(2) Suppose Kamal decides to give each child at least one rupee. So answer is again different than previous one.

How would we possibly answer these kinds of questions?

Syllabus Topic : Combinatorics and Graph Theory

1.2 Combinatorics and Graph Theory

☞ How combinatorics and graph theory relate with each other ? Give an example.

A graph G consists of vertex set V and edge set E of two elements set of V.

Representation of Graph

(1) First number represent number of vertices

(2) After number of vertices each pair of number represent an edge of the graph

Example : Graph G is defined as,

$$G(V, E) = \{(6,2)(1,5)(1,7)(6,8)(1,9)(3,4)(7,5)(3,1)(9,5)(9,7)\}$$

So G is graph with 9 vertices as 10 pair of number is there. It has 10 edges.

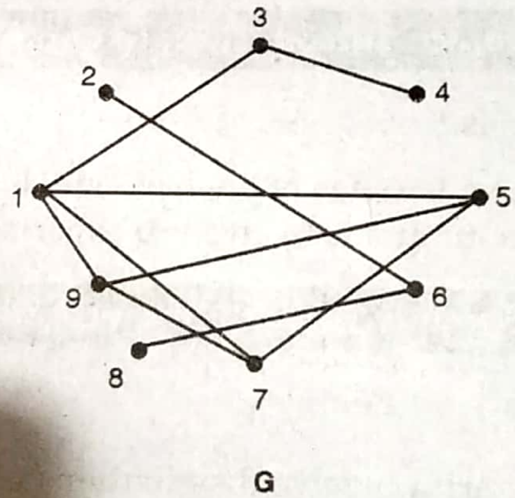


Fig. 1.2.1(a)

We observe following points in graph G, (Refer Fig. 1.2.1(a))

(1) G has 9 vertices and 10 edges. Edge is nothing but line connected two vertices.

(2) {2, 6} is an edge. As this pair given in the notation. Similarly you can write remaining edges, but {4, 5} is not an edge.

(3) Vertices 1 and 9 are adjacent. All vertices those are connected by edge are adjacent to each other.

(4) Vertices 7 and 3 are not adjacent because they are not directly connected by edge.

(5) P = (5, 9, 7, 1, 3, 4) is a path of length 5 from vertex 5 to vertex 4. Because it contain 5 edges (5, 9), (9, 7), (7, 1), (1, 3) and (3, 4).

(6) C = (1, 7, 9, 5) is cycle of length 4 because it contain 4 edges. Also start and end vertex is same i.e. '1'. Four edges are (1,7),(7,9),(9,5) and (5,1).

(7) G is disconnected and has two components. One of the components has vertex set {8, 6, 2}. Components is nothing but connected Graph.

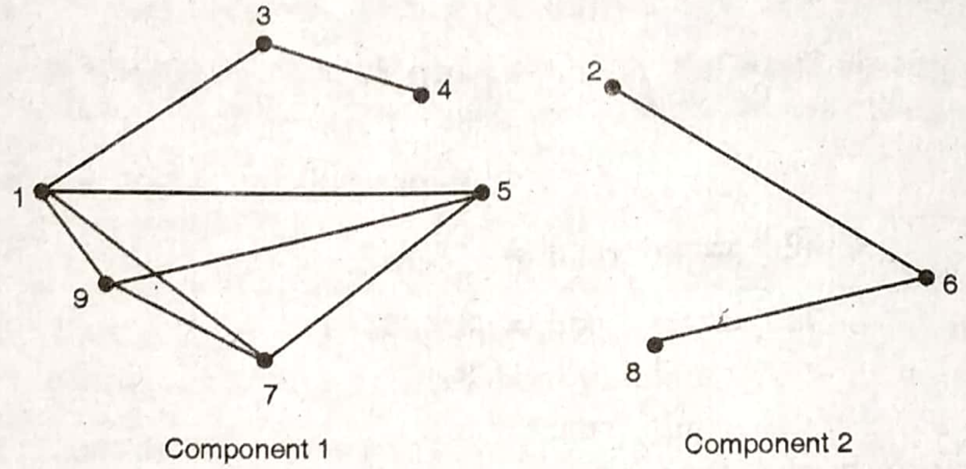


Fig. 1.2.1(b)

If a graph has a single component then it is connected graph.

(8) {9, 1, 7} is a triangle.

(9) {9, 5, 7, 1} is a clique of size 4. If you observe you will get following Fig. 1.2.1(c).

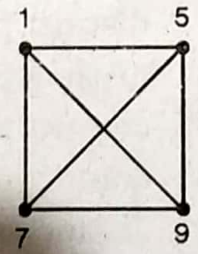


Fig. 1.2.1(c)

(10) {5, 8, 2, 4} is an independent set of size 4. You can observe that 5 and 8 are not connected also 2 and 4 is not connected.

Syllabus Topic : Combinatorics and Number Theory

1.3 Combinatorics and Number Theory

How combinatorics and number theory relate with each other ? Give an example

- Number theory is related with properties of positive integer.

– Few times combinatorial techniques are used in number theory.

Example : Collatz sequence

Form a sequence with a positive integer $n > 1$. If n is odd then next number is $3n + 1$ and if n is even then next number is $\frac{n}{2}$. Stop the sequence one we get 1.

$$\begin{aligned}
\text{Consider } n &= 14 \implies \text{even} = \frac{14}{2} = 7 \\
n &= 7 \implies \text{odd} = 3n + 1 \\
&\qquad = 3 \times 7 + 1 \\
&\qquad = 22 \\
&= 22 \implies \text{even} = \frac{22}{2} = 11 \\
&= 11 \implies \text{odd} = 3 \times 11 + 1 \\
&\qquad = 34
\end{aligned}$$

and so on.

So we get 14, 7, 22, 11, 34, 17

Now suppose we start with 29 then we get 29, 88, 44, 22.

We observe that, 22 appear in the both sequence. So both will agree from this point.

Example 1.3.1 : Find integer partition of 8.

Solution :

(1)	8	distinct parts
(2)	6 + 2	distinct parts
(3)	5 + 2 + 1	distinct parts
(4)	4 + 3 + 1	distinct parts
(5)	7 + 1	distinct, odd parts
(6)	5 + 3	distinct, odd parts
(7)	5 + 1 + 1 + 1	odd parts
(8)	3 + 3 + 1 + 1	odd parts
(9)	3 + 1 + 1 + 1 + 1 + 1	odd parts
(10)	1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	odd parts
(11)	6 + 1 + 1	
(12)	4 + 2 + 2	
(13)	3 + 3 + 2	
(14)	3 + 2 + 1 + 1 + 1	
(15)	2 + 2 + 2 + 1 + 1	
(16)	4 + 4	

(17) $4 + 2 + 1 + 1$
(18) $2 + 2 + 1 + 1 + 1 + 1$
(19) $4 + 1 + 1 + 1 + 1$
(20) $3 + 2 + 2 + 1$
(21) $2 + 2 + 2 + 2$
(22) $2 + 1 + 1 + 1 + 1 + 1 + 1$

There are 22 partition, exactly 6 of them are odd parts and distinct parts

Syllabus Topic : Combinatorics and Geometry

1.4 Combinatorics and Geometry

☞ How combinatorics and geometry relate with each other ? Give an example

Many problems in geometry are naturally a combinatorial or for which combinatorial techniques give clear idea about problem.

Example 1.4.1 : Consider family of 4 lines in the plane such that, each pair of lines intersects and each intersect point intersects with only two lines. Find number of regions.

Solution :

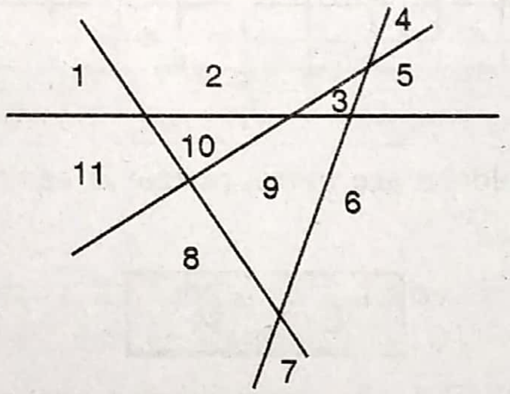


Fig. P.1.4.1

Number of regions are 11.

Can different arrangements of lines determine different number of regions?

Syllabus Topic : Combinatorics and Optimization

1.5 Combinatorics and Optimization

☞ Where we use optimization ?

On daily base problems like to optimize our way to go to our work , to reduce transportation cost in industries we use Optimization.

CHAPTER 2

Strings

Syllabus

Strings Sets, and Binomial Coefficients: Strings- A First Look, Combinations, Combinatorial, The Ubiquitous Nature of Binomial Coefficients, The Binomial, Multinomial Coefficients.

Syllabus Topic : Strings, Sets and Binomial Coefficients : Strings- A First Look

2.1 String

☞ What is string ?

Function $S : [n] \rightarrow X$

Where,

$[n]$ is n element set $\{1, 2, ..., n\}$, n be a positive integer and X be a set, then function S is called X-string of length n.

- Elements of X are called as characters and S (i) is i^{th} character of S.

  $S(1) = x_1, S(2) = x_2, ..., S(n) = x_n$

  $S =$ "$x_1\ x_2\ x_3 .... x_n$"

  S (i) also denoted by S_i.

- If 'X' is a set of numbers then strings are called as **"sequences".**

  For example :

  Sequence of odd integers is define by $S_i = 2_i - 1$.

- Strings are called words then the set X is called alphabet and elements of X are called **letters.**

For example :

rrsrssaaa is a 9 letter word on the 3 letter alphabet {a, r, s}.

- In computing language, strings are called arrays.
- Strings can be n-tuple like $(x_1, x_2 \ldots x_n) \in x_1 \times x_2 \times x_3 \ldots \times x_n$ i.e. [n].

Solved Examples

Example 2.1.1 : In a state of Maharashtra, license number consists of 2 digits followed by a space followed by 2 capital letters. The first digit cannot be a 0. How many licence numbers are possible ?

Solution :

$$X = \{0, 1, \ldots, 9\}$$

$$X - \{0\} = \{1, 2, \ldots, 9\} \text{ for first digit}$$

$$Y = \{\text{Space}\}$$

$$Z = \{\text{Set of 26 capital letters}\}$$

Number of different licence number is a string from $(X - \{0\}) \times X \times Y \times Z \times Z$

$$= 9 \times 10 \times 1 \times 26^2$$

$$= 60840$$

Remark

If $X = \{0, 1\}$, an X-string is called binary string or bit string or 0 – 1 string.

If $X = \{0, 1, 2\}$, an X string is called a ternary string.

Example 2.1.2 : Suppose that, a website allows users to set password with condition, the first character must be a lower case letter in the English alphabet. Second and third letters may be upper or lower case alphabet or decimal digit (0 to 9). Fourth place must be @ . Fifth and sixth are lower case English letter, *, %, and # and seventh place must be a digit. How many different password can user set ?

Solution : Consider, a string of length 7

L : Lower case letters.

U : Upper case letters

D : Digits (0 – 9)

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$
L	L	L	@	L	L	D
	U	U		*	*	
	D	D		%	%	
				#	#	
26	62	62	1	29	29	10

Below each position we write number of options.

For second position, 26 lower case + 26 upper case + 10 digit = 62 ways.

Each choice is independent of other. We multiply all numbers to get different possibility.

$26 \times 62 \times 62 \times 1 \times 29 \times 29 \times 10 = 840529040$

Syllabus Topic : Combinations

2.2 Combinations

What is combinations ? and prove the formula.

- Selection from set of samples is nothing but combination,
- Selection of k from n, denoted by $\binom{n}{k}$ here, $0 \le k \le 1$ |x| 1 = n
- It is also denoted by C (n, k).
- Number of combinations of n things, taken k at a time.

Proposition

If n and k are integers with $0 \le k \le n$, then $\binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!}$

$= \frac{n!}{k!\,(n-k)!}$

Proof

P (n, k) is an arrangement of k from n element and C (n, k) is an selection of k from n element.

k! multiple of C (n, k) is arrangement as for arrangement of k from k is

$k \times (k-1) \times (k-2) \times \ldots \times 3 \times 2 \times 1 = k!$

$$\therefore k!\, C(n, k) = P(n, k)$$

$$C(n, k) = \frac{P(n, k)}{k!}$$

Solved Examples

Example 2.2.1 : A Maharashtrian restaurant list 11 items in the "vegetable" category of its menu. A one vegetable plate contain 4 different vegetables. How many different ways can customer order ?

Solution :

- Out of 11 items we have to select 4 different ways for a customer to order a vegetable plate at the restaurant is C (11, 4) = 330.

Example 2.2.2 : How many ways can a committee can be formed from four men and six women with

(a) At least 2 men and at least twice as many women as men ?

(b) Four members, at least two of which are women, and Mr. and Mrs XYZ will not serve together

Solution :

(a) We prepare following table

men (4)	Women (6)	Selection from men	Selection from women	Total selections
2	4	C (4, 2)	C (6, 4)	C (4, 2) × C (6, 4)
2	5	C (4, 2)	C (6, 5)	C (4, 2) × C (6, 5)
2	6	C(4, 2)	C (6, 6)	C(4, 2) × C (6, 6)
3	6	C(4, 3)	C (6, 6)	C(4, 3) × C (6, 6)

∴ Total number of committees are

$C(4, 2) \times C(6, 4) + C(4, 2) \times C(6, 5) + C(4, 2) \times C(6, 6) + C(4, 3) \times C(6, 6)$

$= 90 + 36 + 6 + 4 = 136$

(b) Number of 4-member committees $= n_1$

At least 2 of which are women and let n_2 be the number of 4 member committees, atlest 2 of which are women and Mr. And Mrs. X.Y.Z are included in each. Now clearly

$$n_1 = C(4, 2) \times C(6, 2) + C(4, 1) \times C(6, 3) + C(4, 0) \times C(6,4)$$

$$n_2 = C(3, 1) \times C(5, 1) + C(5, 2)$$

∴ Number of committees is $n_1 - n_2$.

Example 2.2.3 : A committee of 5 is to be selected among 6 boys and 5 girls.

Determine the number of ways of selecting the committee, if it is to consist of at least one boy and one girl.

Solution :

Boys (6)	Girls (5)	Selection from boys	Selection from girls	Total selection
1	4	C (6, 1)	C (5, 4)	C (6, 1) × C (5, 4)
2	3	C (6, 2)	C (5, 3)	C (6, 2) × C (5, 3)
3	2	C (6, 3)	C (5, 2)	C (6, 3) × C (5, 2)
4	1	C (6, 4)	C (5, 1)	C (6, 4) × C (5, 1)

∴ Total number of committees are

$C(6, 1) \times C(5, 4) + C(6, 2) \times C(5, 3) + C(6, 3) \times C(5, 2) + C(6, 4) \times C(5, 1)$

Example 2.2.4 : A menu card in a restaurant displays, four soups, five ice-creams, three cold drinks and five fruits juices. How many different menus can a customer select if

(i) He selects one item from each group without omission.

(ii) He chooses to omit the fruit juices, but selects one each from the other groups.

(iii) He chooses to omit the cold drinks but decides to take a fruit juices and one item each from the remaining groups.

Solution :

(1) The customer can select the soup in 4 ways, ice creams in 5 ways, cold drinks in 3 ways and fruit juices in 5 ways. By product rule the number of ways in which he can select one item each, without omission is

$C(4, 1) \times C(5, 1) \times C(3, 1) \times C(5, 1) = 4 \times 5 \times 3 \times 5 = 300$

$\therefore$ Such selections are 300 ways.

(ii) The customer can select an ice-cream in 5 ways, cold drinks in 3 ways and soups in 4 ways.

By product rule number of ways of selection

$$= C(5, 1) \times C(3, 1) \times C(4, 1) = 5 \times 3 \times 4 = 60$$

$\therefore$ Such selections are 60 ways.

(iii) The number of ways to make the required selection is

$$= C(4, 1) \times C(5, 1) \times C(5, 1) = 4 \times 5 \times 5 = 100$$

$\therefore$ Such selections are 100 ways.

Example 2.2.5 : In how many ways can one select a captain, vice-captain and leader from the members of a committee consisting of 9 men and 11 women, if the leader must be a women, and the vice captain a man.

Solution : Selection of captain may be from men or women, hence there are following two cases.

Sr. No.	Case	Captain (men/women)	Vice captain (men)	Leader (women)	Total
1	Captain from men	C (9, 1)	C (8, 1)	C (11, 1)	$= 9 \times 8 \times 11$
					+
2	Captain from women	C (11, 1)	C (9, 1)	C (10, 1)	$= 11 \times 9 \times 10$

Total number of selections are

$$9 \times 8 \times 11 + 11 \times 9 \times 10 = 1782$$

$\therefore$ Total number of ways to make the selection is 1782.

Example 2.2.6 : How many numbers are there between 100 and 1000 in which all the digits are distinct ?

Solution : Numbers between 100 and 1000 are 3 digited and total numbers available to fill in that are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

9	9	8
100^{th}	10^{th}	1^{st}

For 100^{th} place we choose any one among

1, 2, 3, 4, 5, 6, 7, 8, 9

i.e. for 100^{th} place we have 9 choices. For 10^{th} place we choose any are among 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 but one no. already used in 100^{th} place.

$\therefore$ For 10^{th} place we have 9 choice.

Similarly in unit place we have 8 choice.

$\therefore$ Total counting of such numbers = $9 \times 9 \times 8 = 648$

Example 2.2.7 : How many different numbers can be formed from the digits 0, 2, 3, 4, 5, 6 lying between 100 and 1000 in which no digit being repeated ? How many of them are not divisible by 5 ?

Solution :

↓	↓	↓
100^{th}	10^{th}	1^{st}

For 100^{th} place we have 2, 3, 4, 5, 6 choices. i.e. Total 5 choices are available.

For 10^{th} place we have 0, 2, 3, 4, 5, 6, but one number already used in 100^{th} place

$\therefore$ Total 5 choices are available.

Similarly for unit place we have 4 choices

$\therefore$ Total numbers are $5 \times 5 \times 4 = 100$

Now if 0 or 5 appears at unit place then that number is divisible by 5.

First if 0 is in unit place than 100^{th} and 10^{th} places have 5 and 4 choices respectively.

These numbers are $5 \times 4 = 20$

If 5 appears in unit place then 100^{th} place has 4 choices (2, 3, 4, 6) and 10^{th} place has again 4 choices since it can be 0.

$\therefore$ These numbers are $4 \times 4 = 16$

$\therefore$ Total numbers divisible by 5 are $20 + 16 = 36$. Hence total numbers which are not divisible by 5 are $100 - 36 = 64$.

Example 2.2.8 : How many strings of three decimal digits with repetitions allowed.

(a) That begin with an odd digit.

(b) Have exactly two digit that are 4's.

Solution : Total number of digits are (0, 1, 2, ..., 9) = 10 out of that (1, 3, 5, 7, 9) = 5 are odd.

(a) For three decimal digits 100^{th} place digit may be any one out of 5 odd digits. Similarly unit place and 10^{th} place digit may be any one digit out of total 10 digit.

100^{th}	10^{th}	Unit
5	10	10

$\therefore$ Total number of such three decimal digits are $5 \times 10 \times 10 = 500$.

(b) Three decimal number having exactly two digits that are 4's that is only one digit is other than 4. There exist 3 different cases.

1. $4 \square 4 \Rightarrow$ Other than4 all alternatives from 0 to 9 = 9
2. $4 4 \square \Rightarrow$ Other than 4 all alternatives from 0 to 9 = 9
3. $\square 4 4 \Rightarrow$ Other than 0 and 4 all alternatives from 0 to 9 = 8

$\therefore$ Total number of such digits. = 9 + 9 + 8 = 26

Example 2.2.9 : How many four digit numbers can be formed from the digits 1, 2, 3, 4 and 5, with repetition possible, which are divisible by 5.

Solution :

1000^{th}	100^{th}	10^{th}	1^{st}

Four digit number is divisible by 5 only when we fix unit place by 5.

Now we are left with 1, 2, 3, 4 but repetition is allowed 50.

1000^{th}	100^{th}	10^{th}	1^{st}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
5 way	5 way	5way	1way

$= 5 \times 5 \times 5 \times 1 = 125$

Example.2.2.10 : In how many ways can 25 late admitted students be assigned to three practical batches if the first batch can accommodate 10 students, the second 8 and third only ?

Solution : The first batch can assigned 10 students from 25 in C (25, 10) ways.

The second batch can assigned 8 students from 15 remaining in C (15, 8) ways. The third batch can assigned 7 students from remaining 7 students in (7, 7) ways. By product rule total number of assigning students.

$$= C(25, 10) \times C(15, 8) \times C(7, 7)$$

$$= \frac{25!}{15!\ 10!} \times \frac{15!}{7!\ 8!} \times \frac{7!}{7!\ 0!}$$

Example.2.2.11 : In how many ways one right and one left shoe be selected from six pairs of shoes without obtaining a pair.

Solution : Suppose there are $(a_1\ b_1)$, $(a_2\ b_2)$, $(a_3\ b_3)$, $(a_4\ b_4)$, $(a_5\ b_5)$ and $(a_6\ b_6)$ such six pairs of shoes. So we can select one out of this ${}^6C_1$.

In this for without obtaining pair a_1 can match to b_2, b_3, b_4, b_5, b_6 shoes

$\therefore$ a_1 has such 5 pairs. We select one out of this ${}^5C_1$.

Similarly a_2, a_3, a_4, a_5, a_6 each have such 5 pairs.

$\therefore$ Total without obtaining pairs are ${}^6C_1 \times {}^5C_1 = 30$.

$\therefore$ 30 such pairs exists.

Example.2.2.12 : A farmer buys 3 cows, 2 goats and 4 hens from a man who has 4 cows, 3 goats, and 8 hens. How many choices does the farmer have ?

Solution :

$$\text{No. of choices for cows} = {}^4C_3 = 4$$

$$\text{No. of choices for goats} = {}^3C_2 = 3$$

$$\text{No. of choices for hens} = {}^8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70$$

$$\text{Total No. of choices} = 4 \times 3 \times 70$$

$$= 840 \text{ ways}$$

Example 2.2.13 : Find the number of strings of eight English letters.

(a) That contains no vowels,

If letters can be repeated. (b) That starts with the letter x and contains at least one vowel, if letters can be repeated.

Solution : Clearly there are 26 alphabets in English. Out of which 5 vowels and 21 consonants.

a) Arrange 21 non vowels in string of eight English letters with repetition in 21^8 ways.

Also if string contain 1 vowel and 7 consonants in ${}^5C_1 \times {}^{21}C_7$ ways.

If string contains 2 vowels and 6 consonants in ${}^5C_2 \times {}^{21}C_6$ ways

If string contains 3 vowels and 5 consonants in ${}^5C_3 \times {}^{21}C_5$ ways.

If string contains 4 vowels and 4 consonants in ${}^5C_4 \times {}^{21}C_4$ ways.

If string contains 5 vowels and 3 consonants in ${}^5C_5 \times {}^{21}C_3$ ways.

$\therefore$ Total number of strings are

$21^8 + {}^5C_1 \times {}^{21}C_7 + {}^5C_2 \times {}^{21}C_6 + {}^5C_3 \times {}^{21}C_5 + {}^5C_4 \times {}^{21}C_4 + {}^5C_5 \times {}^{21}C_3$.

b) Out of 8 letters first x is fixed, only 7 letters remains to assign.

If string contain 1 vowel and 6 consonants in ${}^5C_1 \times {}^{21}C_6$ ways.

If string contains 2 vowels and 5 consonants in ${}^5C_2 \times {}^{21}C_5$ ways.

If string contains 3 vowels and 4 consonants in ${}^5C_3 \times {}^{21}C_4$ ways.

If string contains 4 vowels and 3 consonants in ${}^5C_4 \times {}^{21}C_3$ ways.

If string contains 5 vowels and 2 consonants in ${}^5C_4 \times {}^{21}C_2$ ways.

$\therefore$ Total number of strings are

$21^7 + {}^5C_1 \times {}^{21}C_6 + {}^5C_2 \times {}^{21}C_5 + {}^5C_3 \times {}^{21}C_4 + {}^5C_4 \times {}^{21}C_3 + {}^5C_5 \times {}^{21}C_2$ ways.

Example 2.2.14 : Amit and Vijay are members of a club with a membership of 30. In how many ways can a committee of 10 be formed if

(1) Amit must be include in the committee ?

(2) Amit or Vijay should be in committee but not both ?

Solution. :

(1) Selection of Amit is fixed in C (1, 1) ways and remaining 9 members can be choosen in C (29, 9) ways.

Total number of formation of such committees

$$= C(1, 1) \times C(29, 9) = \frac{29!}{9!\,(29-9)!} = \frac{29!}{9!\,20!}$$

(2) If selection of Amit is fixed then Vijay is restricted

Remaining 9 members can be choosen in C (28, 9) ways.

Similarly selection of Vijay is fixed then Amit is restricted.

Remaining 9 members can be choosen in C (28, 9) ways.

Total number of formation of committees = $2 \times C(28, 9)$.

Example 2.2.15 : Mathematics students have to attempt six out of ten questions in an examination in any order.

(i) How many choices have they

(ii) How many choices do they have if they must answer atleast three out of the first five ?

Solution. :

(i) Selection of 6 questions out of ten can be made in

C (10, 6) = 210 ways.

Such selection of questions are 210 ways.

(ii) Selection of 3 questions out of 5 can be made in

C (5, 3) ways and

Selection of next 3 questions out of remaining 5 questions can be made in

C (5, 3) ways

Similarly 4 questions from first 5 can be made in

C (5, 4) ways

Selection of next 2 questions out of remaining 5 can be made in

C (5, 2) ways

Otherwise 5 questions from first 5 can be made in

C (5, 5) ways and

Remaining 1 question from next 5 can select in

C (5, 1) ways

Total such selections are

$$C(5, 3) \times C(5, 3) + C(5, 4) \times C(5, 2) + C(5, 5) \times C(5, 1)$$
$$= 100 + 50 + 5 = 155$$

All such selection of 6 questions from 10 are 155 ways.

Example 2.2.16 : There are 10 points in a plane of which 4 are collinear. Find the number of triangles that can be formed with vertices at these point.

Solution. : 4 points are collinear hence 6 are non-collinear.

Triangles by 6 non collinear points are C (6, 3) = 20.

Triangles by two collinear and one non collinear points = C (4, 2) × C (6, 1)

Similarly triangle by one collinear and two non collinear points

$$= C(4, 1) \times C(6, 2)$$

Total number of triangles

$$= C(6, 3) + C(4, 2) \times C(6, 1) + C(4, 1) \times C(6, 2)$$
$$= 20 + 36 + 60 = 116$$

∴ There exists 116 triangles from such 10 points.

Example 2.2.17 : In how many ways can 5 balls be selected from 8 identical red balls and 8 identical white balls.

Solution. : Selection of 5 balls from 2 different coloured balls.

If n = 2 and r = 5

Such selection of balls can be made in

$$C(n + r - 1, n - 1) \text{ way} = C(2 + 5 - 1, 2 - 1)$$
$$= C(6, 1) = 6$$

Such selections are 6 ways.

Actual selections

$$\left.\begin{array}{cc} \text{red} & \text{white} \\ 5 & 0 \\ 0 & 5 \\ 4 & 1 \\ 1 & 4 \\ 2 & 3 \\ 3 & 2 \end{array}\right\} \begin{array}{c} 6 \\ \text{Selections} \end{array}$$

Example 2.2.18 : Ten balls are picked from a pile of red, blue and white balls. How many such selections contain less than 5 red balls.

Solution. : Selection of 10 balls from 3 different coloured balls

i.e $n = 3, r = 10$

Such selection of balls can be made in

$$C(3 + 10 - 1, 3 - 1) = C(12, 2) = 66$$

Such selections are 66 ways.

But selection of at least 5 red balls out of 3 different coloured balls are

$$C(5 + 3 - 1, 3 - 1) = C(7, 2) = 21. \quad r = 5, n = 3$$

Number of selection of 10 balls containing at least 5 red balls.

$$= 66 - 21 = 45.$$

$\therefore$ 45 such selection exists.

Example 2.2.19 : How many non-negative integer solutions are there in the equation $x + y + z + u + v = 10{,}000$.

Solution. : Here distribution of 10,000 objects $\Rightarrow r = 10{,}000$

Number of variable are 5 $\Rightarrow n = 5$

Number of solutions

$$= C(n + r - 1, n - 1) = C(5 + 10000 - 1, 5 - 1) = C(10004, 4)$$

There are C (100004, 4) number of solutions exists.

Example 2.2.20 : In how many ways can 15 different books be distributed among three students A, B, C. So that A and B together receive twice as many books as C.

Solution. : A, B, C receive x, y, z number of book respectively.

$\therefore \quad x + y + z = 15 \quad$ and $\quad x + y = 2z$

Solution of above equations gives $z = 5$. i.e. unique selection of z.

Hence remaining 10 book have to distribute in A and B.

i.e. $\quad x + y = 10$

Here distribution of 10 books $\Rightarrow r = 10$

Number of variables are 2 $\Rightarrow n = 2$

Number of solutions

$$= C(n + r - 1, n - 1) = C(2 + 10 - 1, 2 - 1) = C(11, 1) = 11$$

11 ways in which the distribution can be done.

Example 2.2.21 : How many non-negative integer solutions are there in the equation $x + y + z + w = 10$?

Solution. :

Number of variables in linear equation $= n = 4$,

Constant to RHS $= r = 10$

∴ Number of non-negative integer solutions

$$= C(n+r-1, n-1) = C(4+10-1, 4-1) = C(13, 3) = 286.$$

Example 2.2.22 : Find coefficient of x^{23} in $(x^2 + x^3 + x^4 + \ldots)^5$.

Solution. :

Consider $(x^2 + x^3 + x^4 + \ldots)^5 = x^{10}(1 + x + x^2 + \ldots)^5$

To find coefficient of x^{23} from $(x^2 + x^3 + x^4 + \ldots)^5$ is same as to find coefficient of x^{13} from $(1 + x + x^2 + \ldots)^5$.

As coefficient of x^n from $(1 + x + x^2 + \ldots)^r$ is

$$^{n+r-1}C_{n-1}$$

∴ Coefficient of x^{13} from $(1 + x + x^2 + \ldots)^5$ is

$$= {}^{13+5-1}C_{5-1} = {}^{17}C_4$$

Example 2.2.23 : Use a generating function for finding the number of distributions of 27 identical balls, into five distinct boxes if each box has between 3 and 8 balls.

Solution. :

Generating function of each box becomes $(x^4 + x^5 + x^6 + x^7)$ because it contains 4, 5, 6, or 7 balls only.

∴ Generating function of such 5 boxes is $(x^4 + x^5 + x^6 + x^7)^5$

∴ Number of distribution of 27 identical balls in 5 boxes is same as coefficient of x^{27} from expression $(x^4 + x^5 + x^6 + x^7)^5$

i.e. coefficient of x^{27} from $x^{20}(1 + x + x^2 + x^3)^5$

i.e. coefficient of x^7 from $(1 + x + x^2 + x^3)^5$

Coefficient of x^7 from $(1 + x + x^2 + x^3)^5$ is 155

∴ 155 different distributions of 27 identical balls into five distinct boxes.

Example 2.2.24 : In how many ways are there to pick 2 different cards from a standard 52-card deck such that

(i) The first is Ace and the second card is not a queen ?

(ii) The first is space and second card is not a queen ?

Solution. :

(i) First card Ace can be select in C (4, 1) = 4 ways.
Second can be select from 52 – 1 – 4 = 47 in C (47, 1) = 47 ways.
There are 4 × 47 ways to choose the pair.

(ii) First space card can be select from C (13, 1) = 13 ways
Second can be select from 48 cards. If first card is not queen then there are C (47, 1) ways and if first card is queen then there are C (48, 1) ways.
Total selections = (1 × 48) + (12 × 47) = 612 ways.

Syllabus Topic : The Ubiquitous Nature of Binomial Coefficients

2.3 The Ubiquitous Nature of Binomial Coefficients

Examples based on The Ubiquitous Nature of Binomial Coefficients are as follows.

Solved Examples :

Example 2.3.1 : In how many ways can be office assistant distribute 12 identical folders among four office employees. Each employee receive at least one folder ?

Solution :

1. Consider,

$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}\ \boxed{12}$

to break above into 4 parts, we need to place 3 divider.

To place divider we have 11 places.

$\therefore$ Number of ways C (11, 3) = 165

For above example, if we remove condition that each employee receive at least one folder, then how many ways can the distribution be made ?

2. Consider a dummy folder which is actually has no presence. It is for if any one not receive a single folder.

Now row contain total 12 + 4 = 16 folder so to place divider we have 15 places.

$\therefore$ Number of ways C (15, 3) = 455.

Example 2.3.2 : For Example 2.3.1, if we put condition that 2 employees will get guaranteed a folder and for remaining 2 not necessary to get folder. Find the number of ways.

Solution : Now consider only two dummy folders,

So in a row total 12 + 2 = 14 folders. To place divider we have 13 places.

$\therefore$ Number of ways C (13, 3) = 286

Example 2.3.3 : Find number of solutions to the inequality $x_1 + x_2 + x_3 \leq 358$..

Solution :

1. **If all $x_i > 0$ and equality hold**

i.e. $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 = 358$.

Number of solution is C (357, 2).

Here we fix any one number and remaining 2 we select from 358 – 1 = 357.

2. **If act $x_i \geq 0$ and equality hold**

i.e. $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 358$

as $x_i \geq 0$ it may take all $x_1 = x_2 = x_3 = 0$

It means we have total 358 + 3 = 361 choices out of this we fix one and remaining 2 select from 360.

$\therefore$ Number of solution is C (360, 2).

3. If $x_1, x_3 > 0$ and $x_2 = 51$ and equality hold

i.e. $x_2 = 51$ and $x_1 + x_2 + x_3 = 358$

$\Rightarrow x_1 + x_3 = 307$

$\therefore$ Selection of one from 306 keeping one fix out of x_1 and x_3.

$\therefore$ Number of solution is C (306, 1).

4. If all $x_i > 0$ and inequality is strict

i.e. all $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 < 358$

Now to keep always inequality we consider x_4 such that,

$x_1 + x_2 + x_3 + x_4 = 358$.

So it is selection of 3 from 357.

$\therefore$ Number of solution is C (357, 3).

5. If all $x_i \geq 0$ and the inequality is strict

It may take all $x_1 = x_2 = x_3 = 0$

We have total 358 + 3 = 361.

Now to keep inequality consider x_4.

It is selection of 3 from 360.

$\therefore$ Number of solution is C (360, 3)

Syllabus Topic : The Binomial Theorem and Binomial Coefficients

2.4 The Binomial Theorem

State and prove Binomial theorem.

Theorem

Let x and y are real numbers with x, y and x + y non-zero, then for every non-negative integer n,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Proof

$$(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{\text{n factor}}$$

Here we choose either x or y from $(x + y)^n$.

If we choose x, n – i times, then y has to be chosen i times.

$\therefore$ Resulting product is $x^{n-i} y^i$

Out of this n factor we choose y, from i of them.

So number of such terms is C (n, i).

Solved Example

Example 2.4.1 : Find coefficient of $x^5 y^7$ in $(3x-2y)^{12}$

Solution :

$$(3x-2y)^{12} = \sum_{i=0}^{12} \binom{12}{i} (3x)^{12-i} (-2y)^i$$

To find coefficient of $x^5 y^7$

Put i = 7,

$$= \binom{12}{7} (3x)^5 (-2y)^7$$

$\therefore$ Coefficient of $x^5\ y^7$ is

$$= \binom{12}{7} 3^5 (-2)^7$$

Syllabus Topic : Multinomial Coefficients

2.5 Multinomial Coefficient

Explain Multinomial coefficients.

- Let A be a set of n elements. Now suppose we want to paint n items by colours red and blue.
- If we paint k items with red, then remaining (n – k) with blue.

  i.e. $\binom{n}{k}\binom{n-k}{n-k} = \frac{n!}{k!\,(n-k)\,!}$
- If we consider $k = k_1$ and $(n-k) = k_2$

  then,

  Number of ways = $\frac{n!}{k_1!\ k_2!}$

  Number of this form are called multinomial coefficients.

  General notation is,

  $$\binom{n}{k_1, k_2, \ldots k_r} = \frac{n!}{k_1!\ k_2! \ldots k_r!}$$

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

6. $\binom{m+n}{n} = \binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \ldots + \binom{m}{n}\binom{n}{n}$

Proof :

Consider set A containing m + n elements then number of subsets of A having n elements is $\binom{m+n}{n}$ above n can be done. Consider $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$ where A_1 containing m elements and A_1 containing n elements.

Now we select zero elements from A_1 and $\cap$ elements from A_2 which can be done $\binom{m}{0}\binom{n}{n}$ one elements from A_1 and n – 1 elements from A_2 which can be done $\binom{m}{1}\binom{n}{n-1}$ 2 elements from A_1 and n – 2 elements from A_2 which can be done $\binom{m}{2}\binom{n}{n-2}$ and so on......

Total numbers of ways of subsets with n elements of A is

$$\binom{m}{0}\binom{n}{n} + \binom{m}{1}\binom{n}{n-1} + \binom{m}{2}\binom{n}{n-2} + \ldots..$$

We also know that

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{n} = \binom{n}{0}, \binom{n}{n-1} = \binom{n}{1}, \binom{n}{n-2}$$

$$= \binom{n}{2}, \ldots..$$

$$\therefore \quad \binom{m+n}{n} = \binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \ldots + \binom{m}{n}\binom{n}{n}$$

Syllabus Topic : Solving Combinatorial Problems Recursively

3.5 Solving Combinatorial Problems Recursively

Example 3.5.1 : Draw n lines in a plane with conditions :

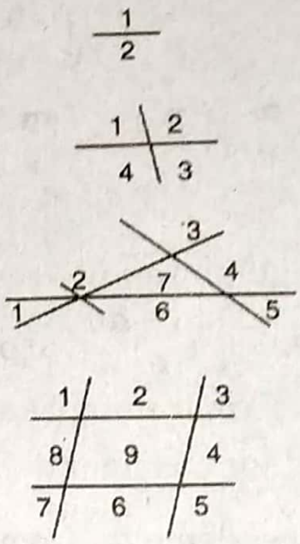
1. Each pair of lines cross each other
2. Each intersecting point

Passes through maximum 2 lines. Find recursive formula for number of region.

Solution :

Consider,

r (n) = Number of regions generated due to n lines.



For n = 1, r (1) = 2

n = 2, r (2) = 4

n = 3, r (3) = 7

n = 4, r (4) = 11

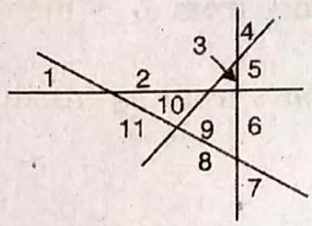


Fig. P. 3.5.1

Here each pair is not intersect

We can write,

r (n) = n + r (n – 1) for n > 1 and r (1) = 1.

This is how we get solution to combinatorial problem by recursively.

Syllabus Topic : Mathematical Induction and Inductive Definitions Proofs by Induction

3.6 Mathematical Induction and Inductive Definition Proofs by Induction

Mathematical Induction is the process of proving a general formula from particular cases.

3.6.1 First Principle of Mathematical Induction

☞ **Define first principle of mathematical induction.**

Let p(n) be a statement involving a natural number n.

If i) p(1) is true. ii) If p(k) is true.

Then p (k + 1) is true. For k > 1

Then p(n) is true for all n ≥ 1.

Working rule :

Let p(n) : given statement to be proved.

Step I : Prove statement for n = 1.

Step II : Assume that statement is true for n = k; n ∈ ℕ.

Step III : Prove statement for n = k + 1.

Then by first principle of mathematical induction result is true for all n ≥ 1.

Solved Examples :

Example 3.6.1 : Prove that sum of first n natural numbers in $\frac{n(n+1)}{2}$.

Solution : Let $p(n) : 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = 1$$

$$\text{RHS} = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k+1$

i.e. to prove $1 + 2 + 3 \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

Consider, $\text{LHS} = 1 + 2 + 3 + \ldots + k + (k+1) = [1 + 2 + 3 + \ldots + k] + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1) \quad \ldots(\text{from } (1))$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \text{RHS}$$

$\therefore$ Result is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.2 : Prove that by Mathematical induction, $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$.

Solution :

Let $p(n) : 2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = 3n - 1 = 3 - 1 = 2$$

$$\text{RHS} = \frac{n(3n+1)}{2} = \frac{1(3+1)}{2} = \frac{4}{2} = 2$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 2 + 5 + 8 + \ldots.. + (3k - 1) = \frac{k(3k+1)}{2} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $2 + 5 + 8 + \ldots.. + (3k - 1) + (3k + 2) = \dfrac{(k+1)(3k+4)}{2}$

Consider,

$$\begin{aligned} \text{LHS} &= 2 + 5 + 8 + \ldots.. + (3k - 1) + (3k + 2) \\ &= [2 + 5 + 8 + \ldots.. + (3k - 1)] + (3k + 2) \\ &= \frac{k(3k+1)}{2} + (3k+2) \qquad \ldots\text{from Equation (1)} \\ &= \frac{k(3k+1) + 2(3k+2)}{2} = \frac{3k^2 + k + 6k + 4}{2} \\ &= \frac{3k^2 + 7k + 4}{2} = \frac{(k+1)(3k+4)}{2} = \text{RHS} \end{aligned}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.3 : Prove that sum of squares of first n natural numbers is $\dfrac{n(n+1)(2n+1)}{6}$.

Solution : Let, $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

i.e. to prove $1^2 + 2^2 + 3^2 + \ldots.. + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Step I : For $n = 1$.

$$\text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 1^2 + 2^2 + 3^2 + \ldots.. + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1^2 + 2^2 + 3^2 + \ldots.. + k^2 + (k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$

Consider,

$$\begin{aligned} \text{LHS} &= 1^2 + 2^2 + 3^2 + \ldots.. + k^2 + (k+1)^2 \\ &= (1^2 + 2^2 + 3^2 + \ldots.. + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \end{aligned}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction statement is true for all natural numbers n.

Example 3.6.4 : Prove that, $1^2 + 3^2 + 5^2 + \ldots.. + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ $\forall n \geq 1$.

Solution : Let $p(n) : 1^2 + 3^2 + 5^2 + \ldots.. + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Step I : For $n = 1$.

$$\text{LHS} = (2n-1)^2 = (2-1)^2 = 1^2 = 1$$

$$\text{RHS} = \frac{n(2n-1)(2n+1)}{3} = \frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove

$$1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

Consider, $\text{LHS} = 1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 + (2k+1)^2$

$$= [1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2] + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \ldots(\text{from (1)})$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} = \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3}$$

$$= \frac{(2k+1)[2k^2 - k + 6k + 3]}{3} = \frac{(2k+1)[2k^2 + 5k + 3]}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.5 : Prove that, $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$ $\forall n \in \mathbb{N}$.

Solution : Let p(n) : $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

Step I : For n = 1.

$$\text{LHS} = (-1)^{n+1} n^2 = (-1)^{1+1} (1)^2 = (1)(1) = 1$$

$$\text{RHS} = \frac{(-1)^{n+1} n(n+1)}{2} = \frac{(-1)^{1+1}(1)(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, k ∈ ℕ.

$$\therefore \quad 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 = \frac{(-1)^{k+1} k(k+1)}{2} \qquad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= \frac{(-1)^{k+2}(k+1)(k+2)}{2}$$

Consider, LHS $= 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 + (-1)^{k+2}(k+1)^2$

$$= [1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2] + (-1)^{k+2}(k+1)^2$$

$$= \frac{(-1)^{k+1} k(k+1)}{2} + (-1)^{k+2}(k+1)^2 \qquad \ldots\text{from Equation (1)}$$

$$= \frac{(-1)^{k+1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}(-1)^{-1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}(k+1)[-k + 2(k+1)]}{2} = \frac{(-1)^{k+2}(k+1)(k+2)}{2} = \text{RHS}$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction p (n) is true ∀ n ∈ ℕ.

Example 3.6.6 : Prove that by mathematical induction,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4} \ \forall n \geq 1.$$

Solution : Let p(n) : $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Step I : For n = 1.

$$\text{LHS} = n^3 = 1^3 = 1$$

$$\text{RHS} = \frac{n^2(n+1)^2}{4} = \frac{1(1+1)^2}{4} = \frac{4}{4} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, k ∈ ℕ.

$$\therefore 1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 = \frac{k^2(k+1)^2}{4} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

Consider,

$$\text{LHS} = 1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 + (k+1)^3 = (1^3 + 2^3 + 3^3 + \ldots\ldots + k^3) + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \ldots\text{from Equation (1)}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2[k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2[k^2 + 4k + 4]}{4} = \frac{(k+1)^2(k+2)^2}{4} = \text{RHS}$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for all natural numbers n.

Example 3.6.7 : Prove that, $3 + 3^2 + 3^3 + \ldots\ldots + 3^n = \frac{3}{2}(3^n - 1)$

Solution : Let p(n) : $3 + 3^2 + 3^3 + \ldots\ldots + 3^n = \frac{3}{2}(3^n - 1)$

Step I : For n = 1.

LHS = 3

$$\text{RHS} = \frac{3}{2}(3^n - 1) = \frac{3}{2}(3 - 1) = \frac{3}{2}2 = 3$$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, k ∈ ℕ.

$$\therefore 3 + 3^2 + 3^3 + \ldots\ldots + 3^k = \frac{3}{2}(3^k - 1) \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $3 + 3^2 + 3^3 + \ldots\ldots + 3^k + 3^{k+1} = \frac{3}{2}(3^{k+1} - 1)$

Example 3.6.11 : Prove that for each natural number, $1\cdot3 + 2\cdot4 + 3\cdot5 + 4\cdot6 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$.

Solution : Let $p(n) : 1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Step I : For n = 1.

$$\text{LHS} = n(n+2) = (1)(1+2) = 3$$

$$\text{RHS} = \frac{n(n+1)(2n+7)}{6} = \frac{1(2)(9)}{6} = 3$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) = \frac{k(k+1)(2k+7)}{6} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}$

Consider,

$$\begin{aligned} \text{LHS} &= 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) + (k+1)(k+3) \\ &= [1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2)] + (k+1)(k+3) \\ &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \quad \ldots\text{from Equation (1)} \\ &= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6k + 18]}{6} = \frac{(k+1)[2k^2 + 13k + 18]}{6} \\ &= \frac{(k+1)(k+2)(2k+9)}{6} = \text{RHS} \end{aligned}$$

$\therefore$ Result is true for n = k + 1

$\therefore$ By principle of mathematical induction result is true for any natural number.

Example 3.6.12 : Prove that,

$$1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Solution :

Let $p(n) : 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Step I : For $n = 1$.

$$\text{LHS} = n(n+1)(n+2) = 1(1+1)(1+2) = 1\cdot2\cdot3 = 6$$

$$\text{RHS} = \frac{n(n+1)(n+2)(n+3)}{4} = \frac{(1)(2)(3)(4)}{4} = 6$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots\ldots + k(k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots\ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Consider,

$$\begin{aligned}
\text{LHS} &= 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots\ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= [1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots\ldots + k(k+1)(k+2)] + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{RHS}
\end{aligned}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number $n \geq 1$.

Example 3.6.13 : Prove by mathematical induction, $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots\ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Solution : Let $p(n) : \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots\ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Step I : For $n = 1$.

$$\text{LHS} = \frac{1}{1(1+1)} = \frac{1}{1\cdot2} = \frac{1}{2}$$

$$\text{RHS} = \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

$\therefore$ Result is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{k\,(k+1)}+\frac{1}{(k+1)\,(k+2)}=\frac{k+1}{k+2}$

Consider, LHS $= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{k\,(k+1)}+\frac{1}{(k+1)\,(k+2)}$

$$= \left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{k\,(k+1)}\right]+\frac{1}{(k+1)\,(k+2)}$$

$$= \frac{k}{k+1}+\frac{1}{(k+1)\,(k+2)} \qquad \ldots\text{from Equation (1)}$$

$$= \frac{k\,(k+2)+1}{(k+1)\,(k+2)}=\frac{k^2+2k+1}{(k+1)\,(k+2)}=\frac{(k+1)^2}{(k+1)\,(k+2)}=\frac{k+1}{k+2}=\text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.14 : Prove that $\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2n+1)\,(2n+3)}=\frac{n}{3(2n+3)}$

Solution : Let p(n) : $\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2n+1)\,(2n+3)}=\frac{n}{3(2n+3)}$

Step I : For $n = 1$.

$$\text{LHS} = \frac{1}{(2n+1)\,(2n+3)}=\frac{1}{(2+1)\,(2+3)}=\frac{1}{3\cdot 5}=\frac{1}{15}$$

$$\text{RHS} = \frac{n}{3(2n+3)}=\frac{1}{3(2+3)}=\frac{1}{3\cdot 5}=\frac{1}{15}$$

$\therefore$ Result is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2k+1)\,(2k+3)}=\frac{k}{3(2k+3)} \qquad \ldots(1)$$

Step III : To prove statement for $n = k + 1$

i.e. to prove,

$$\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2k+1)\,(2k+3)}+\frac{1}{(2k+3)\,(2k+5)}=\frac{(k+1)}{3(2k+5)}$$

Consider, LHS $= \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2k+1)\,(2k+3)}+\frac{1}{(2k+3)\,(2k+5)}$

$$= \left[\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots+\frac{1}{(2k+1)\,(2k+3)}\right]+\frac{1}{(2k+3)\,(2k+5)}$$

$\therefore$ Result is true for $n = k + 1$.

$\therefore$ By Mathematical induction result is true for all n.

Example 3.6.16 : Show that $1(1!) + 2(2!) + 3(3!) + \ldots\ldots + n(n!) = (n+1)! - 1$

Solution : Let $p(n) : 1(1!) + 2(2!) + 3(3!) + \ldots\ldots + n(n!) = (n+1)! - 1$

Step I : For $n = 1$.

$$\text{LHS} = n(n!) = 1(1!) = 1$$

$$\text{RHS} = (n+1)! - 1 = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$$

$$\therefore \text{LHS} = \text{RHS}.$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1(1!) + 2(2!) + 3(3!) + \ldots\ldots + k(k!) = (k+1)! - 1 \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1(1!) + 2(2!) + 3(3!) + \ldots\ldots + k(k!) + (k+1)(k+1)! = (k+2)! - 1$

Consider,
$$\begin{aligned}
\text{LHS} &= 1(1!) + 2(2!) + 3(3!) + \ldots\ldots + k(k!) + (k+1)(k+1)! \\
&= [1(1!) + 2(2!) + 3(3!) + \ldots\ldots + k(k!)] + (k+1)(k+1)! \\
&= (k+1)! - 1 + (k+1)(k+1)! \quad \ldots(\text{from } (1)) \\
&= (k+1)! [1 + k + 1] - 1 = (k+1)! (k+2) - 1 \\
&= (k+2)(k+1)! - 1 = (k+2)! - 1 = \text{RHS}
\end{aligned}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction p(n) is true $\forall\, n \in \mathbb{N}$.

Example 3.6.17 : Prove that, $n(n^2 - 1)$ is divisible by 3.

Solution : Let $p(n) : n(n^2 - 1)$ is divisible by 3.

Step I : For $n = 1$.

$$\text{LHS} = n(n^2 - 1) = 1(1^2 - 1) = 1(1 - 1)$$
$$= 0 \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore$ $k(k^2 - 1)$ is divisible by 3.

$$\therefore \quad k(k^2 - 1) = 3m \text{ for some } m \in \mathbb{Z}$$

$$\therefore \quad k^3 - k = 3m$$

$$\therefore \quad k^3 = 3m + k \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $(k+1)[(k+1)^2 - 1]$ is divisible by 3.

Consider, $\text{LHS} = (k+1)[(k+1)^2 - 1] = (k+1)(k^2 + 2k + 1 - 1)$

$$= (k+1)(k^2+2k) = k^3 + 2k^2 + k^2 + 2k$$
$$= k^3 + 3k^2 + 2k$$
$$= 3m + k + 3k^2 + 2k \text{...from Equation (1)}$$
$$= 3m + 3k^2 + 3k$$
$$= 3(m + k^2 + k) \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.18 : Prove that, $2^{2n} - 1$ is divisible by 3.

Solution : Let $p(n) : 2^{2n} - 1$ is divisible by 3.

Step I : For $n = 1$.

Consider, $2^{2n} - 1 = 2^2 - 1 = 4 - 1 = 3$ which is divisible by 3.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore 2^{2k} - 1$ is divisible by 3.

$\therefore 2^{2k} - 1 = 3m$ for some $m \in \mathbb{Z}$

$\therefore 2^{2k} = 3m + 1$ ...(1)

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $2^{2(k+1)} - 1$ is divisible by 3.

Consider,

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$$
$$= (3m+1) \cdot 4 - 1 \quad \text{...from Equation (1)}$$
$$= 3 \times 4 \times m + 4 - 1 = 3 \times 4 \times m + 3.$$
$$= 3(4m + 1) \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction result is true for any natural number n.

Example 3.6.19 : $3^{2n} + 7$ is divisible by 8, Prove by induction $\forall\, n \in \mathbb{N}$.

Solution : Let $p(n) : 3^{2n} + 7$ is divisible by 8.

Step I : For $n = 1$.

Consider, $3^{2n} + 7 = 3^2 + 7 = 9 + 7 = 16$ which is divisible by 8.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore 3^{2k} + 7$ is divisible by 8.

$\therefore 3^{2k} + 7 = 8m$ for some $m \in \mathbb{Z}$

$\therefore 3^{2k} = 8m - 7$ ...(1)

Step III : Now to prove statement for $n = k + 1$

Consider,

$$3^{2(k+1)} + 7 = 3^{2k+2} + 7 = 3^{2k}.3^2 + 7 = 3^{2k}.9 + 7$$

$$= (8m - 7)\,9 + 7 \qquad \text{...from Equation (1)}$$

$$= 8 \times 9 \times m - 63 + 7 = 8 \times 9 \times m - 56.$$

$$= 8\,(9m - 7) \text{ which is divisible by 8.}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of Mathematical induction statement is true for all $n \in \mathbb{N}$.

Example 3.6.20 : Prove $\forall\, n \in \mathbb{N}$ $n\,(n-1)\,(2n-1)$ is divisible by 6.

Solution : Let $p(n) : n\,(n-1)\,(2n-1)$ is divisible by 6.

Step I : For $n = 1$.

Consider, $n\,(n-1)\,(2n-1) = 1\,(1-1)\,(2-1)$

$= 0$ which is divisible by 6.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore k\,(k-1)\,(2k-1)$ is divisible by 6.

$$\therefore k\,(k-1)\,(2k-1) = 6m \qquad \text{for some } m \in Z^+$$

$$\therefore (k^2 - k)\,(2k-1) = 6m$$

$$2k^3 - k^2 - 2k^2 + k = 6m$$

$$2k^3 = 6m + 3k^2 - k \qquad \text{...(1)}$$

Step III : Now to prove statement for $n = k + 1$

Consider, $(k+1)\,[(k+1)-1)]\,[2(k+1)-1]$

$$= (k+1)\,[k]\,(2k+2-1) = (k^2+k)\,(2k+1)$$

$$= 2k^3 + k^2 + 2k^2 + k = (6m + 3k^2 - k) + 3k^2 + k$$

$$= 6m + 6k^2 = 6\,(m + k^2)$$

Which is divisible by 6.

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction statement is true for every natural number. n.

Example 3.6.21 : Show that $n^3 + 2n$ is divisible by 3 for $n \geq 1$.

Solution : Let $p(n) : n^3 + 2n$ is divisible by 3.

Step I : For $n = 1$.

Consider, $n^3 + 2n = (1)^3 + 2\,(1) = 1 + 2 = 3$ which is divisible by 3.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \geq 1$.

$\therefore k^3 + 2k$ is divisible by 3.

$\therefore k^3 + 2k = 3m$ for some $m \in \mathbb{Z}^+$

$\therefore k^3 = 3m - 2k$ ...(1)

Step III : Now to prove statement for $n = k + 1$

Consider,

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (3m - 2k) + 3k^2 + 3k + 2k + 3$ ...from Equation (1)

$= 3m + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$

which is divisible by 3.

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.22 : $8^n - 3^n$ is multiple of 5 $\forall n \geq 1$, Prove by mathematical induction.

Solution : Let $p(n) : 8^n - 3^n$ is multiple of 5.

Step I : For $n = 1$.

Consider,

$8^n - 3^n = 8 - 3 = 5$ which is multiple of 5.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 8^k - 3^k$ is multiple of 5.

$\therefore 8^k - 3^k = 5m$ for some $m \in \mathbb{Z}$

$\therefore 8^k = 5m + 3^k$ ...(1)

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $8^{k+1} - 3^{k+1}$ is multiple of 5.

Consider,

$8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 3^k$

$= 8(5m + 3^k) - 3 \cdot 3^k$ ...from Equation (1)

$= 40m + 8 \cdot 3^k - 3 \cdot 3^k = 40m + (8 - 3)3^k = 40m + 5 \cdot 3^k = 5(8m + 3^k)$

Which is multiple by 5.

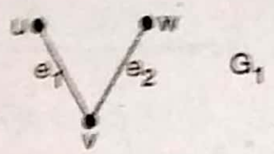
$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction statement is true for all natural numbers.

Example 3.6.23 : $n^4 - 4n^2$ is divisible by 3 $\forall$ $n \geq 2$, Prove by mathematical induction.

Solution : Let $p(n) : n^4 - 4n^2$ is divisible by 3.

e.g. (1)

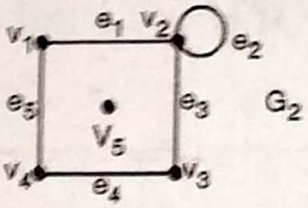


here $V = \{u, v, w\}$

$E = \{e_1, e_2\}$

Fig. 4.1.1

(2)



here $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

Fig. 4.1.2

Observe in G_2 edge $e_2 = (v_2, v_2)$ has end vertices v_2 and v_2 i.e. same and vertices.

a. Adjacency :

Two vertices v_1 and v_2 in a graph G are said to be **adjacent** to each other iff they are end vertices of the **same** edge e.

b. Incidence :

If the vertex u is an end vertex of the edge e then the edge e is said to be incident on vertex u.

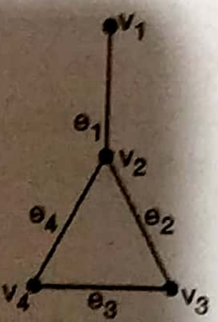


Fig. 4.1.3

here adjacent pairs of vertices : $\{v_1, v_2\}$

$\{v_2, v_3\}$

$\{v_3, v_4\}$

$\{v_1, v_4\}$

and edge e_1 is incident on vertices v_1 and v_2

edge e_2 is incident on vertices v_2 and v_3

edge e_3 is incident on vertices v_4 and v_3

edge e_4 is incident on vertices v_2 and v_4.

Solved Examples :

Example 4.1.1 : Consider a circle centered (1, 1) and having radius 3 units. Is it a graph ? Justify.

Solution :

Circle with centre (1, 1) and radius 3 is not graph because circumference of circle is not forms an edge.

Example 4.1.2 : Draw all possible graph with 3 vertices.

Solution :

graph with zero edges

graph with 1 edge

graph with 2 edges

graph with 3 edges

Fig. P. 4.1.2

These are all possible graphs on three vertices.

Example 4.1.3 : How many edges are there in a graph with 10 vertices each of degree 6 ?

Solution :

In graph of 10 vertices with each of degree 6.

$$d(G) = 10 \times 6.$$

Each edge gives two degree.

$\therefore$ Total number of edges in such graph

$$= \frac{60}{2} = 30.$$

$\therefore$ G has 30 edges.

4.1.4 Representation of Graph

(A) Ordered pair representation
(B) Structure representation
(C) Matrix representation

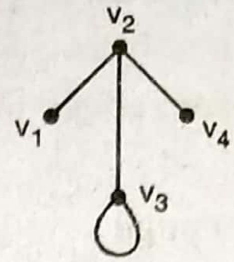
(A) Ordered pair representation :

A graph G can be represent in ordered pair (V, E) where V is non-empty set of vertices, And E is set of edges.

e.g. G (V, E) with $V = \{ v_1, v_2, v_3, v_4 \}$

$E = \{ (v_1, v_2), (v_2, v_4), (v_3, v_3), (v_2, v_3) \}$

(B) Structure representation :



A graph can be represented by structure with points and lines each line has two end points.

e.g. above order form of graph represented by structure.

(C) Matrix representation :

Depending on adjacency of vertices and incidency of an edges, a graph can be represented in two types of matrices :

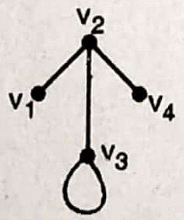
(i) Adjacency matrix
(ii) Incidence matrix

(I) Adjacency matrix :

If G is a graph on n vertices say $v_1, v_2, \ldots v_n$ then adjacency matrix of G is the $n \times n$ matrix $A(G) = [a_{ij}]_{n \times n}$

where, a_{ij} = number of edges between v_i and v_j.
= 1 for self loop

e.g. Consider above graph



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Observations :

(1) The sum of the entries in the i^{th} column or i^{th} row gives the degree of vertex v_i (diagonal entries counted twice).

(2) $a_{ii} = 0$ iff there is no loop at v_i.

(3) Since the number of edges joining v_i to v_j = no. of edges joining v_j to v_i.
i.e. $a_{ij} = a_{ji}$ $\therefore$ Adjacency matrix is **symmetric**.

(4) If G is simple graph then A(G) is a matrix containing 0^s and 1^s. (Binary matrix in which each diagonal entry is 0).

Solved Examples :

Example 4.1.4 : Draw the graph represented by adjacency matrix

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 2 & 1 & 1 \\ v_2 & 2 & 0 & 2 & 1 \\ v_3 & 1 & 2 & 1 & 0 \\ v_4 & 1 & 1 & 0 & 1 \end{array}$$

Solution : There are 4 row and column so we consider four vertex say v_1, v_2, v_3, v_4. Then we check all diagonal entries. In third and fourth entry we have one so there is one loop at v_3 and v_4 and remaining entries are nothing but number of edges between corresponding vertices.

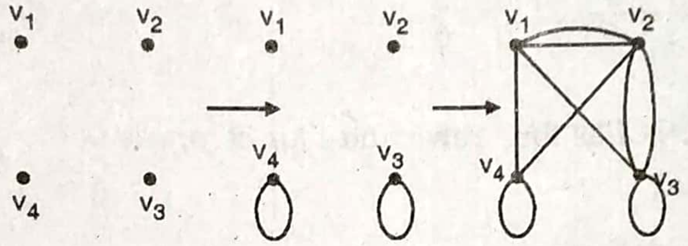


Fig. P. 4.1.4

Example 4.1.5 : Find adjacency matrix of following graph.

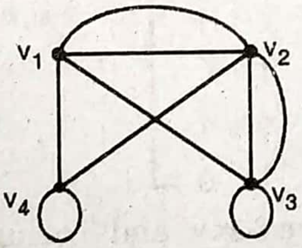


Fig. P. 4.1.5

Solution : Graph G has 4 vertices. Adjacency matrix of it is.

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 2 & 1 & 1 \\ v_2 & 2 & 0 & 2 & 1 \\ v_3 & 1 & 2 & 1 & 0 \\ v_4 & 1 & 1 & 0 & 1 \end{array}$$

Example 4.1.6 : Find adjacency matrix of graph G.

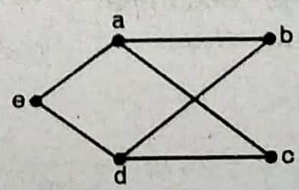


Fig. P. 4.1.6

Solution :

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 4.1.7 : Draw the graph represented by the adjacency matrix.

$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array}\begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Solution :

Adjacency matrix has four rows and four columns :

∴ Graph of it is

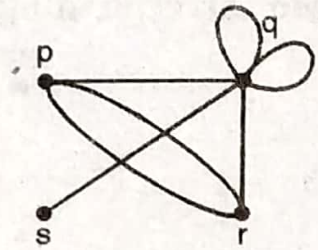


Fig. P. 4.1.7

Example 4.1.8 : Draw a graph with adjacency matrix A.

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution : Adjacency matrix have 5 row and 5 columns.

∴ Graph of matrix have 5 vertices (p, q, r, s, t)

$$\Rightarrow A = \begin{array}{c} \\ p \\ q \\ r \\ s \\ t \end{array}\begin{array}{c} \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 2 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

∴ Graph becomes,

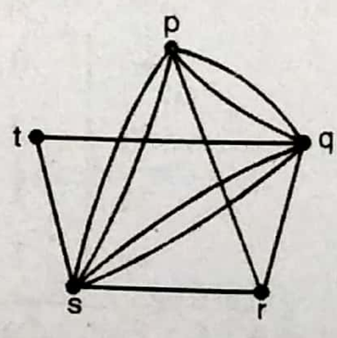


Fig. P. 4.1.8

Example 4.1.9 : Define adjacency matrix of a graph and draw graph whose adjacency matrix is given below :

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

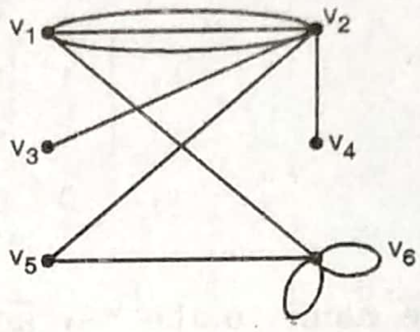


Fig. P. 4.1.9

(ii) Incidence matrix :

Let G be a graph in structural form with n vertices $v_1, v_2, \ldots\ldots v_n$ and m edges $e_1, e_2, \ldots\ldots, e_m$ then the incidence matrix of G denoted by I(G) is,

$I(G) = [a_{ij}]_{n \times m}$ where,

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not incident with } e_j \\ 1 & \text{if } v_j \text{ is incident with } e_j \\ 2 & \text{if } e_j \text{ is a loop. at } v_i \end{cases}$$

e.g.

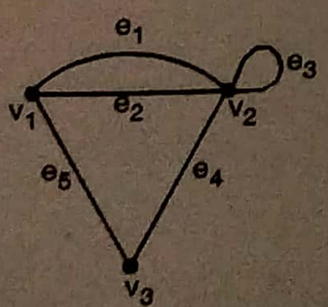


Fig. 4.1.4

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{3 \times 5} \\ \begin{array}{ccccc} \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ 2 & 2 & 2 & 2 & 2 \end{array} \end{array} \begin{array}{l} \\ = 3 = d(v_1) \\ = 5 = d(v_2) \\ = 2 = d(v_3) \\ = 10 = d(G) \end{array}$$

Observations :

(1) Any column sum must be 2.

(2) The sum of the entries in a row indicates the degree of the corresponding vertex.

(3) If two columns are identical, then the corresponding edges are parallel edges.

(4) If a row contains only zero's then the corresponding vertex must be an isolated vertex.

(5) If a row contains a single '1' the corresponding vertex must be pendant vertex.

(6) If an edge is a loop there will be a single '2' in the column remaining entries being zero's.

Example 4.1.10 : Draw the graph represented by incident matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Solution :

First give name to the row and column, then find entry with 2 i.e. there is loop.

See column to find end vertex of corresponding edge.

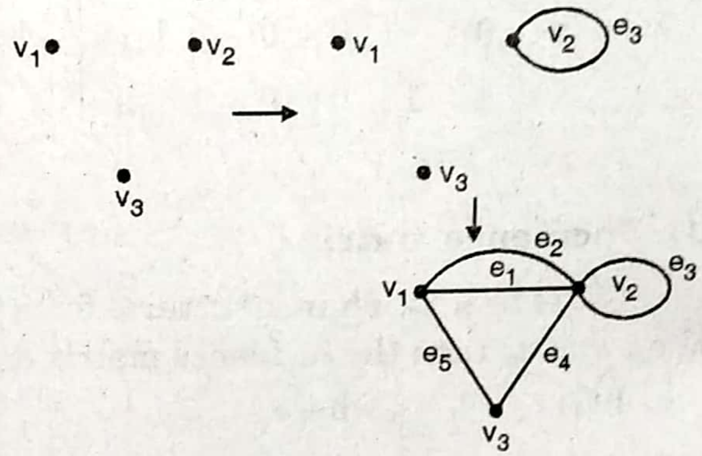


Fig. P. 4.1.10

Example 4.1.11 : Find incidence and adjacent matrices of following graphs :

Solution :

(A)

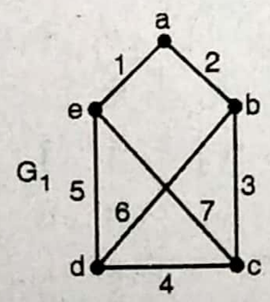


Fig. P. 4.1.11

(1) Incidence matrix

$$I(G_1) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

(2) Adjacency matrix

$$A(G_1) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(B)

(1) Incidence matrix

$$I(G_2) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

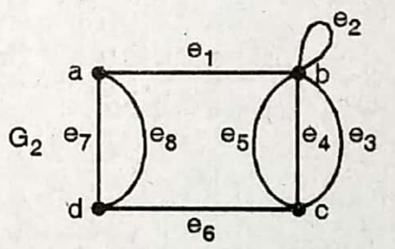


Fig. P. 4.1.11(a)

(2) Adjacency matrix

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 4.1.12 : Write incidence and adjacency matrix of the following graph.

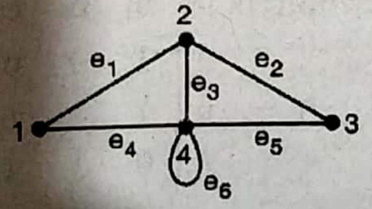


Fig. P. 4.1.12

Solution : Adjacency matrix is'

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Incidency matrix is,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

Example 4.1.13 : Write adjacency matrix and incidency matrix of the following graph:

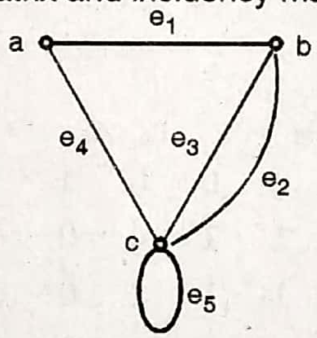


Fig. P. 4.1.13

Solution :

Numbers of vertices = 3, Numbers of edges = 5

$\therefore$ Adjacency matrix is

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \end{array}$$

Incidency matrix is

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

4.1.5 Isomorphism :

ISO ≈ same   morph ≈ structure

Sometimes two graphs look different but they represent the same graph, then these two graphs are **isomorphic graphs.**

Definition :

Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two graphs. The graphs G_1 and G_2 are said to be isomorphic if there is bijective functions $f_V : V_1 \to V_2$ and $f_E : E_1 \to E_2$ such that if u and v are end vertices of some edge $e \in E_1$ then $f_v(u)$, $f_v(v)$ are end vertices of $f_E(e)$.

Solved Examples :

Example 4.1.14 : Draw all possible non isomorphic simple graphs with 4 vertices.

Solution : graphs of zero edges

graphs of 1 edge

graph of 2 edges

graph of 3 edges

graph of 4 edges

graph of 5 edges

graph of 6 edges

There are such 10 graphs exists.

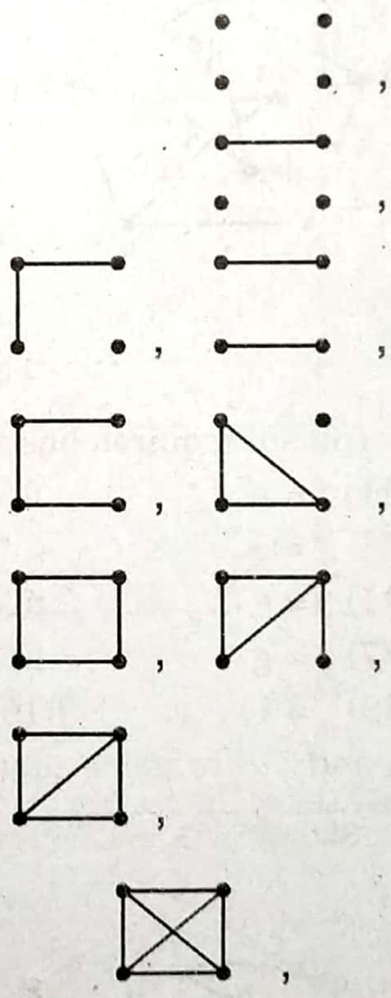


Fig. P. 4.1.14

Example 4.1.15 : Draw all possible non isomorphic graph with 4 edges and 4 vertices.

Solution :

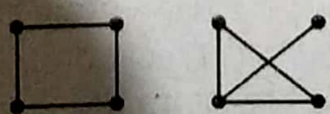


Fig. P. 4.1.15

Example 4.1.16 : Define isomorphic graphs.

Determine whether given graphs G_1 and G_2 be isomorphic.

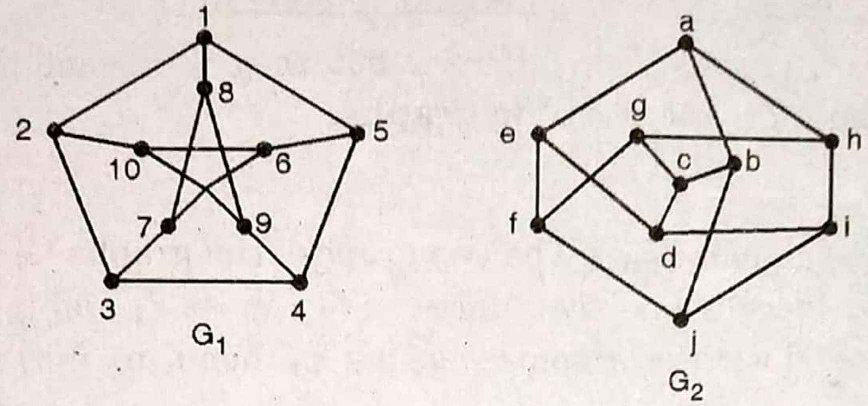


Fig. P. 4.1.16

Solution : We label the vertices as follows,

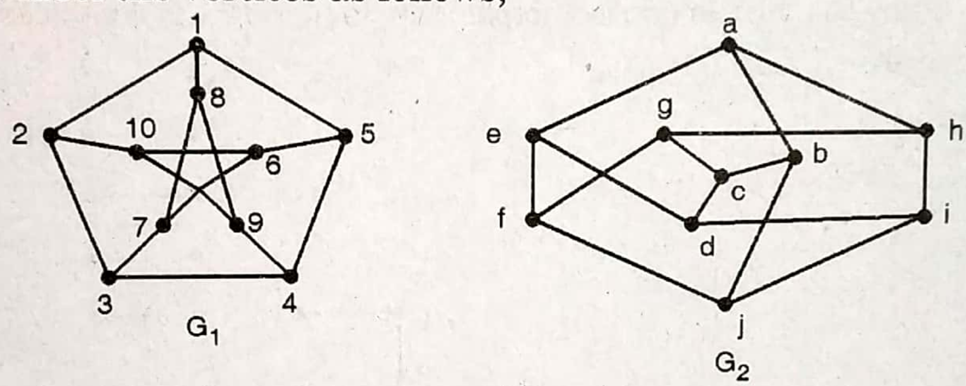


Fig. P. 4.1.16(a)

For isomorphism required bijection is

f(1) = a	f(2) = b
f(3) = c	f(4) = d
f(5) = e	f(6) = f
f(7) = g	f(8) = h
f(9) = i	f(10) = j

Hence G_1 and G_2 are isomorphic.

Example 4.1.17 : Show that G_1 and G_2 are isomorphic.

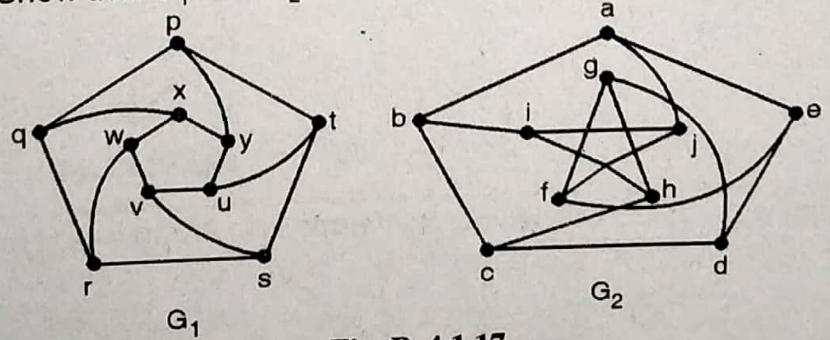


Fig. P. 4.1.17

Solution : We label the vertices of G_1 and G_2 as above for isomorphism the required bijection is :

$f(p) = a$

$f(q) = b$, $f(r) = c$, $f(s) = d$

$f(t) = e$, $f(u) = f$, $f(v) = g$

$f(w) = h$, $f(x) = i$, $f(y) = j$

Hence G_1 and G_2 are isomorphic.

Note : Adjacency between vertices is preserved.

e.g. Following two graphs G_1 and G_2 are isomorphic.

Here number of vertices in $G_1 = 4 =$ number of vertices in G_2

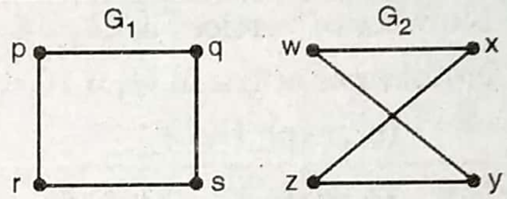


Fig. P. 4.1.17(a)

number of edges in $G_1 = 4 =$ number of edges in G_2

In graph G_1

vertex	degree	degree of adjacent vertices
p	2	2, 2
q	2	2, 2
r	2	2, 2
s	2	2, 2

In graph G_2

vertex	degree	degree of adjacent vertices
w	2	2, 2
x	2	2, 2
y	2	2, 2
z	2	2, 2

i.e. adjacency of vertices preserves.

$\Rightarrow f(p) = w$, $f(q) = x$, $f(r) = y$, $f(s) = z$ is bijection.

$\therefore$ G_1 and G_2 are isomorphic graphs.

Example 4.1.18 : Show that the following graphs are isomorphic:

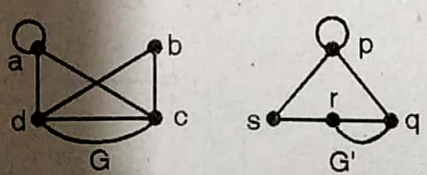


Fig. P. 4.1.18

Solution : Number of vertices in $G = 4 =$ number of vertices in G'

Number of edges in $G = 7 \neq$ number of edges in $G' = 0$

$\therefore$ G and G' are not isomorphic.

Example 4.1.19 : Determine if the following graphs are isomorphic.

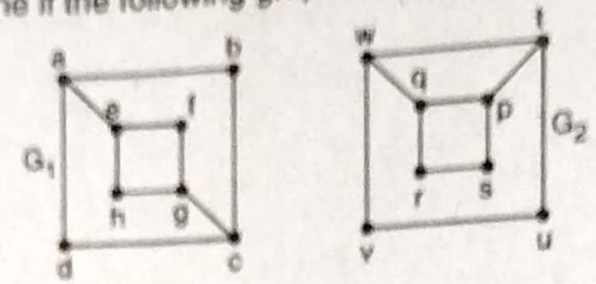


Fig. P. 4.1.19

Solution :

Here Number of vertices in G_1 = 4 = number of vertices in G_2

Number of edges in G_1 = 10 = number of edges in G_2

In graph G_1

Vertex	Degree	Degree of adjacent vertices
a	3	3, 2, 2
b	2	3, 3
c	3	2, 3, 2
d	2	3, 3
e	3	3, 2, 2
f	2	3, 3
g	3	3, 2, 2
h	2	3, 3

In graph G_2

Vertex	Degree	Degree of adjacent vertices
p	3	3, 3, 2
q	3	2, 3, 3
r	2	3, 2
s	2	2, 3
t	3	2, 3
u	2	3, 2
v	2	3, 2
w	3	2, 3, 2

From above table we say that in G_1 vertex with degree 3 having degree of adjacent vertex are 3, 2, 2 dose not exist in G_2.

$\therefore$ G_1 is not isomorphic with G_2.

Example 4.1.20 : Determine whether following graphs are isomorphic or not.

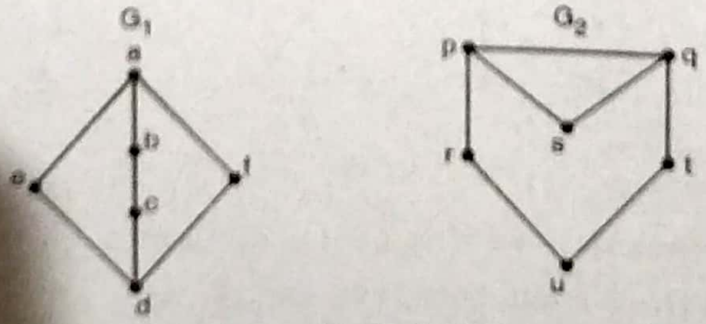


Fig. P. 4.1.20

Solution :

Number of vertices in G_1 = 6 = number of vertices in G_2.

Number of edges in G_1 = 7 = number of edges in G_2.

In graph G_1

vertex	degree	degree of adjacent vertices
a	3	2, 2, 2
b	3	2, 2, 2
c	2	3, 2
d	2	3, 2
e	2	3, 3
f	2	3, 3

In graph G_2

vertex	degree	degree of adjacent vertices
p	3	2, 2, 3
q	3	2, 2, 3
r	2	3, 3
s	2	3, 2
t	2	3, 2
u	2	2, 2

From above discussion, vertex 'a' is of degree 3 with adjacency 2, 2, 2 in G_1. But there is no vertex in G_2 with degree 3 having adjacency 2, 2, 2.

G_1 is not isomorphic with G_2 by the bijection.

Example 4.1.21 : Are the two graph isomorphic ? Justify.

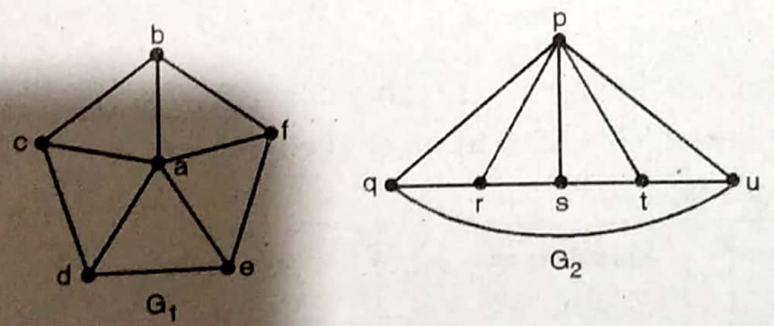


Fig. P. 4.1.21

Solution : number of vertices in G_1 = 6 = number of vertices in G_2.

number of edges in G_2 = 7 = number of edges in G_2.

Adjacency in G_1

Vertex	Degree	Degree of adjacent vertices
a	5	3, 3, 3, 3, 3
b	3	5, 3, 3
c	3	5, 3, 3

Adjacency in G_2

Vertex	Degree	Degree of adjacent vertices
p	5	3, 3, 3, 3, 3
q	3	5, 3, 3
r	3	5, 3, 3

Adjacency in G_1

Vertex	Degree	Degree of adjacent vertices
d	3	5, 3, 3
e	3	5, 3, 3
f	3	5, 3, 3

Adjacency in G_2

Vertex	Degree	Degree of adjacent vertices
s	3	5, 3, 3
t	3	5, 3, 3
u	3	5, 3, 3

from above two tables adjacency of vertices preserves.

$\therefore$ G_1 and G_2 are isomorphic with bijection,

$f(a) = p$, $f(b) = q$, $f(c) = r$, $f(d) = s$, $f(e) = t$, $f(f) = u$.

Example 4.1.22 : Show that following two graphs are isomorphic.

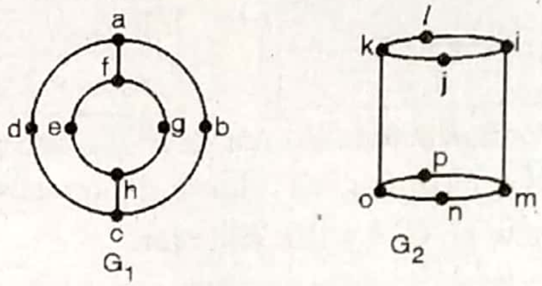


Fig. P. 4.1.22

Solution : Here

Number of vertices in G_1 = 8 = number of vertices in G_2

Number of edges in G_1 = 10 = number of edges in G_2

In graph G_1

Vertex	Degree	Degree of adjacent vertices
a	3	2, 3, 2
b	2	3, 3
c	3	2, 3, 2
d	2	3, 3
e	2	3, 3
f	3	3, 2, 2
g	2	3, 3
h	3	2, 2, 3

In graph G_2

Vertex	Degree	Degree of adjacent vertices
j	2	3, 3
k	3	3, 2, 2
l	2	3, 3
i	3	2, 2, 3
m	3	2, 2, 3
n	2	3, 3
o	3	2, 2, 3
p	2	3, 3

From above table adjacency of vertices preserves.

$\Rightarrow f(a) = p, \quad f(b) = j, \quad f(c) = k, \quad f(d) = l$

$f(f) = m, \quad f(g) = n, \quad f(h) = 0, \quad f(e) = p$

$\therefore$ G_1 and G_2 are isomorphic graphs.

Example 4.1.23 : Out of the three graphs which two are isomorphic ? Justify.

Solution :

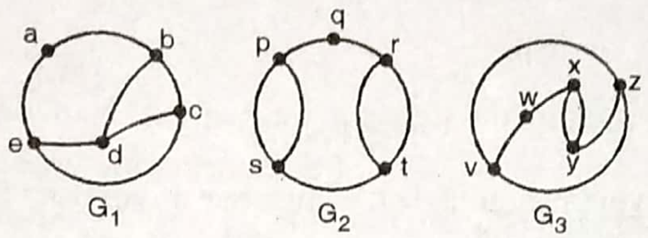


Fig. P. 4.1.23

In G_1

number of vertices = 5

number of edges = 7

In G_2

number of vertices = 5

number of edges = 7

In G_3

number of vertices = 5

number of edges = 7

vertex	degree	degree of adjacent vertices	vertex	degree	degree of adjacent vertices	vertex	degree	degree of adjacent vertices
a	2	(3, 3)	p	3	(3, 2)	v	3	(3, 2)
b	3	(3, 3, 2)	q	2	(3, 3)	w	2	(3, 3)
c	3	(3, 3, 3)	r	3	(3, 2)	x	3	(3, 2)
d	3	(3, 3, 3)	s	3	(3, 3)	y	3	(3, 3)
e	3	(3, 3, 2)	t	3	(3, 3)	z	3	(3, 3)

In above discussion

See in G_2 and G_3 adjacency and non-adjacency of vertices preserves. $\therefore$ G_2 is isomorphic with G_3.

i.e. $G_2 \cong G_3$. by bijection $f(p) = v$, $f(q) = w$, $f(r) = x$ $f(s) = y$, $f(t) = z$.

But in G_1 vertex b which has three adjacent vertices.

$\therefore$ $G_2 \cong G_3$ but $G_1 \not\cong G_2$ and $G_1 \not\cong G_3$

Example 4.1.24 : Show that the following graphs are isomorphic :

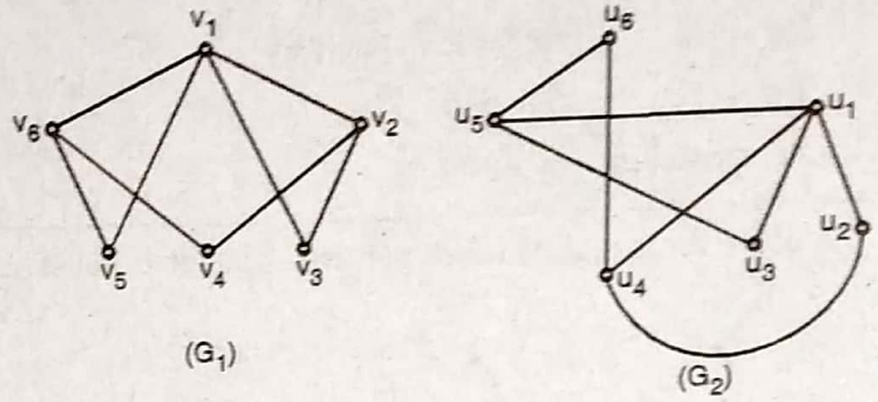


Fig. P. 4.1.24

Solution : Number of vertices in G_1 = 6 = number of vertices in G_2

Number of edges in G_1 = 8 = number of edges in G_2

G_1 Vertices	Degree	Degree adj	G_2 Vertices	Degree	Degree adj
v_1	4	3, 2, 2, 3	u_1	4	2, 2, 3, 3
v_2	3	4, 2, 2	u_2	2	4, 3
v_3	2	4, 3	u_3	2	4, 3
v_4	2	3, 3	u_4	3	4, 2, 2
v_5	2	4, 3	u_5	3	4, 2, 2
v_6	3	4, 2, 2	u_6	2	3, 3

From above table a bijective function between vertices of G_1 and G_2 s. t.

$v_1 \rightarrow u_1$ $\quad$ $v_2 \rightarrow u_4$

$v_3 \rightarrow u_2$ $\quad$ $v_4 \rightarrow u_6$

$v_5 \rightarrow u_3$ $\quad$ $v_6 \rightarrow u_5$

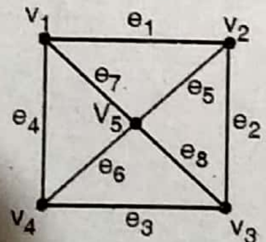
$\therefore$ G_1 and G_2 are isomorphic.

4.1.6 Spanning Subgraph

Subgraph H (V(H), E(H)) is called spanning subgraph of G (V(G), E(G)) if the vertex set of H is same as the vertex set of G i.e. if V(H) = V(G).

e.g. Consider,

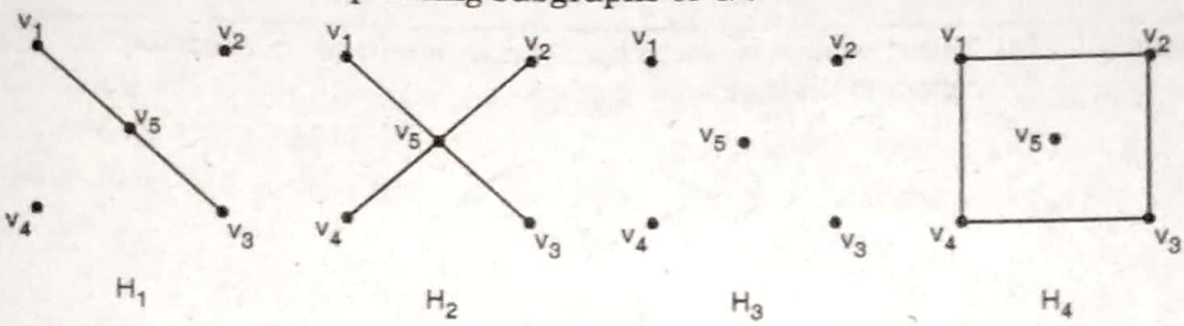
G



$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

Following are some spanning subgraphs of G :

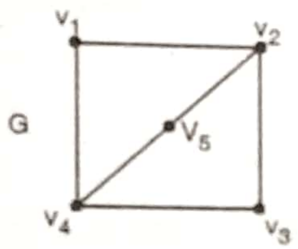
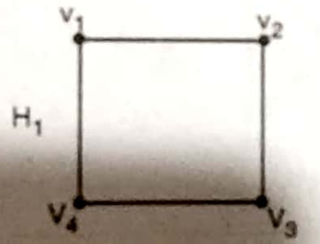
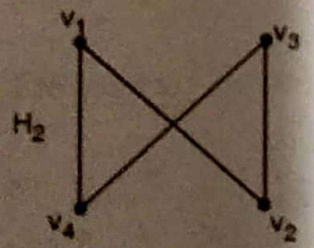


Remarks :

(1) Any simple graph on n vertices is subgraph of complete graph k_n. (infact it is spanning subgraph of k_n)

(2) Any graph isomorphic to a subgraph of G is also subgraph of G.

e.g.

	
Then 	H_1 is a subgraph of G
But 	H_2 is isomorphic to H_1

$\therefore$ H_2 is also subgraph of G.

(3) A single vertex in a graph G is also subgraph of G.

(4) A single edge together with its end vertices in a graph G is also subgraph of G.

(5) Subgraph of a graph G is also a subgraph of G.

(6) Any graph G is subgraph of itself.

Solved Examples :

Example 4.1.25 : Define spanning subgraph. Draw any two non-isomorphic spanning subgraphs of the following graph.

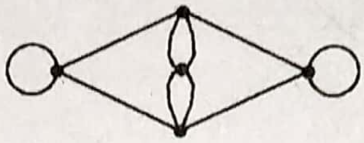


Fig. P. 4.1.25

Solution : Subgraph H (V(H), E(H)) is called spanning subgraph of G (V(G), E(G)) if the vertex set of H is same as the vertex set of G i.e. if V(H) = V(G).

e.g. Consider,

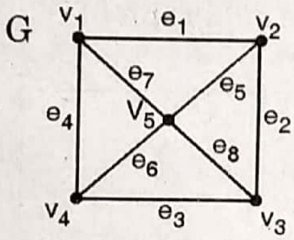


Fig. P. 4.1.25(a)

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

2 spanning non isomorphic subgraphs of above graph are,

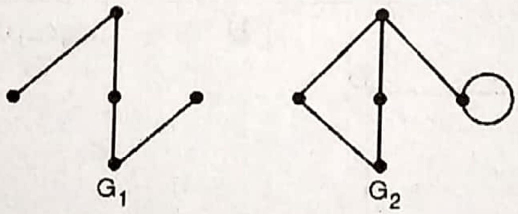


Fig. P. 4.1.25(b)

Example 4.1.26 : Draw any four non-isomorphic spanning subgraphs of the graph k_4.

Solution : k_4 is complete graph of 4 vertices.

Spanning subgraph is subgraph containing all vertices of k_4.

∴ Non isomorphic spanning subgraphs are :

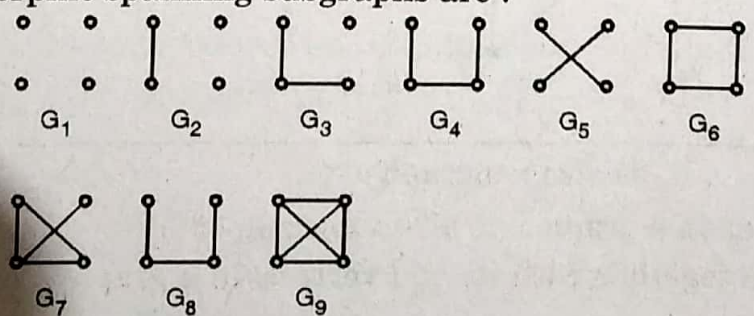


Fig. P. 4.1.26

All above are non isomorphic spanning subgraphs of k_4.

4.1.7 Connected and Disconnected Graph

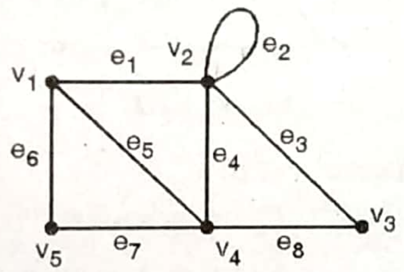
1. **Walk :**

Let G be a graph. A finite alternating sequence W : $\{v_1\ e_1\ v_2\ e_2\ v_3\ e_3 \ldots v_i\ e_i \ldots v_{i+1} \ldots v_n\}$ of vertices and edges of G beginning and ending with a vertex such that every edge in W is incident on a vertex which precedes and succeeds it is called a walk in G.

Note :

1. The walk is called $v_1 - v_n$ walk.
2. v_1 and v_n are called terminal vertices of the walk, and $v_2, v_3, \ldots, v_{n-1}$ are called intermediate vertices.
3. If terminal vertices are same in a walk then it is called as closed walk.
4. In a walk an edge or vertex may be repeated more than once.
5. The number of edges in the walk (including repetition) is called the length of the walk.

e.g. Consider,



Then following are some walks :

1. W_1 : $v_1\ e_1\ v_2\ e_2\ v_2\ e_3\ v_3\ e_8\ v_4$ It is $v_1 - v_4$ walk; length = 4
2. W_2 : $v_5\ e_7\ v_4\ e_5\ v_1\ e_1\ v_2$ It is $v_5 - v_2$ walk; length = 3
3. W_3 : $v_2\ e_2\ v_2$ It is $v_2 - v_2$ closed walk; length = 1
4. W_4 : $v_2\ e_2\ v_2\ e_1\ v_1\ e_5\ v_4\ e_7\ v_5\ e_6\ v_1\ e_1\ v_2$ It is $v_2 - v_2$ closed walk; length = 6

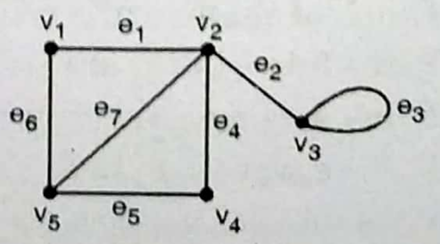
2. **Trail :**

A walk in which no edge is repeated is called a **trail.**

Note :

1. If terminal vertices in a trail are same it is called a closed trail, otherwise it is called an open trail.

e.g. Consider,



Now,

1. $v_3\ e_3\ v_3\ e_2\ v_2\ e_7\ v_5\ e_5\ v_4\ e_4\ v_2\ e_1\ v_1$
   here edges are not repeated therefore it is trail. ($\therefore$ it is trail)
2. $v_2\ e_2\ v_3\ e_3\ v_3\ e_2\ v_2$: not a trail ($\because$ e_2 is repeated twice)

3. Path :

The walk in which no vertex is repeated more than once is called a **path**.

Note :

1. Since vertex does not repeat, edge cannot repeat more than once.
   ∴ 'every path is trail'.
2. Loop cannot be included in a path.
3. The number of edges in a path is called the length of the path.
4. Any two paths with the same number of vertices are isomorphic.

Solved Examples :

Example 4.1.27 : Find all u – v path in the following graph G.

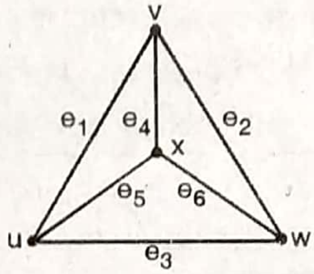


Fig. P. 4.1.27

Solution : Following are paths from u to v.

$P_1 : u\ e_1\ v$

$P_2 : u\ e_5\ x\ e_4\ v$

$P_3 : u\ e_3\ w\ e_2\ v$

$P_4 : u\ e_5\ x\ e_6\ w\ e_2\ v$

$P_5 : u\ e_3\ w\ e_6\ x\ e_4\ v$

Example 4.1.28 : Write all $v_1 - v_4$ paths in the graph G having adjacency matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution :

Graph of above matrix contains 6 vertices.

Paths from v_1 to v_4 are

$P_1 : v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4$

$P_2 : v_1\ e_6\ v_6\ e_5\ v_5\ e_4\ v_4$

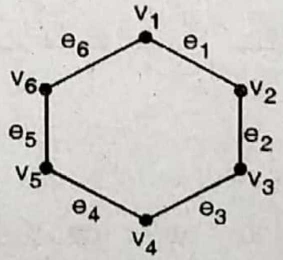


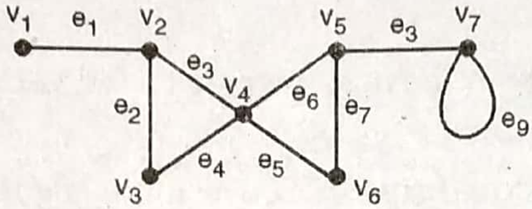
Fig. P. 4.1.28

4. **Cycle (Circuit) :**

A closed path is called a **cycle or circuit.**

Note : The number of edges in a cycle is called the length of the cycle.

e.g. Consider,



here,

1. $v_1\, e_1\, v_2\, e_2\, v_3\, e_4\, v_4$ It is path length = 3
2. $v_7\, v_5\, v_6\, v_4\, v_2\, v_3$ It is path length = 5
3. $v_5\, v_4\, v_2\, v_3\, v_4\, v_6$ Not a path
4. $v_2\, e_3\, v_4\, e_4\, v_3\, e_2\, v_2$ It is circuit length = 3
5. $v_7\, e_9\, v_7$ It is circuit length = 1
6. $v_2\, e_3\, v_4\, e_6\, v_5\, e_7\, v_6\, e_5\, v_4\, e_4\, v_3\, e_2\, v_2$ Neither path nor circuit.
7. $v_1\, e_1\, v_2\, e_2\, v_3\, e_4\, v_4\, e_5\, v_6\, e_7\, v_5\, e_8\, v_7$ It is longest path from v_1 to v_7 length = 6

Solved Example :

Example 4.1.29 : If G is complete graph on 10 vertices, then find number of cycles in G.

Solution : Any three vertices forms cycle. Such different cycles are ${}^{10}C_3$.

Any three vertices forms cycle. Such different cycles are ${}^{10}C_4$.

Similarly for 5, 6, 7, 8, 9, 10 vertices forms

${}^{10}C_5$, ${}^{10}C_6$, ${}^{10}C_7$, ${}^{10}C_8$, ${}^{10}C_9$, ${}^{10}C_{10}$ different cycles respectively.

$\therefore$ total number of cycles are :

$${}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$$

$$= \sum_{r=3}^{10} {}^{10}C_r$$

Syllabus Topic : Loops and Multiple Edges

4.2 Loops and Multiple Edges

Define loops and multiple edges.

(1) **Loop (self loop) :** If both the end vertices of an edge are same then the edge is called a **loop**.

e.g.

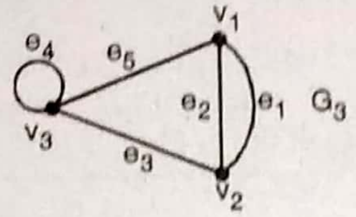


Fig. 4.2.1

here $V = \{v_1, v_2, v_3\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

Hence edge e_4 is a loop.

Observe in G_3 edges e_1 and e_2 have same pair of end vertices

i.e. $e_1 = (v_1, v_2)$ $e_2 = (v_1, v_2)$

(2) Parallel edges (multiple edges): If two or more edges have same terminal vertices then these edges are called as **parallel edges.**

e.g.

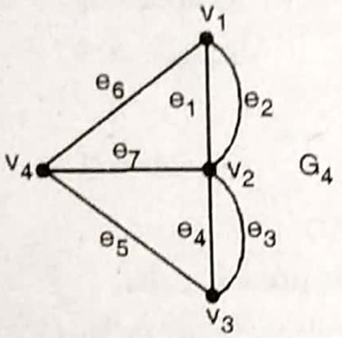


Fig. 4.2.2

here $V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

here e_1 and e_2 are parallel edges

also e_3, e_4 are parallel edges.

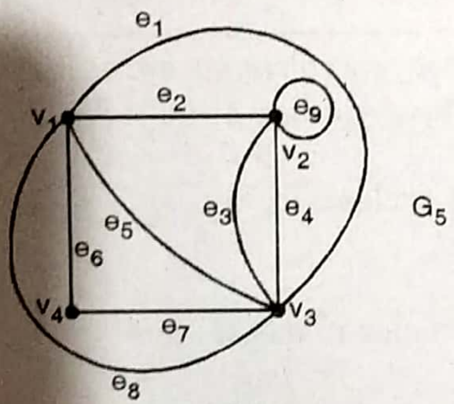


Fig. 4.2.3

here $e_1 = (v_1, v_3)$

$e_8 = (v_1, v_3)$

$e_5 = (v_1, v_3)$

i.e. e_1, e_8, e_5 are **parallel edges.**

Also e_4 and e_3 are **parallel edges.**

and e_9 is a **loop.**

(3) Simple graph : A graph without loops and parallel edges is called simple graph.

Syllabus Topic : Multigraphs

4.3 Multigraph

Define multigraphs with the help of example.

(1) Multigraph :

A graph with parallel edges but not loop is called a multigraph.

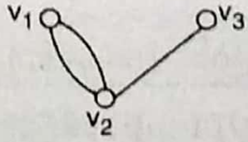


Fig. 4.3.1

(2) Compound graph (Pseudograph) : A graph which contains loops or parallel edges is called compound graph or multigraph.

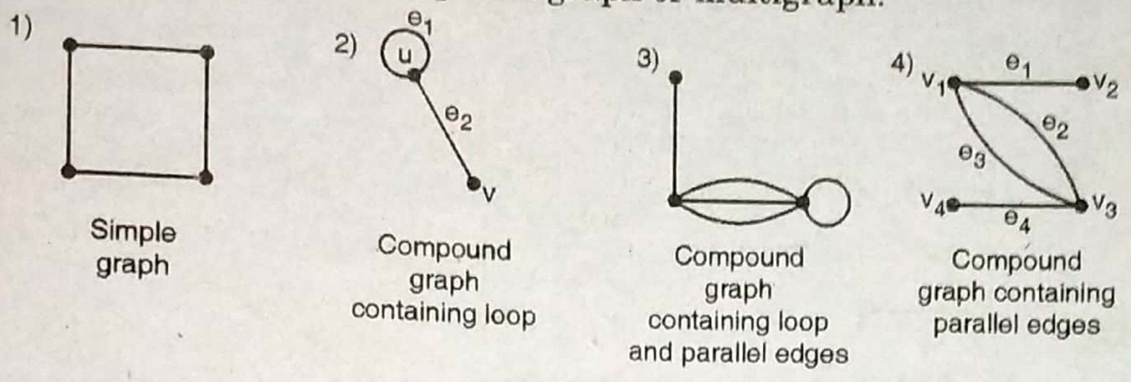


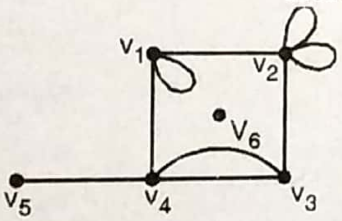
Fig. 4.3.2

Graph	Loop	Parallel edges
Simple graph	No	No
Multigraph	No	Yes
Compound graph	Yes	Yes

(3) Degree of a vertex : The number of edges incident on a vertex v is called **degree** of vertex v, with loop being counted twice.

Notation: Degree of v =d(v)

e.g.



here $d(v_1) = 4$

$d(v_2) = 6$

$d(v_3) = 3$

$d(v_4) = 4$

$d(v_5) = 1$

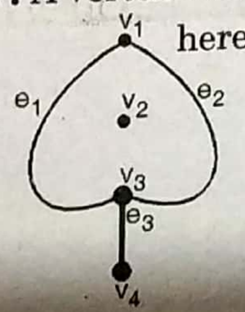
$d(v_6) = 0$

Fig. 4.3.3

(4) Isolated vertex : A vertex with degree zero is called as **isolated** vertex.

(5) Pendant vertex : A vertex with degree one is called as **pendant** vertex.

e.g.



here

v_4 is pendant vertex.

v_2 is isolated vertex.

Fig. 4.3.4

(6) Degree of the graph :

Sum of the degrees of all vertices of graph G is called degree of the graph G.

Notation : d(G) – degree of graph G.

i.e. $$d(G) = \sum_{v_i \in G} d(v_i)$$

e.g. (1)

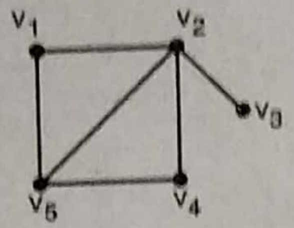


Fig. 4.3.5

$d(v_1) = 2$
$d(v_2) = 4$
$d(v_3) = 1$
$d(v_4) = 2$
$d(v_5) = 3$

$d(G) = 2 + 4 + 1 + 2 + 3$

$\Rightarrow d(G) = 12$

$$\Rightarrow \sum_{v_i \in G} d(v_i) = 12$$

(2)

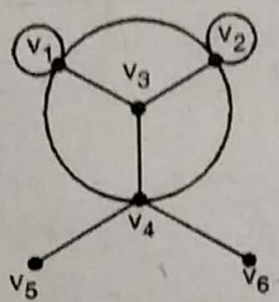


Fig. 4.3.6

$d(v_1) = 5$
$d(v_2) = 5$
$d(v_3) = 3$
$d(v_4) = 5$
$d(v_5) = 1$

$d(v_6) = 5 + 5 + 3 + 5 + 1$

$$\Rightarrow d(G) = 12 \quad \Rightarrow \quad \sum_{v_i \in G} d(v_i) = 19$$

Syllabus Topic : Eulerian and Hamiltonian Graphs

4.4 Eulerian and Hamiltonian Graphs

4.4.1 Eulerian Trail

If every edge of graph G is included in a trail, then that trail 1 is called **Eulerian trail.**

4.4.2 Eulerian Circuit

If every edge of graph G is included in a circuit then, that circuit is called **Eulerian circuit.**

4.4.3 Eulerian Graph

> Define Eulerian graph.

Eulerian graph is a graph that contains an Eulerian circuit.

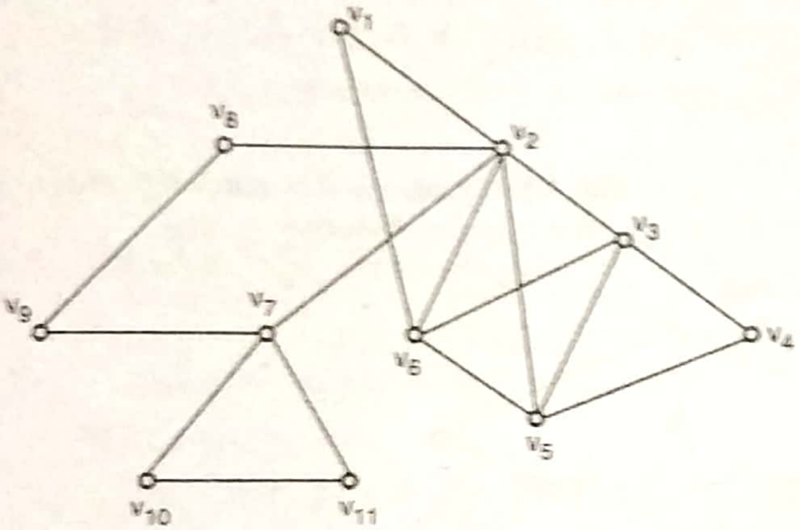


Fig. 4.4.1 : An Eulerian graph

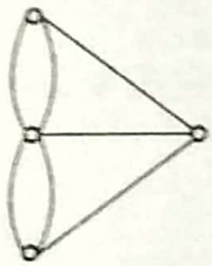


Fig. 4.4.2 : Not an Eulerian graph

(The multigraph of konigsberg's bridges).

4.4.4 Hamiltonian Path

If a vertex set of G is same as vertex set of path P i.e. a path P spans the vertices of G, then P is said to be Hamiltonian path P of G.

Remark :

A graph containing Hamiltonian path is called traceable.

4.4.5 Hamiltonian Cycle

As a cycle C spans the vertices of graph G then such cycle C is called Hamiltonian cycle of graph G.

4.4.6 Hamiltonian Graph

> Define Hamiltonian Graph.

A graph G is called Hamiltonian graph if it contains Hamiltonian cycle.

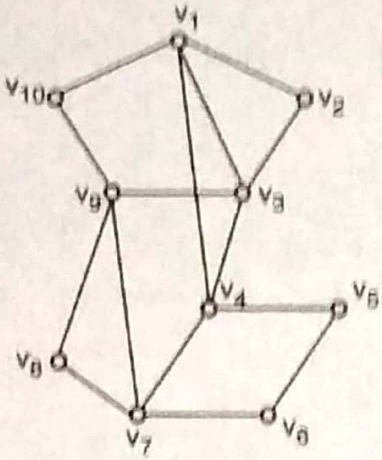


Fig. 4.4.3 : Graph is Eulerian and Hamiltonian

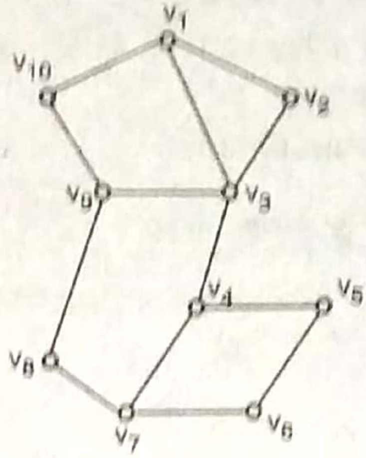


Fig. 4.4.4 : Graph is Hamiltonian but not Eulerian

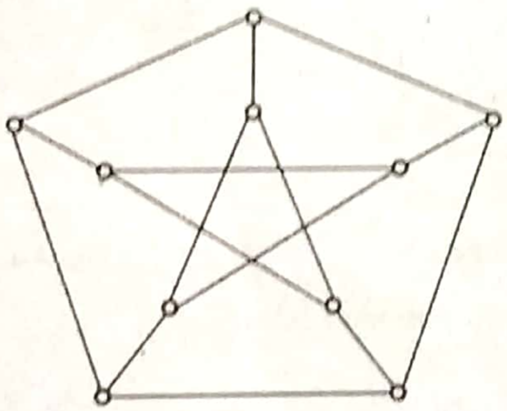


Fig. 4.4.5 : Not Hamiltonian graph. The Petersen graph

Syllabus Topic : Graph Colouring

4.5 Graph Colouring

What is Graph Colouring ?

- Let V (G) be a vertex set of a graph G. Consider, set of positive integers [1, 2,...k], then function
  k : V (G) → {1, 2, ...k} is called a k-colouring.
- Then each integer from {1, 2,...k} is consider to be a colour.

4.5.1 Proper k-colouring

Define proper k-colouring.

- If adjacent vertices are coloured differently i.e. for adjacent vertices V_1 and V_2.

- If k (V_1) ≠ k (V_2), then k is called a proper k colouring of G and G is called as k- colourable.

For example :

Consider graph C_5 such that,

$P(V_1) = P(V_3) = 1$, $P(V_2) = P(V_4) = 2$ and $P(V_5) = 3$.

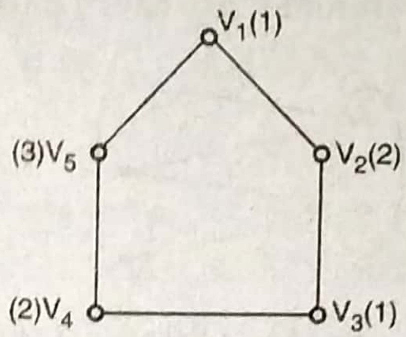


Fig. 4.5.1

Just by observation we say that, C_5 is 3-colourable.

Now what is the minimum number of colours required to graph become colourable ?

4.5.2 Chromatic Number

Define chromatic number.

Chromatic number of a graph G is the smallest integer k such that, G will become k-colourable. It is denoted by χ (G).

For example :

If you observe C_5, it is impossible that C_5 is 2-colourable.

It is necessary that minimum 3 colours required.

So $\chi(C_5) = 3$

List of chromatic number for some graphs

$$\chi(C_n) = \begin{cases} 2 \text{ ; if n is even} \\ 3 \text{ ; if n is odd} \end{cases}$$

C_n = A cycle on n vertices.

P_n = Simply a path on n vertices

$$\chi(P_n) = \begin{cases} 2 \text{ if } n \geq 2 \\ 1 \text{ if } n = 1 \end{cases}$$

$\chi(k_n) = n$ ...k_n complete graph

$\chi(E_n) = 1$ ...E_n is empty graph

$\chi(k_{m,n}) = 2$

$k_{m,n}$ is complete bipartite graph.

4.5.3 Bipartite Graphs

Define Bipartite graph.

- A graph G is said to be bipartite graph, if its vertex set can be partion into two sets say A and B in a such a way that every edge of G has one end point in A and another in B.

- A and B are called partite sets.

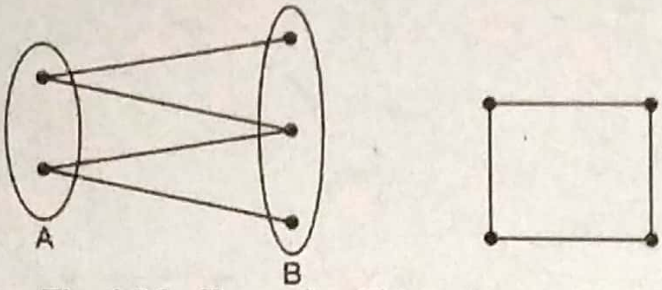


Fig. 4.5.2 : Example of bipartite graph

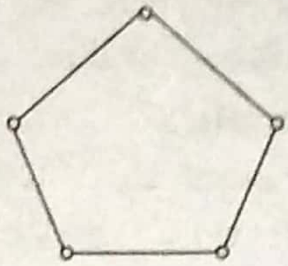


Fig. 4.5.3 : Example of non bipartite graph

Complete Bipartite Graph

Define complete Bipartite graph.

- A graph G is bipartite graph with A and B are partite set.
- If there is every possible connection of a vertex of A with vertex of B.

  Such graph is denoted by $k_{|A||B|}$.

  For example :

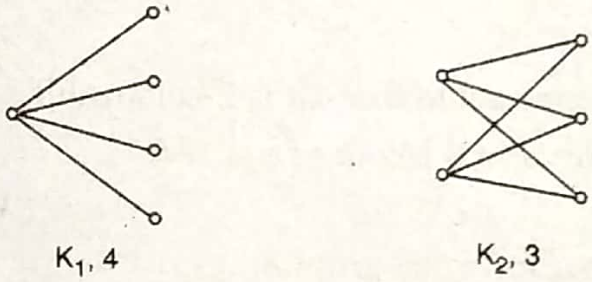


Fig. 4.5.4

4.5.4 Clique

Define Clique.

- A clique in a graph G = (V, E)
- Let $k \subseteq V$ such that, sub graph induced by k is isomorphic ($\cong$) to the complete graph $k_{|k|}$.
- Every pair of vertices in k are adjacent.
- The maximum clique size is called clique number of a graph G.
- It is denoted by w (G).

Proposition

1. If $F : X \rightarrow Y$ is a function and $|x| \geq (m-1)|y| + 1$, then $\exists$ an element $y \in Y$ and distinct elements $x_1, x_2 \ldots x_m \in X$,

   So that, $F(x_i) = y$ for $i = 1 \ldots m$.
2. For every $n \geq 3$ $\exists$ a graph G_n so that, $\chi(G_n) = n$ and $w(G_n) = 2$.

Remark

If χ (0) = w (0) for every induced subgraph 0 of G, then G is said to be perfect graph G.

4.6 Planar Graphs

☞ **Define planar graph.**

- A graph G in which pairs of edges intersect only at vertices is called **planar graph.** Otherwise graph is called non planar.
- A drawing of a planar graph G is called as planar representation or a planar embedding.

For example :

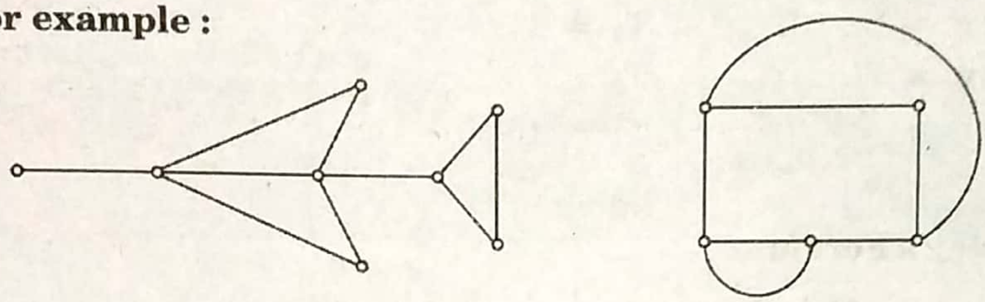
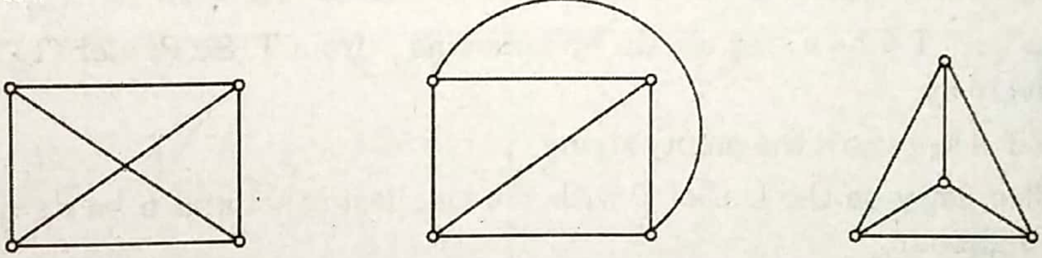


Fig. 4.6.1

Solved Example :

Example 4.6.1 : Represent k_4 in a planar representation.

Solution :



(a) k_4 **(b) Planar representation of k_4** **(c)**

Fig. P. 4.6.1

A face : A region bounded by vertices and edges and not containing any other edges and vertices is called a face of planar drawing of a graph.

Euler's Theorem :

Let G be a connected planar graph with n vertices, m edges and f faces then it satisfy,

$n - m + f = 2$

i.e. number of vertices – number of edges + number of faces = 2

CHAPTER 5

Unit II

Applying Probability to Combinatorics

Syllabus

Applying Probability to Combinatorics, Small Ramsey Numbers, Estimating Ramsey Numbers, Applying Probability to Ramsey Theory, Ramsey's Theorem the Probabilistic Method.

Syllabus Topic : Applying Probability to Combinatorics

Introduction

If you need good command of statistics. You must know basic knowledge of combinatorics. In combinatorics we study permutations and combinations.

5.1 The Pigeon-Hole Principle

Define the pigeon-hole principle.

If n + 1 Pigeons are entered in n holes then there exists at least one hole containing more than one pigeons.

Generalised Pigeon-hole principle

If (kn + 1) pigeons entered in n holes then at least one hole contains k + 1 or more Pigeons. (k is positive integer).

For example :

(i) If 11 notebooks have to distribute in 10 students then at least one student get more than one notebooks.

(ii) Among 13 friends there is one month in which 2 or more friends will have their birth day during that month.

Solved Examples :

Example 5.1.1 : Show that at a party of 20 people, there are two people who have the same number of friends.

Solution : In a party of 20 person each has at least one and at most 19 friends.

Person having n friends we assign him in room labelled n. i.e. we require 19 rooms to allocate 20 persons. Therefore by pigeon hole principle, there is at least one room containing two persons. i.e. there are at least two persons has same number of friends.

Example 5.1.2 : If 10 points are chosen in a square whose sides have length 3. Show that there must be at least two points which are at most $\sqrt{2}$ distance apart.

Solution :

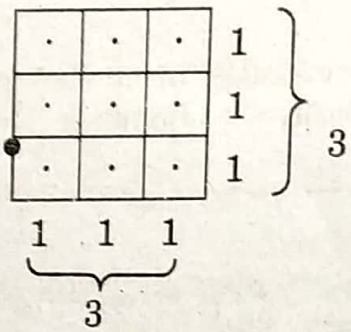


Fig. P. 5.1.2

Divide square of length 3 into small squares of length 1. We get 9 squares. If we assign one point in one small square then condition is fulfilled. But to assign 10 points into 9 squares by pigeon hole principle at least one of the square contains two points and distance between them is less than $\sqrt{2}$.

Example 5.1.3 : Prove that if given a set of any seven distinct integers, there must exist two integers in the set whose sum or difference is multiple of 10.

Solution :

To hold condition of divisibility we require numbers whose unit placed elements are distinct and which are from one of the set { 0, 1, 2, 3, 4, 5 } or { 9, 8, 7, 6, 5, 0 } or { 2, 3, 4, 5, 9, 0 } ...etc. Such each set contains exactly 6 numbers. By pigeon hole principle if selection of 7 numbers for which 6 unit placed elements. Then there exist at least one repeated unit placed element or elements from other sets maintained above. Hence such 7 numbers does not exists.

Example 5.1.4 : The circumference of a wheel is divided into 36 sectors and the numbers 1, 2, ..., 36 are assigned to them in arbitrary manner. Show that there are three consecutive sectors such that the sum of their assigned number is at least 56.

Solution :

Let $a_1, a_2, \ldots, a_{36}$ be arbitrary assignment of numbers 1, 2, 3, ... 36 to the 36 sectors. We group them into the collection of three consecutive and find 36 sums as below :

$S_1 = a_1 + a_2 + a_3$, $S_2 = a_2 + a_3 + a_4$,

$S_3 = a_3 + a_4 + a_5$ ..... $S_{36} = a_{36} + a_1 + a_2$

The sum of these 36 numbers is 3 times the sum $1 + 2 + 3 + \ldots + 36$ because each a_j is counted three times.

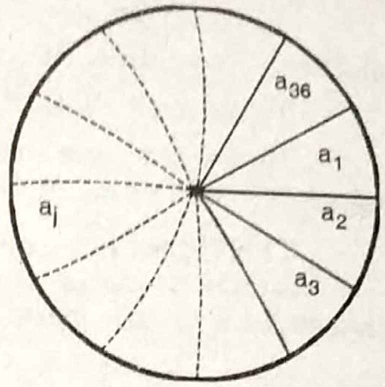


Fig. P. 5.1.4

$$\therefore S_1 + S_2 + \ldots + S_{36} = 3(1 + 2 + \ldots + 36) = 3(666) = 1998$$

$\therefore$ The sum of 36 positive integers is 1998 and 1998/36 = 55.5

This implies that at least one $S_i \geq 56$.

Hence there are three consecutive sectors, the sum of whose assigned numbers is at least 56.

Example 5.1.5 : If 10 points are chosen within the equilateral triangle of side length 3. Prove that the selection includes at least two points which are at most 1 unit farthest apart from each other.

Solution :

Here we can find 9 small triangles of given triangle, of which there are two points of two triangle are 1 unit. farthest apart but we have 10 points hence by pigeon hole principle one of the triangle contains at least two points hence their distance reduces less than 1 unit.

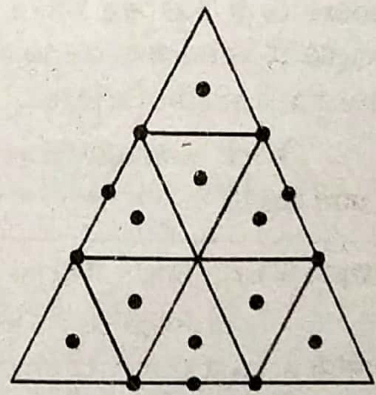


Fig. P. 5.1.5

Example 5.1.6 : Given n consecutive integers, prove that there is one which is divisible by n.

Solution :

After dividing n consecutive integers by n we get 1, 2, 3, 4, ... (n – 1) remainders. By pigeon hole principle for n numbers and (n – 1) remainders there exist at least two numbers having same remainders. But these numbers are not consecutive integers.

$\therefore$ For n consecutive integers one of which is divisible by n.

Example 5.1.7 : Show that there does not exist 7 lectures each of 30 minutes from 10 am to 1 pm.

Solution : Period from 10 am to 1 pm is 3 hours.

∴ There are 6 slots of 30 minutes each from 10 am to 1 pm.

∴ By pigeon hole principle there exist at least two lectures in same slot of time.

∴ There does not exist 7 lectures from 10 am to 1 pm.

Example 5.1.8 : Show that in a town having population of 6 lakh there are at least two persons who are born on same date, month and time.

Solution : Number of minutes for one year = 525600 as pigeons holes and 6 lakh people as the pigeons. So by pigeon hole principle it follows that at least two persons are born on the same date, month and time.

Example 5.1.9 : Show that if any five numbers from the set {1, 2, ...8} are chosen, then two of them will add up to 9.

Solution : The possible collections of two numbers from set { 1, 2, .. 8} having sum is 9 are

$$S_1 = \{1, 8\}, S_2 = \{2, 7\}$$
$$S_3 = \{3, 6\}, S_4 = \{4, 5\}.$$

If we select one number from each set then we cannot have sum of any two numbers is 9 but we have to select 5 numbers from 4 sets then by pegion hole principle It is necessary to select both numbers from one of the set S_1 or S_2 or S_3 or S_4, but then sum of these numbers becomes 9.

∴ There does not exist five numbers from {1, 2,.....8}. Such that two of them will add up to 9.

Example 5.1.10: Show that at a party of 20 persons, there are two persons who have same number of friends.

Solution : In a party there are 20 persons. Each person have at least 1 friend and at most 19 friends (A person not a friend of himself).

Allocate room to the person so that he has i friends such that $1 \le i \le 19$.

∴ We require at most 19 rooms for allocations.

By Pigeon hole principle there exist one of the room containing two persons and that two persons have same number of friends.

Example 5.1.11: Given a group of n women and their husbands, how many people must be chosen from this group to guarantee that the set contains married couple ?

Solution : Number of women = n.

If we choose n women then there is no change to have a married couple but if we add one more person which is man then he must be husband of one of the woman.

Hence n + 1 people must be chosen so that its guarantee to have married couple.

Example 5.1.12 : How many friends must you have to guarantee that atleast of them will have birthday in the same month ?

Solution : There are 12 months suppose there are 12 Friends each having birthday in different month. If we required 9 friend having birthday in same month. So required at least 12×9 friends.

But we required at least 10 of them will have birthday in same month so we add 1 more in that so at least in one month there are 10 friend having birthday.

$\therefore$ No. of friends $= 12 \times 9 + 1 = 108 + 1 = 109$

Example 5.1.13 : How many students do you need in a college to guarantee that there are atleast two students who have the same first two initials.

Solution : Number of students having different first initial $= 26 \times 25 = 650$

Similarly number of students having different second initial $= 26 \times 25 = 650$

$\therefore$ Total number of students having first two different initial $= 2(26 \times 25)$ but among $2(26 \times 25) + 1$ number of students gets at least one pair of students has same first two initials.

i.e. 1301 students required in college to have same first two initials.

Syllabus Topic : Small Ramsey Numbers

5.2 Small Ramsey Numbers

☞ Define small Ramsey numbers.

It is little bit difficult to find Ramsey number R (m, n).

Following table gives us Ramsey number R (m, n) when $3 \leq m \leq 9$, $3 \leq n \leq 9$.

Single number in a cell represent exact answer and two numbers represent upper and lower bounds of Ramsey number R (m, n)

- The flow is illustrated in Fig. 6.1.2. In the Fig. 6.1.2, the numbers associated with each edge are its capacity and the sum of flow that φ places on that edge. The first conservation law is determined here.
- For example, the edge (E, D) has capacity 20 and currently carries a flow of 8. The value of this flow is

$$30 = \varphi(S, F) + \varphi(S, B) + \varphi(S, E) = \varphi(A, T) + \varphi(C, T)$$
$$= 8 + 9 + 13 = 16 + 14$$

- To observe that the second conservation law holds at, for example, vertex B, note that the flow into B is $\varphi(S, B) + \varphi(E, B) + \varphi(D, B) = 20$ and the flow out of B is $\varphi(B, F) + \varphi(B, A) + \varphi(B, C) = 20$.
- As we have seen in Fig. 6.1.1 that it is easy to find a flow just by assigning $\phi(e) = 0$ for each edge e.

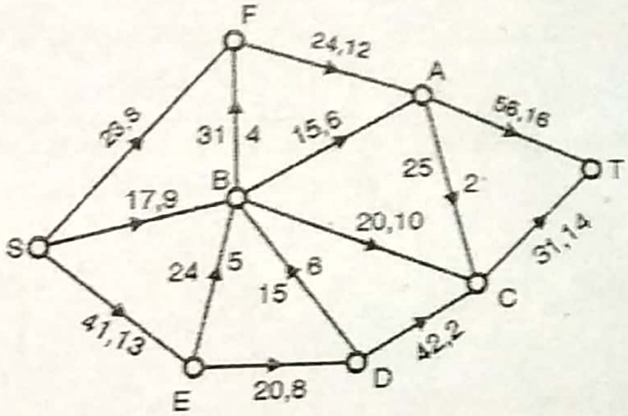


Fig. 6.1.2 : Network flow

- It is very easy to misjudge the importance of this observation. Network flow problems are a special case of a more general class of optimization problems known as linear programs, and in general, it may be very difficult to find a feasible solution to a linear programming problem.
- In fact, conceptually, finding a feasible solution - any solution - is just as hard as finding an optimal solution.

Syllabus Topic : Flows and Cuts

6.2 Flows and Cuts

Explain flows and cuts.

- Asking the maximum value of a flow in a given network is a natural task. In another way, we want to find the largest number v_0 so that there exists a flow φ of value v_0 in the network.
- Along with the maximum value v_0, we also want to find a flow φ having this value. We are going to develop an efficient algorithm which finds :

(a) A flow of maximum value

(b) A certificate verifying the claim of optimality.

The certificate makes use of the following important concept.

- A partition $V = L \cup U$ of the vertex set V of a network with $S \in L$ and $T \in U$ is called a **cut**. The capacity of a cut $V = L \cup U$, denoted c (L, U), and it is defined by

$$c(L, U) = \sum_{x \in L,\, y \in U} c(x, y)$$

- In other words, the capacity of the cut $V = L \cup U$ is the total capacity of all edges from L to U. keep in mind that we add only the capacities of the edges from L to U while computing the capacity of the cut $V = L \cup U$. The edges from U to L are not included in this sum

Example 6.2.1 : Consider the network diagram in Fig. P. 6.2.1 as given below. Let's first consider the cut $V = L_1 \cup U_1$ with $L_1 = \{S, F, B, E, D\}$ and $U_1 = \{A, C, T\}$.

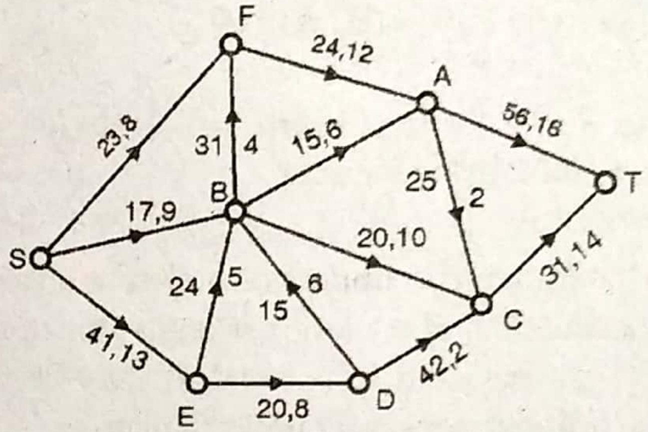


Fig. P. 6.2.1

Solution :

Here, the capacity of the cut is :

$$c(L_1, U_1) = c(F, A) + c(B, A) + c(B, C) + c(D, C)$$
$$= 24 + 15 + 20 + 42 = 101.$$

When we consider the cut $V = L_2 \cup U_2$ with $L_2 = \{S, F, B, E\}$ and $U_2 = \{A, D, C, T\}$.

Here the capacity of the cut is,

$$c(L_2, U_2) = c(F, A) + c(B, A) + c(B, C) + c(E, D)$$
$$= 24 + 15 + 20 + 20 = 79.$$

Here, we have not included c (D, B) in the calculation as the directed edge (D, B) is from U_2 to L_2.

The following theorem shows the **relationship between flows and cuts.**

☞ **Explain the relation between flows and cuts.**

Theorem

Let G = (V, E) be a network, let ϕ be a flow in G and let $V = L \cup U$ is a cut. Then the value of the flow is at most as large as the capacity of the cut.

Proof

In this proof, the convention adopted is that $\phi(x, y) = 0$ if (x, y) is not a directed edge of a network G.

$$v_0 = \sum_{y \in V} \phi(S, y) - \sum_{z \in V} \phi(z, S),$$

As the second summation is 0. Moreover, as per the second of our flow conservation laws, we have for any vertex other than the source and the sink,

$$\sum_{y \in V} \phi(x, y) - \sum_{z \in V} \phi(z, x), = 0$$

Now we have,

$$v_0 = \sum_{y \in V} \phi(S, y) - \sum_{z \in V} \phi(z, S)$$

$$= \sum_{y \in V} \phi(S, y) - \sum_{z \in V} \phi(z, S) + \sum_{\substack{x \in L \\ x \neq S}} \left[\sum_{y \in V} \phi(x, y) - \sum_{z \in V} \phi(z, x)\right]$$

$$= \sum_{x \in L} \left[\sum_{y \in V} \phi(x, y) - \sum_{z \in V} \phi(z, x)\right]$$

Here, we take a pause and check for the last line. Observe that if (a, b) is a directed edge with both endpoints in L, then when the outer sum is conducted for $x = a$, we get an overall contribution of $\phi(a, b)$. Conversely, when it is conducted for $x = b$, we get a contribution of $-\phi(a, b)$. Thus, the terms cancel out and everything simplifies to,

$$\sum_{\substack{x \in L \\ y \in U}} \phi(x, y) - \sum_{\substack{x \in L \\ z \in U}} \phi(z, x) \leq \sum_{\substack{x \in L \\ y \in U}} \phi(x, y) \leq \sum_{\substack{x \in L \\ y \in U}} c(x, y) = c(L, U)$$

Hence, $v_0 \leq c(L, U)$.

Syllabus Topic : Augmenting Paths

6.3 Augmenting Paths

Explain Augmenting path with example.

In this section we are going to develop the classic labelling algorithm of Ford and Fulkerson. This algorithm starts with any flow in a network and continued to modify the flow. It always increases the value of the flow until it reaches a step where no further improvements are possible. The labelling algorithm makes use of some natural and descriptive terminology.

Suppose we have a network G = (V, E) with a flow φ of value v. The φ is the current flow and check for ways to augment φ by making a relatively small number of changes.

Step IV : Output = F

Step V : Stop

Example 6.4.1 : Find maximum flow of above network.

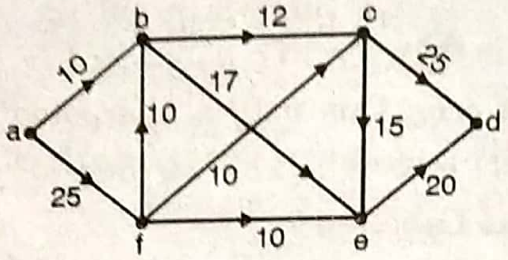


Fig. P. 6.4.1

Solution : Clearly in given network.

Source = a Sink = d

Assign 0 flow for each edge.

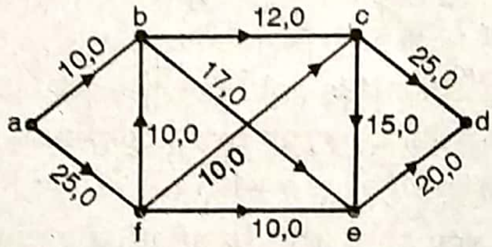


Fig. P. 6.4.1 (a)

First unsaturated path is

$$P_1 : a \xrightarrow[\Delta = 10-0]{10,0} b \xrightarrow[\Delta = 12-0]{12,0} c \xrightarrow[\Delta = 25-0]{25,0} d$$

$\Rightarrow \Delta = \min \{10, 12, 25\} = 10$

Add Δ into the flow of each edge in defined unsaturated path P, hence new flow becomes.

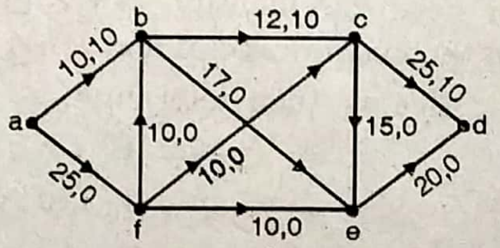


Fig. P. 6.4.1 (b)

Next unsaturated path is

$$P_2 : a \xrightarrow[25]{25,0} f \xrightarrow[10]{10,0} b \xrightarrow[2]{12,10} c \xrightarrow[15]{25,10} d$$

$\Delta = \min \{25, 10, 2, 15\} = 2$

Add Δ = 2 into flow of each edge in P_2

New flow becomes.

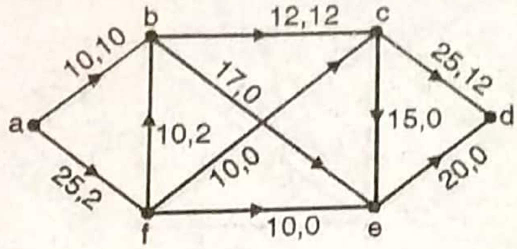


Fig. P. 6.4.1(c)

Next unsaturated path is

$$P_3 : a \xrightarrow[23]{25,2} f \xrightarrow[8]{10,2} b \xrightarrow[17]{17,0} e \xrightarrow[20]{20,0} d$$

Δ = min {23, 8, 17, 20} = 8

Add Δ = 8 in each edge of P_3 new flow becomes.

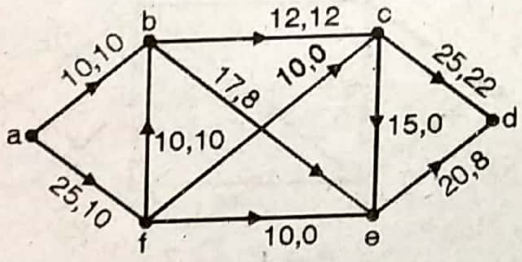


Fig. P. 6.4.1 (d)

Next unsaturated path is

$$P_4 : a \xrightarrow[10]{25,10} f \xrightarrow[10]{10,0} c \xrightarrow[13]{25,12} d$$

Δ = min {10, 10, 13} = 10

Add Δ = 10 into flow of each edge in P_4 new flow becomes

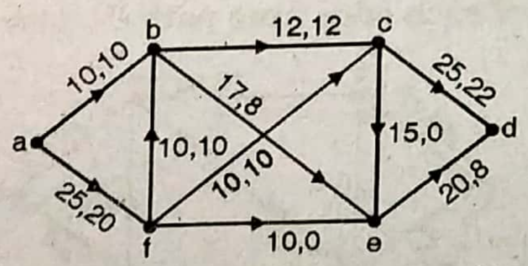


Fig. P. 6.4.1 (e)

Next unsaturated path is

$$P_5 : a \xrightarrow[5]{25,20} f \xrightarrow[10]{10,0} e \xrightarrow[12]{20,8} d$$

Δ = min {5, 10, 12} = 5

Add Δ = 5 into flow of each edge in P_5 new flow becomes

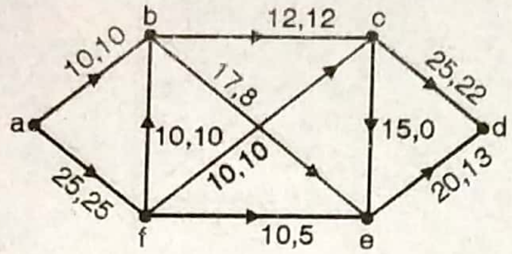


Fig. P. 6.4.1 (f)

All paths from source to sink are saturated hence maximum flow occurred and it is Maximum flow = 22 + 13 = 35

Example 6.4.2 : Capacity of each edge is given. Find maximum flow from a to d in the network. What is the value of maximum flow.

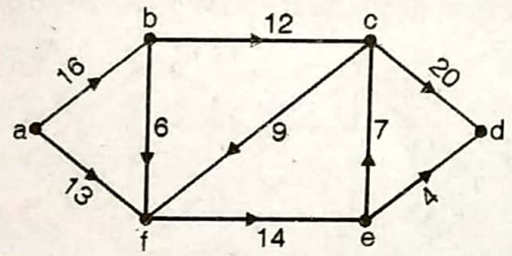


Fig. P. 6.4.2

Solution : For given network source is a and sink is d assign zero flow for each edge then first unsaturated path from a to d is

$$P_1 = a \xrightarrow[16]{16,0} b \xrightarrow[12]{12,0} c \xrightarrow[20]{20,0} d$$

$\Delta ij = \{16, 12, 20\}$

$$\therefore \Delta = \min\{16, 12, 20\} = 12$$

Add $\Delta = 12$ in to flow of each edge from path P_1, new network becomes

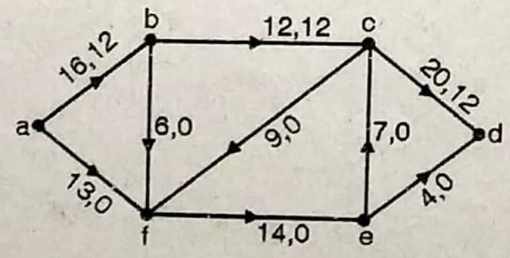


Fig. P. 6.4.2 (a)

Next unsaturated path is

$$P_2 = a \xrightarrow[4]{16,12} b \xrightarrow[6]{6,0} f \xrightarrow[14]{14,0} e \xrightarrow[4]{4,0} d$$

$\Delta ij = \{4, 6, 14, 4\}$

$$\Delta = \min\{4, 6, 14, 4\} = 4$$

Add $\Delta = 4$ into flow of each edge from path P_2, new network becomes.

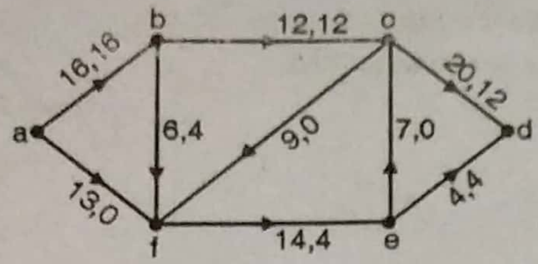


Fig. P. 6.4.2 (b)

Next unsaturated path is

$$P_3 = a \xrightarrow[13]{13,0} f \xrightarrow[10]{14,4} e \xrightarrow[7]{7,0} c \xrightarrow[8]{20,12} d$$

$\Delta ij = \{13, 10, 7, 8\}$

$$\Delta = \min\{13, 10, 7, 8\} = 7$$

Add $\Delta = 7$ into flow of each edge of path P_3, new network becomes.

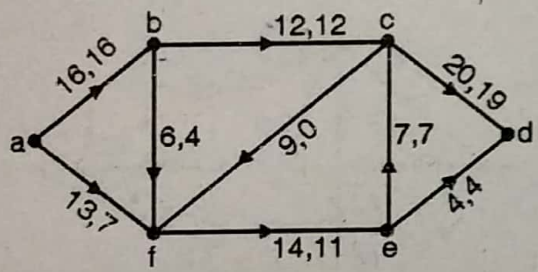


Fig. P. 6.4.2 (c)

All directed paths from a to d are saturated $\therefore$ maximum flow is occurred and it is

$$\text{Maximum flow} = 19 + 4 = 23$$

Example 6.4.3 : Find the maximum flow in the following network by using ford and fulkerson's algorithm.

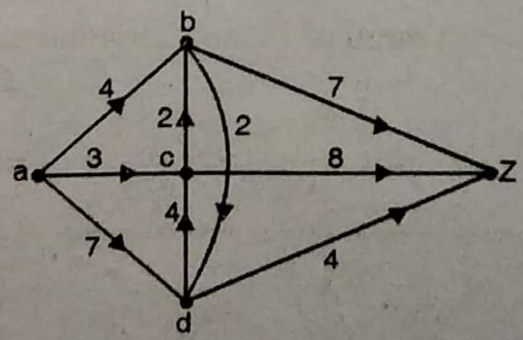


Fig. P. 6.4.3

Solution :

Source is a and sink is z. Assign zero flow for each of the edge in network then flow becomes.

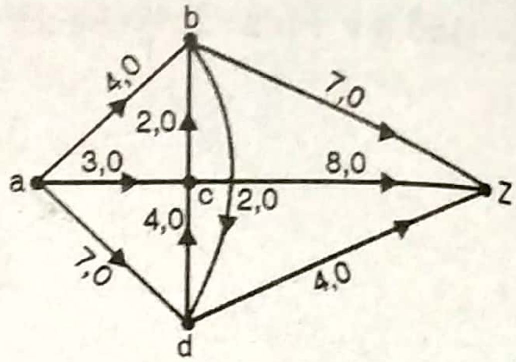


Fig. P. 6.4.3 (a)

First unsaturated directed path from a to z is

$$P_1 = a \xrightarrow[4]{4,0} b \xrightarrow[7]{7,0} z$$

$$\Delta ij = \{4, 7\}$$

$$\Rightarrow \Delta = \min\{4, 7\} = 4$$

Add $\Delta = 4$ into flow of each edge in the P_1 new flow becomes.

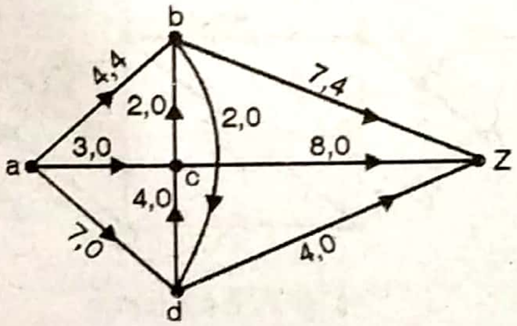


Fig. P. 6.4.3 (b)

Next unsaturated path is

$$P_2 = a \xrightarrow[3]{3,0} c \xrightarrow[8]{8,0} z$$

$$\Delta ij = \{3, 8\}$$

$$\Rightarrow \Delta = \min\{3, 8\} = 3$$

Add $\Delta = 3$ into flow of each edge of P_2 new flow becomes

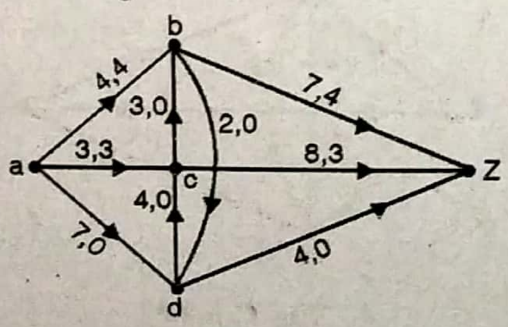


Fig. P. 6.4.3 (c)

Next unsaturated path is,

$$P_3 = a \xrightarrow[7]{7,0} d \xrightarrow[4]{4,0} z$$

$$\Delta ij = \{7, 4\}$$

$$\Delta = \min\{7, 4\} = 4$$

Add $\Delta = 4$ in flow of each edge in P_3 then new network is

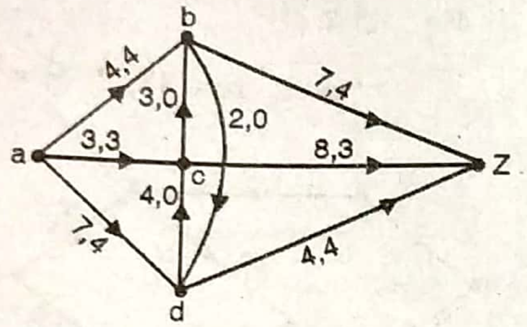


Fig. P. 6.4.3 (d)

Next unsaturated path is

$$P_4 = a \xrightarrow[3]{7,4} d \xrightarrow[4]{4,0} c \xrightarrow[5]{2,3} z$$

$\Delta ij = \{3, 4, 5\}$

$$\Delta = \min\{3, 4, 5\} = 3$$

New flow becomes

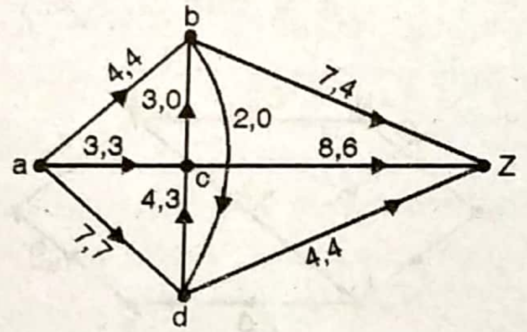


Fig. P. 6.4.3 (e)

All directed paths from a to z are saturated hence maximum flow is occurred hence maximum flow is

$$4 + 6 + 4 = \mathbf{14}$$

Example 6.4.4 : Determine maximal flow in the given network by using ford-fulkerson algorithm find value of the maximal flow.

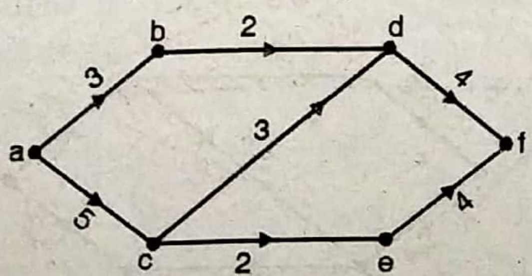


Fig. P. 6.4.4

Solution : Source is a

Sink is f assign flow of each edge is zero.

Scanned by CamScanner

First unsaturated path from a to f is

$$P_1 = a \xrightarrow[3]{3,0} b \xrightarrow[2]{2,0} d \xrightarrow[4]{4,0} f$$

$$\Delta ij = \{3, 2, 4\}$$

$$\Rightarrow \Delta = \min\{3, 2, 4\} = 2$$

New flow becomes

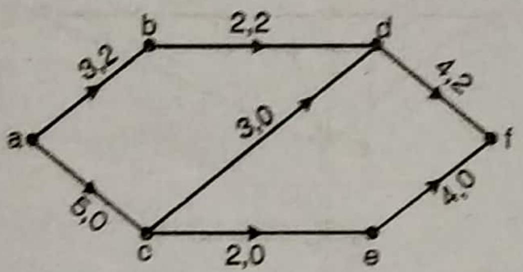


Fig. P. 6.4.4 (a)

Next unsaturated path is

$$P_1 = a \xrightarrow{5,0} c \xrightarrow{3,0} d \xrightarrow{4,2} f$$

$$\Delta ij = \{5 \quad 3 \quad 2\}$$

$$\Delta = \min\{5, 3, 2\} = 2$$

New flow becomes

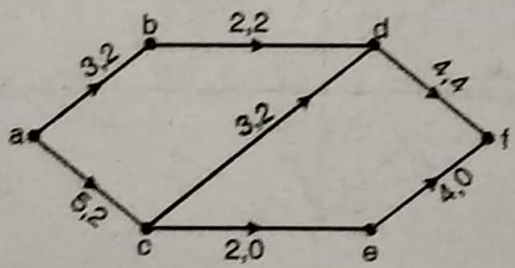


Fig. P. 6.4.4 (b)

Next unsaturated path is

$$P_3 = a \xrightarrow[3]{5,2} c \xrightarrow[2]{2,0} e \xrightarrow[4]{4,0} f$$

$$\Delta ij = \{3, 2, 4\}$$

$$\Delta = \min\{3, 2, 4\} = 2$$

New flow becomes

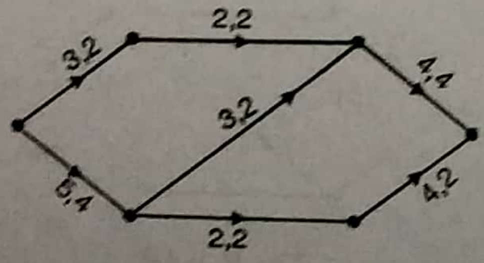


Fig. P. 6.4.4 (c)

All directed paths are saturated ∴ maximum flow occurred ∴ maximum flow is 4 + 2 = 6.

Example 6.4.5 : Define flow and Network Find the maximal flow of the following network by listing at least 6 S to D cuts.

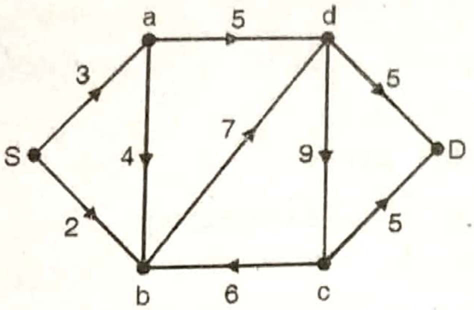


Fig. P. 6.4.5

Solution :

Network :

Connected directed weighted simple graph satisfying following properties is called network.

1. There is only one vertex with indegree zero, called source.
2. There is only one vertex with outdegree zero, called sink.
3. There is weight of each edge, called capacity of the edge.

Flow :

Let N is network and C_{ij} is capacity of directed edge $e_{ij} = (V_i, V_j)$. A flow F in N is allocation of real number f_{ij} to each directed edge e_{ij} such that.

1. $F_{ij} \leq C_{ij}$
2. For each vertex V_j which is neither source nor sink the equality $\sum\limits_i F_{ij} = \sum\limits_k F_{ik}$ holds.

i.e. for each vertex V_i sum of in weight equals to sum of out weight.

Maximal flow of the given network :

For given network source is S and sink is D. Assign zero flow for each edge then first unsaturated path from S to D is

$$P_1 : S \underset{3}{\xrightarrow{3,0}} a \underset{5}{\xrightarrow{5,0}} d \underset{5}{\xrightarrow{5,0}} D$$

$$\Delta_{ij} = \{3, 5, 5\}$$

$$\therefore \quad \Delta = \min \{3, 5, 5\} = 3$$

$\therefore$ Add $\Delta = 3$ in to flow of each edge from path P_1 new network becomes.

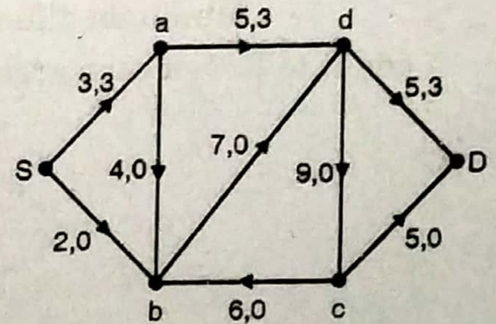


Fig. P. 6.4.5(a)

Next unsaturated path is

$$S \xrightarrow[2]{2,0} b \xrightarrow[7]{7,0} d \xrightarrow[2]{5,3} D$$

$$\Delta_{ij} = \{2, 7, 2\}$$

$$\Delta = \min\{2, 7, 2\} = 2$$

$\therefore$ Add $\Delta = 2$ in to flow of each edge from path P_2 new network becomes.

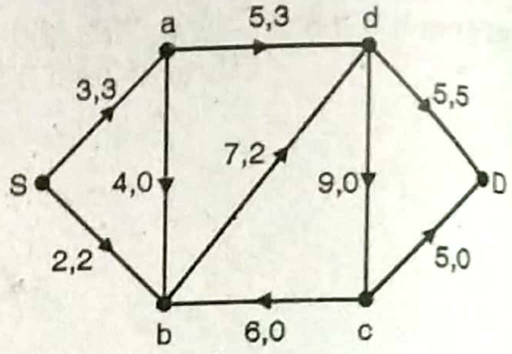


Fig. P. 6.4.5(b)

$\therefore$ There does not exist on saturated path from S – D hence max flow is occurred and it is **5 + 0 = 5.**

Example 6.4.6 : Using Ford-Fulkerson's Algorithm, find a maximal flow in the following network :

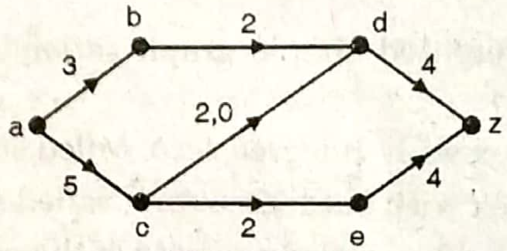


Fig. P. 6.4.6

Solution : Assign flow of each edge is zero then

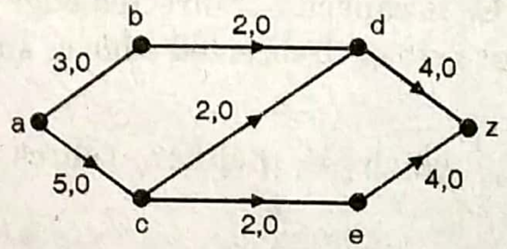


Fig. P. 6.4.6 (a)

First unbalanced path is

$$a \xrightarrow[3]{3,\,0} b \xrightarrow[2]{2,\,0} d \xrightarrow[4]{4,\,0} Z$$

$$\text{Minimum difference} = 2$$

$\therefore$ add it to flow of each edge in the path we get

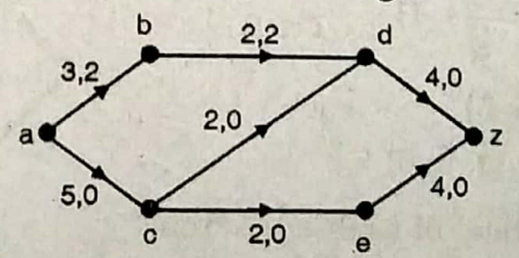


Fig. P. 6.4.6(b)

Next unbalanced path is, $a \xrightarrow[5]{5,\,0} c \xrightarrow[2]{2,\,0} d \xrightarrow[4]{4,\,0} Z.$

Minimum difference = 2

$\therefore$ add it to flow of each edge in the above path we get

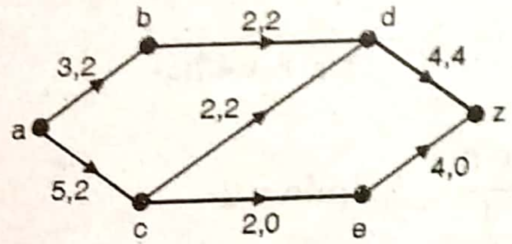


Fig. P. 6.4.6(c)

Next unbalanced path is, $a \xrightarrow[3]{5,\,2} c \xrightarrow[2]{2,\,0} e \xrightarrow[4]{4,\,0} Z.$

Minimum difference = 2

$\therefore$ add it to flow of each edge in the above path we get

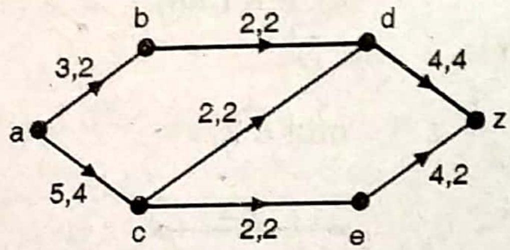


Fig. P. 6.4.6(d)

All the paths from a to z are satisfactory

$\therefore$ maximum flow is obtained and it is 4 + 2 = **6**

Example 6.4.7 : Determine maximal flow in the following network by using Ford-Fulkerson Algorithm :

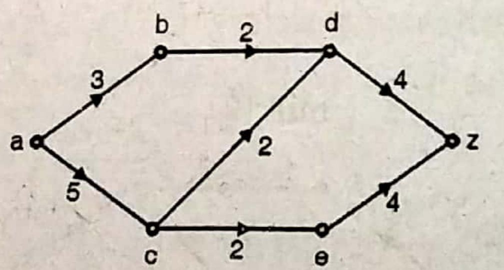


Fig. P. 6.4.7

Solution : Initially we assign 0 for each edge.

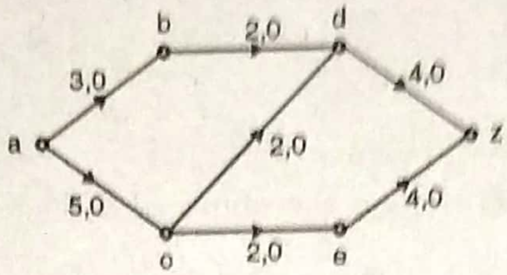


Fig. P. 6.4.7(a)

Unsaturated path is

$$a \xrightarrow[3]{3,\,0} b \xrightarrow[2]{2,\,0} d \xrightarrow[4]{4,\,0} z \qquad \min = 2$$

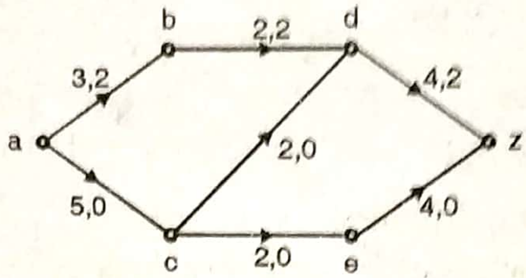


Fig. P. 6.4.7(b)

Next unsaturated path is.

$$a \xrightarrow[5]{5,\,0} c \xrightarrow[2]{2,\,0} d \xrightarrow[2]{4,\,2} z \qquad \min = 2$$

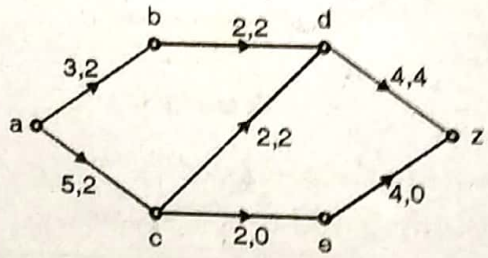


Fig. P. 6.4.7(c)

Next unsaturated path is.

$$a \xrightarrow[3]{5,\,2} c \xrightarrow[2]{2,\,0} e \xrightarrow[4]{4,\,0} z \qquad \min (2)$$

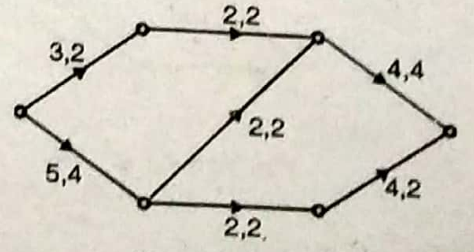


Fig. P. 6.4.7(d)

All paths are saturated

$\therefore$ Maximum flow is occurred

and maximum flow is 4 + 2 = 6

A.1.2 Trail

A walk in which no edge is repeated is called a trail.

Note : If terminal vertices in a trail are same it is called a closed trail, otherwise it is called an open trail.

e.g. Consider,

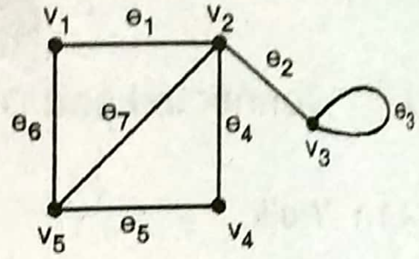


Fig. A.1.2

Now,

1. $v_3\, e_3\, v_3\, e_2\, v_2\, e_7\, v_5\, e_5\, v_4\, e_4\, v_2\, e_1\, v_1$

   here edges are not repeated

   therefore it is trail. ($\therefore$ it is trail)

2. $v_2\, e_2\, v_3\, e_3\, v_3\, e_2\, v_2$: not a trail ($\because$ e_2 is repeated twice)

A.1.3 Path

An open walk in which no vertex is repeated more than once is called a path.

Note :

1. Since vertex does not repeat, edge cannot repeat more than once.

   $\therefore$ 'every path is trail'.

2. Loop cannot be included in a path.
3. The number of edges in a path is called the length of the path.
4. Any two paths with the same number of vertices are isomorphic.

Ex. A.1.1 : Find all u – v path in the following graph G.

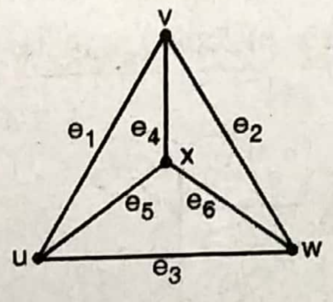


Fig. P. A.1.1

Soln. : Following are paths from u to v.

P_1 : u e_1 v　　　P_2 : u e_5 x e_4 v

P_3 : u e_3 w e_2 v　　　P_4 : u e_5 x e_6 w e_2 v

P_5 : u e_3 w e_6 x e_4 v

A.1.4 Cycle (Circuit)

A closed walk in which no vertex (except initial and final) appears more than once is called a cycle or circuit.

Note :

1. Self loop is a circuit.
2. A regular graph of degree 2 is a circuit.

e.g. Consider,

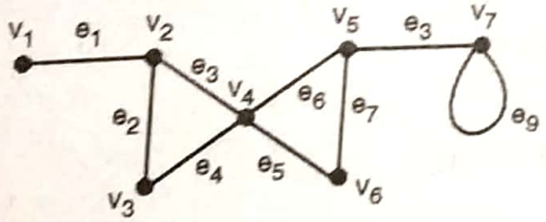


Fig. A.1.3

here,

1.	$v_1 e_1 v_2 e_2 v_3 e_4 v_4$	It is path	length = 3
2.	$v_7 v_5 v_6 v_4 v_2 v_3$	It is path	length = 5
3.	$v_5 v_4 v_2 v_3 v_4 v_6$	Not a path	
4.	$v_2 e_3 v_4 e_4 v_3 e_2 v_2$	It is circuit	length = 3
5.	$v_7 e_9 v_7$	It is circuit	length = 1
6.	$v_2 e_3 v_4 e_6 v_5 e_7 v_6 e_5 v_4 e_4 v_3 e_2 v_2$	Neither path nor circuit.	
7.	$v_1 e_1 v_2 e_2 v_3 e_4 v_4 e_5 v_6 e_7 v_5 e_8 v_7$	It is longest path from v_1 to v_7	length = 6

A.1.5 Distance between Pair of Vertices

Length of shortest path between pair of vertices is called as distance between them.

Notation : distance between two vertices u and v is d (u, v).

Ex. A.1.4 : For the following graph

Find :

1. Any two paths from u_1 to v_5.
2. Any two walks from u_1 to u_5.
3. Any two cycles (circuits) containing u_3.
4. A Cycle of length 4.
5. A Cycle of length 5.
6. Path of length 6 from u_1 to u_4.
7. Distance between u_1 to all other vertices.
8. Vertex of distance 3 from u_4.

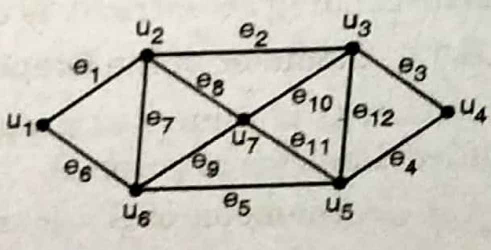


Fig. P. A.1.4

Soln. :

1. Two paths from u_1 to u_5.

   $P_1 : u_1 e_6 u_6 e_5 u_5$

   $P_2 : u_1 e_1 u_2 e_7 u_6 e_9 u_7 e_{10} u_3 e_3 u_4 e_4 u_5$

2. Two walks from u_1 to u_5

$W_1 : u_1\ e_6\ u_6\ e_9\ u_7\ e_{10}\ u_3\ e_2\ u_2\ e_8\ u_7\ e_{11}\ u_5$

$W_2 : u_1\ e_1\ u_2\ e_8\ u_7\ e_{11}\ u_5$

3. Two cycles containing u_3

   $C_1 : u_3\ e_2\ u_2\ e_8\ u_7\ e_{10}\ u_3$

   $C_2 : u_7\ e_{11}\ u_5\ e_4\ u_4\ e_3\ u_3\ e_{10}\ u_7$

4. Cycle of length 4

   $C_1 : u_2\ e_7\ u_6\ e_5\ u_5\ e_{12}\ u_3\ e_2\ u_2$

5. Cycle of length 5

   $C_1 : u_2\ e_1\ u_1\ e_6\ u_6\ e_9\ u_7\ e_{10}\ u_3\ e_2\ u_2$

6. Path of length 6 from u_1 to u_4

   $P : u_1\ e_1\ u_2\ e_7\ u_6\ e_9\ u_7\ e_{11}\ u_5\ e_{12}\ u_3\ e_3\ u_4$

7. Distance between u_1 to all other, vertices is distance of shortest path between u_1 and any other vertex.

   $\therefore\ d(u_1 u_2) = 1 \qquad d(u_1 u_3) = 2$

   $d(u_1 u_4) = 3 \qquad d(u_1 u_5) = 2$

   $d(u_1 u_6) = 1$

8. Only u_1 is the vertex at distance 4 from u_4.

A.1.6 Eccentricity of Vertex

Eccentricity of vertex is denoted by e(u) for vertex u.

It is defined as $e(u) = \max \{d(u, v) / v \in V, v \neq u\}$

A.1.7 Centre and Radius of Graph

The vertex with minimum eccentricity is called centre of graph. Corresponding eccentricity is called radius of graph.

A.1.8 Diameter of the Graph

Let G is connected graph. Distance between farthest vertices of graph G is called diameter of graph G.

i.e. Diameter of G = max {d(u, v) / u and v are vertices of G}

e.g. consider,

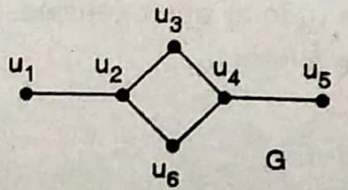


Fig. A.1.4

(i) Find distance between u_1 and u_4

$d(u_1, u_4) = 3$

(ii) Find eccentricity of u_1,

$d(u_1, u_2) = 1$ $d(u_1, u_3) = 2$ $d(u_1, u_4) = 3$ $d(u_1, u_5) = 4$

$d(u_1, u_6) = 2$

$\therefore e(u_1) = 4$

Similarly $e(u_2) = 3$, $e(u_3) = 2$, $e(u_4) = 3$, $e(u_5) = 4$, $e(u_6) = 2$

(iii) Find centre and radius of graph

Minimum eccentricity is 2

$\therefore$ Radius is 2.

Vertex with eccentricity 2 are v_3, v_6

$\therefore$ Centres are v_3, v_6

(iv) Find diameter of graph.

Maximum eccentricity is 4.

$\therefore$ Diameter is 4.

Ex. A.1.5 : In the following graph G find :

a) All paths from v_1 to v_3

b) Distance between v_1 to v_3

c) Eccentricity of vertex v_4

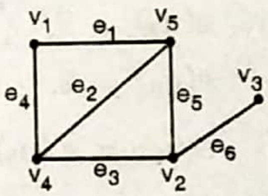


Fig. P. A.1.5

Soln. :

a) All paths from v_1 to v_3

$P_1 : v_1\ e_1\ v_5\ e_5\ v_2\ e_6\ v_3$

$P_2 : v_1\ e_1\ v_5\ e_2\ v_4\ e_3\ v_2\ e_6\ v_3$

$P_3 : v_1\ e_4\ v_4\ e_3\ v_2\ e_6\ v_3$

$P_4 : v_1\ e_4\ v_4\ e_2\ v_5\ e_5\ v_2\ e_6\ v_3$

b) Distance between v_1 to v_3 is length of shortest path between them and it is and it is 3.

i.e. Distance between v_1 to v_3 is 3.

c) Eccentricity of vertex v_4 : Eccentricity is the length of farthest vertex.

From v_4 farthest vertex is v_3. It is of length 2.

$\therefore$ Eccentricity of v_4 is 2.

Ex. A.1.6 : Find eccentricities of all vertices of the graph shown in Fig. P. A.1.6 :
Hence find centre of the given graph.

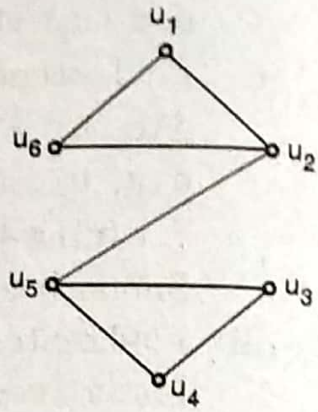


Fig. P. A.1.6

Soln. : Eccentricities of all vertices of given graph are :

$$e(u_1) = \max\{d(u_1, u_2), d(u_1, u_6), d(u_1, u_3), d(u_1, u_5), d(u_1, u_4)\}$$
$$= \max\{1, 1, 3, 2, 3\}$$
$$\therefore \quad e(u_1) = 3$$

Similarly, $e(u_2) = 2$, $e(u_3) = 3$, $e(u_4) = 3$,
$e(u_5) = 2$, $e(\mu_6) = 3$

$\because \quad e(u_2) = e(u_5) = 2$ (Minimum eccentricity)

$\therefore$ u_2 and u_5 are centres of the given graph.

Ex. A.1.7 : Find eccentricities of all vertices of the following graph and hence find its centre and radius.

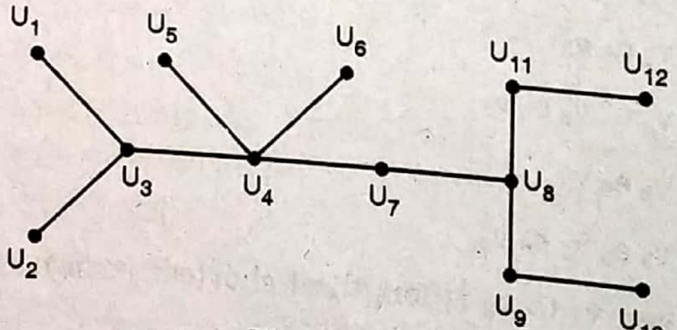


Fig. P. A.1.7

Soln. :

$E(v_1) = 6$ $\quad E(v_7) = 3$
$E(v_2) = 6$ $\quad E(v_8) = 4$
$E(v_3) = 5$ $\quad E(v_9) = 5$
$E(v_4) = 4$ $\quad E(v_{10}) = 6$
$E(v_5) = 5$ $\quad E(v_{11}) = 5$

$E(v_6) = 5$ $E(v_{12}) = 6$

Radius = 3, Centre = u_7 Diameter = 6

Ex. A.1.8 : Find centre, radius and diameter of the following tree :

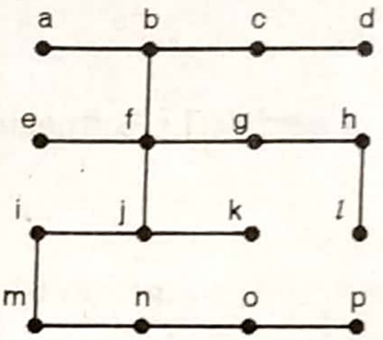


Fig. P. A.1.8

Soln. : First to find eccentricities of all vertices

E(a) = 8	E(I) = 5
E(b) = 7	E(j) = 5
E(c) = 8	E(k) = 5
E(d) = 9	E(l) = 9
E(e) = 7	E(m) = 6
E(f) = 6	E(n) = 7
E(g) = 7	E(o) = 8
E(h) = 8	E(p) = 9

Radius = 5

Centre = i , j ,k

Diameter = 9

A.2 Connectedness

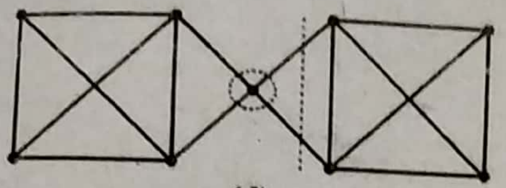
A.2.1 Connected Graph and Disconnected Graph

Definition

A graph G is said to be connected if there is at least one path between every pair of vertices in G, otherwise G is said to be disconnected graph.

Thus a graph G is disconnected if there is a pair of vertices in G such that there is no path between them.

4. $k(G) = 1, \lambda(G) = 2, \delta(G) = 3$



(d)

Fig. P. A.3.5

$k(G) = 1$

$\lambda(G) = 2$

$\delta(G) = 3$

Ex. A.3.6 : Construct a simple graph G, such that $\lambda(G)$ 1 = K(G).

Soln. :

This graph may disconnect by deletion of single vertex b.

$\therefore$ vertex connectivity $K(G) = 1$

Similarly graph can disconnect by deletion of single edge e_1

$\therefore$ edge connectivity $\lambda(G) = 1$.

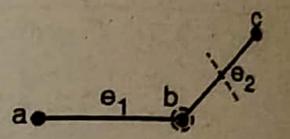


Fig. P. A.3.6

Ex. A.3.7 : Give example of a connected graph G such that $k(G) < \lambda(G) < \delta(G)$.

Soln. :

We construct graph of

$k(G) = 1, \lambda(G) = 2, \delta(G) = 3$

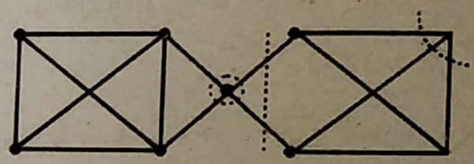


Fig. P.A.3.7

Ex. A.3.8 : With usual notation construct a simple graph G. Such that $k(G) = \lambda(G) < \delta(G)$

Soln. : We construct graph G with

$k(G) = 1, \lambda(G) = 1$ and $\delta(G) = 2$

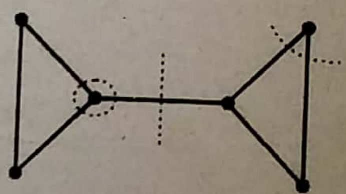


Fig. P. A.3.8

A.4 Weighted Graph :

Define Weight graph.

Definition :

Weighted graph is a graph in which each edge of a graph is assigned a positive real number.

Positive real number is called as weight of edge 'e'.

e.g.

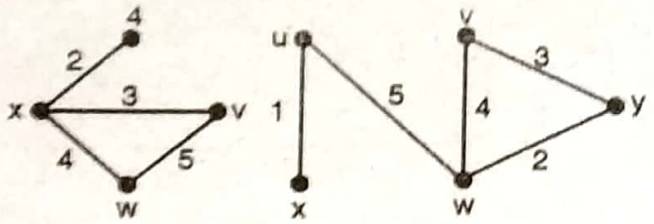


Fig. A.4.1

* Weight of path is sum of weights of all edges in the path.
* Weight of minimum weighted path is called as shortest path.

A.4.1 Dijkstra's Algorithm :

Dijkstra's algorithm is used to find shortest path from a vertex to all other vertices of connected graph.

Steps of Dijkstra's algorithm :

Step 1 : Consider given connected weighted graph. G.V is the set of vertices of G. s and t are any two vertices of G.

Step 2 : Initially $\lambda(s) = 0$ and $\lambda(v) = \infty$. P_v denotes shortest path from s to v. Assign T = V.

Step 3 : Select vertex u in T for which $\lambda(u)$ is minimum.

Step 4 : If u = t then STOP. And P_t is desired path.

Step 5 : For every vertex $V \in T$ which is adjacent to u, Consider edge $e = \{u, v\}$

If $\lambda(u) + W(e) < \lambda(v)$ then

$$\lambda(v) = \lambda(u) + W(e) \text{ and } P_v = P_u \cup \{e\}$$

Step 6 : Modify T by T = T (u) and goto step 2.

Ex. A.4.1 : Use Dijkstra's algorithm to obtain shortest path from vertex 'a' to all the remaining vertices.

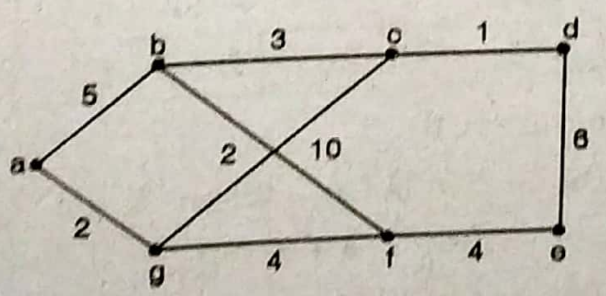


Fig. P. A.4.1

Soln. :

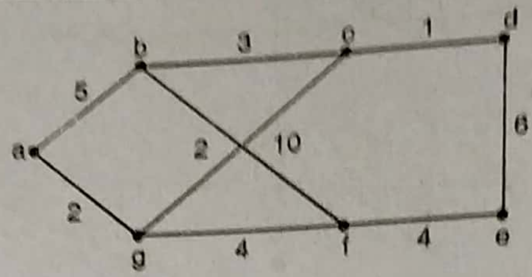


Fig. P. A.4.1(a)

Initially λ (a) = 0

Vertex	a	B	c	D	e	f	g
λ (v)	0	∞	∞	∞	∞	∞	∞
T	a	B	c	D	e	f	g

Vertices adjacent to a are b. g

$$\lambda(b) = \min\{\lambda(b), \lambda(a) + \omega(a, b)\}$$
$$= \min\{\infty, 0 + 5\} = 5$$
$$\lambda(g) = \min\{\lambda(g), \lambda(a) + \omega(a, g)\}$$
$$= \min\{\infty, 0 + 2\} = 2$$

Vertex	a	b	c	D	e	f	g
λ (v)	0	5	∞	∞	∞	∞	2
T	–	b	c	D	e	f	g

Vertices adjacent to g are c, f

$$\lambda(c) = \min\{\lambda(c), \lambda(g) + \omega(g, c)\}$$
$$= \min\{\infty, 2 + 2\} = 4$$
$$\lambda(f) = \min\{\lambda(f), \lambda(g) + \omega(g, f)\}$$
$$= \min\{\infty, 2 + 4\} = 6$$

Vertex	a	b	c	D	e	f	g
λ (v)	0	5	4	∞	∞	6	2
T	–	b	c	D	e	f	–

Vertices adjacent to c are b, d

$$\lambda(b) = \min\{\lambda(b), \lambda(c) + \omega(c, b)\}$$
$$= \min\{5, 4 + 3\} = 5$$
$$\lambda(d) = \min\{\lambda(d), \lambda(c) + \omega(c, d)\}$$
$$= \min\{\infty, 4 + 1\} = 5$$

Vertex	a	b	c	D	e	f	g
λ (v)	0	5	4	5	∞	6	2
T	–	b	–	D	e	f	–

Vertices adjacent to d is e

$$\lambda(e) = \min\{\lambda(e), \lambda(d) + \omega(d, e)\}$$
$$= \min\{\infty, 5 + 6\} = 11$$

Vertex	a	b	c	D	e	f	g
λ (v)	0	5	4	5	11	6	2
T	–	b	–	–	e	f	–

Vertices adjacent to b is f

$$\lambda(f) = \min\{\lambda(f)g, \lambda(b) + \omega(b, f)\}$$
$$= \min\{6, 5 + 10\} = 6$$

Vertex	a	b	c	D	e	f	g
λ (v)	0	5	4	5	11	6	2
T	–	–	–	–	e	f	–

Vertices adjacent to f is e

$$\lambda(e) = \min\{\lambda(e), \lambda(f) + \omega(f, e)\} = \min\{11, 6 + 4\} = 10$$

Vertex	a	b	c	d	e	f	G
λ (v)	0	5	4	5	10	6	2
T	–	–	–	–	e	–	–

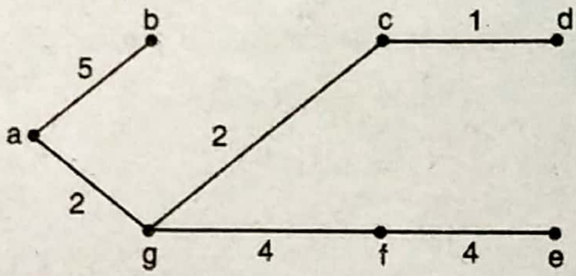


Fig. P. A. 4.1(b)

Ex. A.4.2 : Prove that a graph with n vertices and vertex connectivity k has at least $\frac{k \times n}{2}$ edges.

Soln. : Let G is connected graph with n vertices and e edges.

$$\therefore \; k(G) \le \lambda(G) \le \frac{2 \times e}{n}$$

$$\therefore \; k \le \frac{2e}{n} \qquad \therefore e \ge \frac{k \times n}{2}$$

i.e. Graph with n vertices and vertex connectivity k has at least $\frac{k \times n}{2}$ edges.

Ex. A.4.5 : Use Dijkstra's algorithm to obtain shortest path from a to f in the following graph :

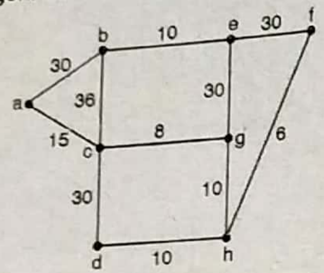


Fig. P. A.4.5

Soln. :

Initially λ (a) = 0

Vertex	a	b	c	d	e	f	g	h
λ (V)		∞	∞	∞	∞	∞	∞	∞
T	a	b	c	d	e	f	g	h

Adjacent vertices of a are b and c

$$\lambda(b) = \min\{\lambda(b), \lambda(a) + w(a, b)\}$$
$$= \min\{\infty, 0 + 30\}$$
$$= 30$$
$$\lambda(c) = \min\{\lambda(c), \lambda(a) + w(a, c)\}$$
$$= \min\{\infty, 0 + 15\}$$
$$= 15$$

Vertex	a	b	c	D	e	f	g	h
λ (V)	0	30	15	∞	∞	∞	∞	∞
T	–	b	c	D	e	f	g	h

Minimum λ value vertex is 'c' adjacent vertices of c are b, d, g

$$\lambda(b) = \min\{30, 15 + 30\} = 30$$
$$\lambda(d) = \min\{\infty, 15 + 30\} = 45$$
$$\lambda(g) = \min\{\infty, 15 + 8\} = 23$$

Vertex	a	b	c	D	e	f	g	h
λ (V)	0	30	15	45	∞	∞	23	∞
T	–	b	–	D	e	f	g	h

From set T, vertex with minimum λ value is 'g' adjacent vertices of g from set T are e, h.

$$\lambda(e) = \min\{\infty, 23 + 30\} = 53$$
$$\lambda(h) = \min\{\infty, 23 + 10\} = 33$$

Vertex	a	b	c	d	e	f	g	h
λ (V)	0	30	15	45	53	∞	23	33
T	–	b	–	d	e	f	–	h

Adjacent of b is e

$$\lambda(e) = \min\{53, 30 + 10\} = 40$$

Vertex	a	b	c	d	e	f	g	h
λ (V)	0	30	15	45	40	∞	23	33
T	–	–	–	d	e	f	–	h

Adjacent of h are d and f.

$$\lambda(d) = \min\{45, 33 + 10\} = 43$$
$$\lambda(f) = \min\{\infty, 33 + 6\} = 39$$

Vertex	a	b	c	d	e	f	g	h
λ (V)	0	30	15	43	40	39	23	33
T	–	–	–	d	e	f	–	–

Adjacent of is e

$$\lambda(e) = \min\{40, 39 + 30\} = 40$$

Vertex	a	b	c	d	e	f	g	h
λ (V)	0	30	15	43	40	39	23	33
T	–	–	–	d	e	–	–	–

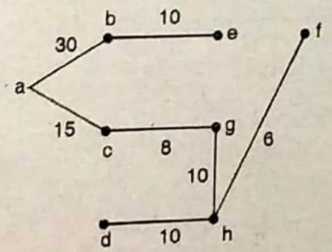


Fig. P. A.4.5(a)

Ex. A.4.6 : By using Dijkstra's algorithm, find the shortest path from vertex 'a' to all vertices of the graph given below.

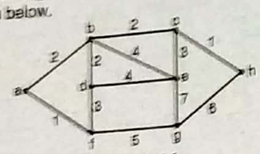


Fig. P. A.4.6

Soln. : Initially λ (a) = 0

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	∞	∞	∞	∞	∞	∞	∞
T	a	b	c	d	e	f	g	h

Vertices b and f are adjacent to a

$\therefore$ λ (b) = min {∞, 2} = 2

λ (f) = min {∞, 1} = 1

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	∞	∞	∞	1	∞	∞
T	–	b	c	d	e	f	g	h

λ (f) = 1 is minimum

Vertex d and g are adjacent to f

$\therefore$ λ (d) = min {∞, 4} = 4

λ (g) = min {∞, 6} = 6

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	∞	4	∞	1	6	∞
T	–	b	c	d	e	–	g	h

λ (b) = 2 is minimum

The vertex in T adjacent to b are c, d, e.

$\therefore$ λ (c) = min {∞, 4} = 4

λ (d) = min {4, 4} = 4

λ (e) = min {∞, 6} = 6

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	∞
T	–	–	c	d	e	–	g	h

λ (c) = 4 is minimum

The vertex in T adjacent to be are c are e, h

λ (e) = min {6, 7} = 6

λ (h) = min {∞, 5} = 5

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	5
T	–	–	–	d	e	–	g	h

λ (d) = 4 is minimum

The vertices in T adjacent to d is e only

λ (e) = min {6, 8} = 6

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	5
T	–	–	–	–	e	–	g	h

λ (h) = 5 is minimum

The vertices in T adjacent to h is g only

λ (g) = min {6, 11} = 6

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	5
T	–	–	–	–	e	–	g	–

λ (e) = λ (g) = 6

Suppose λ (e) = 6 is minimum and g $\in$ T is adjacent to e

$\therefore$ λ (g) = min {6, 13} = 6

Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	5
T	–	–	–	–	–	–	g	–

Finally

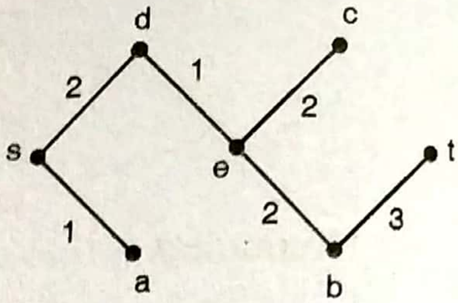
Vertex	a	b	c	d	e	f	g	h
λ (v)	0	2	4	4	6	1	6	5
T	–	–	–	–	–	–	–	–

T	–	–	–	t	b	–	–

Vertices adjacent to b is t

$\lambda(t) = \min\{10, 5 + 3\} = 8$

Vertex	s	d	c	t	b	a	E
λ (v)	0	2	5	8	5	1	3
T	–	–	–	t	–	–	–



B. Trees

B.1 Introduction :

Special type of graph like tree is tree graph. It has various applications in

(1) Technology.

(2) Classifications and distribution study.

(3) Managerial problems etc.

B.2 Tree :

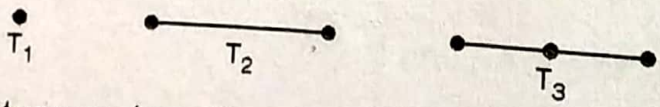
Definition :

A connected graph without any cycle is called a tree.

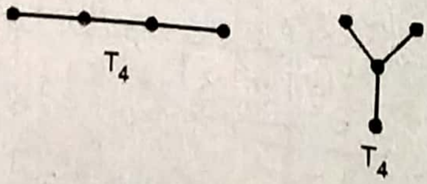
Notation : Tree graph on n vertices is denoted by T_n.

e.g.

(1) Trees on 1, 2 and 3 vertices

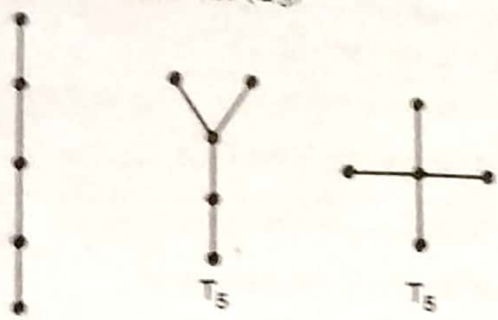


(2) Non isomorphic tree on 4 vertices (T_4)



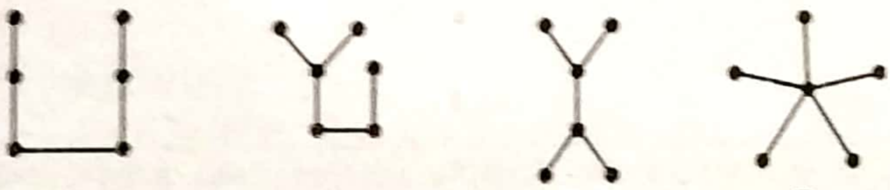
Scanned by CamScanner

(3) Non isomorphic tree on 5 vertices. (T_5)



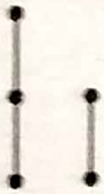
T_5 T_5 T_5

(4) Non isomorphic tree 6 vertices (T_6)

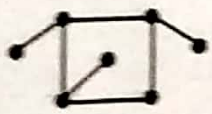


T_6 T_6 T_6 T_6

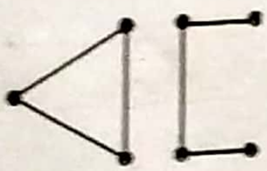
Following are not trees.



Disconnected hence not tree.



Connected but contains cycle hence not tree.



Disconnected as well as contains cycle $\therefore$ not tree.

Fig. B.2.1

B.2.1 Properties of Tree Graph :

(1) Tree is simple graph, not contains loops as well as parallel edges.

(2) Each edge of tree is isthmus.

(3) Tree on n vertices has (n – 1) edges.

(4) There exist unique path between any pair of vertices.

(5) Joining any two non-adjacent vertices of a tree by an edge then there exist only one circuit in resulting graph.

(6) Other than pendent vertices, each vertex is cut vertex.

(7) Every tree contains at least one pendent vertex.

Ex. B.2.1 : Draw all non isomorphic trees on 6 vertices.

Soln. : All non isomorphic trees on 6 vertices are

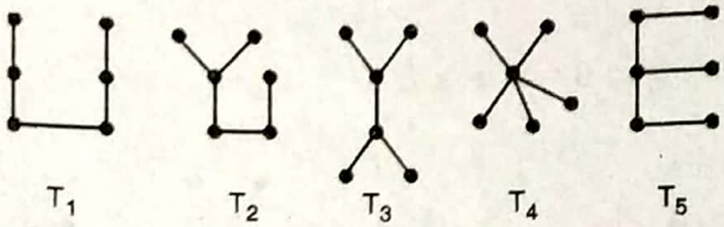


Fig. P. B.2.1

Ex. B.2.2 : Draw all possible non-isomorphic spanning trees of the following graph.

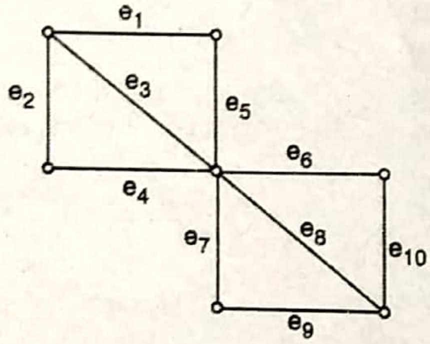


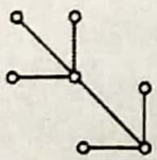
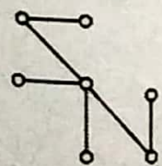
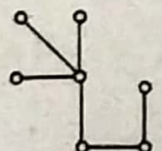
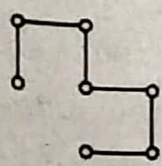
Fig. P. B.2.2

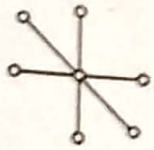
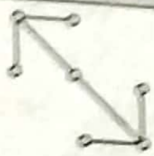
Soln. :

Here, $n = 7$, $e = 10$

Each spanning tree of G contains $7 - 1 = 6$ edges.

$\therefore$ Out of 10 edges delete 4 edges in such way that the remaining graph is tree on 7 vertices.

Deletion of 4 edges	Spanning tree	Deletion of 4 edges	Spanning tree
e_1, e_2, e_6, e_7		e_2, e_5, e_6, e_9	
e_1, e_2, e_6, e_8		e_3, e_4, e_7, e_8	

Deletion of 4 edges	Spanning tree	Deletion of 4 edges	Spanning tree
e_1, e_2, e_9, e_{10}		e_4, e_5, e_6, e_7	

B.3 Centre of a Tree :

B.3.1 Eccentricity of Vertex :

Definition :

Let T is tree. v is vertex in T. Distance of farthest vertex from v is called eccentricity of v in T.

Notation : Eccentricity of v = E(v).

i.e. $E(v) = \text{Max}\{d(u,v)/u \text{ is vertex in } T\}$

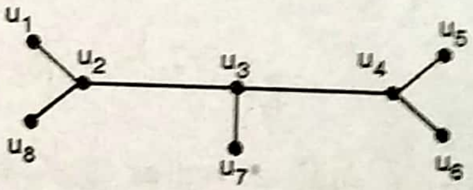
e.g.(1) : Eccentricity of each vertex in T_1 is given below.

$E(u_1) = 4,$ $E(u_2) = 3$

$E(u_3) = 2,$ $E(u_4) = 3$

$E(u_5) = 4,$ $E(u_6) = 4$

$E(u_7) = 3,$ $E(u_8) = 4$



T_1

e.g.(2) : Eccentricities of all vertices in T_2 are as follows :

$E(v_1) = 4,$ $E(V_7) = 4,$

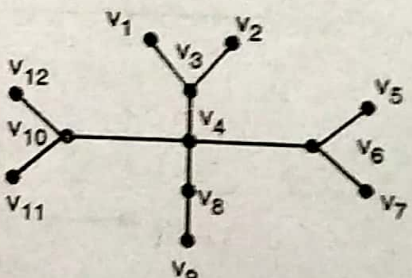
$E(V_2) = 4,$ $E(V_8) = 3,$

$E(V_3) = 3,$ $E(V_9) = 4,$

$E(V_4) = 2,$ $E(V_{10}) = 3,$

$E(V_5) = 4,$ $E(V_{11}) = 4,$

$E(V_6) = 3,$ $E(V_{12}) = 4.$



T_2

Ex. B.3.1 : Find eccentricity of each vertex in the following tree, hence find centre of tree.

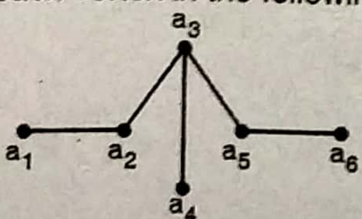


Fig. P. B.3.1

Ex. B.3.2 : For given graph T. Find centre of T.

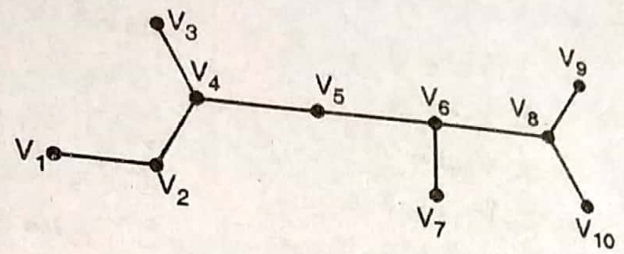


Fig. P. B.3.2

Soln. :

$E(v_1) = 6$ $\quad$ $E(v_6) = 4$

$E(v_2) = 5$ $\quad$ $E(v_7) = 5$

$E(v_3) = 5$ $\quad$ $E(v_8) = 5$

$E(v_4) = 4$ $\quad$ $E(v_9) = 6$

$E(v_5) = 3$ $\quad$ $E(v_{10}) = 6$

Here vertex of minimum eccentricity is v_5

$\therefore$ v_5 is centre of the given tree.

B.3.3 Radius of a Tree :

Definition :

The eccentricity of the centre in a tree is called radius of tree T.

Notation : R(T) ≡ Radius of tree.

B.3.4 Diameter of a Tree :

Definition :

Diameter of a tree T is defined as the length of the longest path in T. OR

Let T is tree · V is vertex of maximum eccentricity. Eccentricity of vertex v is called as diameter of tree.

Notation : Diameter of tree T ≡ D(T).

i.e. $D(T) = \max \{E(v) / E(v) \text{ is eccentricity of vertex } v\}$.

Ex. B.3.3 : Find centre, radius and diameter of the following tree.

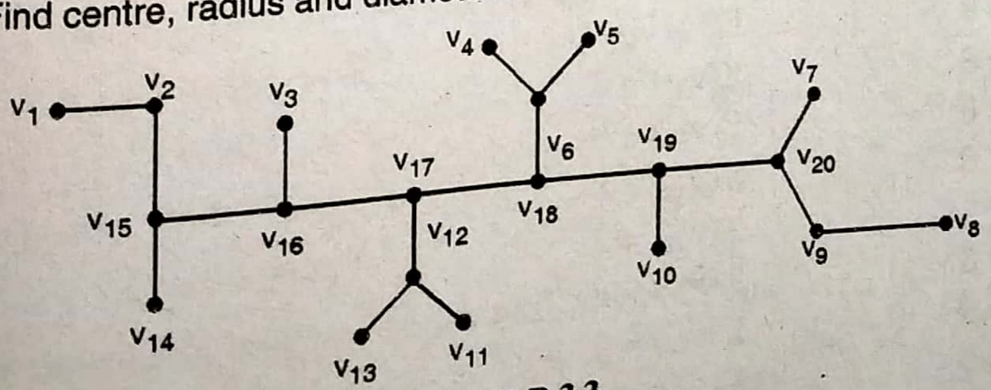


Fig. P.B.3.3

Soln. :

$E(V_1) = 9$
$E(V_2) = 8$
$E(V_3) = 7$
$E(V_4) = 7$
$E(V_5) = 7$
$E(V_6) = 6$
$E(V_7) = 8$
$E(V_8) = 9$
$E(V_9) = 8$
$E(V_{10}) = 7$
$E(V_{11}) = 7$
$E(V_{12}) = 8$
$E(V_{13}) = 7$
$E(V_{14}) = 8$
$E(V_{15}) = 7$
$E(V_{16}) = 6$
$E(V_{17}) = 5$
$E(V_{18}) = 5$
$E(V_{19}) = 6$
$E(V_{20}) = 7$

Two adjacent vertices v_{17} and v_{18} has minimum eccentricity

i.e. $E(v_{17}) = E(v_{18}) = 5$

$\therefore$ V_{17} and v_{18} are two adjacent vertices are centres of given tree.

i.e. C(T) are v_{17} and v_{18}

Radius of T : Centres v_{17} and v_{18} has eccentricity 5

$\therefore$ radius of given **tree** is 5.

$\therefore R(T) = 5$

Diameter of T : Maximum eccentricity from all vertices occures at v_1, v_8 and it is 9.

$\therefore$ Diameter of T is 9.

$\therefore D(T) = 9$

Ex. B.3.4 : Find centre, radius and diameter of the tree.

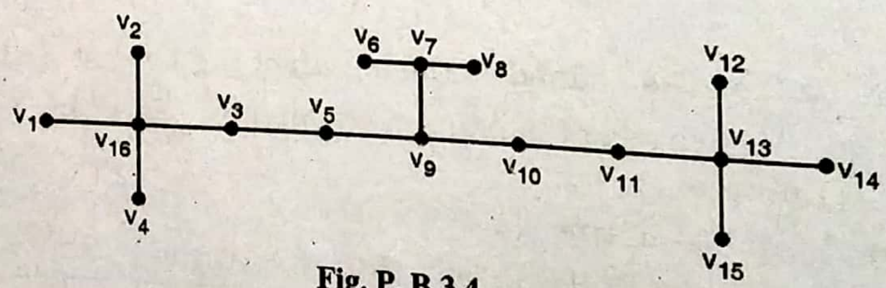


Fig. P. B.3.4

Soln. :

$E(v_1) = 8$
$E(v_2) = 8$
$E(v_3) = 6$
$E(v_4) = 8$
$E(v_5) = 5$
$E(v_6) = 6$
$E(v_7) = 5$
$E(v_8) = 6$
$E(v_9) = 4$
$E(v_{10}) = 5$
$E(v_{11}) = 6$
$E(v_{12}) = 8$
$E(v_{13}) = 7$
$E(v_{14}) = 8$
$E(v_{15}) = 8$
$E(v_{16}) = 7$

B.4 Spanning Tree :

B.4.1 Definition (Spanning Tree) :

Q. Define spanning tree of a graph.

Let G be a connected graph. A subgraph T of G is said to be a spanning tree if T is a tree and T passes through each vertex of G.

OR

Let G is connected graph. T is tree which is spanning subgraph of G. Then T is called as spanning tree of G.

- **Spanning subgraph** ⇒ Let G is graph. H is subgraph of G containing all vertices of G, then H is called as spanning subgraph of G.

B.4.1.1 How to Find Spanning Tree ?

Step 1 : First draw all vertices from original graph.

Step 2 : Select some or all edges from original graph such that those does not create circuit.

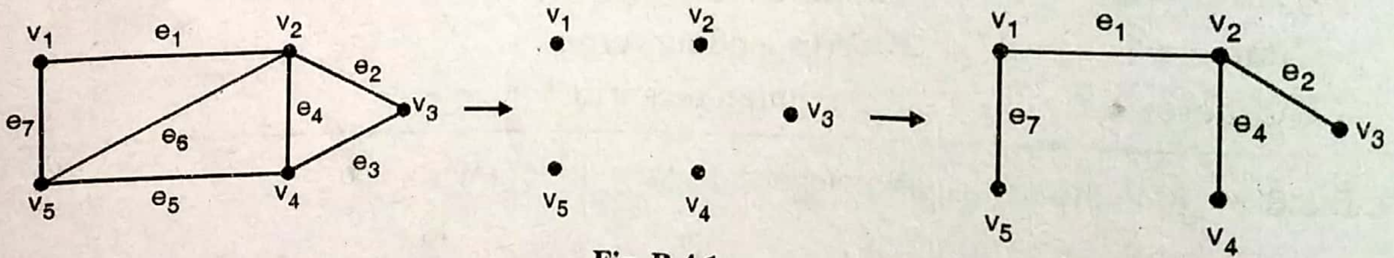


Fig. B.4.1

Step 3 : Number of edges must be n – 1. n number of vertices.

For better understanding and online tutorial click on www.mathkamal4all.webs.com

Ex. B.4.1 : Find all spanning trees of graph given below.

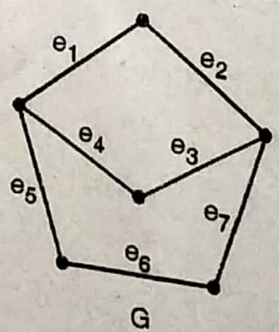


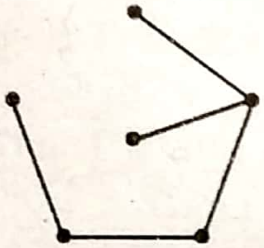
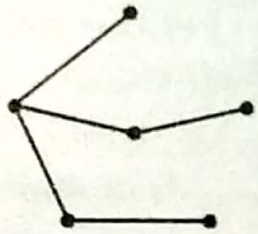
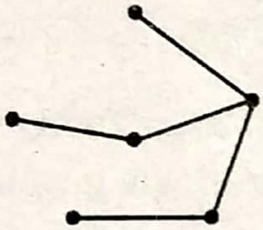
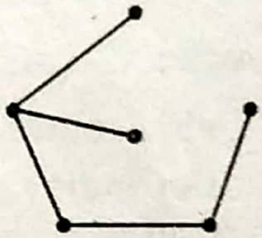
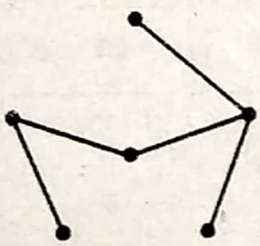
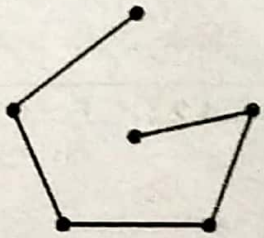
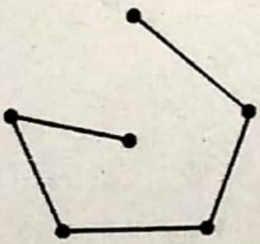
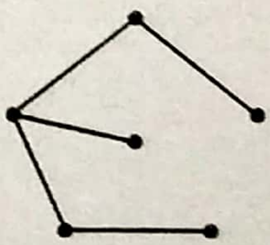
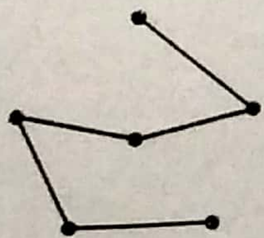
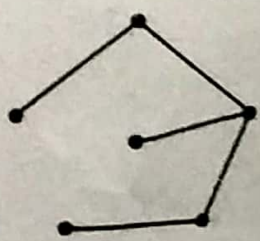
Fig. P. B.4.1

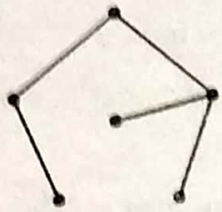
Soln. : Here number of vertices in G = 6

Number of edges in G = 7

Each spanning tree of G contains 6 – 1 = 5 edges.

Therefore out of 7 edges delete 2 edges in such a way that the remaining graph is tree on 6 vertices.

Deletion of 2 edges	Spanning tree	Deletion of two edges	Spanning tree
e_1, e_4		e_2, e_7	
e_1, e_5		e_2, e_3	
e_1, e_6		e_2, e_4	
e_1, e_3		e_4, e_7	
e_1, e_7		e_4, e_5	

Deletion of 2 edges	Spanning tree	Deletion of two edges	Spanning tree
e_4, e_6			

B.4.2 Definition (Branches of Spanning Tree) :

Let G is connected graph. T is spanning tree of G, then edges of G which are in T are called as branches of T in G.

i.e. edges of spanning tree are called branches of T in G.

B.4.3 Definition (Chords of Spanning Tree) :

Let G is connected graph. T is spanning tree of G, then edges of G which are not in T are called as chords of T in G.

Remark :

(1) For connected graph G there exists one or more than one spanning trees.

(2) Depending on various spanning trees for single graph G, set of branches as well as set of chords changes.

(3) Let G is connected graph with n vertices and m edges, then any spanning tree of G has $n - 1$ branches and $(m - (n - 1)) = m - n + 1$ chords.

Ex. B.4.2 : For following graph G, find any two spanning trees of G. List all branches and chords for corresponding spanning tree.

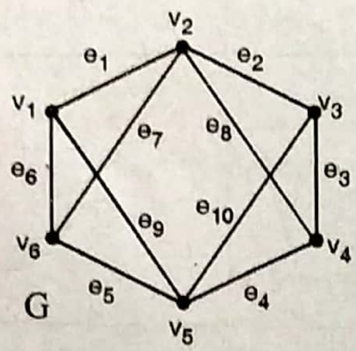


Fig. P.B.4.2

Ex. B.5.2 : Using Kruskal's algorithm find shortest spanning tree of following graph.

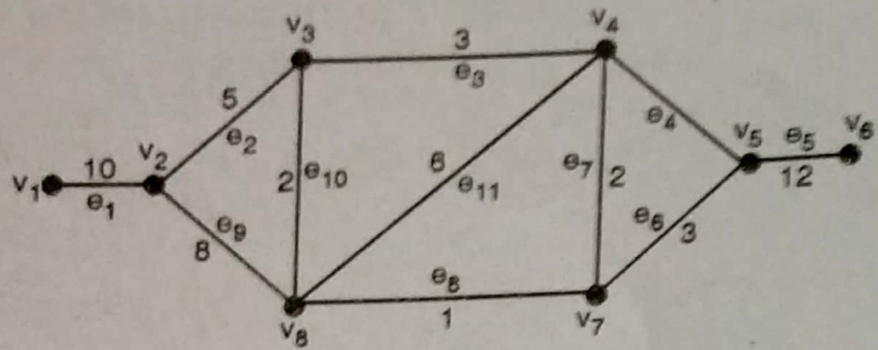


Fig. P. B.5.2

Soln. :

Plot all vertices of graph G.

Increasing sequence of weights of given graph is

1, 2, 2, 3, 3, 4, 5, 6, 8, 10, 12

Corresponding edge sequence of increasing weights of graph is

Vertex	e_8	e_7	e_{10}	e_3	e_6	e_4	e_2	e_{11}	e_9	e_1	e_5
Weight	1	2	2	3	3	4	5	6	8	10	12

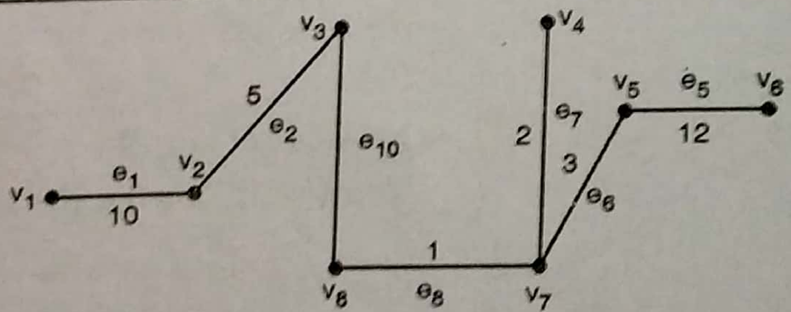


Fig. P. B.5.2(a)

Minimum weighted spanning tree. Trace edge e_8 of minimum weight 1.

Trace next edge e_7 of next minimum weight 2.

Trace next edge e_{10} of next minimum weight 2.

Next minimum weighted edge is e_3 of weight 3, but it forms circuit. Hence select next minimum weight edge e_6 of weight 3.

Next minimum weighted edge is e_4 of weight 4, but it forms circuit, avoid it.

Next minimum weighted edge e_2. It is of weight 5, trace it. Next minimum weighted edges e_{11} and e_9 forms circuit, avoid it.

Trace next edge e_1 of next minimum weight 10. Trace next edge e_5 of next minimum weight 12.

Graph becomes connected, without any cycle, contains all vertices of G i.e. spanning tree.

Weight of spanning tree is 1 + 2 +2 +3 + 5 + 10 + 12 = 35

Weight of minimum weighted spanning tree = 35

Ex. B.5.3 : Using Kruskal's Algorithm find the minimum weighted spanning tree in the following graph.

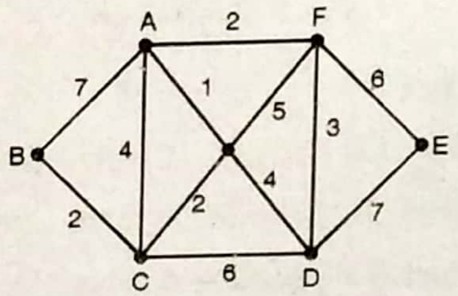


Fig. P. B.5.3

Soln. : Total number of vertices = 7

So we stop the process at 6 edges.

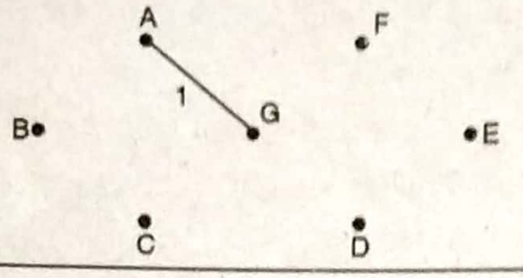
Collect all edges with weight

Weight	7	2	4	1	2	2	6	5	4	3	6	7
Edge	BA	BC	AC	AG	AF	CG	CD	GF	GD	FD	FE	DE

In non-decreasing weight

Weight	1	2	2	2	3	4	4	5	6	6	7	7
Edge	AG	AF	CG	BC	FD	AC	GD	GF	CD	FE	BA	DE

1st we select AG with minimum weight 3.

	Number of edges = 1 Total weight = 1

Select AF with minimum weight in remaining edges and does not form circuit.

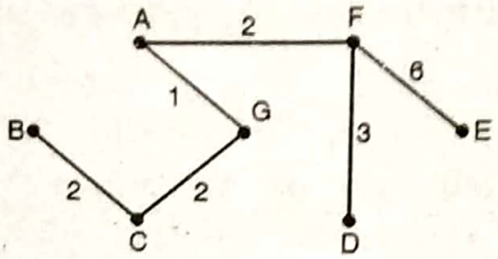


Fig. P. B.5.3(a)

Select CG, then BC. Next FD

After this we cannot select AC because it form circuit

After this we cannot select GD because it form circuit

After this we cannot select GF because it form circuit

After this we cannot select CD because it form circuit

Select FD. Total number of edges 6. We stop the process.

Total minimum weight 2 + 2 + 1 + 2 + 3 + 6 = **16.**

Ex. B.5.4 : Using Kruskal's algorithm find the shortest spanning tree in the following graph :

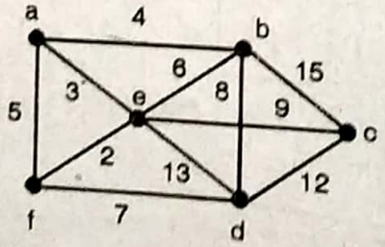


Fig. P. B.5.4

Soln. :

Total number of vertices = 6

We stop the process at 5. edges

Collect all edges with weight

Weight	5	3	4	2	7	6	9	13	15	8	12
Edge	af	ae	ab	fe	fd	eb	ec	ed	bc	bd	dc

In non-decreasing weight

Weight	2	3	4	5	6	7	8	9	12	13	15
Edge	fe	ae	ab	af	eb	fd	bd	ec	dc	ed	bc

1st we select fe

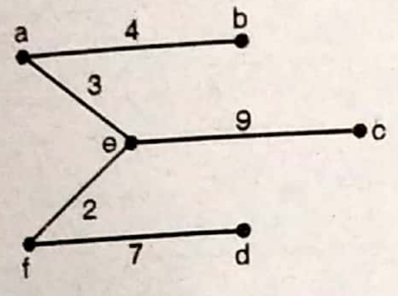


Fig. P.B.5.4(a)

Step	select	Do not select form circuit
1	fe	–
2	ae	–
3	ab	af
4	–	af
5	–	eb
6	fd	–
7	–	bd
8	ec	–

Number of edges 5 stop process

Total minimum weight = 4 + 3 + 9 + 2 + 7 = 25

Ex. B.5.5 : Using Kruskal's Algorithm, Find shortest spanning tree of the following graph. Also find weight of the shortest spanning tree.

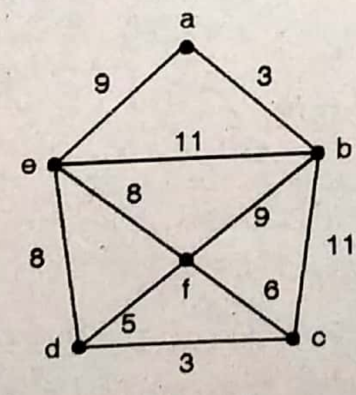


Fig. P. B.5.5

Soln. :

Total number of vertices = 6

We stop the process at 5 edges

Collect all edges with weight

$$\leq \left(\frac{5}{2}\right)^{m-1} \times \frac{5}{2} \leq \left(\frac{5}{2}\right)^{m}$$

$\therefore$ i.e. $a_m \leq \left(\frac{5}{2}\right)^m$

$\therefore$ Statement is true for $n = m$.

$\therefore$ By second principle of mathematical induction statement is true for all $n \geq 2$.

P.3 Solving problems on Eulerian and Hamiltonian graphs.

Example 1 : Give an example of graph which is Eulerian and Hamiltonian

Solution :

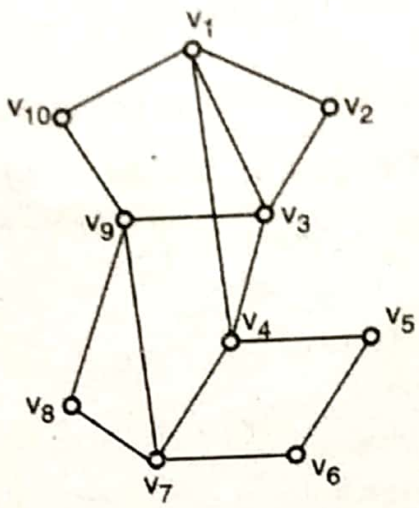


Fig. P.1 : Graph is Eulerian and Hamiltonian

Example 2 : Give an example of graph which is Hamiltonian but not Eulerian.

Solution :

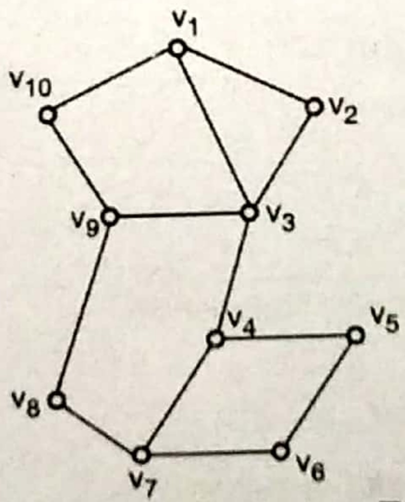


Fig. P.2 : Graph is Hamiltonian but not Eulerian

Example 3 : Give an example of a graph which is not Hamiltonian graph.

Solution :

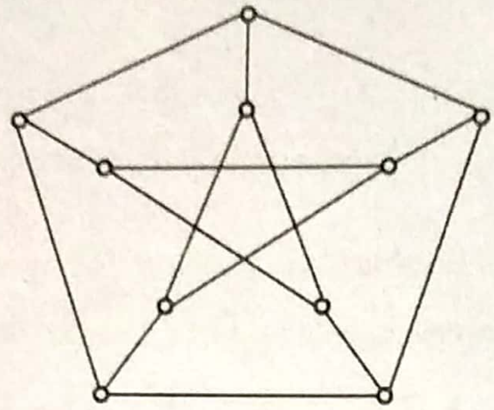


Fig. P.3 : Not Hamiltonian graph. The Petersen graph

P.4 Solving Problems on Chromatic Number and Coloring

Example 1 : What is Chromatic number of C_5

Solution :

$$\chi = (C_n) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases}$$

As n = 5 as n is odd

$$\chi(C_5) = 3$$

Example 2 : What is chromatic number of P_4.

Solution :

$$\chi(P_4) = \begin{cases} 2 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

As $n = 4$ is ≥ 2

$$\chi(P_4) = 2$$

Example 3 : What is Chromatic Number of $\chi(K_{3,2})$?

Solution :

Chromatic number of $K_{3,2}$ is 2

P.5 Solving problems Kruskal's Algorithm

Before going for this we have to study basic topic treefor this topic Please refer **Annexure (B. Trees).**

P.6 Solving problems using Prim's Algorithm

Example 1 : Find minimum spanning tree of following graph.

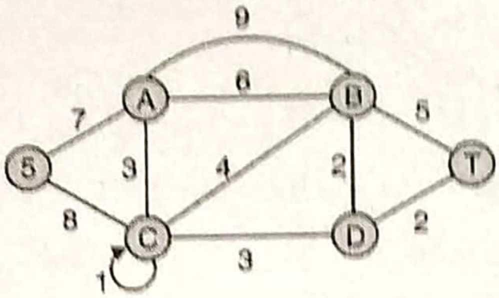


Fig. P.1

Solution :

Step I : Remove all loops and parallel edges.

From parallel edges delete edge with highest weight

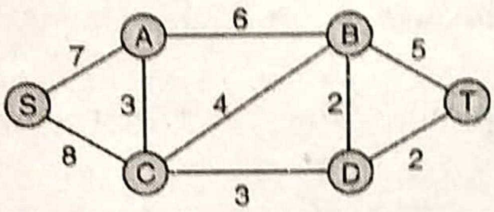


Fig. P. 1(a)

Step II : Choose any arbitrary node as root

As in a spanning tree all vertices are included we can consider any vertex as node we choose s.

Step III : Check outgoing edges from (5) and select one with lowest cost

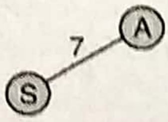


Fig. P. 1(b)

Check outgoing edges from S – 7 – A

Outgoing edges are SC, AC, AB out of this minimum cost is 3 of AC

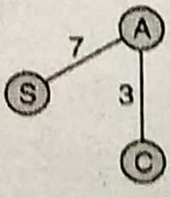


Fig. P. 1(c)

Check outgoing edges from S – 7 – A – 3 – C

Here we don't consider CS become it comple cycle.

Remaining edges are AB, CB, CD out off this minimum cost is 3 of CD.

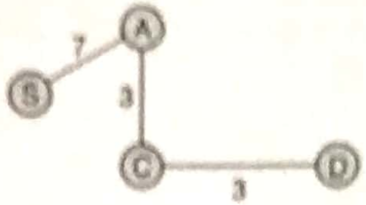


Fig. P. 1(d)

Check out going edges from S – 7 – A – 3 – C – 3 – D

Outgoing edges AB, CB, DB, DT

Out off this minimum cost are of DB and DT so we add both edges

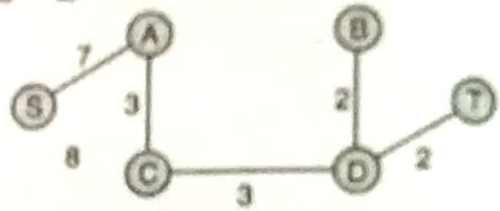


Fig. P. 1(e)

Number of vertices in tree = 6 which is equal to number of vertices in graph G.

∴Fig. P. 1(e) is shortest spanning tree with weight

$7 + 3 + 3 + 2 + 2 = 17$

Example 2 : Find minimum spanning tree of following graph.

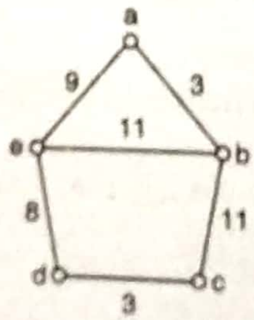


Fig. P. 2

Solution :

Minimum spanning tree set	Outgoing edges	Weight of outgoing edges	Minimum cost edge
a	ab ae	3 9	ab
a – 3 – b	ae be bc	9 11 11	ae

P.9 Solving problems on network flows using Ford-Fulkerson Labeling Algorithm

Example 1 : Capacity of each edge is given. Find maximum flow from a to d in the network.

What is the value of maximum flow.

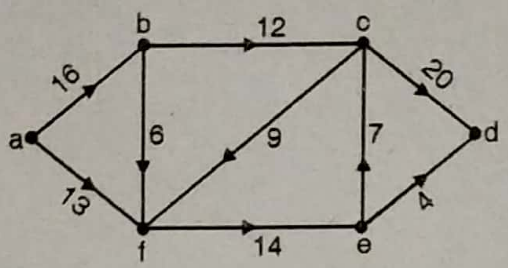


Fig. P. 1

Solution : For given network source is a and sink is d assign zero flow for each edge then first unsaturated path from a to d is

$$P_1 = a \xrightarrow[16]{16,0} b \xrightarrow[12]{12,0} c \xrightarrow[20]{20,0} d$$

$\Delta ij = \{16, 12, 20\}$

$$\therefore \Delta = \min \{16, 12, 20\} = 12$$

Add $\Delta = 12$ in to flow of each edge from path P_1, new network becomes

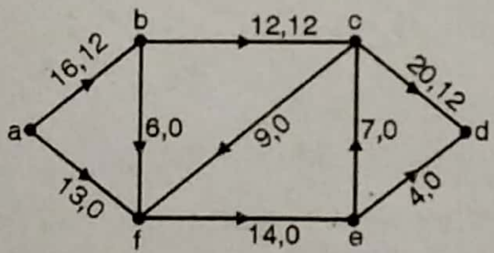


Fig. P. 1(a)

Next unsaturated path is

$$P_2 = a \xrightarrow[4]{16,12} b \xrightarrow[6]{6,0} f \xrightarrow[14]{14,0} e \xrightarrow[4]{4,0} d$$

$\Delta ij = \{4, 6, 14, 4\}$

$$\Delta = \min \{4, 6, 14, 4\} = 4$$

Add $\Delta = 4$ into flow of each edge from path P_2, new network becomes.

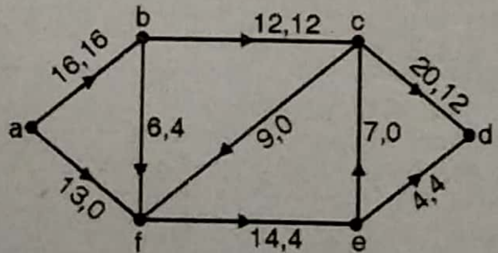


Fig. P. 1(b)

Next unsaturated path is

$$P_3 = a \xrightarrow[13]{13,0} f \xrightarrow[10]{14,4} e \xrightarrow[7]{7,0} c \xrightarrow[8]{20,12} d$$

$\Delta ij = \{13, 10, 7, 8\}$

$$\Delta = \min\{13, 10, 7, 8\} = 7$$

Add $\Delta = 7$ into flow of each edge of path P_3, new network becomes.

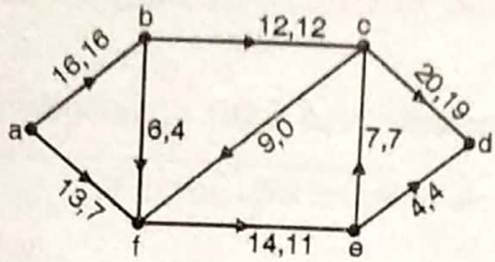


Fig. P. 1(c)

All directed paths from a to d are saturated ∴ maximum flow is occurred and it is

$$\text{Maximum flow} = 19 + 4 = 23$$

Example 2 : Consider the following network flow and find the maximum flow using Ford-Fulkerson labeling algorithm. The edges are labeled with *f,c*.

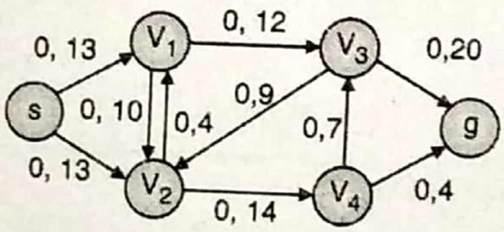


Fig. P. 2

Solution : Path which augments the flow is

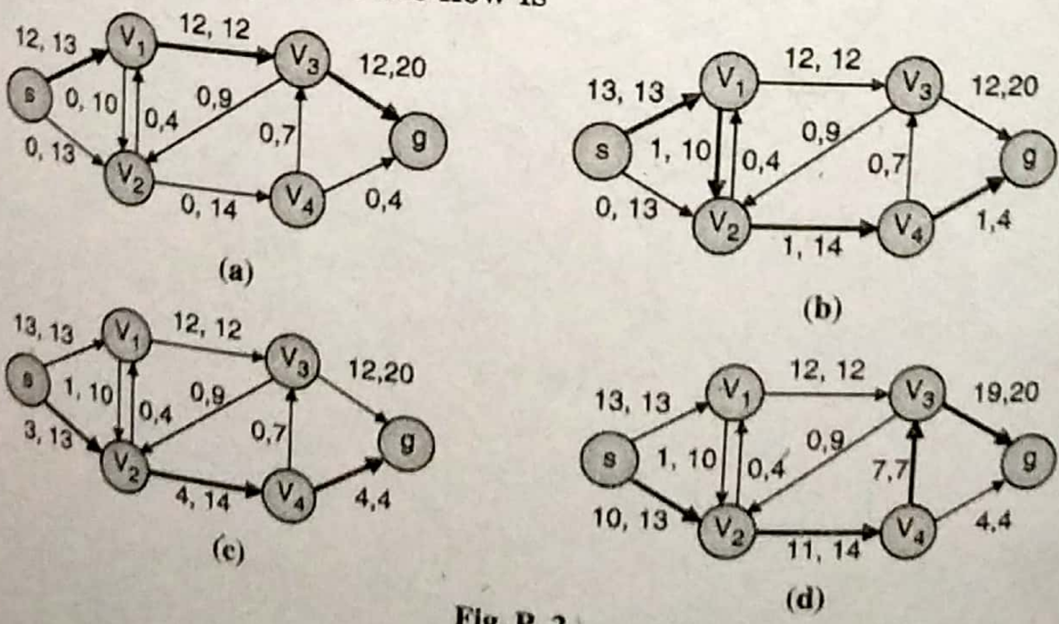


Fig. P. 2

At this point, there is no way of transmitting more flow through the graph, since edges v_1v_3, v_4v_3 and v_4g are all saturated (N.B. those edges form a minimum cut.) The magnitude of the maximum flow through the network is therefore the amount of flow leaving the source, which is equal to the flow entering the sink, which is equal to the flow across the minimum cut. In this case, that value is 23. If we chose the paths in a different order, the values on edges not on the minimum cut might have been different, but the flow magnitude would still have been the same.

P.10 Solving problems on posets and their associated networks

Example 1 : Give an example of Poset with height 5.

Solution :

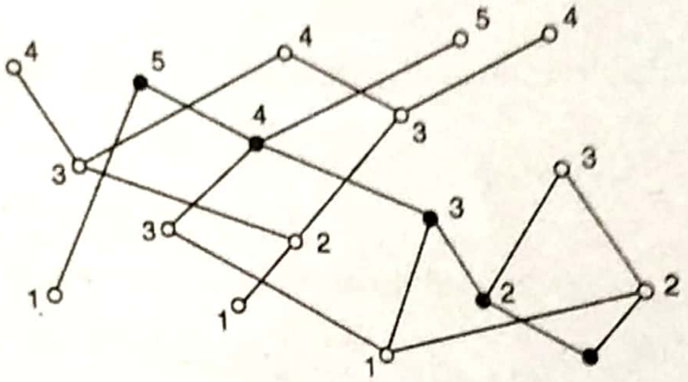


Fig. P. 1

Example 2 : Give an example of width 7.

Solution :

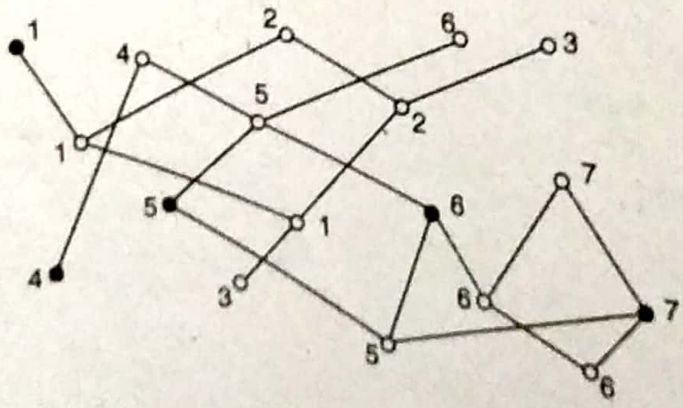


Fig. P. 2

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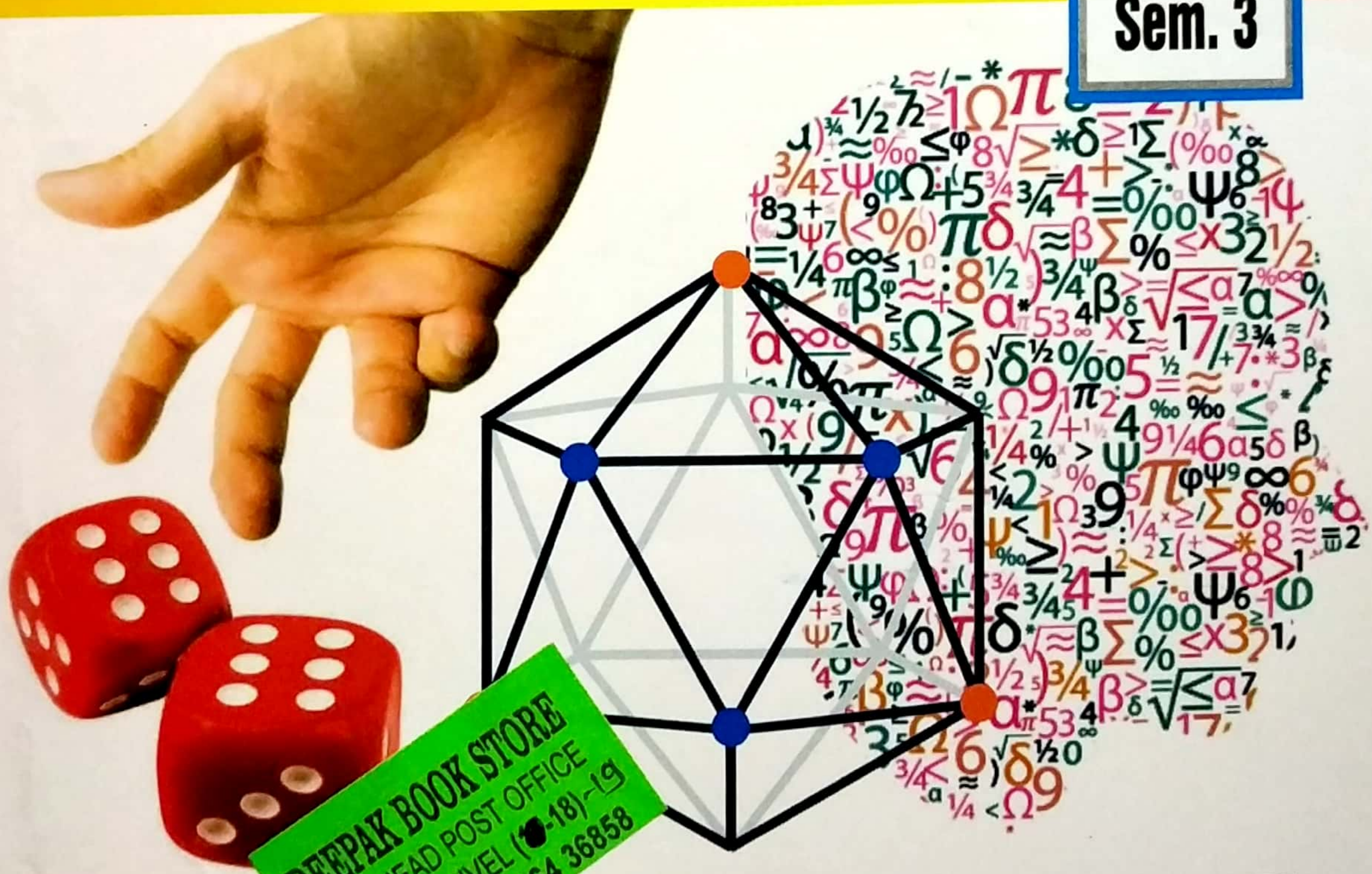

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