Unit II

CHAPTER 2

Matrices & Basis

Syllabus Topics

Matrix : Matrices as vectors, Transpose, Matrix-vector and vector-matrix multiplication in terms of linear combinations, Matrix-vector multiplication in terms of dot-products, Null space, Computing sparse matrix-vector product, Linear functions, Matrix-matrix multiplication, Inner product and outer product, From function inverse to matrix inverse

Basis : Coordinate systems, Two greedy algorithms for finding a set of generators, Minimum Spanning Forest and GF(2), Linear dependence, Basis, Unique representation, Change of basis, first look, Computational problems involving finding a basis Dimension : Dimension and rank, Direct sum, Dimension and linear functions, The annihilator.

Syllabus Topic : Matrix – Matrices as Vectors

2.1 Matrix

Matrix is rectangular arrangement of elements in 'm' rows and 'n' columns.

In general, a matrix is denoted by capital letters say A, B, ...etc. and its corresponding elements are denoted by a_{ij}, b_{ij}, ... etc.

Thus we say matrix $A = [a_{ij}]_{m \times n}$

Where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, \ldots, n$.

$m \times n$ is known as the order of matrix and if $m = n$ then the matrix is known as square matrix.

In python we denote matrix by A [i, j] .

Note: The rows and columns of any matrix are vectors.

Thus in Python, i^{th} row vector is,

[A[i, 0], A[i, 1], A[i, 2], . . ., A[i, m – 1]]

j^{th} column vector is

[A[0, j], A[1, j], A[2, j], . . ., A[n – 1, j]]

Matrix by listing

A matrix can be represented by listing its rows or by listing its columns.

Representing matrix by its row list we say,

A [i, j] = L [i] [j], for every $0 \le i \le m$, $0 \le j \le n$

Example 1

$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -4 & -5 \end{bmatrix}$ will be represented as [[1, 2, – 3], [1, – 4, – 5]]

Representing matrix by its columns

A [i, j] = L [j] [i], $0 \le i < m$, $0 \le j < n$

For the same example

[[1, 1], [2, – 4], [– 3, – 5]]

Example 2

Write a nested comprehension whose value is list of - row - list representation of matrix given below.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Output

Python 3.6.0 Shell

File Edit Shell Debug Options Window Help

```
>>> [[0 for j in range(4)] for i in range(3)]
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0]]
>>> |
```

Ln: 110 Col: 4

A matrix has rows and columns; let the rows be denoted by R and columns C. Now say R will be a set of rows $\{R_1, R_1, \ldots, R_m\}$ these R_i's are row vectors and C will be a set of columns $\{C_1, C_2, \ldots, C_n\}$ these C_j's are column vectors thus the matrix can be seen as the Cartesian product of $R \times C$.

	C_1	C_2	C_3
R_1	1	2	3
R_2	10	20	30
R_3	100	200	300

Vectors are nothing but one dimensional array. The following python code illustrates the row vectors R_1, R_2 and R_3 of the matrix.

Code

```
In [26]: import numpy as np

In [27]: a=np.array([[1,2,3],[10,20,30],[100,200,300]])

In [28]: a
Out[28]:
array([[  1,   2,   3],
       [ 10,  20,  30],
       [100, 200, 300]])

In [29]: a[0,:]
Out[29]: array([1, 2, 3])

In [30]: a[1,:]
Out[30]: array([10, 20, 30])

In [31]: a[2,:]
Out[31]: array([100, 200, 300])
```

For column vector C_1, C_2, C_3 the following is the python code.

Code

```
In [32]: import numpy as np

In [33]: a=np.array([[1,2,3],[10,20,30],[100,200,300]])

In [34]: a
Out[34]:
array([[  1,   2,   3],
       [ 10,  20,  30],
       [100, 200, 300]])

In [35]: a[:,0]
Out[35]: array([  1,  10, 100])

In [36]: a[:,1]
Out[36]: array([  2,  20, 200])

In [37]: a[:,2]
Out[37]: array([  3,  30, 300])
```


To retrieve the matrix dimensions you can use shape attribute.

```
In [41]: import numpy as np

In [42]: a=np.array([[1,2,3],[10,20,30],[100,200,300]])

In [43]: a.shape
Out[43]: (3, 3)

In [44]:
```

In python you can implement the matrix by reshaping it. The following example illustrates it.

```
In [44]: a = np.array([1,1,2,3,5,8,13,21,34]).reshape(3,3)

In [45]: a
Out[45]:
array([[ 1,  1,  2],
       [ 3,  5,  8],
       [13, 21, 34]])

In [46]: a.shape
Out[46]: (3, 3)
```

If you want to create a zero matrix then you can create is as follows

```
In [49]: np.zeros((3,4))
Out[49]:
array([[ 0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.]])

In [50]: np.zeros((3,4),dtype=int)
Out[50]:
array([[0, 0, 0, 0],
       [0, 0, 0, 0],
       [0, 0, 0, 0]])
```

Dictionary is also used to implement the matrix. In the following example the key and the value assigned to the key is given. The key is the row and the column number.

```
In [83]: matrix = {(0, 0): 1, (0, 1): 2, (1, 0): 3, (1,1): 4}

In [84]: matrix
Out[84]: {(0, 0): 1, (0, 1): 2, (1, 0): 3, (1, 1): 4}

In [85]: matrix[(1,1)]
Out[85]: 4

In [86]: matrix[(0,0)]
Out[86]: 1
```

Identity matrix

Identity matrix is a square matrix whose diagonal elements are 1.

Example

$$\begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Code

```
#program for identity matrix
n=int(input("Enter a number: "))
for i in range(0,n):
    for j in range(0,n):
        if(i==j):
            print("1",sep=" ",end=" ")
        else:
            print("0",sep=" ",end=" ")
    print()
```

Output

```
RESTART: C:/Users/Administrator/AppData/Local/Programs/Python/Python36-32/ide.py
Enter a number: 3
1 0 0
0 1 0
0 0 1
```


You can also create the identity matrix by using eye () function as follows

```
In [54]: eye(3,dtype=int)
Out[54]:
array([[1, 0, 0],
       [0, 1, 0],
       [0, 0, 1]])
```

Syllabus Topic: Transpose

2.1.1 Transpose of a Matrix

If $A = [a_{ij}]_{m \times n}$ then the transpose of A is given as $A^t = [a_{ji}]_{n \times m}$, i.e. rows are written as columns and columns are written as rows.

Transpose of matrix is also denoted by A^T or A'.

Example 1

$$\text{If, } B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Code

```
#program of transpose of Matrix
X = [[12,7],
     [4 ,5],
     [3 ,8]]

t = [[0,0,0],
     [0,0,0]]

print("original matrix")
print(X)
print("transpose of matrix")
for i in range(len(X)):
    for j in range(len(X[0])):
        t[j][i] = X[i][j]

for r in t:
    print(r)
```

Output

```
Python 3.6.0 Shell
File Edit Shell Debug Options Window Help
original matrix
[[12, 7], [4, 5], [3, 8]]
transpose of matrix
[12, 4, 3]
[7, 5, 8]
>>>
```

☞ **Symmetric matrix**

A square matrix A is symmetric if $A^T = A$.

For Example, $A = \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$

The following is the Python code which checks the given matrix is symmetric or not.

```
symmetric.py - C:/Users/Administrator/AppData/Local/Programs/Python/Python36-32/symmetric.py (3.6.0)
File Edit Format Run Options Window Help
#Program to check matrix is symmetric or not
def isSym(mat, N):
    for i in range(N):
        for j in range(N):
            if (mat[i][j] != mat[j][i]):
                return False
    return True

mat = [ [ 1, 3, 5 ], [ 3, 2, 4 ], [ 5, 4, 1 ] ]
if (isSym(mat, 3)):
    print("Yes")
else:
    print("No")
```


Output

```
Python 3.6.0 Shell
File Edit Shell Debug Options Window Help
Yes
>>>
```

Syllabus Topic : Matrix - Vector and Vector - Matrix Multiplication in terms of Linear Combinations

2.1.2 Matrix - Vector and Vector - Matrix Multiplication in terms of Linear Combination

Now mathematically two matrices $A_{m \times n}$ and $B_{n \times k}$ can be multiplied if the number of columns of to A is same as the number rows of B. i.e. AB is possible but BA is not possible.

Linear combination definition of matrix vector multiplication let M be R × C matrix over F. Let v be C - vector over F. Then M ∗ v is the linear combination.

$\sum_{c \in C} v[c]$ (Column c of M)

If M is an R × C matrix but v is not ∞ a C-vector then product M ∗ v is not defined

☞ **Example 1**

As per above definition

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} * [7\ 0\ 1]$$

$$= 7\,[1, 1] + 0\,[2, 2] + 1\,[3, 4]$$

$$= [7, 7] + [0, 0] + [3, 4]$$

$$= [10, 11]$$

Python code for the above example is,

Code

```
In [128]: import numpy as np

In [129]: a=np.array([[ 1, 2,3],[1,2,4]])

In [130]: b=np.array([7,0,1])

In [131]: print(np.dot(a,b))
[10 11]
```

☞ Linear Combinations Definitions of Vector Matrix Multiplication

Let M be an R × C matrix. Let w be an R-vector

Then w * M is the linear combination

$$\sum_{r \in R} w[r], \quad (\text{row } r \text{ of } M)$$

If M is an R × C matrix but w is not an R-vector then the product w*M illegal (not valid).

☞ Example 2

As per definition

$$[1, 2] * \begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 1 \end{bmatrix}$$

$$= 1\,[4, 6, 8] + 2\,[5, 7, 1] = [4, 6, 8] + [10, 14, 2]$$

$$= [14, 20, 10]$$

Python code for above example is

Code

```
In [136]: import numpy as np

In [137]: a=np.array([1,2])

In [138]: b=np.array([[ 4, 6,8],[5,7,1]])

In [139]: print(np.dot(a,b))
[14 20 10]
```


Syllabus Topic : Matrix Vector Multiplication in terms of Dot Products

2.1.3 Matrix Vector Multiplication in terms of Dot Products

☞ Dot Product Definition of Matrix – Vector Multiplication

If M is an R × C matrix and u is a C-vector then M * u is the R-vector v such that v[r] is the dot product of row r of M with u.

Example 2.1.1:

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \\ -1 & -10 \end{bmatrix} * [2, -1]$$

Soln.:

$$= [[1, 2] \cdot [2, -1], [4, 9] \cdot [2, -1], [-1, -10] \cdot [2, -1]]$$

$$= [(1)(2) + (2)(-1), (4)(2) + (9)(-1), (-1)(2) + (-10)(-1)]$$

$$= [2 - 2, 8 - 9, -2 + 10]$$

$$= [0, -1, 8]$$

Code

```
In [117]: import numpy as np

In [118]: a= np.array([[ 1, 2],[ 4, 9],[ -1,-10]])

In [119]: b = np.array([2, -1])

In [120]: np.dot(a,b)
Out[120]: array([ 0, -1,  8])
```

☞ Dot Product Definition of Vector-Matrix Multiplication

If M is an R × C matrix and u is a R-vector then u * M is the C-vector v such that v[c] is the dot product of u with column C of M.

Example 2.1.2 : $\begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 4 & 9 \\ -1 & -10 \end{bmatrix}$

Soln. :

$$= [0[1, 2] + 4\ [4, 9] + 6\ [-1, -10]\ ]$$
$$= [[0, 0] + [16, 36] + [-6, -60] = [10, -24]$$

Code

```
In [147]: import numpy as np
In [148]: a=np.array([0,4,6])
In [149]: b = np.array([[ 1, 2],[ 4, 9],[ -1,-10]])
In [150]: print(np.dot(a,b))
[ 10 -24]
```

Linear Equation

Consider $a_1 x_1 + a_2 x_2 = b_1$ an linear equation in two variables x_1 and x_2. This can be written in vector notation as $a \cdot x = \beta$, where $a = [a_1, a_2]$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \beta = b_1$$

In general if we have

$$a_1 \cdot x = \beta_1$$
$$a_2 \cdot x = \beta_2$$
$$\vdots = \vdots$$
$$a_m \cdot x = \beta_m$$

Then such a collection of linear equation is called a **linear system.** Where vector X is the vector of unknowns, which are to be found. A solution vector x that satisfies all the equations is the solution for the system.

We can also generalize the above concept

Consider,
$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

then we can write the above system of linear equations in terms of matrix as

$$AX = B$$

Where, $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Algebraic Properties of Matrix - Vector Multiplication

Let, M be $R \times C$ matrix

For any C - vector v and a scalar α

$$M * (\alpha v) = \alpha (M * v)$$

For any C - vectors u and v

$$M * (u + v) = M * u + M * v$$

Syllabus Topic : Null Space

2.1.4 Null Space

The null space of a matrix A is the set of all those vectors v such that $A * v = 0$ and this set is denoted by Null Space (A)

or Null (A) or ker(A)

i.e. $\text{Null}(A) = \{v \mid A * v = 0\}$

Note that : Null space (A) is also a vector space.

Example 2.1.3 : Find the null space of matrix A

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 3 & 4 & 7 \end{bmatrix}$$

n.: By definition

$$\text{Null (A)} = \{ v \mid Av = 0 \}$$

$$= \left\{ v \left| \begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right. \right\}$$

Perform $R_2 \to R_3 (R_1 + R_2)$ and $R_2 \to R_2 \to R_1$

$$= \left\{ v \left| \begin{bmatrix} 1 & 5 & 6 \\ 0 & -4 & -4 \\ 0 & -7 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right. \right\}$$

$R_2/(7)$ and $R_2/(-4)$

$$= \left\{ v \left| \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right. \right\}$$

Now we observe that we have two equations and 3 unknowns i.e.

$$v_1 + 5v_2 + 6v_3 = 0$$

$$v_2 + v_3 = 0 \Rightarrow v_2 = -v_3$$

$$\therefore v_1 - 5v_3 + 6v_3 = 0$$

$$v_1 = -v_3$$

$$\therefore \text{Vector } v = [v_1, v_2, v_3] = [-v_3, -v_3, v_3]$$

$$= v_3 [-1, -1, 1]$$

$$\therefore \text{Null (A)} = \{v \mid v_3 (-1, -1, 1)\}$$

☞ **Lemma**

For any matrix A of $R \times C$ and C-vector v a vector z is in the null (A) iff

$A * (v + z) = A * v$

☞ **Corollary**

Suppose u_1 is a solution to the matrix equation $A * x = B$, Then u_2 is also a solution iff $u_1 - u_2$ belongs to the null space of A.

☞ **Corollary**

Suppose a matrix vector equation $AX = B$ has a solution. The solution is unique iff the null space of A consists of only the zero vector.

Syllabus Topic : Computing Sparse Matrix - Vector Product

2.1.5 Computing Sparse Matrix - Vector Product

Consider M and $R \times C$ matrix and u is a C-vector then $M * u$ is the R-vector v such that for each $r \in R$.

$$v[r] = \sum_{c \in C} M[r, c]\, u[c] \quad \text{...(A)}$$

The most straight forward way to implement matrix vector multiplication based on this definition is

1. for each i in R
2. $v[i] = \sum_{j \in C} M[i, j]\, u[j]$

However, this does not take advantages of the fact that many entries of M are zero and do not even appear in our sparse representation of M.

The trick is to initialize the output vector v to the zero vector and then iterate over the non zero entries of M, adding the terms as specified in equation (A)

1. Initialize v to zero vectors
2. For each pair (i, j) such that the sparse representation specifies M[i, j].
3. v [i] = v[i] + M[i, j] u [j].

Syllabus Topic : Linear Functions

2.1.6 Linear Functions

Let U and V be vector spaces over a field $\mathbb{F}$ then we call $f : U \to V$ a linear function if (i) $f(u + v) = f(u) + f(v)$, for all u, $v \in U$ and (ii) $f(\alpha u) = \alpha f(u)$, for any scalar $\alpha \in \mathbb{F}$. In mathematics linear functions are called linear transformations.

Let, M be $R \times C$ matrix over a field $\mathbb{F}$. Then $f : \mathbb{F}^C \to \mathbb{F}^R$ given by $f(x) = M * x$ is a linear function.

☞ **Example 1**

$f : \mathbb{F}^2 \to \mathbb{F}$ given by $f(x, y) = x + y$ because

$$f(x_1, y_1) = x_1 + y_1 \quad \text{and} \quad f(x_2, y_2) = x_2 + y_2$$

$\therefore$ $f((x_1, y_1) + (x_1, y_2)) = f((x_1 + x_2, y_1 + y_2)) = (x_1 + x_2) + (y_1 + y_2)$

$= (x_1 + y_1) + (x_2 + y_2) = f(x_1, y_1) + f(x_2, y_2)$

Also, $f(\alpha(x, y)) = f(\alpha x, \alpha y) = \alpha x + \alpha y$

$= \alpha(x + y) = \alpha f(x, y)$

$\therefore$ f is linear function.

Example 2

The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (-y, x)$ is a linear map.

This map, geometrically rotates the image by 90° in clockwise direction.

Example 3

The map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, y) = (-x, y)$ is a linear map.

This map, geometrically reflects the points about y-axis.

Note : If $f : U \to V$ is a linear function then the zero vector of U i.e. $\overline{0_u}$ is mapped onto the zero vector of V i.e. $\overline{0_v}$ ...(B)

Kernel (f)

Kernel of a linear function is defined as the set of all those vectors v whose images under f is 0 i.e.

Kernel (f) $= \{u \in U \mid f(u) = \overline{0v}\}$

Kernel (f) is also known as Nullity of f or Nullity f.

Note 1 : Kernel (f) is a vectorspace.

Note 2 : If ker (f) = {0} i.e. it is a trivial vector space then f is one-one (injective) and vice versa.

Now consider $f : U \to V$ is a linear function and the linear combination.

$\alpha_1 u_1 + \alpha_2 u_2$, where $u_1, u_2 \in U$ then $f(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 f(u_1) + \alpha_2 f(u_2)$

if $\alpha_1 + \alpha_2 \in \mathbb{R}$ and $\alpha_1 + \alpha_2 = 1$ then set of all affine combinations

$\{\alpha_1 u_1 + \alpha_2 u_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 + \alpha_2 = 1\}$

Under the linear function is mapped onto

$\{\alpha_1 f(u_1) + \alpha_2 f(u_2) + \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 + \alpha_2 = 1\}$

i.e. on to the affine combination of $f(u_1)$ and $f(u_2)$

Thus image under f of the line through u_1 and u_2 is the line through $f(u_1)$ and $f(u_2)$.

Now we give two proofs,

1. If $f : U \to V$ is a linear transformation, ker (f) = { 0 } iff f is injective.

Proof

Part I

Let, ker (f) = { 0 }

To show f is injective i.e. $f(x_1) = f(x_2)$ then $x_1 = x_2$

Let, $v_1, v_2 \in$ ker (f)

$f(v_1) = 0v$ and $f(v_2) = 0v$ ...By definition of ker (f)

Now, $f(v_1) - f(v_2) = 0v - 0v$

$\therefore$ $f(v_1) - (-v_2) = 0v$ ($\because$ f is linear)

$\therefore$ $f(v_1 - v_2) = 0v$

$v_1 - v_2 \in$ ker (f)

But, ker (f) = { 0 }

$v_1 - v_2 = 0$

$\therefore$ $v_1 = v_2$

$\therefore$ f is injective

Part II

Assume f is injective

To show that ker(f) = {0}

Suppose that ker(f) ≠ {0}

∃ a non zero vector v other than $\overline{0_v}$ in ker (f).

∃ v ∈ ker (f) ⇒ f(v) = 0

Now we know that since f is linear function 0 is mapped onto 0.

$\therefore \; f(0_u) = 0_u$

$\therefore$ but $v \neq 0_u$

$\therefore$ f is not injective.

(Part II is proved using contra positive argument.)

2. The image of a linear function $f : U \rightarrow V$ given by Im(f) is subspace of V.

Proof

To show Im(f) is a subspace of V

$0_v \in$ Im(f) because $0_v = f(0_u)$

Let $v_1, v_2 \in I_m(f)$

$$\therefore v_1 = f(u_1) \text{ and } v_2 = f(u_2)$$

$$\text{Consider } v_1 + v_2 = f(u_1) + f(u_2)$$

$$= f(u_1 + u_2) \quad \text{...}(\because \text{ f is line}$$

$$\therefore v_1 + v_2 \in I_m(f)$$

Also $\alpha v_1 = \alpha f(u_1) = f(\alpha u_1)$, for any $\alpha \in \mathbb{F}$.

$\therefore \; \alpha v_1 \in$ Im(f)

$\therefore$ Im(f) is a subspace of V.

Now to any linear function we can associate a matrix

Suppose $f : \mathbb{F}^C \rightarrow \mathbb{F}^R$ is a linear function.

Then there is an $R \times C$ matrix M over $\mathbb{F}$ such that $f(x) = M * x$ for every vec $x \in \mathbb{F}^C$.

Diagonal Matrices

Let $d_1, d_2, \ldots d_n$ be real numbers.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the function such that

$f([x_1, x_2, \ldots, x_n]) = [d_1x_1, d_2x_2, \ldots, d_nx_n]$ then the corresponding matrix is

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \vdots & \\ & & & d_n \end{bmatrix}$$

In order to simplify our understanding we have a vector $(d_1, d_2, \ldots, d_n)$ as an arrangement of n components of an $d_i e_i$ vectors where e_i is the standard basis vector i.e. $e_i = (0, 0, \ldots, 1, \ldots. 0)$, here1 on the i^{th} position. Thus in a diagonal matrix every row vector is of the form $d_i e_i$.

Following is the python code for the diagonal matrix.

Code

```
In [159]: import numpy as np

In [160]: a = np.array([1,2,3,4])

In [161]: d = np.diag(a)

In [162]: print (d)
[[1 0 0 0]
 [0 2 0 0]
 [0 0 3 0]
 [0 0 0 4]]
```

Syllabus Topic : Matrix - Matrix Multiplication

2.1.7 Matrix - Matrix Multiplication

Let $A_{m \times n}$ and $B_{n \times r}$ matrices then we know that AB is defined and its order will be $m \times r$ where as BA is not defined.

This can be seen in two ways.

(i) Vector matrix definition of matrix-matrix multiplication each row r of A,

row r of AB = (row r of A) $*$ B

Assuming multiplication conditions are satisfied

$$\text{Consider, } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then, row 1 of AB = $1\,b_1 + 2\,b_2 + 3\,b_3$

row 2 of AB = $4\,b_1 + 5\,b_2 + 6\,b_3$

row 3 of AB = $7\,b_1 + 8\,b_2 + 9\,b_3$

...here b_1, b_2, b_3 are row vectors of B.

The following is the python code to multiple 3∗3 matrix with 3∗4 matrix:

Code

```
# Program to multiply two matrices using nested loops
X = [[3,2,2],
    [4 ,1,5],
    [7 ,2,7]]                    # 3x3 matrix

Y = [[5,3,1,2],
    [1,7,3,0],
    [2,5,2,1]]                   # 3x4 matrix

MUL = [[0,0,0,0],
       [0,0,0,0],
       [0,0,0,0]]               # result is 3x4
print("MULTIPLICATION OF TWO MATRIX IS:")
for i in range(len(X)):
    for j in range(len(Y[0])):
        for k in range(len(Y)):
            MUL[i][j] += X[i][k] * Y[k][j]

for r in MUL:
    print(r)
```

Output

```
MULTIPLICATION OF TWO MATRIX IS:
[21, 33, 13, 8]
[31, 44, 17, 13]
[51, 70, 27, 21]
>>>
```


(ii) Matrix vector definition of matrix multiplication each column label S of B.

Column s of AB = A ∗ (columns of B)

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = [S_1 \mid S_2 \mid S_3]_{2\times 3}$$

$$AB = [As_1 \mid As_2 \mid As_3]$$

Note : For $(AB)^T = B^T A^T$.

Syllabus Topic: Inner Product and Outer Product

2.1.8 Inner Products and Outer Product

Consider $$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 3 \end{bmatrix}_{3\times 2}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{2\times 1}$$

then $$AB = \begin{bmatrix} (1)(2) + (1)(1) \\ (2)(2) + (4)(1) \\ (0)(2) + (3)(1) \end{bmatrix}$$

Every entry of the matrix so obtained is obtained using a rule given known as inner product.

Let u and v are D-vectors then the $u^t \cdot v$ is known as inner product of two vectors and it is denoted by <u , v>.

- **Outer product**

If $$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{3\times 1}$$

$$v = [v_1\ v_2\ v_3\ v_4]_{1\times 4}$$

then $$uv^t = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 & u_1v_4 \\ u_2v_1 & u_2v_2 & u_2v_3 & u_2v_4 \\ u_2v_1 & u_3v_2 & u_3v_3 & uu_3v_4 \end{bmatrix}$$

The following is the python code to find out the inner and outer product of matrix de

```
import numpy as np
x = np.array([1, 4, 0], float)
y = np.array([2, 2, 1], float)
print("Matrices and vectors.")
print("x:")
print(x)
print("y:")
print(y)
print("Inner product of x and y:")
print(np.inner(x, y))
print("Outer product of x and y:")
print(np.outer(x, y))
```

Output

```
Matrices and vectors.
x:
[ 1.  4.  0.]
y:
[ 2.  2.  1.]
Inner product of x and y:
10.0
Outer product of x and y:
[[ 2.  2.  1.]
 [ 8.  8.  4.]
 [ 0.  0.  0.]]
```

Syllabus Topic : From Function Inverse to Matrix Inverse

2.1.9 Function from Inverse to Matrix Inverse

Let, $f : U \to V$ be a linear function, and

$g : V \to U$ be a linear function such that

$g \circ f : U \to U$ is identity function on U

$f \circ g : V \to V$ is identity function on V

then g is inverse function of f and denoted as f^{-1}.

Moreover g is also linear (here).

Since to linear function f we associated a matrix M then the associated matrix of $g = f^{-1}$ is M^{-1}.

Matrix Inverse

Let A be $R \times C$ matrix over $\mathbb{F}$ and B be $C \times R$ matrix over $\mathbb{F}$.

Define $f : \mathbb{F}^C \to \mathbb{F}^R$ such that.

$f_A(x) = A \cdot x$ and $g : \mathbb{F}^R \to \mathbb{F}^C$ such that

$f_B(y) = B \cdot y$

If f and g are inverses of each other then $A^{-1} = B$ and $B^{-1} = A$

If A^{-1} has exist then $\det(A) \neq 0 \Rightarrow$ A is said to be non singular other wise if is called singular matrix and so is the function.

Note : (1) $(AB)^{-1} = B^{-1} A^{-1}$, if A and B are invertible.

1. If $D = \begin{bmatrix} d_1 & & O \\ & d_2 & \ddots \\ O & & d_n \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/d_1 & & O \\ & 1/d_2 & \ddots \\ O & & 1/d_n \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $f_A(x_1, x_2, x_3) = (x_1 + \alpha x_3, x_2, x_3)$

Then, $A^{-1} = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $f_{A^{-1}}(x_1, x_2, x_3) = (x_1 - \alpha x_3, x_2, x_3)$

Lemma

If A is $R \times C$ matrix and A^{-1} is $C \times R$ matrix then AA^{-1} is $R \times R$ identity matrix and $A^{-1}A$ is $C \times C$ identity matrix.

Lemma

Suppose A is upper triangular matrix (or lower triangular matrix) then A is invertible iff all its diagonal entries are non zero.

Result

If A is invertible then for any vector B the equation $AX = B$ has exactly one solution i.e. $X = A^{-1} B$.

...hon code for inverse of a matrix is as follows

Code

```
In [184]: from numpy.linalg import inv

In [185]: a = np.array([[1., 2.], [3., 4.]])

In [186]: inv(a)
Out[186]:
array([[-2. ,  1. ],
       [ 1.5, -0.5]])
```

Syllabus Topic : Basis - Co-ordinate Systems

2.2 Basis

2.2.1 Co-ordinate Representation

In vector analysis, a co-ordinate system for a vector space V is specified generators $a_1, a_2, \ldots, a_n$ of V.

Every vector v in V can be written as a linear combination.

$$v = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n$$

$\therefore$ The vector v is represented by the vector $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ of coefficients

In this context, the coefficients are called the co-ordinate representation of v in terms of $a_1, a_2, \ldots, a_n$.

Example 2.2.1: Find the co-ordinate representation of $v = [1, 3, 5, 3]$ in terms of $a_1 = [1, 1, 0, 0]$, $a_2 = [0, 1, 1, 0]$, $a_3 = [0, 0, 1, 1]$

Soln. :

Let $v = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$

$$[1, 3, 5, 3] = \alpha_1 [1, 1, 0, 0] + \alpha_2 [0, 1, 1, 0] + \alpha_3 [0, 0, 1, 1]$$

$$[1, 3, 5, 3] = [\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3]$$

$$\therefore \alpha_1 = 1$$

$$\alpha_1 + \alpha_2 = 3, \alpha_2 + \alpha_3 = 5, \alpha_3 = 3$$

$$\therefore \alpha_2 = 3 - 1 = 2$$

$\therefore$ v = [1, 2, 3] in terms of vector a_1, a_2, a_3.

Example 2.2.2 : Find the co-ordinate representation of vector. v = [0, 0, 0, 1] in terms of the vectors [1, 1, 0, 1], [0, 1, 0, 1] and [1, 1, 0, 0] in GF (2).

Soln. :

Let $a_1 = [1, 1, 0, 1]$

$a_2 = [0, 1, 0, 1]$

$a_3 = [1, 1, 0, 0]$

$\therefore v = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$

$[0, 0, 0, 1] = [\alpha_1 + \alpha_3, + \alpha_2 + \alpha_3, 0, \alpha_1 + \alpha_2]$

$\alpha_1 + \alpha_3 = 0$ ...(1)

$\alpha_1 + \alpha_2 + \alpha_3 = 0$ ...(2)

$\alpha_1 + \alpha_2 = 1$ ...(3)

From (1) (2), $\alpha_2 + 0 = 0 \Rightarrow \alpha_2 = 0$

$\therefore$ From 3 $\alpha_1 = 1$

Now from (1) $1 + \alpha_3 = 0$

In GF (2) $\alpha_3 = 1$

$\therefore$ Co-ordinate representation of vector

v = [0, 0, 0, 1] will be [1, 0, 1] in terms of a_1, a_2, a_3 in GF(2)

Co-ordinate Representation and matrix vector multiplication

Suppose the co-ordinate axes are $a_1, a_2, \ldots, a_n$.

we form a matrix $A = [a_1 \mid a_2 \mid \ldots \mid a_n]$ whose columns are generators

- We can write the statement "u is the co-ordinate representation of v" in $a_1, a_2, \ldots, a_n$ as the matrix vector equation.

$$Au = v$$

- Transfer to go from a co-ordinate representation u to the vector being represented we multiply A times u.

...ver, to go from a vector v to co-ordinate representation, we can solve th matrix vector equation $A \cdot x = v$.

Because the columns of A are generators for V and v belongs to V, th equation must have at least one solution.

Syllabus Topic : Two Greedy Algorithms for Finding a Set of Generators

2.2.2 Two Greedy Algorithms for Finding a Set of Generators

These algorithms help us to find the minimum number of vectors whose spa equals the vector space V.

(1) Grow Algorithm

def Grow (V)

B = ϕ repeat while possible

Find a vector v in V that is not in span B and put in B.

The algorithm stops when there is not vector to add, at which time B spans a of V. Thus if the algorithm stops, it will have found a generating set.

(2) Shrink Algorithm

def shrink (V)

B = some finite set of vectors that span V repeat while possible:

find a vector v in B such that the span $(B - \{v\}) = V$ and remove v from B.

The algorithm stops when there is no vector whose removal would leave spanning set. At every point during the algorithm, B span V, so its spans at th end. Thus algorithms certainly find the generating.

Grow and shrink algorithms are called greedy algorithms because in each st the algorithm makes a choice without giving thought to the future.

The grow and shrink algorithms for finding the smallest generating set fo vector space are remarkable, they do in fact find the smallest set.

Syllabus Topic : Minimum Spanning Forest and GF(2)

2.2.3 Minimum Spanning Forest and GF(2)

Consider the following graph with weights assigned on its edges.

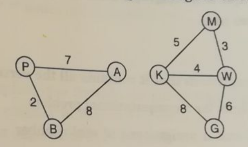

Fig. 2.2.1

We understand the above Graphs as the water-supply network amongst the different buildings of a society.

The weights on the edges represent the cost of installation of pipe lines.

Our Goal is the select a set of pipes to install so every pair of areas are connected in the above graph by the edges which represent pipes and the cost is minimum.

We define few terms from graph theory which will be helpful here.

☛ Terms in Graph Theory

1. Path

For a Graph G,

A path is defined as a sequence of vertices and edges in which no vertex is repeated i.e. if u, and v are distinct vertices then $u = u_0 - e_1 - u_1 - e_2 - u_2 \ldots - e_n - v_n = v$. We say it is a $u - v$ path. Here u and v are starting and end vertices, u_i and v_j are intermediate vertices and e is the edges between the vertices $u_j - u_i$.

2. Cycle

A graph is cyclic or said to be a cycle if $v_1 - e_1 - v_2 - e_2 - \ldots v_{n-1} - e_n - v_n = v_1$

i.e. the starting and the ending vertex are the same.

3. **Forest**

A graph is said to be forest if it has no cycles in it.

4. **Spanning subgraph**

A spanning subgraph is a subgraph containing all vertices of G. It need no contain all the edges in G.

5. **Spanning forest**

It would be a subgraph such that it contains all the vertices but not cycles.

We give two algorithms for a computational problem.

- **Input:** A graph G, and a assignment of real number – weights to the edge of G.
- **Output:** A minimum - weight set B of edges that is spanning and a forest.

1. **Grow Algorithm for MSF**

Def Grow (G)

B : ϕ

Consider the edges in order, from lowest weight to highest weight.

For each edge e if e's endpoints are not yet connected via edges in B:

Add e to B

This algorithm exploits the freedom we have in grow algorithm to select whic vector to add.

The weights in increasing order are : 2, 3, 4, 5, 7, 8, 9. Thus the solutio obtained which consists of the edges with 2, 3, 4, 5, 7 is

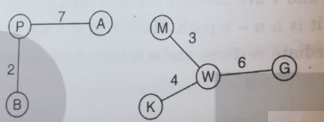

Fig. 2.2.2

2. **Shrink Algorithm for MSF**

Def SHRINK (G)

B = {all edges}

Consider the edges in order, from the highest weight to lowest weight for each edge e if every pair of nodes are connected via.

B – {e}

Remove e from B

This algorithm exploits the freedom in the shrink algorithm to select which vector to remove the weights in decreasing order are : 9, 8, 7, 6, 5, 4, 3, 2. The solution consists of the edges with 7, 6, 4, 3 and 2

We get the same solution here.

Syllabus Topic : Linear Dependence

2.2.4 Linear Dependence

Lemma : (superfluous – Vector Lemma)

For any set S and any vector $v \in S$, if v can be linear combination of the other vectors in S then span $(S - \{v\})$ = span S.

Proof

Let $S = \{v_1, v_2, \ldots, v_n\}$

and Let $v_n = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-1} v_{n-1}$ ...(2.2.1)

To show span (S) = span $(S - \{v\})$

Now for every $v \in$ span (S)

$$v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n \quad \text{...(2.2.2)}$$

Now,

$$\begin{aligned} v &= \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-1} v_{n-1}) \\ &= \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n \alpha_1 v_1 + \beta_n \alpha_2 v_2 + \ldots + \beta_n \alpha_{n-1} v_n \\ v &= (\beta_1 + \beta_n \alpha_1) v_1 + (\beta_2 + \beta_n \alpha_2) v_2 + \ldots + (\beta_n + \beta_n \alpha_n v_{n-1}) \end{aligned}$$

Which shows that an arbitrary vector in span S can be written as linear combination.

Linear Dependence

We again define, a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be **linearly independent if**

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0 \text{ then every } c_i = 0$$

Otherwise the set is **linearly dependent**. Which means that in the set $\{v_1, v_2, \ldots, v_n\}$ there is a vector which can be expressed as a linear combination of other vectors in that set.

For Example $v_2 = C_1' v_1 + C_3' v_3 + \ldots + C_n' v_n$

OR

We **say that set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if**

$C_1v_1 + C_2 v_2 + \ldots + C_n v_n = 0$ then for some i, $C_i \neq 0$

☞ **Example**

The following set of vectors is linearly independent

(i) $\{(1, 1), (0, 2)\}$

(ii) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

The following set of vector is linearly dependent

(i) $\{(1, 2), (2, 3), (3, 5)\}$

(ii) $\{(0, 1, 0), (0, -2, 0)\}$

Note : Any set containing a zero vector is always linearly dependent.

Construct a linearly independent set from $\{(1, 2, 3), (4, 4, 6)\}$

For space $\mathbb{R}^3$.

Short trick is that since the space is third 3 dimensional we need third vector for it which is can be found as adding the two vector and by increasing or decreasing any one co-ordinate by a small increment.

For Example $\{(1, 2, 3), (4, 4, 6), (5, 6, 10)\}$

Third vector is obtained by adding the first two vectors and the third co-ordinate is increased by 1.

Note : Any subset of linearly independent set is also linearly independent.

☞ **Lemma : (Span lemma)**

Let $v_1, v_2, \ldots, v_n$ be vectors. A vectors v_i is in the span of other vectors iff the zero vector can be written as linear combination of $v_1, v_2, \ldots, v_n$ in which the coefficient of vi is non zero.

☞ **Corollary : (Grow Algorithm Corollary)**

The vectors obtained by grow algorithm are linearly independent.

☞ **Proof**

For n = 1, 2, . . . let v_n be the vector added to B in the n^{th} iteration of the grow algorithm. We show by the mathematical induction that $v_1, v_2, \ldots, v_n$ are linearly independent.

For n = 0, there no vectors so claim holds true, trivially.

Assume for $n = k - 1$, the claim holds we prove the claim for $n = k$

The vector v_k added to B in the k^{th} iteration is not in the span of $v_1, \ldots, v_{k-1}$

$\therefore$ by the span lemma, for any coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1} + \alpha_k v_k \text{ it must be that } \alpha_k = 0$$

$$0 = \alpha_1 v_1 + \ldots + \alpha_{k-1} v_{k-1}$$

By the claim that holds for $n = k - 1$

$v_1, v_2, \ldots, v_{k-1}$ are linearly independent, so $\alpha_1, \ldots, \alpha_{k-1}$ are all zero. We have proved that the only linear combination of $v_1, v_2, \ldots, v_k$ are equals to the zero vector is the trivial combination. i.e. that $v_1, v_2, \ldots, v_k$ are linearly independent.

This prove the claim for n = k.

☞ **Corollary**

The vectors obtained by shrink algorithm are linearly independent.

Syllabus Topic : Basis

2.2.5 Basis

☞ **Definition**

Let V be a vector space

The set $B = \{v_1, v_2, \ldots, v_n\}$ is said to the basis set of vector space V if

(1) span (B) = V

(2) set B is linearly independent set.

Dimension of vector space V

The number of elements in the Basis set B is the dimension of vector space V and we denote the dimension of a vector space by dim (V).

Consider the vector space $\mathbb{R}^3$,

Its standard basis vectors are

$$B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$$

$$\therefore \dim (\mathbb{R}^3) = 3$$

Similarly $\dim (\mathbb{R}^n) = n$

We note a fact without proof that every vectorspace has a basis.

The standard generators of $\mathbb{F}^D$ from a basis of $\mathbb{F}^D$.

Lemma

Any finite set T of vectors contains a subset B that is a basis for span T.

Theorem

Let V be a finite dimensional vectorspace and let dim (V) = n.

Then

(a) Any subset of V which contains more than n vectors is linearly dependent.

(b) No subset of V which contains fewer than n vectors can span V.

Lemma

Let S be a linearly independent subset of vector space V. Suppose β is a vector in V which is not in the subspace spanned by S then the set S obtained by adjoining β to S is linearly independent.

Theorem

If W is a subspace of finite dimensional vector space V, every linearly independent subset of W is finite and is a part of a finite basis for W.

Syllabus Topic : Unique Representation

2.2.6 Unique Representation in terms of Basis

Lemma

Let $a_1, a_2, \ldots, a_n$ be a basis for a vectorspace V. For any vector $v \in V$; there is exactly one representation of v in terms of the basis vectors.

Proof

Let $v \in V$, Also $\{a_1, a_2, \ldots, a_n\}$ is the basis set for V.

$$\therefore v = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n$$

To show that $v = [\alpha_1, \ldots, \alpha_n]$ representation is unique

$$\text{Let } v = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n$$

$$v - v = (\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n) - (\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n)$$

$$0 = (\alpha_1 - \beta_1)\, a_1 + (\alpha_2 - \beta_2)\, a_2 + \ldots + (\alpha_n - \beta_n)\, a_n$$

Now $\{a_1, \ldots, a_n\}$ is basis set, thus by definition it is linearly independent

$$\therefore \alpha_i - \beta_i = 0 \quad \text{for } i = 1, 2, \ldots, n$$

$$\therefore \alpha_i = \beta_i \quad \text{for } i = 1, 2, \ldots, n$$

Hence the representation is unique.

Syllabus Topic : Change of Basis, First Look

2.2.7 Change of Basis and First Look

First look at Lossy Commpression

In information technology, lossy compression or irreversible compression is the class of data encoding methods that uses inexact approximations and partial data discarding to represent the content. These techniques are used to reduce data size for storage, handling and transmitting the content.

Suppose we need to store many 2000 × 1000 grayscale images. To store these images we have various strategies:

Strategy 1 : Replace vector with closest sparse vector

It is not easy to have an image with few nonzero; therefore we replace that image with a different image, one that is spare and expect that it will be perceptually similar. Such a compression method is lossy since information in the original image is lost.

When we replace the vector v by k-sparse vector which is obtained from v by replacing all but k largest magnitude entries by zeros. Thus the image can be represented more compactly.

Strategy 2 : Represent image vector by its coordinate representation:-

Step 1 : Select a collection of vectors { $a_1, a_2, \ldots, a_n$ }

Step2 : For each image vector find and store its coordinate representation u in terms of $a_1, a_2, \ldots\ldots, a_n$.

Step 3 : To recover the original image from the coordinate representation, compute the corresponding linear combination.

Using this strategy there no loss of fidelity of the original image.

Strategy 3 : An Hybrid approach:

Step1 : Select a collection of vectors { $a_1, a_2, \ldots, a_n$ }.

Step2 : For each image that is required to be compressed, take the corresponding vector v and find its coordinate representation u in terms of $a_1, a_2, \ldots, a_n$.

Step3 : replace u with the closest k-spare vector $\tilde{u}$ and store it.

Step4 : To recover an image from $\tilde{u}$, calculate the corresponding linear combination of $a_1, a_2, \ldots, a_n$.

(This strategy works when in step 2 when we can express u in terms of $a_1, a_2, \ldots, a_n$. And in step 4 the image whose coordinate representation is $\tilde{u}$ should not differ much from the original image, the image whose coordinate representation is u.)

2.2.7(A) Matrix Representation of Linear Function

Let T be a linear function from vector space V into itself and suppose that $S = \{u_1, u_2, \ldots, u_n\}$ is a basis of V. Now,

$T(u_1), T(u_2), \ldots, T(u_n)$ are vectors in V so each is a linear combination of the vectors in the basis the basis S say.

$$\begin{aligned}
T(u_1) &= a_{11} u_1 + a_{12} u_2 + \ldots + a_{1n} u_n \\
T(u_2) &= a_{21} u_1 + a_{22} u_2 + \ldots + a_{2n} u_n \\
&\vdots \\
T(u_n) &= a_{n1} u_1 + a_{n2} u_2 + \ldots + a_{nn} u_n
\end{aligned}$$

The transpose of the above coefficient matrix w.r.t basis S is called matrix representation of T and it is denoted as

$$[T]_s = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}$$

So the columns of $[T]_s$ are the co-ordinates vectors of $T(u_1), T(u_2), \ldots, T(u_n)$ respectively.

Example

$F : \mathbb{R}^2 \to \mathbb{R}^2$ is linear function.

$F(x, y) = (2x + 3y, 4x - 5y)$ and the Basis $S = \{u_1, u_2\} = \{(1, 2), (2, 5)\}$

$$\begin{aligned}
F(u_1) &= F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}8\\-6\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} \\
\therefore\ 8 &= x + 2y \\
-6 &= 2x + 5y \\
\therefore\ x &= 52, y = -22 \\
\therefore\ F(u_1) &= 52 u_1 - 22 u_2
\end{aligned}$$

Now, $F(u_2) = F\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}19\\-17\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix}$

$$\begin{aligned}
x + 2y &= 19 \\
2x + 5y &= -17 \\
x = 129, y &= -55 \\
\therefore\ F(u_2) &= 129 u_1 - 55 u_2 \\
\therefore\ [F]_s &= \left[\, [F(u_1)]_s \;\middle|\; [F(u_2)]_s \,\right] \\
&= \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}
\end{aligned}$$

2.2.7(B) Change of Basis

Let V be as n – dimensional vectorspace over a field $\mathbb{F}$.

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a basis of a vector space V and $S' = \{v_1, v_2, \ldots, v_n\}$ be another basis.

Because S is a basis, each vector in the new basis S′ can be written uniquel as linear combination of the vectors in S, say

$$v_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n$$

$$v_2 = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n$$

$$\vdots$$

$$v_n = a_{n1}u_1 + a_{n2}u_2 + \ldots + a_{nn}u_n$$

Let P be the transpose of the above matrix of coefficient that is $P = [P_{ij}]$ whe $P_{ij} = a_{ji}$. Then P is called the matrix of change of basis (or transition matrix) fr old basis S to new basis S′

☞ **Remark 1**

The above matrix P may also be viewed as the matrix whose columns a respectively, the co-ordinate column vectors the 'new; basis vectors v_i relative the old basis S; namely.

$$P = [[v_1]_s, [v_2]_s, \ldots, [v_n]_s]$$

☞ **Remark 2**

Analogously, there is a change of basis matrix Q from the new basis S′ to th old basis S. Similarly Q may be viewed as the matrix whose columns a respectively the co-ordinate column vectors of the old basis vectors u_i relative the "new basis" S′ namely.

$$Q = [[u_1]_{S'}, [u_2]_{S'}, \ldots, [u_n]_{S'}]$$

☞ **Remark 3**

Because the vectors $v_1, v_2, \ldots, v_n$ in the basis S′ are linearly independent, t matrix P is invertible and similarly Q is also invertible.

☞ **Example**

Consider in $\mathbb{R}^2$ basis as

$$S = \{u_1, u_2\} ; S' = \{v_1, v_2\}$$

$$= \{(1, 2), (3, 5)\} = \{(1, -1) ; (1, -2)\}$$

$$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix}; \quad \begin{matrix} x + 3y = 1 \\ 2x + 5y = 1 \end{matrix}$$

$$x = -8, y = 3$$

$$\text{Also, } \begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \begin{matrix} x + 3y = 1 \\ 2x + 5y = -1 \end{matrix}$$

$$x = -11, y = 4$$

$$\therefore v_1 = -8u_1 + 3u_2$$

$$v_2 = -11u_1 + 4u_2$$

$$\therefore P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$$

$$\text{Now, } Q = P^{-1} = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$$

2.2.7(c) Applications of Change of Basis Matrix

☞ **Theorem**

Let P be the change of basis matrix from S to S′ in a vectorpsace V.

Then for any vector $v \in V$, we have

$$P[v]_{S'} = [v]_S$$

And Hence, $P^{-1}[v]_S = [v]_{S'}$

(without Proof)

☞ **Theorem**

Let P be the change of basis matrix from a basis S to a basis S′ in a vector space V. Then for any linear function T on V.

$$[T]_{S'} = P^{-1}[T]_S P$$

That is, if A and B are matrix representation of T relative to S and to S′ respectively then

$$B = P^{-1}AP$$

Consider two basis of $\mathbb{R}^3$.

$$E = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$S = \{u_1, u_2, u_3\} = \{(1, 0, 1), (2, 1, 2), (1, 2, 2)\}$$

$$\text{Now } u_1 = (1, 0, 1) = e_1 + 0e_2 + e_3$$

$$u_2 = (2, 1, 2) = 2e_1 + e_2 + 2e_3$$

$$u_3 = (1, 2, 2) = e_1 + 2e_2 + 2e_3$$

$$\text{Hence, } P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$Q = P^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

Let $v = [1, 3, 5]$ to find $[v]_S$

$$v = x\,u_1 + y\,u_2 + z\,u_3$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore x = 7, \quad y = -5, \quad z = 4$$

$$v = 7u_1 - 5u_2 + 4u_3$$

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}$$

$$v = 7u_1 - 5u_2 + 4u_3$$

(b) Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(Find matrix B represents that represent A relative to basis S.)

$$A(u_1) = (-1, 3, 5) = 1u_1 + 5u_2 + 6u_3$$

$$A(u_2) = (1, 2, 9) = 21u_1 - 14u_2 + 8u_3$$

$$A(u_3) = (3, -4, 5) = 17u_1 - 8u_2 + 2u_3$$

$$\therefore B = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}$$

$$B = P^{-1}AP$$

$$= \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 21 & \\ -5 & -14 & \\ 6 & 8 & \end{bmatrix}$$

Syllabus Topic : Computational Problems

2.2.8 Computational Problem Involving Finding a Basis

Bases are quite useful. It is important for us to have implementable algorithms to find a basis for a given vectorspace. But a vectorspace can be huge even infinite how can it be input to a procedure?

There a 2 natural ways to specify a vectorspace V.

1. Specifying generators for V. This is equivalent to specifying a matrix A such that V = col (A)
2. Specifying a homogenous linear system whose solution set is V. This is equivalent to specifying matrix A such that V = null (A).

For each of these ways to specify V, we consider the computational problem of finding a basis.

Computational problem 1

Finding a basis of the vector space spanned by given vectors.

- **Input :** a list [$v_1, v_2, \ldots, v_n$] of vectors
- **Output :** a list of vectors that form a basis for span { $v_1, v_2, \ldots, v_n$ }

Computational problem 2

Finding a basis of the solution of a homogeneous linear system

- **Input :** a list $[a_1, a_2, \ldots, a_m]$ of vectors
- **Output :** a list of vectors that form a basis for the set of solutions to the system $a_1 \cdot x = 0, \ldots, a_m \cdot x = 0$

Exchange lemma (without proof)

Suppose S is a set of vectors and A is a subset of S. Suppose z is a vector in span S and not in A such that A∪ { z } is linearly independent.

Then there is a vector $\omega \in S - A$ such that Span (S) = Span $(\{z\} \cup S - \{\omega\})$

Syllabus Topic : Dimension

2.3 Dimension

We have already introduced the basic idea of dimension along with the basis of a vectorspace V. Here we deal with it in detail.

Lemma : Morphing lemma

Let V be a vectorspace over a field $\mathbb{F}$.

Suppose S is a set of generators for V and B is a linearly independent set vectors belonging to V then $|S| \geq |B|$.

i.e. number of basis elements is always less than or equal to the number generators.

Basis theorem

Let V be the vectorspace over $\mathbb{F}$. All bases for V have the same size (if $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is basis for V and $\beta' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is also basis for V then m = n) (if V is finite dimensional).

Theorem

Let V be a vectorspace over $\mathbb{F}$. Then a set of generators for V is smallest set generators iff the set is a basis for V.

> **Note :** |S| denotes the number of elements in set S.

Syllabus Topic : Dimensions and Rank

2.3.1 Dimensions and Rank

2.3.1(A) Dimension of Vectorspace V : (dim V)

The number of elements in the basis set of a vector space V over a field F the dimesion of the vectorspace V.

The dimension of a vectorspace is finite if the basis set is finite otherwise infinite.

Example

$\dim(\mathbb{R}^n) = n$ and its basis set is the standard basis set $\{e_1, e_2, \ldots, e_n\}$.

Example

Any field $\mathbb{F}$ and D is any finite set then, basis for $\mathbb{F}^D$ is the standard b which consist of $|D|$ vectors $\dim(\mathbb{F}^D) = |D|$ If V is a finite dimens vectorspace and W is subspace of V then $\dim(W) \leq \dim(V)$

2.3.1(B) Rank of set of vectors

The rank of set of vectors S is defined as the dimension of span of S. We denote it by rank (S) or $\rho(S)$.

> **Note :** For any set S of vectors, rank $(S) \leq |S|$

Rank for a matrix

1. **Row rank of a matrix**

   If $A = [a_{ij}]_{m \times n}$ is a matrix over $\mathbb{F}$ then row rank is defined as the number of linear by independent rows of matrix A.

2. **Column Rank of a Matrix**

   If $A = [a_{ij}]_{m \times n}$ is a matrix over $\mathbb{F}$ then the column rank of A is defined as the number of linearly independent columns of matrix A.

Rank of Matrix A

Now we define rank of matrix A as if row rank (A) = column rank (A) then rank (A) = row rank (A) = column rank (A)

We shall see other definitions for rank of matrix also.

1 If matrix $A_{n \times n}$ square matrix and $\det(A) \neq 0$ then rank $(A) = n$.

Rank of matrix is a very important concept we introduce an advance definition for the same.

Row echelon form of a matrix

A matrix $A_{m \times n}$ with entries row a_{i*} and column a_{*j} is said to be in row echelon form provided the following conditions hold:

1 If a_{i*} consist entirely of zeros then all rows below a_{i*} are also entirely zeros i.e. all zero rows are at the bottom.

2 If the first non zero entry in a_{i*} lies in the j^{th} position then all entries below the i^{th} position in columns $a_{*1}, a_{*2}, \ldots, a_{*j}$ are zero.

The pivots are the first non zero entries in each row.

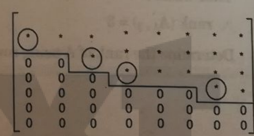

Fig. 2.3.1

Rank $(A_{m \times n})$ = rank (row echelon form A)

= number of non zero row in row echelon form of A.

Row echelon form of a matrix is obtained only by using row transformation i

$R_i \to R_i \pm k\, R_j,\ k \in \mathbb{F},\ R_i \leftrightarrow R_j,\ R_i \to k\, R_i\ k \in \mathbb{F}$

☛ **Example**

Find rank $A = \begin{bmatrix} 1 & 5 & 10 \\ 2 & 6 & 10 \\ 3 & 8 & 10 \\ 4 & 12 & 20 \end{bmatrix}$

Note : A is 4×3 matrix

To find its rank we use the row echelon method. Perform the following R transformations.

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3\,R_1,\ R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 5 & 10 \\ 0 & -4 & -10 \\ 0 & -7 & -20 \\ 0 & -8 & -20 \end{bmatrix}$$

Now using $R_2,\ R_3 \to R_3 - \frac{7}{4} R_2,\ R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 5 & 10 \\ 0 & -4 & -10 \\ 0 & 0 & -10/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally we obtain row echelon form

Thus number of non zero rows = 3

$\therefore$ rank $(A_{4 \times 3}) = 3$

Determine the rank of A using row echelon form.

1. $A = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix}$

2. $A = \begin{bmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{bmatrix}$

The following is the python code for row reduced echelon form:

rowech.py - C:/Users/Administrator/AppData/Local/Programs/Python/Python36-32/rowech.py (3.6.0)

File Edit Format Run Options Window Help

```
def RE( M):
    if not M: return
    lead = 0
    rowCount = len(M)
    columnCount = len(M[0])
    for r in range(rowCount):
        if lead >= columnCount:
            return
        i = r
        while M[i][lead] == 0:
            i += 1
            if i == rowCount:
                i = r
                lead += 1
                if columnCount == lead:
                    return
        M[i],M[r] = M[r],M[i]
        lv = M[r][lead]
        M[r] = [ mrx / float(lv) for mrx in M[r]]
        for i in range(rowCount):
            if i != r:
                lv = M[i][lead]
                M[i] = [ iv - lv*rv for rv,iv in zip(M[r],M[i])]
        lead += 1
mtx = [[ 1,5,10],[ 2,6,10],[3,8,10],[4,12,20],]
RE( mtx )

for rw in mtx:
  print (', '.join( (str(rv) for rv in rw) ))
```

Ln: 25 Col: 1

Output

```
Python 3.6.0 Shell
File  Edit  Shell  Debug  Options  Window  Help
1.0, 0.0, 0.0
0.0, 1.0, 0.0
-0.0, -0.0, 1.0
0.0, 0.0, 0.0
>>> |
                               Ln: 48  Col: 4
```

Syllabus Topic : Direct Sum

2.3.2 Direct - Sum

If U and V are two vectorspace consisting D - vectors over $\mathbb{F}$ and if $U \cap V = \{0\}$ (i.e. only common element in both the vectorspace is the zero vector) then direct sum is denoted by

$U \oplus V$ and is defined as

$U \oplus V := \{u + v \mid u \in U \text{ and } v \in V\}$

Note : $U \oplus V$ is a vectorspace (Prove it)

Example

1. In GF (2)

$$u = \text{Span}\{1000, 0100\},\ V = \text{Span}\{0010\}$$

Note : $\{0000\} \in U \cap V$

$$\therefore\ U \oplus V = \{0000 + 0000, 1000 + 0000, 0100 + 0000, 1100 + 0000, 0000 + 0010, 1000 + 0010, 0100 + 0010, 1100 + 0010\}$$

$$= \{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110\}$$

2.3.2(A) Generators for Direct Sum

The union of set of generators of V and union of set of generators of W is a set of generators for $V \oplus W$

Proof

Suppose $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$

$$W = \text{Span}\{\omega_1, \omega_2, \ldots, \omega_n\}$$

Then every vector $v \in V$ can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$$

And every vector $\omega \in W$ can be written as

$$\omega = \beta_1 \omega_1 + \beta_2 \omega_2 + \ldots + \beta_n \omega_n$$

So every vector in $V \oplus W$ can be written as

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$$

2.3.2(B) Basis Direct Sum Lemma

The union of a basis U and a basis of V is a basis of $U \oplus V$.

Proof

Let $\{u_1, u_2, \ldots, u_m\}$ be a basis for U.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

Since a basis is a set of generators. We already know from previous lemma that $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ is a set of generators for $U \oplus V$.

To show it is a basis, we need to show it is linearly independent.

Consider $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n = 0$ ...(1)

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m = (-\beta_1) v_1 + (-\beta_2) v_2 + \ldots + (-\beta_n) v_n$$

Now $u_1, u_2, \ldots, u_m$ are linearly independent vectors in U and $v_1, v_2, \ldots, v_n$ are linearly independent vectors in V.

$$\therefore\ \alpha_i = 0 \text{ and } \beta_j = 0$$

From (1) we get that,

$$\alpha_i \text{ and } \beta_j = 0$$

Hence the claim

Note : $\dim(U \oplus V) = \dim U + \dim V$

Any vector in $U \oplus V$ has unique representation as $u + v$ where $u \in U$ and $v \in V$.

Proof

Let $\{u_1, u_2, \ldots, u_m\}$ be a basis for u.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

Then $\{u_1, u_2, \ldots, u_m, v_1, \ldots, v_n\}$ is a basis for $u \oplus V$.

Let ω be any vector in $U \oplus V$, write ω as

$$\omega = \underbrace{\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m}_{\text{in U}} + \underbrace{\beta_1 v_1 + \ldots + \beta_n v_n}_{\text{in V}}$$

consider $\omega = u + v, u \in U, v \in V$

$$\omega = \gamma_1 u_1 + \gamma_2 u_2 + \ldots + \gamma_m u_m + \xi_1 v_1 + \xi_2 v_2 + \ldots + \xi_n v_n$$

thus by uniqueness lemma

$$\gamma_1 = \alpha_1, \; \alpha_2 = \alpha_2, \ldots, \gamma_m = \alpha_m$$

$$\xi_1 = \beta_1, \; \xi_2 = \beta_2, \ldots, \xi_n = \beta_n$$

Hence, the claim.

Complementary subspaces

If $U \oplus V = \omega$ then we say that U and V are complentary subspaces of ω

Proposition (without proof)

For any vector space ω and any subspace U of W there is a subspace V of W such that $W = V \oplus W$.

Syllabus Topic : Dimension and Linear Functions

2.3.3 Dimension and Linear Functions

Let V and W be vectorspace over $\mathbb{F}$.

Then $f : V \to W$ is invertible if f is one-one and onto.

i.e. f is one – one $\Leftrightarrow$ ker (f) = {0}

and f is onto $\Leftrightarrow I_m(f) = W$

Now we know that

$I_m(f)$ is a subspace of W

$\Rightarrow$ If f is onto then dim Im(f) = dim (w)

Thus f is invertible if dim (ker(f)) = 0

and dim (Im(f)) = dim (W)

Now ,

Let $f : V \to \omega$ be a linear function

Define $f^* : V^* \to W^*$ where $V^* \subseteq V$ and $W^* \subseteq W$ and define $f^*(x) = f(x)$

Let $\omega_1, \omega_2, \ldots, \omega_r$ be the preimages of $\omega_1, \omega_2, \ldots, \omega_r$

Basis of ω^* and $v_1, v_2, \ldots v_r$ be the

Now we say span of $\{v_1, v_2, \ldots, v_r\} = V^*$

Then f^* define is one – one and onto more over $v_1, v_2, \ldots, v_r$ is basis of V^*

Lemma

$$V = \ker f \oplus V^*$$

Where $f^* : V^* \to W^*$ is a linear function which is invertible.

Kernel image theorem : (without proof)

For any linear function $f : V \to W$ dim (ker f) + dim $(I_m f)$ = dim V

Linear function invertiblety theorem

Let $f : V \to w$ be a linear function then f is invertible iff dim (ker f) = 0

and dim V = dim W

Rank Nullity theorem

Let V and W be vectorspace over the field $\mathbb{F}$ and let T be linear function from V into W. Suppose that V is finite – dimensional then rank (T) + nullity (T) = dim (V) i.e. rank (T) + dim (ker T) = dim (V)

Proof

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis for the nullspace of T = ker T. There $\alpha_{(k+1)}, \ldots, \alpha n$ in V such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V.

Prove that $\{T\alpha_{k+1}, \ldots, T\alpha_n\}$ is basis for the range of T.

The vectors $T\alpha_1, \ldots, T\alpha_n$ certain span the range of T and $T\alpha_j = 0$ for $j \le k$, we see that $T(\alpha_{k+1}), \ldots, T(\alpha_n)$ span the range

To check the are independent.

$$\sum_{i=k+1}^{n} c_i T(\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

$\Rightarrow$ vector $\alpha = \sum_{i=k+1}^{n} c_i \alpha_i \in$ Null space of T.

Since $\alpha_1, \alpha_2, \ldots, \alpha_k$ form a basis of null space

There must be scalars $b_1, b_2, \ldots, b_k$ such that

$$\alpha = \sum_{i=1}^{k} b_i \alpha_i$$

Thus $\sum_{i=1}^{k} b_i \alpha_i - \sum_{j=k+1}^{n} c_j \alpha_j = 0$

and since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent we must have

$b_1 = b_2 = \ldots = b_k = c_{k+1} = \ldots = c_n = 0$

if r = rank (T), the fact that $T(\alpha_{k+1}), \ldots, T(\alpha_n)$ form a basis for the range of T tells us that $r = n - k$. Since k is nullity of Tρ n is the dim of V, we are done.

$\therefore$ dim V $= n = r + k =$ rank (T) + Nullity T

Syllabus Topic : The Annihilator

2.3.4 Annihilator

If V is a vectorspace over F and S is a subset of V the **annihilator** of S is S° and is defined as the set of all $f : V \to F$ such that

$f(v) = 0 \; \forall \, v \in S$

i.e. $S^\circ = \{f \mid f : V \to F, f(v) = 0, \forall \, v \in S\}$

Note: S° is subspace of V^* but S may or may not be subpace of V.

2.3.4(A) Annihilator of Vectorspace

For a subspace V of F^n, annihilator of V given as

$V^\circ = \{u \in F^n \mid u \cdot v = 0, \forall \, v \in V\}$

Lemma

Let $a_1, a_2, \ldots, a_m$ be generator for V and

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \text{ then } V^\circ = \text{Null } (A)$$

i.e. a_i are row vectors

Theorem

If V and V° are subspace of F^n then dim V + dim $V^\circ = n$

Hint : prove using rank nullity theorem.

Corollary

If W_1, W_2 are subspace of finite dimensional vectorspace V then

$$W_1 = W_2 \text{ iff } W_1^\circ = W_2^\circ$$

Also $(W^\circ)^\circ = W$

Exercise

Q. 1 Compute the following matrix-vector products

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ * [0.5, 0.5]

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ * [1.2, 4.44]

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ * [1, 2, 3]

Q. 2 Calculate $\begin{bmatrix} 4 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Q. 3 Let a, b be real numbers and let

$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

(a) What is AB?

(b) Without actually calculating A^2, find A^2 from AB

(c) What is A^n, when n is positive integer?

Q. 4 Compute

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ (b) $[2\ 4\ 1] \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

Q. 5 Solve the system

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Q. 6 Find $M_{2 \times 2}$ such that $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} * M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q. 7 Demonstrate for each of the pair given below, whether they are inverses of each other or not?

(a) $\begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -9 & 5 \end{bmatrix}$ Over $\mathbb{R}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ Over GF (2)

(c) $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/6 \\ -2 & 1/2 \end{bmatrix}$ Over $\mathbb{R}$

Basis

Q. 8 Let V = span $\{v_1 = [2, 0, 4, 0], v_2 = [0, 1, 0, 1], v_3 = [0, 0, -1, -1]\}$ For each of the following vectors, show it belongs to V by expressing it as a linear combination of the generators of V

(a) [2, 1, 4, 1] (b) [1, 1, 1, 0] (c) [0, 1, 1, 2]

Q. 9 Let V = Span {[0, 0, 1], [2, 0, 1], [4, 1, 2]}, for each of the following vectors show it belongs to V by writing it is a linear combination of the generator of V.

(a) [2, 1, 4] (b) [1, 1, 1] (c) [5, 4, 3] (d) [0, 1, 1]

Q. 10 Let V = Span {[0, 1, 0, 1], [0, 0, 1, 0], [1, 0, 0, 1], [1, 1, 1, 1]} where the vectors are over GF(2). For each of the following vectors over GF(2), show it belongs to V by writing it as a linear combination of the generators of V.

(a) [1, 1, 0, 0] (b) [1, 0, 1, 0] (c) [1, 0, 0, 0]

Q. 11 Show that the following set vectors over $\mathbb{R}$ are linearly dependent

(a) {(1, 2, 0), (2, 4, 1), (0, 0, – 1)}

(b) {(2, 4, 0), (8, 16, 4), (0, 0, 7)}

(c) {(1, 2, 3), (4, 5, 6), (1, 1, 1)}

Q. 12 Give four vectors that are linearly dependent but such that any three are linearly independent.

Q. 13 For each of the following problems, show that the given vectors over GF(2) are linearly dependent

(a) (1, 1, 1, 1) , (1, 0, 1, 0), (0, 1, 1, 0), (0, 1, 0,1)

(b) (0, 0, 0, 1), (0, 0, 1, 0), (1, 1, 0, 1), (1, 1, 1, 1)

(c) (1, 1, 0, 1, 1), (0, 0, 1, 0, 0), (0, 0, 1, 1, 1), (1, 0, 1, 1, 1), (1, 1, 1, 1, 1)

Q. 14 Let S = {[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0,1, 0], [0, 0, 0, 0, 1]}

A = {[1, 0, 0, 0, 0], [0, 1, 0, 0, 0]} For each of following vecto z. Find a vector w in S / A such that span (S) = Span (S U {z} – {ω})

(a) z = [1, 1, 1, 1, 1] (b) z = [0, 1, 0, 1, 0] and (c) z = [1, 0, 1, 0, 1]

Q. 15 For each of the following matrices

(a) Give a basis for the row space

(b) Give a basis for the column space

(c) Verify that the row rank and equals the column rank

(1) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$

(5) $\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 5 & 6 & 4 & 5 \\ 3 & 7 & 6 & 11 & 6 & 9 \\ 1 & 5 & 10 & 8 & 9 & 9 \\ 2 & 6 & 8 & 11 & 9 & 12 \end{bmatrix}$

Q. 16 Let W be the subspace of $\mathbb{R}^5$ spanned by the following vectors.

$u_1 = (1, 2, 1, 3, 2)$, $u_2 = (1, 3, 3, 5, 3)$

$u_3 = (3, 8, 7, 13, 8)$, $u_4 = (1, 4, 6, 9, 7)$

$u_5 = (5, 13, 13, 25, 19)$

Find the basis of W consisting of the original vectors and find dim (W).

Q. 17 Let $V = M_2(\mathbb{R})$, the vectorspace of 2×2 matrices

Where $U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$, $W = \left\{ \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \middle| \, c, d \in \mathbb{R} \right\}$ find dim (U + W)

Q. 18 Consider $\mathbb{R}^3$. The basis S of $\mathbb{R}^3$ is given by $S = \{u_1 = (1, -1, 0), u_2 = (1, 1, 0)$, $u_3 = (0, 1, 1)\}$. Find the co-ordinate representation of $v = (5, 3, 4)$ relative to the basis S.

(Ans. : 3, 2, 4)

Q. 19 Express M as the linear combination of A, B and C.

$$M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

Q. 20 Find a basis and dimension of $W \subseteq \mathbb{R}^3$ where

(a) $W = \{(a, b, c) \mid a + b + c = 0\}$ (b) $W = \{(a, b, c) \mid a = b = c\}$

Q. 21 Find the dimension and a basis of the solution space W of each of the homogeneous system.

(a)
$$\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ x + 2y + 3z + s + t &= 0 \\ 3x + 6y + 8z + s + 5t &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x + 2y + z - 2t &= 0 \\ 2x + 4y + 4z - 3t &= 0 \\ 3x + 6y + 7z - 4t &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x + y + 2z &= 0 \\ 2x + 3y + 3z &= 0 \\ x + 3y + 5z &= 0 \end{aligned}$$

Q. 22 Relative to the basis $S = \{u_1, u_2\} = \{(1, 1), (2, 3)\}$ of $\mathbb{R}^2$, find the co-ordinate vector of where, (a) $v = (4, -3)$ (b) $v = (a, b)$

Q. 23 Consider subspace $U = \{(a, b, c, d) \mid b - 2c + d = 0\}$ and $W = \{(a, b, c, d) \mid a = d, b = 2c\}$ of $\mathbb{R}^4$. Find a basis and dimension of

(a) U (b) W (c) $U \cap W$

Q. 24 Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be linear mapping defined by $f(x, y, z, t) = (x - y + z + t, x + 2z - \ldots, x + y + 3z - 3t)$ Find a basis and a dimension of (a) the image of f (b) ker (f)

Q. 25 Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$ show that g is linear mapping. Also find a basis and the dimension of

(a) image of (g) and (b) ker (g)

Q. 26 Find the matrix of f relative to the standard basis $E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ for linear mapping $F(x, y) = (2x + 3y, 4x - 5y)$

Q. 27 Consider the vector space V of the functions whose basis set $S = \{\sin(t), \cos(t), e^{3t}\}$ let $D : V \to V$ be linear mapping defines as $D(f(t)) = \frac{d}{dt}(f(t) = f'(t))$

Compute the matrix representing D in the basis S.

Q. 28 Let $G : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ be a linear mapping $G(x, y) = (2x - 7y, 4x + 3y)$ and the basis $S = \{u_1, u_2\} = \{(1, 3), (2, 5)\}$

(a) Find the matrix representation $[G]_s$ of G relative to S.

(b) Verify $[G]_s [v]_s = [G([v]_s)]$ for vector $v = (4, -3)$ is $\mathbb{R}^2$

Q. 29 Consider the following bases of $\mathbb{R}^2$

$S = \{u_1, u_2\} = \{(1, -2), (3, -4)\}$

$S' = \{v_1, v_2\} = \{(1, 3), (3, 8)\}$

Find

(a) The co-ordinates of $v = (a, b)$ relative to the basis S

(b) Find the change of basis matrix P from S to S′

(c) Find the co-ordinates of $v = (a, b)$ relative to the basis S′

(d) Find the change of basis matrix Q from S′ back to S

(e) Verify $Q = P^{-1}$

(f) Show that for any vector

$v = (a, b)$ in $\mathbb{R}^2$, $P^{-1}[v]_S = [v]_{S'}$

Q. 30 Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map defined by $f(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$

(a) Find the matrix of f in the following bases of $\mathbb{R}^3$ and $\mathbb{R}^2$:

$S = \{w_1, w_2, w_3\} = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$S' = \{u_1, u_2\} = \{(1, 3), (2, 5)\}$

Annihilators

Q. 31 Let W be the subspace of $\mathbb{R}^4$ spaced by $(1, 2, -3, 4), (1, 3, -3, 6), (1, 4, -1, 8)$. Find a basis of the annihilator of W.

Q. 32 Let W be the subspace of $\mathbb{R}^3$ spanned by (1, 1, 0) and (0, 1, 1). Find a basis of the annihilator of W.

Q. 33 Prove that if U and W are subspace of V then show $(U + W)^\circ = U^\circ \cap W^\circ$

□□□