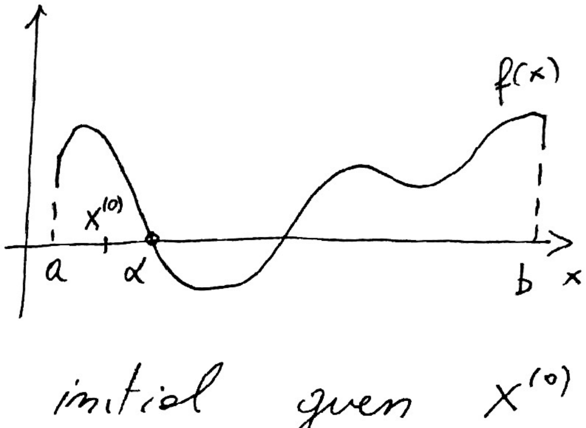
Nonlinear Equations

We are interested in computing the ZEROS of a real function f, i.e., the ROOTS of the equation

$$f(x) = 0 \qquad x \in [a,b]$$

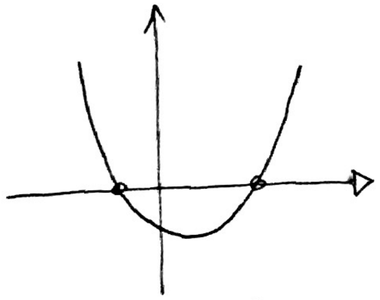
Methods for the numerical approximation of a zero of f are usually iterative. Let α be a zero, i.e., $f(\alpha)=0$



Given an initial guess $x^{(0)}$ (eventually sufficiently close to α), the aim is to generate a sequence of numbers $x^{(k)}$ $(k=1,2,\dots)$ that converges to α, i.e.,

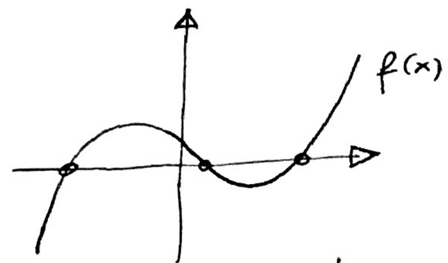
$$\lim_{k\to\infty} x^{(k)} = \alpha$$

<u>Remark</u>: (Zeros of polynomial functions): can be computed analytically ~~up~~ for polynomials up to order 4 (Abel-Ruffini theorem)



(order 2)

$f(x) = ax^2 + bx + c$



cubic (order 3)

<u>Remark</u>: Roots of polynomials of any order can be computed by determining the eigenvalues of the companion matrix: (numerically)

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & -b_{n-1} \end{bmatrix}$$

$$f(x) = x^n + b_{n+1} x^{n-1} + \dots + b_1 x + b_0$$ (monic polynomial)

Indeed, the characteristic polynomial of G that defines the eigenvalues coincides with $f(\lambda)$, i.e.,

$$\det(G - \lambda I) = x^n + b_{n-1}x^{n-1} + \dots + b_1 x + b_0 = 0$$

$\Rightarrow$ λ is a zero of $f(x)$ $\Leftrightarrow$ λ is an eigenvalue of G

For functions $f(x)$ that are more general than polynomials we follow an iterative approach, i.e., ~~we~~ given $x^{(0)}$ we define an algorithm that generates $x^{(1)}, x^{(2)}, \dots$ hopefully convergent to the root α.

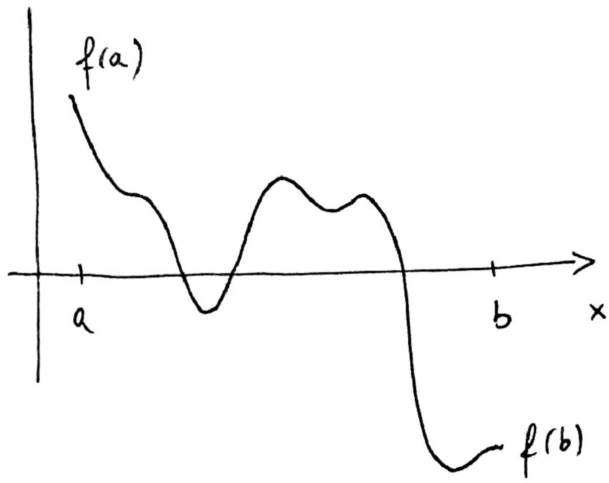
First of all, how can we be sure that such root exists?

<u>Theorem</u>: Let $f(x) \in C^0([a,b])$ with $f(a)f(b) < 0$. Then it exists $\alpha \in]a,b[$ such that

$$f(\alpha) = 0$$

$\Rightarrow$ continuous functions with the property $f(a)f(b) < 0$ must have AT LEAST one zero in $[a,b]$

Example:



$f(a)f(b)<0$

Definition A sequence of numbers $x^{(0)}, x^{(1)}, \dots$ is said to converge to α with order $p \geq 1$ if it exists $C>0$ such that

$$\frac{|x^{(k+1)}-\alpha|}{|x^{(k)}-\alpha|^{p}} \leq C \qquad \forall\ k > k_0$$

If $p=1$ (first order convergence) we must have $C<1$ (otherwise the sequence does not converge). In this case $(p=1)$, C is called convergence factor.