CHAPTER 1

Unit I

Introduction

Syllabus Topics

Field : Introduction to complex numbers, numbers in Python , Abstracting over fields, Playing with GF(2), Vector Space : Vectors are functions, Vector addition, Scalar-vector multiplication, Combining vector addition and scalar multiplication, Dictionary-based representations of vectors, Dot-product, Solving a triangular system of linear equations. Linear combination, Span, The geometry of sets of vectors, Vector spaces, Linear systems, homogeneous and otherwise.

Notations and Definitions of Terms Used

(1) $\mathbb{R}$ = Set of real numbers.

(2) $\mathbb{C}$ = Set of complex numbers = $\{(a + bi) \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

(3) GF (2) = A field that consists of 0 and 1

[GF (2) = Gatois field with 2 elements]

(4) $\mathbb{N}$ = Set of natural numbers

Syllabus Topic : Introduction to Complex Numbers

1.1 Complex Numbers

Mathematically set of complex numbers is denoted by $\mathbb{C}$ and it is given as

$$\mathbb{C} = \{ a + bi \mid a, b \text{ are real nos. }, i = \sqrt{-1}\}$$

Now addition in complex number $\mathbb{C}$ is defined as,

1. $(a + bi) + (c + di) = (a + c) + (b + d)\, i$

Example

$$(2 + 3i) + (4 + 7i) = 6 + 10\, i$$

Now multiplication in complex number $\mathbb{C}$ is defined as,

2. $(a + bi) \cdot (c + di) = ac + adi + bci - bd$

$$= (ac - bd) + (ad + bc)\, i$$

Example

$$(2 + 3i)(4 + 7i) = 8 + 14i + 12i - 21$$
$$= (8 - 21) + (14 + 12)\, i$$
$$= -13 + 26\, i$$

We note that addition of complex numbers

(i) Is closed i.e. (addition of complex numbers is complex number)

(ii) Is associative i.e. $((a + bi) + (c + di)) + (x + iy) = (a + bi) + ((c + di) + (x + iy))$

(iii) Has additive identity as zero (i.e.) $((a + bi) + (0 + 0i) = a + bi)$ and it is unique.

(iv) Has additive inverse i.e. if $\alpha \in \mathbb{C}$, then $\exists\, x \in \mathbb{C}$ such that $\alpha + x = 0 = x + \alpha$, thus $x = -\alpha$

(v) Is commutative (i.e. $(a + bi) + (c + di) = (c + di) + (a + bi)$) Also we note that multiplication in $\mathbb{C}$

(vi) Is closed (i.e. $(a + bi)(c + di)$ is also a complex number)

(vii) Is associative (i.e. $((a + bi)(c + di)(x + iy) = (a + bi)((c + di)(x + iy)))$

(viii) Has multiplicative identity in $\mathbb{C}$ i.e. $(1 + 0i)$ i.e. $((a + bi)(x + iy)$

$$= (a + bi) \Rightarrow x + iy = 1 + 0i$$

(ix) Has multiplicative inverse in $\mathbb{C}$ i.e. $(a + bi)(x + iy) = (1 + 0i)$

$$\left(\Rightarrow x + iy = \frac{1}{a + bc} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a2 + b^2}\right)^c\right)$$

(x) Also we note that multiplication in $\mathbb{C}$ is commutative (i.e. $(a + bi)(c + di)$

$$= (c + di)(a + bi))$$

Thus due to (i) – (x) the set of complex numbers $\mathbb{C}$ is called a field in mathematics.

Syllabus Topic : Numbers in Python

1.2 Numbers in Python

In python, 'i' which is complex number, it is represented by j i.e. a complex number a + bi – in python is represented as a + bj

Mathematically, $z = a + bi$

Real part of z = Re (z) = a and Imaginary part of z = Im (z) = b

In python, the code we declare as,

Code

```
Python 3.6.0 Shell
File Edit Shell Debug Options Window Help
>>> c=(3+2j)
>>> c
(3+2j)
>>> c.real
3.0
>>> c.imag
2.0
>>>
Ln: 30 Col: 4
```

One can perform all operations like +, – , * , / , ** in python on complex numbers. The python code for addition, subtraction, multiplication and division of two complex numbers is given as follows

Code

```
asmd.py - C:/Users/Administrator/AppData/Local/Programs/Python/Python36-32/asmd.py (3.6.0)
File Edit Format Run Options Window Help
#Python Program for adiition, substraction, multiplication and division
a=(4+2j)
b=(3-6j)
print("Addition of two complex numbers : ", a+b)
print("Subtraction of two complex numbers : ",a-b)
print("Multiplication of two complex numbers : ", a*b)
print("Division of two complex numbers : ",a/b)
Ln: 7 Col: 47
```

Output

```
Python 3.6.0 (v3.6.0:41df79263a11, Dec 23 2016, 07:18:10) [MSC v.1900 32 bit (In
tel)] on win32
Type "copyright", "credits" or "license()" for more information.
>>>
 RESTART: C:/Users/Administrator/AppData/Local/Programs/Python/Python36-32/asmd.
py
Addition of two complex numbers :  (7-4j)
Subtraction of two complex numbers :  (1+8j)
Multiplication of two complex numbers :  (24-18j)
Division of two complex numbers :  (-0+0.6666666666666666j)
```

Mathematically, If $z \in \mathbb{C}$, $z = x + iy$ then $|z|^2 = x^2 + y^2$

Also $|z|^2 = z\bar{z} = (x + iy)(x - iy)$ ...(1)

In python, we obtain $|z|$ instead of $|z|^2$ and it is called by function abs (z)

Thus

$z = 5 + 4j$

```
>>> a=(5+4j)
>>> abs(a)
6.4031242374328485
```

So in python formula (1) will become

$$|z|^2 = (z\,.\,\text{real})^2 + (z\,.\,\text{imag})^2$$

We also note that

If $z \in \mathbb{C}$, $z = x + iy$

Then $z = r.\,e^{i\theta}$, where $r^2 = x^2 + y^2$ and $\theta = \arg(z)$

Now to find θ we locate z in complex plane and then as per its position we find θ using one of the formula.

(i) z in I quadrant $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(ii) z in II quadrant $\theta = \pi - \tan^{-1}\left(\frac{y}{x}\right)$

(iii) z in III quadrant $\theta = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$

(iv) z in IV quadrant $\theta = -\tan^{-1}\left(\frac{y}{x}\right)$

Syllabus Topic : Abstracting over Fields and Playing with GF(2)

1.3 Abstracting over Fields and Playing with GF(2)

Galois field with two elements is denoted by GF(2). It has two elements 0 and 1. Since it is a field there are two binary operations defined ('+' addition and '·' multiplication). The behaviour of the elements '0' and '1' w.r.t '+' and '·' is given in these two table.

+	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

These tables are called composition tables.

In addition, observe that 1 + 1 = 0 because addition is module 2. It is equivalent to Exclusive OR.

In python there is a module GF2 import the GF2 module. If GF2 module is not there then type the following code and save it to in python library and then import it. The name given to the following code is GF2.py, so whenever you want to perform any operation regarding fields you simply import it.

Code

```
GF2.py - C:/Python27/Lib/GF2.py
File Edit Format Run Options Windows Help
from numbers import Number
class One:
    def __add__(self, other): return self if other == 0 else 0
    __sub__ = __add__
    def __mul__(self, other):
        if isinstance(other, Number):
            return 0 if other == 0 else self
        return other
    def __div__(self, other):
        if other == 0: raise ZeroDivisionError
        return self
    __truediv__ = __div__
    def __rdiv__(self,other): return other
    __rtruediv__ = __rdiv__
    __radd__ = __add__
    __rsub__ = __add__
    __rmul__ = __mul__
    def __lt__(self,other): return False
    def __eq__(self, other):
        if isinstance(other, self.__class__) or other==0:
            return other != 0
        else:
            raise TypeError
    def __hash__(self): return 1
    def __str__(self): return 'one'
    __repr__ = __str__
    def __neg__(self): return self
    def __bool__(self): return True
    def __format__(self, format_spec): return format(str(self),format_spec)
one = One()
zero = 0
Ln: 18 Col: 4
```


Output

```
Python Shell
File Edit Shell Debug Options Windows Help
>>> from GF2 import one
>>> one*one
one
>>> one+one
0
>>> one+0
one
>>> one*0
0
>>> -one
one
>>>
Ln: 51 Col: 4
```

Example 1.3.1: For each of the following problems calculate the answer over GF(2)

a. 1 + 1 + 1 + 0

Soln. :

```
Python Shell
File Edit Shell Debug Options Windows Help
>>> one+one+one+0
one
>>> (one+one+one)*(0+one+one)
0
>>>
Ln: 61 Col: 4
```

Syllabus Topic : Vector Space - Vectors are Functions

1.4 Vector Space

Mathematically vector space is a set V over a field $\mathbb{F}$, with binary operation '+' (addition) and '·' a scalar multiplication satisfying the following properties:

(i) If u and $v \in V$, then $u + v \in V$

(ii) $\alpha \in \mathbb{F}$, $v \in V$, then $\alpha v \in V$.

(iii) $u + v = v + u$,

(iv) $(u + v) + \omega = u + (v + \omega)$

(v) $\exists\, 0 \in V$, such that $v + 0 = v = 0 + v$

$\exists\, \bar{x} \in V$ such that $v + \bar{x} = 0 = \bar{x} + v$ (i.e. $\bar{x} = -v$)

$\alpha \cdot (u + v) = \alpha u + \alpha v$, where $\alpha \in \mathbb{F}$.

$(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$, where $\alpha, \beta \in \mathbb{F}$.

(ix) $(\alpha\beta) \cdot v = \alpha (\beta \cdot v)$, where $\alpha, \beta \in \mathbb{F}$

(x) $1 \cdot v = v$

Note : Elements belonging to field are known as scalars.

When any set satisfies above properties (i) to (x) over $\mathbb{F}$ then it is called a vector space. Generally $\mathbb{F}$, for us will be $\mathbb{R}$ or $\mathbb{C}$ or GF(2).

Note that $\mathbb{R}, \mathbb{C}$, $\mathbb{F}$ they are also vectorspaces.

Any vector $v = (v_1, v_2, \ldots, v_n) \in V$ and $v_1, v_2, \ldots, v_n$ are called its components.

Note : $\mathbb{F}^D$: = The set of functions from set D to the field $\mathbb{F}$.

$\mathbb{F}^d$: = The set of functions from $\{0, 1, 2, \ldots, d-1\}$ to $\mathbb{F}$.

1.4.1 Vectors are Functions

We shall treat vectors as functions because it helps us to build further concepts.

Definition

For a finite set D and field $\mathbb{F}$, a D-vector over $\mathbb{F}$ is a function from D to $\mathbb{F}$.

Thus, $\mathbb{F}^D$ will denote the set of functions with domain D and co domain $\mathbb{F}$.

Example

(a) $\mathbb{R}^{CLUB}$: The set of all CLUB - vectors over $\mathbb{R}$.

(b) $GF(2)^{\{0, 1, 2, \ldots, n-1\}}$ is defined as the set of n - vectors over GF(2).

Sparse vectors

A vector, most of whose components are zero is called a sparse vector. If no more than k of the components of a vector are zero, we say the vector is k-sparse. A k-sparse vector can be represented using space proportional to k.

In the following code you can see that there are four zero elements in the vector.

Code

```
In [186]: import numpy as np
In [187]: v=np.array([2,0,0,0,4,0,1])
In [188]: v
Out[188]: array([2, 0, 0, 0, 4, 0, 1])
```

What can be represented with vectors?

1 **Binary string**

An n - bit binary string 100011101 can be represented as an n-vector over GF(2) [1, 0, 0, 0, 1, 1, 1, 0, 1]

2 **Probability distribution**

If a die is tossed six times than every output of die $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ can be a vector.

3 **Image**

A B-W image of size (1024×768) can be seen as a function from the set of pairs $\{(i, j) \mid 0 \le i < 1024, 0 \le j < 768\}$ to real numbers and hence a vector.

The following code illustrates the vector representation for binary number, probability distribution and for character.

Code

```
In [176]: import numpy as np

In [177]: v=np.array([1,0,0,0,0,0,1])

In [178]: v
Out[178]: array([1, 0, 0, 0, 0, 0, 1])

In [179]: v1=np.array(['a','b','c','d'])

In [180]: v1
Out[180]:
array(['a', 'b', 'c', 'd'],
      dtype='<U1')

In [181]: v2=np.array([2.3,4.4,5.6,0.5])

In [182]: v2
Out[182]: array([ 2.3,  4.4,  5.6,  0.5])
```

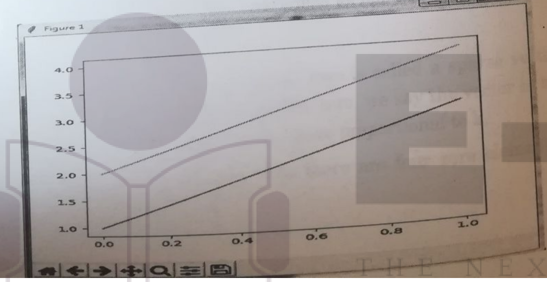
The vectors are plot as follows in python

```
In [16]: import matplotlib.pyplot as plt

In [17]: a=[[1,2],[3,4]]

In [18]: plt.plot(a)
Out[18]:
[<matplotlib.lines.Line2D at 0x5d2d270>,
 <matplotlib.lines.Line2D at 0x5d2dd10>]
```

Output



Syllabus Topic : Vector Addition

1.5 Vector Addition

If v, u are vectors in Vector space V over $\mathbb{F}$

And $v = (v_1, v_2, \ldots\ldots, v_n)$

$u = (u_1, u_2, \ldots, u_n)$

Then $v + u = (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$

Addition of vectors means adding their corresponding components to obtain a new vector.

Here all the properties (i), (iii), (iv), (v) defined in vector space definition hold. (are valid).

Python code for vector addition is given as follows

Code

```
In [162]: import numpy as np

In [163]: u=np.array([4,5,10])

In [164]: v=np.array([3,4,5])

In [165]: c=u+v

In [166]: c
Out[166]: array([ 7,  9, 15])
```

You can also write function for vector addition. It is given as follows

```
def addvector(v,w):
    return [v[0]+w[0], v[1]+w[1]]
```

tput

```
>>> addvector([1,1],[2,2])
[3, 3]
```

Syllabus Topic : Scalar-Vector Multiplication

1.6 Scalar Multiplication

If $v = (v_1, v_2, \ldots, v_n) \in V, \alpha \in \mathbb{F}$

Then $\alpha \cdot v = \alpha (v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$

Example

$\alpha = 3$

$v = [4, 5, 10]$

$\alpha \cdot v = [3.4, 3.5, 3.10]$

$\alpha \cdot v = [12, 15, 30]$

Python code for the above example is given as follows

Code

```
In [150]: import numpy as np

In [151]: v=np.array([4,5,10],dtype=int)

In [152]: v
Out[152]: array([ 4,  5, 10])

In [153]: a=np.array([3.4,3.5,3.10],dtype=int)

In [154]: a
Out[154]: array([3, 3, 3])

In [155]: c=a*v

In [156]: c
Out[156]: array([12, 15, 30])
```


Syllabus Topic : Combining Vector Addition and Scalar Multiplication

1.7 Combination of Vector Addition and Scalar Multiplication

Before we begin this topic,

If point $P = [3, 2], Q = [2, 4]$

then vector $PQ = Q - P = [2, 4] - [3, 2]$

$= [2 - 3, 4 - 2]$

$\overrightarrow{PQ} = [-1, 2]$

Calculate $\overrightarrow{QP}$?

If $O = [0, 0]$ $\overrightarrow{OP} = [3 - 0, 2 - 0] = [3, 2]$

Thus the points forming segment from

$[0, 0]$ to $[3, 2]$ are $\{\alpha [3, 2] \mid \alpha \in \mathbb{R}, 0 \le \alpha \le 1\}$

Now if add $[0.5, 1]$ to $[3, 2]$ Then we have $[3.5, 3]$

Thus $\{\alpha [3, 2] + [0.5, 1] \mid \alpha \in \mathbb{R}, 0 \le \alpha \le 1\}$

$\{[3\alpha + 0.5, 2\alpha + 1] \mid \alpha \in \mathbb{R}, 0 \le \alpha \le 1\}$

Meaning that first the vector $\overrightarrow{OP}$ will be scaled by α times then it will be translated by vector $[0.5, 1]$.

Convex Combination

An expression of the form $\alpha u + \beta v$, where $\alpha, \beta \ge 0$, $v \in V$ and $\alpha + \beta = 1$ is called convex combination of u and v.

Example

Consider, $\alpha [3, 2] + [0.5, 1]$

$= \alpha [[3.5, 3] - [0.5, 1]] + [0.5, 1] = \alpha [3.5, 3] - \alpha [0.5, 1] + [0.5, 1]$

$= \alpha [3.5, 3] + (1 - \alpha) [0.5, 1]$

Put $1 - \alpha = \beta$

$\therefore \alpha [3, 2] + [0.5, 1] = \alpha [3.5, 3] + \beta [0.5, 1]$

The following is true for any pair of u and v over $\mathbb{R}$. The u to v line segment consist of the set of convex combinations of u and v.

Affine combination

An expression of the form $\alpha u + \beta v$, where $\alpha + \beta = 1$, $\alpha, \beta \in \mathbb{R}$ is called an affine combination.

Syllabus Topic : Dictionary-based Representations of Vectors

1.8 Dictionary-based Representations of Vectors

The dictionary in python is a key value pair. The values are separated from the key by using the colon (:). The vector representation using dictionary in python is done in the following way :

```
In [32]: a={ 'on': 1, 'Spain': 1, 'in': 1, 'plain': 1, 'the': 2, 'mainly': 1, 'rain': 1, 'falls': 1}

In [33]: a
Out[33]:
{'Spain': 1,
 'falls': 1,
 'in': 1,
 'mainly': 1,
 'on': 1,
 'plain': 1,
 'rain': 1,
 'the': 2}

In [34]:
```

```
In [34]: a={ '0': [1,2], '1': [1,1], '2': [1,3]}

In [35]: a
Out[35]: {'0': [1, 2], '1': [1, 1], '2': [1, 3]}

In [36]:
```

Syllabus Topic : Dot Product

1.9 Dot Product

If, $v = [v_1, v_2, \ldots, v_n] \in V$

$u_1 = [u_1, u_2, \ldots, u_n] \in V$

Then dot product of u and v is,

$$u \cdot v = u_1 v_2 + u_2 v_2 + \ldots + u_n v_n$$

$$= \sum_{i=1}^{n} u_i v_i$$

Note : $u \cdot v = v \cdot u$. Also (u·v) is a scalar

Example

(1) $u = (-4, 5),\ v = \left(\frac{2}{3}, 4\right)$

Then $u \cdot v = \left(-4 \cdot \frac{2}{3} + 5 \cdot 4\right) = \frac{-8}{3} + 20 = \frac{60-8}{3} = \frac{52}{3}$

(2) Consider vectors u and v over GF(2)

Let, $v = 11111$, $u = 10101$

$u \cdot v = 10101 \cdot 1111 = (1 \cdot 1) + (0 \cdot 1) + (1 \cdot 1) + (0 \cdot 1) + (1 \cdot 1)$

$= 1 + 0 + 1 + 0 + 1$

$u \cdot v = 1$

(3) Find $u \cdot v$, if $u = (u_1, u_2, \ldots, u_n)$

$v = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$

Python code for the dot product is given as follows

Code

```
In [157]: import numpy as np

In [158]: u=np.array([4,5,10])

In [159]: v=np.array([3,4,5])

In [160]: c=np.dot(u,v)

In [161]: c
Out[161]: 82
```

Syllabus Topic : Solving a Triangular System of Linear Equations

1.10 Solving a System of Linear Equation

Here, we consider a system

$$AX = B$$

Where, $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

We consider the easiest case of the system to solve

Case 1 : Let A be upper triangular matrix

i.e. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix}$

Thus the system looks as follows

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & & a_{n-i\,n-1} & a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

This system is solved by backward substitution method i.e.

First we find (i) $x_n \Rightarrow x_n = b_n / a_{nn}$

(ii) x_{n-1} is obtained by solving the 2nd last equation

(iii) x_{n-2} is obtained by solving 3rd last equation using x_{n-1} and x_n

So we proceed up to first equation.

Where we obtain x_1 by substituting all the values of $x_n, x_{n-1}, \dots, x_2$

Example

Consider the following system

$$1x_1 - 3x_2 - 2x_3 = 7$$
$$2x_2 + 4x_3 = 4$$
$$-10x_3 = 12$$

In matrix notation

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 12 \end{bmatrix}$$

$$x_3 = \frac{-12}{10} = \frac{-6}{5}$$

Now $2x_2 + 4x_3 = 4$

$$\therefore x_2 = \frac{4 - 4x_3}{2} = \frac{4 - 4\left(\frac{-6}{5}\right)}{2}$$

$$= 2 + 2\left(\frac{6}{5}\right)$$

$$= \frac{10 + 12}{5} = \frac{22}{5}$$

$$\therefore x_1 - 3x_2 - 2x_3 = 7$$

$$x_1 - 3\left(\frac{22}{5}\right) - 2\left(\frac{-6}{5}\right) = 7$$

$$x_1 - \frac{66}{5} + \frac{12}{5} = 7$$

$$x_1 - \frac{54}{5} = 7$$

$$x_1 = 7 + \frac{54}{5}$$

$$x_1 = \frac{35 + 54}{5} = \frac{89}{5}$$

Code

```
In [106]: import numpy as np

In [107]: a=np.array([[1,-3,-2],[0,2,4],[0,0,-10]])

In [108]: a
Out[108]:
array([[  1,  -3,  -2],
       [  0,   2,   4],
       [  0,   0, -10]])

In [109]: b=np.array([7,4,12])

In [110]: c=np.linalg.solve(a,b)

In [111]: c
Out[111]: array([ 17.8,   4.4,  -1.2])
```

☞ **Caution**

If diagonal elements are zero then the method does not work.

The following is the python code for the linear equation $3x - 9y = -42$ and $2x + 4y = 2$.

Code

```
In [95]: import numpy as np

In [96]: a=np.array([[3,-9],[2,4]])

In [97]: a
Out[97]:
array([[ 3, -9],
       [ 2,  4]])

In [98]: b=np.array([-42,2])

In [99]: c=np.linalg.solve(a,b)

In [100]: print(c)
[-5.  3.]
```


Syllabus Topic : Linear Combination

1.11 Linear Combination of Vector

A vector v is said to the linear combination of vector $v_1, v_2, \ldots, v_n$ if v can be expressed as :

$$v = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n$$

Where k_i's are scalars.

☞ **Example**

1. Consider $u_1 = [3, 4, 5, 6]$

$u_2 = [0, 1, 0, -1]$

Then $3u_1 + 2u_2 = [9, 14, 15, 16]$

The python code for above example is

Code

```
In [87]: a=np.array([3,4,5,6])

In [88]: b=np.array([0,1,0,-1])

In [89]: x=3*a

In [90]: y=2*b

In [91]: z=x+y

In [92]: z
Out[92]: array([ 9, 14, 15, 16])
```

2. In face recognition or image recognition

Average image $= \frac{1}{3}(\text{image}_1) + \frac{1}{3}(\text{image}_2) + \frac{1}{3}(\text{image}_3)$

Syllabus Topic : Span

1.12 Span

The set of all linear combinations of vectors $v_1, v_2, \ldots v_n$ is called the span of the vectors i.e.

$$\text{Span}\{v_1, v_2, \ldots v_n\} = \left\{ \sum_{i=1}^{n} k_i v_i \;\middle|\; \begin{matrix} n \in \mathbb{N} \text{ and} \\ k_i \in \mathbb{F} \end{matrix} \right\}$$

☞ Generators of vector space

Consider the vector space V and $B = \{v_1, v_2, \ldots v_n\}$, $v_i \in V$ then B is said to the generator set of V if every vector v in the vector space V belongs to the Span $\{v_1, v_2, \ldots v_n\}$

i.e. $\text{Span}\{v_1, v_2, \ldots, v_n\} = V$

Then we say the set B is a generator set of V or basis for V.

(1) It is not unique set but if there are two sets B_1 and B_2 which are generators of V then they have same number of elements.

☞ Example

(1) For $\mathbb{R}^2$, $B = \{[1, 0], [0, 1]\}$ this is the set of standard generators.

(2) $\mathbb{R}^n$, $B = \{[1, 0, \ldots, 0], [0, 1, 0, 0, \ldots 0], [0, 0, 1, 0, \ldots, 0] \ldots\ldots, [0, \ldots, 0, 1]\}$

These n vectors are standard generators

By generator $B = \{[1, 0], [0, 1]\}$ of $\mathbb{R}^2$ we mean that any vector can be expressed as linear combination of these vectors.

☞ Example

Consider $[4, 3] = 4 [1, 0] + 3 [0, 1]$

Similarly for $\mathbb{R}^3$ $B = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ is a set of standard generators.

Consider $(-2, -7, 47) = -2(1, 0, 0) + (-7)(0, 1, 0) + 47(0, 0, 1)$

We made the remark that B is not a unique set.

Consider $B = \{(2, 3), (0, 1)\}$ this set is also a set of generator vectors for $\mathbb{R}^2$ then

$$(4, 3) = 2(2, 3) + (-3)(0, 1)$$

The above set B is non standard generators of $\mathbb{R}^2$

A stronger notion of Basis shall be dealt in further topics.

Syllabus Topic : The Geometry of Sets of Vectors

1.13 Geometry of Sets of Vectors

Consider $\mathbb{R}$, let v be a non zero vector in $\mathbb{R}$ span $\{v\} = \{\alpha v \mid \alpha \in \mathbb{R}\}$

Meaning of this is that every vector in $\mathbb{R}$ can be expressed as some multiple of the vector v.

The above set forms the line from origin to the point v. A line is one dimensional object.

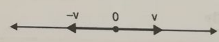


Fig. 1.13.1

Consider $\mathbb{R}^2$, span $\{v_1 = [1, 0], v_2 = [0, 1]\}$ mean of the span is that every vector can be expressed a some linear more over v_1, v_2 are generators.

Thus geometrically span $\{v_1, v_2\} = \mathbb{R}^2$

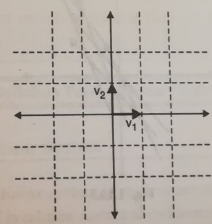


Fig. 1.13.2

What is span of $\{[1, 2], [3, 4]\}$ to check $v_1 = [1, 2]$, $v_2 = [3, 4]$ are generators or not?

$$\begin{aligned} \alpha v_1 + \beta v_2 &= 0 \\ \alpha(1, 2) + \beta(3, 4) &= (0, 0) \\ (\alpha + 3\beta, 2\alpha + 4\beta) &= (0, 0) \\ \alpha + 3\beta &= 0 \end{aligned}$$

$2\alpha + 4\beta = 0$

On solving we get,

$$\beta = 0 \text{ and } \alpha = 0$$

Now,

$$(x, y) = \alpha(1, 2) + \beta(3, 4)$$
$$(x, y) = (\alpha + 3\beta, 2\alpha + 4\beta)$$
$$x = \alpha + 3\beta$$
$$y = 2\alpha + 4\beta$$
$$2x = 2\alpha + 6\beta$$
$$\left(\frac{2x-4}{2}\right) = \beta ; \quad \alpha = -x + 3\left(\frac{2x-y}{2}\right) = \frac{-2x + 6x - 3y}{2}$$
$$\alpha = \frac{4x - 3y}{2}$$

Thus any vector of $(x, y) \in \mathbb{R}^2$

$$(x, y) = \alpha[v_1] + \beta[v_2]$$

Hence span of $\{[1, 2], [3, 4]\} = \mathbb{R}^2$

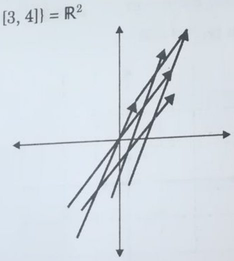


Fig. 1.13.3

Syllabus Topic : Vector Spaces

1.14 Vector Spaces

☞ **Subspace**

Let V be a vectorspace over $\mathbb{F}$ and W be subset of V then we say W is a subspace of vector space V.

if (i) $0 \in W$

(ii) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$.

(iii) For $\alpha \in \mathbb{F}$ and $w \in W$, then $\alpha w \in W$.

☞ **We define**

1. **Affine Hull**

The set of all affine combinations of a collection of vectors is called the affine hull of that collection.

2. **Affine space**

It is a set obtained by translating every vector $v \in V$ by α i.e.

$$\mathcal{A} = \{\alpha + v \mid v \in V\}$$

☞ **Points to remember**

- Span of zero vectors forms a point – a zero dimensional object which is origin.
- Span of a non zero vector forms a line through the origin – a 1-dimensional object - or a point, the origin.
- The span of 2 vectors forms a plane through the origin – a 2- dimensional object or a line passing through the origin or a point, the origin.

Syllabus Topic : Linear Systems

1.15 Homogeneous Linear System

We already saw the linear system in matrix notation looks like AX = B where A is coefficient matrix, X is column matrix of unknowns and B is constant matrix.

In homogenous system of equation B = O matrix. To any homogenous system 0 is a trivial solution. Thus all solutions pass through origin.

Generally, set lines, planes are solution to the homogenous system.

☞ **Example**

$2x - 3y = 0$, $(x, y) \in \mathbb{R}^2$, straight line is the solution set of a homogenous linear equation.

☞ **Understanding their solutions**

Note, $Ax = B$ is a general linear system of equation

Where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{mn} & \cdots & a_{mn} \end{bmatrix}, \quad X \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Now, if $m = n$, i.e. number of equations is same as number of unknowns then A will be a square matrix.

If det (A) ≠ 0 then the system has unique solution

☞ **Example**

$$2x + 3y + 5z = 10$$
$$3x + 6y + 2z = 11$$
$$x + y + 4z = 6$$

If number of variables is more than the number of equations then there is infinity many solutions.

☞ **Examples**

1 $2x + 35y - 39z = -12$

$$6x + 6y - 7z = 8$$
$$12x - 15y + 16z = 28$$

Geometrically three planes intersect in a line.

2 $2x + 3y + 4z = 7$

$$4x + 6y + 8z = 14$$
$$6x + 9y + 12z = 21$$

In short it is only one equation $2x + 3y + 4z = 7$

Geometrically all the three planes are coincident.

3 Consider $2x + 3y + 4z = 5$

$$2x + 3y + 4z = 13$$

Now geometrically these two planes are parallel thus there is no solution to them.

4 Consider $x + 4y - 6z = 1$

$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$

Here we have no solution

In Homogenous system we have $B = O$. If det (A) ≠ 0 then for $Ax = O$ we have trivial solution otherwise we have infinitely many solutions.

Exercise

Q. 1 Express the following in the standard form of complex number $(x + iy)$

(a) $\frac{3+2i}{2-3i}$ (b) $\frac{2-\sqrt{3}\,i}{1+i}$ (c) $\frac{1+i}{1-i}$

Q. 2 Find the complex conjugate of

(a) $\frac{3+5i}{1+2i}$ (b) $\frac{3+2i}{2-3i}$ (c) $\frac{2+3i}{1-i}$

Q. 3 Express the following in polar form and find their arguments.

(a) $\sqrt{3} + i$ (b) $\sin\theta + i\cos\theta$

(c) $\frac{1+2i}{1-3i}$ (d) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$

(e) $\frac{-1}{2} + i\frac{\sqrt{3}}{2}$ (f) $\frac{\sqrt{3}}{2} - \frac{i}{2}$

Vectors

Q. 4 For $u = [0, 4]$ and $v = [-1, 3]$, find vector $u + v$, $v - u$, $u - v$, $3v - 2u$.

Q. 5 For $u = [0, 1, 1]$ and $v = [1, 1, 1]$ over GF(2), find $v + u$ and $v + u + u$.

Q. 6 Find a vector $x = [x_1, x_2, x_3, x_4]$ over GF (2) satisfying the following linear equations

$1100 \cdot x = 1$; $1010 \cdot x = 1$; $1111 \cdot x = 1$

Show that $x + 1111$ also satisfies the equations.

Q. 7 For each of the following pair of vectors over $\mathbb{R}$, evaluate the expression $u \cdot v$:

(a) $u = [1, 0]$, $v = [5, 1616]$

(b) $u = [1, 2, 3]$, $v = [3, 2, 1]$

(c) $u = \left[\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$, $v = \left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right]$

Q. 8 Calculate the magnitude of the following vector and find their unit vector using formula $\hat{u} = \frac{u}{\| u \|}$, where $\| u \|$ is the magnitude of vector given by $\| u \| = \sqrt{\sum_i u_i^2}$

(a) $u = (1, 1, 1)$ (b) $u = (0, 11, 0)$

(c) $u = (1, -2, 3)$ (d) $u = (-1, -1, 4)$

Vector spaces

Q. 9 Prove or give a counter example

(a) "$\{[x, y, z] \mid x, y, z \in \mathbb{R}, x + y + z = 1\}$ is a vectorspace"?

(b) "$\{[x, y, z] \mid x, y, z \in \mathbb{R}, x + y + z = 0\}$ is a vectorspace"?

(c) "$\{[x_1, x_2, x_3, x_4, x_5] \in \mathbb{R}^5 \mid x_2 = 0 \text{ and } x_3 = 0\}$ is a vectorspace"?

Q. 10 Determine which of the following subsets of $\mathbb{R}^n$ are subspace of $\mathbb{R}^n$ $(n > 2)$?

(a) $\{x \mid x_i \geq 0\}$ (b) $\{x \mid x_1 = 0\}$

(c) $\left\{x \mid \sum_{j=1}^{n} x_j = 1\right\}$ (d) $\{x \mid x_1, x_2 = 0\}$

(e) $\{x \mid Ax = b, \text{ where } A_{m \times n} \neq 0 \text{ and } b_{m \times 1} \neq 0\}$

Q. 11 Which of the following is a set of generators for $\mathbb{R}^3$?

(a) $\{(1, 1, 1,)\}$ (b) $\{(1, 0, 0), (0, 0, 1)\}$

(c) $\{(1, 2, 1), (2, 0, -1), (4, 4, 1)\}$ (d) $\{(1, 2, 1), (2, 0, -1), (4, 4, 0)\}$

Q. 12 Let $V = P = \left\{\sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{R}, n \in \mathbb{N}\right\}$ be the set of all polynomials in one with real coefficients. The addition and scalar multiplication is defined as

(i) $\sum_{i=0}^{m} a_i x^i + \sum_{j=0}^{n} b_j x^j = \sum (a_r + b_r) x^r$

Where $a_r = 0$ if $r > m$ and $b_r = 0$ if $r > n$

(ii) $\alpha\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{m} \alpha a_i x^i$

Then prove that V is a vectorspace over the field $\mathbb{R}$.

Q. 13 Prove that the set of all $n \times m$ matrices whose entries are real is a vectorspace over $\mathbb{R}$ with usual matrix addition and scalar multiplication i.e.

$(A = (a_{ij}))$ and $B = (b_{ij})$ then

$A + B := (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

Also, $\alpha A = \alpha(a_{ij}) := (\alpha a_{ij})$

Q. 14. Show that the set of all real symmetric matrices

$$S_n = \{(a_{ij}) \mid aij \in \mathbb{R}, a_{ij} = a_{ji}, \forall 1 \leq i, j \leq n\}$$

is a vectorspace under usual addition of matrices and scalar multiplication as defined above in question 13.

Q. 15 Show that the set of all real skew symmetric matrices

$A_n = \{a_{ij} \mid a_{ij} = -a_{ji} \forall 1 \leq i, j \leq n, a_{ij} \in \mathbb{R}\}$ is also a vectorspace under the matrix addition and scalar multiplication.

Q.16 For each of the following problems calculate the answer over GF(2)

(a) $1 + 1 + 1 + 0$

(b) $1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1$

(c) $(1 + 1 + 1 + 1) \cdot (1 + 1 + 1)$

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