测地线的理论与计算

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测地线的定义

- 是欧氏空间中直线概念的推广
- 微分几何给出的定义
  - 测地曲率为零
  - locally length-minimizing
- 基本功能
  - 定义度量

测地线有很多版本

- 不同的问题用到的版本是不一样的
- 输入的是什么类型的曲面？对于输出，只要距离？还要测地线？
- 两点之间？
- 一点到任何其它点？
- 任意两点之间？
- 一个小范围的测地圆盘？

计算测地线有什么考量?

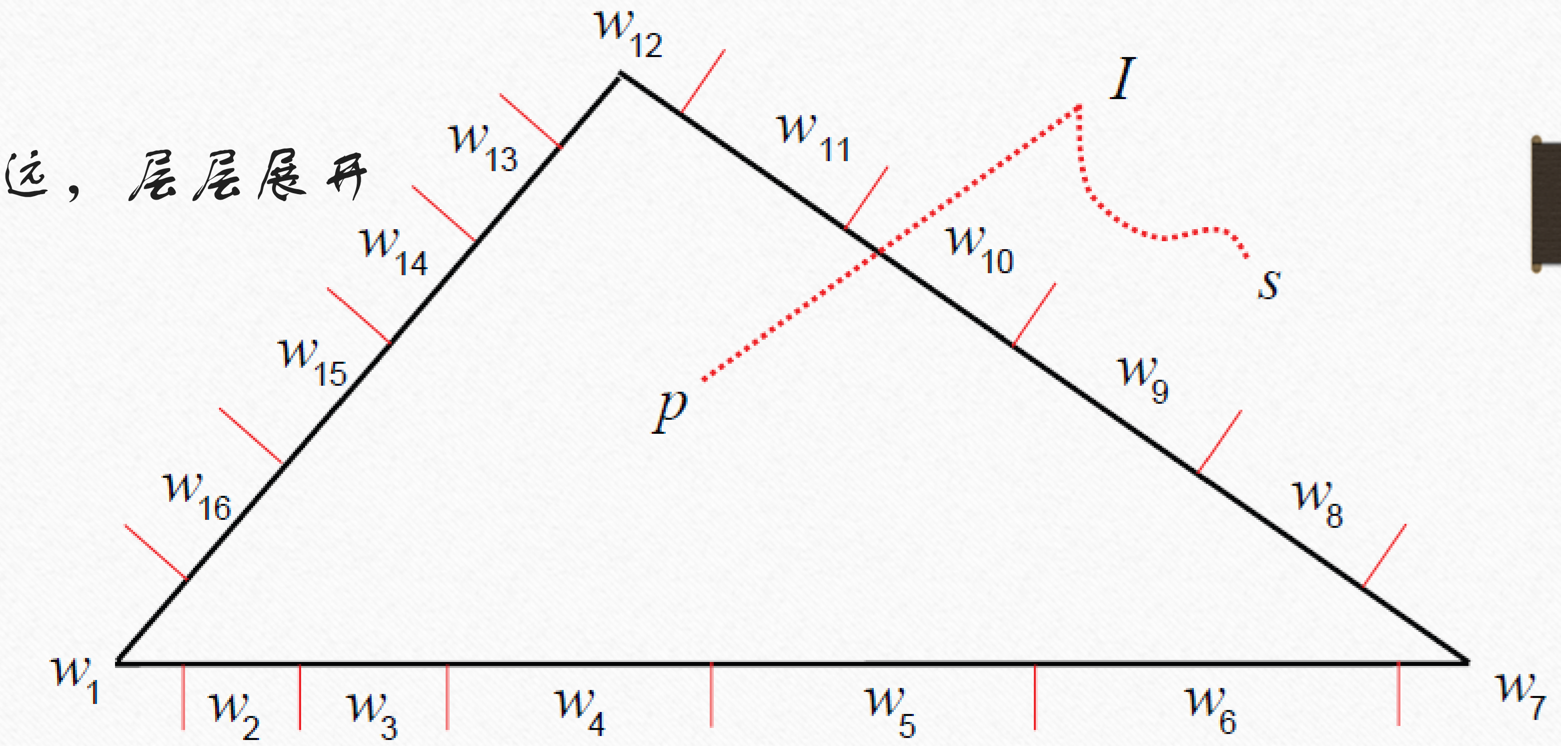
- 速度
- 精度
- 内存占用
- 对称性，三角不等式
- 光滑性
- 普适性

主流的测地线算法

- Single-source geodesic problem
    - Mitchell et al., 1987, SIAM Journal on Computing
    - Chen & Han, 1990, SOCG
    - Surazhsky et al., 2005, ACM TOG
    - Xin & Wang, 2009, ACM TOG
    - Xu et al, 2015, TVCG
    - Qin et al., 2016, ACM TOG

主流算法的思想脉络

- 从Dijkstra算法谈起
- 由内而外，由近及远，层层展开



All-pair geodesic distance query

Motivation

- Fast and accurate query of distance map is central to many computer graphics applications, especially for intrinsic geometry processing
  - Shape signature
  - Sampling on surface
  - Exponential mapping

Motivation

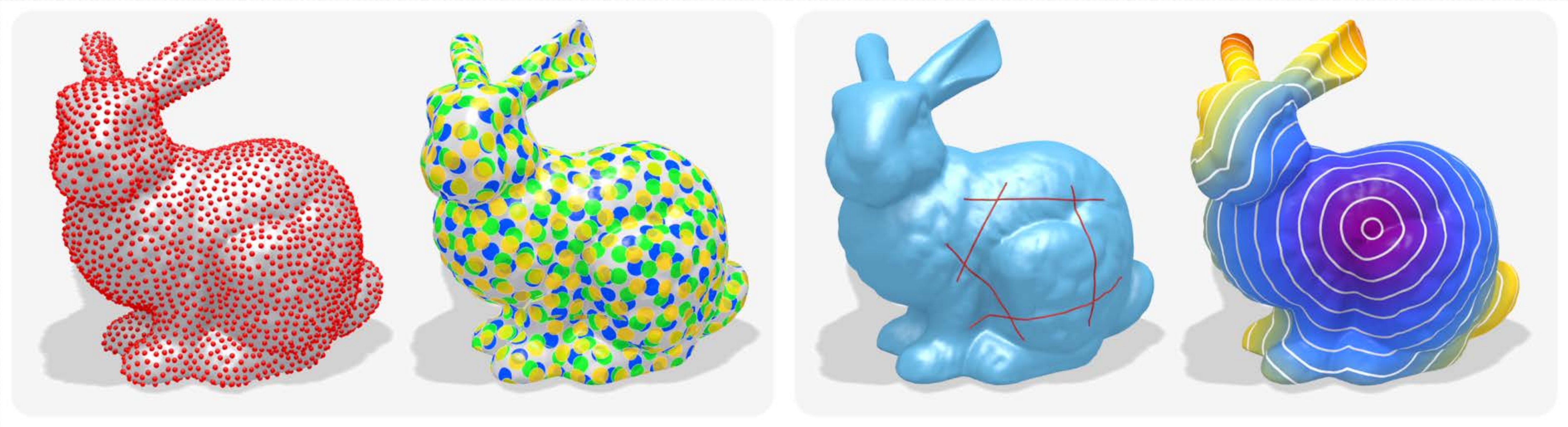
- All-pair geodesic distance problem
  - Popular technique: precomputation + fast distance query
  - Key #1: How to precompute necessary information in a small time/space cost?
  - Key #2: How to develop a fast query algorithm based on the precomputed info?
  - Key #3: Is it possible to support such a query style: Given a tolerance, we report a geodesic with an error less than the given tolerance.
  - Key #4: It seems more useful if we use the relative error instead of the absolute error. How to guarantee this?

Related Research Works

- All-pair geodesic problem
  - Saddle Vertex Graph (SVG): Ying et al. 2013, ACM TOG
  - Geodesic Triangle Unfolding (GTU): Xin et al, 2012, ACM I3D

Algorithm Overview

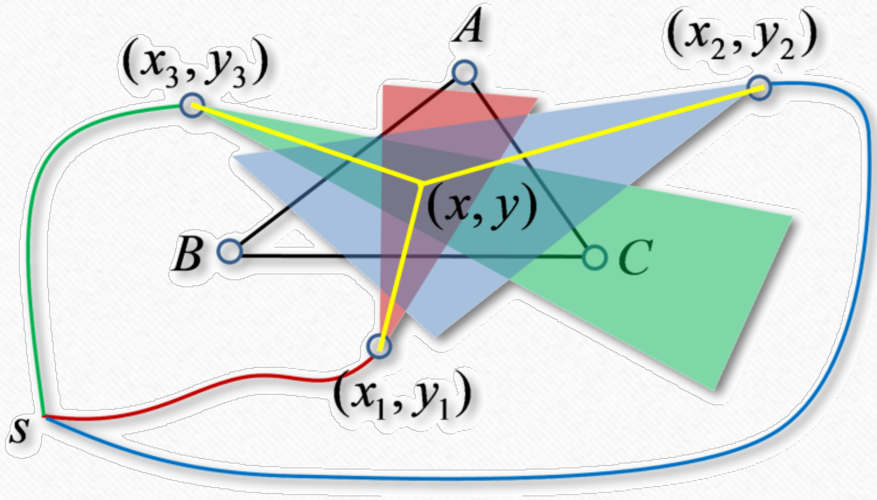
- (a-b): Building geodesic proximity graph

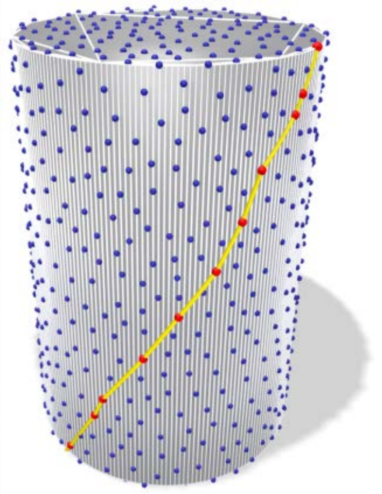


(a) Sampling (b) Proximity (c) Point-to-point (d) Distance field

Proximity Graph

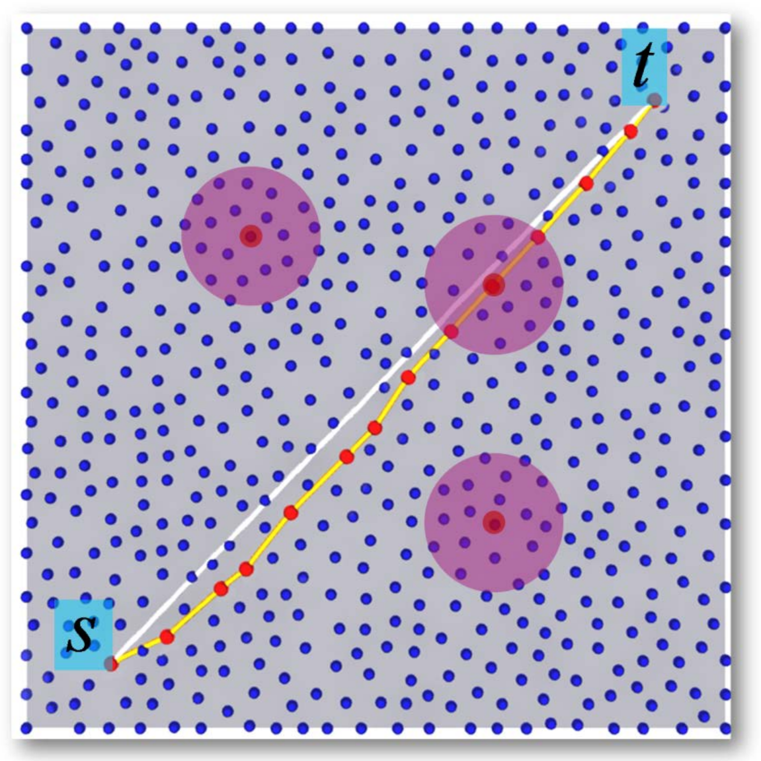
- Farthest point sampling
  - Maintain a geodesic distance field
  - Always locate the farthest point (maybe an interior)





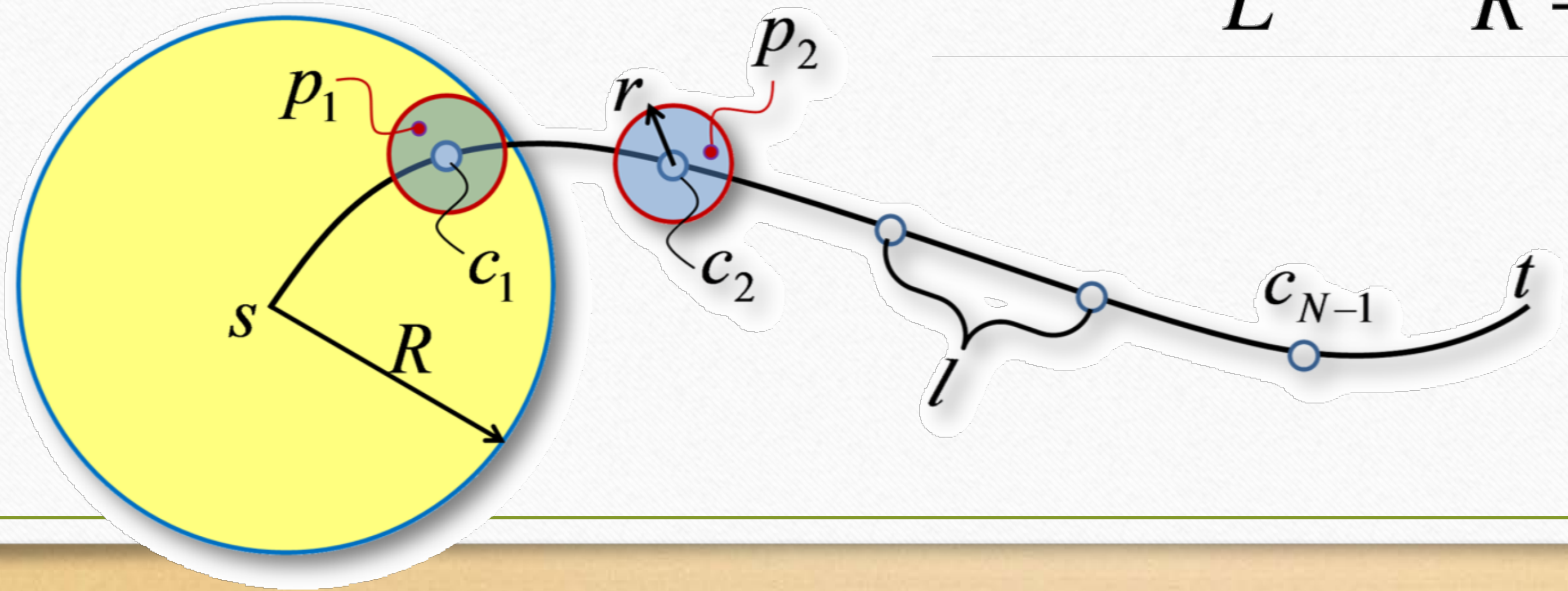
Proximity Graph

- Sampling density
  - Empty circle radius r
- Influence range
  - Pink circle radius R

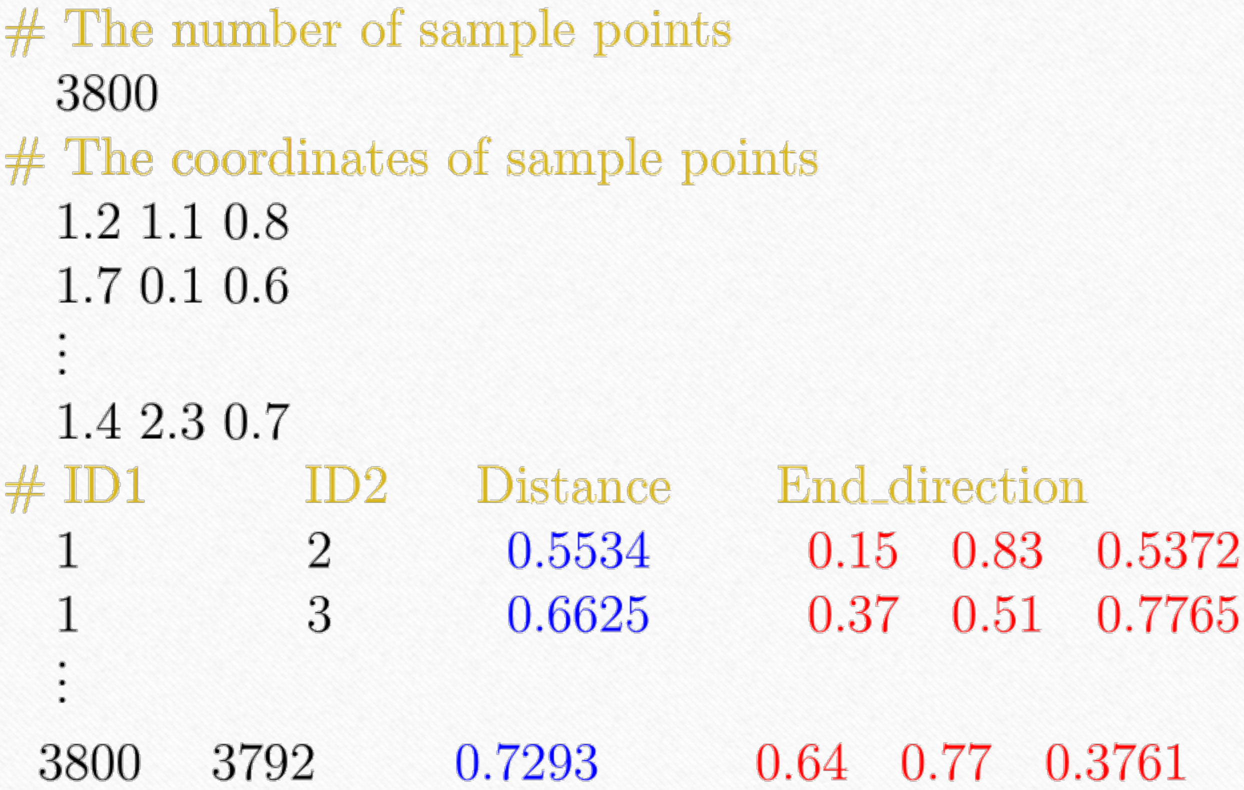


Theorem 4.2. *Let $L \triangleq d_g(s,t)$ be the length of the exact geodesic path between two points s, t, and $\widetilde{L} = \widetilde{d}_g(s,t)$ be our estimate. We have*

$$0 \leq \frac{\widetilde{L} - L}{L} \leq \frac{2r}{R - 2r}.$$

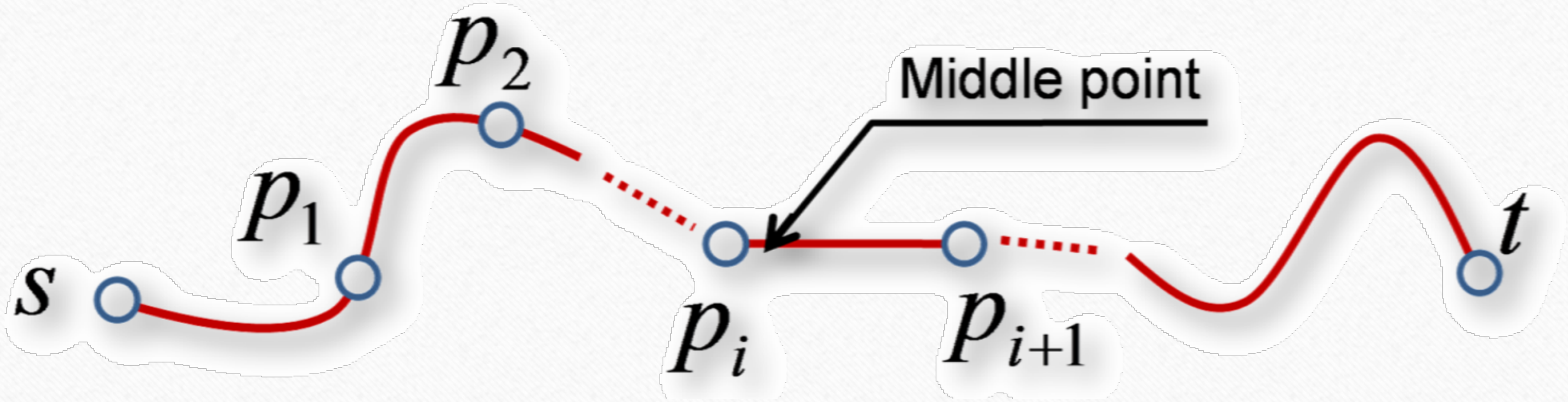


Proximity Graph



```
# The number of sample points
  3800
# The coordinates of sample points
  1.2 1.1 0.8
  1.7 0.1 0.6
  ⋮
  1.4 2.3 0.7
# ID1       ID2     Distance      End_direction
  1         2         0.5534        0.15   0.83   0.5372
  1         3         0.6625        0.37   0.51   0.7765
  ⋮
 3800    3792      0.7293        0.64   0.77   0.3761
```

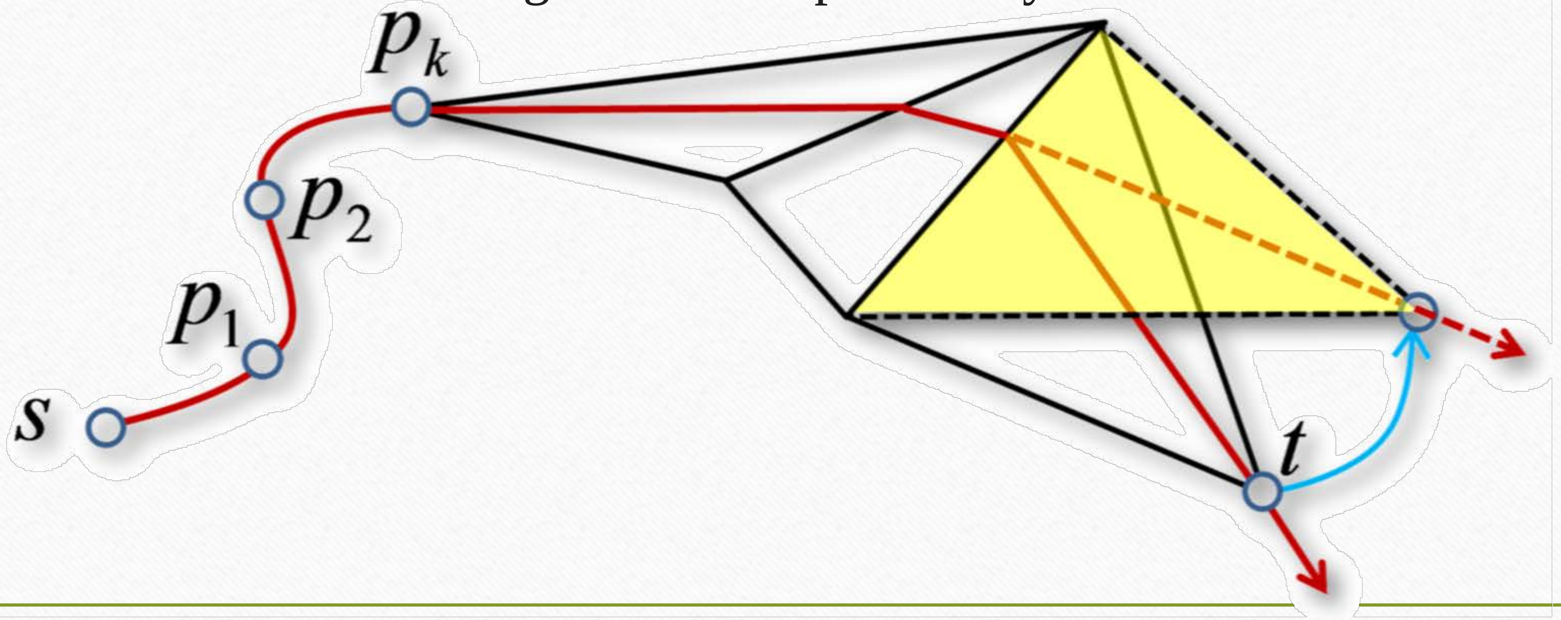
Geodesic distance query



$$\max_{p_i \in \widetilde{\Pi}_g(s,t)} \left(\widetilde{d}_g(s, p_i), \widetilde{d}_g(p_i, t)\right) \leq \frac{\widetilde{d}_g(s,t) + R}{2}$$

Geodesic distance query

- Backtracking a shortest path
  - We can trace different curved segments independently

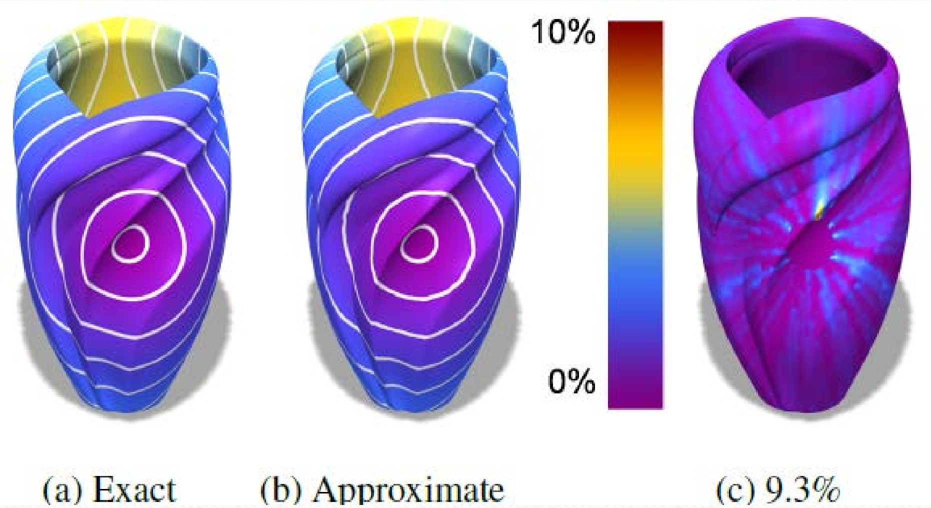


Error Analysis

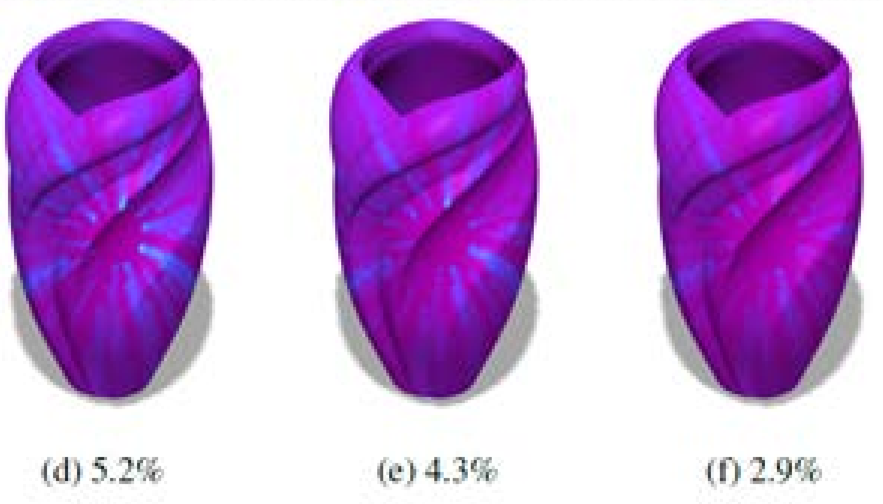
- Alternative error control variables
  - $n_{\wp}$: The total number of samples on a surface
  - $m_{\wp}$: How many nearest samples to define the influence range (or proximity)?

Error Analysis

$m_{\wp}$ =18 $m_{\wp}$ =24 $m_{\wp}$ =30 $m_{\wp}$ =36



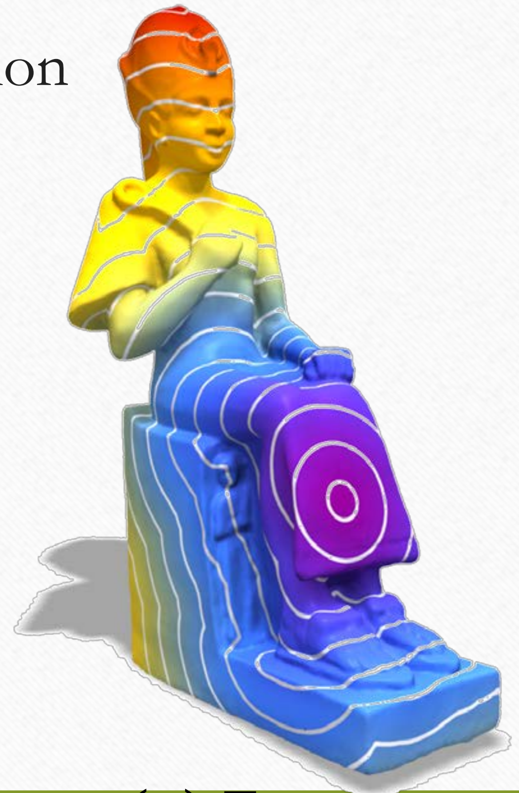
(a) Exact (b) Approximate (c) 9.3%



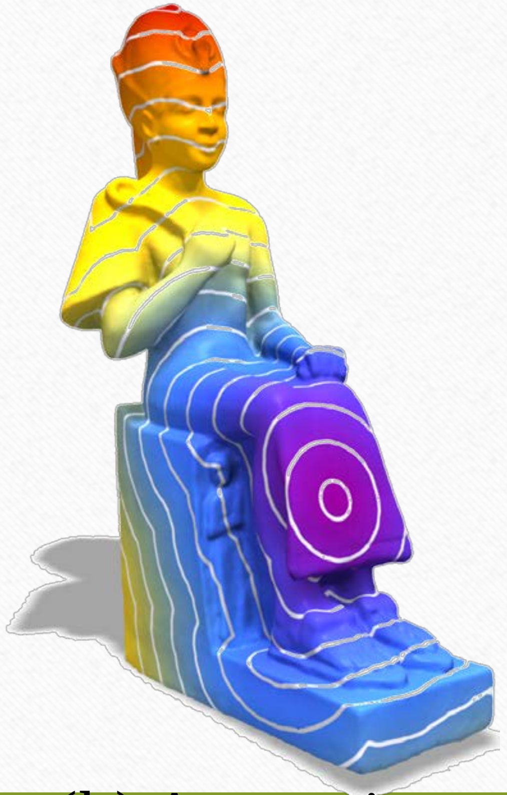
(d) 5.2% (e) 4.3% (f) 2.9%

Error Analysis

- Recommended parameter configuration
  - $n_{\mathcal{P}}$ btw 3K to 6K
  - $m_{\mathcal{P}}$ btw 14 to 28

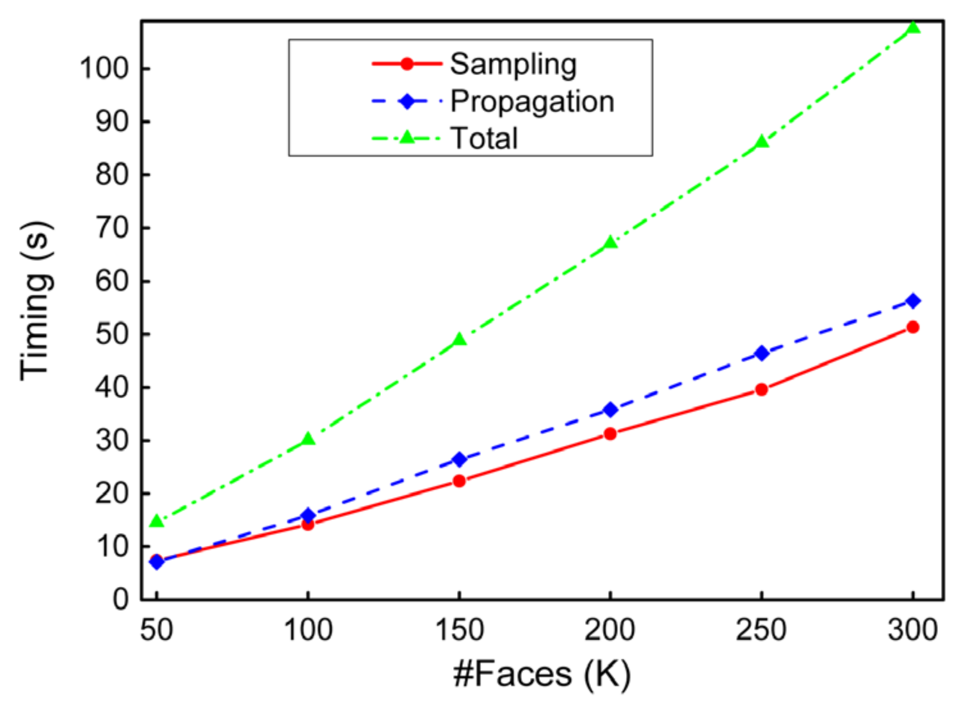


(a) Exact

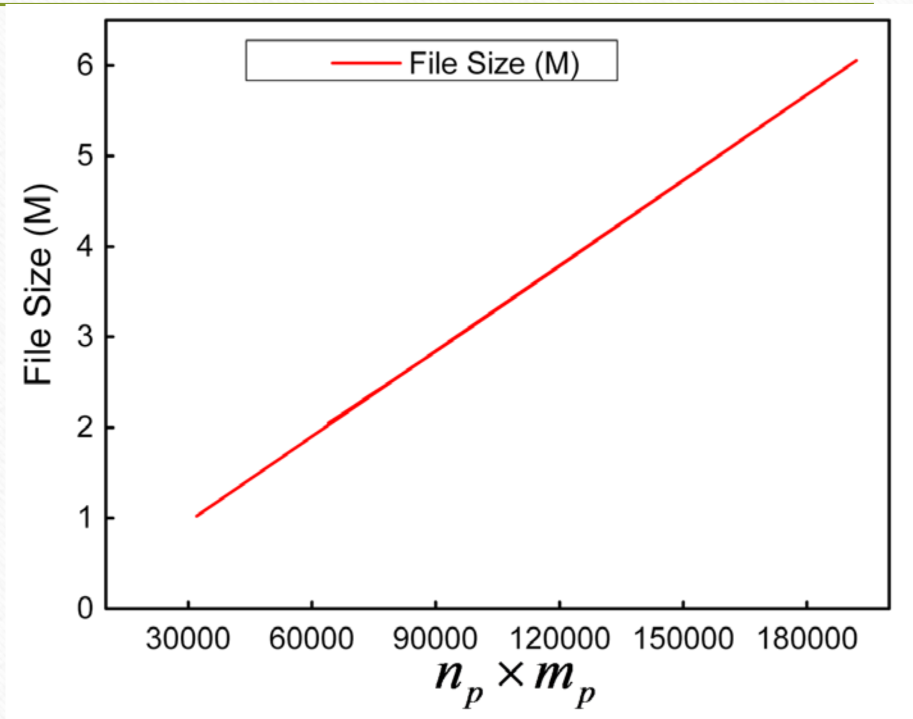


(b) Approximate

Experimental Results



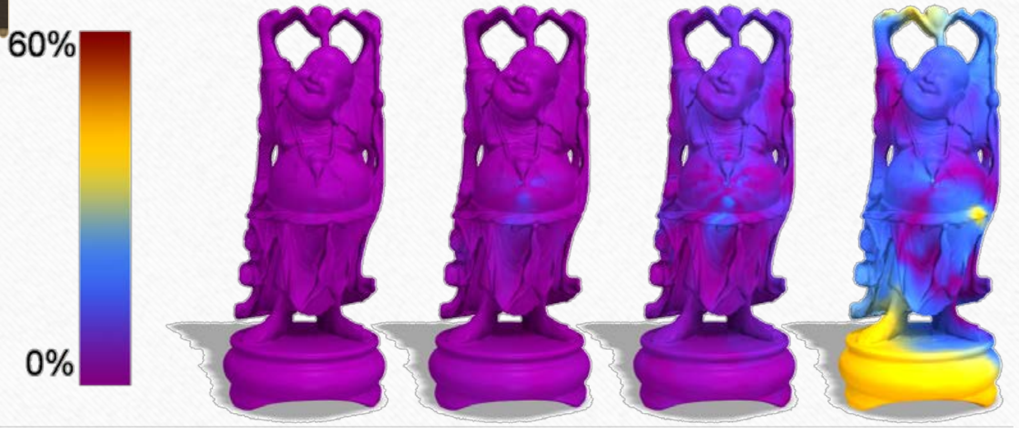
(a) Timing



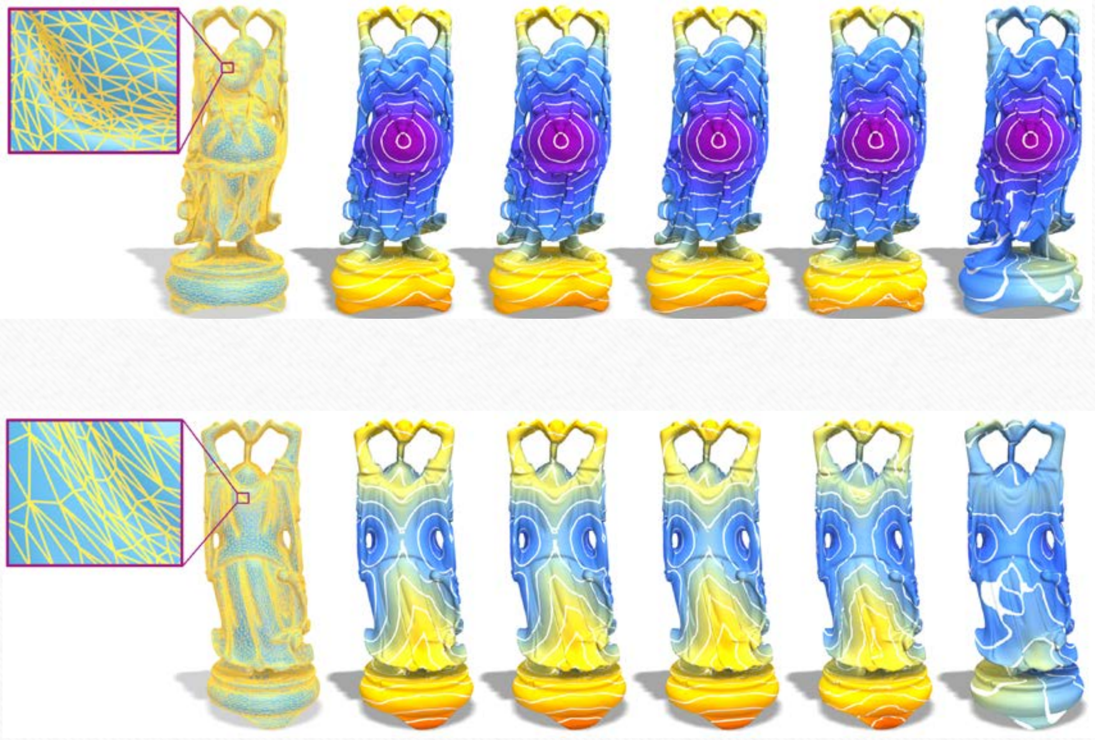
(b) Precomputed file size

Experimental Results

Error map



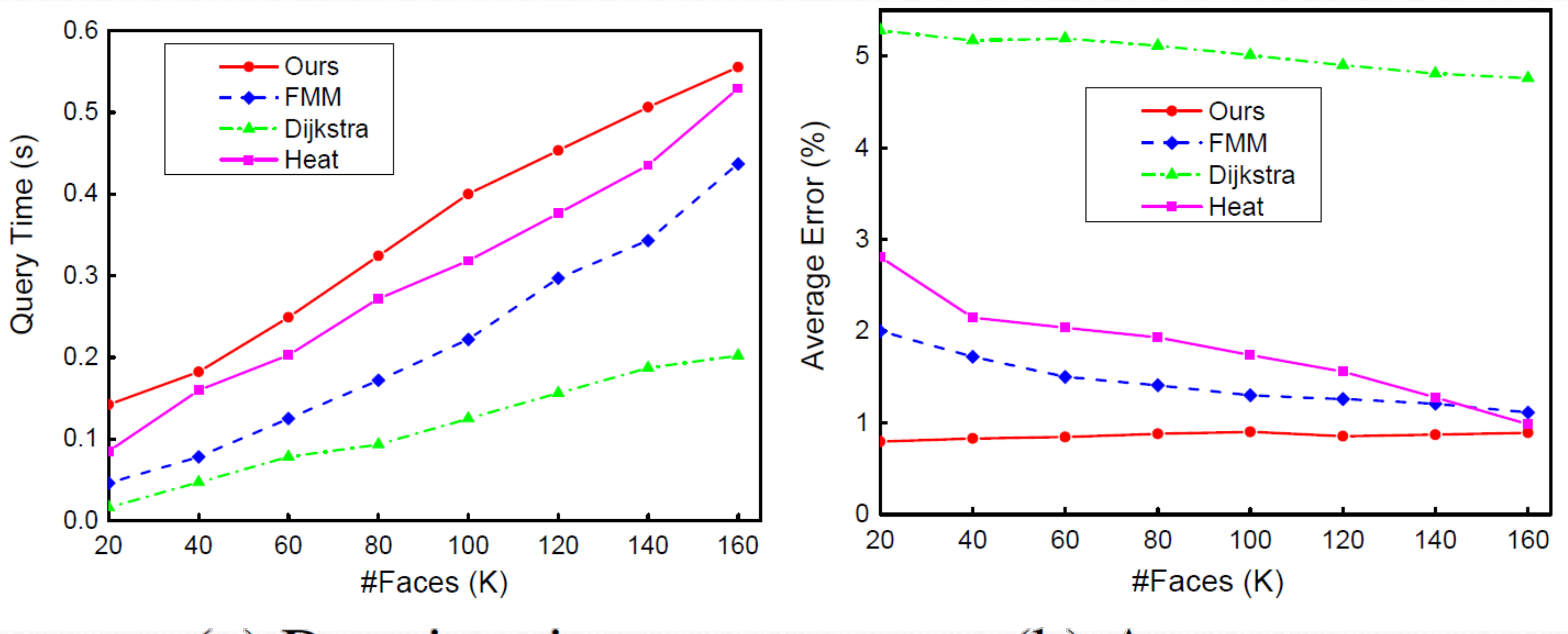
(c) Ours (d) FMM (e)Dijkstra (f) Heat



(a)Buddha (b)Exact (c)Ours (d)FMM (e)Dijkstra (f) Heat

Experimental Results

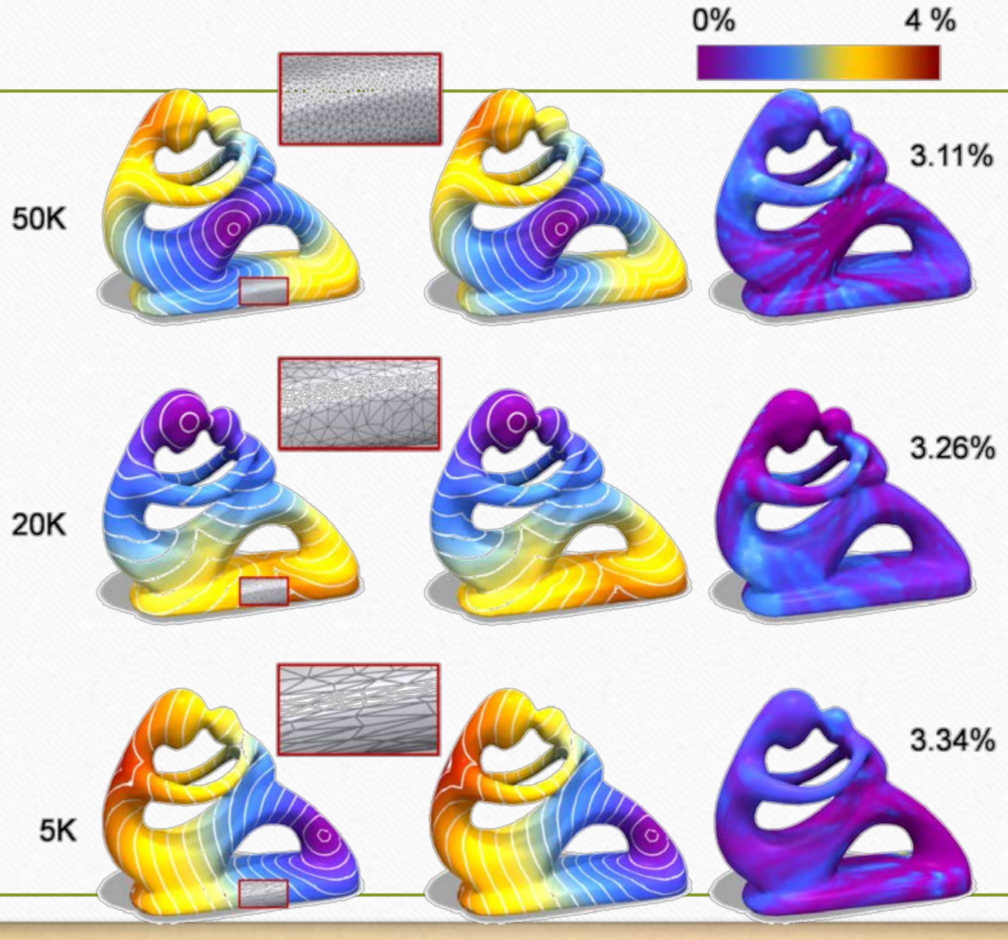
- Quantitative comparison for computing a distance field



(a) Running time

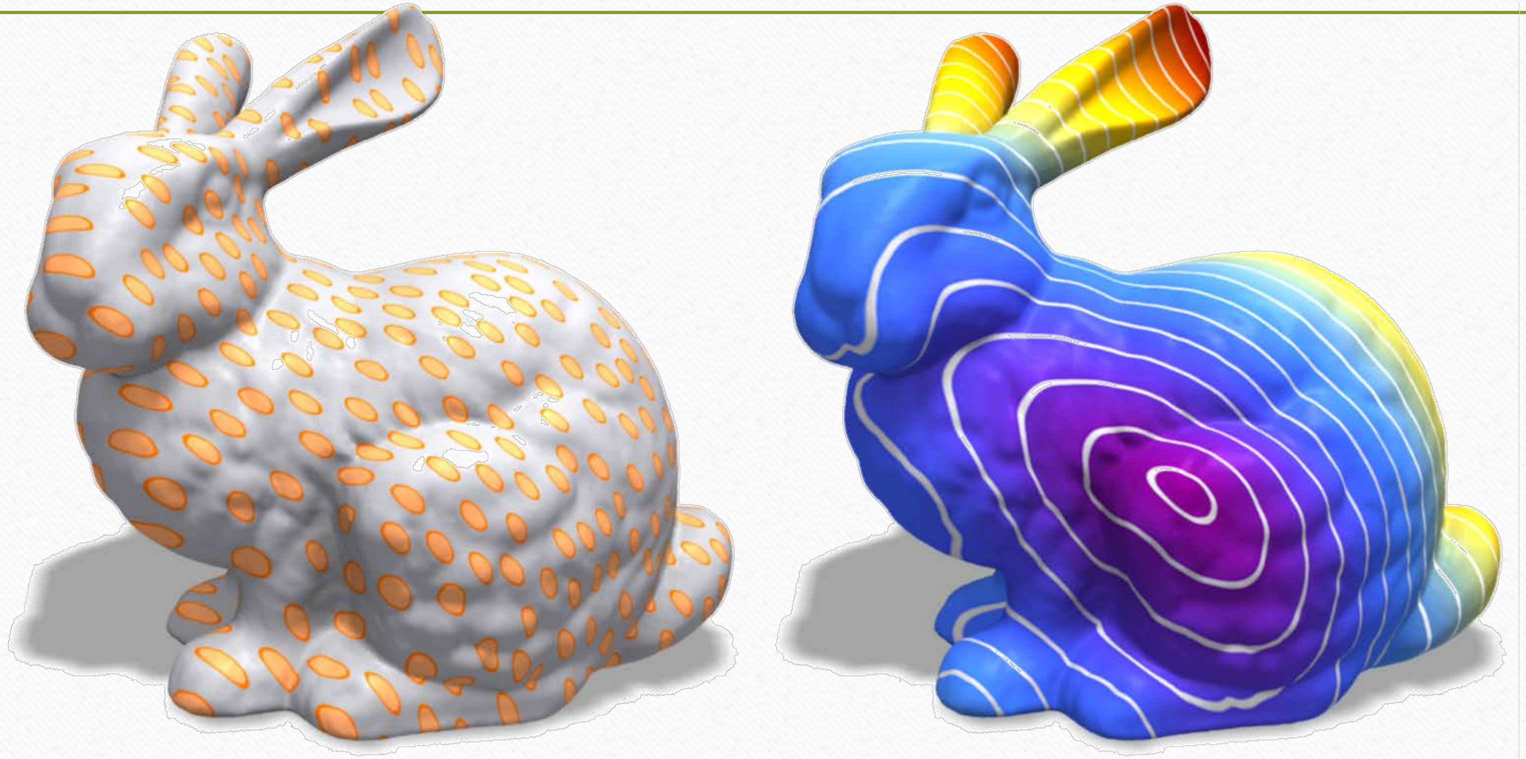
(b) Average error

Experimental Results



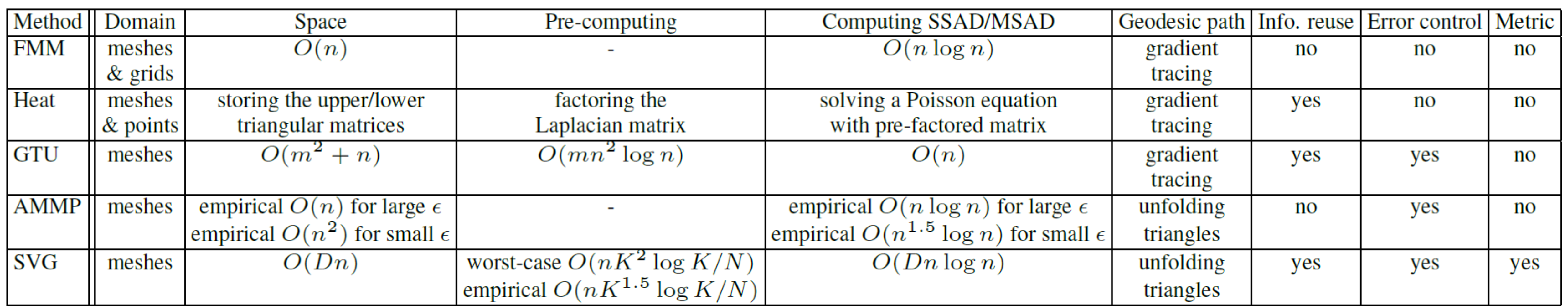
(a) Exact (b) Approximate (c) Difference

Experimental Results



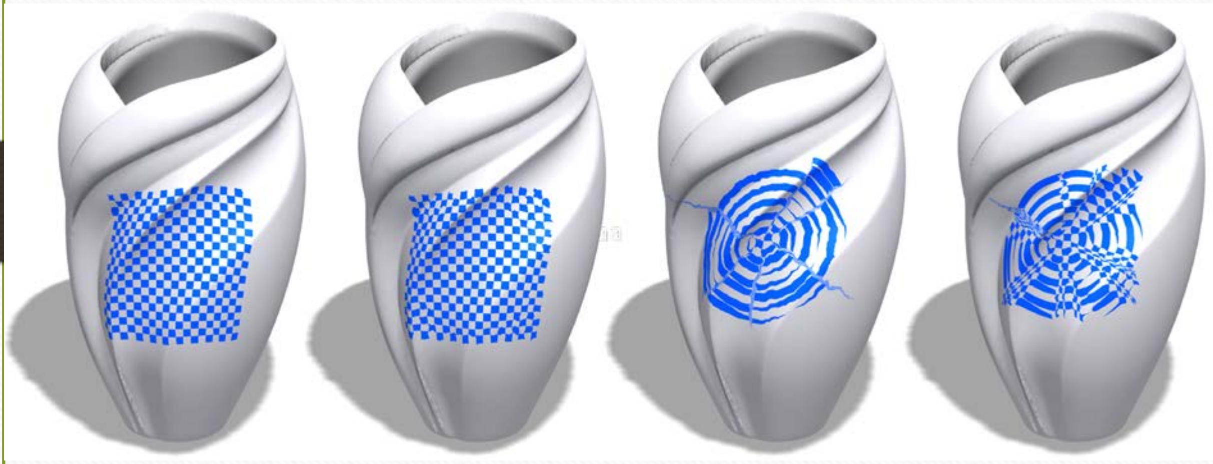
(a) Riemannian metric (b) Geodesic distance field

Experimental Results



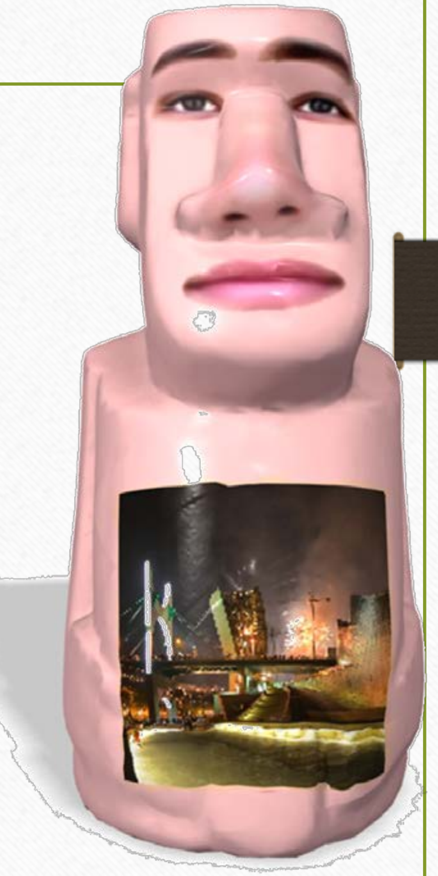
Method	Domain	Space	Pre-computing	Computing SSAD/MSAD	Geodesic path	Info. reuse	Error control	Metric
FMM	meshes & grids	$O(n)$	-	$O(n \log n)$	gradient tracing	no	no	no
Heat	meshes & points	storing the upper/lower triangular matrices	factoring the Laplacian matrix	solving a Poisson equation with pre-factored matrix	gradient tracing	yes	no	no
GTU	meshes	$O(m^2 + n)$	$O(mn^2 \log n)$	$O(n)$	gradient tracing	yes	yes	no
AMMP	meshes	empirical $O(n)$ for large ϵ empirical $O(n^2)$ for small ϵ	-	empirical $O(n \log n)$ for large ϵ empirical $O(n^{1.5} \log n)$ for small ϵ	unfolding triangles	no	yes	no
SVG	meshes	$O(Dn)$	worst-case $O(nK^2 \log K/N)$ empirical $O(nK^{1.5} \log K/N)$	$O(Dn \log n)$	unfolding triangles	yes	yes	yes
Ours	**Meshes**	**O(n)**	**Empirical O(n)**	**O(n log n)**	**Gradient tracing**	**yes**	**yes**	**yes**

Application #1



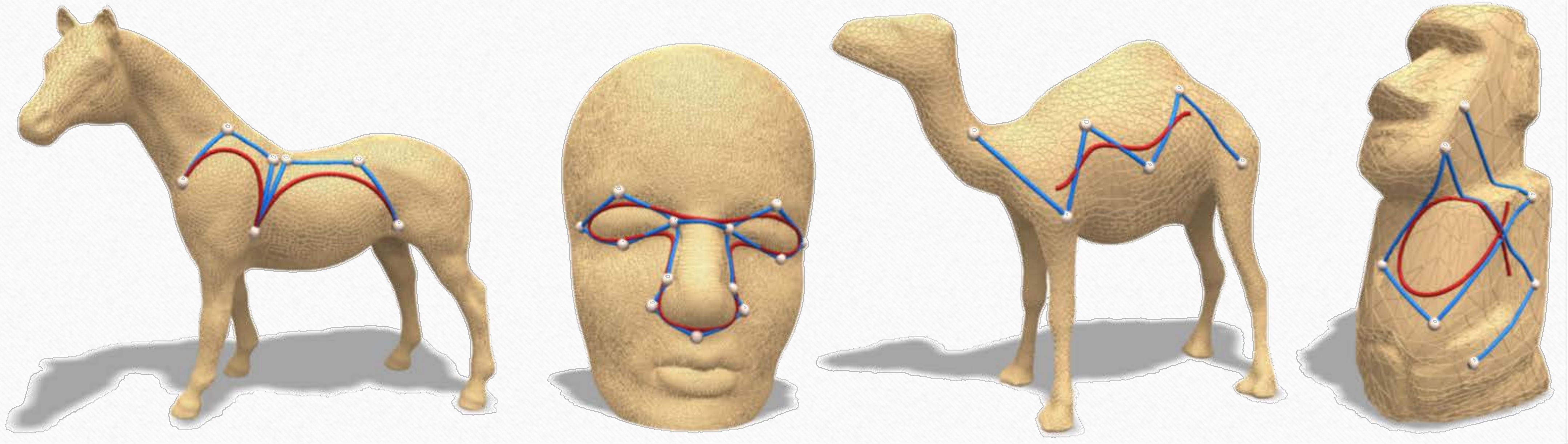
(a) Exact (b) Ours (c) Dijkstra (d) FMM

(e) Texture decals



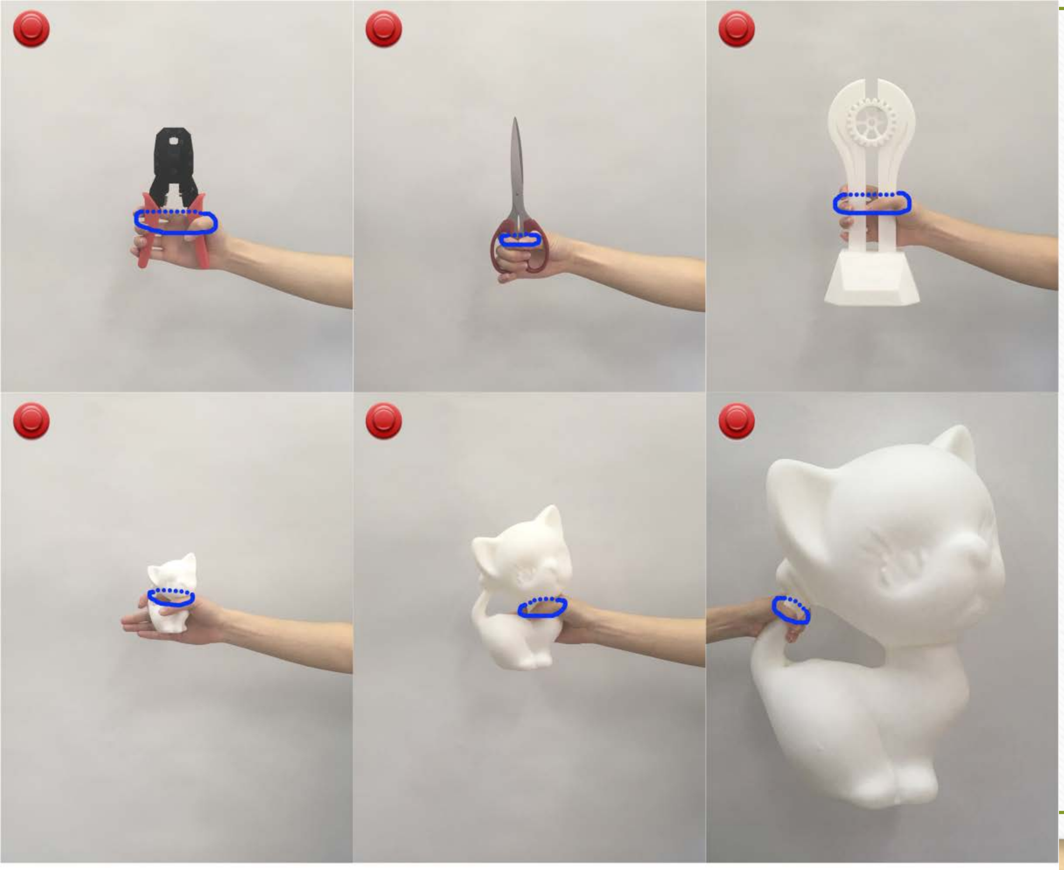
(f) Moai character

Application #2



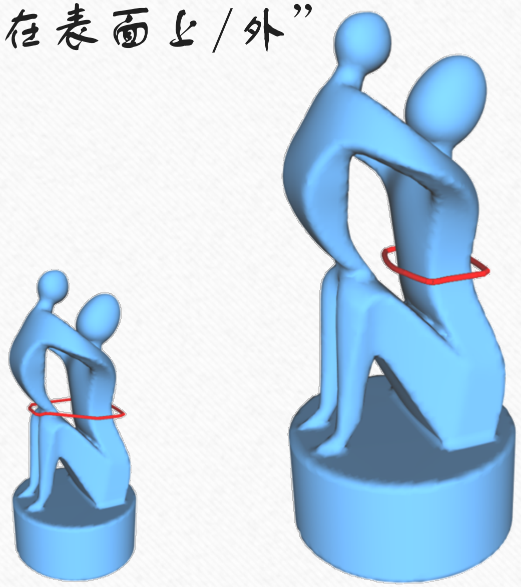
(a) Horse, 40K faces (b) Face, 31K faces (c) Camel, 18K faces (d) Moai, 6K faces

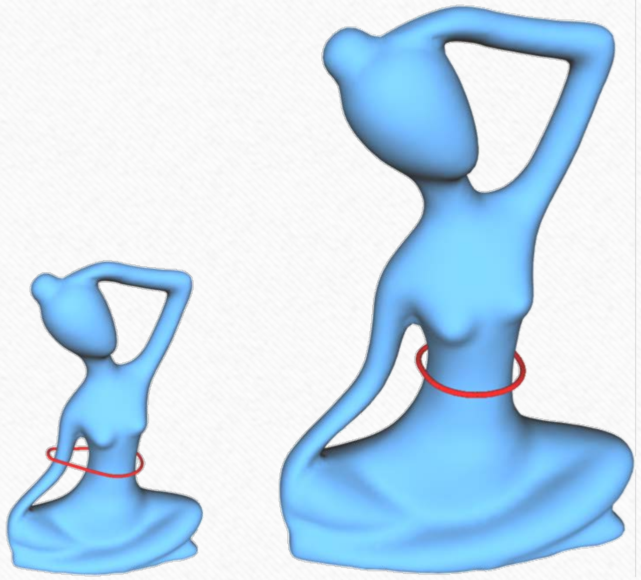
Application #3. 机器人的抓取



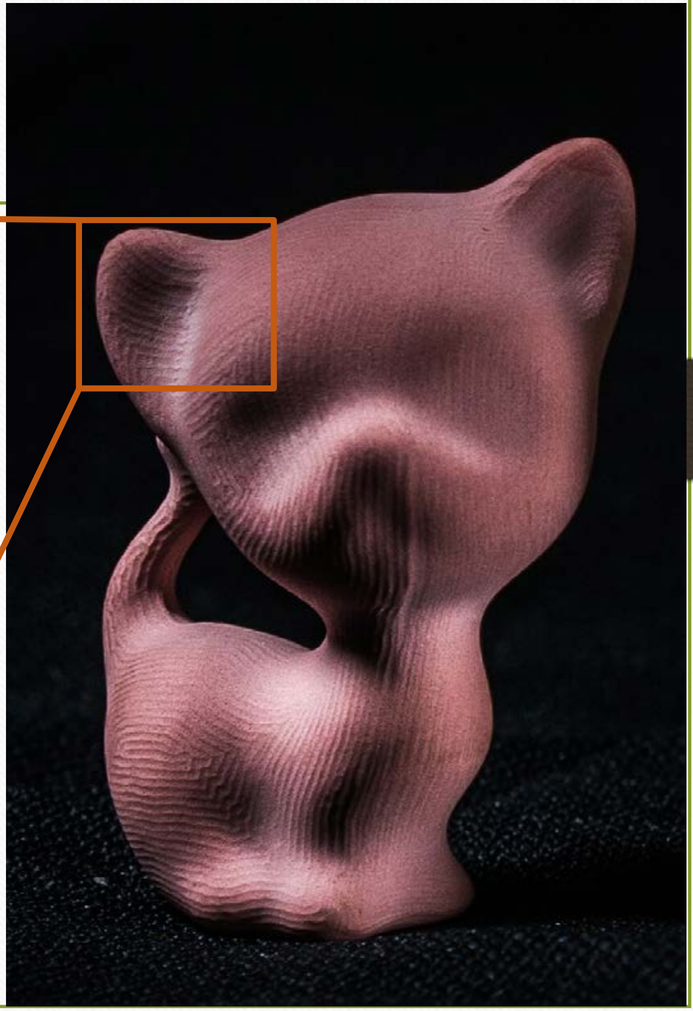
Application #3. 机器人的抓取

- 既能算表面上的测地线，也能算空间中的测地线
  - 约束“在表面上/外”



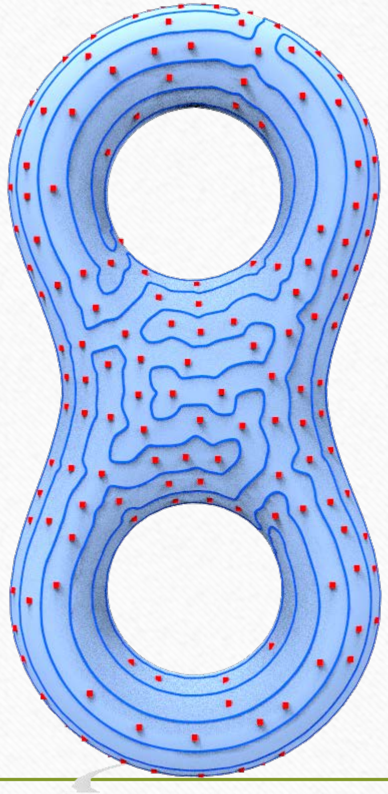


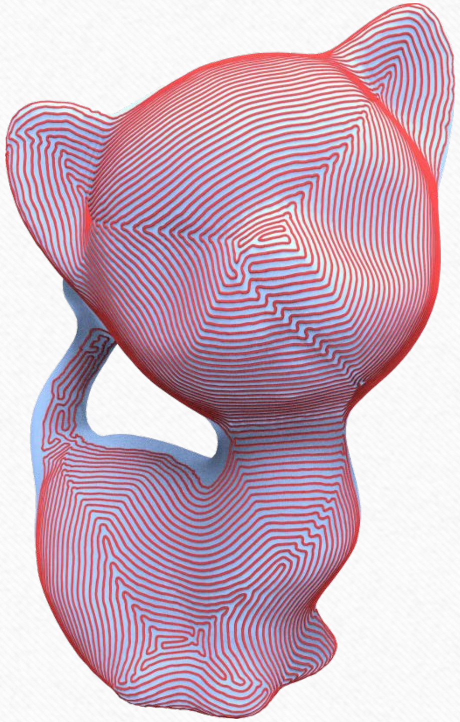
Application #4. CNC加工路径



Application #4. CNC加工路径

- 目标：
  - 比较直（光顺）
- 引力
  - 覆盖表面
- 斥力
  - 保持间距





Q & A

谢谢大家！

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