Local-Global Solvers

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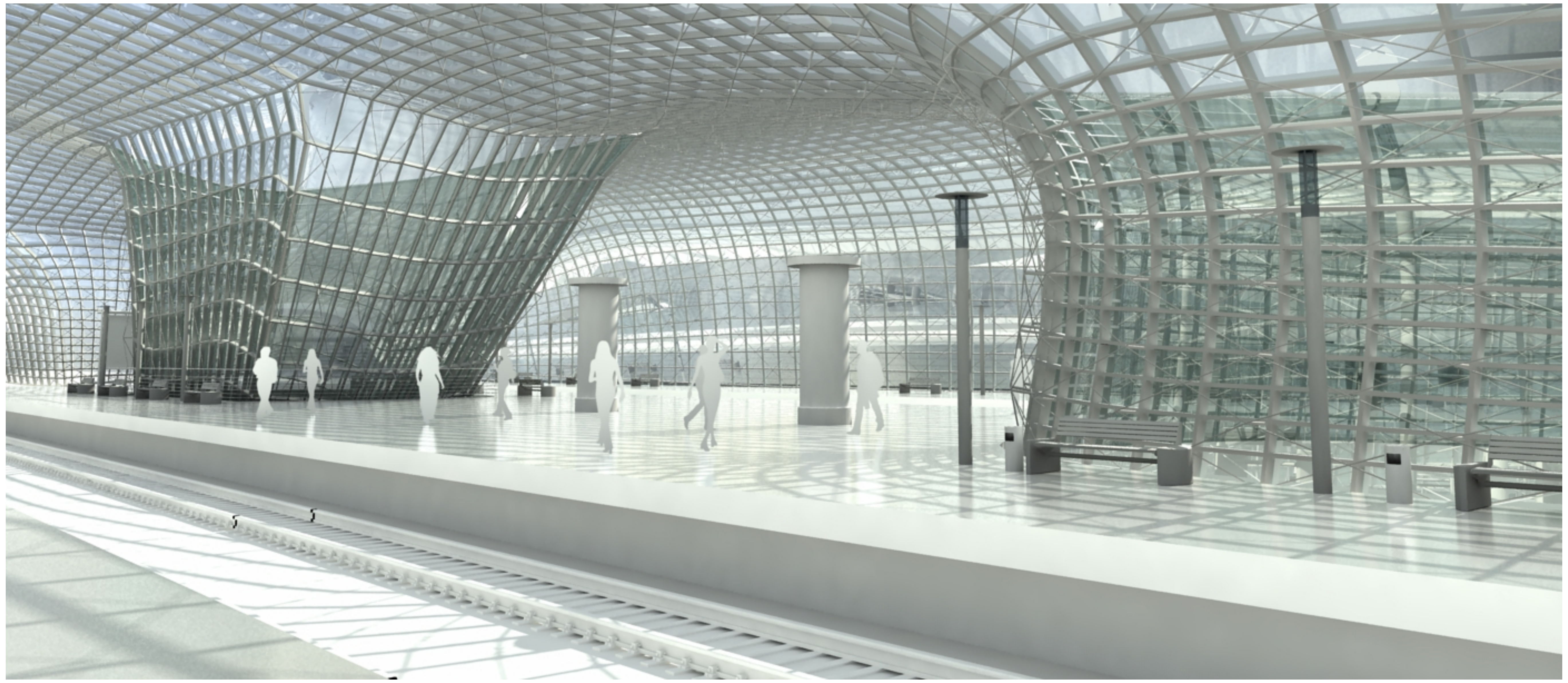
Motivation

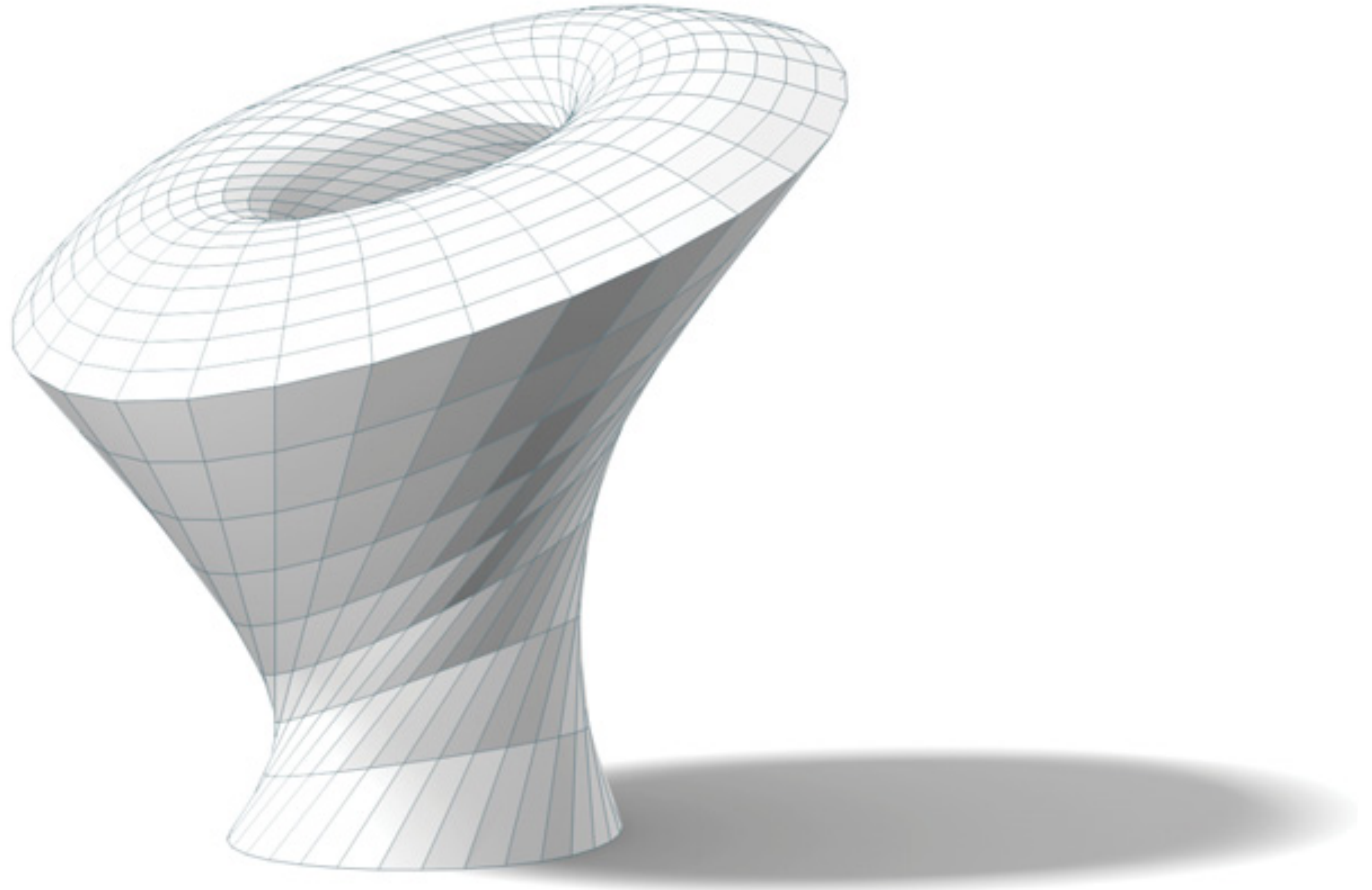
- Shape design subject to geometric constraints



Planar quad meshes
[Liu et al. 2006]

Optimization

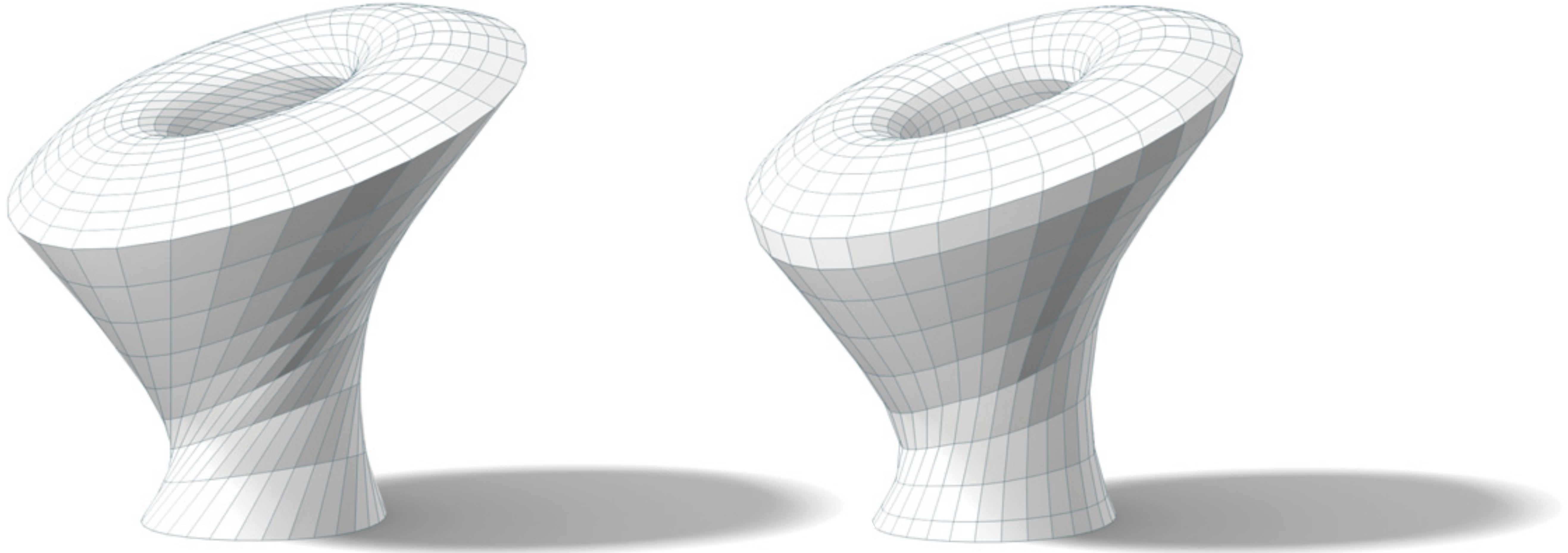
- Input
  - A shape represented with vertex positions (e.g. meshes)
  - Geometric constraints (equality and/or inequality)



Input model

Optimization

- Input
  - A shape represented with vertex positions (e.g. meshes)
  - Geometric constraints (equality and/or inequality)
- Output: optimized shape that best satisfies the constraints

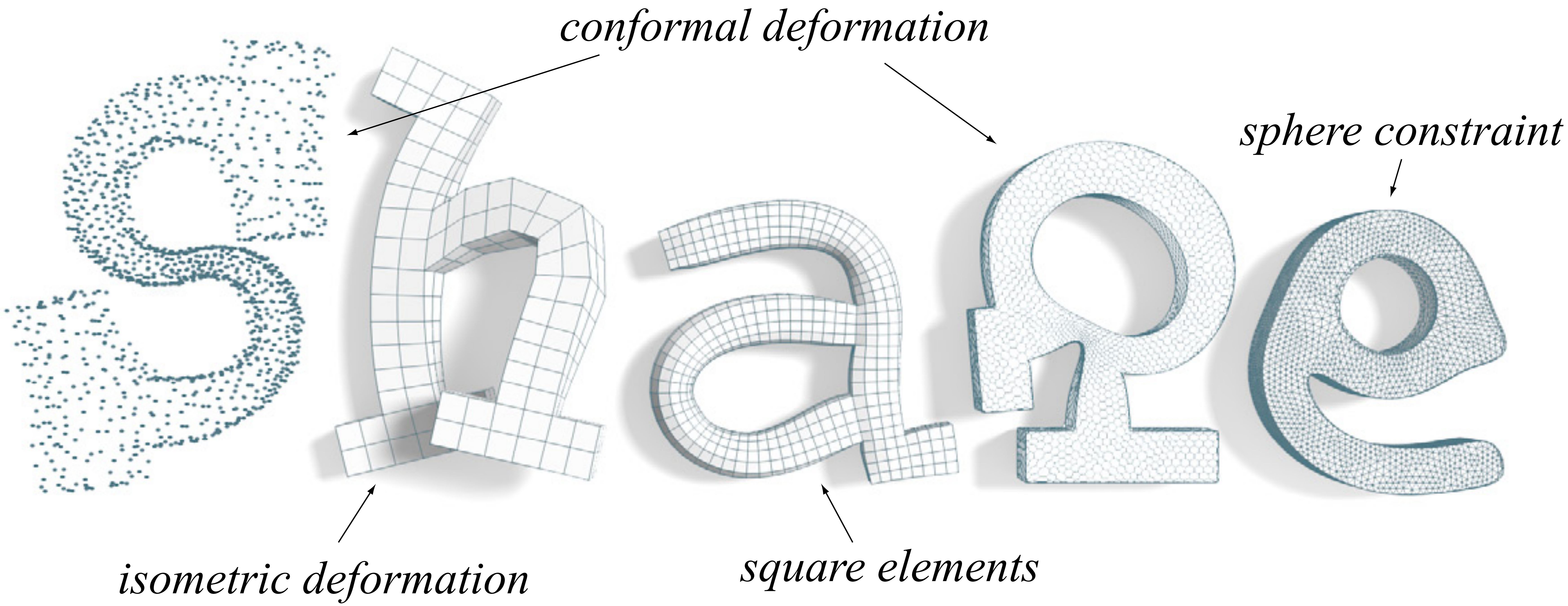


Input model *Optimized PQ mesh* [Bouaziz et al. 2012]

Optimization

- Desirable solver
  - Fast enough for interactive applications
  - Stable
  - Easy to implement

Shape-Up



Bouaziz et al., Shape-Up: Shaping Discrete Geometry with Projections. SGP 2012.

Geometric Perspective

- Vertex positions represented as a high-dimensional vector

$$\mathrm{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_n] \in \mathbb{R}^{3n}$$

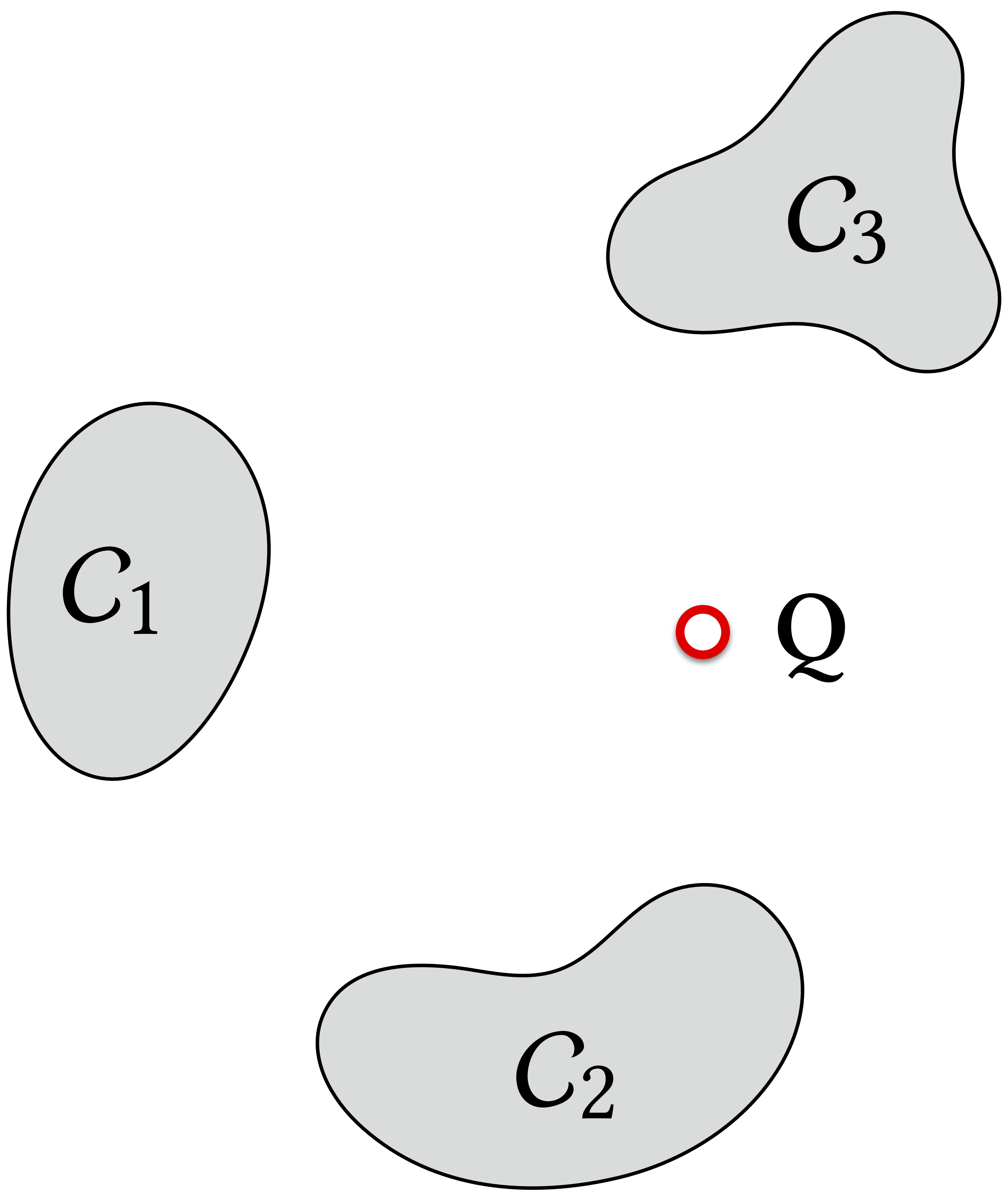
Q

Geometric Perspective

- Vertex positions represented as a high-dimensional vector

$$\mathrm{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_n] \in \mathbb{R}^{3n}$$

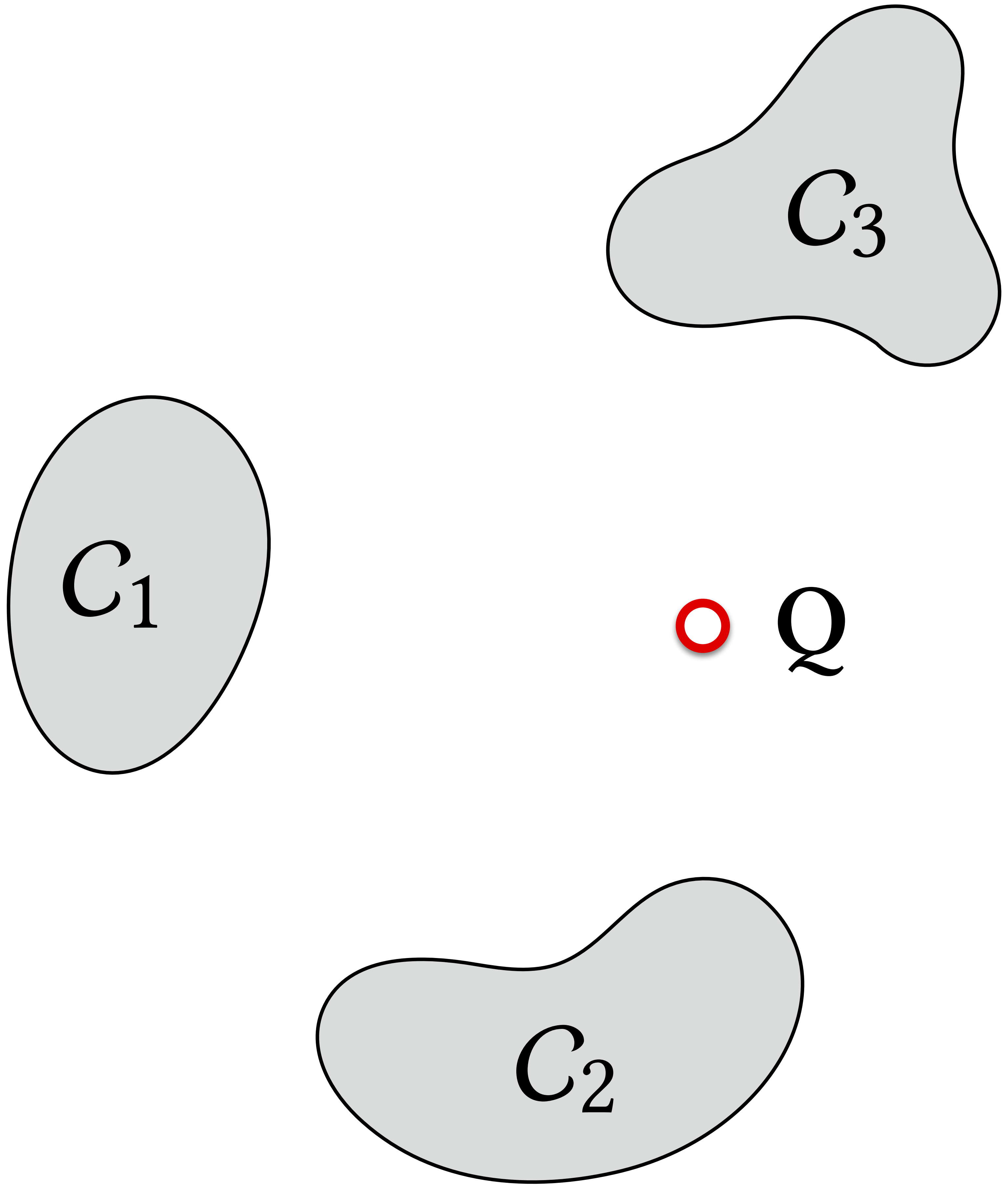
- Each constraint defines a feasible set $\mathcal{C}_i$



Shape-Up

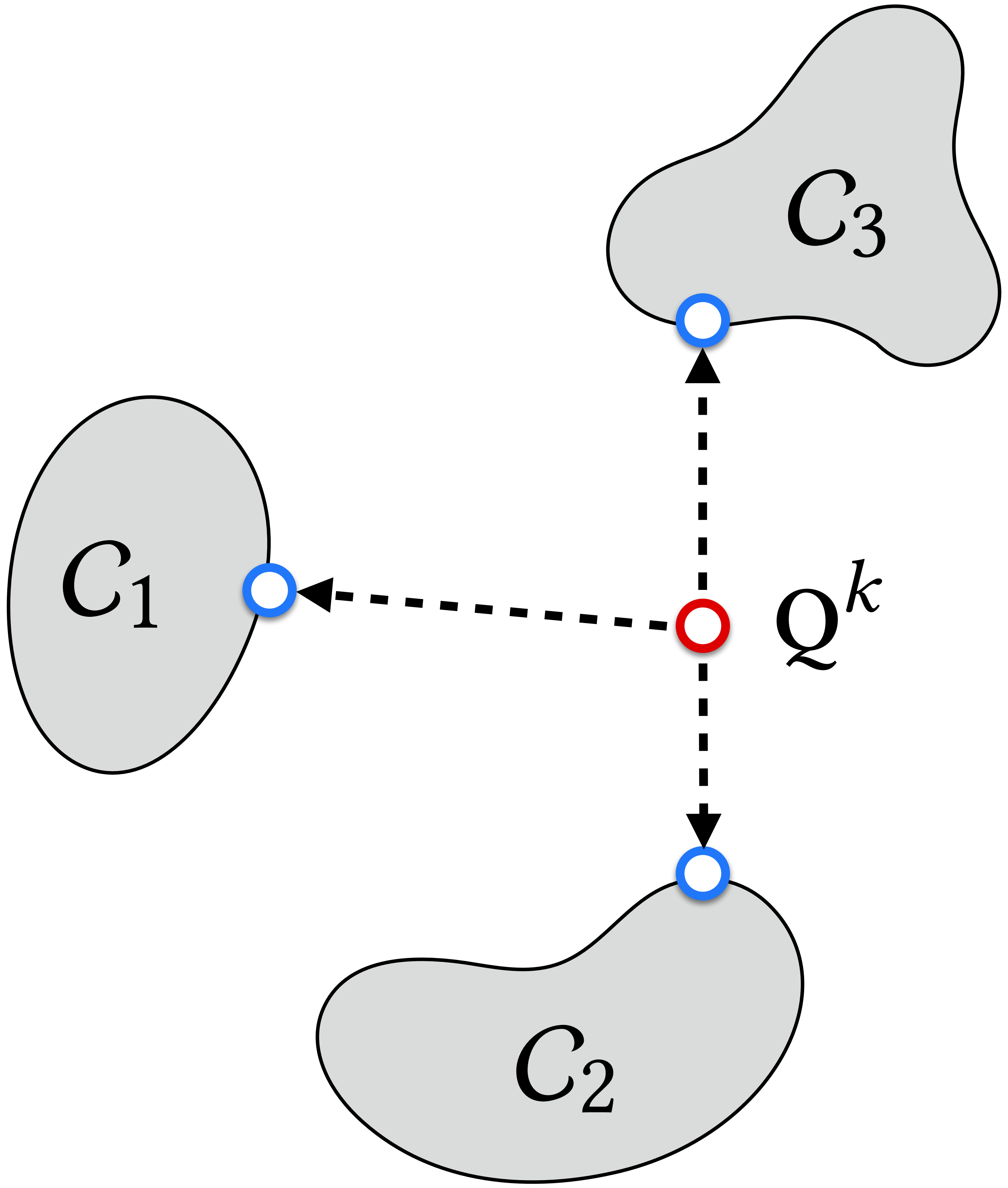
- Optimization: find a point as close as possible to the feasible sets [Bouaziz et al. 2012]

$$\min_{Q} \sum_i w_i \, \mathrm{dist}^2(Q, C_i)$$



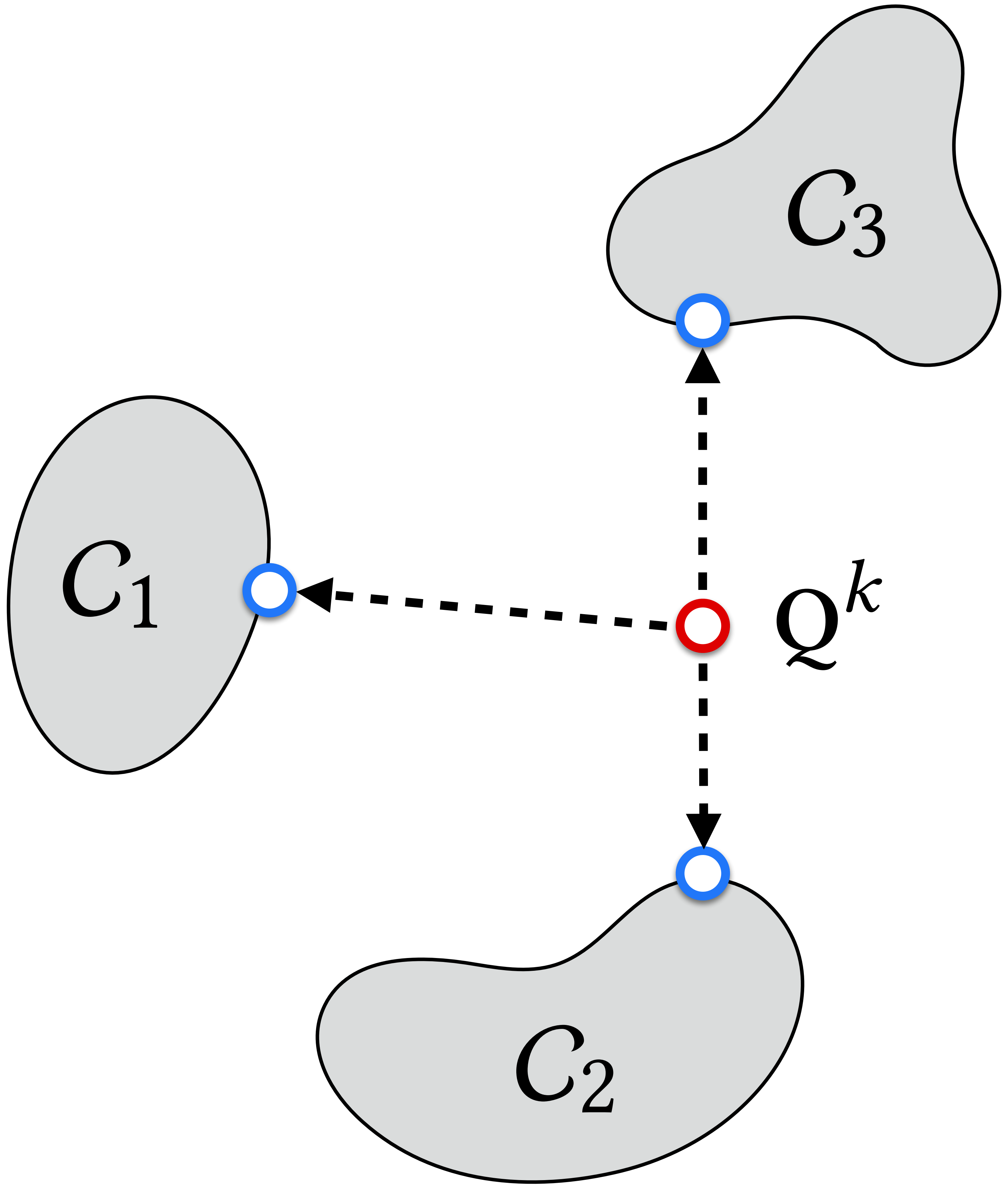
Shape-Up: Alternating Minimization

- Local step: project $\mathbf{Q}$ to feasible sets



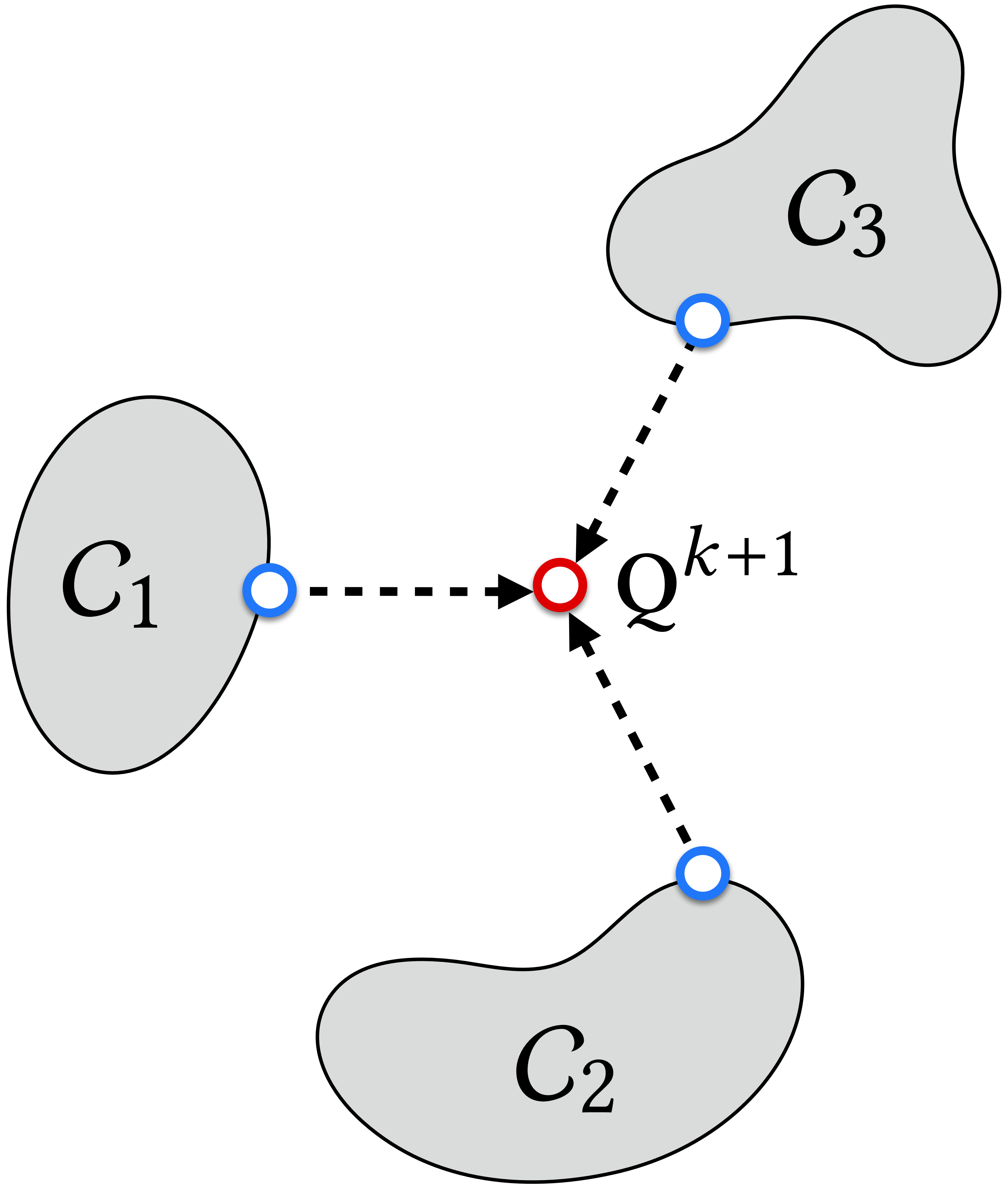
Shape-Up: Alternating Minimization

- Local step: project $\mathbf{Q}$ to feasible sets
  - Correct the shape for individual constraints
  - Closed-form solutions for many constraints
  - Can be done in parallel



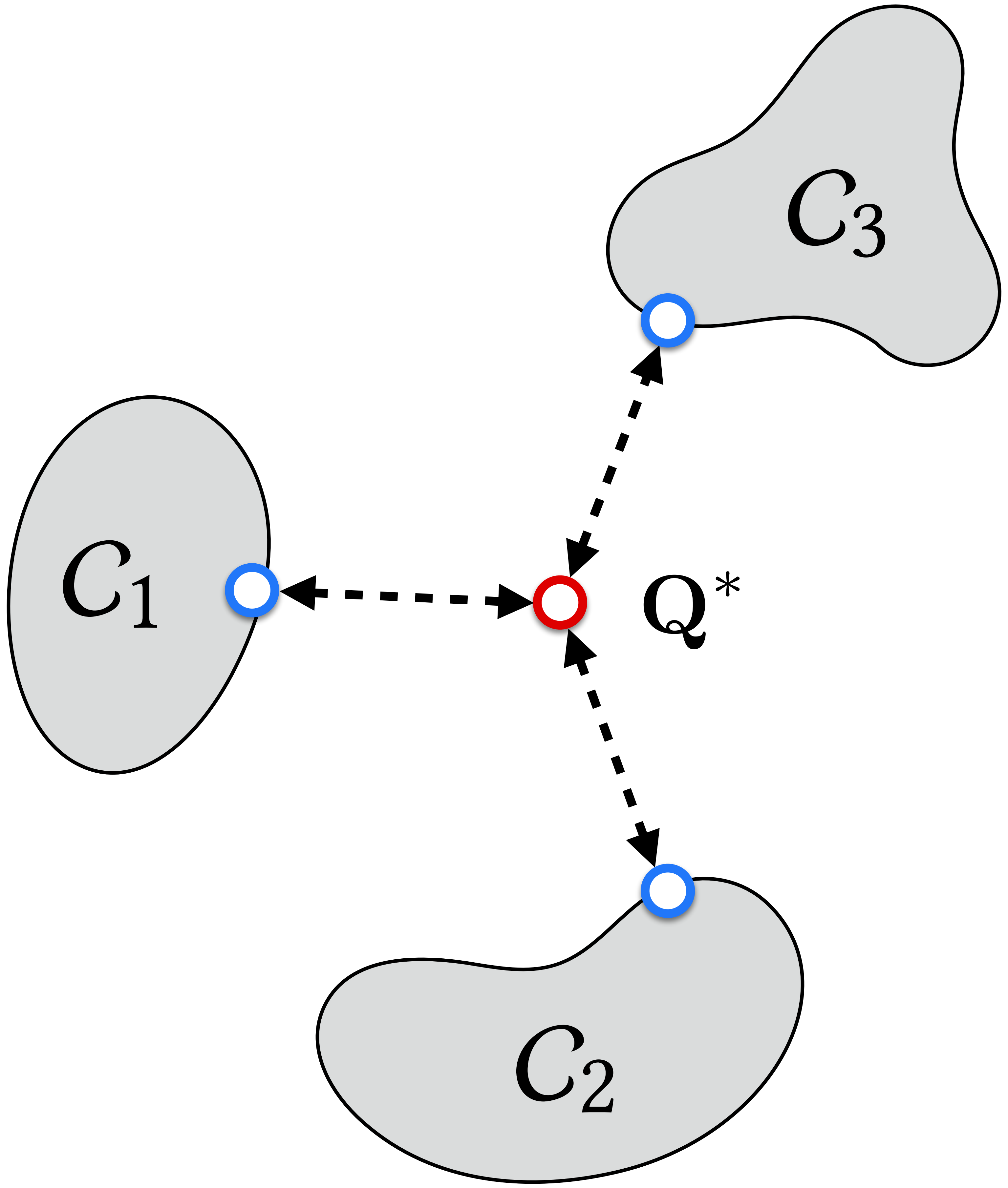
Shape-Up: Alternating Minimization

- Local step: project $\mathbf{Q}$ to feasible sets
- Global step: update $\mathbf{Q}$ by minimizing distance to projections
  - Linear solve with fixed matrix
  - Pre-factorization



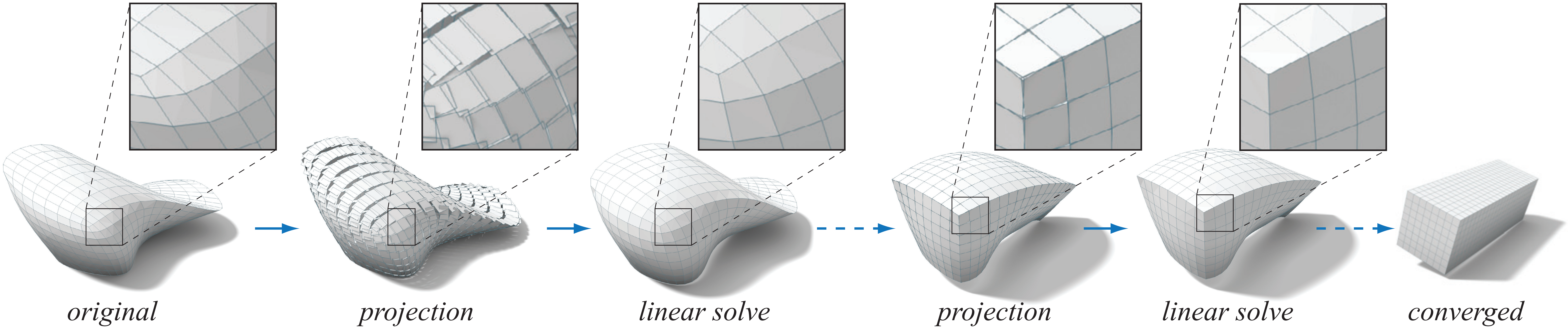
Shape-Up: Alternating Minimization

- Local step: project $\mathbf{Q}$ to feasible sets
- Global step: update $\mathbf{Q}$ by minimizing distance to projections
- Repeat until convergence



Example

- Geometric constraints: square faces



[Bouaziz et al. 2012]

Implementation

- Target energy

$$\min_{\mathbf{Q},\{\mathbf{P}_i\}} \sum_i \frac{w_i}{2}\|\mathbf{A}_i\mathbf{Q}-\mathbf{P}_i\|_F^2+\sigma_i(\mathbf{P}_i).$$

Implementation

- Target energy

$$\min_{\mathbf{Q}, \{\mathbf{P}_i\}} \sum_i \frac{w_i}{2} \|\mathbf{A}_i \mathbf{Q} - \mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

Selection matrix

Auxiliary projection variables

Implementation

- Target energy

$$\min_{\mathbf{Q}, \{\mathbf{P}_i\}} \sum_i \frac{w_i}{2} \|\mathbf{A}_i \mathbf{Q} - \mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

Indicator function:

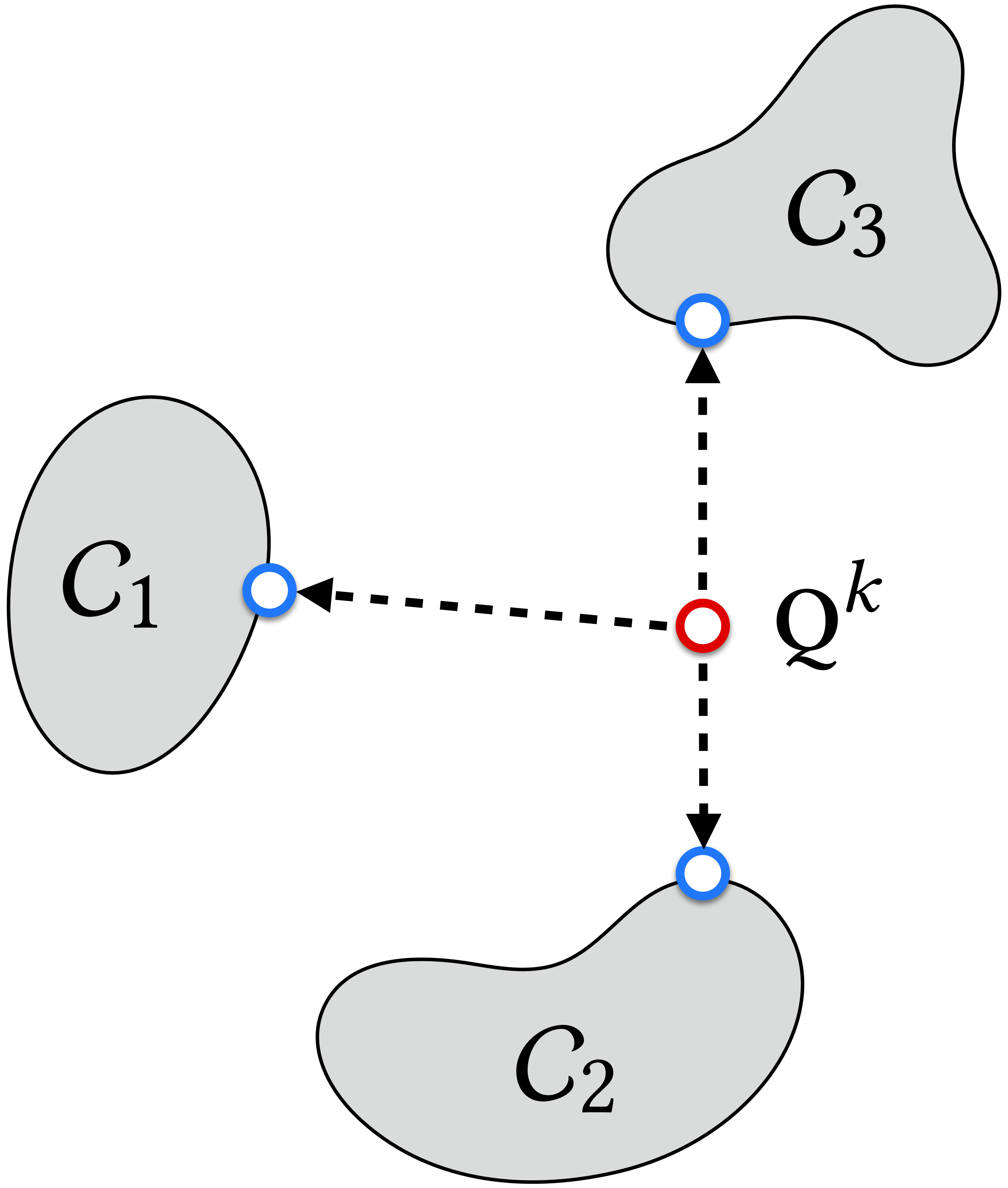
$$\sigma_i(\mathbf{P}_i) = \begin{cases} 0 & \text{if } \mathbf{P}_i \in C_i, \\ +\infty & \text{otherwise.} \end{cases}$$

Implementation

$$\min_{\mathrm{Q}, \{\mathrm{P}_i\}} \sum_i \frac{w_i}{2} \|\mathbf{A}_i \mathbf{Q} - \mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

Alternating Minimization:

- Local step: fix $\mathbf{Q}$, solve for $\mathbf{P}$

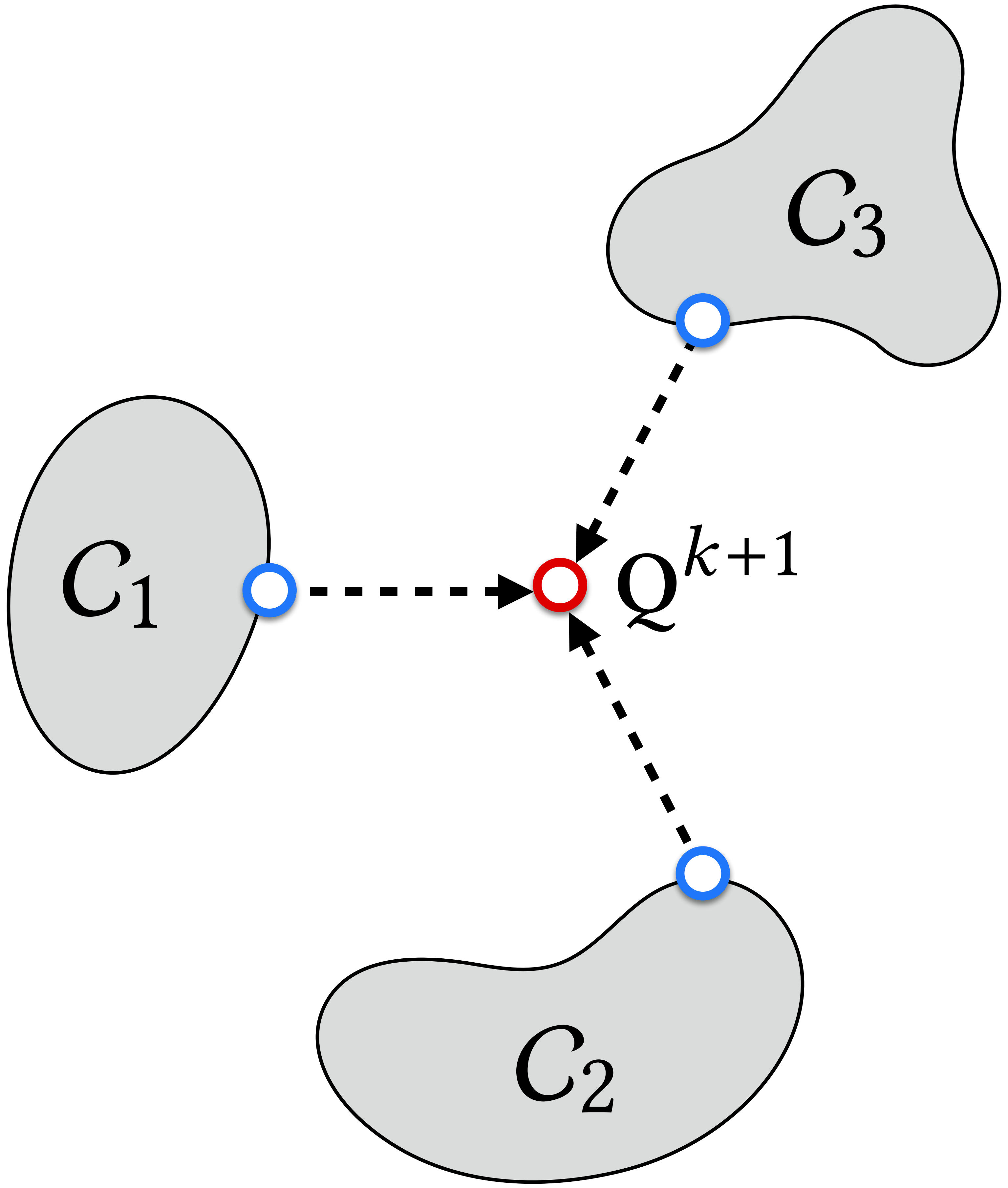


Implementation

$$\min_{\mathbf{Q},\{\mathbf{P}_i\}} \sum_i \frac{w_i}{2} \|\mathbf{A}_i\mathbf{Q} - \mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

Alternating Minimization:

- Local step: fix $\mathbf{Q}$, solve for $\mathbf{P}$
- Global step: fix $\mathbf{P}$, solve for $\mathbf{Q}$

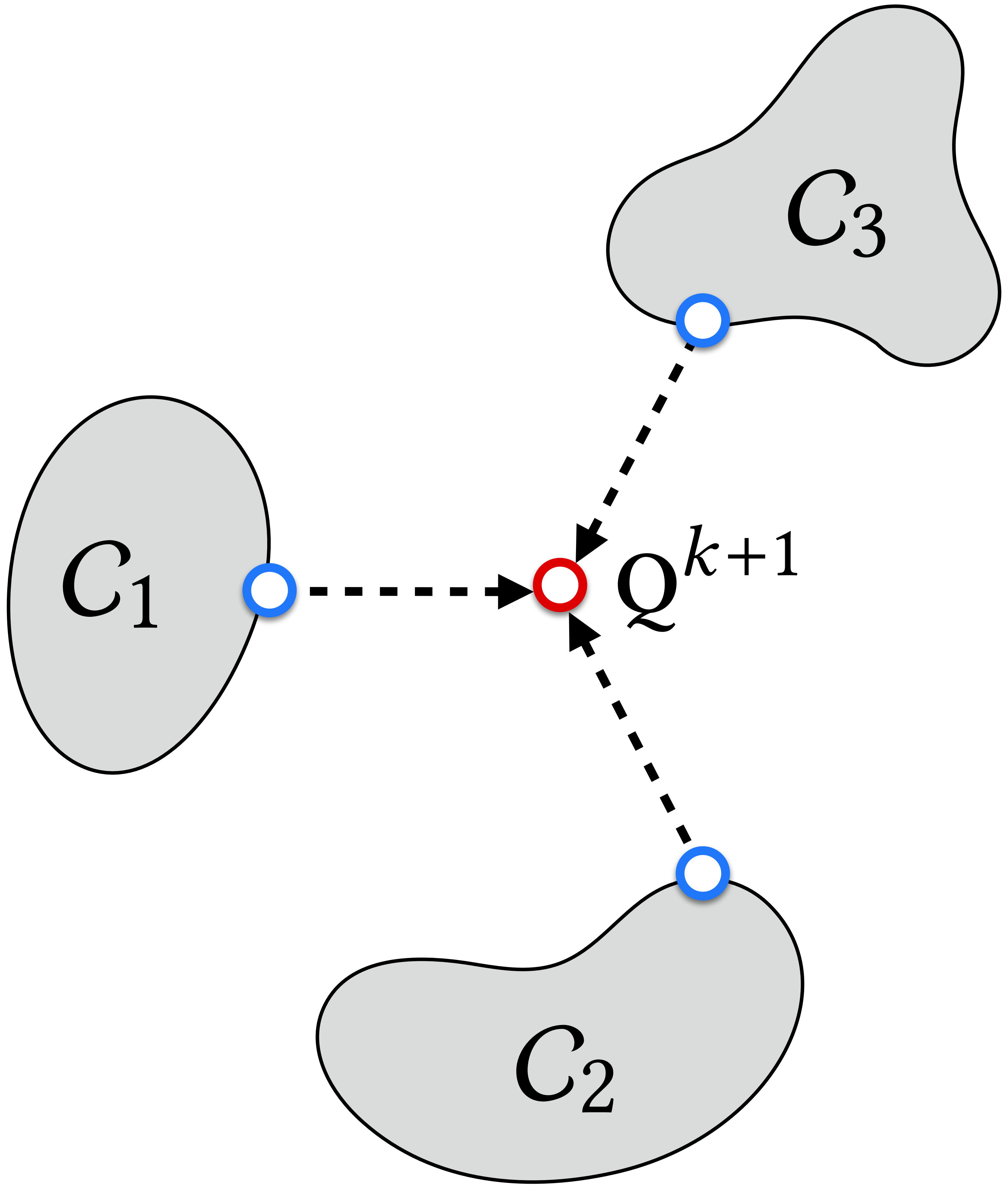


Implementation

$$\min_{\mathrm{Q}, \{\mathrm{P}_i\}} \sum_i \frac{w_i}{2} \|\mathbf{A}_i \mathbf{Q} - \mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

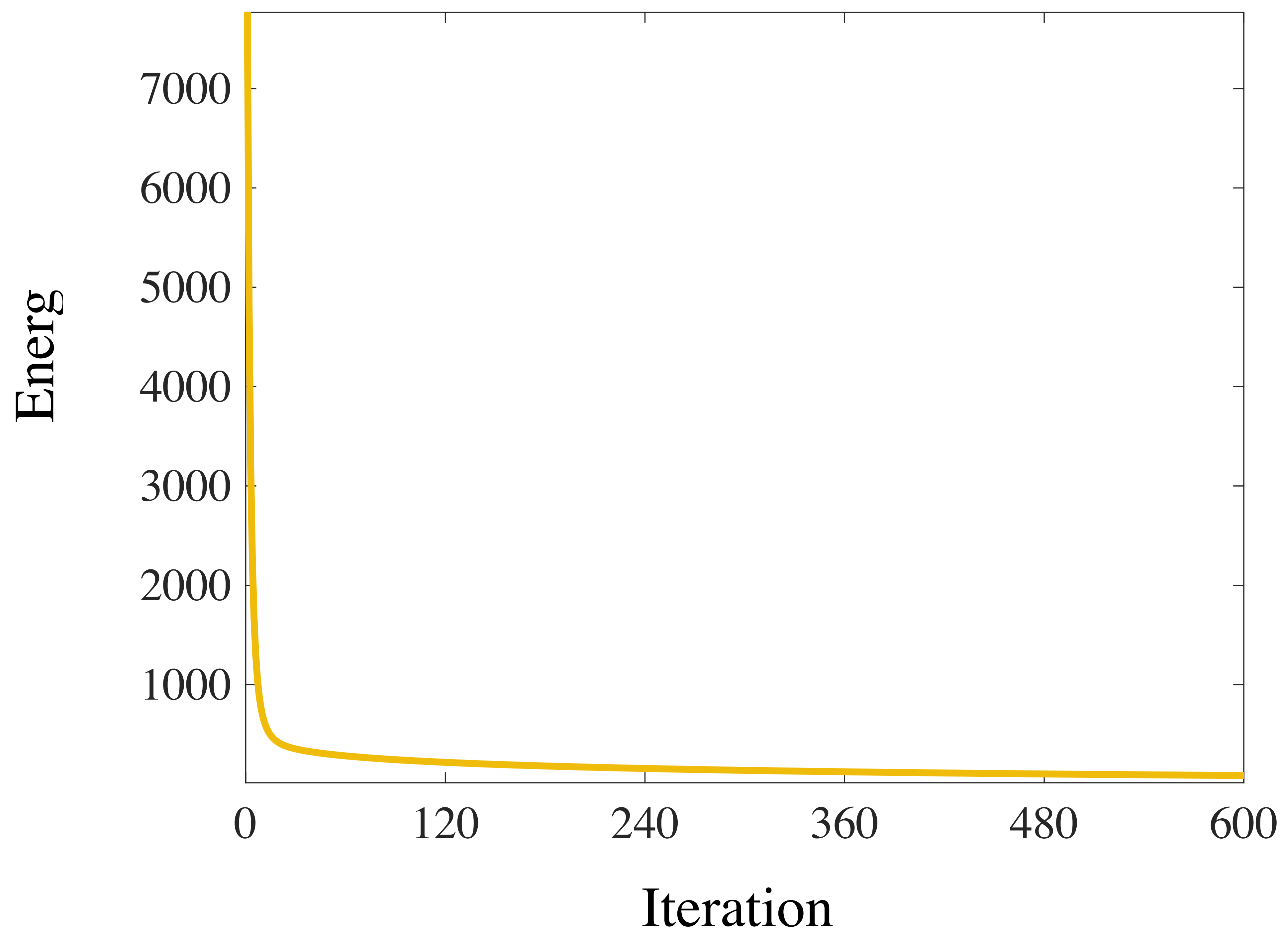
Alternating Minimization:

- Local step: fix $\mathbf{Q}$, solve for $\mathbf{P}$
- Global step: fix $\mathbf{P}$, solve for $\mathbf{Q}$
- Monotonic decrease of energy



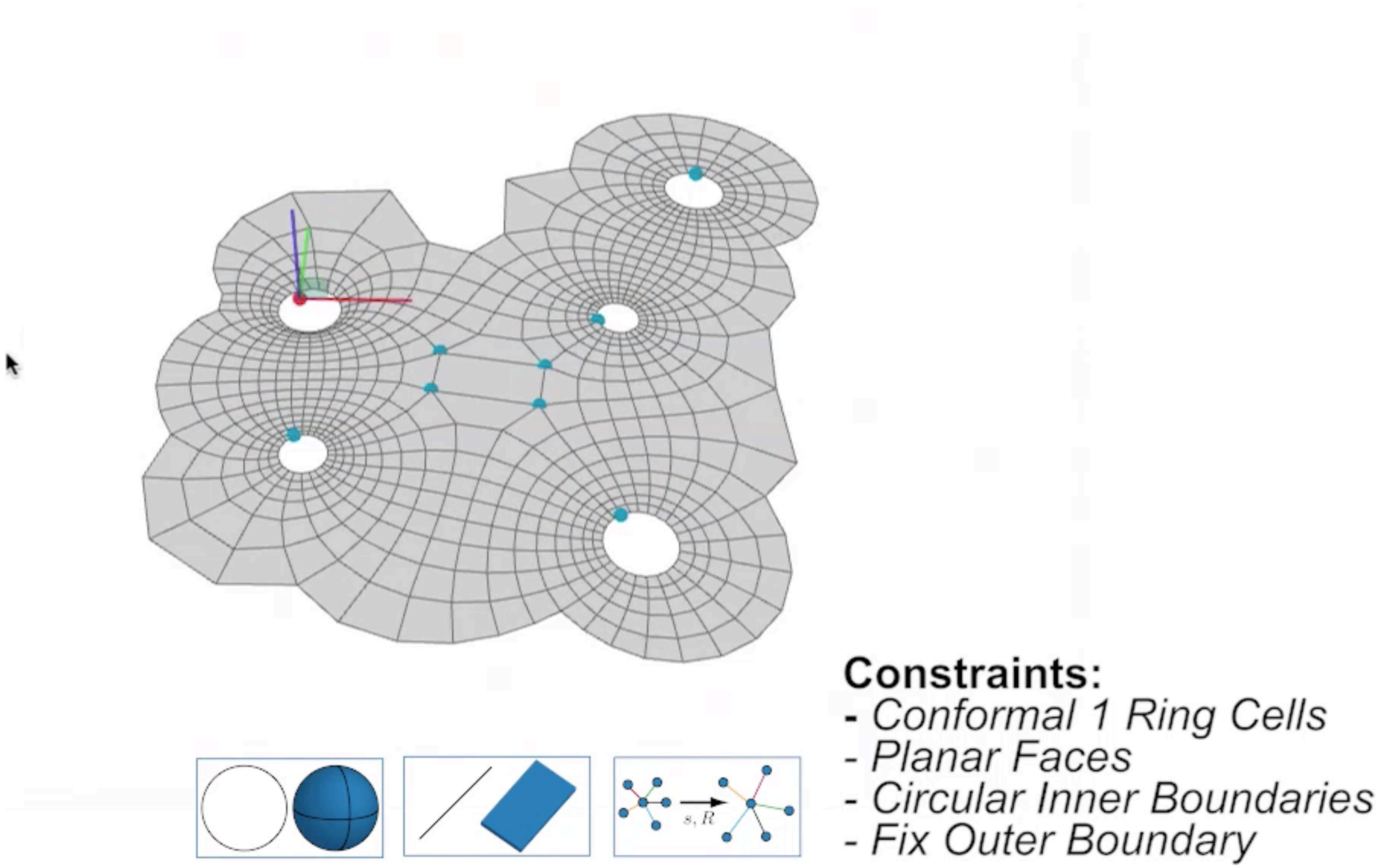
Performance

- Rapid decrease of target energy during initial iterations
  - Suitable for interactive applications that only require approximate solutions
  - Takes much longer time for full convergence



Examples

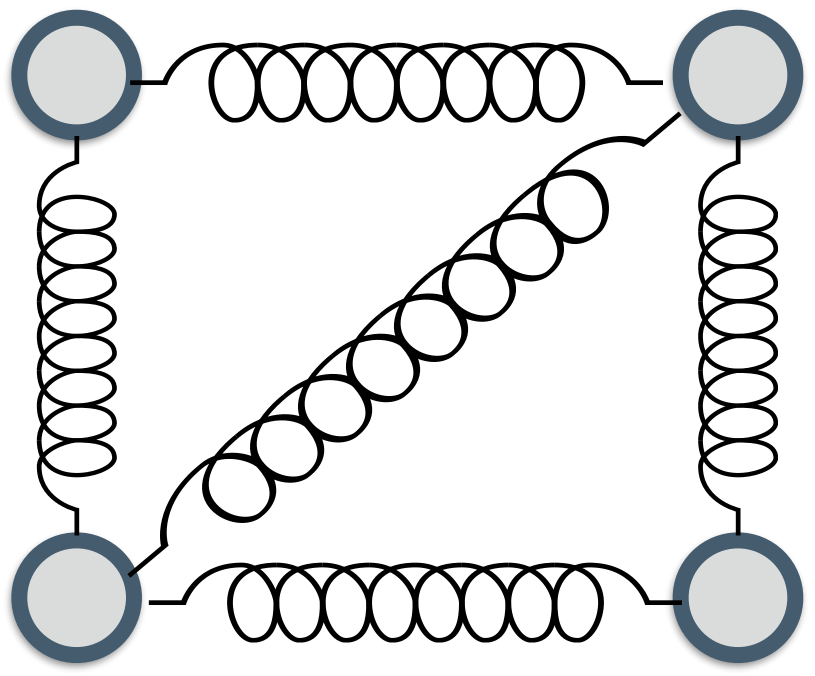
- Interactive deformation with constraints



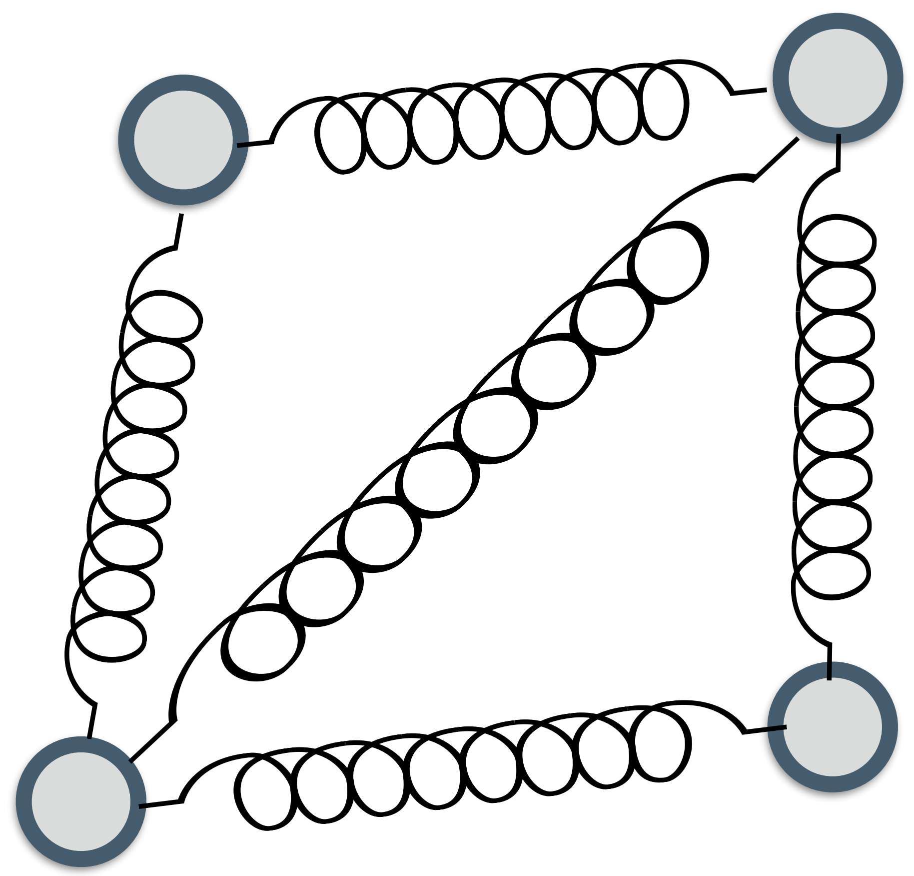
https://www.youtube.com/watch?v=PVdLCWhg7Ic

Physics Simulation

- Input: Node positions and velocities at current time instance
- Output: Node positions and velocities at next time instance



t_k



t_{k+1}

Physics Simulation

- Discretizing Newton's second law of motion for each node

mass

$$F = ma$$

external + internal forces

acceleration

Physics Simulation

- Discretizing Newton's second law of motion for each node

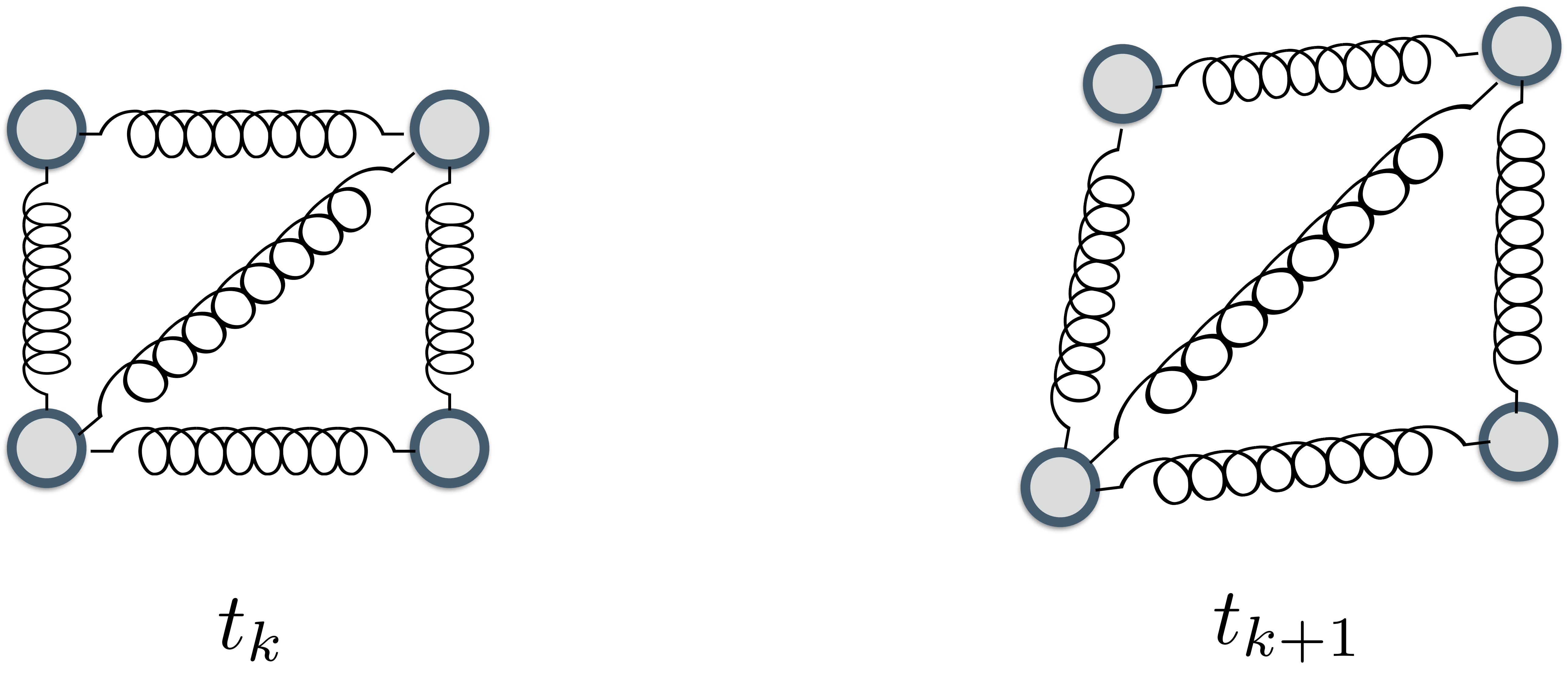
$$F = ma$$

Often nonlinear w.r.t. node positions

Linear w.r.t. node positions

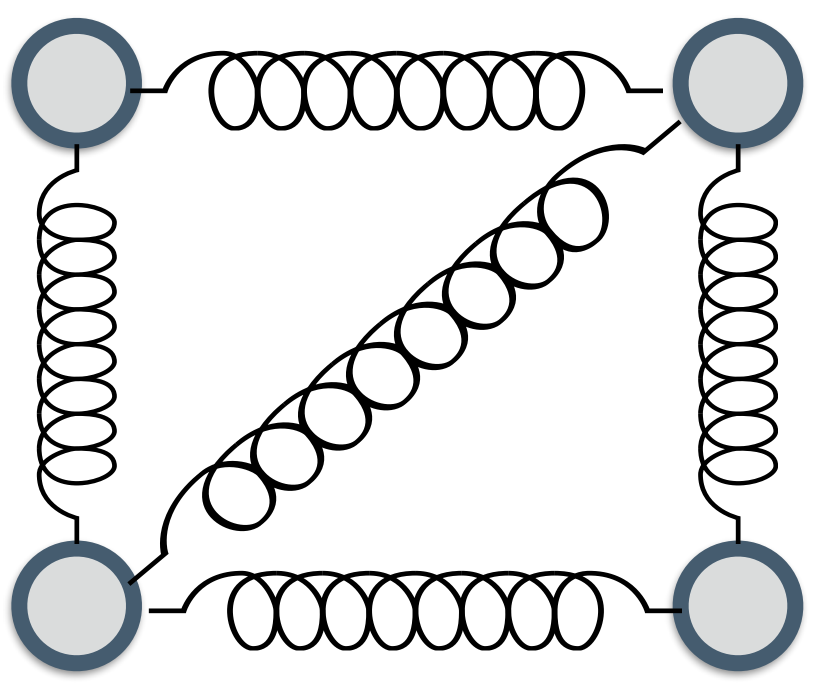
Physics Simulation

- Expilcit Euler integration
  - Evaluate F and a at current instance, update velocity
  - Integrate velocity with time step h to obtain new position
  - Simple implementation; requires small time step for stability

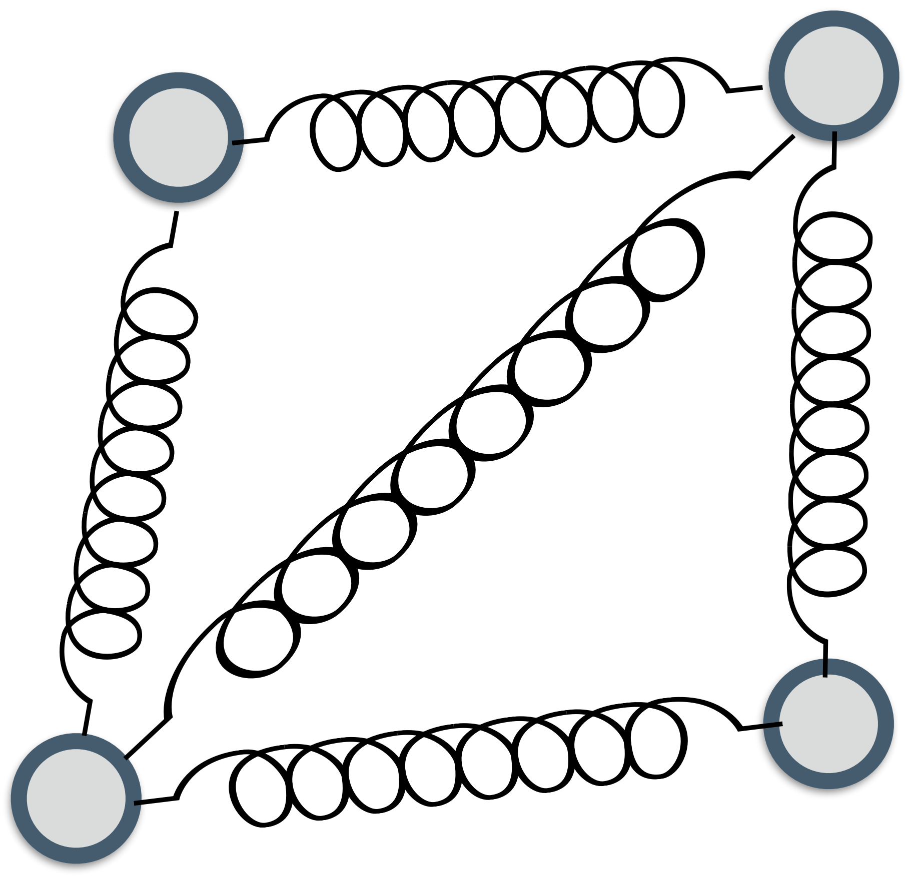


Physics Simulation

- Implicit Euler integration: enforcing Newton's law at next time instance
  - F and a are functions of unknown new position $\mathbf{q}$
  - Nonlinear equation of $\mathbf{q}$
  - Allows for large time step



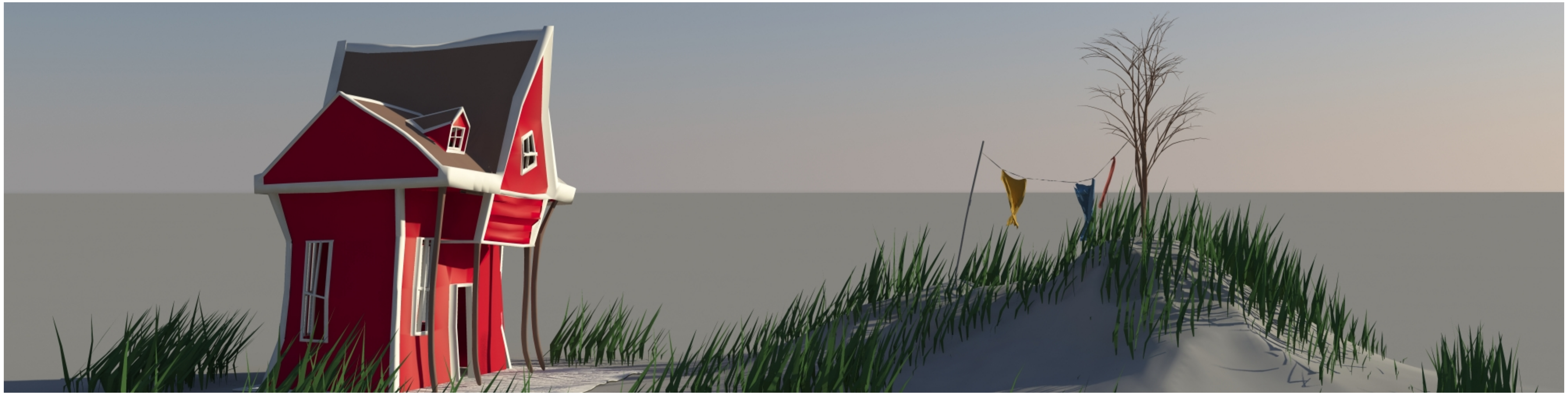
t_k



t_{k+1}

Projective Dynamics

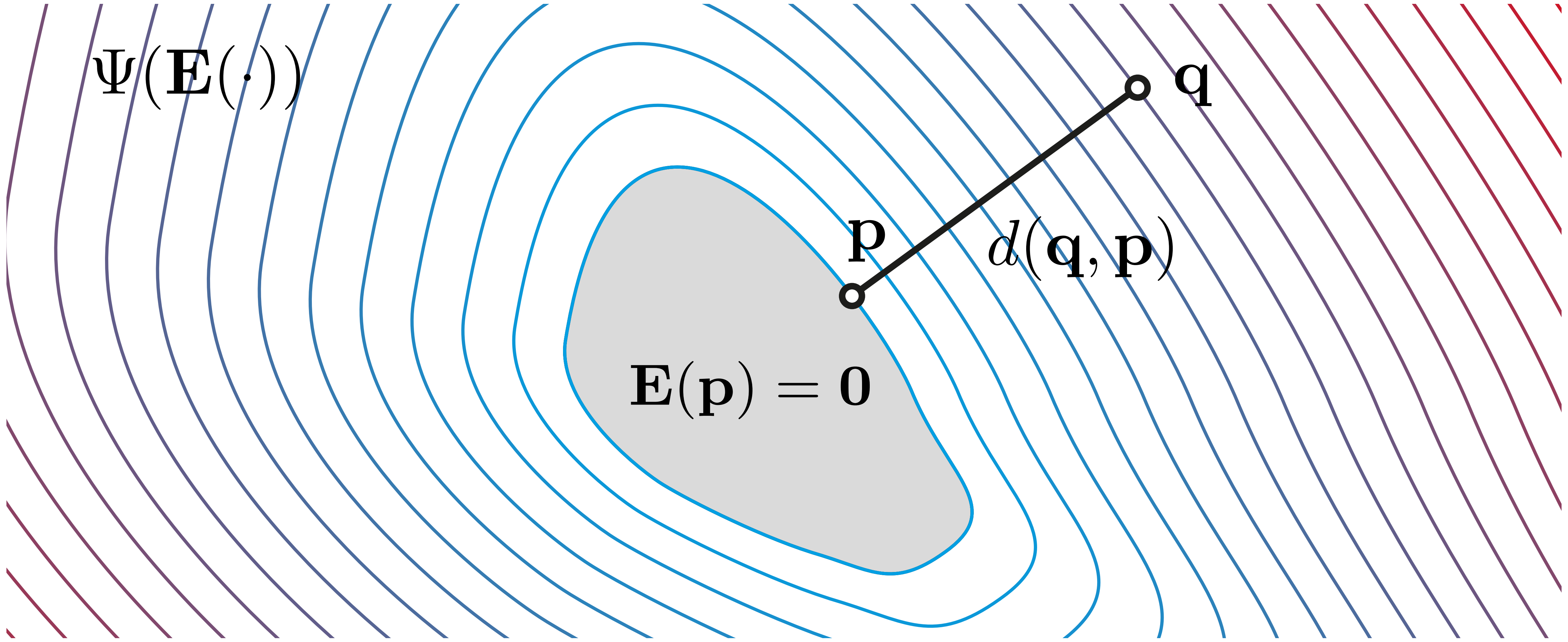
- Efficient implicit solver for physics simulation



Bouaziz et al., Projective Dynamics: Fusing Constraint Projections for Fast Simulation. SIGGRAPH 2014

Projective Dynamics

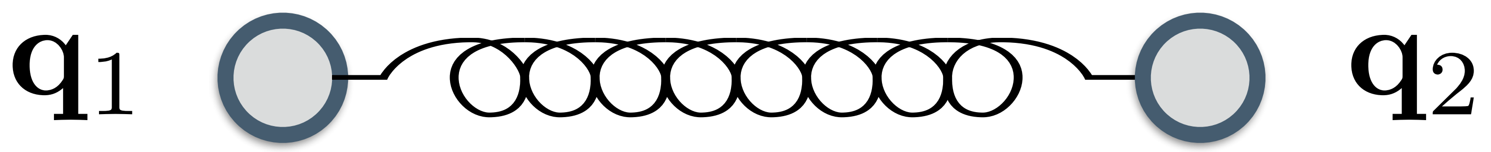
- Key idea: internal force as negative gradient of potential energy
  - Potential energy defined as squared distance to constraint manifold
  - Internal force equals difference between current position $\mathbf{q}$ and projection $\mathbf{p}$



[Bouaziz et al. 2014]

Projective Dynamics

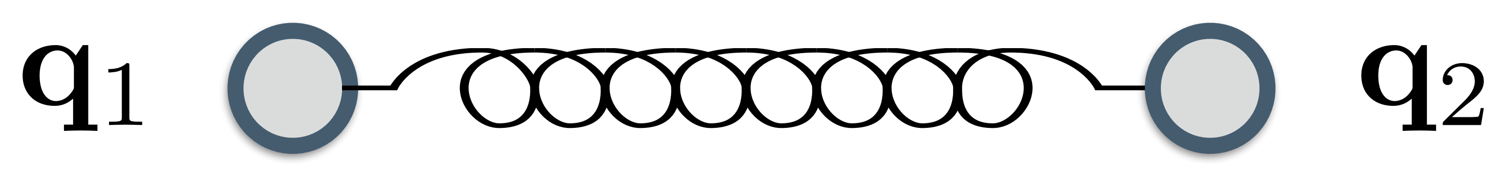
- Example: linear spring force



$$\mathbf{F}_1 = -k(\mathbf{q}_1 - \mathbf{q}_2 - L\frac{\mathbf{q}_1 - \mathbf{q}_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|})$$

Projective Dynamics

- Example: linear spring force



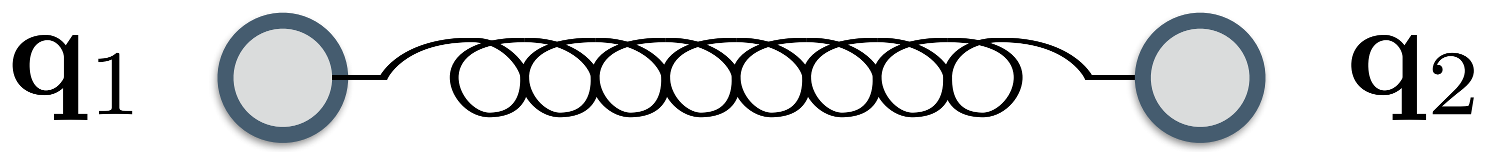
$$\mathbf{F}_1 = -k(\mathbf{q}_1 - \mathbf{q}_2 - L\frac{\mathbf{q}_1 - \mathbf{q}_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|})$$

Elastic coefficient

Rest length

Projective Dynamics

- Example: linear spring force



$$\mathbf{F}_1 = -k(\mathbf{q}_1 - \mathbf{q}_2 - \boxed{L\frac{\mathbf{q}_1 - \mathbf{q}_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|}})$$

Projection of $\mathbf{q}_1 - \mathbf{q}_2$ onto constraint manifold $\{\mathbf{v} \mid \|\mathbf{v}\| = L\}$

Projective Dynamics

- Compute new positions by optimization

$$\min_{\mathbf{Q},\{\mathbf{P}_i\}} \frac{1}{2h^2}\|\mathbf{M}^{\frac{1}{2}}(\mathbf{Q}-\mathbf{R})\|_F^2 + \sum_i \frac{w_i}{2}\|\mathbf{A}_i\mathbf{Q}-\mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

Momentum energy

Potential energy

Projective Dynamics

- Compute new positions by optimization

$$\min_{\mathbf{Q}, \{\mathbf{P}_i\}} \frac{1}{2h^2} \|\mathbf{M}^{\frac{1}{2}}(\mathbf{Q}-\mathbf{R})\|_F^2 + \sum_i \frac{w_i}{2} \|\mathbf{A}_i\mathbf{Q}-\mathbf{P}_i\|_F^2 + \sigma_i(\mathbf{P}_i).$$

- Optimality condition reduces to Newton's second law of motion
- Alternating minimization between $\mathbf{Q}$ and $\mathbf{P}$

Results

https://www.youtube.com/watch?v=TkzmqAgcilU

Summary

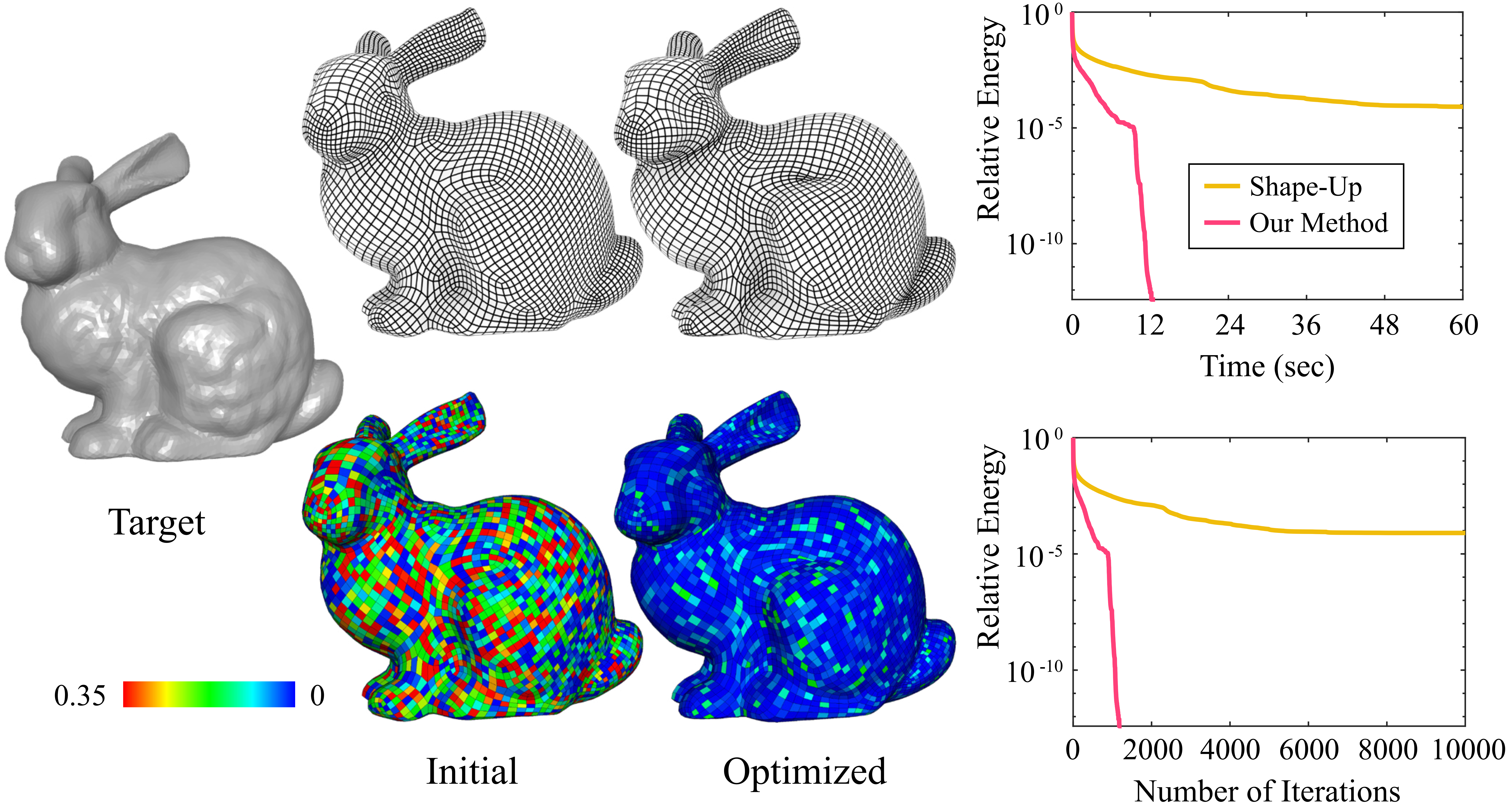
- Geometric perspective of optimization: minimizing distance to feasible sets
  - Efficient alternating minimization
  - Fast for approximate solution

Summary

- Geometric perspective of optimization: minimizing distance to feasible sets
  - Efficient alternating minimization
  - Fast for approximate solution

- Open-source implementation `www.shapeop.org`

Acceleration



Peng et al. Anderon Acceleration for Geometry Optimization and Physics Simulation. SIGGRAPH 2018.