Unit II

Example 3.6.5 : Prove that, $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$ $\forall n \in N$.

Solution : Let $p(n) : 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = (-1)^{n+1} n^2 = (-1)^{1+1} (1)^2 = (1)(1) = 1$$

$$\text{RHS} = \frac{(-1)^{n+1} n(n+1)}{2} = \frac{(-1)^{1+1} (1)(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore \quad 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 = \frac{(-1)^{k+1} k(k+1)}{2} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= \frac{(-1)^{k+2} (k+1)(k+2)}{2}$$

Consider,

$$\begin{aligned}
\text{LHS} &= 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2 \\
&= [1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2] + (-1)^{k+2} (k+1)^2 \\
&= \frac{(-1)^{k+1} k(k+1)}{2} + (-1)^{k+2} (k+1)^2 \qquad \ldots\text{from Equation (1)} \\
&= \frac{(-1)^{k+1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2} \\
&= \frac{(-1)^{k+2} (-1)^{-1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2} \\
&= \frac{(-1)^{k+2} (k+1)[-k + 2(k+1)]}{2} = \frac{(-1)^{k+2} (k+1)(k+2)}{2} = \text{RHS}
\end{aligned}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction $p(n)$ is true $\forall\, n \in \mathbb{N}$.

Example 3.6.5 : Prove that, $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$ $\forall\, n \in \mathbb{N}$.

Solution : Let $p(n) : 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

Step I : For $n = 1$.

$$\text{LHS} = (-1)^{n+1} n^2 = (-1)^{1+1} (1)^2 = (1)(1) = 1$$

$$\text{RHS} = \frac{(-1)^{n+1} n(n+1)}{2} = \frac{(-1)^{1+1}(1)(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore \quad 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 = \frac{(-1)^{k+1} k(k+1)}{2} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= \frac{(-1)^{k+2} (k+1)(k+2)}{2}$$

Consider, LHS $= 1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$

$$= [1^2 - 2^2 + 3^2 - 4^2 + \ldots.. + (-1)^{k+1} k^2] + (-1)^{k+2} (k+1)^2$$

$$= \frac{(-1)^{k+1} k(k+1)}{2} + (-1)^{k+2} (k+1)^2 \qquad \ldots\text{from Equation (1)}$$

$$= \frac{(-1)^{k+1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2} (-1)^{-1} k(k+1) + (-1)^{k+2}\, 2(k+1)^2}{2}$$

$$= \frac{(-1)^{k+2} (k+1)[-k + 2(k+1)]}{2} = \frac{(-1)^{k+2} (k+1)(k+2)}{2} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction $p(n)$ is true $\forall\, n \in \mathbb{N}$.

Example 3.6.6 : Prove that by mathematical induction,

$$1^3 + 2^3 + 3^3 + \ldots.. + n^3 = \frac{n^2 (n+1)^2}{4} \quad \forall\, n \geq 1.$$

Solution : Let $p(n) : 1^3 + 2^3 + 3^3 + \ldots.. + n^3 = \dfrac{n^2 (n+1)^2}{4}$

Step I : For $n = 1$.

$$\text{LHS} = n^3 = 1^3 = 1$$

$$\text{RHS} = \frac{n^2 (n+1)^2}{4} = \frac{1(1+1)^2}{4} = \frac{4}{4} = 1$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$$\therefore 1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 = \frac{k^2 (k+1)^2}{4} \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 + (k+1)^3 = \dfrac{(k+1)^2 (k+2)^2}{4}$

Consider,

$$\text{LHS} = 1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 + (k+1)^3 = (1^3 + 2^3 + 3^3 + \ldots\ldots + k^3) + (k+1)^3$$

$$= \frac{k^2 (k+1)^2}{4} + (k+1)^3 \qquad \ldots\text{from Equation (1)}$$

$$= \frac{k^2 (k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2 [k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2 [k^2 + 4k + 4]}{4} = \frac{(k+1)^2 (k+2)^2}{4} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all natural numbers n.

Example 3.6.7 : Prove that, $3 + 3^2 + 3^3 + \ldots\ldots + 3^n = \frac{3}{2}(3^n - 1)$

Solution : Let $p(n) : 3 + 3^2 + 3^3 + \ldots\ldots + 3^n = \frac{3}{2}(3^n - 1)$

Step I : For $n = 1$.

$$\text{LHS} = 3$$

$$\text{RHS} = \frac{3}{2}(3^n - 1) = \frac{3}{2}(3 - 1) = \frac{3}{2}2 = 3$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore\ 3 + 3^2 + 3^3 + \ldots\ldots + 3^k = \frac{3}{2}(3^k - 1) \qquad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $3 + 3^2 + 3^3 + \ldots\ldots + 3^k + 3^{k+1} = \frac{3}{2}(3^{k+1} - 1)$

Example 3.6.11 : Prove that for each natural number, $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \ldots\ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$.

Solution : Let $p(n) : 1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots\ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Step I : For $n = 1$.

$$\text{LHS} = n(n+2) = (1)(1+2) = 3$$

$$\text{RHS} = \frac{n(n+1)(2n+7)}{6} = \frac{1(2)(9)}{6} = 3$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots\ldots + k(k+2) = \frac{k(k+1)(2k+7)}{6} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots\ldots + k(k+2) + (k+1)(k+3)$

$$= \frac{(k+1)(k+2)(2k+9)}{6}$$

Consider, LHS $= 1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots\ldots + k(k+2) + (k+1)(k+3)$

$$= [1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots\ldots + k(k+2)] + (k+1)(k+3)$$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \quad \ldots\text{from Equation (1)}$$

$$= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6k + 18]}{6} = \frac{(k+1)[2k^2 + 13k + 18]}{6}$$

$$= \frac{(k+1)(k+2)(2k+9)}{6} = \text{RHS}$$

$\therefore$ Result is true for $n = k + 1$

$\therefore$ By principle of mathematical induction result is true for any natural number.

Example 3.6.12 : Prove that,

$$1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Solution :

Let $p(n) : 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

Step I : For $n = 1$.

$$\text{LHS} = n(n+1)(n+2) = 1(1+1)(1+2) = 1\cdot2\cdot3 = 6$$

$$\text{RHS} = \frac{n(n+1)(n+2)(n+3)}{4} = \frac{(1)(2)(3)(4)}{4} = 6$$

$\therefore$ LHS = RHS

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore\ 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Consider,

$$\begin{aligned}
\text{LHS} &= 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= [1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots.. + k(k+1)(k+2)] + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{RHS}
\end{aligned}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural

Example 3.6.13 : Prove by mathematical induction, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$

Solution : Let $p(n):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$

Step I : For n = 1.

$$\text{LHS} = \frac{1}{1(1+1)}=\frac{1}{1\cdot 2}=\frac{1}{2}$$

$$\text{RHS} = \frac{n}{n+1}=\frac{1}{1+1}=\frac{1}{2}$$

$\therefore$ Result is true for n = 1.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$$\therefore \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1} \qquad \text{...(1)}$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Consider, LHS $= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$$= \left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{...from Equation (1)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.14 : Prove that $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots\ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$

Solution : Let p(n) : $\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots\ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$

Step I : For n = 1.

$$\text{LHS} = \frac{1}{(2n+1)(2n+3)} = \frac{1}{(2+1)(2+3)} = \frac{1}{3 \cdot 5} = \frac{1}{15}$$

$$\text{RHS} = \frac{n}{3(2n+3)} = \frac{1}{3(2+3)} = \frac{1}{3 \cdot 5} = \frac{1}{15}$$

$\therefore$ Result is true for n = 1.

Step II : Assume that statement is true for $n = k,\ k \in \mathbb{N}$.

$$\therefore \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots\ldots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad \ldots(1)$$

Step III : To prove statement for n = k + 1

i.e. to prove,

$$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots\dots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{(k+1)}{3(2k+5)}$$

Consider, LHS $= \frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots\dots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

$$= \left[\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\dots\dots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{(2k+3)(2k+5)}$$

$\therefore$ Result is true for n = k + 1.

$\therefore$ By Mathematical induction result is true for all n.

Example 3.6.17 : Prove that, $n(n^2 - 1)$ is divisible by 3.

Solution : Let $p(n) : n(n^2 - 1)$ is divisible by 3.

Step I : For $n = 1$.

$$\text{LHS} = n(n^2 - 1) = 1(1^2 - 1) = 1(1 - 1)$$

$$= 0 \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore$ $k(k^2 - 1)$ is divisible by 3.

$$\therefore \quad k(k^2 - 1) = 3m \text{ for some } m \in \mathbb{Z}$$

$$\therefore \quad k^3 - k = 3m$$

$$\therefore \quad k^3 = 3m + k \qquad \text{...(1)}$$

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $(k + 1)[(k + 1)^2 - 1]$ is divisible by 3.

Consider, $\text{LHS} = (k + 1)[(k + 1)^2 - 1] = (k + 1)(k^2 + 2k + 1 - 1)$

$$= (k + 1)(k^2 + 2k) = k^3 + 2k^2 + k^2 + 2k$$
$$= k^3 + 3k^2 + 2k$$
$$= 3m + k + 3k^2 + 2k \text{...from Equation (1)}$$
$$= 3m + 3k^2 + 3k$$
$$= 3(m + k^2 + k) \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for any natural number n.

Example 3.6.18 : Prove that, $2^{2n} - 1$ is divisible by 3.

Solution : Let $p(n) : 2^{2n} - 1$ is divisible by 3.

Step I : For $n = 1$.

Consider, $2^{2n} - 1 = 2^2 - 1 = 4 - 1 = 3$ which is divisible by 3.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore 2^{2k} - 1$ is divisible by 3.

$\therefore 2^{2k} - 1 = 3m$ for some $m \in \mathbb{Z}$

$\therefore 2^{2k} = 3m + 1$ ...(1)

Step III : Now to prove statement for $n = k + 1$

i.e. to prove $2^{2(k+1)} - 1$ is divisible by 3.

Consider,

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k}.2^2 - 1 = 2^{2k}.4 - 1$$

$$= (3m + 1)\cdot 4 - 1 \quad \text{...from Equation (1)}$$

$$= 3 \times 4 \times m + 4 - 1 = 3 \times 4 \times m + 3.$$

$$= 3\,(4m + 1) \text{ which is divisible by 3.}$$

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction result is true for any natural number n.

Example 3.6.19 : $3^{2n} + 7$ is divisible by 8, Prove by induction $\forall\ n \in \mathbb{N}$.

Solution : Let $p(n) : 3^{2n} + 7$ is divisible by 8.

Step I : For $n = 1$.

Consider, $3^{2n} + 7 = 3^2 + 7 = 9 + 7 = 16$ which is divisible by 8.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 3^{2k} + 7$ is divisible by 8.

$\therefore 3^{2k} + 7 = 8m$ for some $m \in \mathbb{Z}$

$\therefore 3^{2k} = 8m - 7$ ...(1)

Step III : Now to prove statement for n = k + 1

Consider,

$$3^{2(k+1)} + 7 = 3^{2k+2} + 7 = 3^{2k}.3^2 + 7 = 3^{2k}.9 + 7$$

$= (8m - 7)\ 9 + 7$ ...from Equation (1)

$= 8 \times 9 \times m - 63 + 7 = 8 \times 9 \times m - 56.$

$= 8\ (9m - 7)$ which is divisible by 8.

$\therefore$ Statement is true for n = k + 1

$\therefore$ By first principle of Mathematical induction statement is true for all $n \in \mathbb{N}$.

Example 3.6.20 : Prove $\forall n \in \mathbb{N}$ $n(n-1)(2n-1)$ is divisible by 6.

Solution : Let $p(n) : n(n-1)(2n-1)$ is divisible by 6.

Step I : For $n = 1$.

Consider,

$$n(n-1)(2n-1) = 1(1-1)(2-1)$$

$$= 0 \text{ which is divisible by } 6.$$

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k$, $k \in \mathbb{N}$.

$\therefore k(k-1)(2k-1)$ is divisible by 6.

$$\therefore k(k-1)(2k-1) = 6m \quad \text{for some } m \in \mathbb{Z}^+$$

$$\therefore (k^2 - k)(2k-1) = 6m$$

$$2k^3 - k^2 - 2k^2 + k = 6m$$

$$2k^3 = 6m + 3k^2 - k \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

Consider, $(k+1)[(k+1)-1)][2(k+1)-1]$

$$= (k+1)[k](2k+2-1) = (k^2+k)(2k+1)$$

$$= 2k^3 + k^2 + 2k^2 + k = (6m + 3k^2 - k) + 3k^2 + k$$

$$= 6m + 6k^2 = 6(m + k^2)$$

Which is divisible by 6.

$\therefore$ **Statement** is true for $n = k + 1$

$\therefore$ **By first** principle of mathematical induction statement is true for every **natural number.** n.

Example 3.6.21 : Show that $n^3 + 2n$ is divisible by 3 for $n \geq 1$.

Solution : Let $p(n)$: $n^3 + 2n$ is divisible by 3.

Step I : For $n = 1$.

Consider, $n^3 + 2n = (1)^3 + 2(1) = 1 + 2 = 3$ which is divisible by 3.

$\therefore$ Statement is true for $n = 1$.

Step II : Assume that statement is true for $n = k, k \geq 1$.

$\therefore k^3 + 2k$ is divisible by 3.

$\therefore k^3 + 2k = 3m$ for some $m \in \mathbb{Z}^+$

$$\therefore k^3 = 3m - 2k \quad \ldots(1)$$

Step III : Now to prove statement for $n = k + 1$

Consider,

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (3m - 2k) + 3k^2 + 3k + 2k + 3 \quad \ldots\text{from Equation (1)}$$

$$= 3m + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$$

which is divisible by 3.

$\therefore$ Statement is true for $n = k + 1$

$\therefore$ By principle of mathematical induction statement is true for all $n \geq 1$.

Example 3.6.22 : $8^n - 3^n$ is multiple of 5 $\forall n \geq 1$, Prove by mathematical induction.

Solution : Let $p(n) : 8^n - 3^n$ is multiple of 5.

Step I : For n = 1.

Consider,

$$8^n - 3^n = 8 - 3 = 5 \text{ which is multiple of 5.}$$

$\therefore$ Statement is true for n = 1.

Step II : Assume that statement is true for $n = k, k \in \mathbb{N}$.

$\therefore 8^k - 3^k$ is multiple of 5.

$$\therefore 8^k - 3^k = 5m \quad \text{for some } m \in \mathbb{Z}$$

$$\therefore 8^k = 5m + 3^k \quad \ldots(1)$$

Step III : Now to prove statement for n = k + 1

i.e. to prove $8^{k+1} - 3^{k+1}$ is multiple of 5.

Consider,

$$8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 3^k$$

$$= 8(5m + 3^k) - 3 \cdot 3^k \quad \ldots\text{from Equation (1)}$$

$$= 40m + 8 \cdot 3^k - 3 \cdot 3^k = 40m + (8-3)3^k = 40m + 5 \cdot 3^k = 5(8m + 3^k)$$

Which is multiple by 5.

$\therefore$ Statement is true for n = k + 1

$\therefore$ By first principle of mathematical induction statement is true for all natural numbers.