Gauss elimination method and the LU factorization

The Gauss elimination method with pivoting by row induces a factorization of the matrix A in the form:

$$PA = LU$$

where P is a permutation matrix, L is lower triangular and U is upper triangular. P keeps track of all the row permutations we have performed in the pivoting steps, L has all coefficients we used to perform elimination.

__Example__: To illustrate the LU factorization, consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -3 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

This vector keeps track of all permutations we do in the pivoting steps

$$\begin{bmatrix} 1 & 1 & 3 \\ -3 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \xrightarrow[\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}]{\text{pivoting}} \begin{bmatrix} -3 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 2 & -1 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & \frac{10}{3} \\ 0 & 2 & -\frac{1}{3} \end{bmatrix}$$

To perform elimination, we multiplied the first row by $-\frac{1}{3}$ and $-\frac{2}{3}$ and subtracted it respectively from the second and the third row.

We store $-\frac{1}{3}$ and $-\frac{2}{3}$ in the ~~sam~~ spots where we have zeros, i.e.,

$$\begin{bmatrix} -3 & 0 & 1 \\ -\frac{1}{3} & 1 & \frac{10}{3} \\ -\frac{2}{3} & 2 & -\frac{1}{3} \end{bmatrix} \xrightarrow[\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}]{\text{pivoting}} \begin{bmatrix} -3 & 0 & 1 \\ -\frac{2}{3} & 2 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{10}{3} \end{bmatrix}$$

ELIMINATION: Multiply the ~~second~~ first row of the block matrix $\begin{bmatrix} 2 & -\frac{1}{3} \\ 1 & \frac{10}{3} \end{bmatrix}$ by $+\frac{1}{2}$ and subtract it from the third row

$$\begin{bmatrix} -3 & 0 & 1 \\ -\frac{2}{3} & 2 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{21}{6} \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

permutation vector

We claim that $PA = LU$ where:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

permutation matrix corresponding to the permutation vector

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -\frac{1}{3} \\ 0 & 0 & \frac{21}{6} \end{bmatrix}$$

In fact: $PA = \begin{bmatrix} -3 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix}$

$$LU = \begin{bmatrix} -3 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & \frac{18}{6} \end{bmatrix}$$

<u>Remark</u> (Computing determinants of matrices)

The Laplace rule requires

$$\lfloor n!\,e \rfloor - 2 \quad \text{operations} \quad \Rightarrow \text{not viable for } n > 15$$

Napier number

LOWEST INTEGER LESS OR EQUAL THAN $n!e$

If we use the LU factorization we have

$$PA = LU \Rightarrow \det(P)\det(A) = \underbrace{\det(L)}_{=1}\det(U)$$

$$\boxed{\det(A) = (-1)^s \prod_{i=1}^{n} u_{ii}}$$

s is the total number of permutations

The cost of computing $\det(A)$ with LU factorization is:

$$\underbrace{\frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6}}_{\text{LU factorization}} + \underbrace{n+1}_{(-1)^s \prod_{i=1}^{n} u_{ii}} \cong \frac{2}{3}n^3 - \frac{n^2}{2} + \frac{5}{6}n$$

(n is the size of A)

Example: The LU factorization of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -3 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{yields} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

($\Rightarrow s = 2$)
2 permutations of the rows of the identity matrix

$$U = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & -\frac{1}{3} \\ 0 & 0 & \frac{21}{6} \end{bmatrix}$$

$$\Rightarrow \det(A) = (-1)^2 \left(-3 \times 2 \times \frac{21}{6} \right) = -21$$

Remark (Tridiagonal systems)

The LU factorization of a tridiagonal matrix A (nonsingular)

$$A = \begin{bmatrix} a_1 & c_1 & & & 0 \\ e_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & e_n & a_n \end{bmatrix}$$

assume that all principal minors are nonsingular (i.e. we do not do pivoting)

yields bidiagonal matrices L and U

$$L = \begin{bmatrix} 1 & & & \\ \beta_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \beta_n & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \alpha_1 & c_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & c_{n-1} \\ 0 & & & \alpha_n \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. It is clear that L is bi diagonal as well as U.

The coefficients of L and U, β_j and α_k, can be obtained by imposing $LU = A$

This yields:

$$\alpha_1 = a_1 \qquad \alpha_1 \beta_2 = e_2 \qquad \beta_2 c_1 + \alpha_2 = a_2 \qquad (\cdots)$$

i.e.

$$\alpha_1 = a_1 \qquad \beta_i = \frac{e_i}{\alpha_{i-1}} \qquad \alpha_i = a_i - \beta_i c_{i-1} \qquad i = 2, \dots, n$$

$\Rightarrow$ LU for tridiagonal matrices is computed with $O(n)$ OPERATIONS

With these coefficients available we can solve the linear system with $O(n)$ operations

To this end, let $Ax=b$ be the tridiagonal system. The LU factorization of A is computed at linear cost in n

$$LUx = b$$

Define $Ux = y$. This yields the system $Ly = b$, which can be solved for y using forward substitution at linear cost in n.

$$Ly = b \xRightarrow[\text{substitution}]{\text{forward}} y_1 = b_1 \qquad y_i = b_i - \beta_i y_{i-1} \quad i=2,\dots,n$$

$$Ux = y \xRightarrow[\text{substitution}]{\text{backward}} x_n = \frac{y_n}{\alpha_n} \qquad x_i = \frac{1}{\alpha_i}\left(y_i - c_i x_{i+1}\right) \quad i = n-1,\dots,1$$

This method is known as THOMAS ALGORITHM and it allows us to compute the solution to tri-diagonal system with $O(n)$ operations instead of $O(n^3)$.

Remark: The M-continuity system for cubic interpolatory splines is tridiagonal.