Fixed-point iterations

Geometric approaches to rootfinding can be studied and analyzed by using a general framework, i.e., the theory of fixed points. The basic idea is the following. Given a nonlinear function

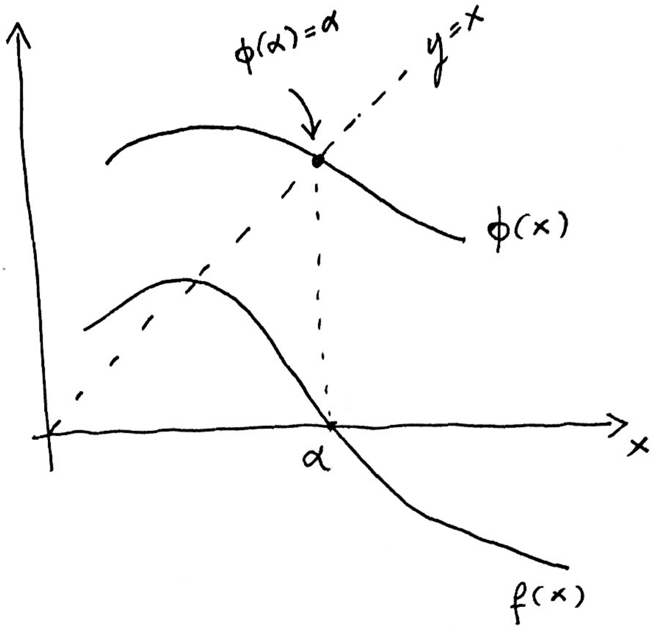
$$f : [a,b] \to \mathbb{R}$$

we transform the problem $f(\alpha) = 0$ into an <u>equivalent</u> problem in the form $\phi(\alpha) = \alpha$, where ϕ is a suitable AUXILIARY FUNCTION.

$$f(\alpha) = 0 \quad \longleftrightarrow \quad \phi(\alpha) = \alpha$$

(α is a ZERO OF $f(x)$) (α IS A FIXED POINT OF $\phi(x)$)

<u>Example</u>: Chord method and Newton's method.



~~Appro~~ Determining the zeros of a function f is thus equivalent to finding the fixed points of a suitable auxiliary function. This can be done by the following algorithm:

$$x^{(k+1)} = \phi(x^{(k)}) \qquad (x^{(0)} \text{ given})$$

<u>Remark</u>: (ϕ is not unique given f). There are many iteration functions ϕ that can be constructed for a given $f(x)$. Consider, for example,

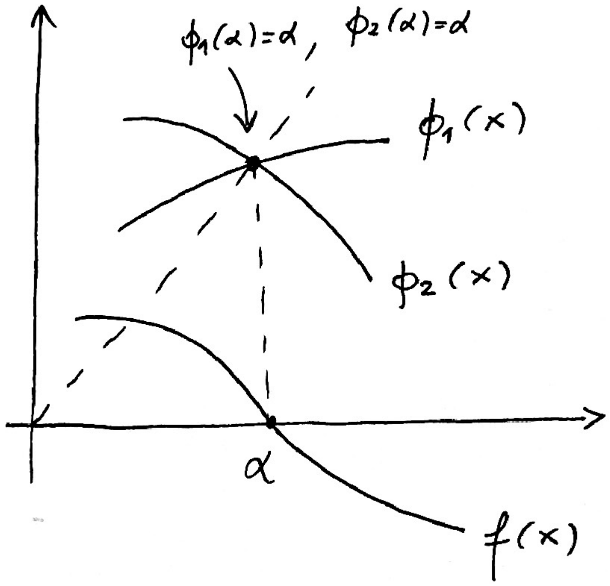
$$\phi(x) = x + F(f(x))$$

where F is any continuous function such that $F(0)=0$. Examples of such function F could be:

$$F(x) = x$$
$$F(x) = e^{-x} - 1$$

This means that ~~the~~ the zeros of a given function $f(x)$ can be (in principle) computed by determining the fixed points of different auxiliary functions. In the example above, we have:

$$\phi_1(x) = x + f(x)$$
$$\phi_2(x) = x + e^{-f(x)} - 1$$



α is a fixed point of both ϕ_1 and ϕ_2

Remark: The choice of the auxiliary function ϕ can influence stability, convergence and convergence rate of fixed point iterations.

Example: $f(x) = x^4 - 4$. has 2 real zeros $\alpha_{1,2} = \pm\sqrt{2}$

Let us consider the following auxiliary functions:

$$\phi_1(x) = x + f(x)$$
$$\phi_2(x) = x - \frac{f(x)}{11}$$

Both are such that $f(\alpha)=0$
$$\Downarrow$$
$$\phi(\alpha)=\alpha$$

Set $x^{(0)} = 1$. Then:

$x^{(k+1)} = \phi_1(x^{(k)})$ diverges

$x^{(k+1)} = \phi_2(x^{(k)})$ converges to $\sqrt{2}$

(Try it in your computer)

Why is this happening?

<u>Theorem</u> (sufficient conditions for convergence of fixed point iterations)

Consider the sequence $x^{(k+1)} = \phi(x^{(k)})$, $x^{(0)}$ given. Then:

① If $\phi : [a,b] \to [a,b]$ is continuous then there exists at least one fixed point in $[a,b]$. Moreover, if $\phi(x)$ is a contraction, i.e.

$$\exists\, L < 1 : \quad |\phi(x_1) - \phi(x_2)| \leq L\, |x_1 - x_2| \quad \forall x_1, x_2 \in [a,b]$$

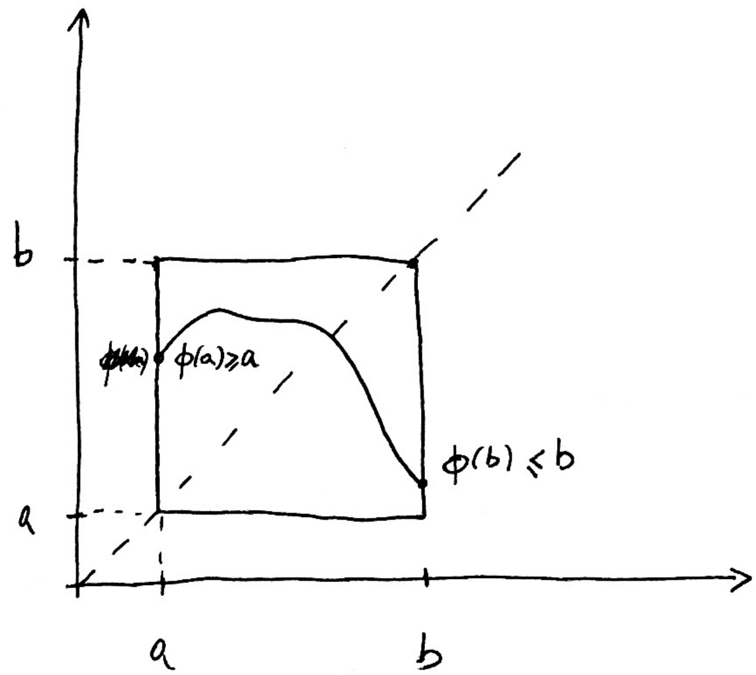
then the fixed point is unique in $[a,b]$ and globally attracting (any $x^{(0)} \in [a,b]$ will converge to such unique fixed point).

② If $\phi \in C^{(1)}([a,b])$ and $\exists\, K < 1 : |\phi'(x)| \leq K$ for all $x \in [a,b]$ then we have a globally attracting unique fixed point in $[a,b]$.

Proof ① Let $g(x) = \phi(x) - x$. g is continuous by our assumptions on ϕ. In addition, since the range of ϕ is bounded, we have:

$$g(a) = \phi(a) - a \geq 0$$
$$g(b) = \phi(b) - b \leq 0$$

The reason for such inequalities is the following:



$\Rightarrow g(a) \geq 0$, $g(b) \leq 0$ with continuous g implies that there exists at least one zero og g in $[a, b]$, i.e., one fixed point of $\phi(x)$.

Now, let us assume that ϕ is a contraction, i.e.,

$$|\phi(x_1)-\phi(x_2)| \leq L\,|x_1-x_2| \qquad x_1, x_2 \in [a,b], \quad L<1$$

We want to prove that in this case the fixed point is unique. Let us proceed by contraddiction. Suppose we have two fixed points α_1 and α_2 in $[a,b]$. Then

$$|\underbrace{\phi(\alpha_1)}_{=\alpha_1}-\underbrace{\phi(\alpha_2)}_{=\alpha_2}| \leq L\,|\alpha_1-\alpha_2| \underset{(L<1)}{<} |\alpha_1-\alpha_2|$$

Therefore we have $|\alpha_1-\alpha_2|<|\alpha_1-\alpha_2|$ which is impossible. Therefore $\alpha_1=\alpha_2$, i.e., the fixed point is unique if ϕ is contracting.

Now, let us prove the globally attracting property, i.e., converge to the fixed point disregarding the initial guess $x^{(0)}$.

$$\begin{aligned}|x^{(k+1)}-\alpha| = |\phi(x^{(k)})-\phi(\alpha)| &\leq L\,|x^{(k)}-\alpha| \\ &\leq L^2\,|x^{(k-1)}-\alpha| \\ &\;\vdots \\ &\leq L^{k+1}\,|x^{(0)}-\alpha|\end{aligned}$$

Therefore

$$\frac{|X^{(k+1)} - \alpha|}{|X^{(0)} - \alpha|} \leq L^{k+1} \qquad (L<1)$$

$\Rightarrow$ $X^{(k)}$ converges to α disregarding $x^{(0)}$ (simply take the limit left and right)

Therefore if ϕ is continuous and contracting we have a unique globally attracting fixed point and the sequence $X^{(k)}$ converges to that point with order 1.

(2) Let $\phi \in C^{(1)}([a,b])$ with $|\phi'(x)|<1$ for all $x \in [a,b]$. Let us prove by contradiction that there exists only one fixed point in $[a,b]$. To this end, assume that α_1, α_2 are two fixed points

$$|\alpha_1 - \alpha_2| = |\phi(\alpha_1) - \phi(\alpha_2)| \underset{\substack{\text{mean value}\\ \text{theorem } (\eta \in [a,b])}}{=} |\phi'(\eta)(\alpha_1 - \alpha_2)| = \underbrace{|\phi'(\eta)|}_{<1}|\alpha_1 - \alpha_2|$$

$\Rightarrow |\alpha_1 - \alpha_2| < |\alpha_1 - \alpha_2|$ impossible, therefore $\alpha_1 = \alpha_2$

Let us prove that $x^{(k)}$ converges to α disregarding $x^{(0)}$.

$$x^{(k+1)} - \alpha = \phi(x^{(k)}) - \phi(\alpha) = \phi'(\eta^{(k)})\,(x^{(k)} - \alpha)$$

$\eta^{(k)}$ is some point between $x^{(k)}$ and α *

$$\Rightarrow \quad \frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|} = |\phi'(\eta^{(k)})|$$

Taking the limit

$$\lim_{k\to\infty} \frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|} = |\phi'(\alpha)|$$

(here we used the continuity of ϕ')

This means that if $\phi'(\alpha) \neq 0$ then the sequence converges with order 1.
Also $|\phi'(\alpha)| < 1$ is the convergence factor.
As we will see, if $\phi'(\alpha) = 0$ then the sequence converges with order higher than one.

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As a corollary of part 2 of the previous theorem we have

Theorem (Ostrowsky) Let α be a fixed point of $\phi \in C^{(1)}([a,b])$. If $|\phi'(\alpha)| < 1$ then there exists a neighborhood of α $I_\alpha^\delta = \{ x \in [a,b] \mid |x-\alpha| \leq \delta \}$ such that

$$\lim_{k \to \infty} x^{(k)} = \alpha \quad \text{for any } x^{(0)} \in I_\alpha^\delta$$

Example: Consider again $f(x) = x^4 - 4$ and the auxiliary functions:

$$\phi_1 = x + f(x) \qquad \alpha_{1,2} = \pm\sqrt{2}$$

(real zeros)

$$\phi_2 = x - \frac{f(x)}{11}$$

It can be shown that $|\phi_1'(x)| > 0 \ \forall x > 0$ while $|\phi_2'(x)| < 1 \ \forall x \in]0, 1.765[$

$\Rightarrow$ if we pick $x^{(0)} = 1$ ϕ_1 diverges while ϕ_2 converges to a unique fixed point in $[0, 1.765]$, i.e., $\sqrt{2}$.