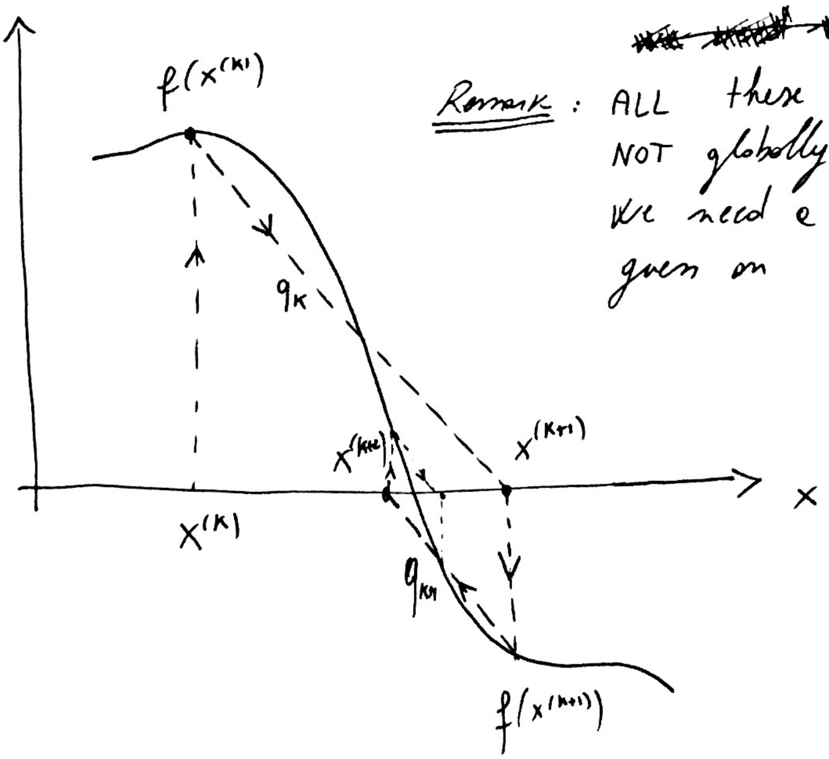
Methods of chord, secant and Newton

All these methods ~~are~~ generate sequences in the form:

(this is actually a DISCRETE DYNAMICAL SYSTEM)

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{q_k}$$

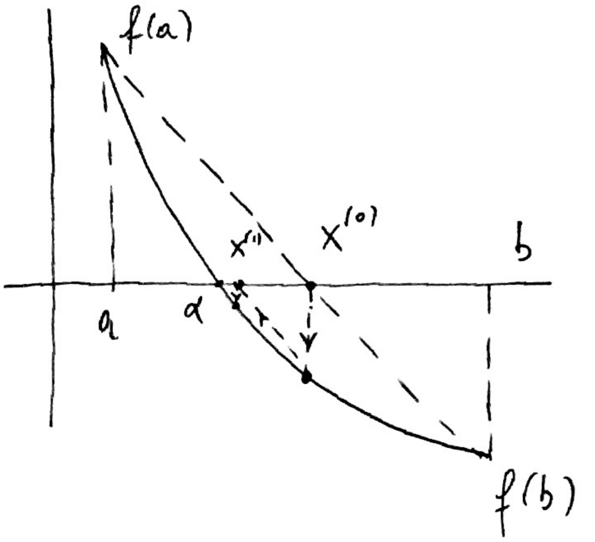
where q_k is the slope of a suitable line that passes through the point $(x^{(k)}, f(x^{(k)}))$ and allows us to determine $x^{(k+1)}$.



Remark: ALL these methods are NOT globally convergent, i.e., we need a good initial guess on the zeros.

How do we choose q_k?

The chord method



$$q_K = \frac{f(b) - f(a)}{b - a}$$

(slope of the chord that passes through $(a, f(a))$ and $(b, f(b))$.)

$$x^{(K+1)} = x^{(K)} - \frac{(b-a)}{f(b) - f(a)} f(x^{(K)})$$

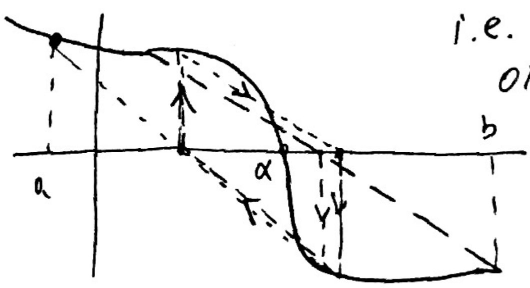
here x^0 is determined with a bisection step but we do not necessarily need such x^0. $\Rightarrow$ x^0 can be anything close to α.

The sequence has <u>convergence order $p = 1$</u> (we will prove this when we will talk about fixed point iterations).

The method is NOT globally convergent, i.e., we need to pick a and b close enough to α. More precisely, we need to make sure that $\text{sign}(q) = \text{sign}(f'(\alpha))$ and that

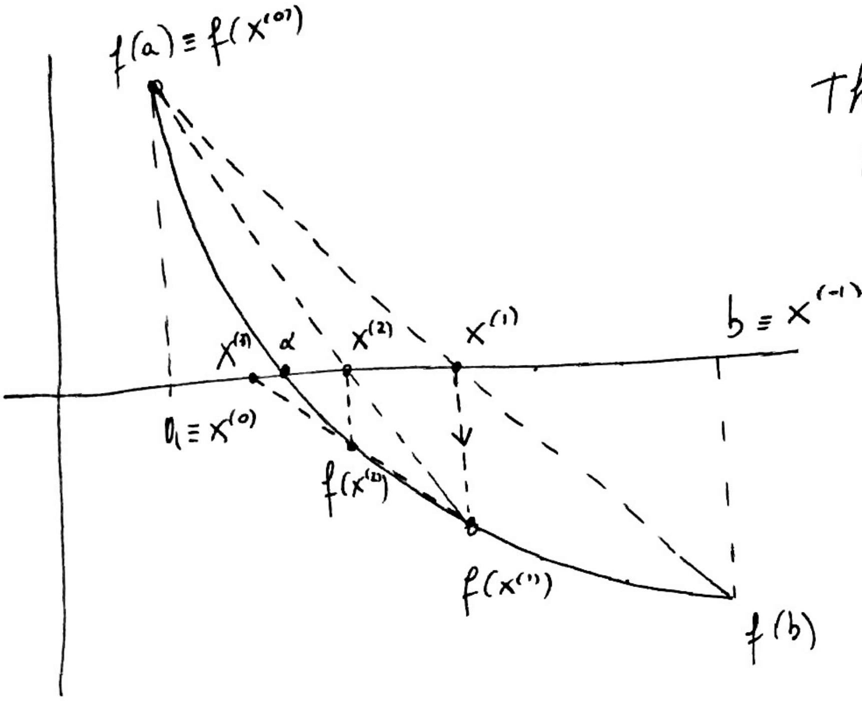
$$|q| > \frac{|f'(\alpha)|}{2}$$

Otherwise we may end up in a situation like this:



i.e. A PERIOD 2 ORBIT (discrete dynamical systems)

Secant Method

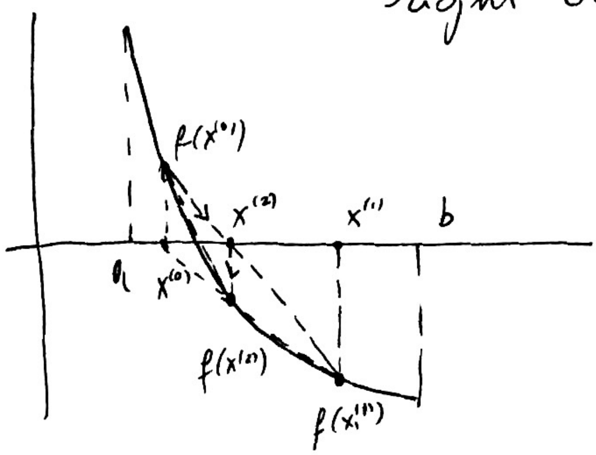


The method is NOT globally convergent

$$q_K = \frac{f(x^{(K)}) - f(x^{(K-1)})}{x^{(K)} - x^{(K-1)}}$$

we need two initial guesses to start the secant method (usually
$x^{(0)} = a$
$x^{(-1)} = b$).

Any two initial guesses close enough to α would do the job. (even if they are both to the right or to the left of α)



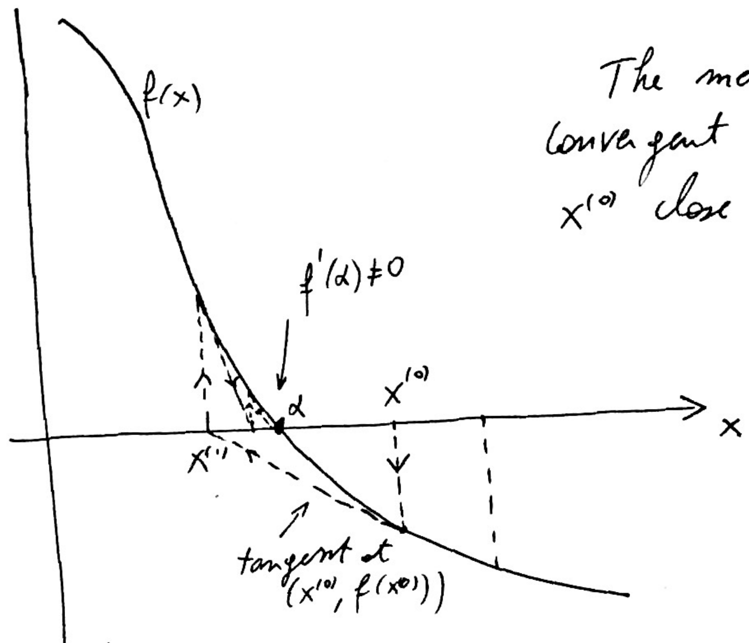
Remark: (Convergence order) If we choose the two initial guesses close enough to α and if $f'(\alpha) \neq 0$ then one can prove that $p = \frac{1+\sqrt{5}}{2} \simeq 1.63$ (GOLDEN RATIO) (SUPERLINEAR CONVERGENCE)

Newton's Method

Suppose that $f(x) \in C^{(1)}([a,b])$ and that $f'(\alpha) \neq 0$ (SIMPLE ROOT).

$$q_K = f'(x_1^{(K)}) \quad \Rightarrow \quad x^{(K+1)} = x^{(K)} - \frac{f(x^{(K)})}{f'(x^{(K)})}$$

(slope of f at $x^{(K)}$)



The method is convergent if we pick $x^{(0)}$ close enough to α

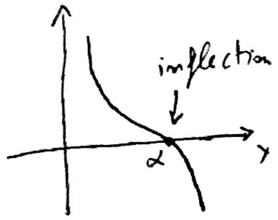
Remark: Alternative derivation:

$$f(x^{(K+1)}) = f(x^{(K)}) + f'(x^{(K)})(x^{(K+1)} - x^{(K)}) + \dots$$

$$f(\alpha) = 0 \quad \Rightarrow \quad x^{(K+1)} = x^{(K)} - \frac{f(x^{(K)})}{f'(x^{(K)})}$$

Remark (Convergence order) Suppose $f \in C^{(2)}$ and $f''(\alpha) \neq 0$ (i.e. α is not an inflection point)

Then: $$\lim_{K\to\infty} \frac{|x^{(K+1)}-\alpha|}{|x^{(K)}-\alpha|^2} = \frac{|f''(\alpha)|}{2\,|f'(\alpha)|}$$



$f'(\alpha) \neq 0$

$f''(\alpha) = 0$

$\Rightarrow$ The Newton method converges with order 2. (proof later).

Remark (stopping criterion for $x^{(K)}$) - ANY METHOD WE DISCUSSED SO FAR

We do not know α, so we can't figure out K such that $|x^{(K)}-\alpha| \leq \varepsilon$ (ε is a threshold). Two criteria:

① Control on the increment (usually this is the condition implemented)

$$|x^{(K)} - x^{(K-1)}| \leq \varepsilon_1$$

② Control on the residual $|f(x^{(K)})| \leq \varepsilon_2$

This can be $\underbrace{\text{excessively optimistic}}$ or $\underbrace{\text{too restrictive}}$

~~[illegible]~~: