Multistep Methods

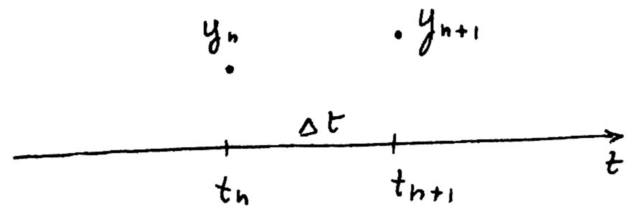
A q-step method is a method for which the solution to the ODE system

$$\frac{dy}{dt} = f(y,t)$$

$$y(0) = y_0$$

at time t_{n+1}, i.e., y_{n+1} depends on $y_n, y_{n-1}, \dots, y_{n-q}$

Example: (1-step method)



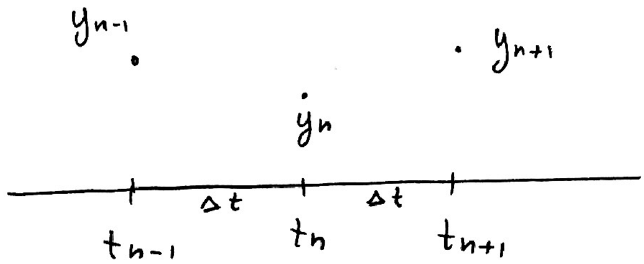
$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(y_n,t_n) + f(y_{n+1},t_{n+1})\right]$$

(Crank-Nicolson)

$$y_{n+1} = y_n + \Delta t\, f(y_n,t_n)$$

(Euler-Forward)

<u>Example</u> (2-step method)



1) $$y_{n+1} = y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} f(y(\tau), \tau)\, d\tau$$

$$= y_{n-1} + 2\Delta t\, f(y_n, t_n)$$

(explicit)

(approximation of the integral with the MIDPOINT RULE)

2) Alternatively, we can interpolate $f(y(\tau), \tau)$ with a second-order polynomial at t_{n-1}, t_n, t_{n+1}, and then integrate. This yields the Simpson quadrature rule, and the scheme:

$$y_{n+1} = y_{n-1} + \frac{\Delta t}{3}\left[f(y_{n-1}, t_{n-1}) + 4 f(y_n, t_n) + f(y_{n+1}, t_{n+1}) \right]$$

(implicit)

Adams-Bashforth Methods

Adams-Bashforth methods are explicit multistep methods to compute the solution to ODE systems in the form

$$\frac{dy}{dt} = f(y,t)$$
$$y(0) = y_0$$

The key idea is the following. Given the time instants $t_n, t_{n-1}, \ldots$ and the corresponding solution vectors $y_n, y_{n-1}, \ldots$ we use EXTRAPOLATION of the interpolant of $f(y(\tau),\tau)$ at t_n, t_{n-1}, t_{n-2}, to compute an approximation of $\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau$.

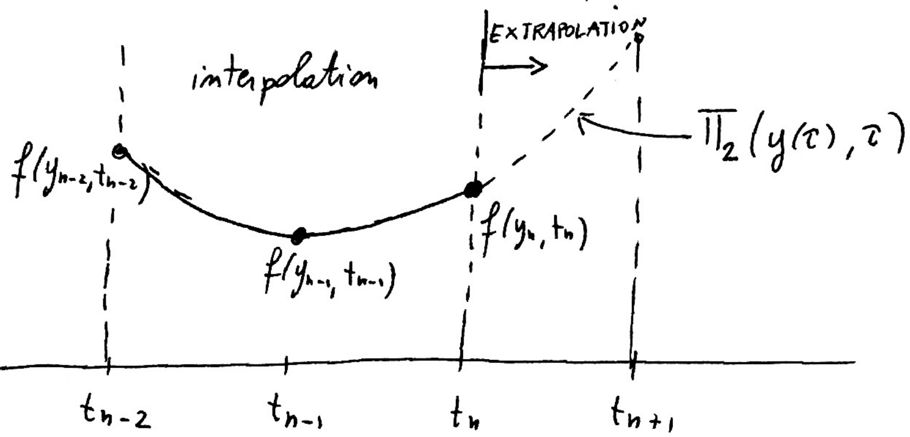
In other words, we consider

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau$$

where $$\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau \simeq \int_{t_n}^{t_{n+1}} \Pi_q(y(\tau),\tau)\, d\tau$$

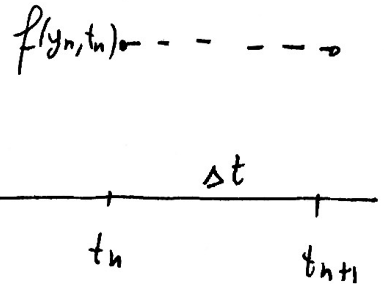
and $\Pi_q(y(\tau),\tau)$ is the polynomial interpolant

of $f(y_n, t_n), f(y_{n-1}, t_{n-1}), \dots, f(y_{n-q}, t_{n-q})$

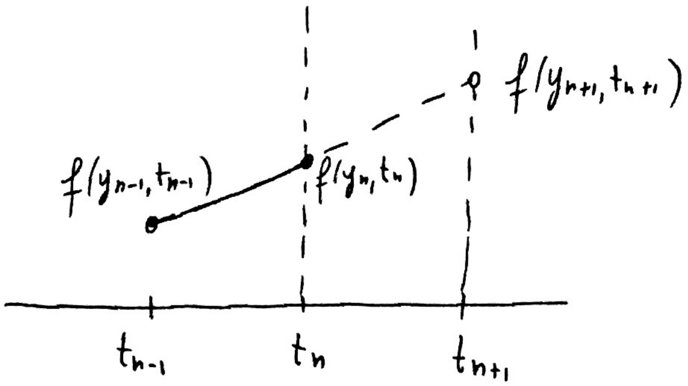


<u>Examples</u>: $q=0 \Rightarrow \Pi_0(y(\tau),\tau) = f(y_n,t_n)$

$y_{n+1} = y_n + \Delta t\, f(y_n,t_n)$ one-step Adams Bashforth coincides with Euler forward



Example : (two-step Adams-Bashforth)



$$\Pi_1(y(\tau),\tau) = f(y_n,t_n) + \frac{t-t_n}{t_{n-1}-t_n}\left(f(y_{n-1},t_{n-1}) - f(y_n,t_n)\right)$$

$$\Rightarrow \Pi_1(y^*_{n+1},t_{n+1}) = 2f(y_n,t_n) - f(y_{n-1},t_{n-1})$$

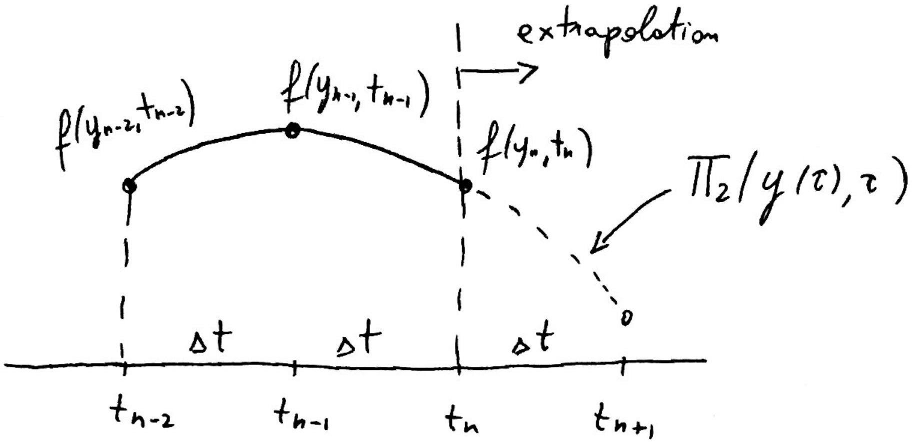
$$\Pi_1(y_n,t_n) = f(y_n,t_n)$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} \Pi_1(y(\tau),\tau)\,d\tau = \frac{\Delta t}{2}\left(3f(y_n,t_n) - f(y_{n-1},t_{n-1})\right)$$

This yields the (second-order) two-step Adams-Bashforth method

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left(3f(y_n,t_n) - f(y_{n-1},t_{n-1})\right)$$

Example: (three step Adams-Bashforth method)

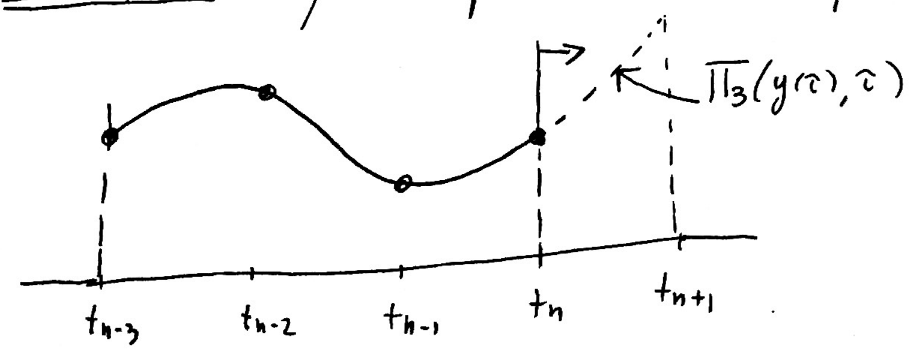


$$\int_{t_n}^{t_{n+1}} \Pi_2(y(\tau), \tau)\, d\tau = \frac{\Delta t}{12}\Big(23 f(y_n, t_n) - 16 f(y_{n-1}, t_{n-1}) + 5 f(y_{n-2}, t_{n-2})\Big)$$

$$\Rightarrow \quad y_{n+1} = y_n + \frac{\Delta t}{12}\Big(23 f(y_n, t_n) - 16 f(y_{n-1}, t_{n-1}) + 5 f(y_{n-2}, t_{n-2})\Big)$$

(third-order explicit)

Example (four-step Adams-Bashforth method)

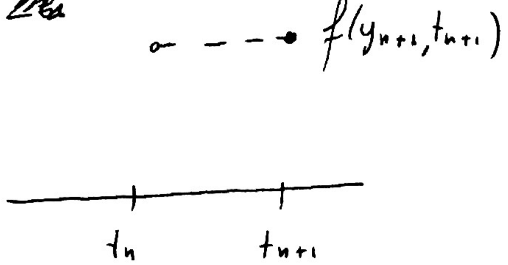


$$y_{n+1} = y_n + \frac{\Delta t}{24}\Big(55 f(y_n, t_n) - 59 f(y_{n-1}, t_{n-1}) + 37 f(y_{n-2}, t_{n-2}) - 9 f(y_{n-3}, t_{n-3})\Big)$$

Adams-Multon Methods

These are implicit multistep methods in which the integral from t_n to t_{n+1} is approximated by replacing $f(y(\tau),\tau)$ with the interpolating polynomial at $y_{n+1}, y_n, y_{n-1}, \ldots$

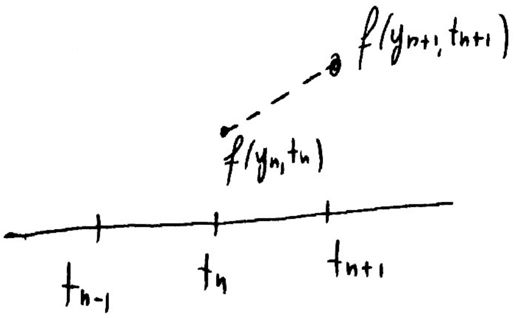
a)



$$y_{n+1} = y_n + \Delta t\, f(y_{n+1}, t_{n+1})$$

(Backward Euler)

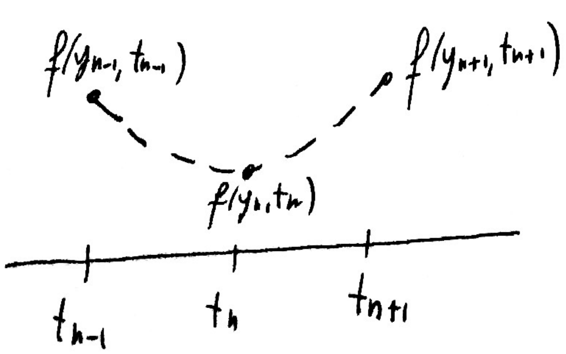
b)



$$y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_{n+1}, t_{n+1}) + f(y_n, t_n)\right)$$

(Crank-Nicolson)

c)



$$y_{n+1} = y_n + \frac{\Delta t}{12}\left(5 f(y_{n+1}, t_{n+1}) + 8 f(y_n, t_n) - f(y_{n-1}, t_{n-1})\right)$$

(third-order Adams-Multon)