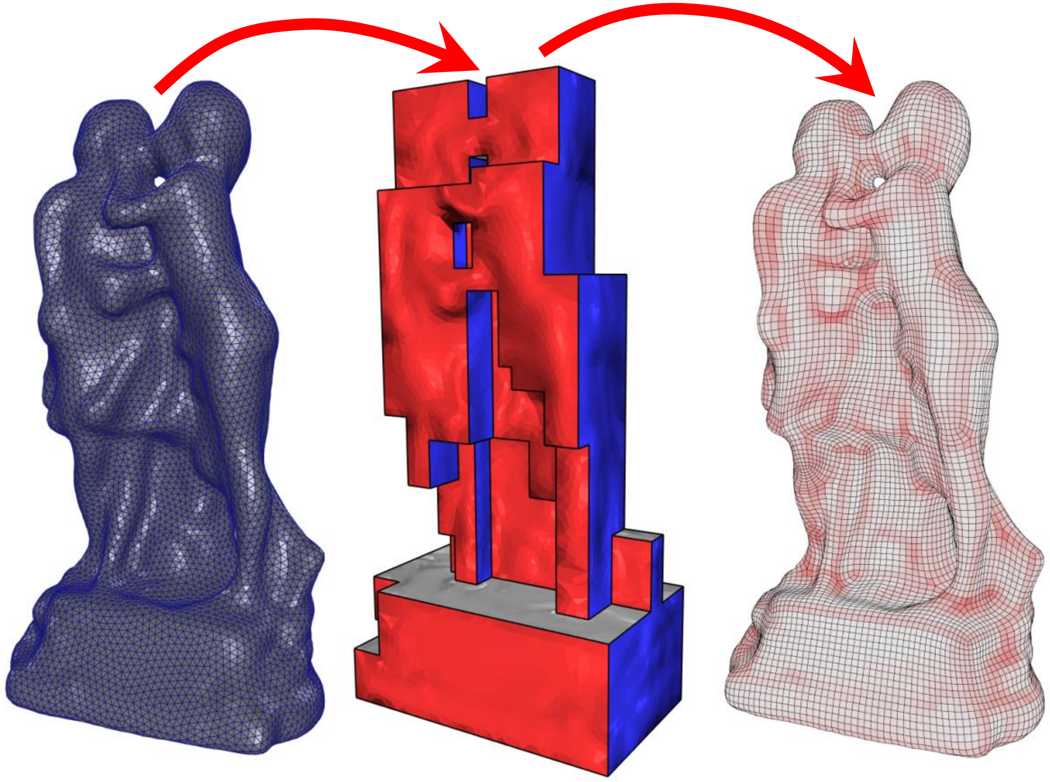


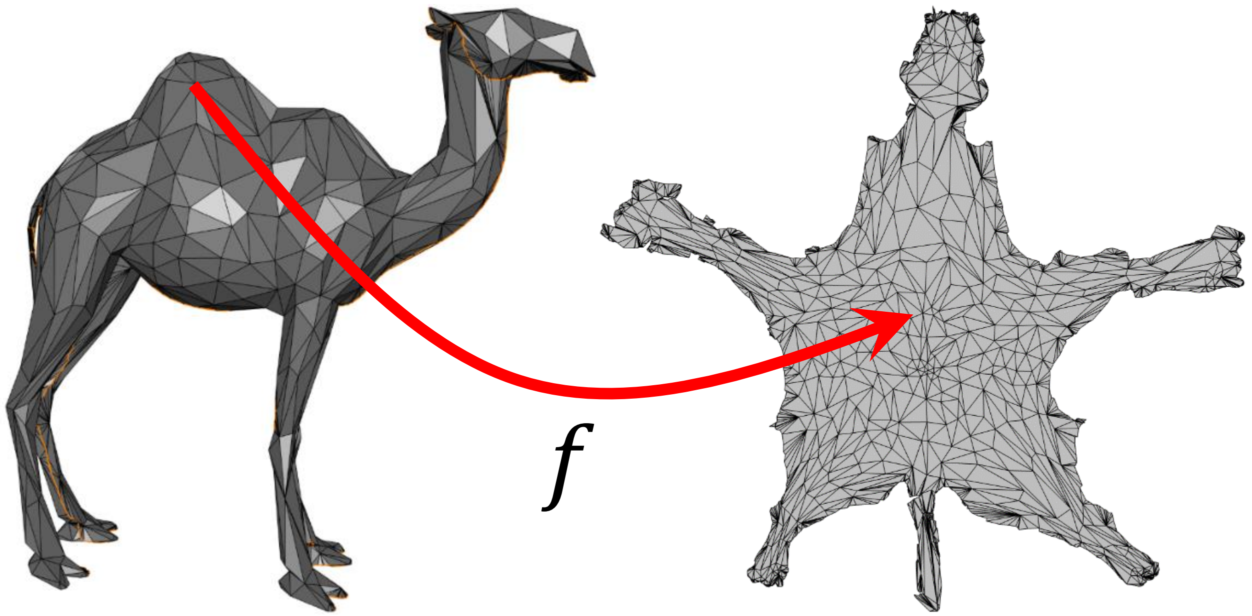
Geometric mappings

- Parameterizations, Deformation, Meshing

Xiao-Ming Fu, GCL, USTC

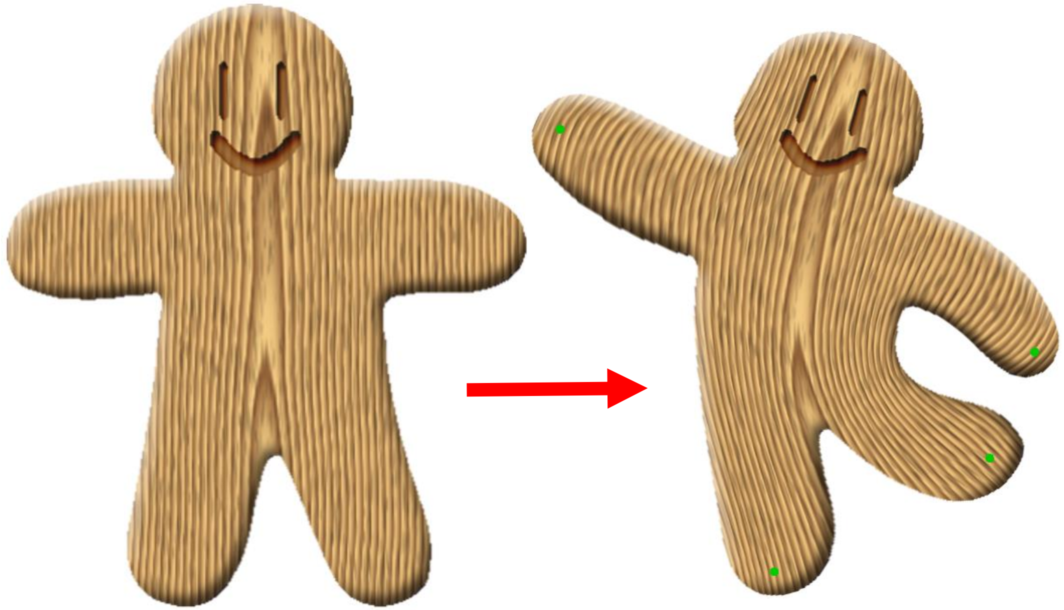
Geometric mappings

- Mesh-based mapping



$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

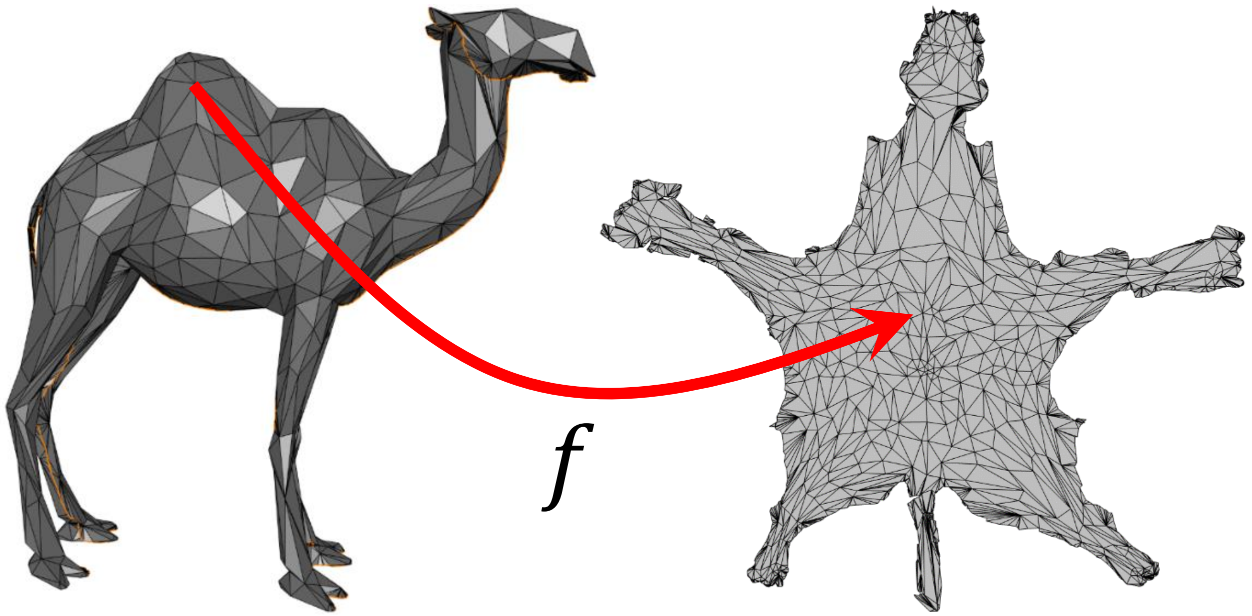
- Meshless (space) mapping



$$f(\boldsymbol{x}) = \sum_{i=1}^{m} \boldsymbol{c}_i B_i(\boldsymbol{x}) + A\boldsymbol{x} + \boldsymbol{b}$$

Applications

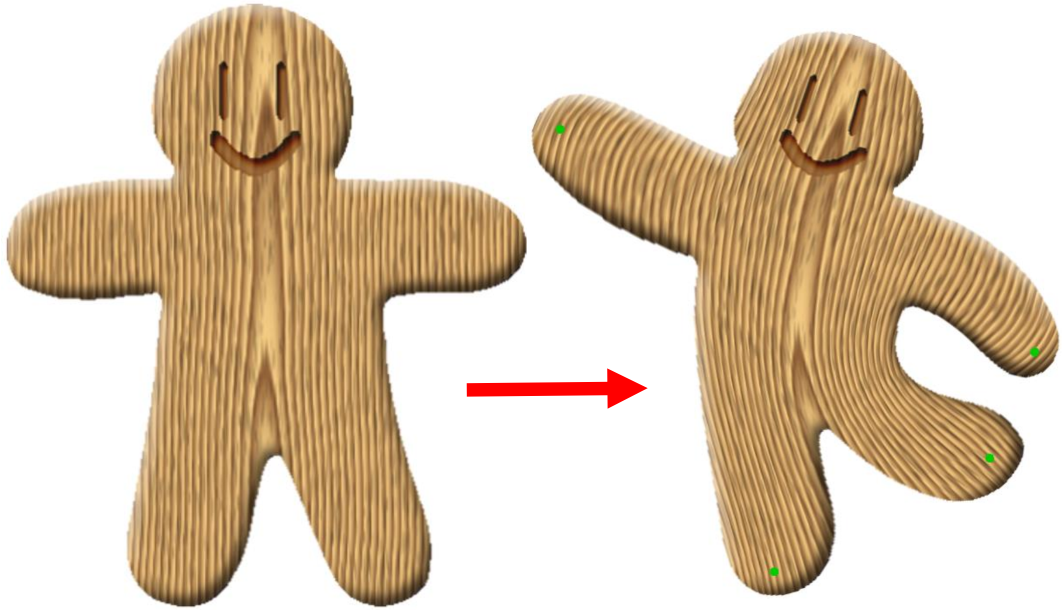
Parameterizations



3D surface 2D region

$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

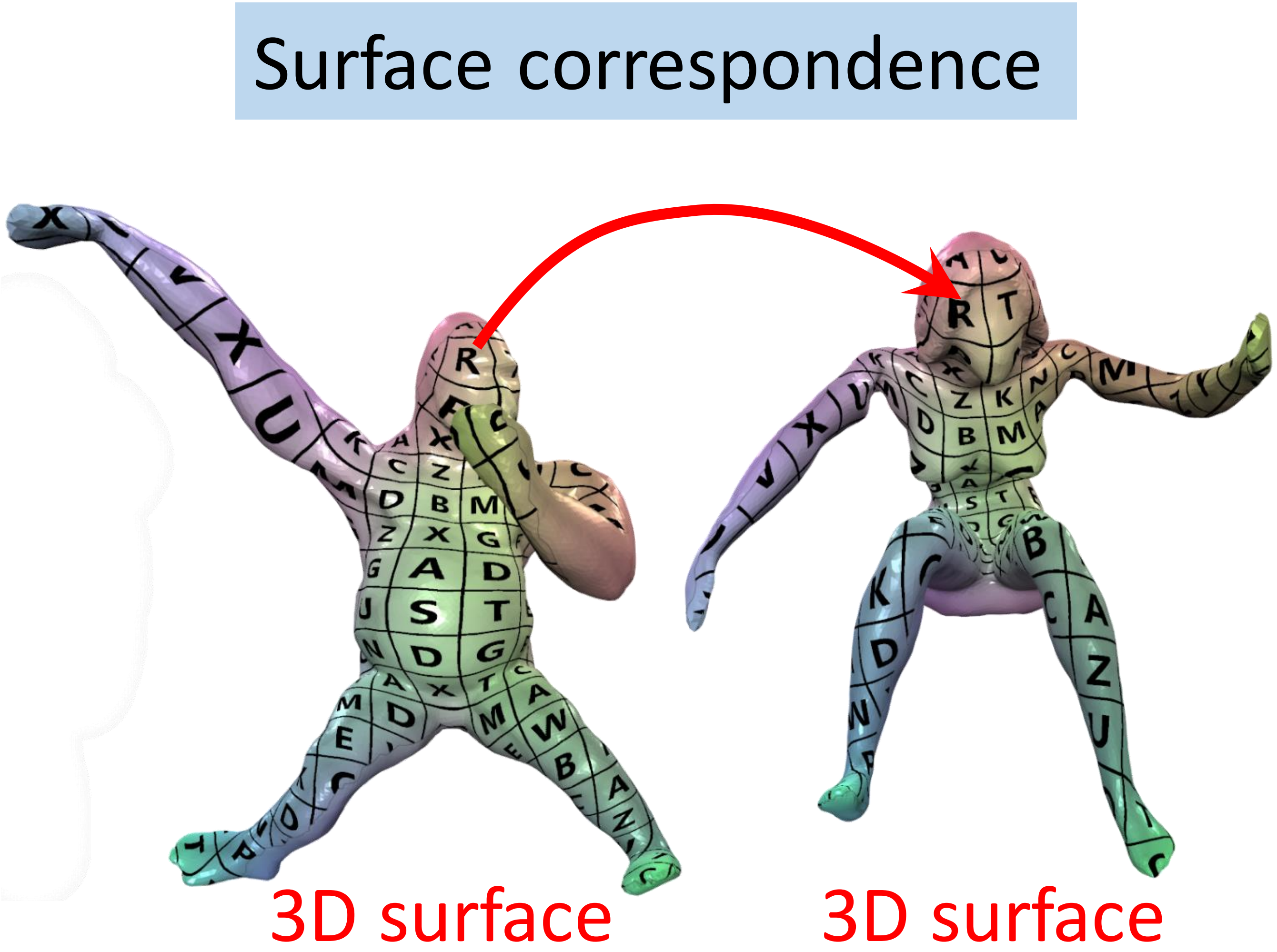
Deformation

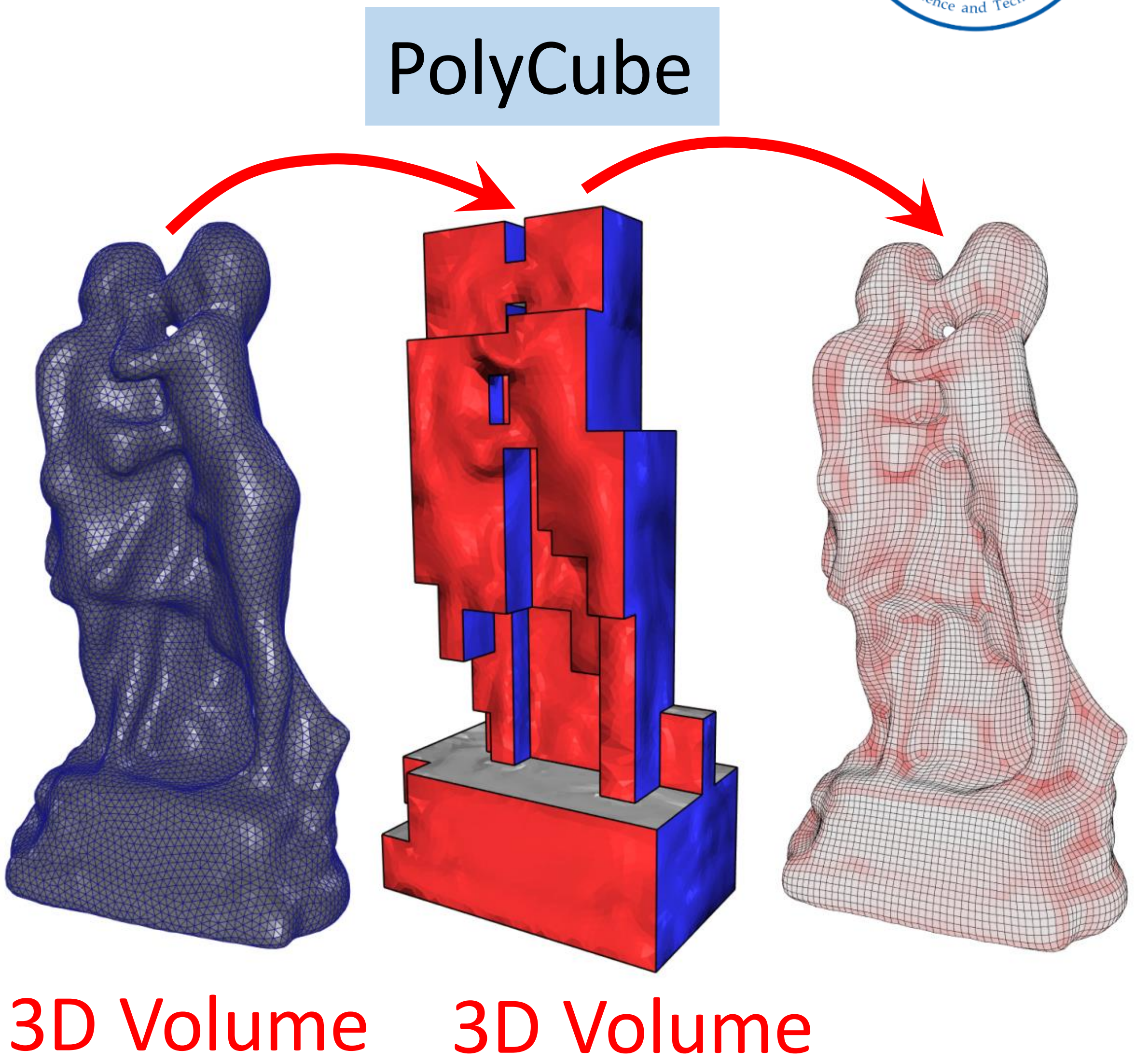


2D region 2D region

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} \boldsymbol{c}_i B_i(\boldsymbol{x}) + A\boldsymbol{x} + \boldsymbol{b}$$

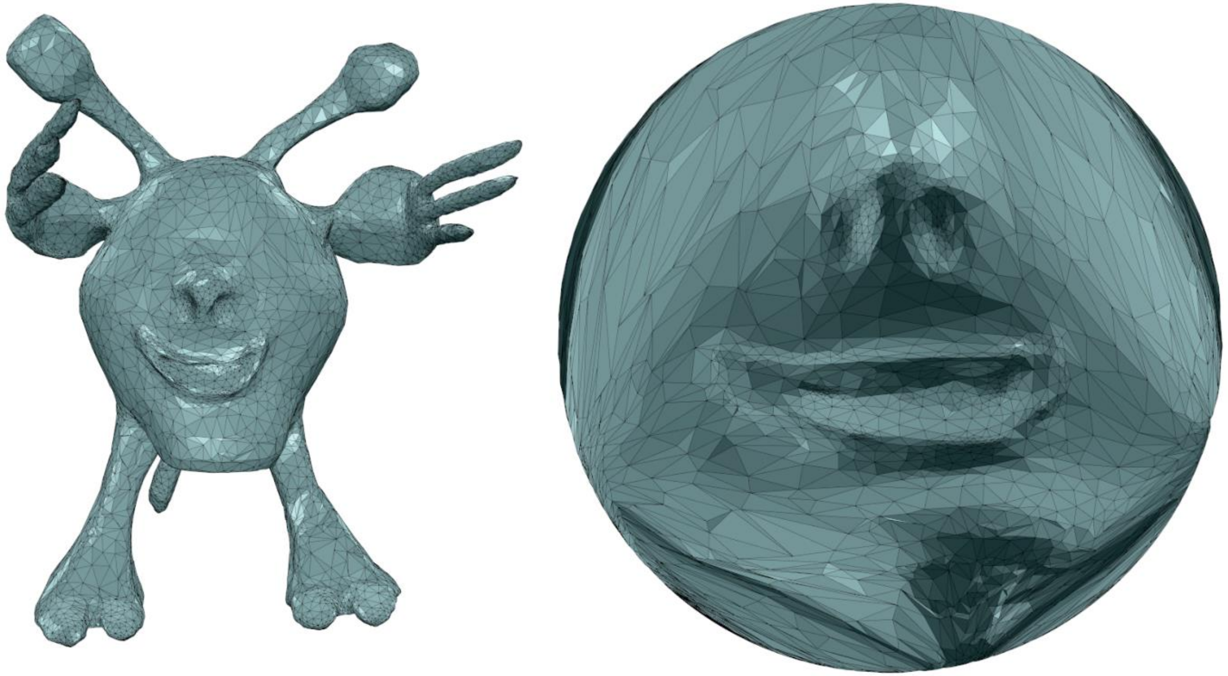
Applications





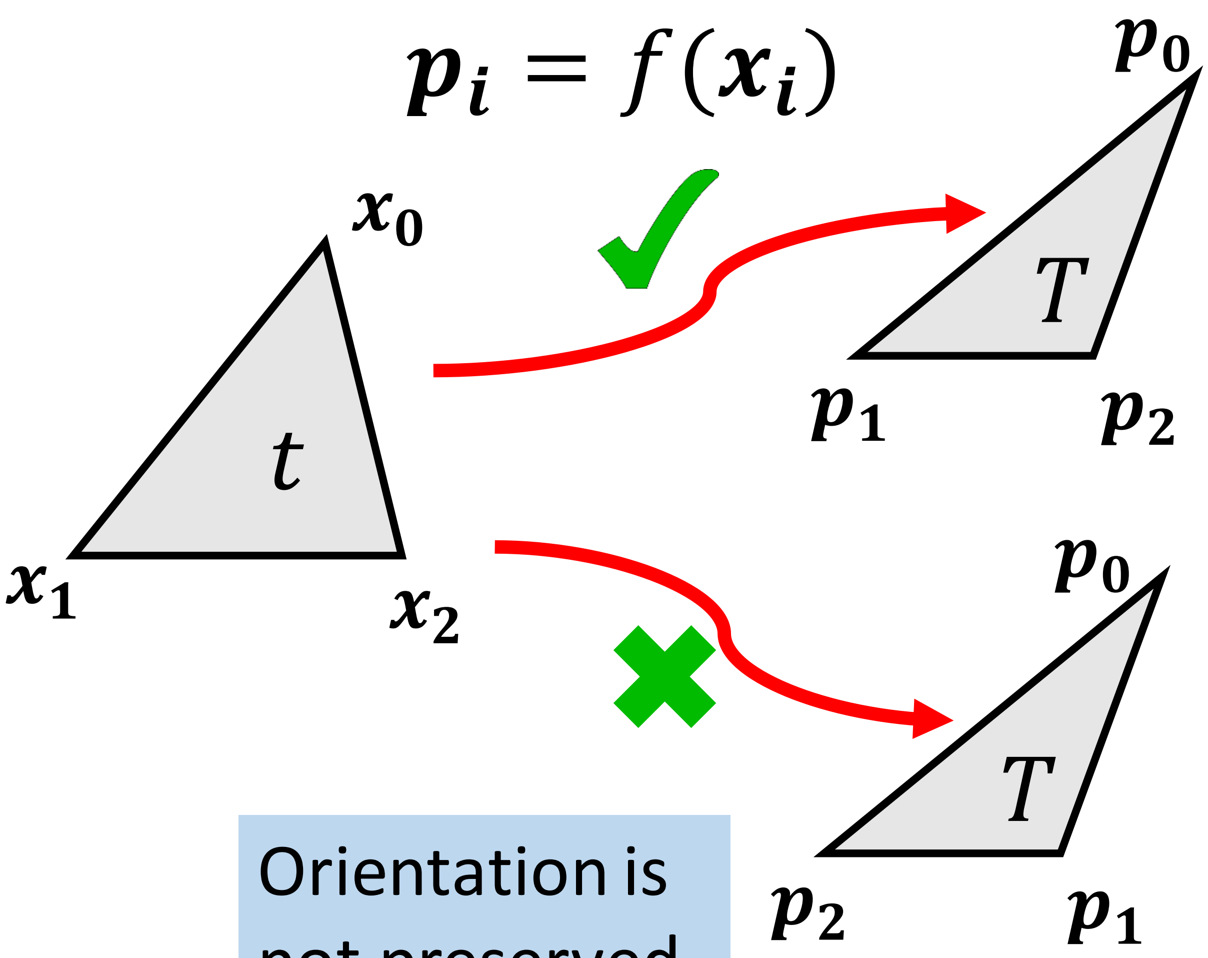
Applications – Geometric processing

- Mesh improvement
- Spherical parameterizations
- Planar and circular meshes
- Physical simulation
- Smoothing
- Registration
- ……



Basic requirements – No foldover

$$\boldsymbol{p}_i = f(\boldsymbol{x}_i)$$



Orientation is not preserved.

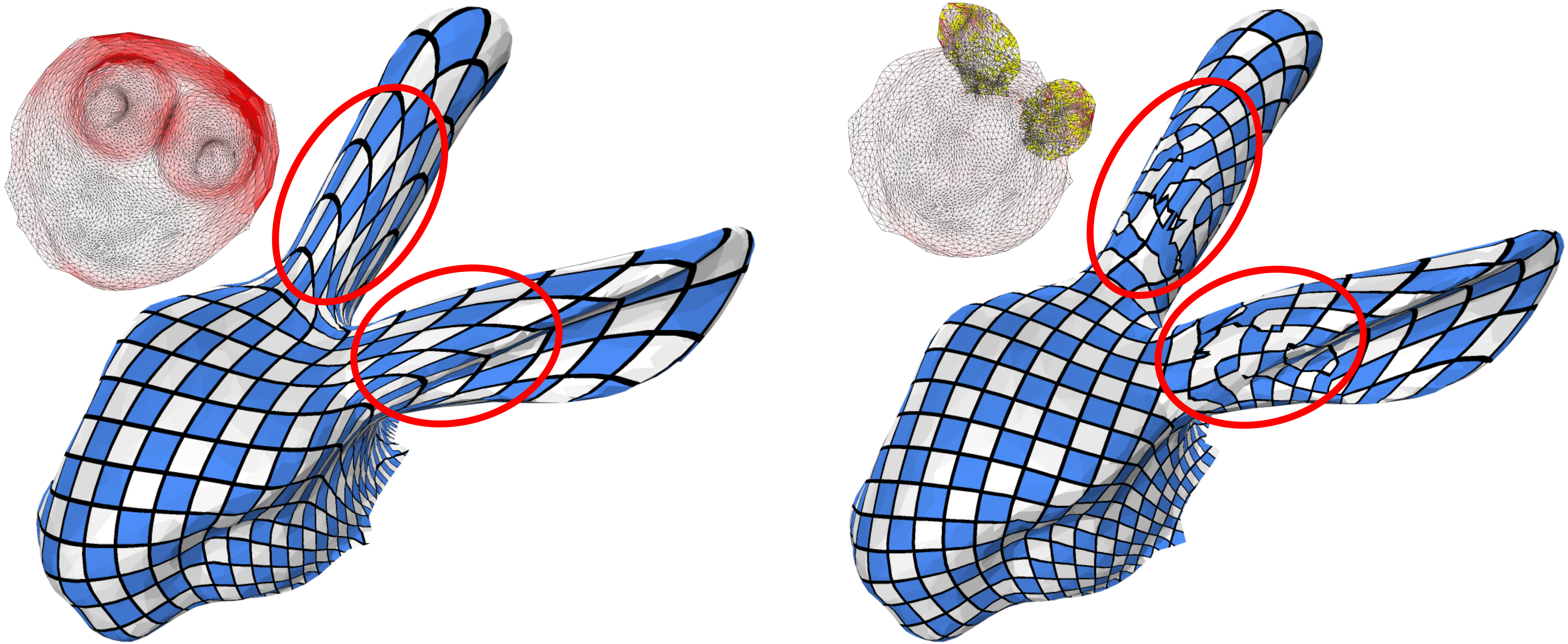
1. No realistic material with negative volume.

2. Physically impossible deformation.

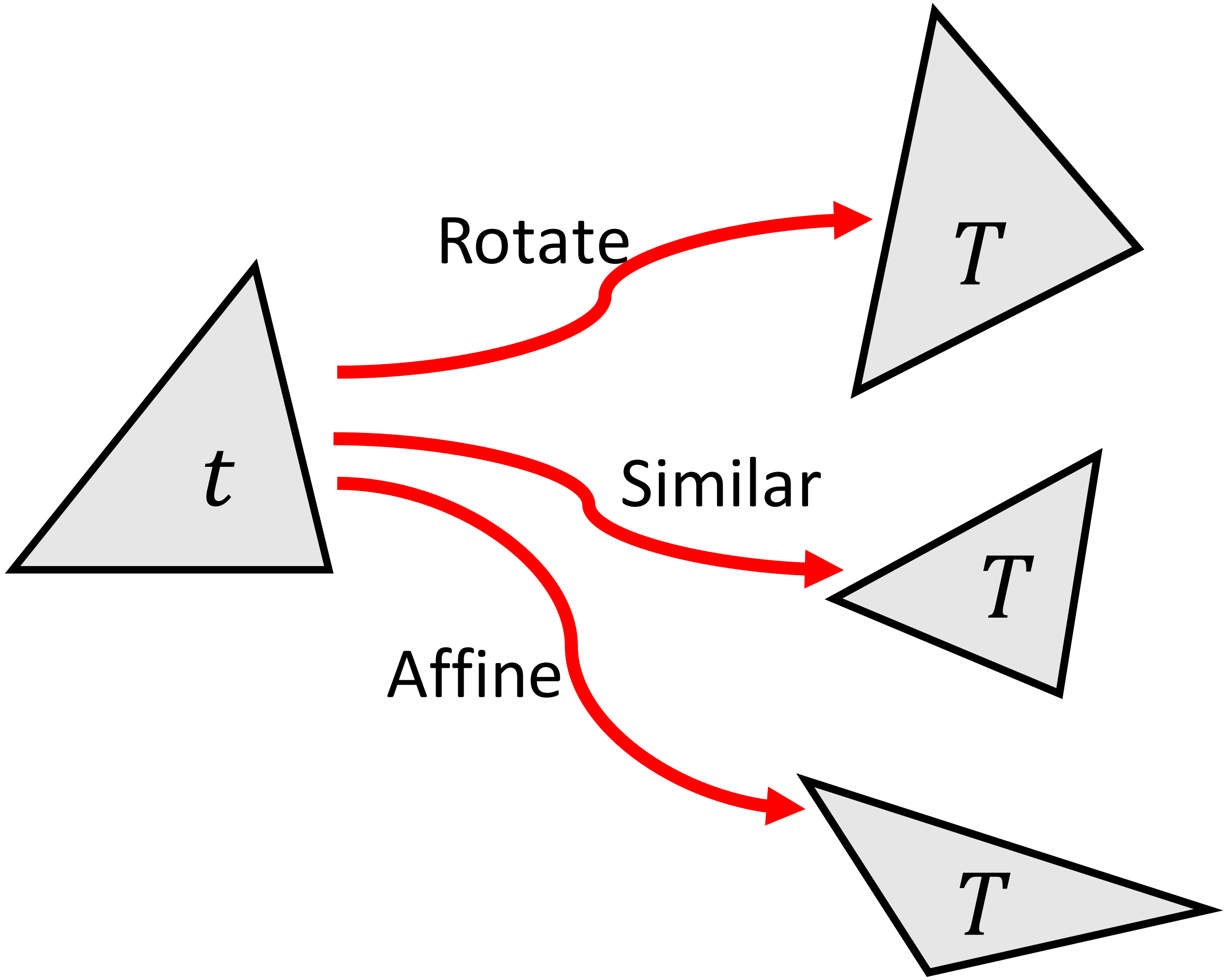
3. Invalidity for following Applications.

$$\det J(f(x)) > 0$$

Basic requirements – No foldover



Basic requirements – Low distortion

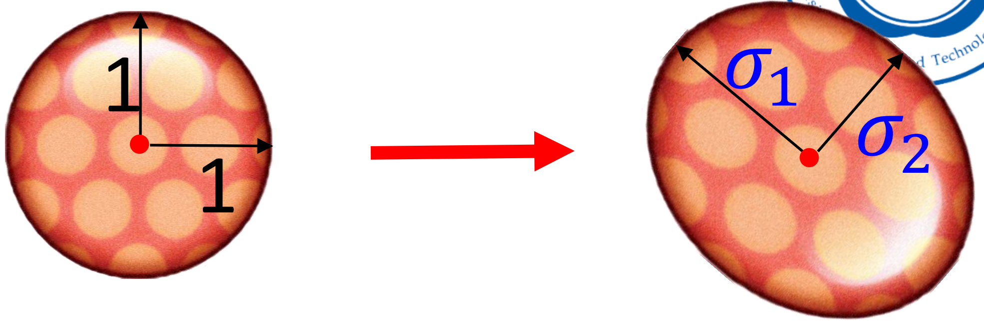


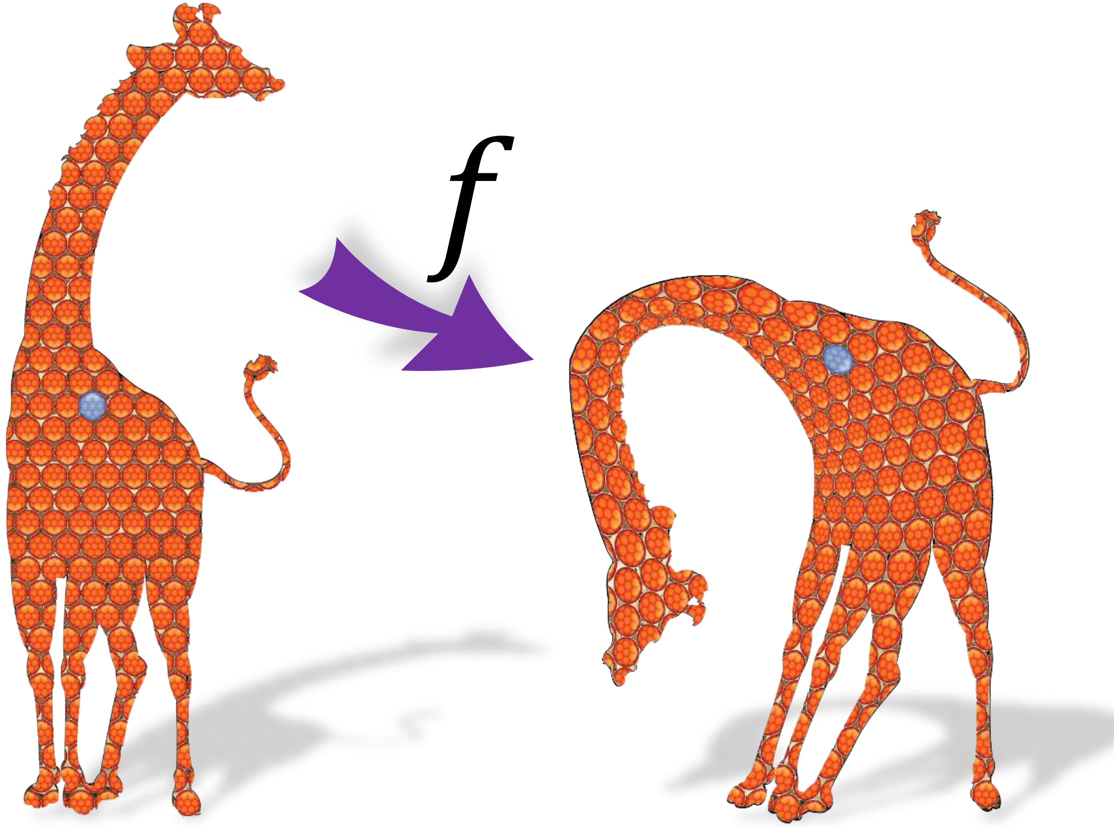
Goal:

- As Rigid As Possible or
- As similar as Possible

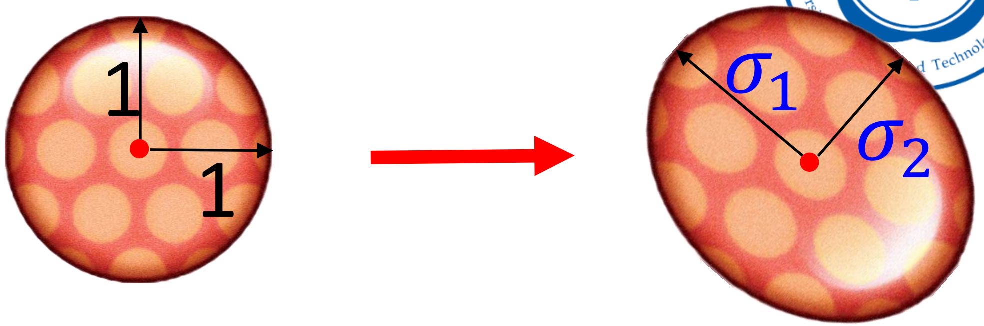
Distortion metrics $D(f)$

- Conformal / Similar
  - $\sigma_1 = \sigma_2$
- Area-preserving
  - $\sigma_1 \sigma_2 = 1$
- Isometric / Rigid
  - $\sigma_1 = \sigma_2 = 1$





Distortion metrics $D(f)$



- Conformal / Similar
  - $\sigma_1 = \sigma_2$
  - LSCM [Lévy et al. 2002]; MIPS [Hormann and Greiner 2000]
- Area-preserving
  - $\sigma_1 \sigma_2 = 1$
- Isometric / Rigid
  - $\sigma_1 = \sigma_2 = 1$
  - ARAP [Liu et al. 2008]; Symmetric Dirichlet [Smith and Schaefer 2015]; AMIPS [Fu et al. 2015]

Formulation

$$\begin{aligned} \min_{f} \ & D(f) \\ \text{s.t.} \ & \det J\big(f(x)\big) > 0, \forall x \in M \\ & S(f) \leq 0 \end{aligned}$$

$D(f)$: distortion metric

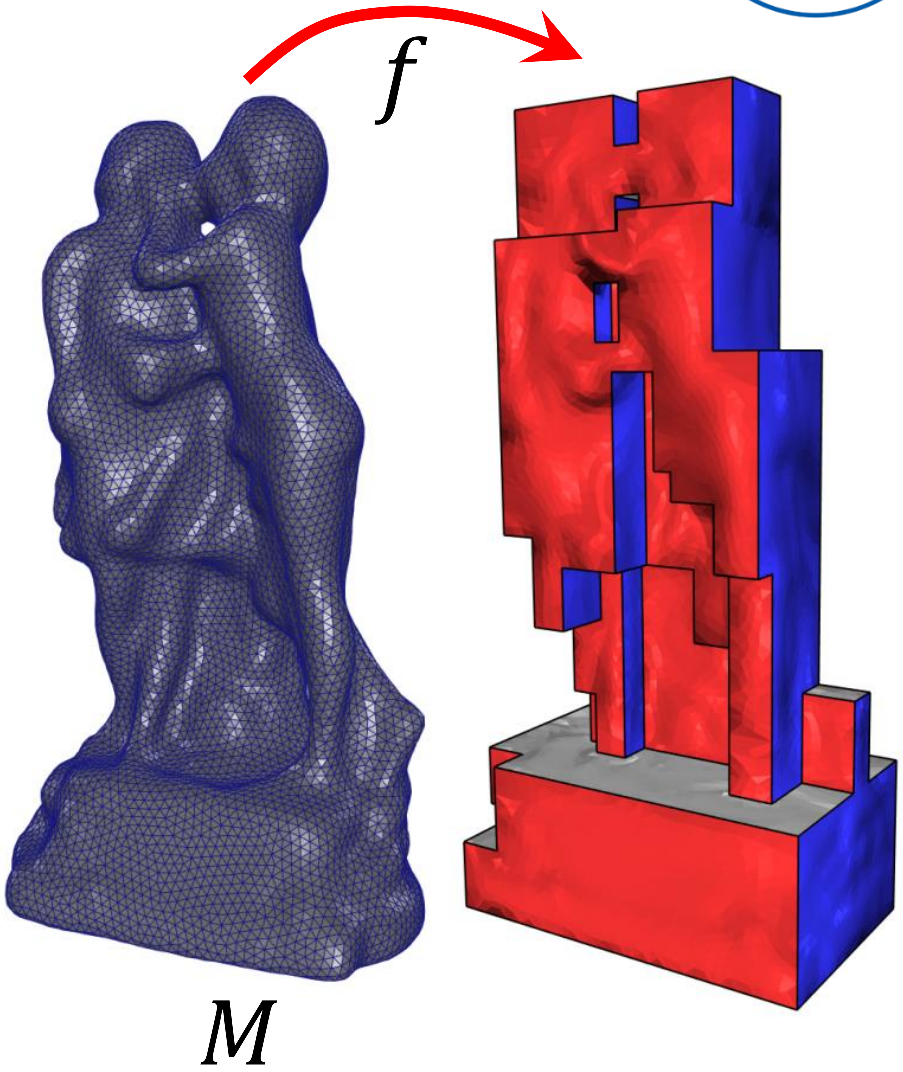
$S(f) \leq 0$: specific constraints from applications

M: input mesh or domain

Specific constraints $S(f)$

- PolyCube
- Constraints:

$\forall t \in M, \boldsymbol{n}\big(f(t)\big) \in \{\pm X, \pm Y, \pm Z\}$



Challenges

$$
\begin{aligned}
&\min_{f} \; D(f) \\
&\text{s.t.} \quad \det J\big(f(x)\big) > 0, \forall x \in M \\
&\qquad\;\; S(f) \leq 0
\end{aligned}
$$

Non-linear objective function.

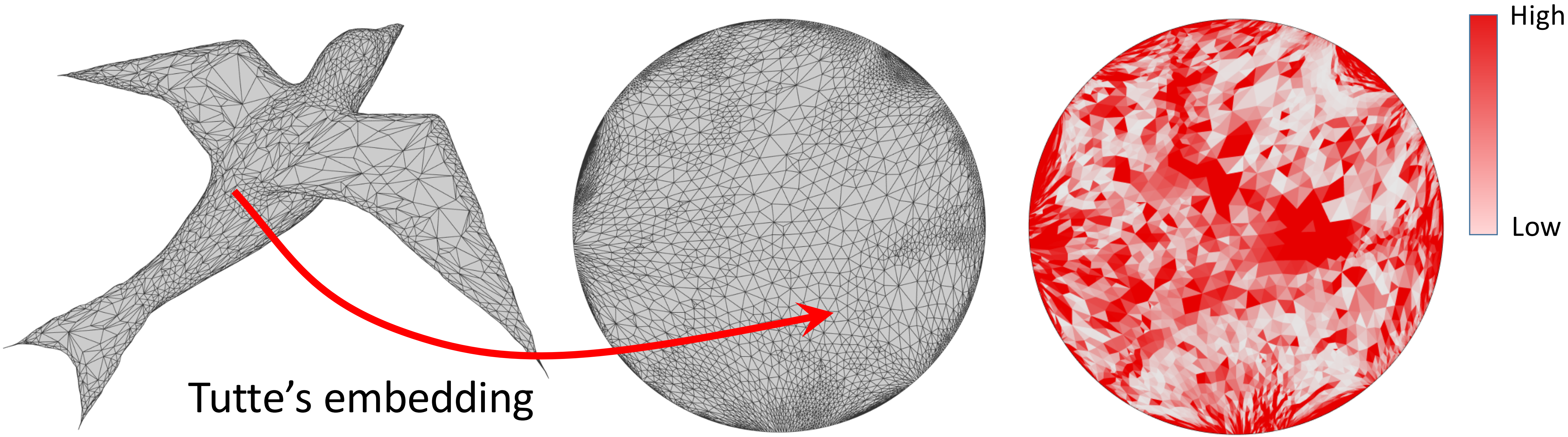
Non-linear and non-convex constraints.

Different applications have different requirements.

Progressive Parameterizations

Foldover-free parameterizations

- Maintenance-based method



Convex boundary

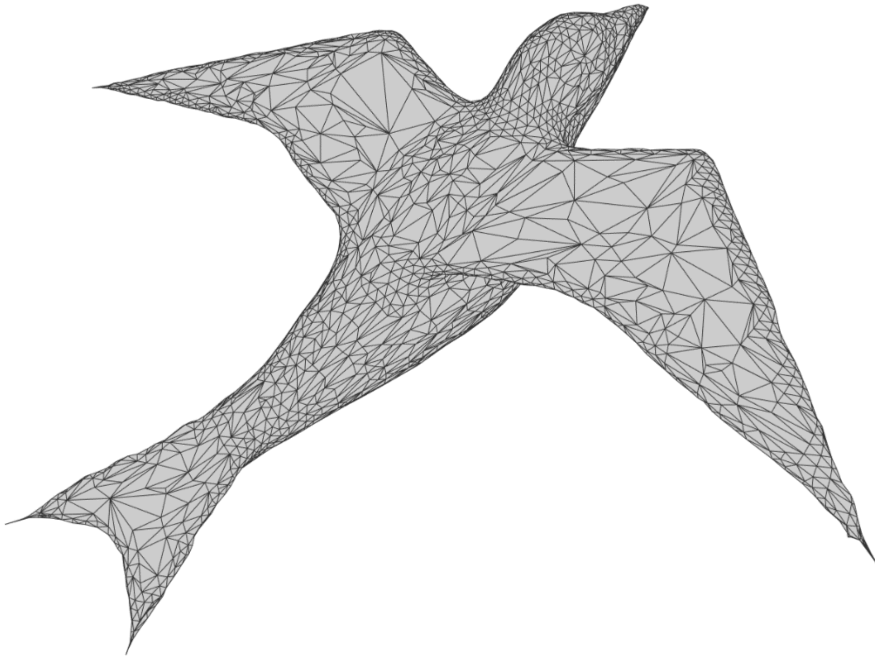
High distortion

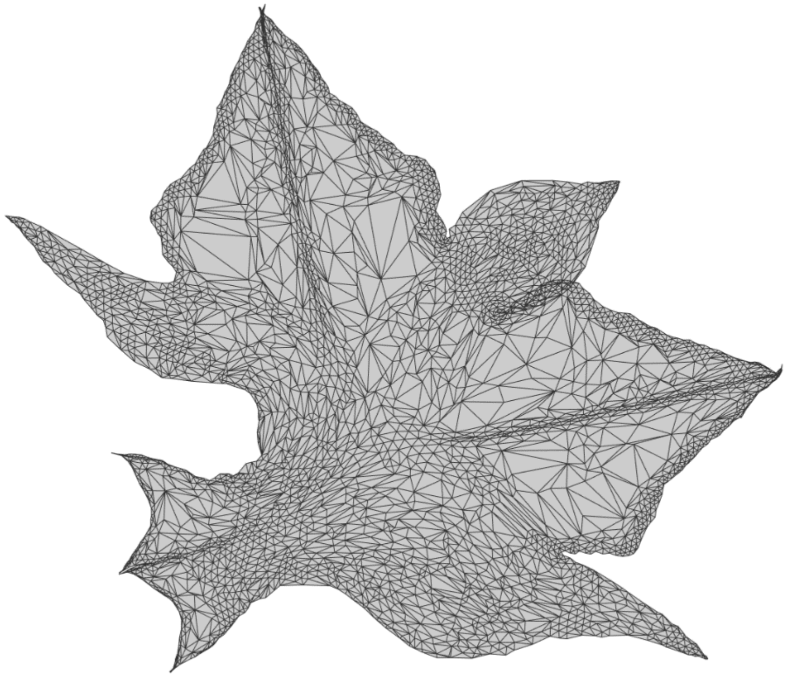
Foldover-free parameterizations

- Maintenance-based method

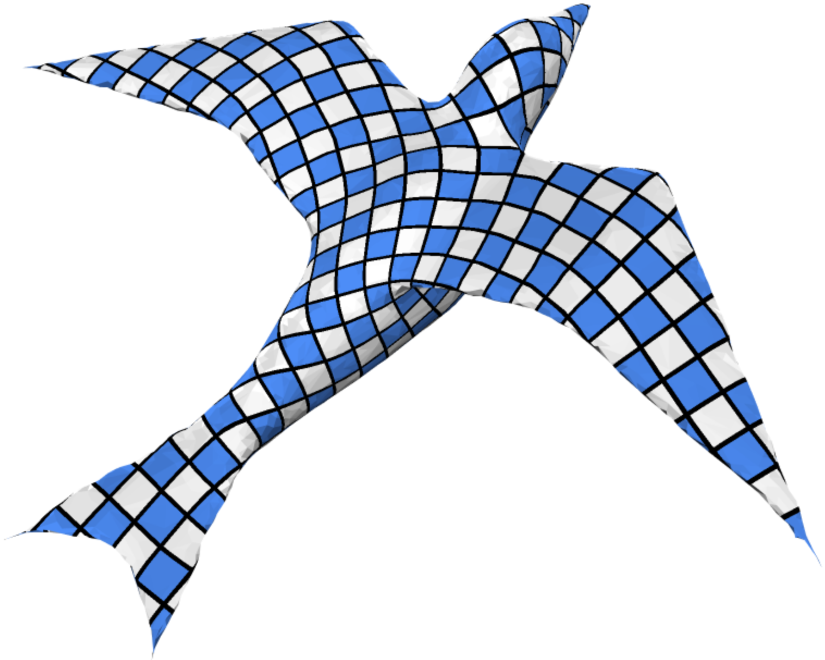
Foldover-free parameterizations

- Maintenance-based method





Parameterization



Texture mapping

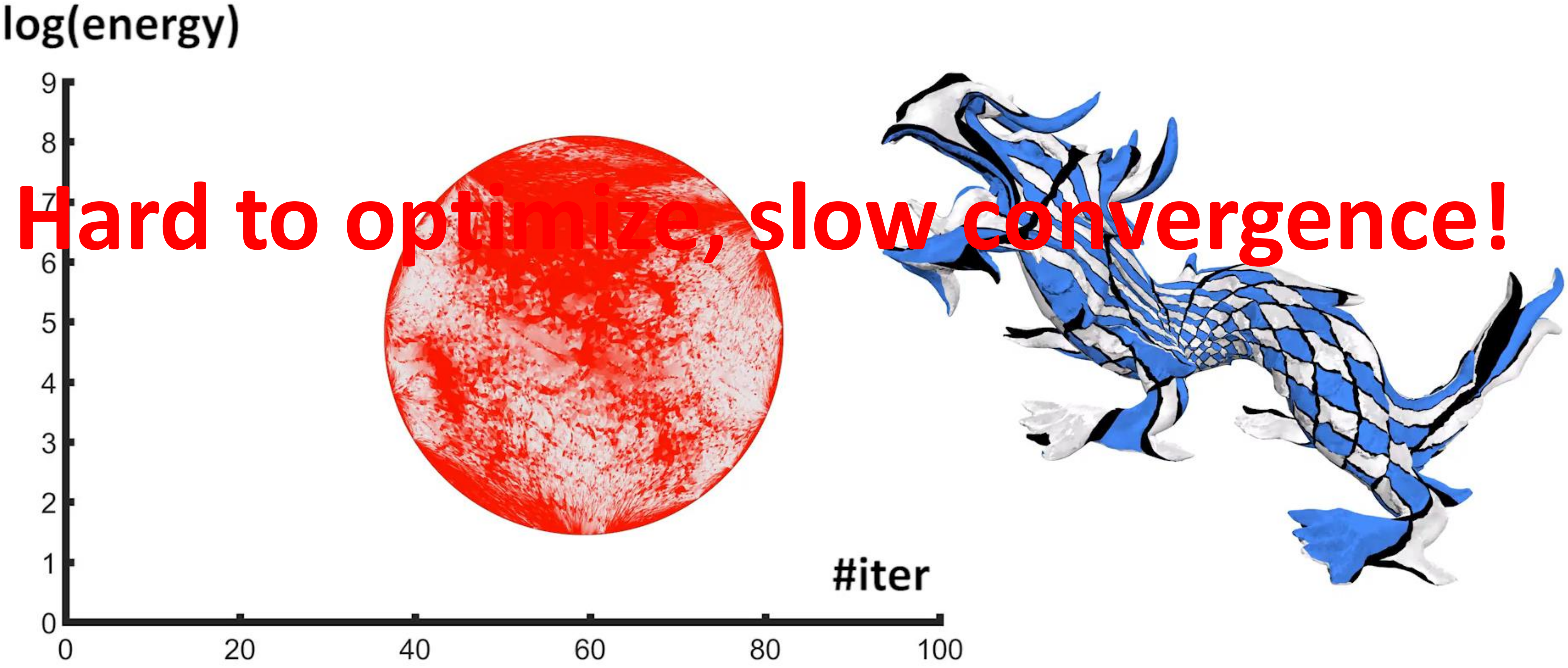
Foldover-free parameterizations

- Maintenance-based method
  - Block coordinate descent methods [Fu et al. 2015; Hormann and Greiner 2000]
  - Quasi-Newton method [Smith and Schaefer 2015]
  - Preconditioning methods [Claici et al. 2017; Kovalsky et al. 2016]
  - Reweighting descent method [Rabinovich et al. 2017]
  - Composite majorization method [Shtengel et al. 2017]

Various solvers!

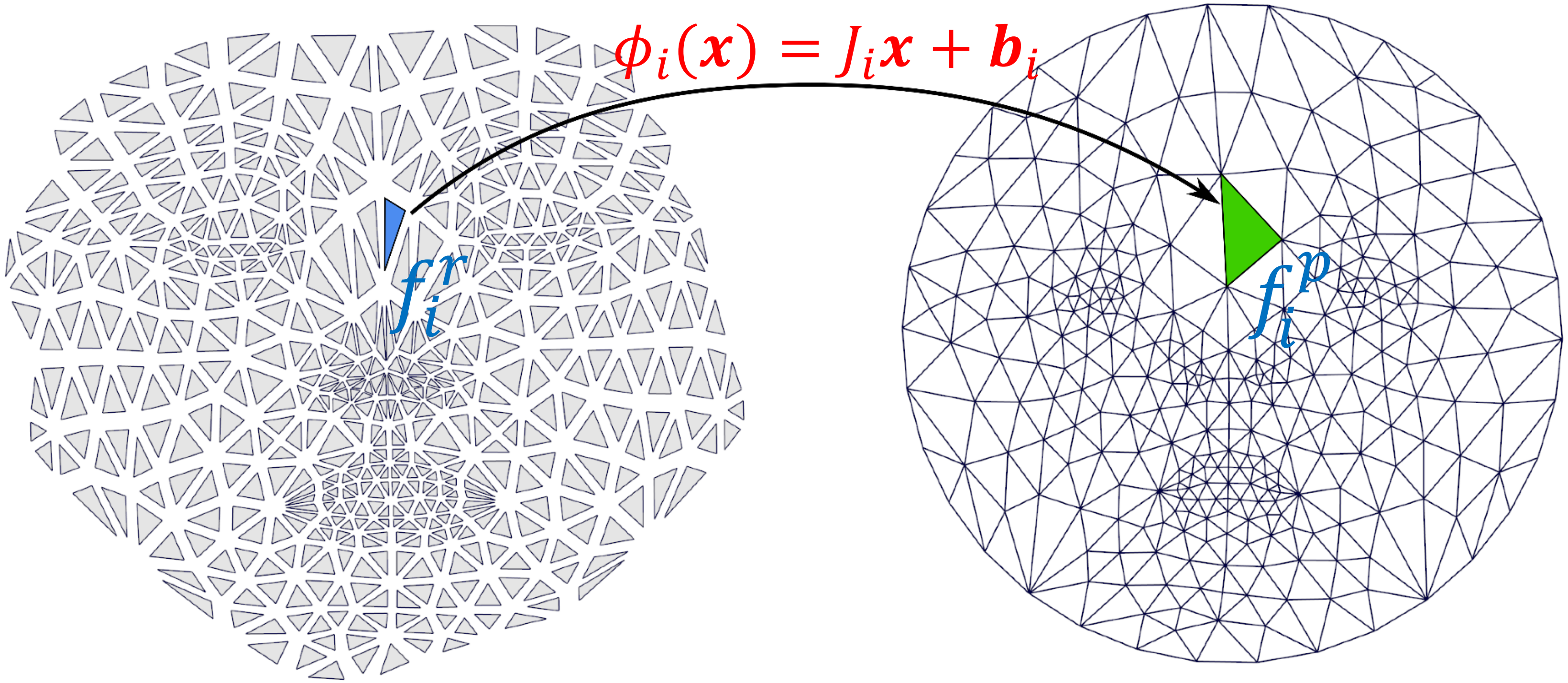
Challenge

Extremely large distortion on initializations



Hard to optimize, slow convergence!

Reference-guided distortion metric



Reference M^r: A set of individual triangles

Parameterized mesh M^p

Symmetric Dirichlet metric:

$$D\left(f_i^r, f_i^p\right) = \frac{1}{4}\left(\|J_i\|_F^2 + \left\|J_i^{-1}\right\|_F^2\right) = \frac{1}{4}\left(\sigma_i^2 + \sigma_i^{-2} + \tau_i^2 + \tau_i^{-2}\right)$$

σ_i, τ_i: singular values of J_i

Opt value = 1 when $\sigma_i = \tau_i = 1$

Formulation

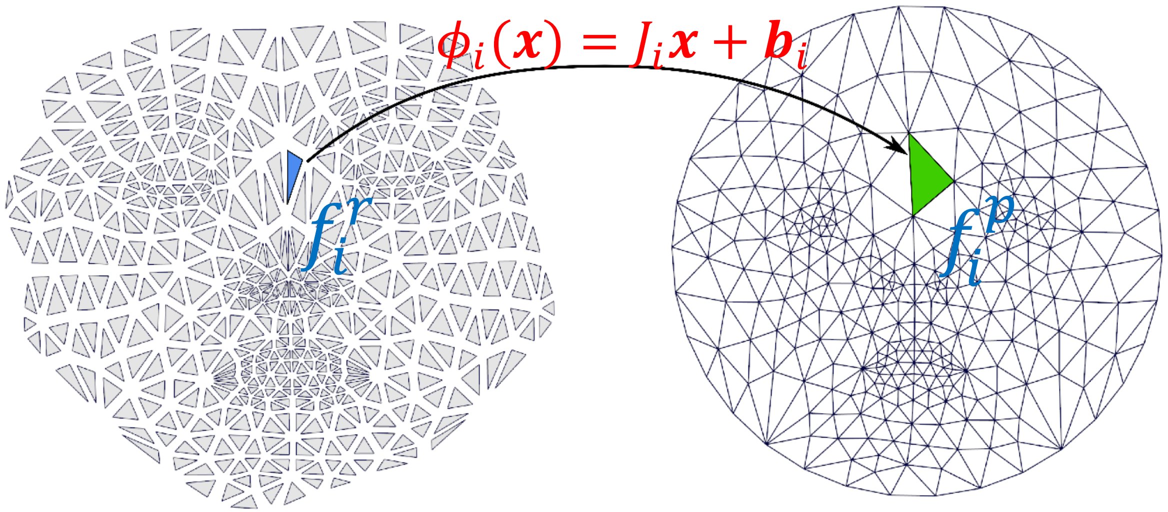
$$\min_{M^p} E(M^r, M^p) = \sum_{i=1}^{N_f} \omega_i D(f_i^r, f_i^p)$$ Low distortion

$$\text{s.t.} \quad \det J_i > 0, \qquad i = 1, \dots, N_f.$$ Foldover-free constraints

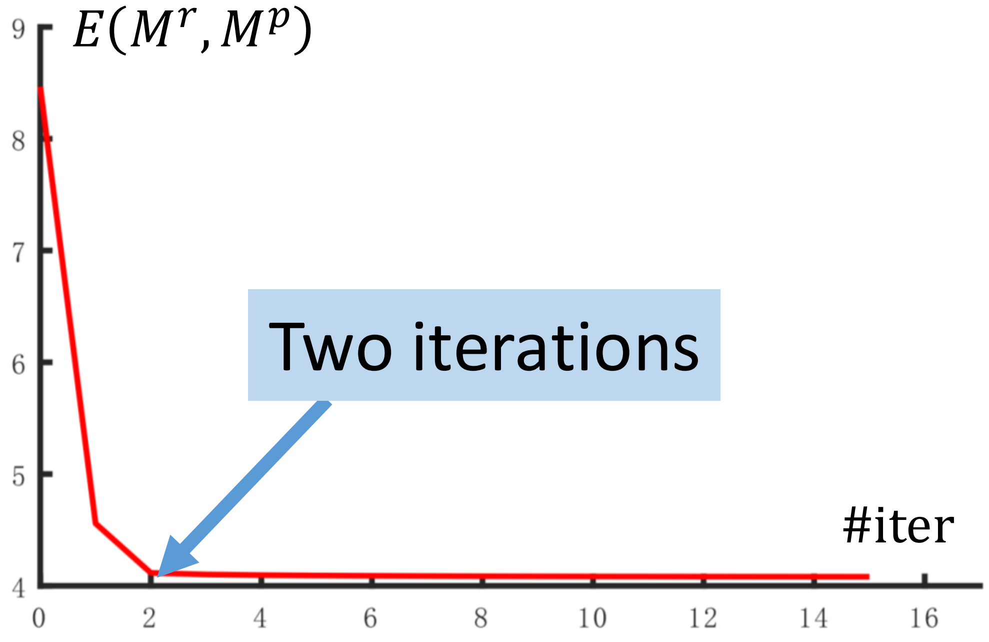
Exsiting methods choose *the triangles f_i of input mesh M* as reference triangles.

The energy is *numerically difficult to optimize*, leading to numerous iterations and high computational cost.

Progressive Parameterizations [Liu et al. 2018]

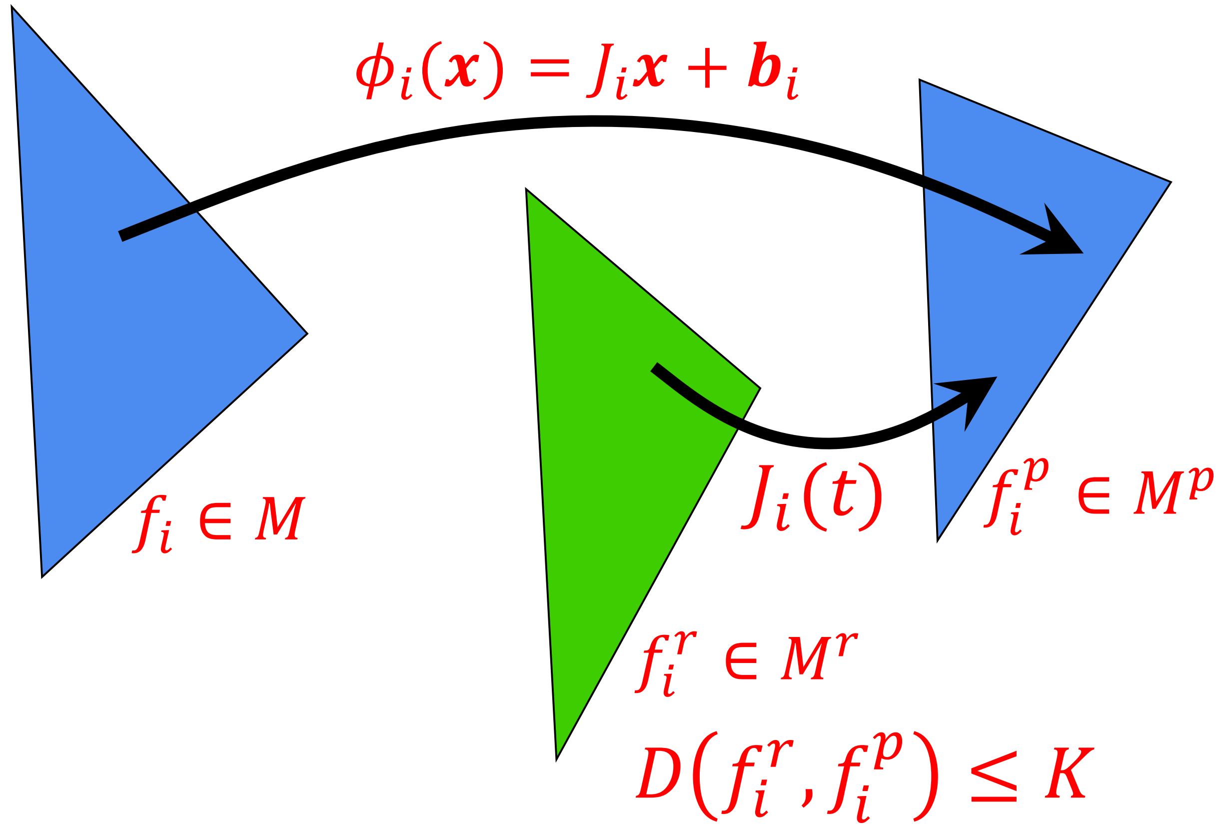


If $D(f_i^r, f_i^p) \leq K, \forall i$, only a few iterations in the optimization of $E(M^r, M^p)$ are necessary.



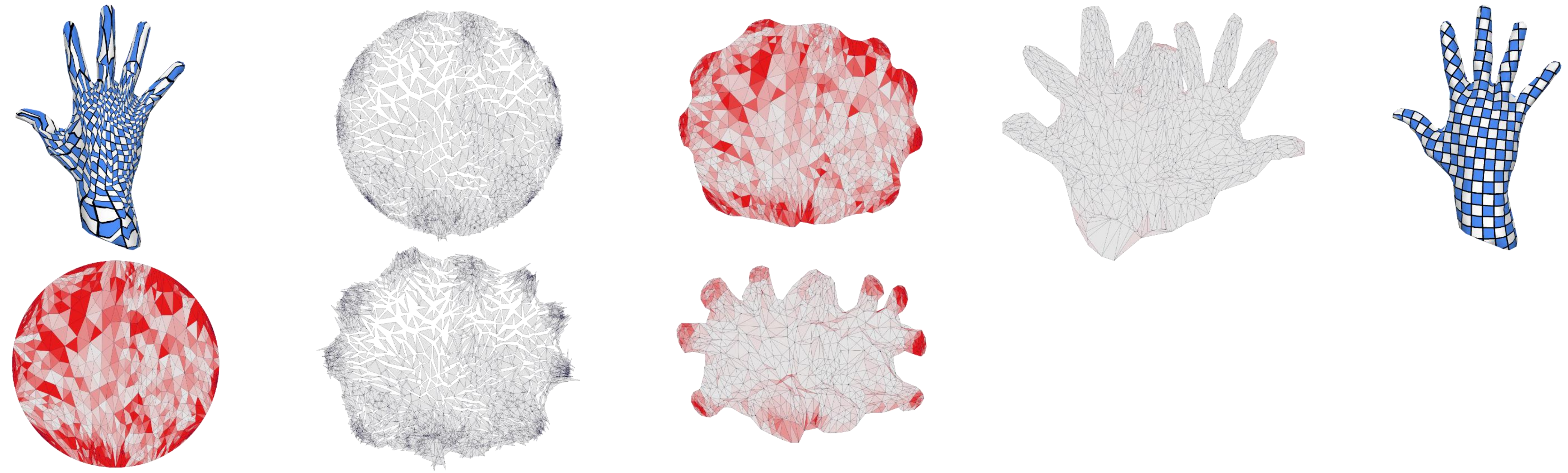
Progressive Parameterizations [Liu et al. 2018]

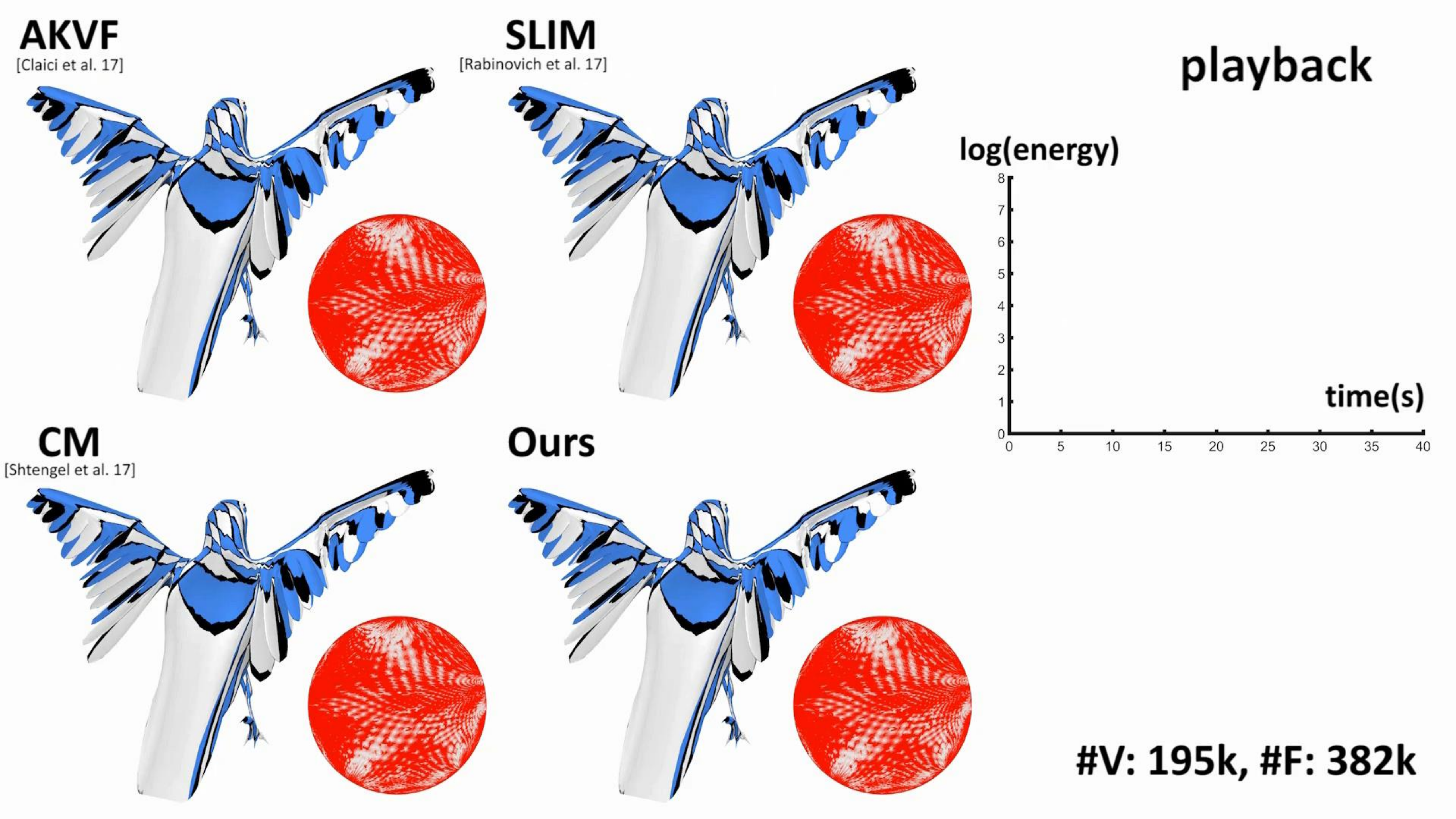
- Progressively approach f_i



Progressive Parameterizations [Liu et al. 2018]

Input: a 3D triangular mesh + initialization → Construct new references → Update Parameterization → Final Optimization → Output 2D parameterization

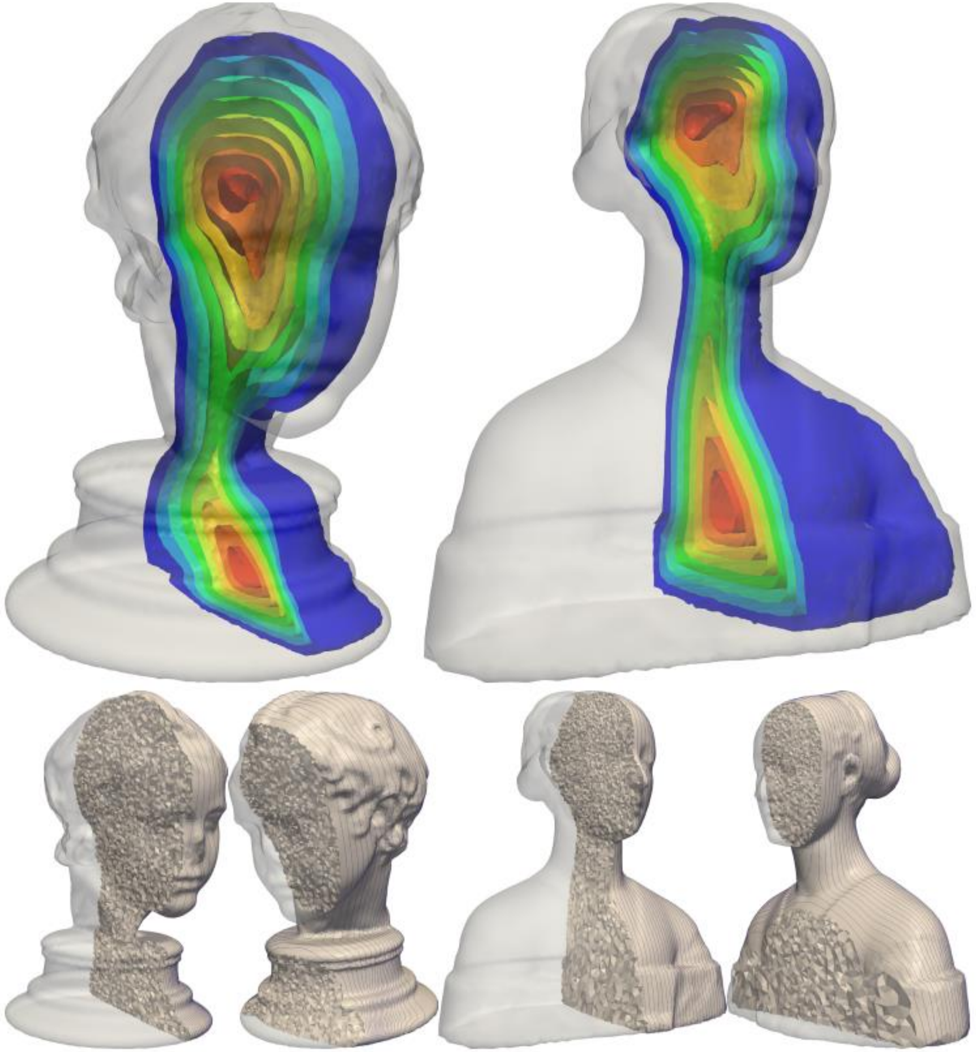




AKVF
[Claici et al. 17]
SLIM
[Rabinovich et al. 17]
playback
log(energy)
8
7
6
5
4
3
2
1
0
0
5
10
15
20
25
30
35
40
time(s)
CM
[Shtengel et al. 17]
Ours
#V: 195k, #F: 382k

Hard constraints

- No initial feasible solutions.
- Given boundary correspondence, how to compute the positions of interior vertices?



[Kovalsky et al. 2015]

Bounded distortion method

$$
\begin{aligned}
\min_{f} \quad & E(f) \\
\text{s.t.} \quad & \det J\big(f(x)\big) > 0, \forall x \in M \\
& D\big(f(x)\big) \leq K, \forall x \in M
\end{aligned}
$$

- Quadric objective function
- Explicitly bound distortion

Convexification

Convex problem

- ✓ Parameterizations
- ✓ Deformation
- ✓ Improvement

Yaron Lipman. “Bounded Distortion Mapping Spaces for Triangular Meshes”. SIGGRAPH 2012.

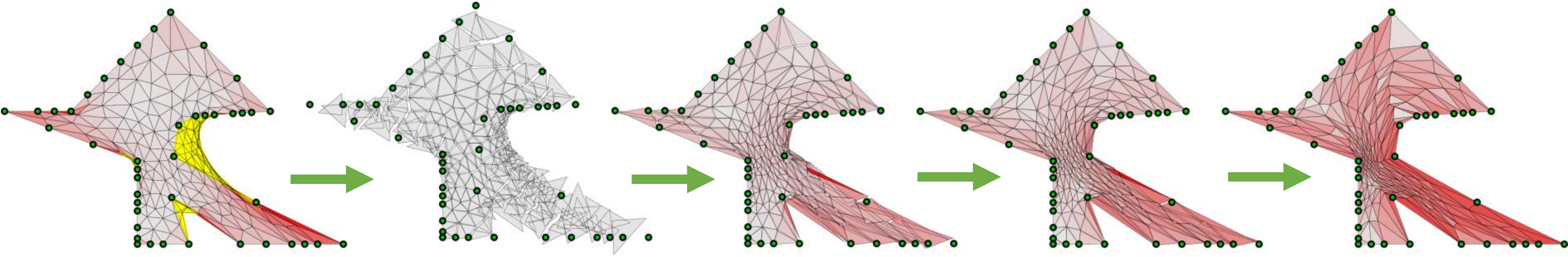
Variable Representations

- Bounded distortion method
  - Constrained problem (QP, SDP)
  - Time-consuming
  - Variables: vertex positions and coefficients

- Angle-based mapping [Sheffer and de Sturler 2001; Sheffer et al. 2005, Paillé et al. 2015]
  - Nonlinear equality constraints

Affine transformations [Fu et al. 2016]

- Disassembly + Assembly
  - Treat affine transformations as variables
  - Unconstrained optimization



Unconstrained optimization problem

Disassembly: project initial A_i^0 into feasible space.

Assembly: unconstrained optimization.

$$\min_{\substack{A_1,\dots,A_N \\ T_1,\dots,T_N}} \lambda E_{assembly} + E_C + \mu E_m$$

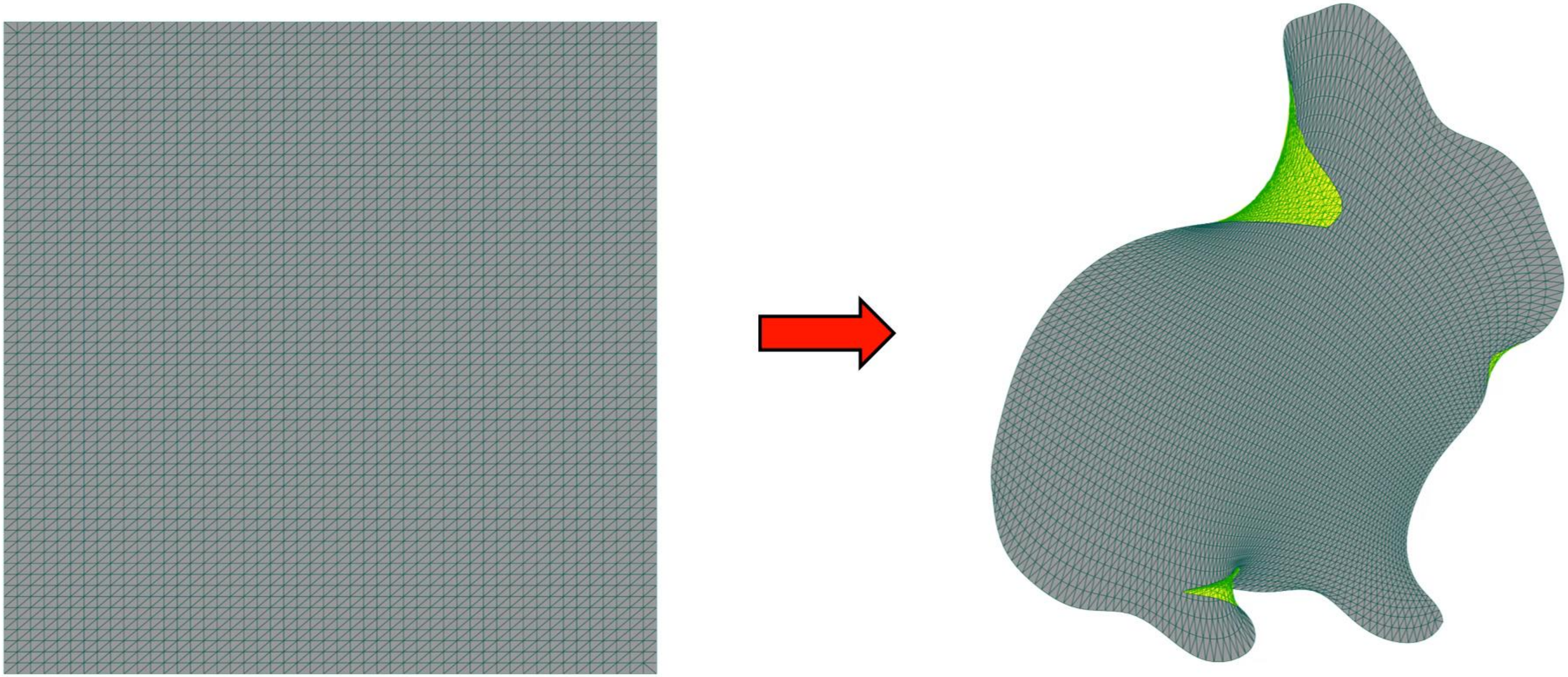
$E_{assembly}$: summation of assembly constraints.

E_C: Barrier function on distortion

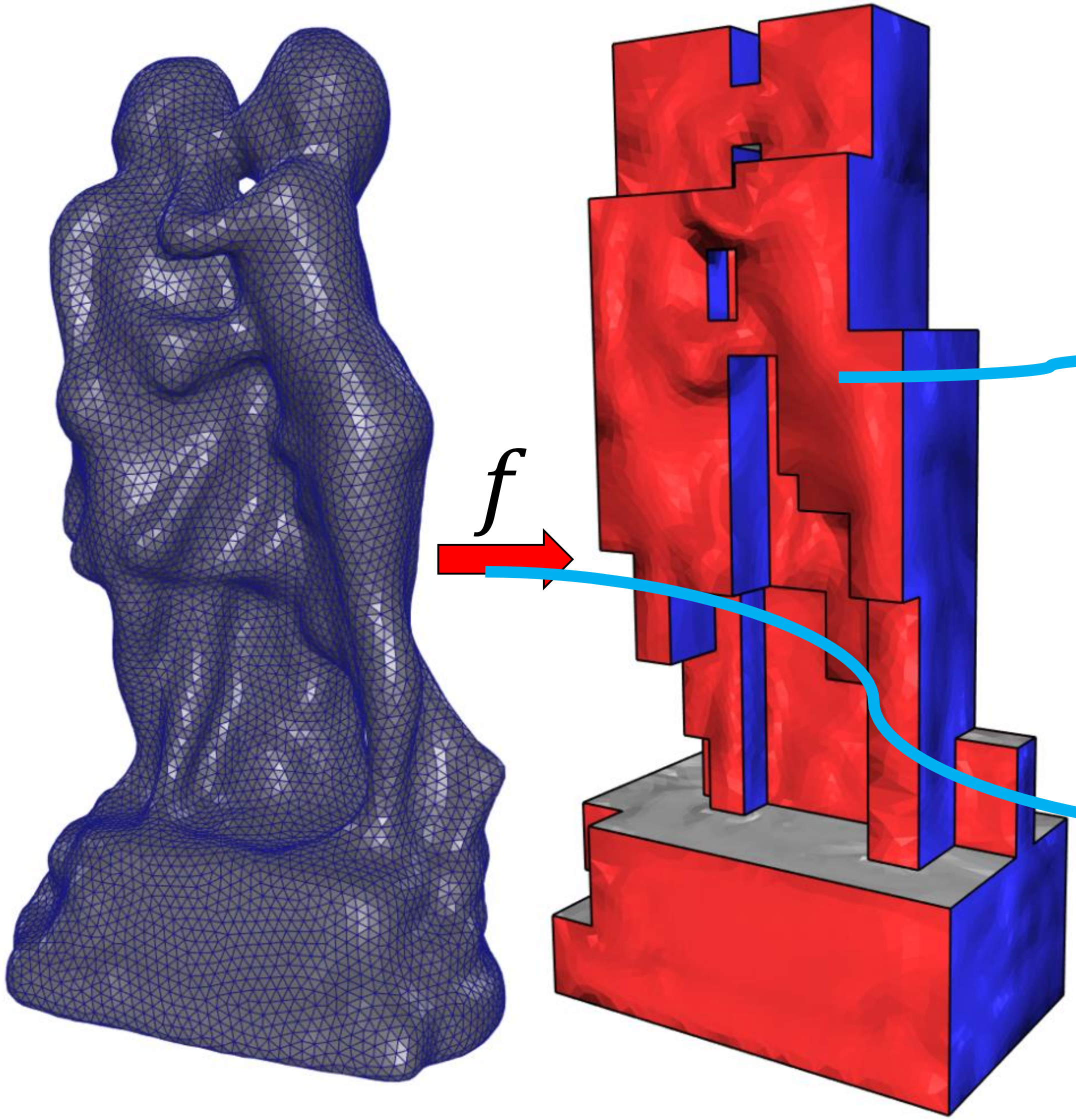
E_m: users' designed energy

$$\lambda_{k+1} = \min\left(\lambda_{\min} \cdot \max\left(\frac{E_{C,k} + \mu E_{m,k}}{E_{assembly,k}}, 1\right), \lambda_{\max}\right)$$

1. $E_{assembly}$ dominates the energy, approach zero;
2. λ_{max}: avoid large distortion.



PolyCube



Tetrahedral Mesh PolyCube

PolyCube:

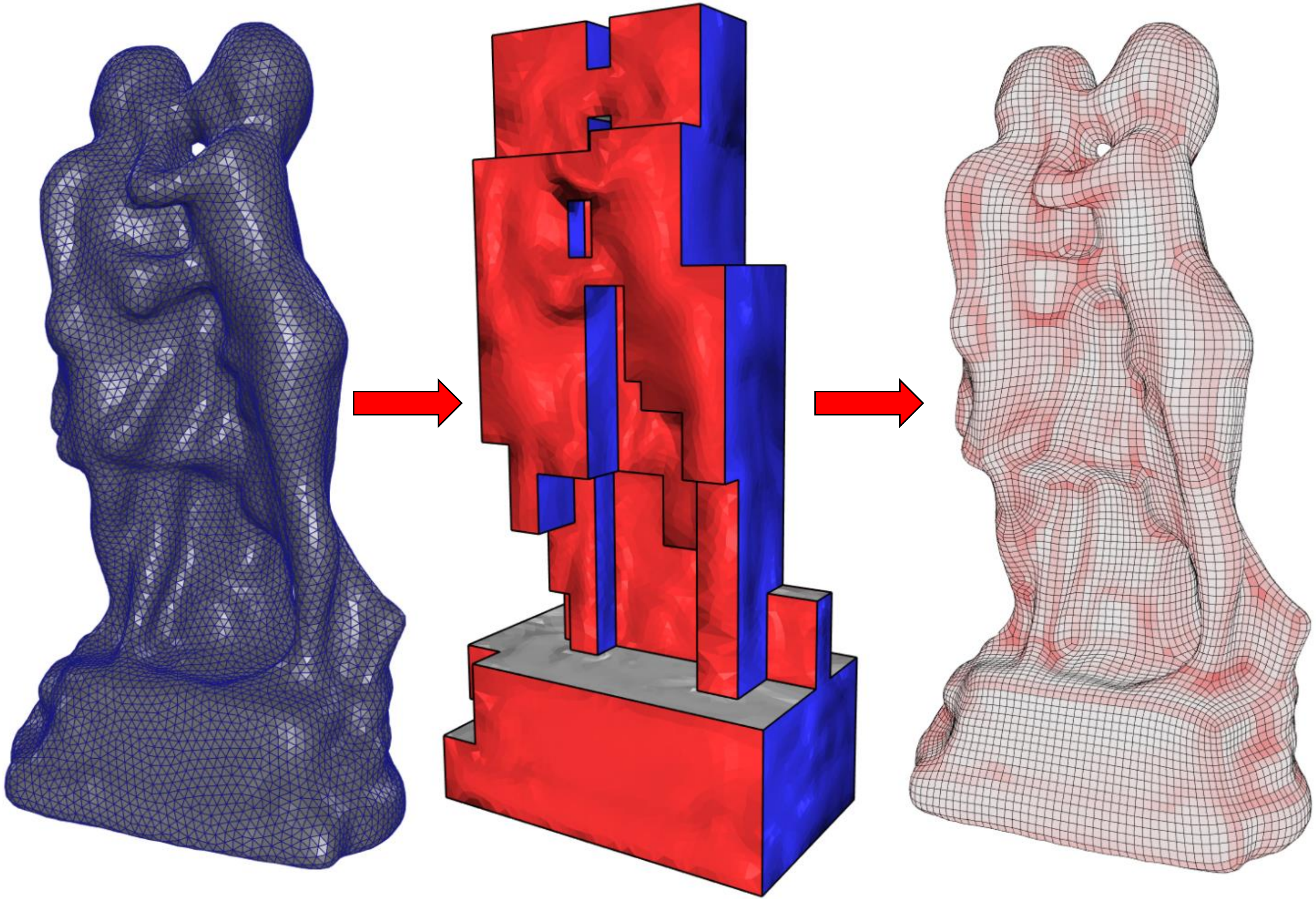
1. Compact representations for closed complex shapes
2. Boundary normal aligns to the axes.
3. Axes: $(\pm 1,0,0)^T, (0, \pm 1,0)^T, (0,0, \pm 1)^T$.

PolyCube-Map f:

1. A mesh-based map.
2. Inversion-free and low distortion.

All-Hex mesh

Tetrahedral Mesh PolyCube All-Hex Mesh

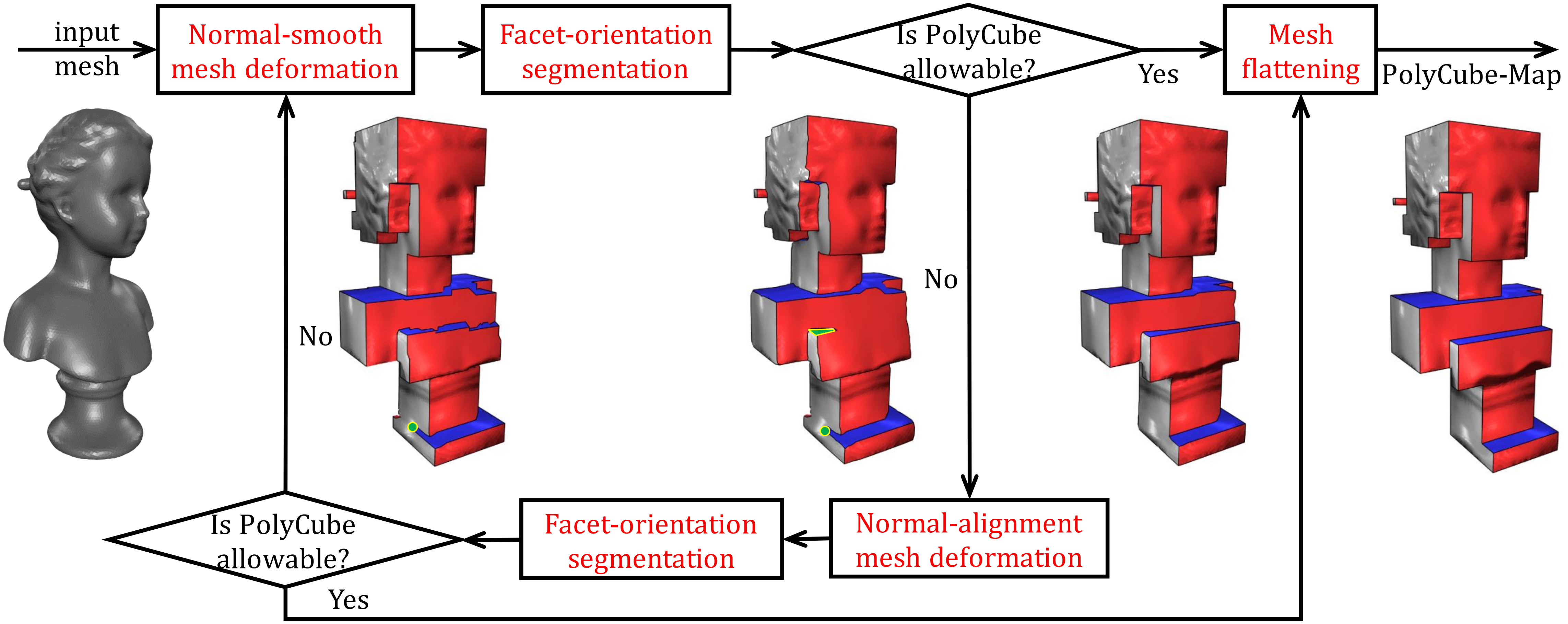


Applications:

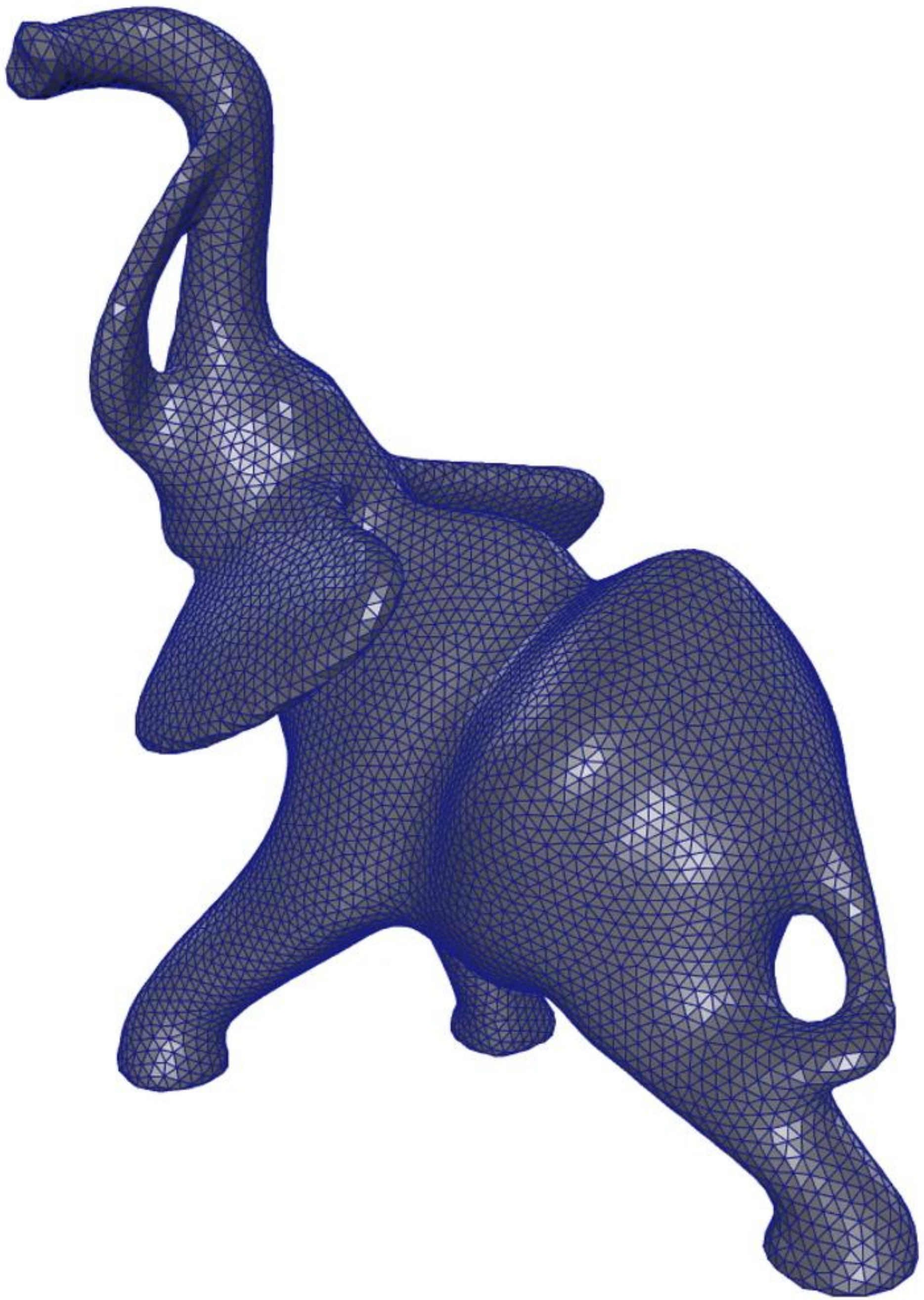
1. All-Hex Mesh generation.
2. Texture Mapping [Tarini et al. 2004].
3. GPU-based subdivision [Xia et al. 2011].

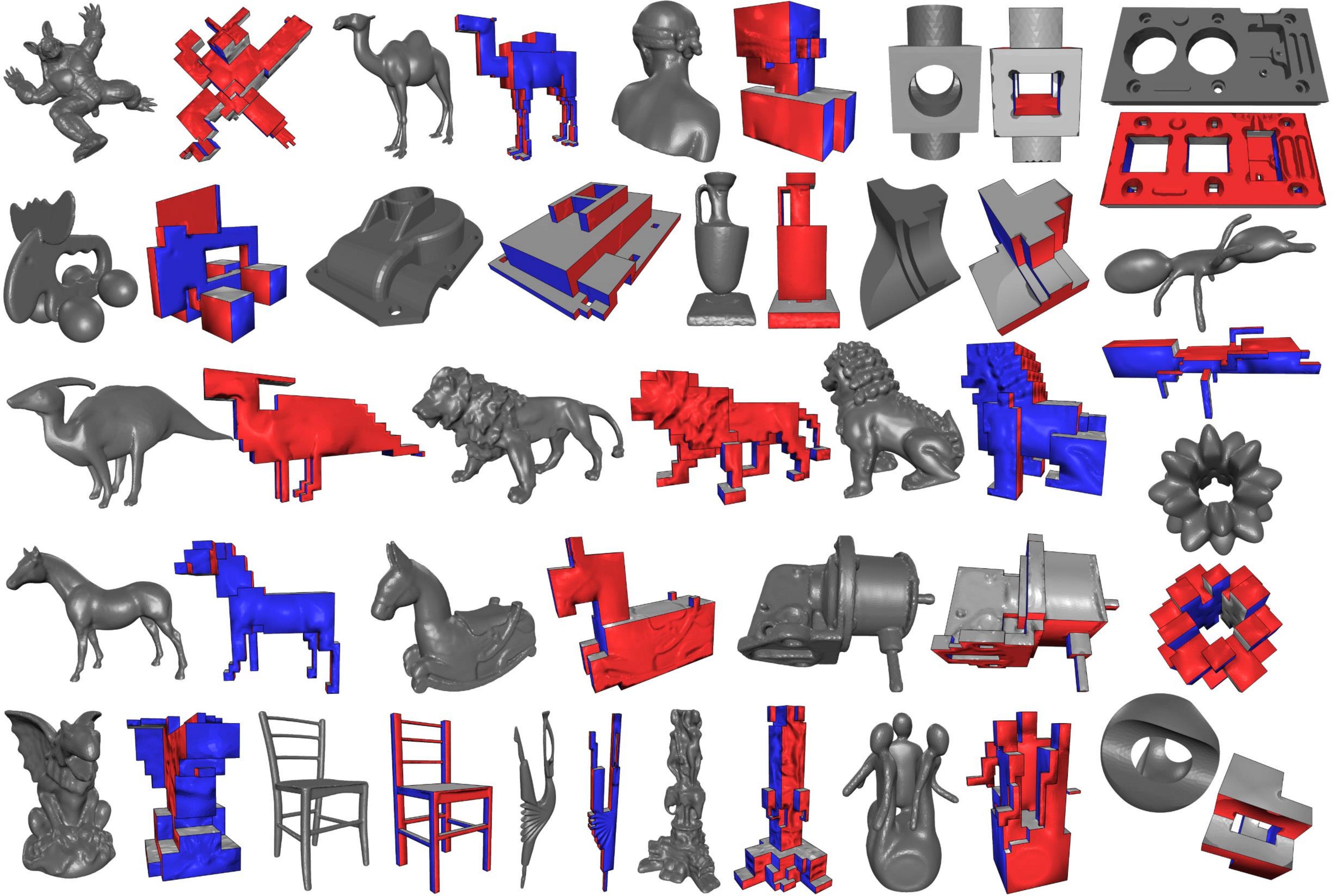
……

Deformation-based Algorithm



An example





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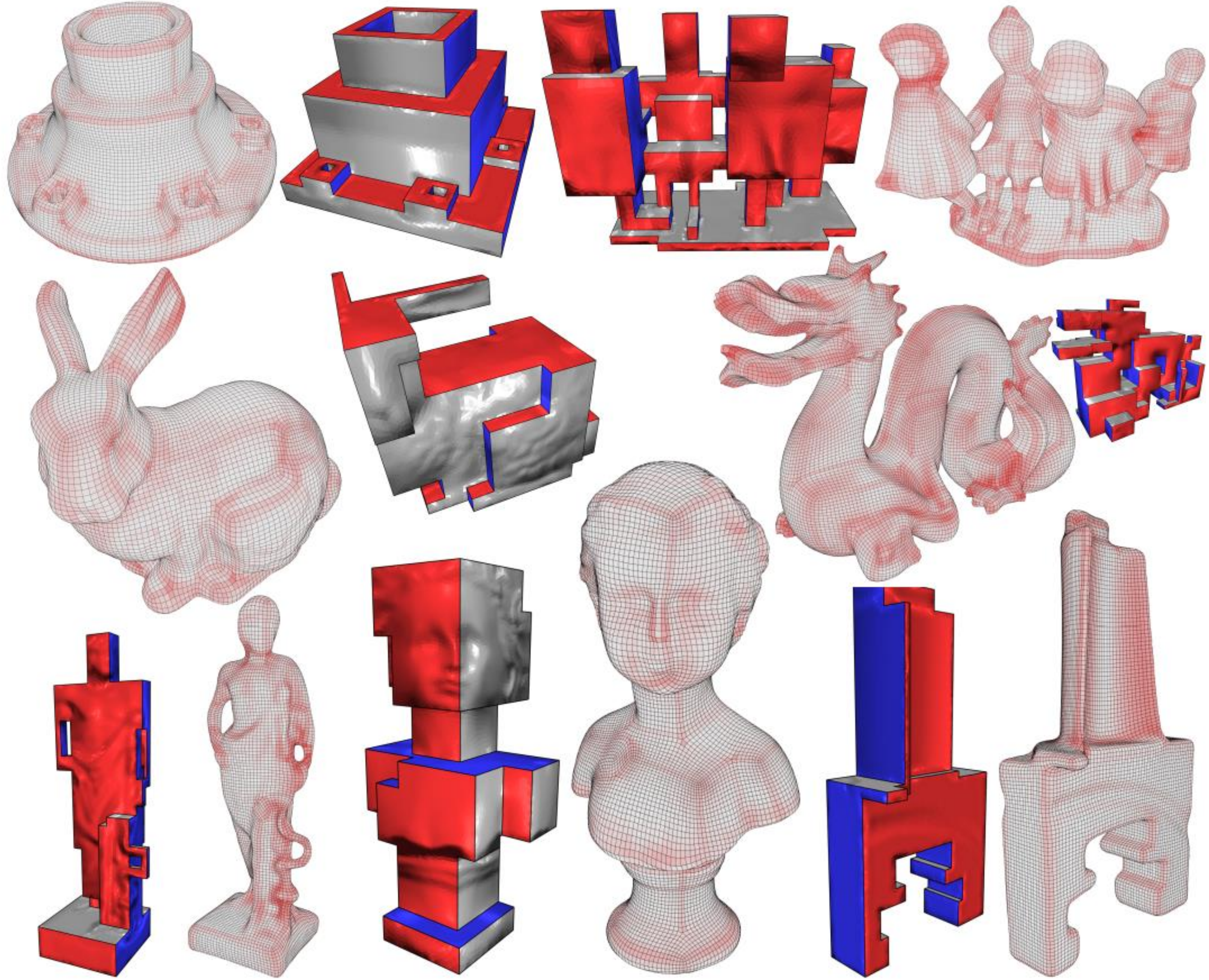
University of Science and Technology of China



Conclusion

- Mappings are everywhere

- A fundamental task in computer graphics
  - Many applications

- Computing effective and efficient solutions is challenging
  - More researches

Thank you!

http://staff.ustc.edu.cn/~fuxm