Gaussian Quadrature

The degree of exactness of a Newton-Cotes formula with $n+1$ points $\{x_0, \cdots, x_n\}$ is

- n if n is odd (trapezoidal)
- $n+1$ if n is even (Simpson)

A natural question is whether suitable choices of nodes exist such that the degree of exactness is $n + M$ for some $M \in \mathbb{N}$, $M > 0$.
Without loss of generality we restrict our attention to nodes defined in the interval $[-1, 1]$.
Any interval $[a,b]$ can be mapped onto $[-1, 1]$ by using a linear transformation:

$$x = \frac{b-a}{2}\eta + \frac{b+a}{2} \qquad x \in [a,b] \quad \eta \in [-1,1]$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{(b-a)}{2}\int_{-1}^{1} f(x(\eta))\,d\eta$$

Theorem (Jacobi). Let $\{x_0, \cdots, x_n\}$ be nodes in $[-1,1]$. For any given $M>0$ $(M\in\mathbb{N})$ the quadrature formula

$$\int_{-1}^{1} f(x)\,w(x)\,dx \simeq \sum_{k=0}^{n} f(x_k)\,w_k$$

has degree of exactness $n+M$ if and only if the polynomial $p_{n+1}(x) = \prod_{i=0}^{n}(x-x_i)$ is orthogonal to all polynomials of order $M-1$ in $L^2_w([-1,1])$ i.e., if

$$(p_{n+1}, q)_{L^2_w} = \int_{-1}^{1} p_{n+1}(x)\,q(x)\,w(x)\,dx = 0 \quad \forall q(x)\in\mathbb{P}_{M-1}$$

Remark: To determine the NODES $\{x_0,\cdots,x_n\}$ we need to identify a polynomial with zeros $\{x_0,\cdots,x_n\}$ that is orthogonal to all polynomials of degree $M-1$.

Example: (Gauss-Legendre quadrature). It is known that Legendre polynomials $L_n(x)$ are defined by the three-term relation

$$L_{n+1}(x) = \frac{2n+1}{n+1}\, x\, L_n(x) - \frac{n}{n-1} L_{n-1}(x)$$

$$L_0(x) = 1$$

$$L_{-1}(x) = 0$$

and are orthogonal in $L^2([-1,1])$.

$$\int_{-1}^{1} L_n(x)\, L_m(x)\, dx = \delta_{nm} \frac{2}{2n+1}$$

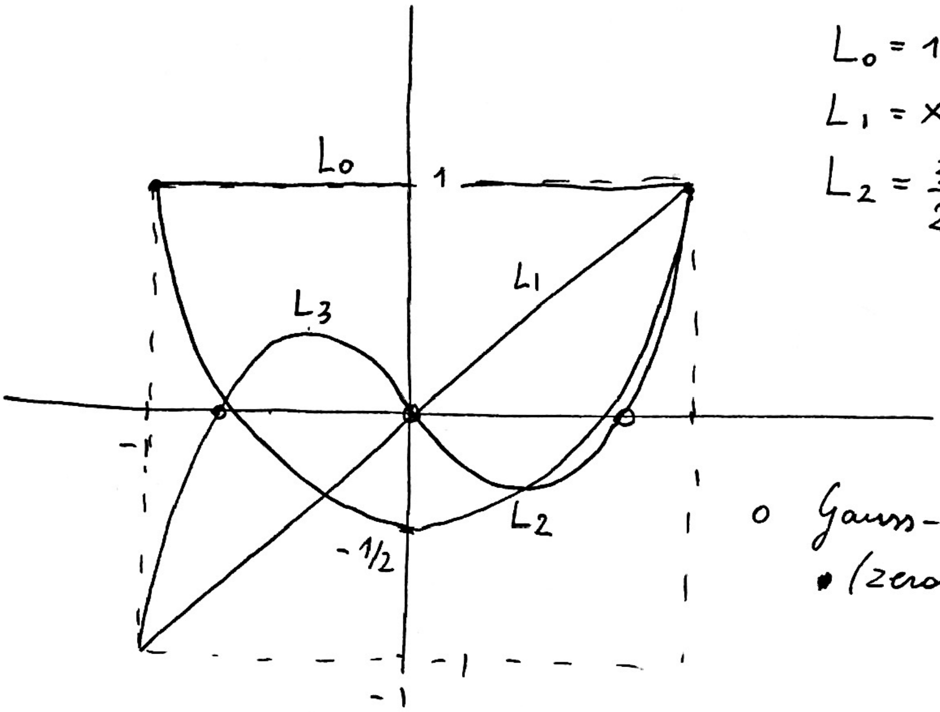
(weight function $w(x) = 1$)

Any polynomial of degree $M-1$ can be represented in terms of Legendre polynomials as

$$q(x) = \sum_{K=0}^{M-1} b_K L_K(x)$$

If we set $p_{n+1}(x) = L_{n+1}(x)$ then $\{x_0, \dots, x_n\}$ are ZEROS OF THE LEGENDRE POLYNOMIAL OF DEGREE $n+1$. (Recall that $p_{n+1} = \prod_{j=0}^{n} (x - x_i)$)

Remark (Legendre polynomials)



$L_0 = 1$

$L_1 = x$

$L_2 = \frac{3}{2} x^2 - \frac{1}{2}$

o Gauss-Legendre points

• (zeros of $L_3(x)$)

with such points we can integrate exactly a polynomial of order 5

By applying Jacobi's theorem we have

$$\int_{-1}^{1} L_{n+1}(x)\, q(x)\, dx = \sum_{K=0}^{M-1} b_K \int_{-1}^{1} L_{n+1}(x) L_K(x)\, dx$$

$$= 0 \iff M-1 = n \quad (\text{or less})$$

$$\Rightarrow M = n+1$$

$\Rightarrow$ Gauss Legendre quadrature has degree of exactness $n + M = n + n + 1 = 2n + 1$.

In other words, with $n+1$ points (ZEROS OF Legendre polynomial $L_{n+1}(x)$) we integrate exactly polynomials of degree $2n+1$

Example : $n = 5 \Rightarrow 6$ points (zeros of $L_6(x)$)
we integrate exactly a polynomial of degree 11.

Remark : The maximum degree of exactness in quadrature formulae of interpolatory type with $n+1$ nodes is $2n+1$. and it is achieved by Gaussian integration.

Now that we have all quadrature nodes $\{x_0, \dots, x_n\}$, how do we construct the Gauss quadrature formula?

$$\int_{-1}^{1} f(x)\,dx \cong \sum_{k=0}^{n} f(x_k) \underbrace{\int_{-1}^{1} l_k(x)\,dx}_{w_k}$$

The Lagrange characteristic polynomials $l_k(x)$ associated with a grid defined by zeros of Legendre polynomials is given by

$$l_k(x) = \frac{L_{n+1}(x)}{(x-x_k)\, L'_{n+1}(x_k)}$$

The integrals of such polynomials gives us the integration weights

$$w_k = \int_{-1}^{1} l_k(x)\,dx = \frac{2}{(1-x_k^2)\, L'_{n+1}(x_k)^2}$$

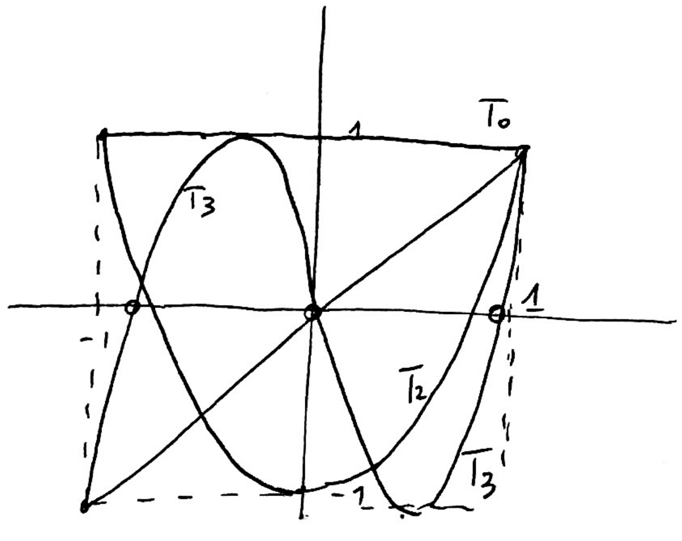
Example (Gauss - Chebyshev quadrature)

$$\int_{-1}^{1} f(x) \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\omega(x)} dx \simeq \sum_{k=0}^{n} f(x_k)\, w_k$$

$$w_k = \int_{-1}^{1} \ell_k(x) \frac{1}{\sqrt{1-x^2}} dx$$

$\{x_0, \dots, x_n\}$ are zeros of the Chebyshev polynomial of degree $(n+1)$, $T_{n+1}(x)$.

Chebyshev polynomials are orthogonal in $L^2_\omega([-1,1])$ relative to the weight $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.



$$\int_{-1}^{1} T_i(x)\, T_j(x) \frac{1}{\sqrt{1-x^2}} dx = \| T_i \|^2_{L^2_w} \delta_{ij}$$

$$\| T_i \|^2_{L^2_w} = \begin{cases} \pi & i = 0 \\ \frac{\pi}{2} & i > 0 \end{cases}$$

① Gauss Chebyshev points $\quad T_{n+1}(x) = 0 \quad$ (left figure : $T_3(x) = 0$)

② Lagrange polynomials $\quad \ell_k(x) = \dfrac{T_{n+1}(x)}{(x - x_k)\, T'_{n+1}(x)}$

③ Integration weights $\quad w_k = \int_{-1}^{1} \ell_k(x) \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{n+1}$

Gauss and Gauss-Lobatto integration
(with $n+1$ nodes)

$$\int_{-1}^{1} f(x)\,w(x)\,dx \simeq \sum_{K=0}^{n} f(x_K)\,w_K \qquad w_K = \int_{-1}^{1} l_K(x)\,w(x)\,dx$$

	GAUSS (EXACTNESS $2n+1$)		GAUSS-LOBATTO (EXACTNESS $2n-1$)	
	LEGENDRE	CHEBYSHEV	LEGENDRE	CHEBYSHEV
x_K	$L_{n+1}(x)=0$	$T_{n+1}(x)=0$	$(1-x^2)\dfrac{dL_n(x)}{dx}=0$	$(1-x^2)\dfrac{dT_n(x)}{dx}=0$
$l_K(x)$	$\dfrac{L_{n+1}(x)}{(x-x_K)L'_{n+1}(x)}$	$\dfrac{T_{n+1}(x)}{(x-x_K)T'_{n+1}(x)}$	$-\dfrac{1}{n(n+1)}\dfrac{(1-x^2)}{(x-x_K)}\dfrac{L'_n(x)}{L_n(x_K)}$	$\dfrac{(-1)^{n+K+1}(1-x^2)T'_n(x)}{d_i\, n^2 (x-x_K)}$
w_K	$\dfrac{2}{(1-x_K^2)L'_{n+1}(x_K)^2}$	$\dfrac{\pi}{n+1}$	$\dfrac{2}{n(n+1)L_n(x_K)^2}$	$\dfrac{\pi}{d_i n}$

$d_0 = d_n = 2$

$d_1 = \cdots = d_{n-1} = 1$

Weight functions:

- Legendre $\quad w(x) = 1$
- Chebyshev $\quad w(x) = \dfrac{1}{\sqrt{1-x^2}}$