Topological sigma models

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This note is an introduction to the topological sigma models, the A-model and the B-model. We start with a $N = 2$ Super symmetric σ-model consisting of maps from a Riemann surface to a Kahler manifold and construct two topological models by the procedure of topological twisting, pioneered by Edward Witten[1, 2]. We'll show that in semiclassical limit, the correlation functions of A-model can be determined by counting holomorphic from Riemann surface to the target manifold (more generally, Euler class of the vector bundle of (-1) ghost zero modes) and that of B-model can be determined by computing the periods of differential forms. Our treatment will closely follow that of Edward Witten [1].

1 $N = 2$ Supersymmetric sigma model

The standard $N = 2$ supersymmetric σ-model in 2 dimensions consists of maps $\Phi : \Sigma_g \to M$, where Σ_g is Riemann surface of genus g and M is a Kahler manifold.
Let ϕ^I denote the local real coordinates on M and ϕ^i are the local complex coordinates on M, with complex coordinates $\phi^{\bar{i}} = \bar{\phi}^i$.
If $(z, \bar{z})$ are the local coordinates on Σ_g, then Φ can be locally described as $\phi^I(z, \bar{z})$ in real coordinates and as ϕ^i in complex coordinates.
These form the bosonic fields of the theory.

The Fermionic fields are given by the sections :

$$\begin{aligned}
\psi^i_+ &\in \Gamma(K^{1/2} \otimes \Phi^*(T^{1,0}M)), \\
\psi^{\bar{i}}_+ &\in \Gamma(K^{1/2} \otimes \Phi^*(T^{0,1}M)) \\
\psi^i_- &\in \Gamma(\bar{K}^{1/2} \otimes \Phi^*(T^{1,0}M)), \\
\psi^{\bar{i}}_- &\in \Gamma(\bar{K}^{1/2} \otimes \Phi^*(T^{0,0}M))
\end{aligned}$$

where K and $\bar{K}$ be the canonical and anti-canonical lines bundles of Σ_g, and $K^{1/2}, \bar{K}^{1/2}$ be the respective square roots.
TM is the complexified tangent bundle of M, $TM = T^{1,0}M \oplus T^{0,1}M$

The Lagrangian of the theory is given by

$$L = 2t \int d^2z(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\psi_-^{\bar{i}}D_z\psi_-^i g_{i\bar{i}} + i\psi_+^{\bar{i}}D_{\bar{z}}\psi_+^i g_{i\bar{i}} + R_{i\bar{i}j\bar{j}}\psi_+^i\psi_+^{\bar{i}}\psi_-^j\psi_-^{\bar{j}}) \tag{1}$$

where, $R_{i\bar{i}j\bar{j}}$ is the Riemann tensor of M and D_z is the ∂ operator on $\bar{K}^{1/2} \otimes \Phi^*(T^{1,0}M)$ obtained by pulling back the Levi-Civita connection on TM. If Γ^i_{jk} is the affine connection on M.

$$D_z\psi_-^i = \frac{\partial}{\partial z}\psi_-^i + \frac{\partial\phi^i}{\partial z}\Gamma^i_{jk}\psi_-^k$$

and similarly for $D_{\bar{z}}$.
The supersymmetric transformations are given as :

$$\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \\
\delta\phi^{\bar{i}} &= i\alpha'_-\psi_+^{\bar{i}} + i\alpha'_+\psi_-^{\bar{i}} \\
\delta\psi_+^i &= -\alpha'_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jk}\psi_+^k \\
\delta\psi_+^{\bar{i}} &= -\alpha_-\partial_z\phi^{\bar{i}} - i\alpha'_+\psi_-^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\psi_+^{\bar{k}} \\
\delta\psi_-^i &= -\alpha'_-\partial_{\bar{z}}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jk}\psi_-^k \\
\delta\psi_-^{\bar{i}} &= -\alpha_+\partial_{\bar{z}}\phi^{\bar{i}} - i\alpha'_-\psi_+^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\psi_+^{\bar{k}}
\end{aligned}$$

Where $\alpha_-, \alpha'_- \in \Gamma(K^{-1/2})$ and $\alpha_+, \alpha'_+ \in \Gamma(\bar{K}^{-1/2})$ are the fermionic infnitesimal parameters.

Following the Noether's theorem, we find 4 conserved currents, $G^\mu_\pm, \bar{G}^\mu_\pm$, where $\mu = 1, 2$ as :

$$\delta\int Ld^2z = \int d^2z(\partial_\mu\alpha_+G^\mu_+ - \partial_\mu\alpha_-G^\mu_- + \partial_\mu\alpha'_-\bar{G}^\mu_+ - \partial_\mu\alpha'\bar{G}^\mu_-)$$

The corresponding supercharges are :

$$Q_{\pm} = \int dx\, G^0_{\pm},$$
$$\bar{Q}_{\pm} = \int dx\, \bar{G}^0_{\pm}$$

The supersymmetric transformation laws can also be expressed in terms of these supersymmetric conserved charges for any field W as :

$$\delta W = -i\alpha_-\{Q_+, W\} + i\alpha_+\{Q_-, W\} - i\alpha'_-\{\bar{Q}_+, W\} + i\alpha'_+\{\bar{Q}_-, W\}$$

We can combine two of these charges to form the so called BRST operator, Q as,

$Q_A = \bar{Q}_+ + Q_-$ or $Q_B = \bar{Q}_+ + \bar{Q}_-$

the other two combinations are just the complex conjugates of these two. We'll take the BRST operator, Q as any of Q_A, Q_B.

The topological theories can be constructed from this supersymmetric σ - model by the procedure known as *Topological Twisting*. The twisting is interpreting the fermionic fields as the sections of bundles different than the ones considered in the original QFT. We can either $+$ twist or $-$ twist the theory, which are :

$+$ **twist :** Take ψ^i_+ to be the section of $\Phi^*(T^{1,0}M)$ and $\psi^{\bar{i}}_+$ to be the section of $K \otimes \Phi^*(T^{1,0}M)$. And similarly for ψ_-.

$-$ **twist :** Take $\psi^{\bar{i}}_+$ to be the section of $\Phi^*(T^{1,0}M)$ and ψ^i_+ to be the section of $K \otimes \Phi^*(T^{1,0}M)$. And similarly for ψ_-

There are two different choices of twisting, when we are twisting both ψ_+ and ψ_-. These are :

- Making a $+$ twist to ψ_+ and a $-$ twist to ψ_-. This is called A-theory.

- Making a $-$ twist to both ψ_+ and ψ_-. This is called B-theory.

Essentially, we are taking $Q = Q_A$ in A-theory and $Q = Q_B$ in B-theory. The topological theory is constructed by restricting to the Q_A - Cohomology

in A-model and to Q_B in B-model.
We'll study these theories in details in subsequent sections.

2 A- Model

Here we are taking $\psi_+^i \in \Gamma(\Phi^*(T^{1,0}M))$ and $\psi_-^{\bar{i}} \in \Gamma(\Phi^*(T^{0,1}M))$, we can combine them to $\chi \in \Gamma(\Phi^*(TM))$ with
$\chi^i = \psi_+^i$ and $\chi^{\bar{i}} = \psi_-^{\bar{i}}$.

We want to look at the action of Q_A, so we'll set $\alpha_- = \alpha'_+ = 0$ and $\alpha_+ = \alpha'_- = \alpha$.
The supersymmetric transformation laws now become,

$$\delta W = -i\alpha\{Q_A, W\}$$

The Lagrangian of the theory can be written as ;

$$L = 2t \int_{\Sigma_g} d^2z \left(\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i\psi_z^{\bar{i}} D_{\bar{z}} \chi^i g_{\bar{i}i} + i\psi_z^i D_z \chi^{\bar{i}} g_{\bar{i}i} - R_{\bar{i}ij\bar{j}} \psi_{\bar{z}}^i \psi_z^{\bar{i}} \chi^j \chi^{\bar{j}} \right) \tag{2}$$

This can be written, modulo the equation of motion of ψ, as :

$$L = it \int_{\Sigma_g} d^2z \{Q, V\} + t \int_{\Sigma_g} \Phi^*(\kappa) \tag{3}$$

where,

$$V = g_{i\bar{j}} \left(\psi_z^{\bar{i}} \partial_{\bar{z}} \psi^j + \partial_z \phi^{\bar{i}} \psi_{\bar{z}}^j \right) \tag{4}$$

and

$$\int_{\Sigma_g} \Phi^*(\kappa) = \int_{\Sigma_g} d^2z \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} g_{i\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} g_{i\bar{j}} \right) \tag{5}$$

is the intergral of the pull back of the Kahler form $\kappa = -ig_{i\bar{j}} dz^i dz^{\bar{j}}$. Thus this is a topological term and it depends only on the cohomology class of κ and the homotopy type of Φ.

Topological nature of the Theory
The A-model is a topological theory in a sense that the partition function doesn't depend on the metric on Σ_g. Let $g_{\mu\nu}$ be an arbitrary metric on Σ_g, the energy - momentum tensor of the theory is given as ;

$$
\begin{aligned}
T_{\mu\nu} &= \int_{\Sigma_g} \frac{\delta L}{\delta g^{\mu\nu}} \\
&= \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\} \\
&= \{Q, b_{\mu\nu}\}
\end{aligned}
$$

The partition function of the theory is given as :

$$
Z = \int D\phi D\psi D\chi e^{L}
$$

Thus

$$
\frac{\delta Z}{\delta g^{\mu\nu}} = -\langle\{Q, b_{\mu\nu}\}\rangle \tag{6}
$$

the brackets denote the vacuum expectation value. Since the theory is invariant under the supersymmetric transformations, the above VEV of energy-momentum tensor vanishes. (The energy is the zeroth component of the energy - momentum tensor and for any classical symmetry of a theory the VEV of $\{Q, H\}$ vanishes). Thus

$$
\frac{\delta Z}{\delta g^{\mu\nu}} = 0
$$

This implies that the partition function is independent of the metric on Σ_g. if $H^{M,\mathbb{Z}} = \mathbb{Z}$, then

$$
\int_{\Sigma_g} \Phi^*(\kappa) = 2\pi n
$$

where n is an interger, known as the Instanton number.

We wish to calculate the path integral for the fields of degree n

$$
\langle \prod_a \mathcal{O}_a \rangle_n = e^{-2\pi n t} \int_{B_n} D\phi \; D\psi \; D\chi \; exp\left(-it\{Q, \int V\}\right) . \prod_a \mathcal{O}_a \tag{7}
$$

Operators of the theory

An immediate question is what are the operators $\mathcal{O}$ to be considered in this theory. Since an important characteristic of the theory is the metric independence of the partition function, we want operators whose correlation obey the metric independence property. Such operators are determined by following two constraints :

- The topological nature of the theory dictates that they should be BRST - invariant., i.e. $\{Q, \mathcal{O}_a\} = 0$. The correlation function involves the term

  $$exp(it\{Q,V\}) \prod \mathcal{O}_a = (1 - it\{Q,V\}) \prod \mathcal{O}_a = \prod \mathcal{O}_a - it\{Q,V\} \prod \mathcal{O}_a$$

  Differentiation with respect to the metric $g^{\mu\nu}$ on Σ_g, we get for each $\mathcal{O}_a$

  $$\begin{aligned}\frac{\delta \mathcal{O}_a}{\delta g^{\mu\nu}} = it\{Q, b_{\mu\nu}\}\mathcal{O}_a - it\{Q,V\}\frac{\delta \mathcal{O}_a}{\delta g^{\mu\nu}} \\ = it\{Q, \mathcal{O}_a\} b_{\mu\nu}\end{aligned}$$

  For the correlation function to be independent of the metric, we require, $\{Q, \mathcal{O}_a\} = 0$.

- The operators cannot be Q-exact, i.e. $\mathcal{O}_a \neq \{Q, S_a\}$ for some S_a for in that case their correlation functions will vanish. To see this consider the expectation value of a product involving a Q-exact operator $\{Q, T\}$

  $$\begin{aligned}\langle \mathcal{O}_1 ......\mathcal{O}_j\{Q,T\}\mathcal{O}_{j+1}...\mathcal{O}_n\rangle = \langle \mathcal{O}_1......\mathcal{O}_j(QT - TQ)\mathcal{O}_{j+1}...\mathcal{O}_n\rangle \\ = \langle \mathcal{O}_1......\mathcal{O}_j QT \mathcal{O}_{j+1}...\mathcal{O}_n\rangle - \langle \mathcal{O}_1......\mathcal{O}_j TQ \mathcal{O}_{j+1}...\mathcal{O}_n\rangle \\ = 0 - 0\end{aligned}$$

  each of these is zeros, (by repeated use of $\{Q, \mathcal{O}_a\} = 0$).

Thus this implies that the operators of the theory are the one in the Q-Cohomology.

Semiclassical Limit

Another important characteristic of the theory is the t independence (if Re$t > 0$ for the path integral to converge) except for the $e^{-2\pi n t}$. The variation of the path integral with respect to t will give irrelevant terms of the form $\{Q, ..\}$. This corresponds to the semiclassical limit.

In the semiclassical limit, the Lagrangian consists of the bosonic part only, which is minimized by the holomorphic maps from $\phi : \Sigma_g \to M$, i.e.

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = 0$$

Thus in semiclassical limit, we've a reduction of the moduli of the theory to the moduli space of holomorphic maps of degree n, $\mathcal{M}_n$ and the path integral is reduced to an integral over this moduli space.

In full (quantum) theory we also have ψ and χ fields.
If a_n be the dimension of the space of χ zero modes, i.e. the space of the solutions of the equations :

$$D_{\bar{z}}\chi^i = D_z\chi^{\bar{i}} = 0$$

and b_n be the dimension of the space of the solutions of ψ equations ;

$$D_{\bar{z}}\psi_z^{\bar{i}} = D_z\psi_{\bar{z}}^i = 0$$

then the Index Theory implies that $w_n = a_n - b_n$ can be given as :

$$w_n = 2d(1-g) + 2\int_{\Sigma_g} \Phi^*(c_1(M)) \tag{8}$$

where d is the dimension of the target manifold and $c_1(M)$ is the first Chern class of M. In particular w_n is a topological invariant.
For Calabi-Yau manifolds, $w_n = 2d(1-g)$, independent of n.

We also have a selection rule coming from the physical laws :

If $g(\mathcal{O}_a)$ is the ghost number of the operator $\mathcal{O}_a$. The ghost number is a number associated to the operators and the at the classical level the theory obey the ghost number conservation law, e.g. $g(\chi) = 1$, $g(\psi) = -1$, $g(\phi) = 0$ and $g(Q) = 1$. At quantum level Index Theory states that the ghost is no longer a symmetry of the theory. The selection rule says that the correlation function, $\langle \prod \mathcal{O}_a \rangle = 0$ unless,

$$w_n = \sum_a g(\mathcal{O}_a)$$

Note χ zero modes are precisely the linearization of the ϕ zero modes. ($\chi \in \Gamma(\Phi^*(TM))$). This implies that the space of χ zero modes is the tangent bundle $T\mathcal{M}_n$. Thus the dimension of $\mathcal{M}_n$ is a_n.
Define *Virtual Dimension* of $\mathcal{M}_n$ to be equal to $w_n = a_n - b_n$.
In sufficiently generic situation, one would expect that if $w_n > 0$, then $b_n = 0$ and $w_n = a_n$. This generic situation may be never attainable in complex geometry.

BRST - de Rham Correspondence :
Let $W = W_{i_1 i_2 \ldots\ldots i_n} d\phi^{i_1} d\phi^{i_2} \ldots\ldots d\phi^{i_n}$ be an n-form on M.
Define a corresponding local operator at $p \in M$:

$$\mathcal{O}_W(P) = W_{i_1 i_2 \ldots\ldots i_n} \chi^{i_1} \chi^{i_2} \ldots\ldots \chi^{i_n}(p) \tag{9}$$

It can be seen that

$$\{Q, \mathcal{O}_W\} = -\mathcal{O}_{dW}$$

where d is the exterior derivative on W.
We thus obtained a natural $1-1$ correspondence :
$\{$ the de Rham cohomology of M $\} \rightarrow \{$ BRST cohomology of the A-theory.$\}$
given by,

$$W \rightarrow \mathcal{O}_{dW}$$

Calculation of Path Integral
Let H_a be some homology cycles (submanifolds) of M, $a = 1, ..., s$ with codimension q_a. Let $W(H_a)$ be the Cohomology duals of H_a and $\mathcal{O}_{H_a}$ be the corresponding local operator.
If p_a are the points in Σ_g, we wish to compute

$$\langle \mathcal{O}_{H_1}(p_1).......\mathcal{O}_{H_s}(p_s)\rangle_n = \int_{B_n} D\phi D\chi D\psi e^{it\int\{Q,V\}}.\prod \mathcal{O}_{H_a}(p_a) \qquad (10)$$

The ghost number of $\mathcal{O}_{H_a}$ is q_a. The selection rule demands that

$$w_n = \sum_a q_a$$

Also based on physical justifications we demand that $\Phi(p_a) \in H_a$.
Thus in semiclassical limit, the path integral reduces to an integral over $\widetilde{\mathcal{M}_n} \subset \mathcal{M}_n$ consisting of ϕ obeying $\Phi(p_a) \in H_a$.
In generic situation, $a_n = w_n$, thus the dimension of $\widetilde{\mathcal{M}_n}$ is

$$w_n - \sum_a q_a = 0$$

Thus $\widetilde{\mathcal{M}_n}$ consists of finite set of points. Let $\#\widetilde{\mathcal{M}_n}$ be the number of such points. Therefore, in generic situation we've :

$$\langle \prod_a \mathcal{O}_{H_a}(p_a)\rangle_n = e^{-2\pi n t}.\#\widetilde{\mathcal{M}_n} \qquad (11)$$

summing over n, this gives

$$\langle \prod_a \mathcal{O}_{H_a}(p_a)\rangle_n = \sum_{n=0}^{\infty} e^{-2\pi n t}.\#\widetilde{\mathcal{M}_n} \qquad (12)$$

But as we remarked earlier, we may not have a generic situation. In such cases, $\widetilde{\mathcal{M}_n}$ may have some positive real dimension, say s. We'll use an

important consequence of Reimann- Roch Theorem, which is :
The space V of ψ zero modes is of dimension s and varies as the fibres of a vector bundle $\mathcal{V}$ of dimension s over $\widetilde{\mathcal{M}_n}$.
Witten [3], based on on the physical arguments, proposed a generalization of the counting the number of points in $\widetilde{\mathcal{M}_n}$ as the evaluation of the Euler class $\chi(\mathcal{V})$ of the bundle $\mathcal{V}$.
Thus, the correlation function is given by, in general situation

$$\langle \prod_a \mathcal{O}_{H_a}(p_a) \rangle_n = \sum_{n=0}^{\infty} e^{-2\pi n t} . \int_{\widetilde{\mathcal{M}_n}} \chi(\mathcal{V}) \tag{13}$$

Morrison and Aspinwall[4], calculated the above Euler class $\chi(\mathcal{V})$ for a Calabi-Yau 3-fold, in genus 0 case. The correlation function of 3 operators in this case can be written as :

$$\langle \mathcal{O}_{H_1} \mathcal{O}_{H_2} \mathcal{O}_{H_3} \rangle = (D_1 \cap D_2 \cap D_3) + \sum_\beta I_{0,3,\beta}(H_1, H_2, H_3) Q^\beta \tag{14}$$

$(D_1 \cap D_2 \cap D_3)$ is the intersection number of the three divisors associated to the 2-forms.

where $Q^i = e^{-t_i}$, $t_i = \int_{S_i} \omega$, S_i is the basis of $H^2(M)$, $i = 1, 2....b_2(M)$.

$\beta \in \sum_i \eta_i S_i$ and $Q^\beta = \prod_i Q_i^{\eta_i}$

The coefficients $I_{0,3,\beta}$ counts the number of holomorphic maps from the sphere ($g = 0$) to the Calabi-Yau 3-fold.
It can be shown that

$$I_{0,3,\beta}(H_1, H_2, H_3) = N_{0,\beta} \int_\beta H_1 \int_\beta H_2 \int_\beta H_3 \tag{15}$$

The $N_{0,\beta}$ are the invariants which encode all the information about the 3-point correlation function. These are known as $g = 0$ *Gromov-Witten Invariants.*
Higher genus *Gromov-Witten Invariants* are obtained by coupling the topological σ-model by $2d$ gravity.
This completes our discussion of the A-model. We'll look at the B-model in the next section.

3 B-Model

Here

$$\begin{aligned}\psi^{\bar{i}}_{\pm} &\in \Gamma(\Phi^*(T^{0,1}M)),\\ \psi^{i}_{+} &\in \Gamma(K\otimes\Phi^*(T^{1,0}M))\\ \psi^{i}_{-} &\in \Gamma(\bar{K}\otimes\Phi^*(T^{1,0}M))\end{aligned}$$

Its convenient to combine, $\psi^i_{\pm}$ into ρ which is a one form with values in $\Phi^*(T^{1,0}M)$, $\rho^i_z = \psi^i_+$ and $\rho^i_{\bar{z}} = \psi^i_-$.
Also set

$$\begin{aligned}\eta^{\bar{i}} &= \psi^{\bar{i}}_+ + \psi^{\bar{i}}_-\\ \theta_i &= g_{i\bar{i}}\left(\psi^{\bar{i}}_+ - \psi^{\bar{i}}_-\right)\end{aligned} \tag{16}$$

In B-model, we want to look at the action of $Q_B = \bar{Q}_+ + \bar{Q}_-$, so we'll set $\alpha_\pm = 0$ and $\alpha'_+ = \alpha'_- = \alpha$.
The supersymmetric transformation laws become :

$$\begin{aligned}\delta\phi^i &= 0\\ \delta\phi^{\bar{i}} &= i\alpha\eta^{\bar{i}}\\ \delta\eta^{\bar{i}} &= 0\\ \delta\theta_i &= 0\\ \delta\rho^i &= -\alpha d\phi^i\end{aligned} \tag{17}$$

We can as well write the transformation laws in terms of the BRST operator $Q = Q_B$ as

$$\delta W = -i\alpha\{Q, W\}$$

But we chose to write the laws in explicit form to point some properties of this theory which are not evident in the Q form.
The Lagrangian of the theory can be written as ;

$$L = t\int_{\Sigma_g} d^2z\left(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{i}}(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_z)g_{i\bar{i}} + i\theta_i(D_{\bar{z}}\rho^i_z + D_z\rho^i_{\bar{z}}) + R_{i\bar{i}j\bar{j}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{i}}\theta_k g^{k\bar{j}}\right) \tag{18}$$

Let's pause here for a while to say a word about the Lagrangian. Note since the forms $\eta^{\bar{i}}$ and $g^{\bar{i}i}\theta_i$ are the sections of $T^{0,1}M$ and the one forms ρ^i take values in $T^{1,0}M$, the fermion determinant in this model is complex. This is in contrast to A-model where $\chi^i, \psi^{\bar{i}}_z$ determinant is complex conjugate of $\chi^{\bar{i}}, \psi^i_{\bar{z}}$ determinant. Thus the B-model does not make sense as a QFT. To circumvent this problem, we need to employ the anomaly cancellation condition, $c_1(M) = 0$. This implies M must be a Calabi-Yau manifold.

We remark here that although we developed the A-model for a Kahler target manifold, the same construction works for any complex manifold, in fact Witten first constructed A-model for general almost complex manifold. But the B-model can be defined for only Calabi-Yau target manifold.
Coming back to the Lagrangian, it can be written as

$$L = t \int_{\Sigma_g} \{Q, V\} + tW \tag{19}$$

where,

$$V = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right) \tag{20}$$

and

$$W = \int_{\Sigma_g} \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{j}} \theta_k g^{k\bar{j}} \right) \tag{21}$$

Here D is the exterior derivative on Σ_g which acts on the 1-forms with values in $T^{1,0}M$.
The W depends on the complex structure of the target manifold, M which is evident from the transformation laws. But under the variation of the metric on M, the W changes by terms of the form $\{Q, ..\}$.
Thus if we consider only operators in the BRST - Cohomology, we have a topological QFT, the B-model.
Thus the A-model depends only on the Kahler class of the target manifold, M and is independent of the complex structure on M, while the B-model depends on the complex structure on M and is independent of the Kahler structure on M. This observation plays a pivotal role in Mirror Symmetry.

Like in any QFT, we are interested in calculating the following path integral

$$\langle \prod_a \mathcal{O}_a \rangle_n = \int_{B_n} D\phi D\eta D\rho e^{-it \int \{Q,V\} + tW} . \prod_a \mathcal{O}_a \tag{22}$$

Semiclassical Limit
Like A-model, the B-model is also characterized by independence of of the coupling constant t, provided $Ret > 0$ so that the path integral converges. The $t\{Q, V\}$ term changes by $\{Q, ..\}$ under the variation of t. For the tW

term, we can remove the t dependence by redefining, θ as :

$$\theta \to \theta/t$$

This redefining gives the trivial dependence on t of the correlation function $\langle \prod_a \mathcal{O}_a \rangle$ as

$$t^{-\sum_a k_a}$$

where k_a is the degree of the BRST invariant operator, $\mathcal{O}_a$.
An important consequence of the trivial t dependence of the path integral is that all computations reduces to the semiclassical limit.
The bosonic Lagrangian is minimized by the constant maps $\Phi : \Sigma_g \to M$, i.e.

$$(\partial_{\bar{z}} + \partial_z)\, \phi^{\bar{j}} = 0$$

The space of constant maps is isomorphic to the target manifold M and thus the path integral reduces to an integral over M.
This simplified behavior of the path integral in B-model is attributed to the trivial t dependence.
Since M is a Calabi-Yau manifold for the B - model, the selection rule says that the $\langle \prod_a \mathcal{O}_a \rangle$ vanishes for the operators of ghost numbers w_a, unless

$$\sum_a w_a = 2d(1-g) \tag{23}$$

$\bar{\partial}$ - BRST correspondence
Similar to A-model, where we developed a correspondence between the de Rham cohomology of M and BRST cohomology of A-model, here we'll show a correspondence between the $\bar{\partial}$ cohomology on M and the BRST cohomology of B-model.
Consider V, a $(0,p)$ with values in bundle $\wedge^q T^{1,0}$ form on M.

$$V = V^{j_1 \ldots\ldots j_q}_{i_1 \ldots\ldots i_p} d\bar{z}^{i_1} \ldots\ldots d\bar{z}^{i_p} \frac{\partial}{\partial z_{j_1}} \ldots\ldots\ldots \frac{\partial}{\partial z_{j_q}} \tag{24}$$

Define a corresponding local operator as

$$\mathcal{O}_V = V^{j_1 \ldots\ldots j_q}_{i_1 \ldots\ldots i_p} \eta^{\bar{i}_1} \ldots\ldots\ldots \eta^{\bar{i}_p} \psi_{j_1} \ldots\ldots \psi_{j_q} \tag{25}$$

This implies

$$\{Q, \mathcal{O}_V\} = -\mathcal{O}_{\bar{\partial} V} \tag{26}$$

We thus obtain a $1-1$ correspondence

$\{\oplus_{p,q} H^p(M, \wedge^q T^{1,0}M)\} \rightarrow \{$ BRST Cohomology of B-Model $\}$

Calculation of Path integral Consider classes $V_a \in H^{p_a}(M, \wedge^{q_a} T^{1,0}M)$ and let $\mathcal{O}_{V_a}$ be corresponding local operators. If P_a are points in Σ_g, we wish to compute

$$\langle \mathcal{O}_{V_a}(P_a) \rangle$$

We'll consider only genus zero ($g = 0$) case. The selection rule implies that

$$\sum_a p_a = \sum_a q_a = d \tag{27}$$

In the semiclassical limit, we've an integral over the space of constant maps $\Phi : \Sigma_0 \rightarrow M$. But in general we've fermionic fields zero modes as well, which are the similarly the constant modes of η and θ.

Using the above selection rule, we can interpret the $\prod \mathcal{O}_{V_a}$ as a d form with values in $\wedge^d T^{1,0}M$ as

$$\prod \mathcal{O}_{V_a} = \prod V_{a,i_1 \ldots\ldots i_p}^{j_1 \ldots\ldots j_q} \eta^{\bar{i}_1} \ldots\ldots \eta^{\bar{i}_p} \psi_{j_1} \ldots\ldots \psi_{j_q}$$

This under the $\bar{\partial}$ - BRST correspondence, corresponds to the tensor product of $(0, p_a)$ forms on M with values in $\wedge^{q_a} T^{1,0}M$ and the usual wedge product implies

$$\otimes_a H^{p_a}(M. \wedge^{q_a} T^{1,0}M) \simeq H^d(M, \wedge^d T^{1,0}M) \tag{28}$$

where we've used the selection rule eq. (27).
Thus we've reduced the path integral over the space of all fields to an integral of elements of $H^d(M. \wedge^d T^{1,0})$ over M.

Remark 3.1. Note that the constraint that M is Calabi-Yau ensures that the $H^d(M, \wedge^d T^{1,0})$ is non-zero and is 1-dimensional.

This completes our discussion of B-model.

4 Conclusions

In this note, following Witten, we constructed two topological sectors from a $N = 2$ supersymmetric σ- model. Witten[1] called these topological theories as a A-model and B-model. We shown explicitly that the correlation functions of A model can be computed by counting the rational curves

and that of B-model can be computed by calculating the periods of the differential forms.
There are important implications of these results in context of Mirror Symmetry. If X and Y are two mirror Calabi-Yau manifolds, in the (physics) sense that the moduli spaces of the $N = 2$ SCFTs associated to X and Y are isomorphic, then geometrically this gives a relation between the complex moduli space of X and Kahler moduli of Y and vice versa. The above construction serves to separate the complex and Kahler moduli spaces of a Calabi-Yau manifold, M. The A-model depends on Kahler structure of M (independent of complex structure) and the B-model depends on the complex structure of M (independent of the Kahler structure). Thus mirror symmetry between X and Y can be interpreted as the relation between A-model of X and the B-model of Y and vice versa.
Kontsevich[5] built on these ideas of Witten and proposed a more mathematically sophisticated version of mirror symmetry in terms of triangulated categories.

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References

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