Analysis of one-step explicit methods

Any one-step explicit method for the numerical approximation of

$$\begin{cases} \dfrac{dy(t)}{dt} = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

can be written in the following general form:

$$y_{n+1} = y_n + \Delta t \underbrace{\Phi(y_n, f(y_n, t_n), t_n, \Delta t)}_{\text{increment function}}$$

<u>Examples</u>:

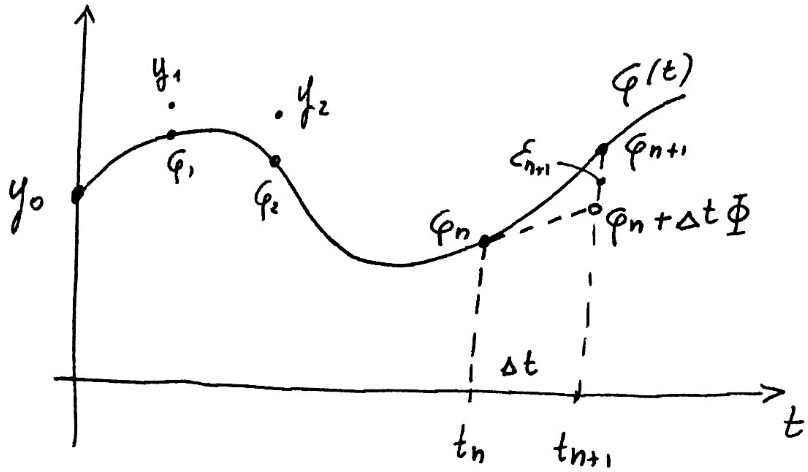
1) $\Phi = f(y_n, t_n)$ (Euler forward method)

2) $\Phi = \dfrac{f(y_n, t_n) + f(y_n + \Delta t\, f(y_n, t_n), \overset{t_n + \Delta t}{t_{n+1}})}{2}$

(Heun method)

Remark (LOCAL TRUNCATION ERROR). Let $\varphi(t)$ be the exact solution to $\frac{dy(t)}{dt} = f(y(t),t)$

$y(0) = y_0$



If we substitute $\varphi_n = \varphi(t_n)$ into the one-step method we obtain

$$\varphi_{n+1} = \varphi_n + \Delta t\,\Phi + \varepsilon_{n+1}$$

We define the <u>local truncation error</u> $\tau_{n+1}(\Delta t)$ as $\tau_{n+1}(\Delta t) = \frac{\varepsilon_{n+1}}{\Delta t}$

$$\Rightarrow \varphi_{n+1} = \varphi_n + \Delta t\left(\Phi + \tau_{n+1}(\Delta t)\right)$$

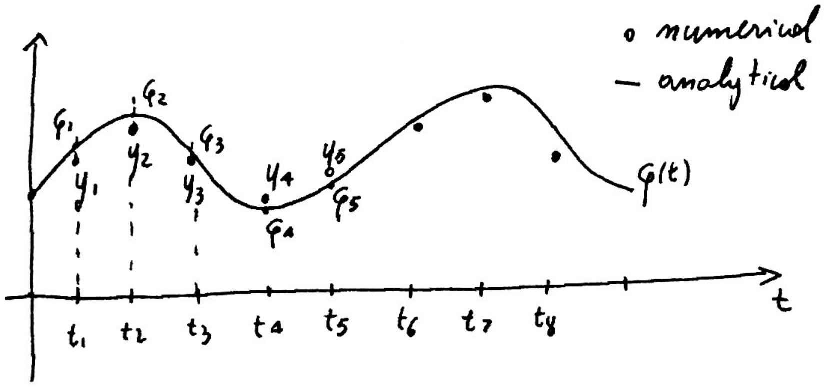
$$\left(\frac{\varphi_{n+1} - \varphi_n}{\Delta t} = \Phi(\varphi_n, f(\varphi_n, t_n), t_n, \Delta t) + \tau_{n+1}(\Delta t) \right)$$

this depends on φ, etc.. as well

Remark (GLOBAL TRUNCATION ERROR)

$$\tau(\Delta t) = \max_{i=1,\dots,N} |\tau_i(\Delta t)|$$

Remark: (Analytical vs numerical solution)



Remark (Consistent and Convergent schemes)

A numerical scheme is said to be CONSISTENT if

$$\lim_{\Delta t \to 0} \tau(\Delta t) = 0$$

If $\tau(\Delta t)$ goes to zero as $(\Delta t)^p$ then the scheme is said to be consistent with order p, ~~or simply order p~~.

A numerical scheme is said to be **convergent** if

$$|y_i - \varphi_i| \leq C_i(\Delta t) \qquad \lim_{\Delta t \to 0} C_i(\Delta t) = 0$$

$$i = 1, \dots, N$$

If $C_i(\Delta t) = O(\Delta t^p)$ then we say that the method converges with order p.

Remark: Euler forward method converges with order 1. Heun method converges with order 2.

Convergence analysis of the Euler forward method

The local truncation error of the Euler forward method is:

$$\tau_{n+1}(\Delta t) = \frac{\varphi_{n+1} - \varphi_n - \Delta t\, f(\varphi_n, t)}{\Delta t} \qquad \begin{array}{l} \varphi_{n+1} = \varphi(t_{n+1}) \\ \varphi_n = \varphi(t_n) \end{array}$$

$$= \frac{\varphi_{n+1} - \varphi_n}{\Delta t} - f(\varphi_n, t)$$

By using the Taylor series expansion of φ_{n+1}

$$\varphi_{n+1} = \varphi_n + \Delta t\, \varphi'(t_n) + \frac{\Delta t^2}{2} \varphi''(\xi_n) \qquad \xi_n \in [t_n, t_n + \Delta t]$$

$$\Rightarrow \frac{\varphi_{n+1} - \varphi_n}{\Delta t} = \underset{f(\varphi_n, t_n)}{\underset{\parallel}{\varphi'(t_n)}} + \frac{\Delta t}{2} \varphi''(\xi_n)$$

Therefore the local truncation error can be expressed as

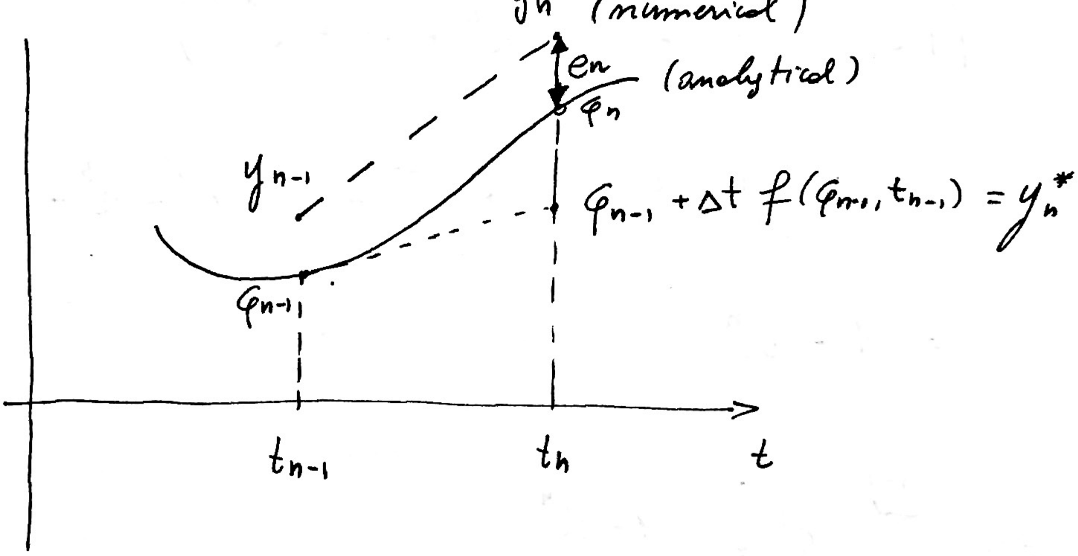
$$\tau_{n+1}(\Delta t) = \frac{\Delta t}{2} \varphi''(\xi_n)$$

This implies that the global truncation error is

$$\tau(\Delta t) = \frac{\Delta t}{2} \max_{t\in[0,T]} |\varphi''(t)| \qquad \left(\frac{\Delta t}{2} \max_{i=1,\dots,N} |\varphi''(\xi_i)| \right)$$

$\Rightarrow$ The Euler forward method is consistent with order 1.

Now let us prove convergence



$$\begin{aligned} e_n &= |y_n - \varphi_n| \\ &= |y_n - y_n^* + y_n^* - \varphi_n| \\ &\le |\varphi_n - y_n^*| + |y_n^* - \overset{y_n}{\cancel{\varphi_n}}| \end{aligned}$$

$$\begin{aligned} y_n^* - y_n &= \varphi_{n-1} - y_n + \Delta t\, f(\varphi_{n-1}, t_{n-1}) \\ &= \varphi_{n-1} - y_{n-1} + \Delta t \left(f(\varphi_{n-1}, t_{n-1}) - f(y_{n-1}, t_{n-1}) \right) \end{aligned}$$

$$\Rightarrow |y_n^* - y_n| \leq e_{n-1} + \Delta t \, | f(\varphi_{n-1}, t_{n-1}) - f(y_{n-1}, t_{n-1}) |$$

$$\leq e_{n-1} + \Delta t \underset{\substack{\downarrow \\ \text{LIPSCHITZ CONSTANT}}}{L} e_{n-1}$$

$$\Rightarrow |y_n^* - y_n| \leq (1 + \Delta t L) e_{n-1}$$

Now we have a recursion:

$$\begin{aligned} e_n &\leq |\varphi_n - y_n^*| + |y_n^* - y_n| \\ &\leq \Delta t |\tau_n(\Delta t)| + (1 + \Delta t L) e_{n-1} \qquad (\tau_n \text{ local truncation error}) \\ &\leq \Delta t |\tau_n(\Delta t)| + (1 + \Delta t L)(\Delta t |\tau_{n-1}| + (1 + \Delta t L) e_{n-2}) \\ &\leq \Delta t |\tau_n(\Delta t)| + (1 + \Delta t L) \Delta t |\tau_{n-1}(\Delta t)| + (1 + \Delta t L)^2 (|\tau_{n-2}(\Delta t)| + (1 + \Delta t L) e_{n-3}) \\ &\leq \left(\sum_{K=0}^{n-1} (1 + \Delta t L)^K \right) \Delta t \underbrace{\tau(\Delta t)}_{\text{global truncation error}} \qquad \underset{\text{initial condition}}{(e_0 = 0)} \end{aligned}$$

Recall that: (GEOMETRIC PROGRESSION)

$$\sum_{K=0}^{n-1} (1 + \Delta t L)^K = \frac{(1 + \Delta t L)^n - 1}{\Delta t L} \qquad (1 + \Delta t L) \leq e^{\Delta t L}$$

$$\Rightarrow e_n \leq \frac{e^{n \Delta t L} - 1}{\cancel{\Delta t} L} \tau(\Delta t) = \frac{e^{n \Delta t L} - 1}{L} \frac{\Delta t}{2} \max_{t \in [0,T]} |\varphi''(t)|$$

Recall that

$$\sum_{K=0}^{n-1} a^K = \frac{1-a^n}{1-a}$$

To prove this consider

$$(1-a)\overbrace{\left(a^0 + a^1 + a^2 + \cdots + a^{n-1}\right)}^{\sum_{K=0}^{n-1} a^K}$$

$$= a^0 + \cancel{a^1} + \cancel{a^2} + \cdots + a^{n-1} - \cancel{a^1} - \cancel{a^2} - a^3 - \cdots a^n$$

$$= 1 - a^n$$

$$\Rightarrow \sum_{K=0}^{n-1} a^K = \frac{1-a^n}{1-a}$$

$$e^{nL\Delta t} - 1 = \cancel{1} + nL\Delta t + \frac{n^2L^2\Delta t^2}{2} - \cancel{1} + \cdots$$

$$= nL\Delta t + \cdots$$

$$= TL + \cdots$$

(where T is the total integration time

Note that $n\Delta t = T$ (total integration time).

Therefore,

$$e_n \leq \frac{e^{TL}-1}{L} \frac{\Delta t}{2} \max_{i=1,\dots,n} |\varphi''(\xi_i)| \qquad \xi_i \in [t_i, t_{i+1}]$$

$\Rightarrow$ The Euler forward method converges with order 1. In fact,

$$\max_{i=1,\dots,n} |y_i - \varphi_i| = \max_{i=1,\dots,n} e_i \leq \frac{e^{TL}-1}{L} \frac{\Delta t}{2} \max_{j=1,\dots,n} |\varphi''(\xi_j)|$$