CGT

Introduction to Combinatorics

- **Combinatorics** is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to ma
- the enumeration (counting) of specified structures, sometimes referred to as arrangements or configurations in a very general sense, associated with finite systems,
- the existence of such structures that satisfy certain given criteria,
- the construction of these structures, perhaps in many ways, and
- optimization, finding the "best" structure or solution among several possibilities, be it the "largest", "smallest" or satisfying some other optimality criterion.
- ny other areas of mathematics

- Permutation: Arrangement of samples
- Combination :Selection of samples from total samples
- String: Combination of characters
- Combinatorics techniques used in Number theory, Geometry, optimization problems.
- Graph are used to solve optimization problem and finding shortest path problems

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Solved Examples

Example 2.1.1 : In a state of Maharashtra, license number consists of 2 digits followed by a space followed by 2 capital letters. The first digit cannot be a 0. How many licence numbers are possible ?

Solution :

$$X = \{0, 1, \ldots, 9\}$$

$$X - \{0\} = \{1, 2, \ldots, 9\} \text{ for first digit}$$

$$Y = \{\text{Space}\}$$

$$Z = \{\text{Set of 26 capital letters}\}$$

Number of different licence number is a string from $(X - \{0\}) \times X \times Y \times Z \times Z$

$$= 9 \times 10 \times 1 \times 26^2$$

$$= 60840$$

Remark

If $X = \{0, 1\}$, an X-string is called binary string or bit string or 0 – 1 string.

If $X = \{0, 1, 2\}$, an X string is called a ternary string.

Example 2.1.2 : Suppose that, a website allows users to set password with condition, the first character must be a lower case letter in the English alphabet. Second and third letters may be upper or lower case alphabet or decimal digit (0 to 9). Fourth place must be @ . Fifth and sixth are lower case English letter, *, %, and # and seventh place must be a digit. How many different password can user set ?

Solution : Consider, a string of length 7

L : Lower case letters.

U : Upper case letters

D : Digits (0 – 9)

1	2	3	4	5	6	7
L	L	L	@	L	L	D
	U	U		*	*	
	D	D		%	%	
				#	#	
26	62	62	1	29	29	10

Below each position we write number of options.

For second position, 26 lower case + 26 upper case + 10 digit = 62 ways.

Each choice is independent of other. We multiply all numbers to get different possibility.

$$26 \times 62 \times 62 \times 1 \times 29 \times 29 \times 10 = 840529040$$

☞ **What is combinations ? and prove the formula.**

- Selection from set of samples is nothing but combination,
- Selection of k from n, denoted by $\binom{n}{k}$ here, $0 \le k \le 1\ |x|\ 1 = n$
- It is also denoted by C (n, k).
- Number of combinations of n things, taken k at a time.

Proposition

If n and k are integers with $0 \le k \le n$, then $\binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!}$

$$= \frac{n!}{k!\,(n-k)!}$$

Proof

P (n, k) is an arrangement of k from n element and C (n, k) is an selection of k from n element.

k! multiple of C (n, k) is arrangement as for arrangement of k from k is

$k \times (k-1) \times (k-2) \times \ldots \times 3 \times 2 \times 1 = k\,!$

$$\therefore k\,!\,C(n, k) = P(n, k)$$

$$C(n, k) = \frac{P(n, k)}{k!}$$

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Example 2.2.1 : A Maharashtrian restaurant list 11 items in the "vegetable" category of its menu. A one vegetable plate contain 4 different vegetables. How many different ways can customer order ?

Solution :

- Out of 11 items we have to select 4 different ways for a customer to order a vegetable plate at the restaurant is C (11, 4) = 330.

Example 2.2.2 : How many ways can a committee can be formed from four men and six women with

(a) At least 2 men and at least twice as many women as men ?

(b) Four members, at least two of which are women, and Mr. and Mrs XYZ will not serve together

Solution :

(a) We prepare following table

men (4)	Women (6)	Selection from men	Selection from women	Total selections
2	4	C (4, 2)	C (6, 4)	C (4, 2) × C (6, 4)
2	5	C (4, 2)	C (6, 5)	C (4, 2) × C (6, 5)
2	6	C(4, 2)	C (6, 6)	C(4, 2) × C (6, 6)
3	6	C(4, 3)	C (6, 6)	C(4, 3) × C (6, 6)

∴ Total number of committees are

C (4, 2) × C (6, 4) + C (4, 2) × C (6, 5) + C (4, 2) × C (6, 6) + C (4, 3) × C (6, 6)

= 90 + 36 + 6 + 4 = 136

(b) Number of 4-member committees = n_1

At least 2 of which are women and let n_2 be the number of 4 member committees, atlest 2 of which are women and Mr. And Mrs. X.Y.Z are included in each. Now clearly

n_1 = C (4, 2) × C (6, 2) + C (4, 1) × C(6, 3) + C(4, 0) × C(6,4)

n_2 = C(3, 1) × C (5, 1) + C(5, 2)

∴ Number of committees is $n_1 - n_2$.

Example 2.2.3 : A committee of 5 is to be selected among 6 boys and 5 girls. Determine the number of ways of selecting the committee, if it is to consist of at least one boy and one girl.

Solution :

Boys (6)	Girls (5)	Selection from boys	Selection from girls	Total selection
1	4	C (6, 1)	C (5, 4)	C (6, 1) × C (5, 4)
2	3	C (6, 2)	C (5, 3)	C (6, 2) × C (5, 3)
3	2	C (6, 3)	C (5, 2)	C (6, 3) × C (5, 2)
4	1	C (6, 4)	C (5, 1)	C (6, 4) × C (5, 1)

∴ Total number of committees are

C (6, 1) × C (5, 4) + C (6, 2) × C (5, 3) + C (6, 3) × C (5, 2) + C (6, 4) × C (5, 1)

Example 2.2.4 : A menu card in a restaurant displays, four soups, five ice-creams, three cold drinks and five fruits juices. How many different menus can a customer select if

(i) He selects one item from each group without omission.

(ii) He chooses to omit the fruit juices, but selects one each from the other groups.

(iii) He chooses to omit the cold drinks but decides to take a fruit juices and one item each from the remaining groups.

Solution :

(1) The customer can select the soup in 4 ways, ice creams in 5 ways, cold drinks in 3 ways and fruit juices in 5 ways. By product rule the number of ways in which he can select one item each, without omission is

C (4, 1) × C (5, 1) × C (3, 1) × C (5, 1) = 4 × 5 × 3 × 5 = 300

∴ Such selections are 300 ways.

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(ii) The customer can select an ice-cream in 5 ways, cold drinks in 3 ways and soups in 4 ways.

By product rule number of ways of selection

$$= C(5, 1) \times C(3, 1) \times C(4, 1) = 5 \times 3 \times 4 = 60$$

$\therefore$ Such selections are 60 ways.

(iii) The number of ways to make the required selection is

$$= C(4, 1) \times C(5, 1) \times C(5, 1) = 4 \times 5 \times 5 = 100$$

$\therefore$ Such selections are 100 ways.

Example 2.2.5 : In how many ways can one select a captain, vice-captain and leader from the members of a committee consisting of 9 men and 11 women, if the leader must be a women, and the vice captain a man.

Solution : Selection of captain may be from men or women, hence there are following two cases.

Sr. No.	Case	Captain (men/women)	Vice captain (men)	Leader (women)	Total
1	Captain from men	C (9, 1)	C (8, 1)	C (11, 1)	$= 9 \times 8 \times 11$
					+
2	Captain from women	C (11, 1)	C (9, 1)	C (10, 1)	$= 11 \times 9 \times 10$

Total number of selections are

$$9 \times 8 \times 11 + 11 \times 9 \times 10 = 1782$$

Example 2.2.6 : How many numbers are there between 100 and 1000 in which all the digits are distinct ?

Solution : Numbers between 100 and 1000 are 3 digited and total numbers available to fill in that are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

9	9	8
100^{th}	10^{th}	1^{st}

For 100^{th} place we choose any one among

1, 2, 3, 4, 5, 6, 7, 8, 9

i.e. for 100^{th} place we have 9 choices. For 10^{th} place we choose any are among 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 but one no. already used in 100^{th} place.

$\therefore$ For 10^{th} place we have 9 choice.

Similarly in unit place we have 8 choice.

$\therefore$ Total counting of such numbers = $9 \times 9 \times 8 = 648$

Example 2.2.7 : How many different numbers can be formed from the digits 0, 2, 3, 4, 5, 6 lying between 100 and 1000 in which no digit being repeated ? How many of them are not divisible by 5 ?

Solution :

100^{th}	10^{th}	1^{st}

For 100^{th} place we have 2, 3, 4, 5, 6 choices. i.e. Total 5 choices are available.

For 10^{th} place we have 0, 2, 3, 4, 5, 6, but one number already used in 100^{th} place

$\therefore$ Total 5 choices are available.

Similarly for unit place we have 4 choices

$\therefore$ Total numbers are $5 \times 5 \times 4 = 100$

Now if 0 or 5 appears at unit place then that number is divisible by 5.

First if 0 is in unit place than 100^{th} and 10^{th} places have 5 and 4 choices respectively.

These numbers are $5 \times 4 = 20$

If 5 appears in unit place then 100^{th} place has 4 choices (2, 3, 4, 6) and 10^{th} place has again 4 choices since it can be 0.

$\therefore$ These numbers are $4 \times 4 = 16$

$\therefore$ Total numbers divisible by 5 are $20 + 16 = 36$. Hence total numbers which are not divisible by 5 are $100 - 36 = 64$.

Example 2.2.8 : How many strings of three decimal digits with repetitions allowed.

(a) That begin with an odd digit.

(b) Have exactly two digit that are 4's.

Solution : Total number of digits are (0, 1, 2, ..., 9) = 10 out of that (1, 3, 5, 7, 9) = 5 are odd.

(a) For three decimal digits 100^{th} place digit may be any one out of 5 odd digits. Similarly unit place and 10^{th} place digit may be any one digit out of total 10 digit.

100^{th}	10^{th}	Unit
5	10	10

$\therefore$ Total number of such three decimal digits are $5 \times 10 \times 10 = 500$.

(b) Three decimal number having exactly two digits that are 4's that is only one digit is other than 4. There exist 3 different cases.

1. $4 \square 4 \Rightarrow$ Other than4 all alternatives from 0 to 9 = 9
2. $4 4 \square \Rightarrow$ Other than 4 all alternatives from 0 to 9 = 9
3. $\square 4 4 \Rightarrow$ Other than 0 and 4 all alternatives from 0 to 9 = 8

$\therefore$ Total number of such digits. $= 9 + 9 + 8 = 26$

Example 2.2.9 : How many four digit numbers can be formed from the digits 1, 2, 3, 4 and 5, with repetition possible, which are divisible by 5.

Solution :

1000^{th}	100^{th}	10^{th}	1^{st}

Four digit number is divisible by 5 only when we fix unit place by 5.

Now we are left with 1, 2, 3, 4 but repetition is allowed 50.

1000^{th}	100^{th}	10^{th}	1^{st}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
5 way	5 way	5way	1way

$= 5 \times 5 \times 5 \times 1 = 125$

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Example.2.2.10 : In how many ways can 25 late admitted students be assigned to three practical batches if the first batch can accommodate 10 students, the second 8 and third only ?

Solution : The first batch can assigned 10 students from 25 in C (25, 10) ways.

The second batch can assigned 8 students from 15 remaining in C (15, 8) ways. The third batch can assigned 7 students from remaining 7 students in (7, 7) ways. By product rule total number of assigning students.

$$= C(25, 10) \times C(15, 8) \times C(7, 7)$$

$$= \frac{25!}{15!\,10!} \times \frac{15!}{7!\,8!} \times \frac{7!}{7!\,0!}$$

Example.2.2.11 : In how many ways one right and one left shoe be selected from six pairs of shoes without obtaining a pair.

Solution : Suppose there are $(a_1\ b_1)$, $(a_2\ b_2)$, $(a_3\ b_3)$, $(a_4\ b_4)$, $(a_5\ b_5)$ and $(a_6\ b_6)$ such six pairs of shoes. So we can select one out of this ${}^6C_1$.

In this for without obtaining pair a_1 can match to b_2, b_3, b_4, b_5, b_6 shoes

$\therefore$ a_1 has such 5 pairs. We select one out of this ${}^5C_1$.

Similarly a_2, a_3, a_4, a_5, a_6 each have such 5 pairs.

$\therefore$ Total without obtaining pairs are ${}^6C_1 \times {}^5C_1 = 30$.

$\therefore$ 30 such pairs exists.

Example.2.2.12 : A farmer buys 3 cows, 2 goats and 4 hens from a man who has 4 cows, 3 goats, and 8 hens. How many choices does the farmer have ?

Solution :

$$\text{No. of choices for cows} = {}^4C_3 = 4$$

$$\text{No. of choices for goats} = {}^3C_2 = 3$$

$$\text{No. of choices for hens} = {}^8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70$$

$$\text{Total No. of choices} = 4 \times 3 \times 70$$

$$= 840 \text{ ways}$$

Example 2.2.13 : Find the number of strings of eight English letters.

(a) That contains no vowels,

If letters can be repeated. (b) That starts with the letter x and contains at least one vowel, if letters can be repeated.

Solution : Clearly there are 26 alphabets in English. Out of which 5 vowels and 21 consonants.

a) Arrange 21 non vowels in string of eight English letters with repetition in 21^8 ways.

Also if string contain 1 vowel and 7 consonants in ${}^5C_1 \times {}^{21}C_7$ ways.

If string contains 2 vowels and 6 consonants in ${}^5C_2 \times {}^{21}C_6$ ways

If string contains 3 vowels and 5 consonants in ${}^5C_3 \times {}^{21}C_5$ ways.

If string contains 4 vowels and 4 consonants in ${}^5C_4 \times {}^{21}C_4$ ways.

If string contains 5 vowels and 3 consonants in ${}^5C_5 \times {}^{21}C_3$ ways.

$\therefore$ Total number of strings are

$21^8 + {}^5C_1 \times {}^{21}C_7 + {}^5C_2 \times {}^{21}C_6 + {}^5C_3 \times {}^{21}C_5 + {}^5C_4 \times {}^{21}C_4 + {}^5C_5 \times {}^{21}C_3$.

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b) Out of 8 letters first x is fixed, only 7 letters remains to assign.

If string contain 1 vowel and 6 consonants in ${}^5C_1 \times {}^{21}C_6$ ways.

If string contains 2 vowels and 5 consonants in ${}^5C_2 \times {}^{21}C_5$ ways.

If string contains 3 vowels and 4 consonants in ${}^5C_3 \times {}^{21}C_4$ ways.

If string contains 4 vowels and 3 consonants in ${}^5C_4 \times {}^{21}C_3$ ways.

If string contains 5 vowels and 2 consonants in ${}^5C_4 \times {}^{21}C_2$ ways.

$\therefore$ Total number of strings are

$21^7 + {}^5C_1 \times {}^{21}C_6 + {}^5C_2 \times {}^{21}C_5 + {}^5C_3 \times {}^{21}C_4 + {}^5C_4 \times {}^{21}C_3 + {}^5C_5 \times {}^{21}C_2$ ways.

Example 2.2.14 : Amit and Vijay are members of a club with a membership of 30. In how many ways can a committee of 10 be formed if

(1) Amit must be include in the committee ?

(2) Amit or Vijay should be in committee but not both ?

Solution. :

(1) Selection of Amit is fixed in C (1, 1) ways and remaining 9 members can be choosen in C (29, 9) ways.

Total number of formation of such committees

$$= C(1,1) \times C(29,9) = \frac{29!}{9!\,(29-9)!} = \frac{29!}{9!\,20!}$$

(2) If selection of Amit is fixed then Vijay is restricted

Remaining 9 members can be choosen in C (28, 9) ways.

Similarly selection of Vijay is fixed then Amit is restricted.

Remaining 9 members can be choosen in C (28, 9) ways.

Total number of formation of committees = $2 \times C(28, 9)$.

Example 2.2.15 : Mathematics students have to attempt six out of ten questions in an examination in any order.

(i) How many choices have they

(ii) How many choices do they have if they must answer atleast three out of the first five ?

Solution. :

(i) Selection of 6 questions out of ten can be made in

C (10, 6) = 210 ways.

Such selection of questions are 210 ways.

(ii) Selection of 3 questions out of 5 can be made in

C (5, 3) ways and

Selection of next 3 questions out of remaining 5 questions can be made in

C (5, 3) ways

Similarly 4 questions from first 5 can be made in

C (5, 4) ways

Selection of next 2 questions out of remaining 5 can be made in

C (5, 2) ways

Otherwise 5 questions from first 5 can be made in

$C(5, 5)$ ways and

Remaining 1 question from next 5 can select in

$C(5, 1)$ ways

Total such selections are

$$C(5, 3) \times C(5, 3) + C(5, 4) \times C(5, 2) + C(5, 5) \times C(5, 1)$$

$$= 100 + 50 + 5 = 155$$

All such selection of 6 questions from 10 are 155 ways.

Example 2.2.16 : There are 10 points in a plane of which 4 are collinear. Find the number of triangles that can be formed with vertices at these point.

Solution. : 4 points are collinear hence 6 are non-collinear.

Triangles by 6 non collinear points are $C(6, 3) = 20$.

Triangles by two collinear and one non collinear points $= C(4, 2) \times C(6, 1)$

Similarly triangle by one collinear and two non collinear points

$$= C(4, 1) \times C(6, 2)$$

Total number of triangles

$$= C(6, 3) + C(4, 2) \times C(6, 1) + C(4, 1) \times C(6, 2)$$

$$= 20 + 36 + 60 = 116$$

$\therefore$ There exists 116 triangles from such 10 points.

Example 2.2.17 : In how many ways can 5 balls be selected from 8 identical red balls and 8 identical white balls.

Solution. : Selection of 5 balls from 2 different coloured balls.

If $n = 2$ and $r = 5$

Such selection of balls can be made in

$$C(n + r - 1, n - 1) \text{ way} = C(2 + 5 - 1, 2 - 1)$$

$$= C(6, 1) = 6$$

Such selections are 6 ways.

Actual selections

red	white
5	0
0	5
4	1
1	4
2	3
3	2

6 Selections

Example 2.2.18 : Ten balls are picked from a pile of red, blue and white balls. How many such selections contain less than 5 red balls.

Solution. : Selection of 10 balls from 3 different coloured balls

i.e $n = 3, r = 10$

Such selection of balls can be made in

$$C(3 + 10 - 1, 3 - 1) = C(12, 2) = 66$$

Such selections are 66 ways.

But selection of at least 5 red balls out of 3 different coloured balls are

$$C(5 + 3 - 1, 3 - 1) = C(7, 2) = 21. \quad r = 5, n = 3$$

Number of selection of 10 balls containing at least 5 red balls.

$$= 66 - 21 = 45.$$

$\therefore$ 45 such selection exists.

Example 2.2.19 : How many non-negative integer solutions are there in the equation $x + y + z + u + v = 10{,}000$.

Solution. : Here distribution of 10,000 objects $\Rightarrow r = 10{,}000$

Number of variable are $5 \Rightarrow n = 5$

Number of solutions

$$= C(n + r - 1, n - 1) = C(5 + 10000 - 1, 5 - 1) = C(10004, 4)$$

There are C (100004, 4) number of solutions exists.

Example 2.2.20 : In how many ways can 15 different books be distributed among three students A, B, C. So that A and B together receive twice as many books as C.

Solution. : A, B, C receive x, y, z number of book respectively.

$\therefore \quad x + y + z = 15 \quad$ and $\quad x + y = 2z$

Solution of above equations gives $z = 5$. i.e. unique selection of z.

Hence remaining 10 book have to distribute in A and B.

i.e. $\quad x + y = 10$

Here distribution of 10 books $\Rightarrow r = 10$

Number of variables are 2 $\Rightarrow n = 2$

Number of solutions

$$= C(n + r - 1, n - 1) = C(2 + 10 - 1, 2 - 1) = C(11, 1) = 11$$

11 ways in which the distribution can be done.

Example 2.2.21 : How many non-negative integer solutions are there in the equation $x + y + z + w = 10$?

Solution. :

Number of variables in linear equation = n = 4,

Constant to RHS = r = 10

$\therefore$ Number of non-negative integer solutions

$$= C(n + r - 1, n - 1) = C(4 + 10 - 1, 4 - 1) = C(13, 3) = 286.$$

Example 2.2.22 : Find coefficient of x^{23} in $(x^2 + x^3 + x^4 + \ldots)^5$.

Solution. :

Consider $(x^2 + x^3 + x^4 + \ldots)^5 = x^{10}(1 + x + x^2 + \ldots)^5$

To find coefficient of x^{23} from $(x^2 + x^3 + x^4 + \ldots)^5$ is same as to find coefficient of x^{13} from $(1 + x + x^2 + \ldots)^5$.

As coefficient of x^n from $(1 + x + x^2 + \ldots)^r$ is

$${}^{n+r-1}C_{n-1}$$

$\therefore$ Coefficient of x^{13} from $(1 + x + x^2 + \ldots)^5$ is

$$= {}^{13+5-1}C_{5-1} = {}^{17}C_4$$

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Example 2.2.23 : Use a generating function for finding the number of distributions of 27 identical balls, into five distinct boxes if each box has between 3 and 8 balls.

Solution. :

Generating function of each box becomes ($x^4 + x^5 + x^6 + x^7$) because it contains 4, 5, 6, or 7 balls only.

$\therefore$ Generating function of such 5 boxes is ($x^4 + x^5 + x^6 + x^7)^5$

$\therefore$ Number of distribution of 27 identical balls in 5 boxes is same as coefficient of x^{27} from expression ($x^4 + x^5 + x^6 + x^7)^5$

i.e. coefficient of x^{27} from $x^{20} (1 + x + x^2 + x^3)^5$

i.e. coefficient of x^7 from $(1 + x + x^2 + x^3)^5$

Coefficient of x^7 from $(1 + x + x^2 + x^3)^5$ is 155

$\therefore$ 155 different distributions of 27 identical balls into five distinct boxes.

Example 2.2.24 : In how many ways are there to pick 2 different cards from a standard 52-card deck such that

(i) The first is Ace and the second card is not a queen ?

(ii) The first is space and second card is not a queen ?

Solution. :

(i) First card Ace can be select in C (4, 1) = 4 ways.

Second can be select from 52 – 1 – 4 = 47 in C (47, 1) = 47 ways.

There are 4×47 ways to choose the pair.

(ii) First space card can be select from C (13, 1) = 13 ways

Second can be select from 48 cards. If first card is not queen then there are C (47, 1) ways and if first card is queen then there are C (48, 1) ways.

Total selections = $(1 \times 48) + (12 \times 47) = 612$ ways.

among four office employees. Each employee receive at least one folder ?

Solution :

1. Consider,

$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}\ \boxed{12}$

to break above into 4 parts, we need to place 3 divider.

To place divider we have 11 places.

$\therefore$ Number of ways C (11, 3) = 165

For above example, if we remove condition that each employee receive at least one folder, then how many ways can the distribution be made ?

2. Consider a dummy folder which is actually has no presence. It is for if any one not receive a single folder.

Now row contain total 12 + 4 = 16 folder so to place divider we have 15 places.

$\therefore$ Number of ways C (15, 3) = 455.

Example 2.3.2 : For Example 2.3.1, if we put condition that 2 employees will get guaranteed a folder and for remaining 2 not necessary to get folder. Find the number of ways.

Solution : Now consider only two dummy folders,

So in a row total 12 + 2 = 14 folders. To place divider we have 13 places.

$\therefore$ Number of ways C (13, 3) = 286

Example 2.3.3 : Find number of solutions to the inequality $x_1 + x_2 + x_3 \le 358$..

Solution :

1. **If all $x_i > 0$ and equality hold**

   i.e. $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 = 358$.

   Number of solution is C (357, 2).

   Here we fix any one number and remaining 2 we select from 358 – 1 = 357.

2. **If act $x_i \ge 0$ and equality hold**

   i.e. $x_1, x_2, x_3 \ge 0$ and $x_1 + x_2 + x_3 = 358$

   as $x_i \ge 0$ it may take all $x_1 = x_2 = x_3 = 0$

It means we have total 358 + 3 = 361 choices out of this we fix one and remaining 2 select from 360.

$\therefore$ Number of solution is C (360, 2).

3. **If $x_1, x_3 > 0$ and $x_2 = 51$ and equality hold**

   i.e. $x_2 = 51$ and $x_1 + x_2 + x_3 = 358$

   $\Rightarrow x_1 + x_3 = 307$

   $\therefore$ Selection of one from 306 keeping one fix out of x_1 and x_3.

   $\therefore$ Number of solution is C (306, 1).

4. **If all $x_i > 0$ and inequality is strict**

   i.e. all $x_1, x_2, x_3 > 0$ and $x_1 + x_2 + x_3 < 358$

   Now to keep always inequality we consider x_4 such that,

   $x_1 + x_2 + x_3 + x_4 = 358.$

   So it is selection of 3 from 357.

   $\therefore$ Number of solution is C (357, 3).

5. **If all $x_i \ge 0$ and the inequality is strict**

   It may take all $x_1 = x_2 = x_3 = 0$

   We have total 358 + 3 = 361.

   Now to keep inequality consider x_4.

   It is selection of 3 from 360.

   $\therefore$ Number of solution is C (360, 3)

Example 2.4.1 : Find coefficient of $x^5 y^7$ in $(3x - 2y)^{12}$

Solution :

$$(3x-2y)^{12} = \sum_{i=0}^{12} \binom{12}{i} (3x)^{12-i} (-2y)^i$$

To find coefficient of $x^5 y^7$

Put i = 7,

$$= \binom{12}{7} (3x)^5 (-2y)^7$$

$\therefore$ Coefficient of $x^5 y^7$ is

$$= \binom{12}{7} 3^5 (-2)^7$$

3.6.1 First Principle of Mathematical Induction

Define first principle of mathematical induction.

Let p(n) be a statement involving a natural number n.

If i) p(1) is true. ii) If p(k) is true.

Then p (k + 1) is true. For k > 1

Then p(n) is true for all $n \ge 1$.

Working rule :

Let p(n) : given statement to be proved.

Step I : Prove statement for n = 1.

Step II : Assume that statement is true for n = k; $n \in \mathbb{N}$.

Step III : Prove statement for n = k + 1.

Then by first principle of mathematical induction result is true for all $n \ge 1$.

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Example 3.6.1 : Prove that sum of first n natural numbers in $\frac{n(n+1)}{2}$.

Solution : Let $p(n): 1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$

Step I : For n = 1.

$LHS = 1$

$RHS = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$\therefore LHS = RHS$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 1+2+3+\ldots\ldots+k = \frac{k(k+1)}{2}$ ...(1)

Step III : Now to prove statement for n = k+1

i.e. to prove $1+2+3\ldots\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

Consider, $LHS = 1+2+3+\ldots\ldots+k+(k+1) = [1+2+3+\ldots\ldots+k]+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ ...(from (1))

$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = RHS$

$\therefore$ Result is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.2 : Prove that by Mathematical induction, $2+5+8+\ldots..+(3n-1) = \frac{n(3n+1)}{2}$.

Solution :

Let $p(n): 2+5+8+\ldots..+(3n-1) = \frac{n(3n+1)}{2}$

Step I : For n = 1.

$LHS = 3n-1 = 3-1 = 2$

$RHS = \frac{n(3n+1)}{2} = \frac{1(3+1)}{2} = \frac{4}{2} = 2$

$\therefore LHS = RHS$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 2+5+8+\ldots..+(3k-1) = \frac{k(3k+1)}{2}$ ...(1)

Step III : Now to prove statement for n = k + 1

i.e. to prove $2+5+8+\ldots..+(3k-1)+(3k+2) = \frac{(k+1)(3k+4)}{2}$

Consider, $LHS = 2+5+8+\ldots..+(3k-1)+(3k+2)$

$= [2+5+8+\ldots..+(3k-1)]+(3k+2)$

$= \frac{k(3k+1)}{2}+(3k+2)$ ...from Equation (1)

$= \frac{k(3k+1)+2(3k+2)}{2} = \frac{3k^2+k+6k+4}{2}$

$= \frac{3k^2+7k+4}{2} = \frac{(k+1)(3k+4)}{2} = RHS$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.3 : Prove that sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

Solution : Let, $P(n): 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

i.e. to prove $1^2+2^2+3^2+\ldots..+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step I : For n = 1.

$LHS = 1^2 = 1$

$RHS = \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$

$\therefore LHS = RHS$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 1^2+2^2+3^2+\ldots..+k^2 = \frac{k(k+1)(2k+1)}{6}$ ...(1)

Step III : Now to prove statement for n = k + 1

i.e. to prove $1^2+2^2+3^2+\ldots..+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Consider, $LHS = 1^2+2^2+3^2+\ldots..+k^2+(k+1)^2$

$= (1^2+2^2+3^2+\ldots..+k^2)+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$$= \frac{(k+1)\,[k\,(2k+1)+6\,(k+1)]}{6} = \frac{(k+1)\,(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)\,(2k^2+7k+6)}{6} = \frac{(k+1)\,(k+2)\,(2k+3)}{6} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction statement is true for all natural numbers n.

Example 3.6.4 : Prove that, $1^2 + 3^2 + 5^2 + \ldots.. + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ $\forall\ n \geq 1$.

Solution : Let $p(n) : 1^2 + 3^2 + 5^2 + \ldots.. + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Step I : For $n = 1$.

$$\text{LHS} = (2n-1)^2 = (2-1)^2 = 1^2 = 1$$

$$\text{RHS} = \frac{n\,(2n-1)\,(2n+1)}{3} = \frac{1\,(2-1)\,(2+1)}{3} = \frac{3}{3} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore\ 1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 = \frac{k\,(2k-1)\,(2k+1)}{3} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove

$$1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 + (2k+1)^2 = \frac{(k+1)\,(2k+1)\,(2k+3)}{3}$$

Consider, LHS $= 1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2 + (2k+1)^2$

$$= [1^2 + 3^2 + 5^2 + \ldots.. + (2k-1)^2] + (2k+1)^2$$

$$= \frac{k(2k-1)\,(2k+1)}{3} + (2k+1)^2 \quad \ldots(\text{from } (1))$$

$$= \frac{k(2k-1)\,(2k+1) + 3\,(2k+1)^2}{3} = \frac{(2k+1)\,[k(2k-1) + 3\,(2k+1)]}{3}$$

$$= \frac{(2k+1)\,[2k^2 - k + 6\,k + 3]}{3} = \frac{(2k+1)\,[2k^2 + 5\,k + 3]}{3}$$

$$= \frac{(2k+1)\,(k+1)\,(2k+3)}{3} = \frac{(k+1)\,(2k+1)\,(2k+3)}{3} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Unit II

Example 3.6.5 : Prove that, $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1}\,n\,(n+1)}{2}$ $\forall\ n \in \mathbb{N}$.

Solution : Let $p(n) : 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1}\,n\,(n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = (-1)^{n+1} n^2 = (-1)^{1+1} (1)^2 = (1)\,(1) = 1$$

$$\text{RHS} = \frac{(-1)^{n+1}\,n\,(n+1)}{2} = \frac{(-1)^{1+1}\,(1)\,(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 = \frac{(-1)^{k+1}\,k\,(k+1)}{2} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= \frac{(-1)^{k+2}\,(k+1)\,(k+2)}{2}$$

Consider, LHS $= 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= [1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2] + (-1)^{k+2} (k+1)^2$$

$$= \frac{(-1)^{k+1}\,k(k+1)}{2} + (-1)^{k+2}\,(k+1)^2 \quad \ldots\text{from Equation (1)}$$

$$= \frac{(-1)^{k+1}\,k(k+1) + (-1)^{k+2}\,2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}\,(-1)^{-1}\,k(k+1) + (-1)^{k+2}\,2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}\,(k+1)\,[-k + 2\,(k+1)]}{2} = \frac{(-1)^{k+2}\,(k+1)\,(k+2)}{2} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction p (n) is true $\forall\ n \in \mathbb{N}$.

Example 3.6.5 : Prove that, $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2} \; \forall \; n \in \mathbb{N}$.

Solution : Let $p(n) : 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

Step I : For n = 1.

$$LHS = (-1)^{n+1} n^2 = (-1)^{1+1} (1)^2 = (1)(1) = 1$$

$$RHS = \frac{(-1)^{n+1} n(n+1)}{2} = \frac{(-1)^{1+1}(1)(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore \quad 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 = \frac{(-1)^{k+1} k(k+1)}{2} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 + (-1)^{k+2}(k+1)^2 = \frac{(-1)^{k+2}(k+1)(k+2)}{2}$

Consider, $LHS = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2 + (-1)^{k+2}(k+1)^2$

$$= [1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k+1} k^2] + (-1)^{k+2}(k+1)^2$$

$$= \frac{(-1)^{k+1} k(k+1)}{2} + (-1)^{k+2}(k+1)^2 \quad \ldots\text{from Equation (1)}$$

$$= \frac{(-1)^{k+1} k(k+1) + (-1)^{k+2} 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}(-1)^{-1} k(k+1) + (-1)^{k+2} 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2}(k+1)[-k+2(k+1)]}{2} = \frac{(-1)^{k+2}(k+1)(k+2)}{2} = RHS$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction p (n) is true $\forall \; n \in \mathbb{N}$.

Example 3.6.6 : Prove that by mathematical induction,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \; \forall \; n \geq 1.$$

Solution : Let $p(n) : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Step I : For n = 1.

$$LHS = n^3 = 1^3 = 1$$

$$RHS = \frac{n^2(n+1)^2}{4} = \frac{1(1+1)^2}{4} = \frac{4}{4} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

Consider,

$$LHS = 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \ldots\text{from Equation (1)}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2[k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2[k^2 + 4k + 4]}{4} = \frac{(k+1)^2(k+2)^2}{4} = RHS$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for all natural numbers n.

Example 3.6.7 : Prove that, $3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3}{2}(3^n - 1)$

Solution : Let $p(n) : 3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3}{2}(3^n - 1)$

Step I : For n = 1.

$$LHS = 3$$

$$RHS = \frac{3}{2}(3^n - 1) = \frac{3}{2}(3 - 1) = \frac{3}{2} 2 = 3$$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore \quad 3 + 3^2 + 3^3 + \ldots + 3^k = \frac{3}{2}(3^k - 1) \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $3 + 3^2 + 3^3 + \ldots + 3^k + 3^{k+1} = \frac{3}{2}(3^{k+1} - 1)$

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Example 3.6.11 : Prove that for each natural number, $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$.

Solution : Let $p(n) : 1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Step I : For n = 1.

$$\text{LHS} = n(n+2) = (1)(1+2) = 3$$

$$\text{RHS} = \frac{n(n+1)(2n+7)}{6} = \frac{1(2)(9)}{6} = 3$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) = \frac{k(k+1)(2k+7)}{6} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}$

Consider,

$$\begin{aligned}
\text{LHS} &= 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2) + (k+1)(k+3) \\
&= [1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + k(k+2)] + (k+1)(k+3) \\
&= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \quad \ldots\text{from Equation (1)} \\
&= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6k + 18]}{6} = \frac{(k+1)[2k^2 + 13k + 18]}{6} \\
&= \frac{(k+1)(k+2)(2k+9)}{6} = \text{RHS}
\end{aligned}$$

$\therefore$ Result is true for n = k + 1

$\therefore$ By principle of mathematical induction result is true for any natural number.

Example 3.6.12 : Prove that,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Solution :

Let $p(n) : 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Step I : For n = 1.

$$\text{LHS} = n(n+1)(n+2) = 1(1+1)(1+2) = 1 \cdot 2 \cdot 3 = 6$$

$$\text{RHS} = \frac{n(n+1)(n+2)(n+3)}{4} = \frac{(1)(2)(3)(4)}{4} = 6$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$

Consider,

$$\begin{aligned}
\text{LHS} &= 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= [1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + k(k+1)(k+2)] + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{RHS}
\end{aligned}$$

Statement is true for n = k + 1

Example 3.6.13 : Prove by mathematical induction, $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$

Solution : Let $p(n) : \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$

Step I : For n = 1.

$$\text{LHS} = \frac{1}{1(1+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$\text{RHS} = \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

$\therefore$ Result is true for n = 1.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k \cdot (k+1)} = \frac{k}{k+1} \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Consider,

$$\begin{aligned}
\text{LHS} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \ldots\text{from Equation (1)} \\
&= \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}
\end{aligned}$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.14 : Prove that $\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Solution : Let p(n) : $\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Step I : For n = 1.

$$\text{LHS} = \frac{1}{(2n+1)(2n+3)}=\frac{1}{(2+1)(2+3)}=\frac{1}{3\cdot5}=\frac{1}{15}$$

$$\text{RHS} = \frac{n}{3(2n+3)}=\frac{1}{3(2+3)}=\frac{1}{3\cdot5}=\frac{1}{15}$$

$\therefore$ Result is true for n = 1.

Step II : Assume that statement is true for n = k, $k \in \mathbb{N}$.

$$\therefore \frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \quad \dots(1)$$

Step III : To prove statement for n = k + 1

i.e. to prove,

$$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{(k+1)}{3(2k+5)}$$

Consider, $\text{LHS} = \frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

$$= \left[\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{(2k+3)(2k+5)}$$

$\therefore$ Result is true for n = k + 1.

$\therefore$ By Mathematical induction result is true for all n.

Example 3.6.17 : Prove that, $n(n^2-1)$ is divisible by 3.

Solution : Let p(n) : $n(n^2-1)$ is divisible by 3.

Step I : For n = 1.

$$\text{LHS} = n(n^2-1) = 1(1^2-1) = 1(1-1)$$
$$= 0 \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, $k \in \mathbb{N}$.

$\therefore$ $k(k^2-1)$ is divisible by 3.

$\therefore \quad k(k^2-1) = 3m$ for some $m \in \mathbb{Z}$

$\therefore \quad k^3 - k = 3m$

$\therefore \quad k^3 = 3m + k \quad \dots(1)$

Step III : Now to prove statement for n = k + 1

i.e. to prove $(k+1)[(k+1)^2-1]$ is divisible by 3.

Consider, $\text{LHS} = (k+1)[(k+1)^2-1] = (k+1)(k^2+2k+1-1)$

$= (k+1)(k^2+2k) = k^3+2k^2+k^2+2k$

$= k^3+3k^2+2k$

$= 3m+k+3k^2+2k$ ...from Equation (1)

$= 3m+3k^2+3k$

$= 3(m+k^2+k)$ which is divisible by 3.

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

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Example 3.6.18 : Prove that, $2^{2n} - 1$ is divisible by 3.

Solution : Let p(n) : $2^{2n} - 1$ is divisible by 3.

Step I : For n = 1.

Consider, $2^{2n} - 1 = 2^2 - 1 = 4 - 1 = 3$ which is divisible by 3.

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, $k \in \mathbb{N}$.

$\therefore 2^{2k} - 1$ is divisible by 3.

$\therefore 2^{2k} - 1 = 3m$ for some $m \in \mathbb{Z}$

$\therefore 2^{2k} = 3m + 1$ ...(1)

Step III : Now to prove statement for n = k + 1

i.e. to prove $2^{2(k+1)} - 1$ is divisible by 3.

Consider,

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1 \\ &= (3m+1) \cdot 4 - 1 \quad \text{...from Equation (1)} \\ &= 3 \times 4 \times m + 4 - 1 = 3 \times 4 \times m + 3. \\ &= 3(4m + 1) \text{ which is divisible by 3.} \end{aligned}$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By principle of mathematical induction result is true for any natural number n.

Example 3.6.19 : $3^{2n} + 7$ is divisible by 8, Prove by induction $\forall n \in \mathbb{N}$.

Solution : Let p(n) : $3^{2n} + 7$ is divisible by 8.

Step I : For n = 1.

Consider, $3^{2n} + 7 = 3^2 + 7 = 9 + 7 = 16$ which is divisible by 8.

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, $k \in \mathbb{N}$.

$\therefore 3^{2k} + 7$ is divisible by 8.

$\therefore 3^{2k} + 7 = 8m$ for some $m \in \mathbb{Z}$

$\therefore 3^{2k} = 8m - 7$ ...(1)

Step III : Now to prove statement for n = k + 1

Consider,

$$\begin{aligned} 3^{2(k+1)} + 7 &= 3^{2k+2} + 7 = 3^{2k} \cdot 3^2 + 7 = 3^{2k} \cdot 9 + 7 \\ &= (8m - 7)\,9 + 7 \quad \text{...from Equation (1)} \\ &= 8 \times 9 \times m - 63 + 7 = 8 \times 9 \times m - 56. \\ &= 8(9m - 7) \text{ which is divisible by 8.} \end{aligned}$$

$\therefore$ Statement is true for n = k + 1

$\therefore$ By first principle of Mathematical induction statement is true for all $n \in \mathbb{N}$.

Example 3.6.20 : Prove $\forall n \in \mathbb{N}$ n (n – 1) (2n – 1) is divisible by 6.

Solution : Let p(n) : n (n – 1) (2n – 1) is divisible by 6.

Step I : For n = 1.

Consider, $n(n-1)(2n-1) = 1(1-1)(2-1)$

$= 0$ which is divisible by 6.

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for n = k, $k \in \mathbb{N}$.

$\therefore k(k-1)(2k-1)$ is divisible by 6.

$$\begin{aligned} \therefore k(k-1)(2k-1) &= 6m \quad \text{for some } m \in \mathbb{Z}^+ \\ \therefore (k^2 - k)(2k - 1) &= 6m \\ 2k^3 - k^2 - 2k^2 + k &= 6m \\ 2k^3 &= 6m + 3k^2 - k \quad \text{...(1)} \end{aligned}$$

Step III : Now to prove statement for n = k + 1

Consider, $(k+1)[(k+1)-1)][2(k+1)-1]$

$$\begin{aligned} &= (k+1)[k](2k+2-1) = (k^2+k)(2k+1) \\ &= 2k^3 + k^2 + 2k^2 + k = (6m + 3k^2 - k) + 3k^2 + k \\ &= 6m + 6k^2 = 6(m + k^2) \end{aligned}$$

Which is divisible by 6.

$\therefore$ Statement is true for n = k + 1

$\therefore$ By first principle of mathematical induction statement is true for every natural number. n.

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Example 3.6.21 : Show that $n^3 + 2n$ is divisible by 3 for $n \geq 1$.

Solution : Let $p(n) : n^3 + 2n$ is divisible by 3.

Step I : For $n = 1$.

Consider, $n^3 + 2n = (1)^3 + 2(1) = 1 + 2 = 3$ which is divisible by 3.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \geq 1$.

$\therefore k^3 + 2k$ is divisible by 3.

$\therefore k^3 + 2k = 3m$ for some $m \in \mathbb{Z}^+$

$\therefore k^3 = 3m - 2k$ ...(1)

Step III : Now to prove statement for $n = k + 1$

Consider,

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= (3m - 2k) + 3k^2 + 3k + 2k + 3 \quad \text{...from Equation (1)}$$
$$= 3m + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$$

which is divisible by 3.

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.22 : $8^n - 3^n$ is multiple of 5 $\forall n \geq 1$, Prove by mathematical induction.

Solution : Let $p(n) : 8^n - 3^n$ is multiple of 5.

Step I : For $n = 1$.

Consider,

$8^n - 3^n = 8 - 3 = 5$ which is multiple of 5.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore 8^k - 3^k$ is multiple of 5.

$\therefore 8^k - 3^k = 5m$ for some $m \in \mathbb{Z}$

$\therefore 8^k = 5m + 3^k$ ...(1)

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $8^{k+1} - 3^{k+1}$ is multiple of 5.

Consider,

$$8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 3^k$$
$$= 8(5m + 3^k) - 3 \cdot 3^k \quad \text{...from Equation (1)}$$
$$= 40m + 8 \cdot 3^k - 3 \cdot 3^k = 40m + (8-3)3^k = 40m + 5 \cdot 3^k = 5(8m + 3^k)$$

Which is multiple by 5.

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By first principle of mathematical induction statement is true for all natural numbers.