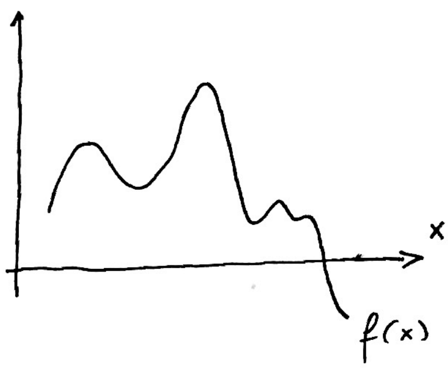
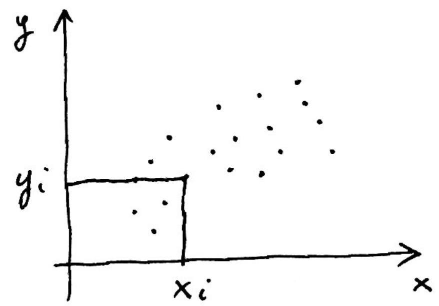
Approximation of functions and data

In many mathematical problems there is the need to approximate a function $f(x)$ or a set of data points $\{x_i, y_i\}_{i=0,\dots,n}$ with another function that is much more simple.





This allows us to perfom operations such as integration, differentiation, solve differential equations, etc. To approximate $f(x)$ we need to identify:

① The class of approximating functions, i.e., in which function space we look for an approximant:

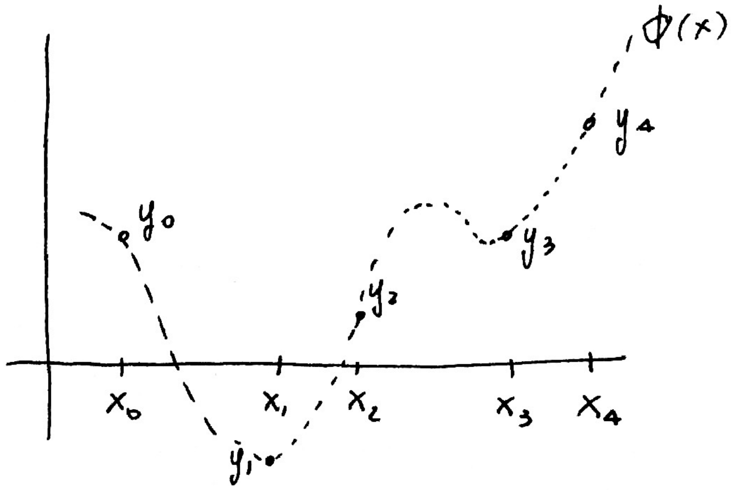
- Polynomials
- Trigonometric functions
- Rational functions

(2) The criterion to select a particular element within the class of approximating functions. For example:

- Interpolation
- Projection
- Least squares

Interpolation

Suppose we have available $n+1$ data points (x_i, y_i) $i=0,\dots,n$, where the nodes x_i are all distinct



We look for an approximant $\phi(x)$ that satisfies the following INTERPOLATION

CONDITIONS

$$\phi(x_i) = y_i \qquad i=0, \dots, n$$

Examples:

1) $\phi(x) = \sum_{K=0}^{n} a_K x^K$ (polynomial interpolants)

2) $\phi(x) = \sum_{K=-\frac{n}{2}}^{n/2} c_K e^{iKx}$ (trigonometric interpolants)

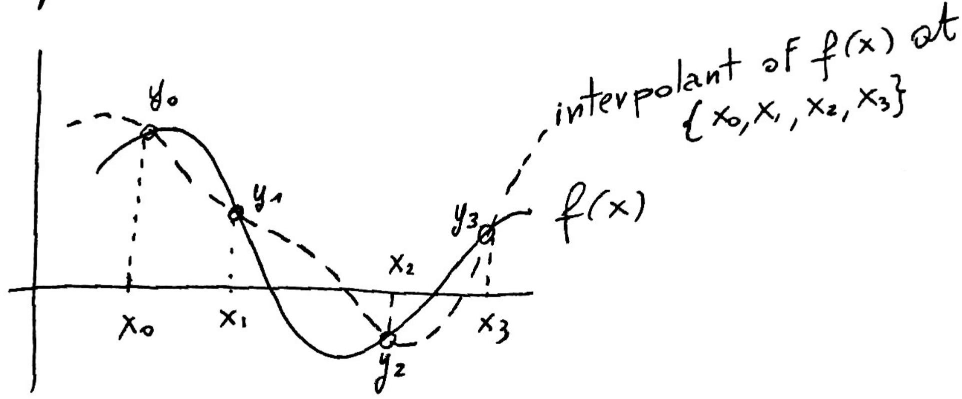
$$= \sum_{K=1}^{n/2} \alpha_K \sin(Kx) + \beta_K \cos(Kx) + \alpha_0$$

3) $\phi(x) = \dfrac{\sum_{i=0}^{h} a_i x^i}{\sum_{J=0}^{q} b_J x^J}$ $\qquad h+q = n-1$

$((h+1)+(q+1) = n+1)$

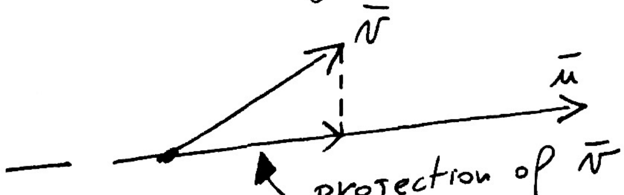
Remark: Interpolation can be generalized to any nonlinear transformation between vector spaces. For example, if X and Y are two linear spaces, e.g., $\mathbb{R}^n$ or $C^{(0)}([a,b])$, and $x_i \in X$ $y_i \in Y$, then we can construct an interpolant $H: X \to Y$ such that $H(x_i) = y_i$

Remark: We can also interpolate a function $f(x)$, provided we sample it at suitable nodes



Projection

The projection method relies on a generalization of the concepts we learned in linear algebra.



$$\frac{\langle \bar{v}, \bar{u} \rangle}{\|\bar{u}\|} \frac{\bar{u}}{\|\bar{u}\|}$$

A function $f(x)$ is indeed a VECTOR in a LINEAR SPACE of infinite dimension

<u>Example:</u> $C^{(0)}([-1,1])$ continuous functions in $[-1,1]$

$L_2([-1,1])$ square integrable functions in $[-1,1]$

$\left(f(x) \in L_2([-1,1]) \text{ if } \int_{-1}^{1} f(x)^2 dx < \infty\right)$

$C^{(0)}$ and L_2 are closed under addition and subtraction, we have the neutral element with respect to addition, we can define scalar multiplication, etc..., In other words C^0 and L_2 are vector spaces.

We can define the inner product (scalar product):

$$(\phi_1(x), \phi_2(x)) = \int_{-1}^{1} \phi_1(x)\, \phi_2(x)\, dx \qquad \phi_1, \phi_2 \in C^{(0)}([-1,1])$$

an the norm:

$$\| \phi_1 \|_2^2 = (\phi_1(x), \phi_1(x)) = \int_{-1}^{1} \phi_1(x)^2 dx$$

(This measures the LENGTH of the function $\phi_1(x)$)

With the INNER PRODUCT $(\ ,\ )$ we can project ~~and~~ the function $f(x)$ onto a BASIS $\{\phi_1(x), \dots, \phi_n(x)\}$ (ϕ_i are simple functions, for example polynomials)

$$f(x) \simeq \sum_{K=1}^{n} a_K \, \phi_K(x)$$

Suppose that $(\phi_i, \phi_J) = \delta_{iJ}$, then

$$a_K = \int_{-1}^{1} \phi_K(x) \, f(x) \, dx \qquad K=1,\dots,n$$

Thus, approximating $f(x)$ through projection reduces to computing integrals

$\boxed{\text{Least squares}}$

Consider the error norm

$$\left\| f(x) - \sum_{K=1}^{n} a_K \phi_K(x) \right\|_2^2 = \int_{-1}^{1} \left(f(x) - \sum_{K=1}^{n} a_K \phi_K(x) \right)^2 dx$$

$$\min_{a_1,\dots,a_n} \left\| f(x) - \sum a_K \phi_K(x) \right\|_2^2 \Rightarrow \sum_{K=1}^{n} M_{JK} a_K = b_J \qquad J=1,\dots,n$$

$$M_{JK} = \int_{-1}^{1} \phi_J(x) \phi_K(x) \, dx \qquad b_J = \int_{-1}^{1} f(x) \phi_J(x) \, dx$$