Geometry of Calabi-Yau manifolds

Janu Verma
Department of Mathematics
Kansas State University

Contents

Introduction

This article provide an introduction to the geometry of Calabi-Yau manifolds. These notes stemmed from my attempts to learn mirror symmetry of Calabi-Yau manifolds. We assume familiarity with the notions of Riemannian geometry and Complex geometry.

1 Calabi-Yau Manifolds

The Calabi-Yau manifolds are an interesting class of complex manifolds, the existence of which was conjectured by E.Calabi and proved by S.T. Yau [1],

hence the name.
The interest in CY manifolds is greatly influenced by string theory in physics where one can associate to a Superconformal Field theory (SCFT) a Calabi-Yau 3-fold. This interaction between string theory and geometry of CY manifolds has been very fruitful, particularly the discovery of a relation between quantum field theories associated to two different CY manifolds which gives rise to beautiful geometric relations between the two CY manifolds. This is known as Mirror Symmetry of CY manifolds.
The goal of this note is to assemble the facts about the geometry of the CY manifolds which are relevant to the mirror symmetry.

Different authors use different definitions of a CY manifold. We'll define a Calabi-Yau manifold as :

Definition 1.1 *A Calabi-Yau Manifold of complex dimension m (real dimension $2m$) is a compact Kahler manifold M with vanishing first Chern class, $c_1(M) = 0$.*

Proposition 1.2 *A Calabi-Yau manifold M as defined above which is simply connected has trivial canonical bundle, $K_M = \Lambda^m T^* M^{(1,0)}$.*

Proof Consider the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

where $\mathcal{O}$ is the sheaf of holomorphic functions, $\mathcal{O}^*$ is the sheaf of nowhere zero holomorphic functions.
The second arrow is the inclusion map and the third arrow is the exponential map $f \to exp(2\pi i f)$. The long exact sequence of the cohomology has a part

$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z})$$

The Picard group $H^1(M, \mathcal{O}^*)$ classifies the bundles line bundles and the trivial line bundles are represented by 0 in Picard group. The last arrow defines the first Chern class of M.
The Dolbeault theorem

$$H^1(M, \mathcal{O}) \simeq H^{(0,1)}(M, \mathbb{C})$$

Hodge decomposition for Kahler manifolds (see section 1.2)

$$H^n(M, \mathbb{C}) = \bigoplus_{p=1}^{n} H^{(n-p,p)}(M, \mathbb{C})$$

and

$$\pi_1(M) = 0$$

imply that

$$H^1(M, \mathcal{O}) = 0$$

This implies that the last arrow is injective. Hence the line bundle with $c_1 = 0$ is trivial ■

Let ω be the Kahler form, then the above proposition implies that there exists a non vanishing holomorphic form of top degree $\Omega^{m,0}$, called *Volume form* which satisfies :

$$\omega^m = (-1)^{\frac{m(m-1)}{2}} (\frac{i}{2})^m (m!) \Omega \wedge \bar{\Omega} \tag{1}$$

Thus we can also define a CY manifold as :

Definition 1.3 *A CY Manifold is a compact, simply connected Kahler manifold manifold with trivial canonical bundle so that there exists a nowhere vanishing holomorphic section of canonical bundle.*

Remark 1.4 *Since the Ricci form of a complex manifold M is $\rho = 2\pi c_1(M)$, this implies that the Ricci form of M, $\rho = 0$ and a thus a CY manifold is naturally Ricci flat.*

There is another differential - geometric definition of a CY manifold in terms of Riemannian holonomy groups.

Definition 1.5 *Let $E \to M$ be a vector bundle over a Riemannian manifold M, with ∇^E a connection on E. Consider a loop $\gamma : [0,1] \to M$ based at a point x i.e. $\gamma(0) = \gamma(1) = x$. If*

$$P_\gamma : E_x \to E_x$$

is the parallel transport map, E_x is the fibre of E over x. We define Holonomy group, $Hol_x(\nabla^E)$ at x as :

$$Hol_x(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset GL(E_x)$$

Definition 1.6 *A compact Kahler manifold (M, g) where g is the Kahler metric is a CY manifold if the holonomy group of the Levi Civita connection on the tangent bundle TM of M is $SU(m)$.*

From now on we'll denote a CY manifold as (M, J, g, ω) where J is the complex structure on M, g is the Kahler metric, ω is the Kahler form i.e. $d\omega = 0$.

1.1 The Calabi Conjecture

Now we've seen the definition of CY manifold, let's say a word about the existence of such manifolds. This existence was conjectured by E. Calabi in 1954 and proved by S.T.Yau in 1976 using the techniques of geometric analysis[1].

The Calabi Conjecture specifies which closed $(1,1)$ forms on a compact manifold are the Ricci forms of the Kahler manifold.
Here I'll give a sketch of the proof of Calabi Conjecture. My treatment will closely follow Joyce [2].

Conjecture 1.7 *Let (M, J, g, ω) be a compact, Kahler manifold. If ρ' is a real, closed $(1,1)$ form on M with $\rho' = 2\pi c_1(M)$, then there exists a unique Kahler metric g' with Kahler form ω' sucth that $[\omega] = [\omega']$ i.e g, g' are in same Kahler class and ρ' is the Ricci form of g'.*

Proof (sketch) Yau translated the problem of finding a unique metric on M to the problem of proving that a nonlinear partial differential equation (Complex Monge - Ampere equation) has a solution. We'll divide the proof in following three steps :
Step I :

Proposition 1.8 *Let (M, J, g, ω) be a Kahler manifold with Ricci form ρ. If ρ' is a real, closed $(1,1)$ form such that $[\rho'] = 2\pi c_1(M)$. Then there exists a unique smooth real valued function $f :\to \mathbb{R}$ such that :*

$$\rho' - i\partial\bar{\partial}f$$

and

$$\int_M e^f \omega^m = \int M\omega^m$$

and a Kahler metric g' on M in same Kahler class as g has the Ricci form ρ' if and only if

$$(\omega')^m = e^f \omega^m$$

Proof Let g, g' be two Kahler metrics on (M, J) with corresponding Kahler forms ω, ω' in the same Kahler class, i.e. $[\omega] = [\omega']$. Define a smooth real valued function

$$f : M \to \mathbb{R}, \qquad (\omega')^m = e^f \omega^m$$

Since the Ricci form is locally given as :

$$\rho = -i\partial\bar{\partial}(log(f))$$

on U, open in M for a smooth function

$$f : U \to (0, \infty)$$

defined by demanding

$$\omega^m = f\frac{(-1)^{m(m-1)/2} i^m m!}{2^m} dz_1 \wedge ... \wedge dz_m \wedge d\bar{z}_1 \wedge ... \wedge d\bar{z}_m$$

this implies,

$$\rho' = \rho - i\partial\bar{\partial}f$$

Also

$$[\omega'] = [\omega]$$

implies

$$\int_M e^f \omega^m = \int_M = \omega^m$$

We're given

$$[\rho] = [\rho'] = 2\pi c_1(M)$$

this implies
$\Rightarrow \rho - \rho'$ is an exact real $(1,1)$ form given as

$$\rho - \rho' = i\partial\bar{\partial} f$$

for a smooth real valued function f.

f is unique upto addition of a constant, but this constant can be fixed by requiring that

$$\int_M e^f \omega^m = \int_M \omega^m$$

Step II:
Here we'll reduce the hypothesis of Calabi Conjecture to an elliptic p.d.e and pose the statement of the conjecture as seeking a solution of the p.d.e.

Since $[\omega] = [\omega']$, we can write

$$\omega' = \omega + i\partial\bar{\partial}\phi$$

where ϕ is a smooth real valued function on M which is unique upto the addition of a constant. This constant can be fixed by requiring that $\int_M \phi = 0$. Using the last proposition, the Calabi Conjecture can be restated as :

Refined Conjecture *Let (M, J, g, ω) be a compact Kahler manifold and let f be a smooth real valued function on M such that $\int_M e^f \omega^m = \int_M \omega^m$. Then there exists a unique smooth real valued function ϕ such that :*

(i)

$$\omega + i\partial\bar{\partial}\phi$$

is a positive $(1,1)$ form that is the Kahler form of some Kahler metric g',
(ii)

$$\int_M \phi \, dV_g = 0$$

(iii)

$$(\omega + i\partial\bar{\partial}\phi)^m = e^f \omega^m$$

Step III :

Thus the Calabi Conjecture essentially boils down to showing that the non linear p.d.e.

$$(\omega + i\partial\bar{\partial}\phi)^m = e^f \omega^m$$

has a solution ϕ for every suitable f.
Yau employed the *Continuity Method* to prove the refined conjecture.
for each $t \in [0,1]$ define

$$f_t = tf + c_t$$

, where c_t is a unique real constant fixed by

$$\int_M e^{tf} \omega^m = \int_M \omega^m$$

f_t varies smoothly with t with $f_0 = 0$ and $f_1 = f$. Define S to be the subset of $[0,1]$ consisting of all t for which there exists a smooth real valued function ϕ on M satisfying the conditions $(i), (ii)$ of the refined conjecture and also
(iv)

$$(\omega + i\partial\bar{\partial}\phi)^m = e^{tf} \omega^m$$

on M.

Idea of Continuity method is to show that S is both open and closed. But since $[0,1]$ is connected, $S = \phi$ or $S = [0,1]$.
But $0 \in S$ since $f_0 = 0$ satisfies $(i), (ii), (iv)$.
Thus $S = [0,1]$. In particular $(i), (ii), (iii)$ admit a solution ϕ for $t = 1$, whence $(iv) = (iii)$.
This completes the proof of the Calabi Conjecture.

Corollary 1.9 *Let (M, J, g, ω) be a compact Kahler manifold with $c_1(M) = 0$. Then there exists a unique Ricci-flat Kahler metric in each Kahler class on M.*

1.2 Hodge Theory on CY manifolds

First we'll give a brief description of the Hodge theory on a Kahler manifold. We'll sketch a proof of the Hodge Theorem.(See [12])
Consider a connected, compact n-dimensional Kahler manifold (M, ω). If Φ is the Volume form, then we define *Inner Product* on $A^{p,q}(M)$ as:

$$(\psi, \eta) = \int_M (\psi(z) \wedge \eta(z))\Phi(z) \tag{2}$$

Definition 1.10 *Define the Hodge Operator, $*$ as a map*

$$* : A^{p,q}(M) \to A^{n-p,n-q}(M) \tag{3}$$

defined by requiring that

$$(\psi, \eta) = \psi(z) \wedge *\eta(z) \tag{4}$$

It is easy to see that $** = (-1)^{p+q}$.

Definition 1.11 *Define the Adjoint of $\bar{\partial}$ as*

$$\bar{\partial}^* := - * \bar{\partial} * \tag{5}$$

It is indeed the adjoint of $\bar{\partial}$ with respect to the inner product.

Definition 1.12 *The Laplacian of $\bar{\partial}$ is defined as :*

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : A^{p,q} \to A^{p,q}$$

The (p,q) - form ψsuch that $\Delta\psi = 0$ is called a Harmonic form of (p,q) type. The space of (p,q) harmonic forms is denoted by $\mathcal{H}^{p,q}(M)$.

Theorem 1.13 *(Hodge) For a compact, connected Kahler manifold M the following holds :*
(i) dim $\mathcal{H}^{p,q} < \infty$.
(ii) There is an orthogonal projection map, $\mathcal{H} : A^{p,q} \to \mathcal{H}^{p,q}$.
(iii) There is a unique operator $G : A^{p,q}(M) \to A^{p,q}(M)$ such that
$G(\mathcal{H}^{p,q}(M)) = 0, \quad \bar{\partial}G = G\bar{\partial}, \quad \bar{\partial}^*G = G\bar{\partial}^*$
and $\mathbf{1} = \mathcal{H} + \Delta G$

Proof The proof is rather involved and uses the machinery of functional analysis and elliptic operators, We'll give an outline of the proof skipping over the details.
Consider a vector bundle E over M and let ∇ be the connection on E. For a smooth section $f \in C^\infty(M, E)$, its Sobolev s-norm is defined as

$$||f||_s^2 = \sum_{k \le s} \int_M ||\nabla^k f||^2 dx \tag{6}$$

The completion of $C^\infty(M, E)$ under the Sobolov norm is called the *Sobolov Space*, $H_s(M, E)$.
We have the following facts in functional analysis.

Lemma 1.14 *(Sobolev) $H_{s+[n/2]+1}(M, E) \subset C^s(M, E)$ and*

$$H_\infty = \cap H_s(M, E) = C^\infty(M, E)$$

Lemma 1.15 *(Rellich) For $s > r$, the inclusion*

$$H_s(M, E) \to H_r(M, E) \tag{7}$$

is compact operator i.e. any bounded sequence in $H_s(M, E)$ has a subsequence whose image converges in $H_r(M, E)$.

Now, denote $H_s^{p,q}(M) = H_s^{p,q}(M, A^{p,q}(M))$. Define the *Dirichlet Inner Product* as :

$$\mathcal{D}(\phi, \psi) = (\phi, \psi) + (\bar{\partial}\phi, \bar{\partial}\psi) + (\bar{\partial}^*\phi, \bar{\partial}\psi) \tag{8}$$

and *Dirichlet norm* $\mathcal{D}(\phi) = \mathcal{D}(\phi, \phi)$.
An estimate from $(\Delta\psi, \psi)$ gives

Lemma 1.16 *(Garding's) There exists $C > 0$ such that for any $\phi \in A^{p,q}(M)$, $||\phi||_1^2 \leq C.\mathcal{D}(\phi)$.*

The ellipticity of the Laplacian gives

Lemma 1.17 *(Regularity) Suppose $\phi \in H_s^{p,q}(M), \psi \in H_0^{p,q}$ is a weak solution of the equation*

$$\Delta\psi = \phi$$

then $\psi \in H_{s+2}^{p,q}(M)$.

The following lemma follows easily from the Garding's lemma.

Lemma 1.18 *There is a map*

$$T : \mathcal{H}_0^{p,q} \to \mathcal{H}_1^{p,q}$$

determined uniquely by

$$(\phi, \eta) = \mathcal{D}(T\phi, eta)$$

for all $\eta \in A^{p,q}(M)$ and $\phi \in \mathcal{H}_0^{p,q}(M)$. Futhermore T is compact and self-adjoint.

Now we have

Theorem 1.19 *(Spectral Theorem) Let T be a campact, self-adjoint operator on Hilbert space H, $E(\rho)$ be the ρ eigenspace. Then all eigenvalues are real, all eigenspaces are finite dimensional. Futhermore H is the completion of $\oplus E(\rho)$ and the only accumulation point of the set of eigenvalues is 0.*

Let T be the compact, self-adjoint operator satisfying

$$(\phi, \eta) = \mathcal{D}(T\phi, \eta) \quad \text{for all} \quad \eta \in A^{p,q}(M)$$

With respect to the spectrum of T, we've the decomposition

$$\mathcal{H}_0^{p,q}(M) = \oplus_m E(\rho) \tag{9}$$

and $\rho_m \to 0$.
Also T is one - one and therefore $\rho_m \neq 0$

$$T\phi = \rho_m\phi$$

implies that

$$\Delta\phi = \frac{1-\rho_m}{\rho_m}\phi$$

Let $\rho_0 = 1$, notice that $\mathcal{H}^{p,q} = E(\rho_0)$.
Define the operator G as

$$G = 0 \text{on} \mathcal{H}^{p,q}$$

$$G\phi = \frac{1-\rho_m}{\rho_m}\phi \tag{10}$$

for $\phi \in E(\rho_m), m \geq 1$.
Obviously G is compact, self-adjoint, satisfies $\mathbf{1} = \mathcal{H} + \Delta G$.
By regularity lemma, G is smooothing operator. It is easy to see that eigenspaces of T are invariant under $\bar{\partial}$ and $\bar{\partial}^*$ in weak sense and hence,

$$\bar{\partial}G = G\partial, \qquad \bar{\partial}^* G = G\bar{\partial}^*$$

This completes our proof of the Hodge Theorem.

Corollary 1.20 *We have the decomposition*

$$A^{p,q}(M) = \mathcal{H}^{p,q} \oplus \bar{\partial}A^{p,q-1}(M) + \bar{\partial}^* A^{p,q+1}(M) \tag{11}$$

and projection map $\mathcal{H}$

$$\mathcal{H}^{p,q}(M) \simeq H^{p,q}_{\bar{\partial}}(M) \tag{12}$$

Corollary 1.21 *If M a Calabi-Yau 3-fold M then all the Hodge numbers vanish except $h^{1,1}$ and $h^{2,1}$.*

Proof Since M is a Kahler manifold, the Hodge theory implies :

$$\overline{H^{p,q}(M,\mathbb{C})} = H^{q,p}(M,\mathbb{C}) \tag{13}$$

thus,

$$h^{p,q} = h^{q,p} \tag{14}$$

Also the Serre duality states :

$$H^{p,q}(M,\mathbb{C}) = H^{m-p,m-q}(M,\mathbb{C}) \tag{15}$$

i.e.

$$h^{p,q} = h^{m-p,m-q} \tag{16}$$

A CY manifold has trivial canonical bundle,

$$h^{m,0} = h^{0,m} = 0 \tag{17}$$

and Serre Duality implies

$$h^{0,n-1} = h^{n-1,0} = 0 \tag{18}$$

As we saw in proposition (1.2),

$$h^{1,0} = h^{0,1} = 0 \tag{19}$$

Thus for a CY 3-fold we have,

1.3 Decomposition Theorem

In this subsection we'll sketch a proof of the Bogomolov decomposition theorem. This gives a crude classification of Kahler manifolds with vanishing first Chern class, in particular CY manifolds.

Theorem 1.22 *Let (M, J, g, ω) be a compact Ricci flat Kahler manifold, then it admits a finite cover isomorphic to the product manifold*

$$(T^{2n} \times M_1 \times ... \times M_k, J_0 \times ... \times J_k, g_0 \times ... \times g_k) \tag{20}$$

where (T^{2n}, J_0, g_0) is a flat Kahler torus and each (M_i, J_i, g_i) is a compact, simply connected, irreducible, Ricci flat Kahler manifold.
Furthermore, i

$$m_i = dim_{\mathbb{C}} M_i$$

then either

$$Hol(g_i) = SU(m_i)$$

for $m_i \geq 2$ or

$$Hol(g_i) = Sp(\frac{m_i}{2})$$

for $m_i \geq 4$.

Proof To start with we have a profound result from Riemannian Geometry : If (M, g) is a compact Riemannian manifold, then M admits a finite cover isomorphic to the product manifold

$$T^n \times N$$

where T^n is flat torus and N is a compact simply connected Riemannian manifold.
Now N is compact, it is complete. Thus the following result of de Rham applies-
de Rham Theorem: A complete, simply connected, Riemannian manifold is isomorphic to a Riemannian product of compact, simply connected, irreducible Riemannian manifolds,

$$(M_1 \times ... \times M_k, g_1 \times ... \times g_k)$$

As g is Kahler, this implies T^n and M_i are also Kahler. Thus $n = 2l$, even dimensional. Since each M_i is simply connected, Ricci flat, we've :

If (M, J, g) is a m-dimensional Ricci-flat Kahler manifold which is simply connected, the $Hol(g) \subset SU(m)$.
Thus from the Berger's theorem on the classification of the Riemannian Holonomy groups, the only possibilities are :

$$Hol(g_i) = SU(m_i)$$

for $m_i \geq 2$ or

$$Hol(g_i) = Sp(m_i/2)$$

for $m_i \geq 4$.
In more algebro-geometric settings,

Corollary 1.23 *If (M, g) is a compact Kahler manifold with $c_M = 0$, then M admits a finite cover which is isomorphic to*

$$Z \times \prod_i S_i \times \prod_i C_i \tag{21}$$

where

- *Z is a complex torus.*
- *each S_i is a simply connected holomorphic symplectic manifold with dim $H^2(S_i, \mathcal{O}_{S_i}) = 1$.*
- *each C_i is a simply connected Calabi - Yau manifold with dim $H^2(C_i, \mathcal{O}_{C_i}) = 0$.*

2 Complex Moduli of CY manifolds

Our main goal in this section is to construct the space of the deformations of the complex structure of a CY manifold (M, J), we'll call this space the Complex moduli space of M denoted $\mathcal{M}_{cx}(M)$. Before delving into the construction complex deformation space for a CY manifold, lets first look at the general theory of deformations of complex structures.

2.1 Deformations of Complex structures

Let (M, J) be a complex manifold. We are interested in the space of all possible complex structures on M varying in a continuos manner quotiented by some equivalence relation.

$$\mathcal{M}_{cx}(M) = \{\text{Complex structures on M}\} \backslash \sim \tag{22}$$

where

$$J \sim J' \tag{23}$$

if there exists a diffeomorphism

$$\phi : M \to M \quad \text{such that} \quad \phi^* J' = J \tag{24}$$

Definition 2.1 *A Deformation of a complex manifold M consists of a surjective, proper, and flat map*

$$p : \chi \to S$$

called a family of Complex manifolds, such that for sS,a distinguished point, $p^{-1}(s)$ is isomorphic to M.

A deformation $\chi \to S$ is called *Universal deformation* if for any other deformation $\chi' \to S'$ there exists a unique analytic function $\phi : S' \to S$ such that

$$\phi(s' = s) \quad \text{and} \quad \phi^*(\chi) \simeq \chi'$$

We'll denote S of the universal deformation as Def(M).
We've the following result from the deformation theory of complex structure of a complex manifold.

Theorem 2.2 *(Kuranishi) If M is a compact complex manifold with $H^0(M, T_M) = 0$, then a universal deformation of X exists.*

The purpose of this section is to show that a CY manifold belongs to Kuranishi family.

Consider a fixed complex structure on M, we've the decomposition

$$T_M \otimes \mathbb{C} = T_M^{1,0} \bigoplus T_M^{0,1} \tag{25}$$

$$T_M^* \otimes \mathbb{C} = \Omega_M^{1,0} \bigoplus \Omega_M^{0,1} \tag{26}$$

Let

$$\pi^{1,0} : T_M^* \otimes \mathbb{C} \to \Omega_M^{1,0} \tag{27}$$

and

$$pi^{0,1} : T_M^* \otimes \mathbb{C} \to \Omega_M^{0,1} \tag{28}$$

be the projection maps.
For a nearby complex structure J' on M, we've

$$T_M^* \otimes \mathbb{C} = \Omega_{M'}^{1,0} \bigoplus \Omega_{M'}^{0,1} \tag{29}$$

here nearby means

$$\pi^{1,0}|_{\Omega_{M'}^{1,0}} : \Omega_{M'}^{1,0} \simeq \Omega_M^{1,0} \tag{30}$$

Define

$$s : \Omega_M^{1,0} \to \Omega_M^{0,1} \tag{31}$$

as

$$s = -\pi^{0,1}(\pi^{1,0}|_{\Omega^{1,0}_{M'}})^{-1} \tag{32}$$

We can also view s as a smooth section of

$$T^{1,0}_M \otimes \Omega^{0,1}_M$$

i.e. the $(1,0)$ vector valued $(0,1)$ form.
Conversely, a given smooth section

$$s \in \Lambda(M, T^{1,0}_M \otimes \Omega^{0,1}_M)$$

determines an almost complex structure on M as follows ;
Take $\Omega^{1,0}_M$ to be the graph of the map

$$s : \Omega^{1,0}_M \to \Omega^{0,1}_M \tag{33}$$

The bundle $\Omega^{0,1}_M$ is the complex conjugate of $\Omega^{1,0}_M$, both the bundles viewed as the subbundles of $T^*_M \otimes \mathbb{C}$. If s is sufficiently small, we obtain a splitting

$$T^*_M \otimes \mathbb{C} = \Omega^{1,0}_{M'} \bigoplus \Omega^{0,1}_{M'} \tag{34}$$

This determines an almost complex structure J by determining the $\pm i$ eigenspaces of J. The almost complex structure thus obtained may not be integrable. Lets now look at the conditions for this almost complex structure to be integrable.

Lemma 2.3 *If $s \in \Lambda(M, T^{1,0}_M \otimes \Omega^{0,1}_M)$ induces an almost complex structure on M, then this almost complex structure is integrable iff s satisfies the Maurer-Cartan equation i.e.*

$$\bar{\partial} s + \frac{1}{2}[s, s] = 0 \tag{35}$$

Proof If $(z_1, z_2, ...z_n)$ are the holomorphic coordinates on M, we can write s as :

$$s = \sum_{i,j} s_{ij} \frac{\partial}{\partial z_i} \otimes d\bar{z}_j \tag{36}$$

then

$$\theta_i := dz_i - s(dz_i) = dz_i - \sum_j s_{ij} d\bar{z}_j$$

form a local basis for the space of $(0,1)$ forms on M'.
Since $\theta_1, \theta_2, ..., \theta_n, d\bar{z}_1, ..., d\bar{z}_n$ form a basis for the space of $1-$forms, we can write ,

$$d\theta_l = \sum_{i,j} A^l_{ij} \theta_i \wedge \theta_j + \sum_{i,j} B^l_{ij} \theta_i \wedge d\bar{z}_j + \sum_{i,j} C^l_{ij} d\bar{z}_i \wedge d\bar{z}_j \tag{37}$$

A standard result in the theory of complex manifolds is that the almost complex structure is integrable if $d\theta_i$ is of type $(2,0) + (1,1)$ for all i.

Thus

$$C_{ij}^l = 0 \tag{38}$$

Since

$$\theta_l = dz_l - \sum_j s_{ij} s\bar{z}_j$$

$$d\theta_l = -\sum \frac{\partial s_{lj}}{\partial z_i} dz_i \wedge d\bar{z}_j - \sum_{i,j} \frac{\partial s_{lj}}{\partial \bar{z}_i} d\bar{z}_i \wedge d\bar{z}_j$$

This implies that

$$A_{ij}^l = 0 \text{ and } B_{ij}^l = -\frac{\partial s_{lj}}{\partial z_i} \tag{39}$$

Thus the almost complex structure is intergrable iff -

$$\sum_i (\frac{\partial_{lk}}{\partial z_i} s_{ij} - \frac{\partial s_{lj}}{\partial z_i} s_{ik}) = \frac{\partial s_{lj}}{\partial \bar{z}_k} - \frac{\partial s_{lk}}{\partial \bar{z}_j} \tag{40}$$

In coordinate free form, we've

$$\bar{\partial} s + \frac{1}{2}[s, s] = 0 \tag{41}$$

Now consider a curve on $Def(M)$ through 0 which we can think of as a continuos family

$$s(t) \in \Lambda(M, T_M^{1,0} \otimes \Omega_M^{0,1}) \tag{42}$$

with $s(0) = 0$, $t \in \mathbb{C}$ is a complex parameter. As we've seen for t close to zero, $s(t)$ induces an almost complex structure on M which is integrable iff

$$\bar{\partial} s(t) + \frac{1}{2}[s(t), s(t)] = 0 \tag{43}$$

$s(t)$ thus determines a family of complex structures on M. We still need to quotient out by the diffeomorphisms of M. Hence the deformation of a complex structure on a complex manifold M produces a T_M valued $(0, 1)$ form satisfying the Maurer-Cartan equation.

In case of Calabi-Yau manifolds,
with each

$$s_i \in \Lambda(M, T_M^{1,0} \otimes \Omega_M^{0,1}) \tag{44}$$

Let's first look at the equation at first order, $n = 1$ i.e. we'll consider the equation modulo t^2

This implies

$$\bar{\partial} s_1 = 0 \tag{45}$$

i.e. s_1 is $\bar{\partial}-$ closed. Also we need to quotient out by the action of the a family of differeomorphisms

$$\phi_t : M \to M, \quad \phi_0 \text{ is the identity.} \tag{46}$$

To first order, we can write ϕ_t locally as

$$z_i \to z_i + tf_i(z, \bar{z}) + ... \tag{47}$$

Taking the action of ϕ_t on $s(t)$ we obtain the space of possible s_1's modulo the action of diffeomorphisms as :

$$Def_1(M) = \{s \in \Lambda(M, T_M^{1,0} \otimes \Omega_M^{0,1} | \bar{\partial} s = 0)\} \backslash \Lambda(M, T_M^{1,0}) = H^1(M, T_M) \tag{48}$$

And for higher orders we've :

$$\bar{\partial} s_n(t) = \frac{1}{2} \sum_{j=1}^{n-1} [s_n, s_{n-j}] = 0, \quad \text{for} \;\; n \geq 2 \tag{49}$$

Now we've the Kodaira-Spencer theorem which says :

Theorem 2.4 *(Kodaira-Spencer[10]) If M is a compact complex manifold with $H^2(M, T_M) = 0$, then a universal deformation exists and the $Def(M)$ can be identified with an open neighborhood of 0 in $H^1(M, T_M)$. In particular the deformation space is unobstructed.*

For M a CY manifold of dimension m, the existence of a non vanishing $m-$form i.e a section of $\Omega_M^{m,0}$ implies the isomorphism:

$$\Omega_M^{m-1} \simeq T_M \tag{50}$$

, this implies,

$$H^0(M, T_M) = H^0(M, \Omega_M^{m-1}) \tag{51}$$

This isomorphism along with the Dolbeault theorem gives :

$$\begin{aligned} H^1(M, T_M) &= H^1(M, \Omega_M^{m-1}) && (52)\\ &= H^{m-1,1}(M, \mathbb{C}) && (53)\\ H^2(M, T_M) &= H^{n-2,1}(M, \mathbb{C}) && (54) \end{aligned}$$

This is a very bad news!
Since $H^2(M, T_M) \neq 0$, we cannot use the Kodaira -Spencer Theory. But fortunately the existence of smooth deformation has been proved for Calabi - Yau manifolds.

2.2 Bogomolov-Todorov-Tian Theorem

The BTT theorem was proved independently by Bogomolov, Todorov and Tian, hence the name. This result establishes that there exists a universal deformation of a Calabi-Yau manifold. In this section, we'll give a proof of the BTT theorem, our treatment will closely follow Tian.

Theorem 2.5 *(BTT [7, 8, 9]) If M is a compact Kahler manifold with $c_1(M) = 0$, then a universal deformation space exists and $Def(M)$ is isomorphic to an open subset in $H^1(M, T_M)$.*

Proof Define a natural map

$$I_k : \Gamma(M, \Omega^{0,k} \otimes T_M) \to \Gamma(M, \Omega^{m-1,k} \otimes K_M^{-1}) \tag{55}$$

using the inner product on the vector field component with

$$\Theta = dz^1 \wedge dz^2 ... \wedge dz^n$$

and then wedging with the $(0, k)$-form component. Let

$$\phi \in \Gamma(M, \Omega^{0,k} \otimes T_M)$$

it can be locally written as

$$\phi = \sum_{i,|J|=k} f_J^i \frac{\partial}{\partial z_i} d\bar{z}^J$$

then,

$$I_k(\phi) = dz^1 \wedge ... \wedge dz^n \left(\sum_{i,|J|=k} f_J^i \frac{\partial}{\partial z_i} d\bar{z}^J \right) = \sum_{i,|J|=k} (-1)^{i-1} f_J^i dz^1 \wedge ... \wedge \hat{dz^i} \wedge ... \wedge dz^n \wedge d\bar{z}^J \tag{56}$$

Since K_M is trivial,

$$I_k(\phi) \in \Gamma(M, \Omega_M^{n-1,k})$$

Using I_k, we can define Lie bracket on $\Gamma(M, \Omega_M^{n-1,k})$.
Before moving to the proof of the theorem, let's first prove the following lemma, which we'll be using in the proof of the theorem.

Lemma 2.6 *If $\beta_i \in \Gamma(M, \Omega_M^{n-1,1})$ for $i = 1, 2$, then*

$$[\beta_1, \beta_2] = \partial(I^{-1}(\beta_1) L \beta_2) - \#(\partial\beta_1) \wedge \beta_2 + \beta_1 \wedge \#\beta_2$$

where, L denotes the inner product, and $\#$ is the inverse of the map

$$\Gamma(M.\Omega_M^{n,q}) \to \Gamma(M, \Omega_M^{0,q})$$

given by wedging with Θ.

Proof In local holomorphic coordinates, β_i can be written as

$$\beta_i = (-1)^{k_i - 1} \phi^i dz^1 \wedge ... \wedge \hat{dz^{k_i}} \wedge ... \wedge dz^n \wedge d\bar{z}^{j_i} \tag{57}$$

then

$$I^{-1}\beta_i = \phi^i \partial_{k_1} d\bar{z}^{j_i}$$

hence

$$[I^{-1}\beta_1, I^{-1}\beta_2] = \left(\phi^1 \frac{\partial \phi^2}{\partial z^{k_1}} \partial_{k_2} - \phi^2 \frac{\partial \phi^1}{\partial z^{k_2}} \partial_{k_1} \right) d\bar{z}^{j_1} \wedge d\bar{z}^{j_2}$$

then

$$I[I^{-1}\beta_1, I^{-1}\beta_2] = (-1)^{k_2-1}\phi^1\frac{\partial\phi^2}{\partial z^{k_1}}dz^1\wedge...\wedge d\hat{z^{k_2}}\wedge...\wedge dz^n\wedge d\bar{z}^{j_1}\wedge d\bar{z}^{j_2}$$
$$-(-1)^{k_1-1}\phi^2\frac{\partial\phi^1}{\partial z^{k_2}}dz^1\wedge...\wedge d\hat{z^{k_1}}\wedge...\wedge dz^n\wedge d\bar{z}^{j_1}\wedge d\bar{z}^{j_2}$$

Now we calculate the other side, we've three cases

$$i)k_1 > k_2$$
$$ii)k_1 = k_2$$
$$iii)k_1 < k_2$$

The case $k_1 = k_2$ is plain. We'll consider the $k_1 < k_2$, the other is similar.

$$I^{-1}\beta_1\mathcal{L}\beta_2 = (-1)^{k_1+k_2}\phi^1\phi^2 dz^1\wedge....\wedge d\hat{z^{k_1}}\wedge....\wedge d\hat{z^{k_2}}\wedge...\wedge dz^n\wedge d\bar{z}^{j_1}\wedge d\bar{z}^{j_2}$$

hence

$$\partial\left(I^{-1}\beta_1\mathcal{L}\beta_2\right) = (-1)^{k_2-1}\left(\phi^1\frac{\partial\phi^2}{\partial z^{k_1}} + \phi^2\frac{\partial\phi^1}{\partial z^{k_1}}\right)dz^1\wedge...\wedge d\hat{z^{k_2}}\wedge....\wedge dz^n\wedge d\bar{z}^{j_1}\wedge d\bar{z}^{j_2}$$
$$-(-1)^{k_1-1}\left(\phi^2\frac{\partial\phi^1}{\partial z^{k_2}} + \phi^1\frac{\partial\phi^2}{\partial z^{k_2}}\right)dz^1\wedge...\wedge d\hat{z^{k_1}}\wedge....\wedge dz^n\wedge d\bar{z}^{j_1}\wedge d\bar{z}^{j_2}$$
$$= [\beta_1,\beta_2] - \#(\partial\beta_1)\wedge\beta_2 + \beta_1\wedge\#(\partial\beta_2)$$

This completes the proof of the lemma.
Now coming back to the proof of the theorem.
Recall,

$$H^{M,T_M} \simeq \{\phi\in\Gamma(M,\Omega_M^{n-1,1}) : \Delta_{\bar{\partial}}\phi = 0\}$$

Also, we've

$$\Delta_\partial = \Delta_{\bar{\partial}} \qquad \text{Since } M \text{ is a Kahler manifold}$$

this means that for any $s_1 \in H^1(M, T_M)$ we've $\partial s_1 = 0$. Also one can check that

$$\bar{\partial}[s_1, s_1] = 0$$

Thus,

$$\frac{1}{2}[s_1, s_1] = \partial\bar{\partial}\phi \tag{58}$$

for

$$\phi\in\Gamma(M,\Omega_M^{n-2,1})$$

Set $s_2 = \partial\phi$, then

$$\bar{\partial}s_2 + \frac{1}{2}[s_1, s_1] = 0 \tag{59}$$

Proceeding by induction, we get $s_1, s_2, s_3, ..., s_n$ such that

$$\partial s_i = 0 \tag{60}$$

$$\bar{\partial}s_i + \frac{1}{2}\sum_{j=1}^{i-1}[s_j, s_{j-1}] = 0 \tag{61}$$

Now from the lemma (2.6), we get

$$\frac{1}{2}\sum_{j=1}^{n}[s_j, s_{n-j}] = \frac{1}{2}\sum_{j=1}^{n}\partial(I^{-1}(s_j)Ls_{n-j}) \tag{62}$$

$$= \frac{1}{2}\partial\left(\sum_{j=1}^{n}I^{-1}(s_j)Ls_{n-j}\right) \tag{63}$$

This implies,

$$\bar{\partial}\left(\sum_{j=1}^{n}[s_j, s_{n-j}]\right) = 0 \tag{64}$$

This gives

$$\frac{1}{2}\sum_{j=1}^{n}[s_j, s_{n-j}] = \partial\bar{\partial}s_{n+1} \tag{65}$$

This completes the induction and thus our proof of the BTT theorem.

Remark 2.7 *There is another more algebraic proof of this theorem due to Kawamata which is based on the Grothendieck-Mumford-Schlessinger deformation theory. Kawamata proved the unobstruction result building on the work of Deligne and Ran on T^1 lifting criterion. A good reference, where a proof is sketched is Gross [2].*

3 Kahler Moduli of a Calabi-Yau

The Kahler moduli space of a Calabi-Yau manifold M is the deformation space of Kahler structure on M. The Kahler structure is given by a closed Hermitian $1,1$ form, ω called the *Kahler form* of M.
For a compact Kahler manifold, M, set

$$H^{1,1}(M,\mathbb{R}) := H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$$

The elements of $H^{1,1}(M,\mathbb{R})$ are real closed $(1,1)$ forms.
The Kahler moduli is given expressed as *Kahler Cone*.

Definition 3.1 *The Kahler Cone $\mathcal{K}(M)$ of a compact Kahler manifold M is defined as :*

$$\mathcal{K}(M) = \{\alpha \in H^{1,1}(M,\mathbb{R}) | \alpha \textit{ is represented by Kahler form on } M\} \tag{66}$$

Lemma 3.2 *$\mathcal{K}(M)$ is an open subset in $H^{1,1}(M,\mathbb{R})$*

Proof Let β be a real closed $(1,1)$ form, then it an be written as :

$$\beta = i \sum_{i,j} \beta_{ij} dz_i \wedge d\bar{z}_j$$

where β_{ij} is a Hermitian matrix. If

$$\omega = i \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j$$

is a Kahler form with g_{ij} positive definite Hermitian.
Since M is a compact manifold, for sufficiently small ϵ, $g_{ij} + \epsilon\beta_{ij}$ is still positive definite Hermitian at each $x \in M$.
Thus we can deform any Kahler form ω to another Kahler form $\omega + \epsilon\beta$ in any direction $\beta \in H^{1,1}(M, \mathbb{R})$ and so $\mathcal{K}(M)$ is open in $H^{1,1}(M, \mathbb{R})$.

In particular $\mathcal{K}(M)$ is a real manifold with tangent space $H^{1,1}(M.\mathbb{R})$.

Now Cor. (1.8) says that each Kahler class is represented by a unique Ricci flat metric, thus we can define the Kahler moduli of a compact Kahler manifold (CY, in particular) M as :

$$\mathcal{M}_{Kah} = \{\text{Space of Ricci flat Kahler metrics on } M\} \tag{67}$$

Motivated by physics and Mirror symmetry, we define *Complexified Kahler Moduli* of a Calabi-Yau 3-fold.

Definition 3.3 *If M is a simply connected Calabi-Yau 3-fold i.e. $H^{M,\mathcal{O}_M} = 0$, then the Complexified Kahler Moduli Space of M, which we'll again denote as $\mathcal{M}_{Kah}(M)$ is defined as :*

$$\mathcal{M}_{Kah}(M) = (H^2(M, \mathbb{R}) + i\mathcal{K}(M))/H^2(M, \mathbb{Z}) \tag{68}$$

Elements of the complexified Kahler moduli space are written as

$$B + i\omega, \qquad B \in H^2(M, \mathbb{R})/H^2(M, \mathbb{Z})$$

where B, known as $B-$field, has its origin in String theory.

If $h^{1,1}(M) = 1$, then

$$\mathcal{K}_{Kah}(M) = (\mathbb{R} + i\mathbb{R}^+)/\mathbb{Z} = \mathcal{H}/\mathbb{Z} \tag{69}$$

the action of Z on upper half plane is

$$z \to z + 1$$

We can naturally identify the Moduli space to the punctured disk,

$$\Delta^* = \{q \in \mathbb{C} | q \neq 0, |q| < 1\} \tag{70}$$

as

$$\mathcal{H}/\mathbb{Z} \to \Delta^* \tag{71}$$

$$z \to q = e^{2\pi iz} \tag{72}$$

If $h^{1,1}(M) > 1$, choose a basis $(e_1, e_2, ..., e_n)$ of $H^2M, \mathbb{Z}$. This defines a cone

$$\Sigma = \{\alpha \in H^2(M, \mathbb{R}) | \alpha = \sum_i t_i e_i, t_i > 0\}$$

Then write the complexified Kahler moduli as :

$$\mathcal{M}_{Kah,\Sigma}(M) = (H^2(M, \mathbb{R}) + i\Sigma)/H^{M,\mathbb{Z}} \tag{73}$$

This we can identify with $n-$ punctured disk,

$$(\Delta^*)^n = \{(q_1, ..., q_n) \in \mathbb{C}^n | \prod_i q_i \neq 0, |q_i| < 1\}$$

where $n = h^{1,1}(M)$, as

$$\mathcal{M}_{Kah,\Sigma} \to (\Delta^*)^n \sum_i z_i e_i \to (e^{2\pi iz_1}, ..., e^{2\pi iz_n}) \tag{74}$$

The Complexified Kahler moduli space is a Complex manifold of dimension $h^{1,1}(M)$.

4 Mirror Symmetry : Hodge Theoretic Version

To each Calabi-Yau 3-fold, M, we can associate a Superconformal Field Theory, $SCFT(M)$ (a QFT which gives the low energy string theory). Properties of $SCFT(M)$ depend on the geometry of M, e.g. the Doldeault cohomology groups of M and the number of holomorphic curves translate to properties of $SCFT(M$.
Sometimes, two different Calabi-Yau 3-folds, X and Y may have the isomorphic SCFTs associated to them and this gives to beautiful geometric relations between the two Calabi-Yau 3-folds, called *Mirror Symmetry of Calabi-Yau 3-folds.*
Mirror Symmetry is a complicated phenomenon and we'll here give the first statement of Mirror Symmetry which was based on the predictions of physicists[11].

Definition 4.1 *Two Calabi-Yau 3-folds X and Y are called a Mirror Pair if there exist a correspondences :*

$$\mathcal{M}_{cx}(X) \quad \leftrightarrow \quad \mathcal{M}_{Kah}(Y) \tag{75}$$

and

$$\mathcal{M}_{cx}(Y) \quad \leftrightarrow \quad \mathcal{M}_{Kah}(X)\} \tag{76}$$

An immediate consequence of this pairing is the relation between the Hodge numbers of X and Y,

$$h^{1,1}(X) = h^{2,1}(Y) \tag{77}$$

and

$$h^{2,1}(X) = h^{1,1}(Y) \tag{78}$$

This is not a complete definition of mirror symmetry, even here we overlooked the subtleties involving the large complex structure limit, variations of Hodge structures and GW invariants.

5 Deformation of Special Lagrangian submanifolds of CY

This section is devoted to the moduli space of special Lagrangian manifolds of Calabi-Yau manifolds. Consider a symplectic manifold X of dimension n.

Definition 5.1 *A submanifold L of X is called Lagrangian if $2dim(L) = dim(X)$ and $\omega|_L = 0$.*

Definition 5.2 *A Lagrangian submanifold L of a Kahler manifold X is called SpecialLagrangian if the volume form $\Omega|_L = 0$.*

Examples of special Lagrangian manifolds are very rare, at present only known special Lagrangian manifolds of a compact Calabi-Yau manifold are the fixed points of real structure on X.
The real structure on X is an antiholomorphic involution σ for which $\sigma^*\omega = -\omega$ and $\sigma^*\Omega = -\Omega$, then the fixed point set of σ form a special Lagrangian submanifold of X.
Stenzel has constructed some special Lagrangian submanifolds for non - compact CY manifolds.

Proposition 5.3 *The normal bundle $N(L)$ of a special Lagrangian manifold L of a Kahler manifold X is isomorphic to cotangent bundle of L.*

Proof We know

$$\omega(v, w) = g(v, Jw)$$

where g is the Hermitian Kahler metric on X and J is the complex structure on X.

$$\omega|_L = 0$$

This implies that J maps tangent vectors of L to normal vectors of L. Hence J induces an isomorphism

$$T(L) \simeq N(L)$$

Using the induced metric on L, we've an isomorphism

$$\beta : T(L) \to T^*(L)$$

Thus we obtain an isomorphism

$$T^*(L) \simeq N(L) \tag{79}$$

.

Explicitly, if $V = V^{i'} \frac{\partial}{\partial w^{i'}}$, then the corresponding 1-form is $\nu = V_i w^i$ with $V^{i'} = V_i$.

We'll now give a result of Mclean which states that the deformation of special Lagrangian manifolds of a CY manifold is unobstructed.

Theorem 5.4 *(Mclean [6]) A normal vector field V to a compact special Lagrangian submanifold L is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the 1-form corresponding to V is harmonic.*

Proof For an open set $U \subset \Gamma(N(L))$, open neighbourhood of 0, define a non-linear map

$$F : U \to \Gamma(L, \Omega_L^n) \oplus \Gamma(L, \Omega_L^2) \tag{80}$$

For a small enough normal vector field,

$$V = V_i \frac{\partial}{\partial \eta_i} \in U$$

defined as

$$F(V) = ((exp_V)^*(\Omega), (exp_V)^*(-\omega)) \tag{81}$$

Here the exponential map exp_V is a diffeomorphism of L onto its image L_V. The map F is the restriction of Ω and $-\omega$ to L_V, then pulled back to L via exp_V^*.

Thus $F^{-1}(0,0)$ is the set of normal vector fields V in U for which the restriction of Ω and ω to L_V is zero.

Hence, L_V is special Lagrangian and $F^{-1}(0,0)$ is the set of nearby special Lagrangian submanifolds.

Consider the linearization of F,

$$F'(t) : \Gamma(N(L)) \to \Gamma(L, \Omega_L^n) \oplus \Gamma(L, \Omega_L^2) \tag{82}$$

where

$$F'(t)(V) = \frac{\partial}{\partial t}(tV)$$

Thus

$$F'(0)(V) = \frac{\partial}{\partial t}(tV)|_{t=0} \tag{83}$$

$$= (\mathcal{L}_V(\Omega)|_L, -\mathcal{L}_V(\omega)|_L) \tag{84}$$

$$= ((i_V(d\Omega) + d(i_V\Omega))|_L, -(i_V(d\omega) + d(i_V\omega))|_L) \tag{85}$$

$$= (d(i_V\Omega)|_L, -d(i_V\omega)|_L) \tag{86}$$

where $\mathcal{L}_V$ denotes Lie derivative, i_V denotes the interior product and we're using the *Cartan's Identity*.
Now, for $V = V^{i'} \frac{\partial}{\partial w^{i'}}$ and $\omega = w^i \wedge w^{i'}$, it can be easily seen that

$$-i_V \omega = V_i \omega^i = v \tag{87}$$

where v is the 1-form corresponding to V under the isomorphism $N(L) \simeq T^*(L)$ and similarly

$$i_V \Omega = *v \tag{88}$$

here $*v$ is the Hodge dual of v. Thus $F'(0)$ can be interpreted as

$$F'(0) = -d * \oplus d \tag{89}$$

Thus first order special Lagrangian deformations, which are given by the kernel of $F'(0)$ corresponds to harmonic 1-forms.
We'll now show that the deformation theory of special Lagrangian manifolds is unobstructed.

Theorem 5.5 *(Mclean [6]) There are no obstructions in extending a first order deformation to an actual special Lagrangian deformation and the Zariski tangent space at L to the moduli space of spacial Lagrangian manifolds is naturally identified with $\mathcal{H}^1(L)$, the space of harmonic 1-forms*

Proof Since F is the pull back of closed forms Ω and ω, its clear that the image of F lies in the space of closed $n-$forms and closed $2-$forms. Also

$$exp_V : L \to M$$

is homotopic to inclusion map

$$i : L \to M$$

the homotopy is given by

$$H(t, V) = tV$$

Thus exp_V^* and i^* give same map in cohomology.Thus

$$[exp_V^*(\Omega)] = [i^*(\Omega)] = [\Omega|_L] = 0$$

Similarly

$$[exp_V^*(\omega)] = 0$$

Thus F is actually a map from $\Gamma(N(L))$ to exact n-forms and exact 2-forms and this implies that $F'(0)$ is surjective.
Now the *Banach Space implicit function theorem* and elliptic regularity implies that $F'(0,0)$ is a smooth manifold with tangent space at 0 is equal to the kernel of $F'(0) = \mathcal{H}^1(L)$.

Corollary 5.6 *The moduli space $\mathcal{M}_{sL}$ of special Lagrangian submanifolds near L is a smooth manifold with dimension $\beta^1(L)$ which carries a Riemannian metric and admits an n-form.*

Proof The Riemannian metric g_{sL} is defined as :
Given $v_1, v_2 \in T_L(\mathcal{M}_{sL})$, denote the corresponding 1-forms as α_1, α_2,

$$g_{sL}(v_1, v_2) = \int_L (\alpha_1 \wedge \alpha_2) dVol_L \tag{90}$$

Also the n-form α^n is defined as :

$$\alpha^n(v_1, v_2, ..., v_n) = \int_L (\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n) \tag{91}$$

Corollary 5.7 *(Joyce [2]) Let $\{(X, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of CY manifolds. Suppose N_0 is a special Lagrangian submanifold of $\{X, J_0, g_0, \Omega_0\}$ and suppose that $(\omega_t|_{N_0}) = 0$ and $(\Omega_t|_{N_0}) = 0$ for all t. Then N_0 extends to a smooth 1-parameter family $N_t : t \in (-\delta, \delta)$, where $0 < \delta \leq \epsilon$ and N_t is a compact special Lagrangian submanifold of (X, J_t, g_t, Ω_t).*

6 Mirror Symmetry : Modern Point of View

There are two modern approaches to mirror symmetry, Homological Mirror Symmetry and SYZ Picture. Both of these are still conjectural and far from being understood. There are certain cases for which the former is proved (K3, Abelian Varieties)and we've some ideas of later in certain limit (tropical limit). We'll state both of these are conjectures.

6.1 Homological Mirror Symmetry (HMS)

This version of Mirror Symmetry was formulated by Maxim Kontsevich and it involves the abstract machinery of triangulated categories and Homological algebra.
Let M be a Calabi-Yau 3-fold, then we can naturally assign to it the bounded derived category of coherent sheaves on M, $D^b(Coh(M))$, which is a triangulated category and it contains the information about the complex structure of M.
Also M has a symplectic structure, so we can associate to M a Fukaya Category, $Fuk(X)$.

Conjecture 6.1 *(Kontsevich [5]) If X and Y are two mirror manifolds, then there exists an equivalence of categories between :*

$$D^b(Coh(X)) \simeq Fuk(Y) \qquad D^b(Coh(Y)) \simeq Fuk(X) \tag{92}$$

6.2 Strominger-Yau-Zaslow (SYZ) Conjecture

This gives the geometric relation between the Mirror manifolds. It is based on the special Lagrangian submanifolds of CY manifolds discussed above.

Conjecture 6.2 *(SYZ [3]) If X and Y are a pair of mirror manifolds, then there exists fibrations :*

$$f : X \to B \qquad g : Y \to B$$

whose fibres are toric special Lagrangian.
The relation between these is that the fibrations are dual to each other, in a sense that

$$X_b = H^1(Y_b, \mathbb{R}/\mathbb{Z}) \qquad Y_b = H^1(X_b, \mathbb{R}/\mathbb{Z})$$

Kontsevich - Soibelman [13] have proved a relation between these two approaches of Mirror Symmetry.

7 Acknowledgements

I am greatly indebted to Prof. Ricardo Castano Bernard for helping me through many problems on the topics discussed in these notes and most importantly for his patience and valuable time.

References

[1] S.T. Yau, *On the Ricci Curvature of Compact Kahler Manifold and the Complex Monge-Ampere Equations I* Comm. on Pure and Applied Mathematics 31(1978) pp 339.

[2] M. Gross,D. Huybrechts, D. Joyce *Calabi-Yau manifolds and related geometries*

[3] A. Strominger, S.T. Yau, E. Zaslow *Mirror Symmetry is T-Duality*(hep-th/9606040)

[4] D. Morrison *Geometry behind Mirror Symmetry*

[5] M. Kontsevich *Homological Algebra of Mirror Symmetry*, Proc. ICM Zurich 1994(alg-geom/9411018)

[6] R.Mcleam *Deformation of Calibrated Submanifolds* Comm. Anal. Geom. 6 (1998) pp 705

[7] G.Tian *Smoothness of Universal Deformation Space of Compact Calabi-Yau Manifolds and its Weil - Peterson Metric* Mathematical Aspects of String Theory, World Scientific, pp 629

[8] Bogomolov *Hamiltonian Kahlerian Manifolds* Dokl. Acad. Nauk. SSR 243(1978) pp 1101

[9] Tododrov *The Weil-Peterson Geometry of the Moduli Space of* $SU(\geq 3)$ *(Calabi-Yau)Manifolds* Comm. Math. Phys. 126(1989) pp 325

[10] Kodaira,Spencer *On the Deformations of Complex Analytic Structure I,II* Ann. Math. 67 (1958) pp 328

[11] P. Candeles, X. de la Ossa, P.Green, L. Parkes *A Pair of Calabi-Yau Manifolds as Exactly Solvable Superconformal Theory.*, Nucl. Phys. B 359 (1991) pp 21

[12] P.Griffiths, J. Harris *Principles of Algebraic Geometry*

[13] M. Kontsevich, Y. Soibelman *homological Mirror Symmetry and Torus Fibrations* (math.SG/0011041)