

CGT

Introduction to Combinatorics

- **Combinatorics** is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to many other areas of mathematics.
- the enumeration (counting) of specified structures, sometimes referred to as arrangements or configurations in a very general sense, associated with finite systems,
- the existence of such structures that satisfy certain given criteria,
- the construction of these structures, perhaps in many ways, and
- optimization, finding the "best" structure or solution among several possibilities, be it the "largest", "smallest" or satisfying some other optimality criterion.
- many other areas of mathematics

- Permutation: Arrangement of samples
- Combination :Selection of samples from total samples
- String: Combination of characters
- Combinatorics techniques used in Number theory, Geometry, optimization problems.
- Graph are used to solve optimization problem and finding shortest path problems

Solved Examples

Example 2.1.1 : In a state of Maharashtra, license number consists of 2 digits followed by a space followed by 2 capital letters. The first digit cannot be a 0. How many licence numbers are possible ?

Solution :

$$X = \{0, 1, \dots, 9\}$$

$$X - \{0\} = \{1, 2, \dots, 9\} \text{ for first digit}$$

$$Y = \{\text{Space}\}$$

$$Z = \{\text{Set of 26 capital letters}\}$$

Number of different licence number is a string from $(X - \{0\}) \times X \times Y \times Z \times Z$

$$= 9 \times 10 \times 1 \times 26^2$$

$$= 60840$$

Remark

If $X = \{0, 1\}$, an X-string is called binary string or bit string or 0 – 1 string.

If $X = \{0, 1, 2\}$, an X string is called a ternary string.

Example 2.1.2 : Suppose that, a website allows users to set password with condition, the first character must be a lower case letter in the English alphabet. Second and third letters may be upper or lower case alphabet or decimal digit (0 to 9). Fourth place must be @ . Fifth and sixth are lower case English letter, *, %, and # and seventh place must be a digit. How many different password can user set ?

Solution : Consider, a string of length 7

L : Lower case letters.

U : Upper case letters

D : Digits (0 – 9)

1	2	3	4	5	6	7
L	L	L	@	L	L	D
U	U		*	*		
D	D		%	%		
			#	#		
26	62	62	1	29	29	10

Below each position we write number of options.

For second position, 26 lower case + 26 upper case + 10 digit = 62 ways.
Each choice is independent of other. We multiply all numbers to get different possibility.

$$26 \times 62 \times 62 \times 1 \times 29 \times 29 \times 10 = 840529040$$



What is combinations ? and prove the formula.

- Selection from set of samples is nothing but combination,
- Selection of k from n , denoted by $\binom{n}{k}$ here, $0 \leq k \leq 1 | x | 1 = n$
- It is also denoted by $C(n, k)$.
- Number of combinations of n things, taken k at a time.

Proposition

$$\text{If } n \text{ and } k \text{ are integers with } 0 \leq k \leq n, \text{ then } \binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!}$$
$$= \frac{n!}{k!(n-k)!}$$

Proof

$P(n, k)$ is an arrangement of k from n element and $C(n, k)$ is an selection of k from n element.

$k!$ multiple of $C(n, k)$ is arrangement as for arrangement of k from k is

$$k \times (k - 1) \times (k - 2) \times \dots \times 3 \times 2 \times 1 = k!$$

$$\therefore k! C(n, k) = P(n, k)$$

$$C(n, k) = \frac{P(n, k)}{k!}$$

Example 2.2.1 : A Maharashtrian restaurant list 11 items in the “vegetable” category of its menu. A one vegetable plate contain 4 different vegetables. How many different ways can customer order ?

Solution :

- Out of 11 items we have to select 4 different ways for a customer to order a vegetable plate at the restaurant is $C(11, 4) = 330$.

Example 2.2.2 : How many ways can a committee can be formed from four men and six women with

- (a) At least 2 men and at least twice as many women as men ?
- (b) Four members, at least two of which are women, and Mr. and Mrs XYZ will not serve together

Solution :

(a) We prepare following table

men (4)	Women (6)	Selection from men	Selection from women	Total selections
2	4	$C(4, 2)$	$C(6, 4)$	$C(4, 2) \times C(6, 4)$
2	5	$C(4, 2)$	$C(6, 5)$	$C(4, 2) \times C(6, 5)$
2	6	$C(4, 2)$	$C(6, 6)$	$C(4, 2) \times C(6, 6)$
3	6	$C(4, 3)$	$C(6, 6)$	$C(4, 3) \times C(6, 6)$

\therefore Total number of committees are

$$\begin{aligned} & C(4, 2) \times C(6, 4) + C(4, 2) \times C(6, 5) + C(4, 2) \times C(6, 6) + C(4, 3) \times C(6, 6) \\ & = 90 + 36 + 6 + 4 = 136 \end{aligned}$$

(b) Number of 4-member committees = n_1

At least 2 of which are women and let n_2 be the number of 4 member committees, atleast 2 of which are women and Mr. And Mrs. X.Y.Z are included in each. Now clearly

$$n_1 = C(4, 2) \times C(6, 2) + C(4, 1) \times C(6, 3) + C(4, 0) \times C(6, 4)$$

$$n_2 = C(3, 1) \times C(5, 1) + C(5, 2)$$

\therefore Number of committees is $n_1 - n_2$.

Example 2.2.3 : A committee of 5 is to be selected among 6 boys and 5 girls.

Determine the number of ways of selecting the committee, if it is to consist of at least one boy and one girl.

Solution :

Boys (6)	Girls (5)	Selection from boys	Selection from girls	Total selection
1	4	$C(6, 1)$	$C(5, 4)$	$C(6, 1) \times C(5, 4)$
2	3	$C(6, 2)$	$C(5, 3)$	$C(6, 2) \times C(5, 3)$
3	2	$C(6, 3)$	$C(5, 2)$	$C(6, 3) \times C(5, 2)$
4	1	$C(6, 4)$	$C(5, 1)$	$C(6, 4) \times C(5, 1)$

\therefore Total number of committees are

$$C(6, 1) \times C(5, 4) + C(6, 2) \times C(5, 3) + C(6, 3) \times C(5, 2) + C(6, 4) \times C(5, 1)$$

Example 2.2.4 : A menu card in a restaurant displays, four soups, five ice-creams, three cold drinks and five fruits juices. How many different menus can a customer select if

- (i) He selects one item from each group without omission.
- (ii) He chooses to omit the fruit juices, but selects one each from the other groups.
- (iii) He chooses to omit the cold drinks but decides to take a fruit juices and one item each from the remaining groups.

Solution :

- (1) The customer can select the soup in 4 ways, ice creams in 5 ways, cold drinks in 3 ways and fruit juices in 5 ways. By product rule the number of ways in which he can select one item each, without omission is

$$C(4, 1) \times C(5, 1) \times C(3, 1) \times C(5, 1) = 4 \times 5 \times 3 \times 5 = 300$$

∴ Such selections are 300 ways.

- (ii) The customer can select an ice-cream in 5 ways, cold drinks in 3 ways and soups in 4 ways.

By product rule number of ways of selection

$$= C(5, 1) \times C(3, 1) \times C(4, 1) = 5 \times 3 \times 4 = 60$$

∴ Such selections are 60 ways.

- (iii) The number of ways to make the required selection is

$$= C(4, 1) \times C(5, 1) \times C(5, 1) = 4 \times 5 \times 5 = 100$$

∴ Such selections are 100 ways.

Example 2.2.5 : In how many ways can one select a captain, vice-captain and leader from the members of a committee consisting of 9 men and 11 women, if the leader must be a women, and the vice captain a man.

Solution : Selection of captain may be from men or women, hence there are following two cases.

Sr. No.	Case	Captain (men/women)	Vice captain (men)	Leader (women)	Total
1	Captain from men	$C(9, 1)$	$C(8, 1)$	$C(11, 1)$	$= 9 \times 8 \times 11$
					+
2	Captain from women	$C(11, 1)$	$C(9, 1)$	$C(10, 1)$	$= 11 \times 9 \times 10$

Total number of selections are

$$9 \times 8 \times 11 + 11 \times 9 \times 10 = 1782$$

\therefore Total number of ways to make the selection is 1782.

Example 2.2.6 : How many numbers are there between 100 and 1000 in which all the digits are distinct ?

Solution : Numbers between 100 and 1000 are 3 digit and total numbers available to fill in that are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

9	9	8
100 th	10 th	1 st

For 100th place we choose any one among

1, 2, 3, 4, 5, 6, 7, 8, 9

i.e. for 100th place we have 9 choices. For 10th place we choose any are among 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 but one no. already used in 100th place.

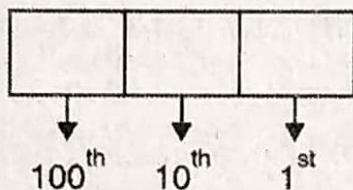
∴ For 10th place we have 9 choice.

Similarly in unit place we have 8 choice.

∴ Total counting of such numbers = $9 \times 9 \times 8 = 648$

Example 2.2.7 : How many different numbers can be formed from the digits 0, 2, 3, 4, 5, 6 lying between 100 and 1000 in which no digit being repeated ? How many of them are not divisible by 5 ?

Solution :



For 100th place we have 2, 3, 4, 5, 6 choices. i.e. Total 5 choices are available.

For 10th place we have 0, 2, 3, 4, 5, 6, but one number already used in 100th place
∴ Total 5 choices are available.

Similarly for unit place we have 4 choices

∴ Total numbers are $5 \times 5 \times 4 = 100$

Now if 0 or 5 appears at unit place then that number is divisible by 5.

First if 0 is in unit place than 100th and 10th places have 5 and 4 choices respectively.

These numbers are $5 \times 4 = 20$

If 5 appears in unit place then 100^{th} place has 4 choices (2, 3, 4, 6) and 10^{th} place has again 4 choices since it can be 0.

\therefore These numbers are $4 \times 4 = 16$

\therefore Total numbers divisible by 5 are $20 + 16 = 36$. Hence total numbers which are not divisible by 5 are $100 - 36 = 64$.

Example 2.2.8 : How many strings of three decimal digits with repetitions allowed.

- (a) That begin with an odd digit.
- (b) Have exactly two digit that are 4's.

Solution : Total number of digits are $(0, 1, 2, \dots, 9) = 10$ out of that $(1, 3, 5, 7, 9) = 5$ are odd.

- (a) For three decimal digits 100^{th} place digit may be any one out of 5 odd digits.
Similarly unit place and 10^{th} place digit may be any one digit out of total 10 digit.

100^{th}	10^{th}	Unit
5	10	10

\therefore Total number of such three decimal digits are $5 \times 10 \times 10 = 500$.

(b) Three decimal number having exactly two digits that are 4's that is only one digit is other than 4. There exist 3 different cases.

1. $4 \square 4 \Rightarrow$ Other than 4 all alternatives from 0 to 9 = 9

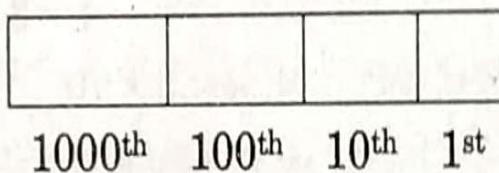
2. $4 4 \square \Rightarrow$ Other than 4 all alternatives from 0 to 9 = 9

3. $\square 4 4 \Rightarrow$ Other than 0 and 4 all alternatives from 0 to 9 = 8

\therefore Total number of such digits. = $9 + 9 + 8 = 26$

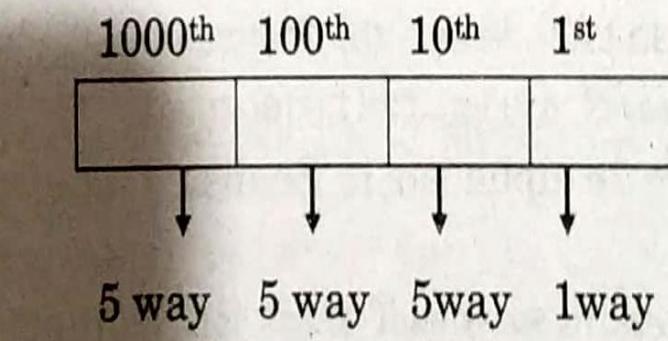
Example 2.2.9 : How many four digit numbers can be formed from the digits 1, 2, 3, 4 and 5, with repetition possible, which are divisible by 5.

Solution :



Four digit number is divisible by 5 only when we fix unit place by 5.

Now we are left with 1, 2, 3, 4 but repetition is allowed 50.



$$= 5 \times 5 \times 5 \times 1 = 125$$

Example 2.2.10 : In how many ways can 25 late admitted students be assigned to three practical batches if the first batch can accommodate 10 students, the second 8 and third only ?

Solution : The first batch can assigned 10 students from 25 in $C(25, 10)$ ways.

The second batch can assigned 8 students from 15 remaining in $C(15, 8)$ ways. The third batch can assigned 7 students from remaining 7 students in $(7, 7)$ ways. By product rule total number of assigning students.

$$= C(25, 10) \times C(15, 8) \times C(7, 7)$$

$$= \frac{25!}{15! 10!} \times \frac{15!}{7! 8!} \times \frac{7!}{7! 0!}$$

Example 2.2.11 : In how many ways one right and one left shoe be selected from six pairs of shoes without obtaining a pair.

Solution : Suppose there are $(a_1 b_1), (a_2 b_2), (a_3 b_3), (a_4 b_4), (a_5 b_5)$ and $(a_6 b_6)$ such six pairs of shoes. So we can select one out of this 6C_1 .

In this for without obtaining pair a_1 can match to b_2, b_3, b_4, b_5, b_6 shoes

$\therefore a_1$ has such 5 pairs. We select one out of this 5C_1 .

Similarly a_2, a_3, a_4, a_5, a_6 each have such 5 pairs.

\therefore Total without obtaining pairs are ${}^6C_1 \times {}^5C_1 = 30$.

\therefore 30 such pairs exists.

Example 2.2.12 : A farmer buys 3 cows, 2 goats and 4 hens from a man who has 4 cows, 3 goats, and 8 hens. How many choices does the farmer have ?

Solution :

$$\text{No. of choices for cows} = {}^4C_3 = 4$$

$$\text{No. of choices for goats} = {}^3C_2 = 3$$

$$\text{No. of choices for hens} = {}^8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70$$

$$\text{Total No. of choices} = 4 \times 3 \times 70$$

$$= 840 \text{ ways}$$

Example 2.2.13 : Find the number of strings of eight English letters.

(a) That contains no vowels,

If letters can be repeated. (b) That starts with the letter x and contains at least one vowel, if letters can be repeated.

Solution : Clearly there are 26 alphabets in English. Out of which 5 vowels and 21 consonants.

a) Arrange 21 non vowels in string of eight English letters with repetition in 21^8 ways.

Also if string contain 1 vowel and 7 consonants in ${}^5C_1 \times {}^{21}C_7$ ways.

If string contains 2 vowels and 6 consonants in ${}^5C_2 \times {}^{21}C_6$ ways

If string contains 3 vowels and 5 consonants in ${}^5C_3 \times {}^{21}C_5$ ways.

If string contains 4 vowels and 4 consonants in ${}^5C_4 \times {}^{21}C_4$ ways.

If string contains 5 vowels and 3 consonants in ${}^5C_5 \times {}^{21}C_3$ ways.

∴ Total number of strings are

$$21^8 + {}^5C_1 \times {}^{21}C_7 + {}^5C_2 \times {}^{21}C_6 + {}^5C_3 \times {}^{21}C_5 + {}^5C_4 \times {}^{21}C_4 + {}^5C_5 \times {}^{21}C_3.$$

b) Out of 8 letters first x is fixed, only 7 letters remains to assign.

If string contain 1 vowel and 6 consonants in ${}^5C_1 \times {}^{21}C_6$ ways.

If string contains 2 vowels and 5 consonants in ${}^5C_2 \times {}^{21}C_5$ ways.

If string contains 3 vowels and 4 consonants in ${}^5C_3 \times {}^{21}C_4$ ways.

If string contains 4 vowels and 3 consonants in ${}^5C_4 \times {}^{21}C_3$ ways.

If string contains 5 vowels and 2 consonants in ${}^5C_5 \times {}^{21}C_2$ ways.

∴ Total number of strings are

$21^7 + {}^5C_1 \times {}^{21}C_6 + {}^5C_2 \times {}^{21}C_5 + {}^5C_3 \times {}^{21}C_4 + {}^5C_4 \times {}^{21}C_3 + {}^5C_5 \times {}^{21}C_2$ ways.

Example 2.2.14 : Amit and Vijay are members of a club with a membership of 30. In how many ways can a committee of 10 be formed if

- (1) Amit must be include in the committee ?
- (2) Amit or Vijay should be in committee but not both ?

Solution. :

- (1) Selection of Amit is fixed in $C(1, 1)$ ways and remaining 9 members can be choosen in $C(29, 9)$ ways.

Total number of formation of such committees

$$= C(1, 1) \times C(29, 9) = \frac{29!}{9!(29-9)!} = \frac{29!}{9! 20!}$$

(2) If selection of Amit is fixed then Vijay is restricted

Remaining 9 members can be chosen in $C(28, 9)$ ways.

Similarly selection of Vijay is fixed then Amit is restricted.

Remaining 9 members can be chosen in $C(28, 9)$ ways.

Total number of formation of committees = $2 \times C(28, 9)$.

Example 2.2.15 : Mathematics students have to attempt six out of ten questions in an examination in any order.

- (i) How many choices have they
- (ii) How many choices do they have if they must answer atleast three out of the first five ?

Solution. :

- (i) Selection of 6 questions out of ten can be made in
 $C(10, 6) = 210$ ways.

Such selection of questions are 210 ways.

- (ii) Selection of 3 questions out of 5 can be made in
 $C(5, 3)$ ways and

Selection of next 3 questions out of remaining 5 questions can be made in

$$C(5, 3) \text{ ways}$$

Similarly 4 questions from first 5 can be made in

$$C(5, 4) \text{ ways}$$

Selection of next 2 questions out of remaining 5 can be made in

$$C(5, 2) \text{ ways}$$

Otherwise 5 questions from first 5 can be made in

$C(5, 5)$ ways and

Remaining 1 question from next 5 can select in

$C(5, 1)$ ways

Total such selections are

$$C(5, 3) \times C(5, 3) + C(5, 4) \times C(5, 2) + C(5, 5) \times C(5, 1)$$

$$= 100 + 50 + 5 = 155$$

All such selection of 6 questions from 10 are 155 ways.

Example 2.2.16 : There are 10 points in a plane of which 4 are collinear. Find the number of triangles that can be formed with vertices at these point.

Solution. : 4 points are collinear hence 6 are non-collinear.

Triangles by 6 non collinear points are $C(6, 3) = 20$.

Triangles by two collinear and one non collinear points = $C(4, 2) \times C(6, 1)$

Similarly triangle by one collinear and two non collinear points

$$= C(4, 1) \times C(6, 2)$$

Total number of triangles

$$= C(6, 3) + C(4, 2) \times C(6, 1) + C(4, 1) \times C(6, 2)$$

$$= 20 + 36 + 60 = 116$$

∴ There exists 116 triangles from such 10 points.

Example 2.2.17 : In how many ways can 5 balls be selected from 8 identical red balls and 8 identical white balls.

Solution. : Selection of 5 balls from 2 different coloured balls.

If $n = 2$ and $r = 5$

Such selection of balls can be made in

$$\begin{aligned} C(n + r - 1, n - 1) \text{ way} &= C(2 + 5 - 1, 2 - 1) \\ &= C(6, 1) = 6 \end{aligned}$$

Such selections are 6 ways.

Actual selections

red	white	
5	0	
0	5	
4	1	
1	4	
2	3	
3	2	

6
Selections

Example 2.2.18 : Ten balls are picked from a pile of red, blue and white balls. How many such selections contain less than 5 red balls.

Solution. : Selection of 10 balls from 3 different coloured balls

$$\text{i.e } n = 3, r = 10$$

Such selection of balls can be made in

$$C(3 + 10 - 1, 3 - 1) = C(12, 2) = 66$$

Such selections are 66 ways.

But selection of at least 5 red balls out of 3 different coloured balls are

$$C(5 + 3 - 1, 3 - 1) = C(7, 2) = 21. \quad r = 5, n = 3$$

Number of selection of 10 balls containing at least 5 red balls.

$$= 66 - 21 = 45.$$

∴ 45 such selection exists.

Example 2.2.19 : How many non-negative integer solutions are there in the equation

$$x + y + z + u + v = 10,000.$$

Solution. : Here distribution of 10,000 objects $\Rightarrow r = 10,000$

Number of variable are 5 $\Rightarrow n = 5$

Number of solutions

$$= C(n + r - 1, n - 1) = C(5 + 10000 - 1, 5 - 1) = C(10004, 4)$$

There are $C(100004, 4)$ number of solutions exists.

Example 2.2.20 : In how many ways can 15 different books be distributed among three students A, B, C. So that A and B together receive twice as many books as C.

Solution. : A, B, C receive x, y, z number of book respectively.

$$\therefore x + y + z = 15 \quad \text{and} \quad x + y = 2z$$

Solution of above equations gives $z = 5$. i.e. unique selection of z.

Hence remaining 10 book have to distribute in A and B.

$$\text{i.e. } x + y = 10$$

Here distribution of 10 books $\Rightarrow r = 10$

Number of variables are 2 $\Rightarrow n = 2$

Number of solutions

$$= C(n + r - 1, n - 1) = C(2 + 10 - 1, 2 - 1) = C(11, 1) = 11$$

11 ways in which the distribution can be done.

Example 2.2.21 : How many non-negative integer solutions are there in the equation
 $x + y + z + w = 10$?

Solution. :

Number of variables in linear equation = $n = 4$,

Constant to RHS = $r = 10$

\therefore Number of non-negative integer solutions

$$= C(n + r - 1, n - 1) = C(4 + 10 - 1, 4 - 1) = C(13, 3) = 286.$$

Example 2.2.22 : Find coefficient of x^{23} in $(x^2 + x^3 + x^4 + \dots)^5$.

Solution. :

$$\text{Consider } (x^2 + x^3 + x^4 + \dots)^5 = x^{10} (1 + x + x^2 + \dots)^5$$

To find coefficient of x^{23} from $(x^2 + x^3 + x^4 + \dots)^5$ is same as to find coefficient of x^{13} from $(1 + x + x^2 + \dots)^5$.

As coefficient of x^n from $(1 + x + x^2 + \dots)^r$ is

$${}^{n+r-1}C_{n-1}$$

\therefore Coefficient of x^{13} from $(1 + x + x^2 + \dots)^5$ is

$$= {}^{13+5-1}C_{5-1} = {}^{17}C_4$$

Example 2.2.23 : Use a generating function for finding the number of distributions of 27 identical balls, into five distinct boxes if each box has between 3 and 8 balls.

Solution. :

Generating function of each box becomes $(x^4 + x^5 + x^6 + x^7)$ because it contains 4, 5, 6, or 7 balls only.

∴ Generating function of such 5 boxes is $(x^4 + x^5 + x^6 + x^7)^5$

∴ Number of distribution of 27 identical balls in 5 boxes is same as coefficient of x^{27} from expression $(x^4 + x^5 + x^6 + x^7)^5$

i.e. coefficient of x^{27} from $x^{20}(1 + x + x^2 + x^3)^5$

i.e. coefficient of x^7 from $(1 + x + x^2 + x^3)^5$

Coefficient of x^7 from $(1 + x + x^2 + x^3)^5$ is 155

∴ 155 different distributions of 27 identical balls into five distinct boxes.

Example 2.2.24 : In how many ways are there to pick 2 different cards from a standard 52-card deck such that

- (i) The first is Ace and the second card is not a queen ?
- (ii) The first is space and second card is not a queen ?

Solution. :

- (i) First card Ace can be select in $C(4, 1) = 4$ ways.
Second can be select from $52 - 1 - 4 = 47$ in $C(47, 1) = 47$ ways.
There are 4×47 ways to choose the pair.
- (ii) First space card can be select from $C(13, 1) = 13$ ways
Second can be select from 48 cards. If first card is not queen then there are $C(47, 1)$ ways and if first card is queen then there are $C(48, 1)$ ways.
Total selections = $(1 \times 48) + (12 \times 47) = 612$ ways.

Example 2.3.1 : In how many ways can be office assistant distribute 12 identical folders among four office employees. Each employee receive at least one folder ?

Solution :

1. Consider,

1 2 3 4 5 6 7 8 9 10 11 12

to break above into 4 parts, we need to place 3 divider.

To place divider we have 11 places.

$$\therefore \text{Number of ways } C(11, 3) = 165$$

For above example, if we remove condition that each employee receive at least one folder, then how many ways can the distribution be made ?

2. Consider a dummy folder which is actually has no presence. It is for if any one not receive a single folder.

Now row contain total $12 + 4 = 16$ folder so to place divider we have 15 places.

$$\therefore \text{Number of ways } C(15, 3) = 455.$$

Example 2.3.2 : For Example 2.3.1, if we put condition that 2 employees will get guaranteed a folder and for remaining 2 not necessary to get folder. Find the number of ways.

Solution : Now consider only two dummy folders,

So in a row total $12 + 2 = 14$ folders. To place divider we have 13 places.

\therefore Number of ways $C(13, 3) = 286$

Example 2.3.3 : Find number of solutions to the inequality $x_1 + x_2 + x_3 \leq 358$.

Solution :

1. If all $x_i > 0$ and equality hold

i.e. $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 = 358$.

Number of solution is $C(357, 2)$.

Here we fix any one number and remaining 2 we select from $358 - 1 = 357$.

2. If all $x_i \geq 0$ and equality hold

i.e. $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 358$

as $x_i \geq 0$ it may take all $x_1 = x_2 = x_3 = 0$

It means we have total $358 + 3 = 361$ choices out of this we fix one and remaining 2 select from 360.

∴ Number of solution is $C(360, 2)$.

3. If $x_1, x_3 > 0$ and $x_2 = 51$ and equality hold

i.e. $x_2 = 51$ and $x_1 + x_2 + x_3 = 358$

$$\Rightarrow x_1 + x_3 = 307$$

∴ Selection of one from 306 keeping one fix out of x_1 and x_3 .

∴ Number of solution is $C(306, 1)$.

4. If all $x_i > 0$ and inequality is strict

i.e. all $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 < 358$

Now to keep always inequality we consider x_4 such that,

$$x_1 + x_2 + x_3 + x_4 = 358.$$

So it is selection of 3 from 357.

∴ Number of solution is $C(357, 3)$.

5. If all $x_i \geq 0$ and the inequality is strict

It may take all $x_1 = x_2 = x_3 = 0$

We have total $358 + 3 = 361$.

Now to keep inequality consider x_4 .

It is selection of 3 from 360.

∴ Number of solution is $C(360, 3)$

Example 2.4.1 : Find coefficient of $x^5 y^7$ in $(3x - 2y)^{12}$

Solution :

$$(3x - 2y)^{12} = \sum_{i=0}^{12} \binom{12}{i} (3x)^{12-i} (-2y)^i$$

To find coefficient of $x^5 y^7$

Put $i = 7$,

$$= \binom{12}{7} (3x)^5 (-2y)^7$$

∴ Coefficient of $x^5 y^7$ is

$$= \binom{12}{7} 3^5 (-2)^7$$

3.6.1 First Principle of Mathematical Induction



Define first principle of mathematical induction.

Let $p(n)$ be a statement involving a natural number n .

If i) $p(1)$ is true. ii) If $p(k)$ is true.

Then $p(k+1)$ is true. For $k > 1$

Then $p(n)$ is true for all $n \geq 1$.

Working rule :

Let $p(n)$: given statement to be proved.

Step I : Prove statement for $n = 1$.

Step II : Assume that statement is true for $n = k$; $n \in \mathbb{N}$.

Step III : Prove statement for $n = k + 1$.

Then by first principle of mathematical induction result is true for all $n \geq 1$.

Example 3.6.1 : Prove that sum of first n natural numbers is $\frac{n(n+1)}{2}$.

Solution : Let $p(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = 1$$

$$\text{RHS} = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$$\therefore \text{LHS} = \text{RHS}$$

\therefore Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \dots(1)$$

Step III : Now to prove statement for $n = k+1$

$$\text{i.e. to prove } 1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Consider, LHS = $1 + 2 + 3 + \dots + k + (k+1) = [1 + 2 + 3 + \dots + k] + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1) \quad \dots(\text{from (1)})$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \text{RHS}$$

\therefore Result is true for $n = k+1$

\therefore By principle of mathematical induction statement is true for any natural number n .

Example 3.6.2 : Prove that by Mathematical induction, $2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}$.

Solution :

Let $p(n) : 2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = 3n - 1 = 3 - 1 = 2$$

$$\text{RHS} = \frac{n(3n + 1)}{2} = \frac{1(3 + 1)}{2} = \frac{4}{2} = 2$$

$$\therefore \text{LHS} = \text{RHS}$$

\therefore Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 2 + 5 + 8 + \dots + (3k - 1) = \frac{k(3k + 1)}{2} \quad \dots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $2 + 5 + 8 + \dots + (3k - 1) + (3k + 2) = \frac{(k + 1)(3k + 4)}{2}$

Consider,

$$\begin{aligned} \text{LHS} &= 2 + 5 + 8 + \dots + (3k - 1) + (3k + 2) \\ &= [2 + 5 + 8 + \dots + (3k - 1)] + (3k + 2) \\ &= \frac{k(3k + 1)}{2} + (3k + 2) \quad \dots \text{from Equation (1)} \\ &= \frac{k(3k + 1) + 2(3k + 2)}{2} = \frac{3k^2 + k + 6k + 4}{2} \\ &= \frac{3k^2 + 7k + 4}{2} = \frac{(k + 1)(3k + 4)}{2} = \text{RHS} \end{aligned}$$

\therefore Statement is true for $n = k + 1$

\therefore By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.3 : Prove that sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

Solution : Let, $P(n) : 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

i.e. to prove $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step I : For $n = 1$.

$$\text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$$

$$\therefore \text{LHS} = \text{RHS}$$

\therefore Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \dots(1)$$

Step III : Now to prove statement for $n = k + 1$

$$\text{i.e. to prove } 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\begin{aligned}\text{Consider, LHS} &= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\&= (1^2 + 2^2 + 3^2 + \dots + k^2) + (k+1)^2 \\&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}\end{aligned}$$

$$\begin{aligned}
 &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
 &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}
 \end{aligned}$$

\therefore Statement is true for $n = k + 1$

\therefore By first principle of mathematical induction statement is true for all natural numbers n .

Example 3.6.4 : Prove that, $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3} \quad \forall n \geq 1.$

Solution : Let $p(n) : 1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$.

Step I : For $n = 1$.

$$\text{LHS} = (2n - 1)^2 = (2 - 1)^2 = 1^2 = 1$$

$$\text{RHS} = \frac{n(2n - 1)(2n + 1)}{3} = \frac{1(2 - 1)(2 + 1)}{3} = \frac{3}{3} = 1$$

$$\therefore \text{LHS} = \text{RHS}$$

\therefore Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3} \quad \dots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove

$$1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 + (2k + 1)^2 = \frac{(k + 1)(2k + 1)(2k + 3)}{3}$$

$$\begin{aligned} \text{Consider, LHS} &= 1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 + (2k + 1)^2 \\ &= [1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2] + (2k + 1)^2 \\ &= \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2 \quad \dots(\text{from (1)}) \\ &= \frac{k(2k - 1)(2k + 1) + 3(2k + 1)^2}{3} = \frac{(2k + 1)[k(2k - 1) + 3(2k + 1)]}{3} \\ &= \frac{(2k + 1)[2k^2 - k + 6k + 3]}{3} = \frac{(2k + 1)[2k^2 + 5k + 3]}{3} \\ &= \frac{(2k + 1)(k + 1)(2k + 3)}{3} = \frac{(k + 1)(2k + 1)(2k + 3)}{3} = \text{RHS} \end{aligned}$$

\therefore Statement is true for $n = k + 1$

\therefore By principle of mathematical induction statement is true for all $n \geq 1$.