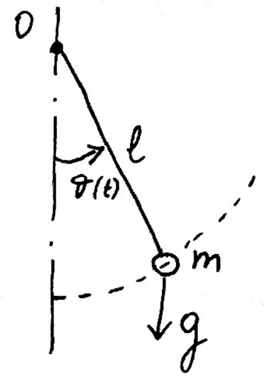
Differential Equations

A differential equation is an equation involving one or more derivatives of an unknown functions

Example : (Nonlinear pendulum)



$$\frac{d^2\vartheta(t)}{dt^2} = -\frac{g}{\ell}\sin(\vartheta(t))$$

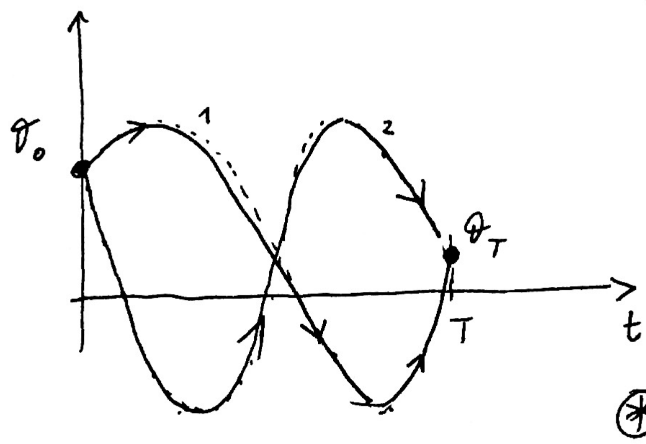
(nonlinear second-order ordinary differential equation)

To compute a unique $\vartheta(t)$ we need two additional conditions. For example, we can set $\vartheta(0) = \vartheta_0$ (initial position of the pendulum) and $\frac{d\vartheta}{dt}(0) = \dot{\vartheta}_0$ (initial velocity of the pendulum). This yields an INITIAL VALUE problem. Note that the pendulum equation can be written as a system of first-order ordinary differential equations. To this end, simply define

$$x_1(t) = \vartheta(t) \qquad x_2(t) = \frac{d\vartheta(t)}{dt} \quad \Rightarrow \quad \begin{cases} \dfrac{dx_1(t)}{dt} = x_2(t) \\ \dfrac{dx_2(t)}{dt} = -\dfrac{g}{\ell}\sin(x_1(t)) \\ x_1(0) = \vartheta_0 \quad x_2(0) = \dot{\vartheta}_0 \end{cases}$$

Nonlinear pendulum (equations in a first-order form)

If we set $x_1(0) = \vartheta_0$ and $x_1(T) = \vartheta_T$ then we have a TWO-POINT BOUNDARY VALUE problem. Basically we are aiming at determining which ~~___~~ trajectory passes through the points $\vartheta(0) = \vartheta_0$ and $\vartheta(T) = \vartheta_T$. (Such trajectory may not be unique)

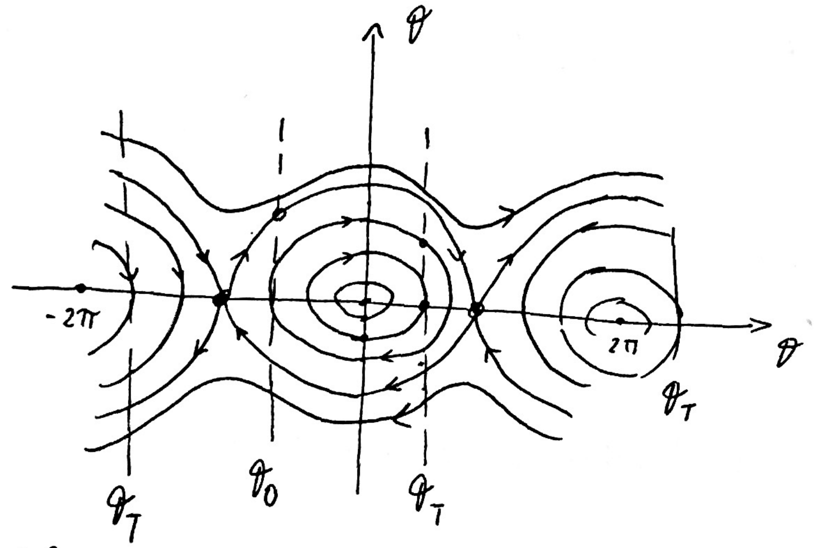


$\Downarrow$

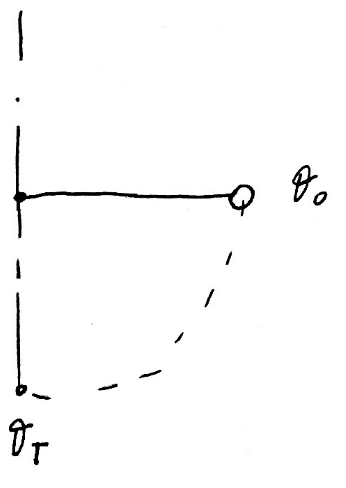
two point boundary value problems for ~~___~~ nonlinear odes might have multiple solutions (*)

(*) in this case the ~~___~~ multiplicity can be due to symmetry. but not necessarily

(*) The phase portrait of the nonlinear pendulum is



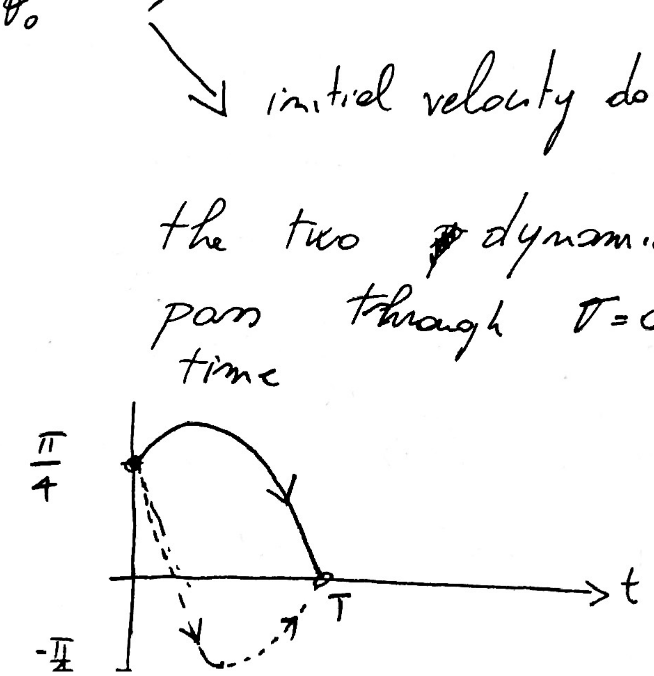
there are many ways we can go from θ_0 to $\theta_T \pm 2k\pi$



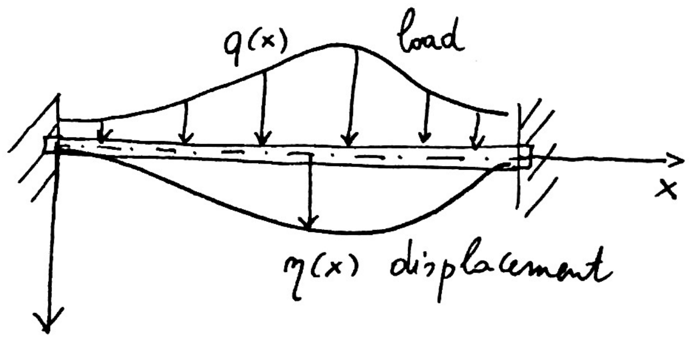
initial velocity upward

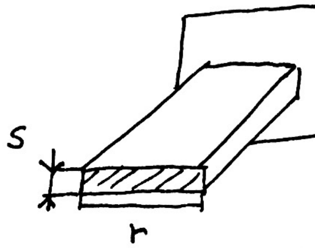
initial velocity downward

the two dynamics can pass through $\theta = 0$ at the same time



Example (Bending of a fully clamped rectangular beam)





$I = \frac{1}{12} r s^3$ (MOMENTUM OF INERTIA)

$E \to$ modulus of elasticity of the material

$$\begin{cases} \dfrac{d^2}{dx^2}\left(E I \dfrac{d^2 \eta(x)}{dx^2} \right) = q(x) \\ \eta(0) = \eta(L) = 0 \\ \dfrac{d\eta(0)}{dx} = \dfrac{d\eta}{dx}(L) = 0 \end{cases}$$

Boundary value problem for a fourth-order LINEAR ordinary differential equation.

Remark: Numerical methods for initial value problems are different from numerical methods for boundary value problems. We will focus mostly on initial value problems (Cauchy problems) for systems of first-order ODEs in a normal form, i.e., systems in the form $\frac{dy}{dt} = f(y,t)$.

Initial value problem for systems of first-order nonlinear ODEs

Consider the following system of first-order ordinary differential equations in a normal form:

$$\frac{dy(t)}{dt} = f(y(t), t) \qquad \begin{array}{l} y \in \mathbb{R}^n \\ f \in \mathbb{R}^n \\ t \in [0, T] \end{array}$$

where $f(y,t)$ is $C^{(1)}$ in $y \in \mathbb{R}^n$. The system can be written in an expanded form as

$$\begin{cases} \dfrac{dy_1(t)}{dt} = f_1(y_1(t), \dots, y_n(t), t) \\ \vdots \\ \dfrac{dy_n(t)}{dt} = f_n(y_1(t), \dots, y_n(t), t) \end{cases}$$

We supplement these equations with the initial condition $y(0) = (y_1(0), \dots, y_n(0))$.

Theorem (Existence and uniqueness of the solution)

Consider the Cauchy problem

$$\begin{cases} \dfrac{dy(t)}{dt} = f(y(t),t) \\ y(0) = y_0 \end{cases}$$

where $\dfrac{\partial f_i}{\partial y_j}$ are continuous in some open set $D \subset \mathbb{R}^n$. Then for $y_0 \in D$ there exists a unique solution $y(t)$ on some time interval about $t=0$. ($y(t)$ is $C^{(1)}$ in such interval)

Remark If $\dfrac{\partial f_i}{\partial y_j}$ are continuous and bounded in $\mathbb{R}^n$ then the solution exists and it is unique for any time interval $[0,T]$.

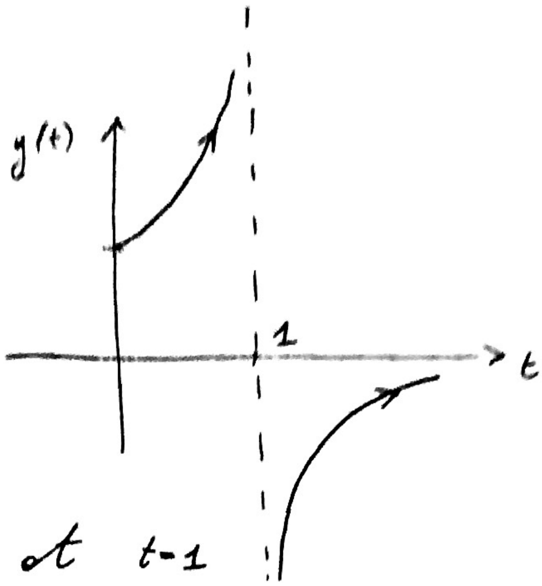
Example: $\begin{cases} \dfrac{dy(t)}{dt} = y(t)^2 \\ y(0) = 1 \end{cases}$

here $f(y,t) = y^2$

$\dfrac{\partial f}{\partial y} = 2y$ which is unbounded and continuous

The solution is

$$y(t) = \frac{1}{1-t}$$



Note that $y(t)$ blows up at $t=1$

This is not surprising since the existence and uniqueness theorem guarantees that $y(t)$ ~~is~~ exists and is unique in some time interval ~~around~~ about $t=0$.

<u>Example</u>: (pendulum equations)

$$\frac{dy_1(t)}{dt} = y_2(t) \qquad y_1(0) = y_{10}$$

$$\frac{dy_2(t)}{dt} = -\sin(y_1(t)) \qquad y_2(0) = y_{20}$$

Here we have $f_1(y_1, y_2) = y_2 \quad f_2(y_1, y_2) = -\sin(y_1)$

All partial derivatives are continuous and bounded. Therefore, the solution to the initial value problem exists and is unique for any finite time.

To compute the numerical solution to the Cauchy problem

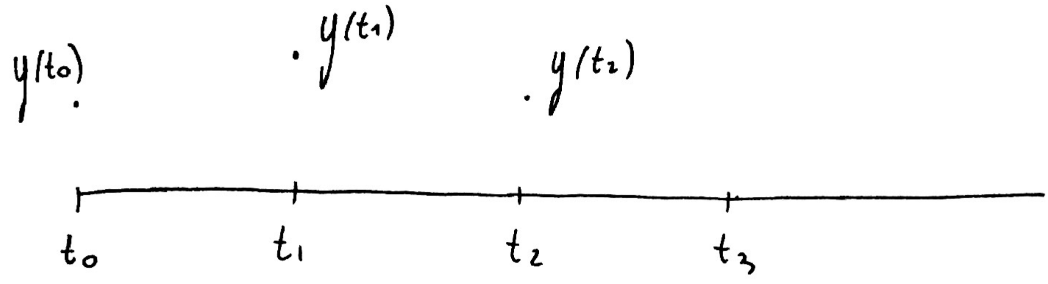
$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(0) = y_0 \end{cases} \qquad y \in \mathbb{R}^n, \ f \in \mathbb{R}^n$$

it is convenient to integrate the ODE in time:

$$\int_0^t \frac{dy(\tau)}{dt} d\tau = \int_0^t f(y(\tau),\tau)\, d\tau$$

$$\Rightarrow \quad y(t) = y(0) + \int_0^t f(y(\tau),\tau)\, d\tau$$

More generally, suppose we have available the solution $y(t)$ at a certain number of time instants $t_0 = 0$, t_1, t_2, t_3, ...
The formula above yields



$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(y(\tau),\tau)\,d\tau$$

$$y(t_2) = y(t_1) + \int_{t_1}^{t_2} f(y(\tau),\tau)\,d\tau$$

$$\vdots$$

$$\Rightarrow \quad y(t_{m+1}) = y(t_m) + \int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\,d\tau$$

A simple way to develop a numerical scheme for the system of ODEs is therefore to approximate the integral $\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\,d\tau$ by ~~any~~ a quadrature rule, for example the trapezoidal rule. As we will see, this yields the Crank-Nicolson method.