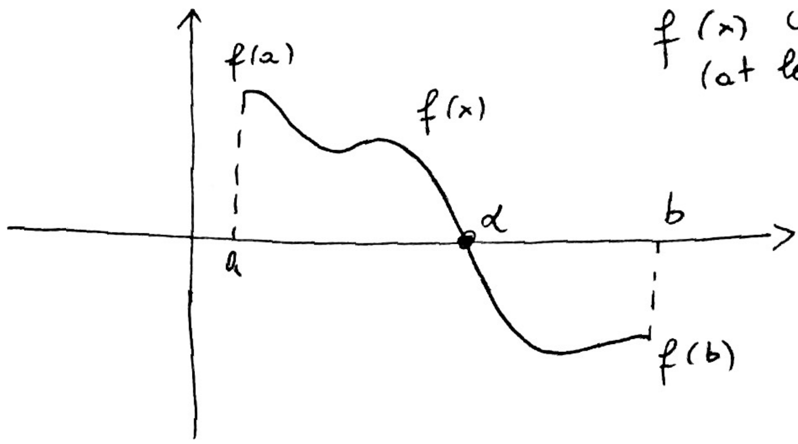
Geometric approaches to root-finding

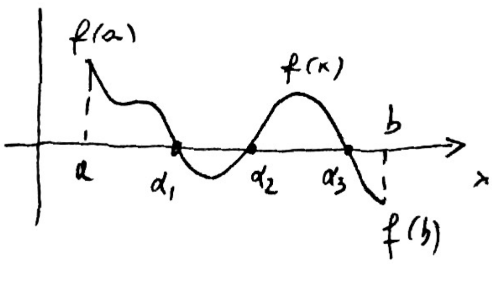
Bisection method

Now we start talking about algorithms to determine zeros of $f(x)$. To this end, suppose we provide two numbers a, b such that $f(a)f(b) < 0$

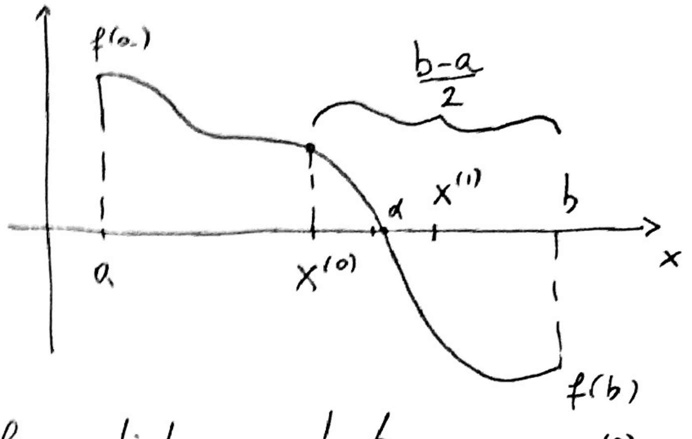


$f(x)$ continuous (at least)

Remark: It is possible that we have more zeros in $[a,b]$



Consider the point $x^{(0)} = \frac{a+b}{2}$



Clearly, the distance between $x^{(0)}$ and α is at most $\frac{b-a}{2}$.

$$|x^{(0)} - \alpha| \leq \frac{b-a}{2}$$

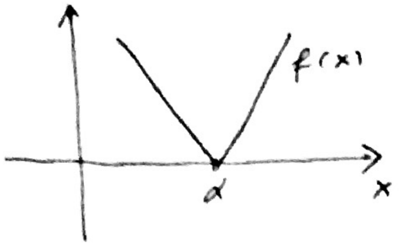
Similarly the distance between $x^{(1)}$ and α is

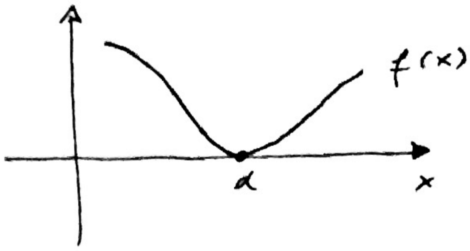
$$|x^{(1)} - \alpha| \leq \frac{b-a}{2^2} \quad \left(\text{half of } \frac{b-a}{2}\right)$$

In general: $$\boxed{|x^{(k)} - \alpha| \leq \frac{(b-a)}{2^{k+1}}}$$

Therefore $\lim_{k \to \infty} |x^{(k)} - \alpha| = 0$ (i.e. $x^{(k)} \to \alpha$), which means that the sequence generated by the bisection method is convergent.

~~Re~~mark There are few unfortunate cases where the bisection method does not work:





We cannot compute such zeros with the bisection methods.

Remark (Stopping criterion)

$$\frac{(b-a)}{2^{k+1}} \leq \varepsilon \qquad (\varepsilon \text{ set by the user})$$

In this way we guarantee that $x^{(k)}$ is close to α to order ε, i.e., $|x^{(k)}-\alpha| \leq \varepsilon$

The minimum number of iterations to achieve such tolerance ε is

$$k_{min} \geq \log_2\left[\frac{(b-a)}{\varepsilon}\right] - 1$$

Remark ("speed" of convergence) How many iterations do we need to "gain" a decimal digit (i.e. to make the error 10 times smaller)?

$$|x^{(k)} - \alpha| \simeq \frac{b-a}{2^{k+1}} \qquad |x^{(j)} - \alpha| \simeq \frac{b-a}{2^{j+1}}$$

Suppose that $|x^{(k)} - \alpha| \cong \dfrac{|x^{(j)} - \alpha|}{10}$ $(k > j)$

This implies:

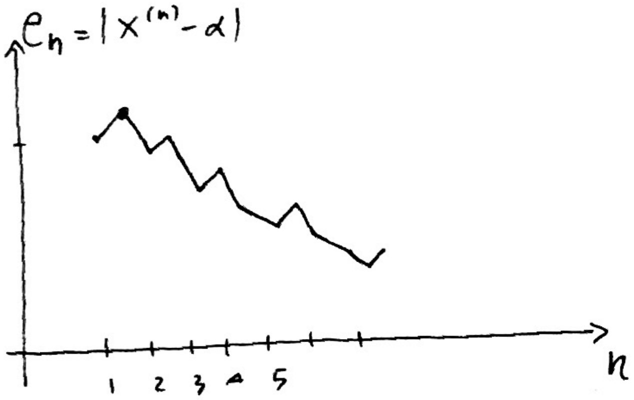
$$\frac{(b-a)}{2^{k+1}} \simeq \frac{(b-a)}{10 \cdot 2^{j+1}} \Rightarrow 2^{k-j} = 10$$

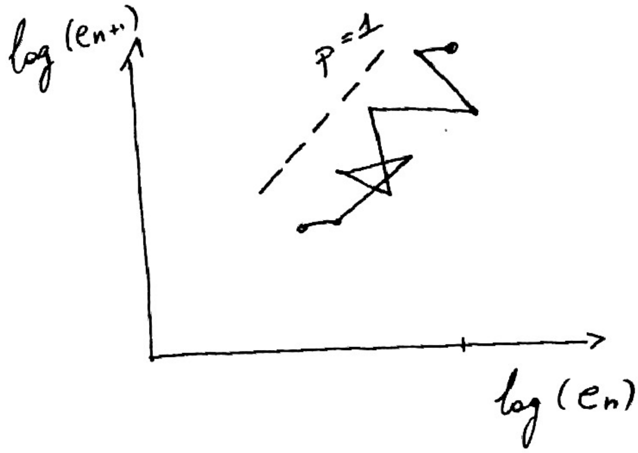
$$\Rightarrow (k-j) = \log_2(10) \simeq 3.3$$

Therefore, on average, every 3.3 iterations we gain a decimal digit in the estimate of the ROOT, i.e., the error becomes 10 times smaller

<u>Remark</u> (Convergence of the bisection method)

We do not have a monotone reduction of the error $\Rightarrow$ we cannot define rigorously a convergence order for the bisection method



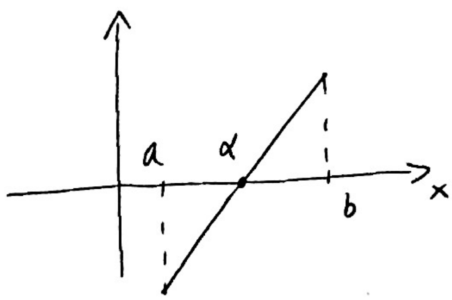


$$\frac{|x^{(n+1)} - \alpha|}{|x^{(n)} - \alpha|^{p}} \leq C$$

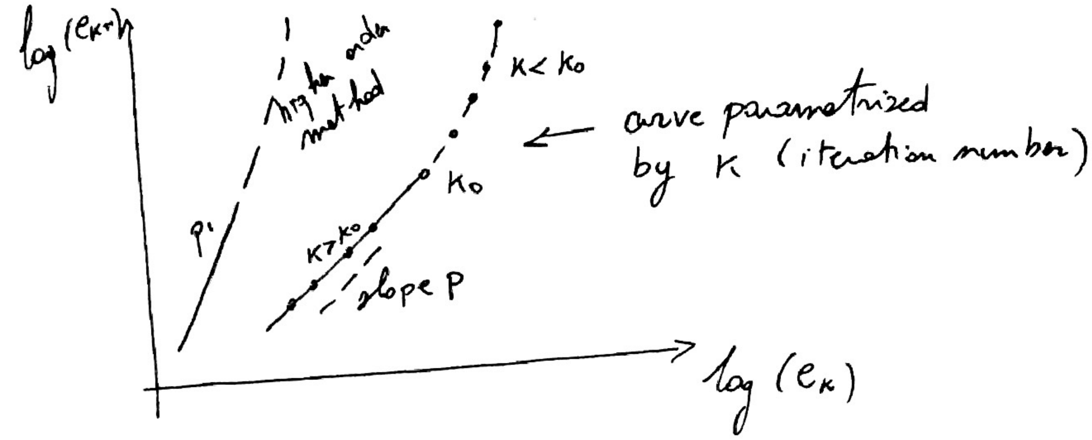
p: convergence order!

on average we get $p=1$

<u>Remark</u>: How many iterations do we need to compute this zero with the bisection method?



Remark (convergence rate of higher-order methods)



$\Rightarrow$ higher order methods converge faster, i.e., in less iterations.

Remark: To construct higher-order methods we need to include information from $f(x)$ at more points, and possibly derivative information (if f is differentiable).