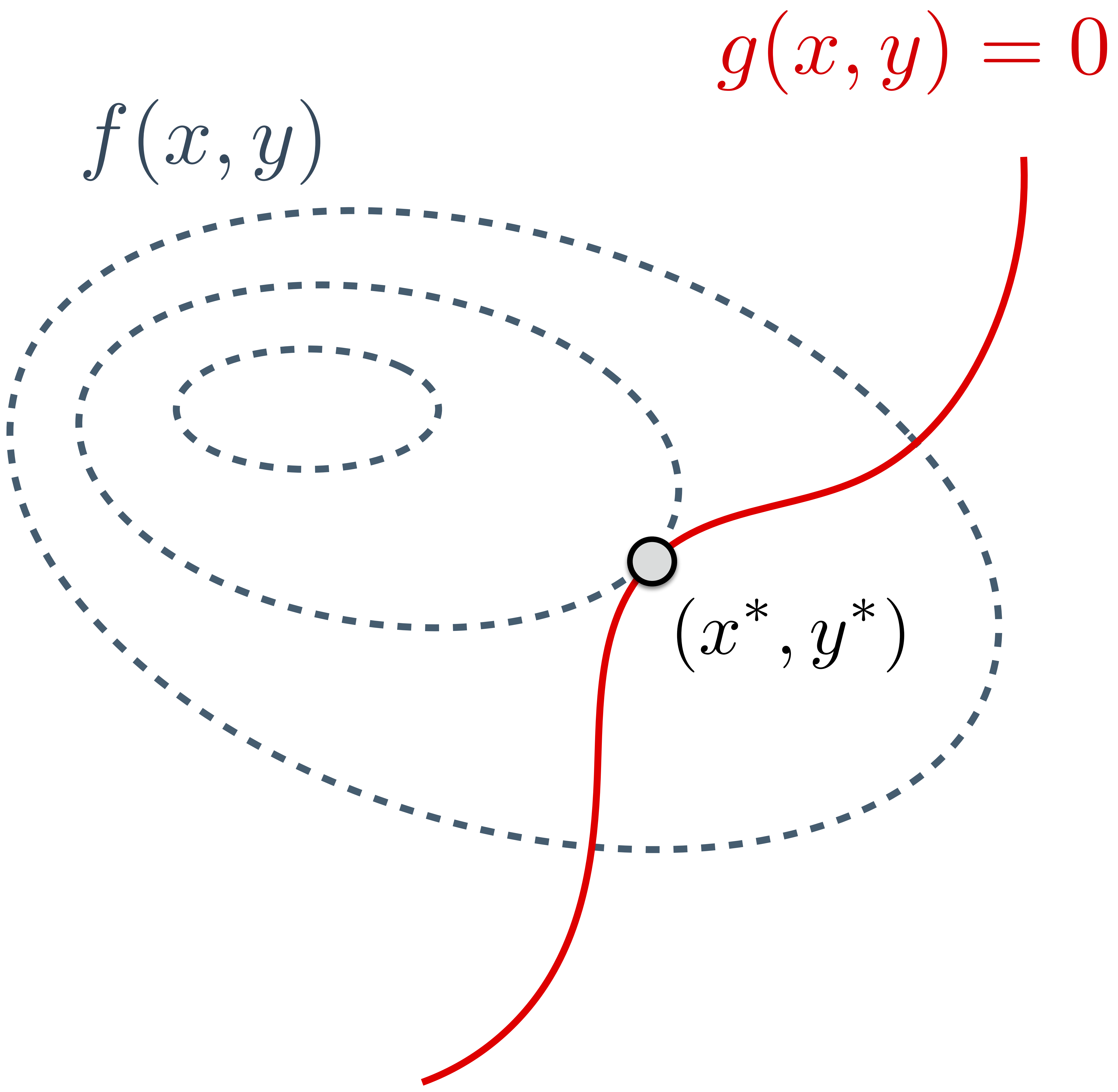
- For $f, g : \mathbb{R}^2 \mapsto \mathbb{R}, \quad c \in \mathbb{R},$

$$\begin{aligned} \min_{x,y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) = 0 \end{aligned}$$

Optimality Conditions

- Feasibility condition:

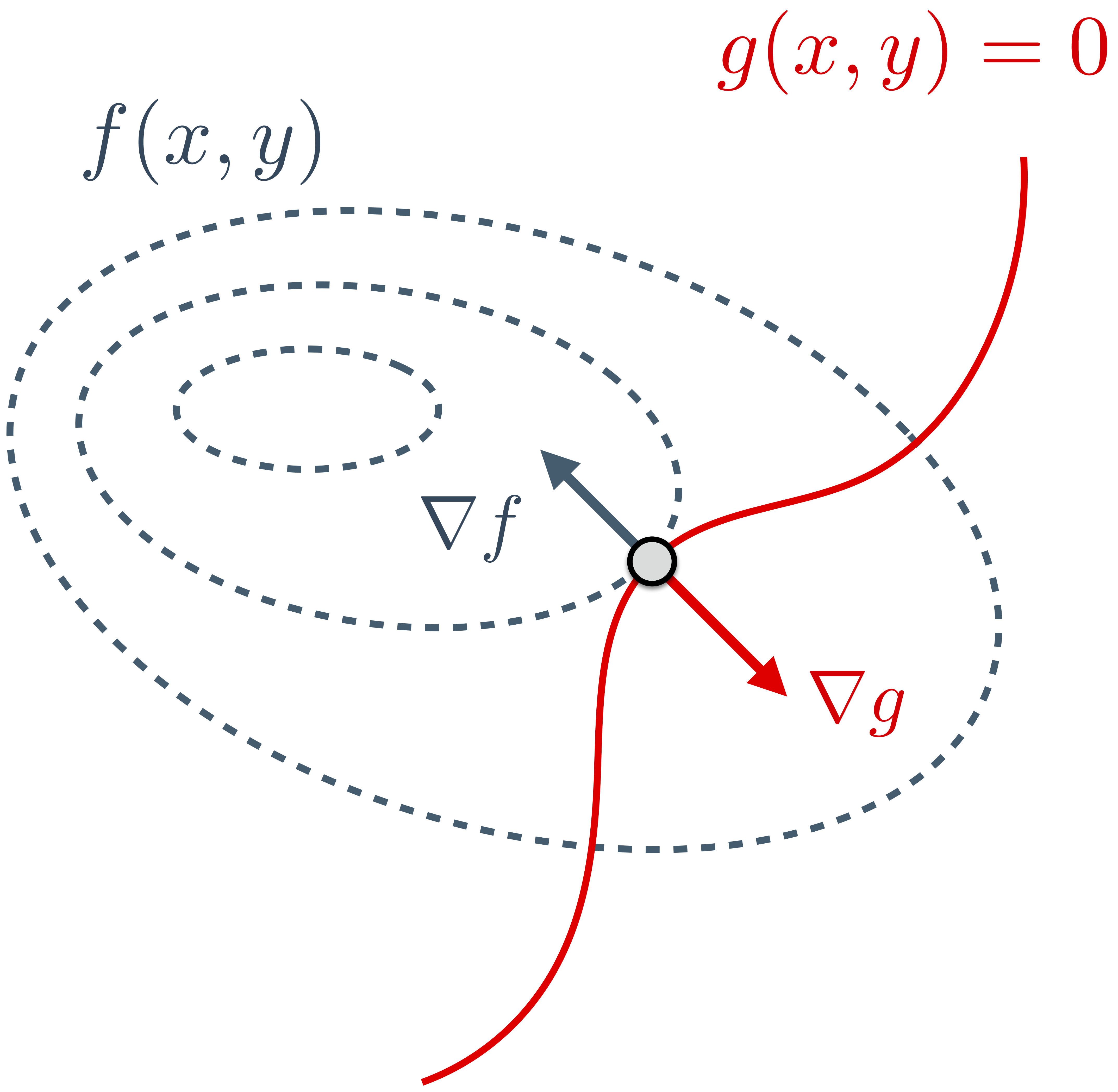
$$g(x^*, y^*) = 0$$



Optimality Conditions

- Necessary condition: gradients $\nabla f, \ \nabla g$ are linearly dependent

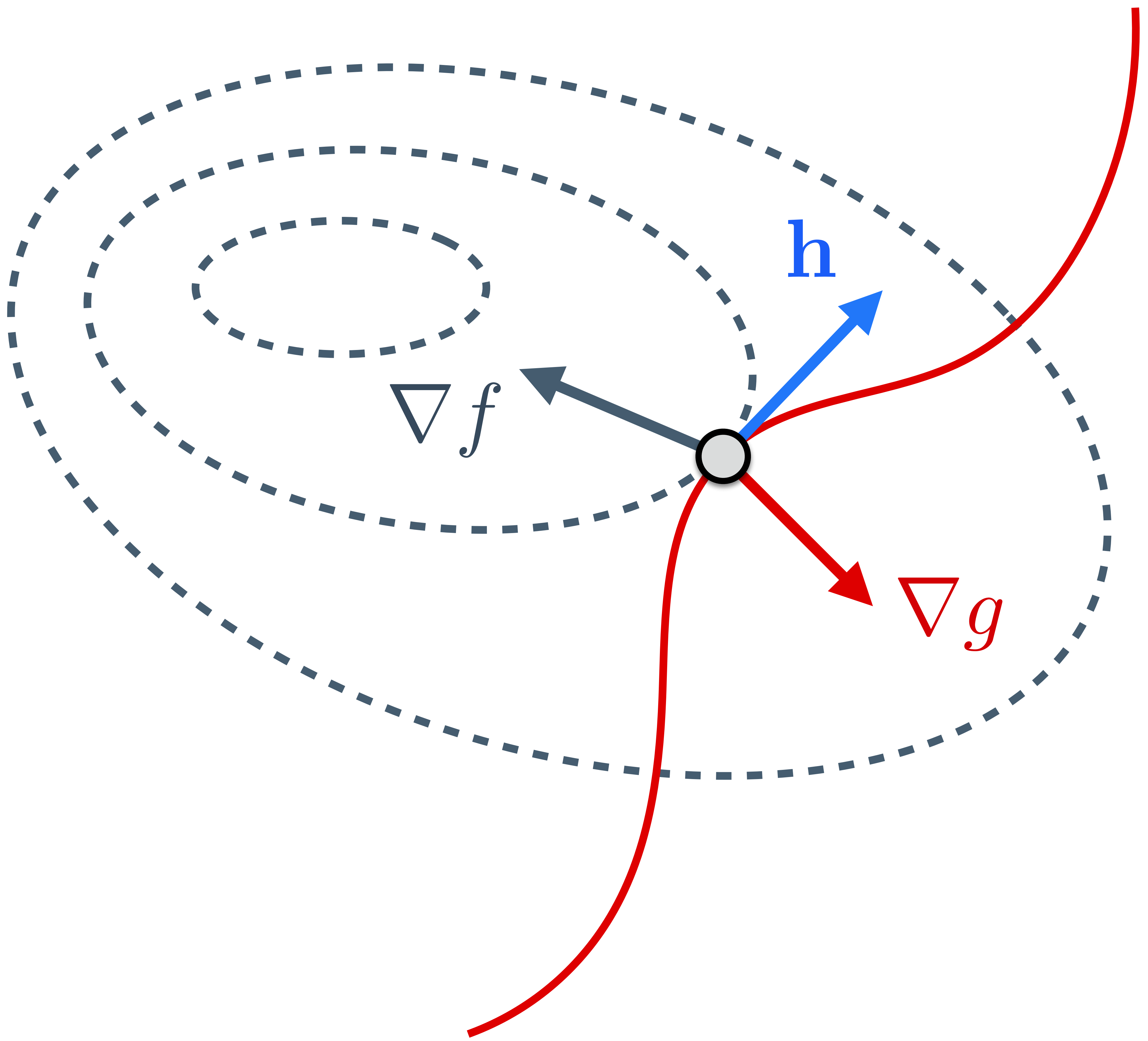
$$\exists \ \lambda : \quad \nabla f(x^*, y^*) = \lambda \nabla g(x^*, y^*)$$



Optimality Conditions

- Proof by contradiction:
  - Suppose $\nabla f, \nabla g$ linearly independent
  - There exists $\mathbf{h}$ such that

$$\nabla f \cdot \mathbf{h} < 0, \quad \nabla g \cdot \mathbf{h} = 0$$



Optimality Conditions

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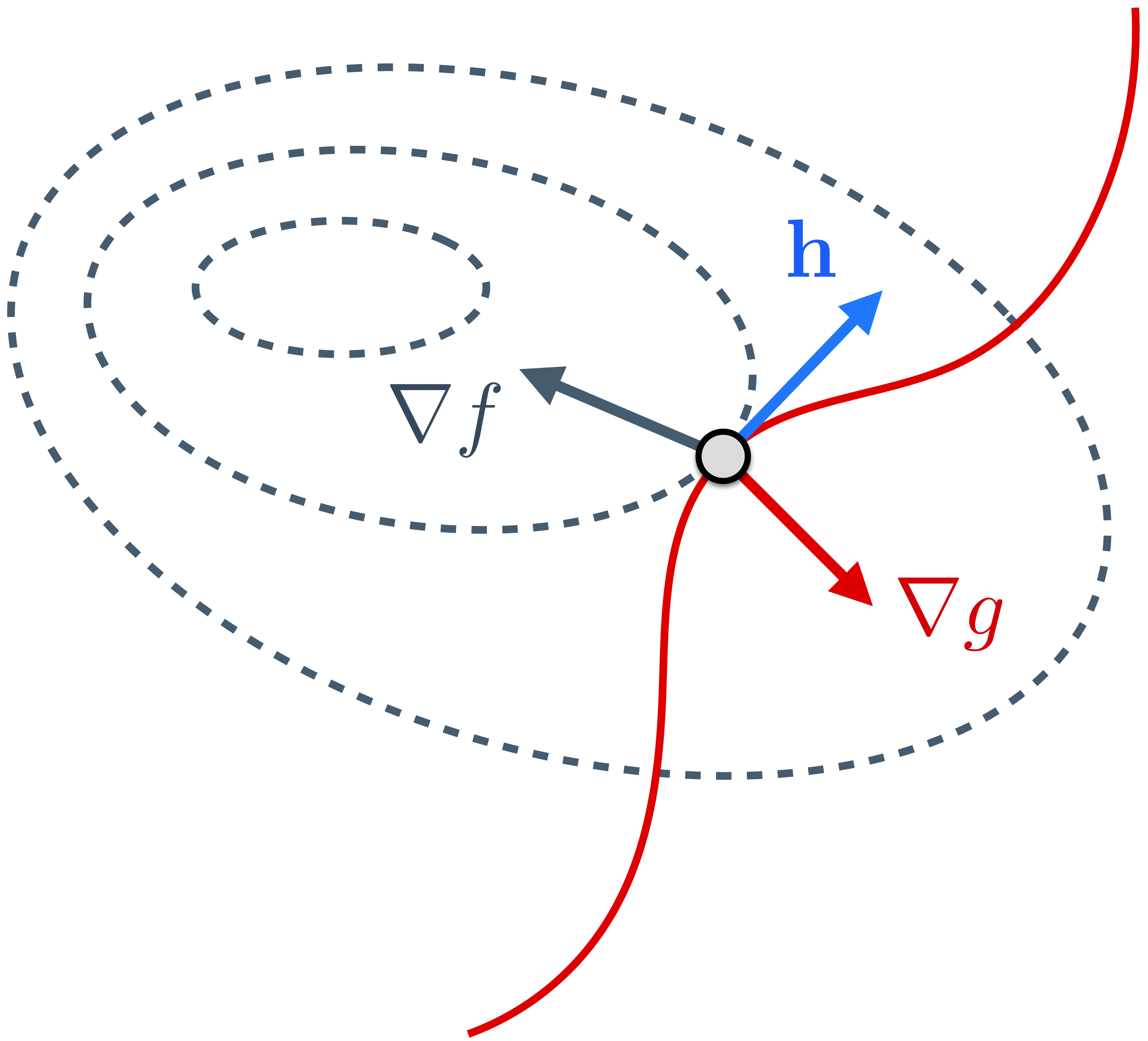
$$\nabla f \cdot \mathbf{h} < 0, \quad \nabla g \cdot \mathbf{h} = 0$$

  - Infinitesimal change along $\mathbf{h}$ satisfies constraint but decreases f



High-Dimensional Case

- For $f, g_1, \ldots, g_m : \mathbb{R}^n \mapsto \mathbb{R}$

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) = 0 \ , \quad i = 1, \ldots, m \end{aligned}$$

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- Necessary condition for solution $\mathbf{x}^*$: linear dependence of $\nabla f, \nabla g_1, \ldots, \nabla g_m$

$$\exists \lambda_1, \ldots, \lambda_m : \quad \nabla f(\mathbf{x}^*) = \sum_{I=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*)$$

Method of Lagrange Multipliers

- Lagrangian for equality-constrained optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \quad f(\mathbf{x})$$
$$\text{s.t.} \quad g_i(\mathbf{x}) = 0\,, \quad i = 1, \ldots, m$$

$$\Downarrow$$

$$\mathcal{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

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Lagrange multipliers

Method of Lagrange Multipliers

- Solution $\mathbf{x}^*$ corresponds to stationary point of Lagrangian

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$$\Downarrow$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = g_i(\mathbf{x}^*) = 0 \qquad \text{(feasibility)}$$

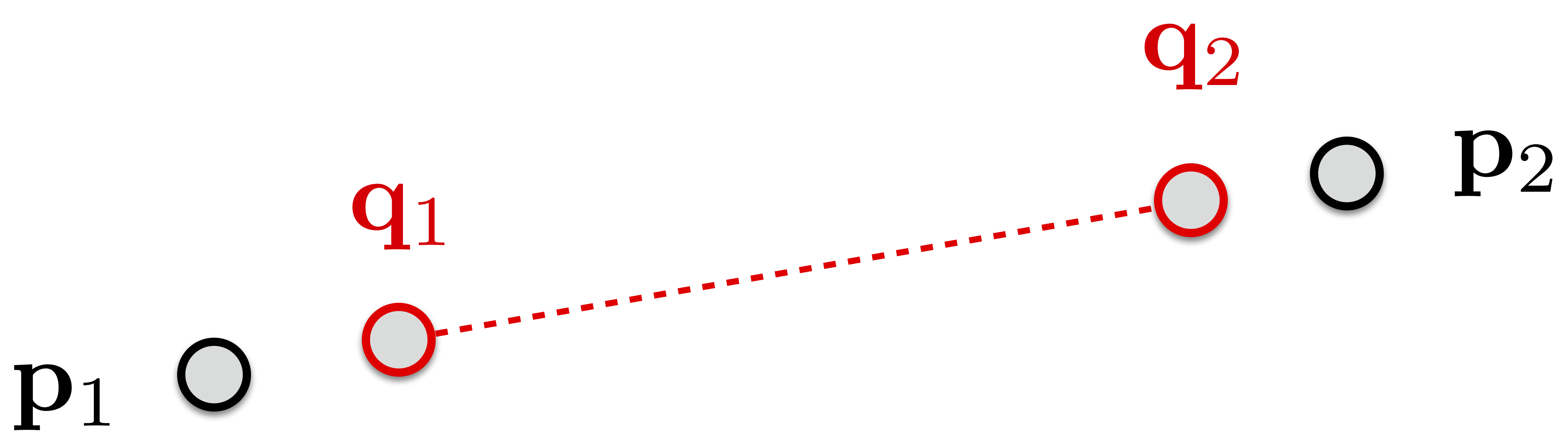
$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = \mathbf{0} \qquad \text{(linear dependence of gradients)}$$

Method of Lagrange Multipliers

- Useful tool for deriving closed-form solutions
  - Example: given $\mathbf{p}_1, \ldots, \mathbf{p}_m \in \mathbb{R}^3$, find closest points $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy certain geometric constraints

Method of Lagrange Multipliers

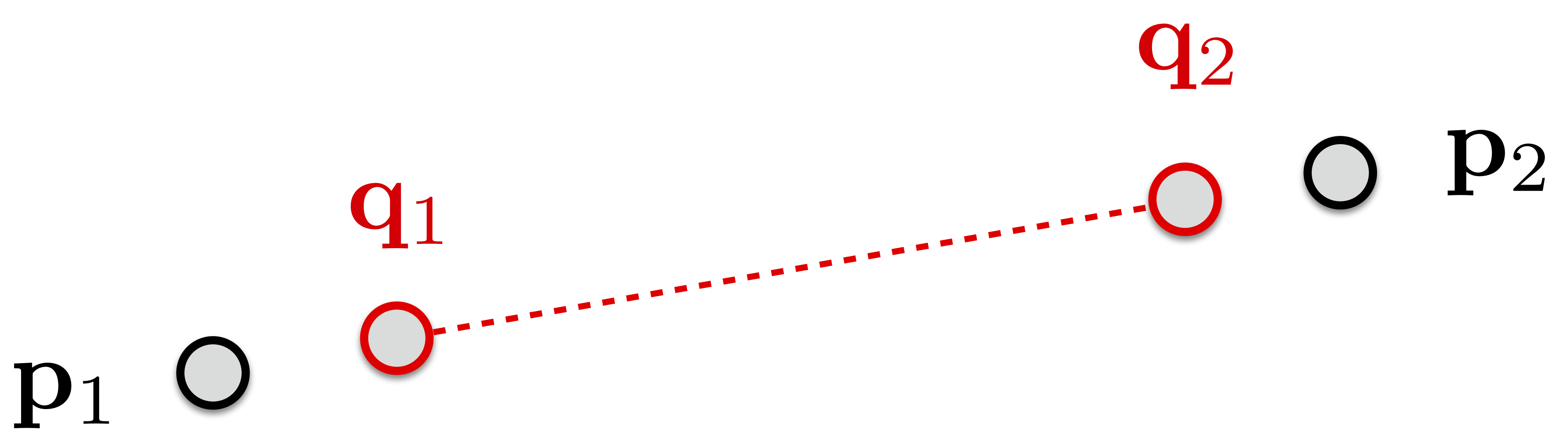
- Given points $\mathbf{p}_1, \mathbf{p}_2$, find closest points $\mathbf{q}_1, \mathbf{q}_2$ such that $\|\mathbf{q}_1 - \mathbf{q}_2\| = L$



Method of Lagrange Multipliers

- Optimization problem

$$\begin{aligned} \min_{\mathbf{q}_1,\mathbf{q}_2} \quad & \|\mathbf{p}_1 - \mathbf{q}_1\|^2 + \|\mathbf{p}_2 - \mathbf{q}_2\|^2 \\ \text{s.t.} \quad & \|\mathbf{q}_1 - \mathbf{q}_2\|^2 = L^2 \end{aligned}$$



Method of Lagrange Multipliers

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$$\Downarrow$$

$$\mathcal{L}(\mathbf{q}_1, \mathbf{q}_2, \lambda) = \|\mathbf{p}_1 - \mathbf{q}_1\|^2 + \|\mathbf{p}_2 - \mathbf{q}_2\|^2 - \lambda(\|\mathbf{q}_1 - \mathbf{q}_2\|^2 - L^2)$$

Method of Lagrange Multipliers

$$\mathcal{L}(\mathbf{q}_1, \mathbf{q}_2, \lambda) = \|\mathbf{p}_1 - \mathbf{q}_1\|^2 + \|\mathbf{p_2} - \mathbf{q_2}\|^2 - \lambda(\|\mathbf{q}_1 - \mathbf{q_2}\|^2 - L^2)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}_1} = 2[\mathbf{q_1} - \mathbf{p}_1 - \lambda(\mathbf{q_1} - \mathbf{q_2})] = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}_2} = 2[\mathbf{q}_2 - \mathbf{p}_2 - \lambda(\mathbf{q_2} - \mathbf{q_1})] = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = L^2 - \|\mathbf{q}_1 - \mathbf{q}_2\|^2 = \mathbf{0}$$

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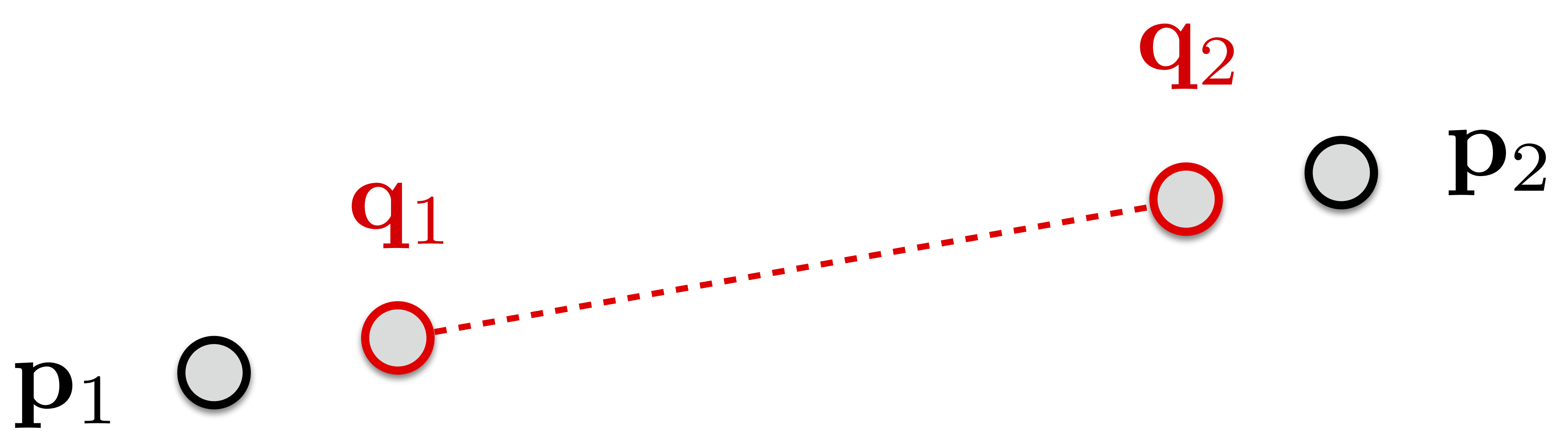
$$\frac{\partial \mathcal{L}}{\partial \lambda} = L^2 - \|\mathbf{q}_1 - \mathbf{q}_2\|^2 = \mathbf{0}$$

1. $\mathbf{p_1}, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2$ colinear
2. $(\mathbf{p}_1 + \mathbf{p}_2)/2 = (\mathbf{q}_2 + \mathbf{q}_2)/2$



Numerical Solver

- Many constrained optimization problems have no closed-form solution
  - Method of Lagrange multipliers amounts to solving nonlinear equations: sometimes challenging even for numerical solvers

Numerical Solver

- Many constrained optimization problems have no closed-form solution
  - Method of Lagrange multipliers amounts to solving nonlinear equations: sometimes challenging even for numerical solvers
  - More sophisticated numerical solvers needed

Penalty Method

- Convert constrained optimization into unconstrained optimization

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) = 0 \ , \quad i = 1, \ldots, m \end{aligned}$$

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$$\Downarrow$$

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \mu \sum_{i=1}^{m} [g_i(\mathbf{x})]^2$$

Penalty Method

- Convert constrained optimization into unconstrained optimization

$$
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\text{s.t.} \quad & g_i(\mathbf{x}) = 0 \ , \quad i = 1, \ldots, m
\end{aligned}
$$

Penalizes constraint violation

$$
\min_{\mathbf{x}} \quad f(\mathbf{x}) + \boxed{\mu} \sum_{i=1}^{m} \boxed{[g_i(\mathbf{x})]^2}
$$

Penalty parameter

Penalty Method

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + \mu \sum_{i=1}^{m} [g_i(\mathbf{x})]^2$$

- Can be solved using standard unconstrained solvers
  - unconstrained solution approaches true solution when $\mu \to +\infty$
  - ill-conditioned when μ becomes too large

Augmented Lagrangian Method

- Solve another problem with the same solution

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & g_i(\mathbf{x}) = 0\,, \quad i = 1, \ldots, m
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}) + \frac{\mu}{2}\sum_{i=1}^{m}[g_i(\mathbf{x})]^2 \\
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- Augmented Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \frac{\mu}{2}\sum_{i=1}^{m}[g_i(\mathbf{x})]^2 - \sum_{i=1}^{m}\lambda_i g_i(\mathbf{x})$$

Augmented Lagrangian Method

- Find a saddle point of the augmented Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \frac{\mu}{2} \sum_{i=1}^{m} [g_i(\mathbf{x})]^2 - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

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Iterative solver:

1. Fix $\{\lambda_i\}$, minimize $\mathcal{L}$ over $\mathbf{x}$: $\mathbf{x}^{k+1} = \operatorname{argmin} \mathcal{L}(\mathbf{x}, \lambda_1^k, \ldots, \lambda_m^k)$
2. Update $\{\lambda_i\}$: $\lambda_i^{k+1} = \lambda_i^k - \mu g_i(\mathbf{x}_i^{k+1})$

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$$\mathcal{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \frac{\mu}{2} \sum_{i=1}^{m} [g_i(\mathbf{x})]^2 - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

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2. Update $\{\lambda_i\}$: $\lambda_i^{k+1} = \lambda_i^k - \mu g_i(\mathbf{x}_i^{k+1})$

Benefit over penalty method: does not require very large μ , avoids ill-conditioning

General Constrained Optimization

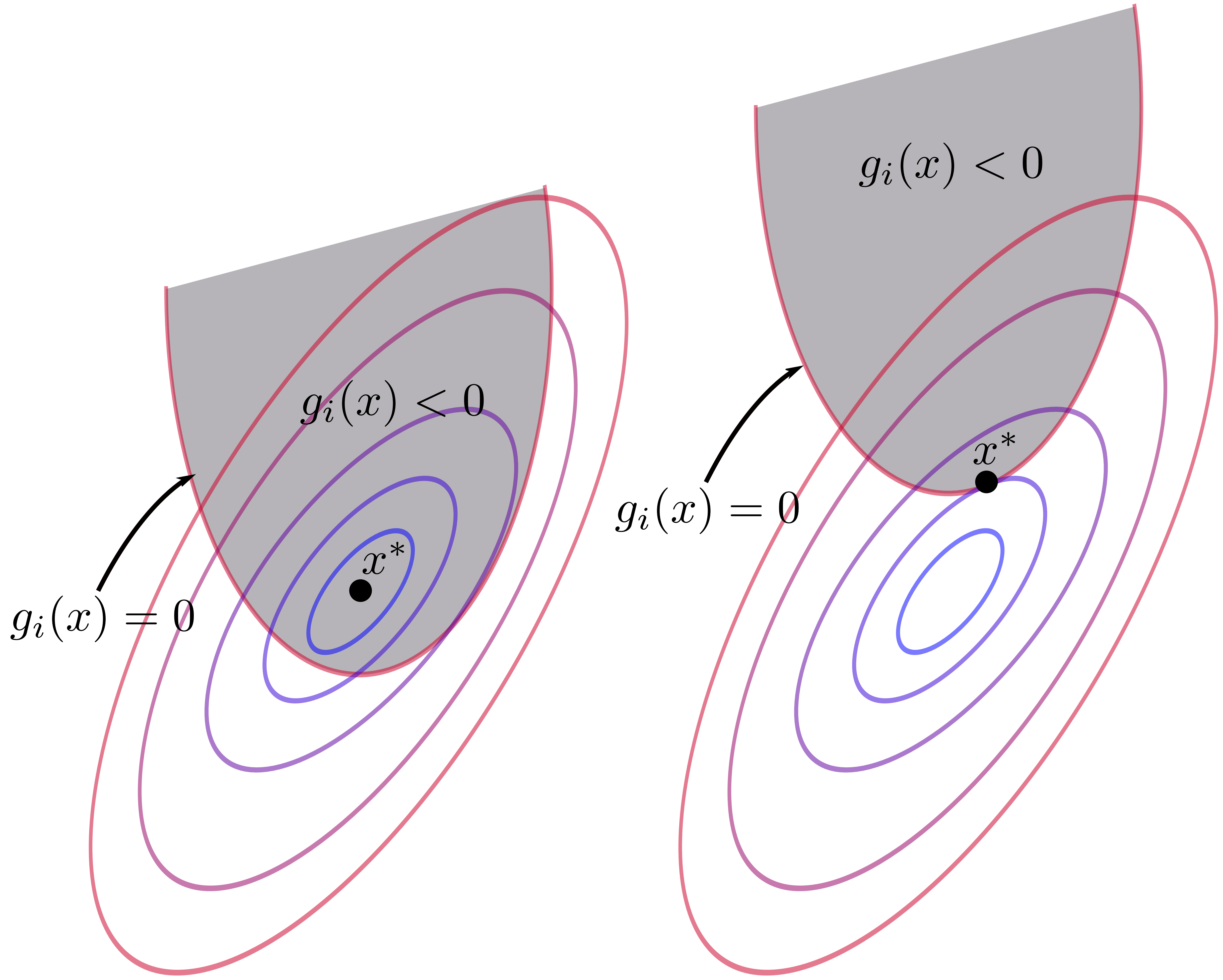
Inequality constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l \end{aligned}$$

Equality constraints

Inequality Constraints

- Inequality constraints only affect the solution when they becomes equality constraints)



Karush–Kuhn–Tucker Conditions

Stationarity

$$-\nabla f(x^*) = \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*),$$

Primal feasibility

$$g_i(x^*) \le 0, \text{ for } i = 1, \ldots, m$$
$$h_j(x^*) = 0, \text{ for } j = 1, \ldots, \ell$$

Dual feasibility

$$\mu_i \ge 0, \text{ for } i = 1, \ldots, m$$

Complementary slackness

$$\mu_i g_i(x^*) = 0, \text{ for } \ i = 1, \ldots, m.$$