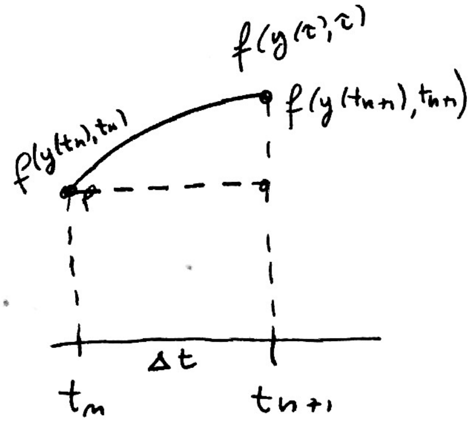
Euler methods

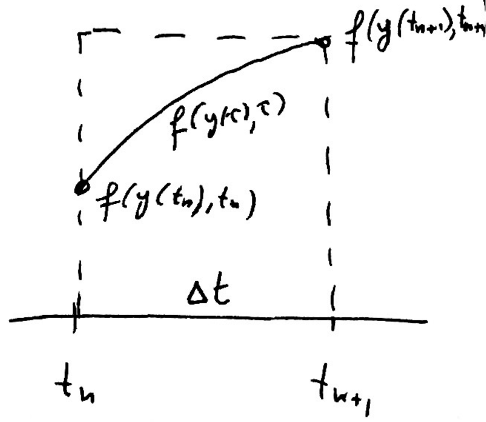
Consider $\quad y(t_{n+1}) = y(t_n) + \displaystyle\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau$

We approximate the integral from t_n to t_{n+1} by assuming that the integrand is constant.
We have different choices:



(1) $$\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau \simeq f(y(t_n),t_n)\, \Delta t$$

this yields $\quad \boxed{y_{n+1} = y_n + \Delta t\, f(y_n, t_n)}$ $\quad$ EULER FORWARD METHOD (EXPLICIT)



(2) $$\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\, d\tau \simeq f(y(t_{n+1}),t_{n+1})\, \Delta t$$

this yields

$$\boxed{y_{n+1} = y_n + \Delta t\, f(y_{n+1}, t_{n+1})}$$

EULER BACKWARD METHOD (IMPLICIT)

Note that in the backward Euler method we need a nonlinear solver to determine y_{n+1} from y_n. In fact, we need to solve the nonlinear algebraic equation

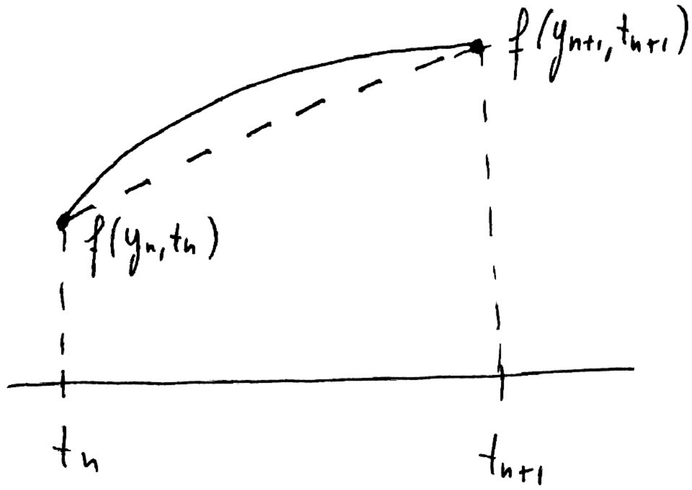
$$y_{n+1} - \Delta t\, f(y_{n+1}, t_{n+1}) - y_n = 0 \quad (\text{for } y_{n+1})$$

<u>Remark</u>: Euler methods are ONE-STEP methods since they require only y_n to determine y_{n+1} (implicitly or explicitly)

Crank-Nicolson and Heun methods

The Crank-Nicolson scheme can be obtained by approximating the integral $\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\,d\tau$ with the trapezoidal rule. This yields:

$$\int_{t_n}^{t_{n+1}} f(y(\tau),\tau)\,d\tau \simeq \frac{\Delta t}{2}\left(f(y_{n+1}, t_{n+1}) + f(y_n, t_n)\right)$$



This yields:

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{2}\Big(f(y_{n+1}, t_{n+1}) + f(y_n, t_n) \Big)}$$

Crank-Nicolson
(implicit - one step)

If we replace $f(y_{n+1}, t_{n+1})$ with $\cancel{f(y_n + \Delta t f(t_n,}$
$f(\underbrace{y_n + \Delta t f(y_n, t_n)}_{}, t_{n+1})$ we obtain the scheme:

we approximated
y_{n+1} with Euler
forward

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{2}\Big(f(y_n + \Delta t f(y_n, t_n), t_{n+1}) + f(y_n, t_n) \Big)}$$

Heun method
(explicit - one step)