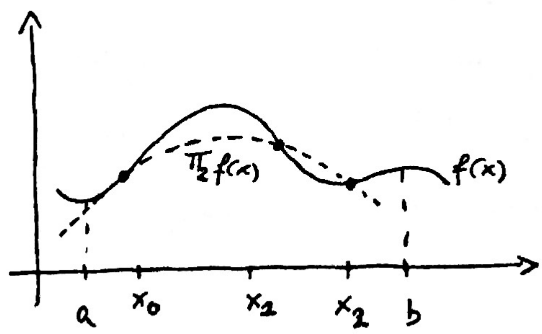
Polynomial interpolation (Global)

Consider a set of nodes $\{x_0, \dots, x_n\}$ in $[a, b]$ and let $y_i = f(x_i)$ be the values of $f(x)$ at x_i. Assume that $x_i \neq x_j$ for $i \neq j$.

<u>Proposition</u> For any set of couples $\{x_i, f(x_i)\}$ $i = 0, \dots, n$ with distinct x_i, there exists a unique polynomial $\Pi_n f(x)$ of degree less or equal than n such that

$$\Pi_n f(x_i) = f(x_i) \qquad i = 0, \dots, n$$

(INTERPOLATION CONDITION)



Proof Consider the $n+1$ pairs $\{x_i, y_i\}$ where $y_i = f(x_i)$ $i=0,\dots,n$ and the polynomial

$$\Pi_n f(x) = a_0 + \dots + a_n x^n$$

By imposing the interpolation conditions $\Pi_n f(x_i) = y_i$ we obtain the following system for $[a_0, \dots, a_n]^T$

$$\underbrace{\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & & x_n^n \end{bmatrix}}_{\text{Vandermonde matrix } V} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

If $x_i \neq x_j$ for $i \neq j$ (distinct nodes) then the Vandermonde matrix is non singular (but very badly conditioned usually)

$$\det(V) = \prod_{0 \le i < j \le n} (x_i - x_j) = \prod_{j=0}^{n-1} \prod_{i=j+1}^{n} (x_i - x_j) \neq 0$$

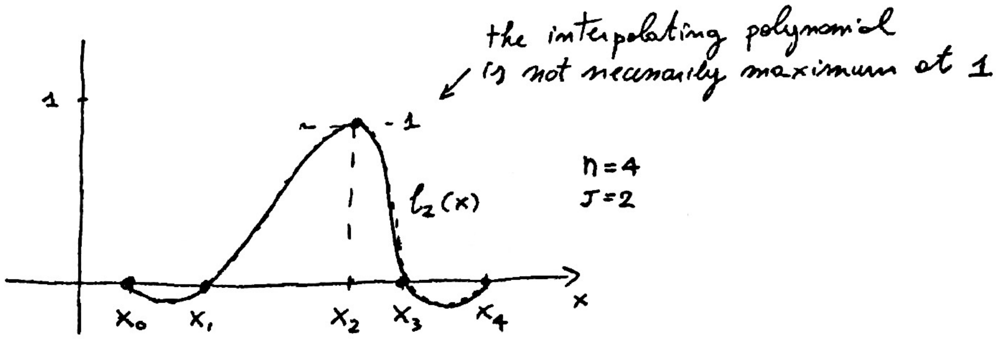
Therefore V can be inverted to obtain $[a_0, \dots, a_n]^T$ uniquely, i.e., $\Pi_n f(x)$ is unique.

Lagrangian polynomial interpolation

To construct $\Pi_n f(x)$ effectively, let us first consider the set of polynomials interpolating

$$\{x_i, \delta_{ij}\} \quad i = 0, \dots, n \qquad j = 0, \dots, n$$

Example



Such polynomials satisfy

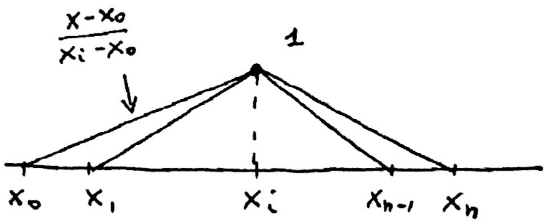
$$l_i(x_j) = \delta_{ij} \qquad \text{(CARDINAL BASIS)}$$

and they can can be written as:

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

LAGRANGE CHARACTERISTIC POLYNOMIALS

Remark (Geometrical interpretation)



$l_i(x)$ is the product of all lines above

Remark (SPAN): Each Lagrange polynomial is of order n, and we have $n+1$ linearly independent ones. This means that

$$\mathbb{P}_n = \text{span}\{l_0(x), \dots, l_n(x)\}$$

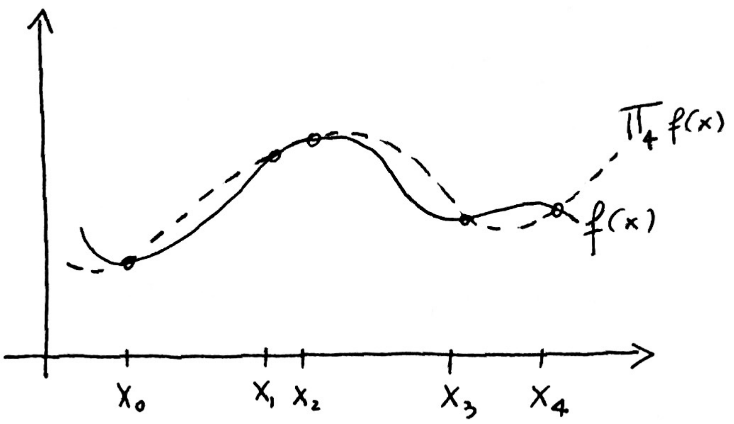
i.e. the set of Lagrange polynomials associated with a set of nodes spans the space of polynomials of order n.

$$\Pi_n f(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x) \quad \left(\begin{array}{c}\text{INTERPOLATION} \\ \text{FORMULA}\end{array}\right)$$

Note that $\Pi_n f(x_s) = \sum_{i=0}^{n} f(x_i) \overbrace{l_i(x_s)}^{\delta_{is}} = \sum_{i=1}^{n} f(x_i)\delta_{is} = f(x_s)$

i.e. $\Pi_n f(x)$ interpolates $f(x)$ at $\{x_0, \dots, x_n\}$.

Interpolation Error



Replacing $f(x)$ with a polynomial interpolant $\Pi_n f(x)$ generates an error that depends on $f(x)$ as well as on the number and LOCATION of the interpolation nodes.

Theorem Let $f \in C^{(0)}([a,b])$ and $\Pi_n f(x)$ the n-th order interpolating polynomial of $f(x)$ at $\{x_0, \dots, x_n\}$. Then,

$$\| f(x) - \Pi_n f(x) \|_\infty \le (1 + \Lambda_n) \inf_{\psi \in \mathbb{P}_n} \| f(x) - \psi(x) \|_\infty$$

best approximating polynomial

$$\Lambda_n = \max_{x \in [a,b]} \lambda_n(x) \qquad \lambda_n(x) = \sum_{j=0}^{n} |l_j(x)|$$

(Lebesgue constant) (Lebesgue function)

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Remark: (optimal interpolation nodes) The theorem tells us that a good set of interpolation nodes $\{x_0, \dots, x_n\}$ MINIMIZES the Lebesgue constant. ~~Thus~~ For a given interval, say $[-1, 1]$, this is a $(n+1)$ dimensional optimization problem

$$\min_{(x_0, \dots, x_n) \in [-1,1]^{n+1}} \Lambda_n (x_0, \dots, x_n)$$

The minimization can be peformed in a greedy way by adding one point at a time.

Remark (lower bound on Λ_n). No matter how well we try to optimaize Λ_n, there exists a lower bound that grows logarithmically with n (SEMINAL RESULT IN APPROXIMATION THEORY)

$$\Lambda_n \geq \frac{2}{\pi} \log (n+1) + C \qquad n \to \infty$$

$\Rightarrow$ Polynomial interpolation somehow tends to diverge from the best approximating polynomial as we increase the number of nodes, no matter where they are.

<u>Remark</u> The Lebesgue constant can provide a guideline to understand whether certain sets of points will result in well behaved interpolating polynomials

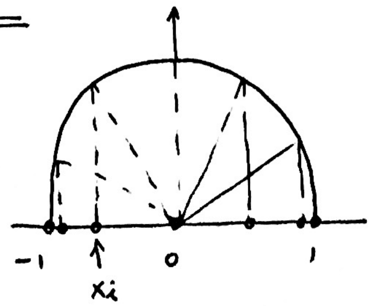
<u>Uniform grids (evenly-spaced) in [-1, 1]</u> (n+1 points)

$$\frac{2^{n-2}}{n^2} \le \Lambda_n \le \frac{2^{n+3}}{n} \qquad \Lambda_n \sim \frac{2^{n+1}}{\underbrace{e}_{\substack{2.7182\\ \text{NAPIER}\\ \text{NUMBER}}}\, n(\log n + \underbrace{\gamma}_{\substack{0.5477\\ \text{EULER}\\ \text{CONSTANT}}})}$$

$$n \equiv 65 \qquad \Lambda_n \simeq 10^{18} \times 4.5$$

<u>Chebyshev - Gauss - Lobatto grids</u>

$$x_i = \cos\left(\frac{\pi}{n} i\right) \quad i = 0, \dots, n$$



$$\Lambda_n \le \frac{2}{\pi}\left(\log(n) + \underbrace{\gamma}_{\to 0.5477} + \log\frac{8}{\pi}\right) + \frac{\pi}{72 n^2}$$

$$n \simeq 65 \qquad \Lambda_n \le 3.6$$