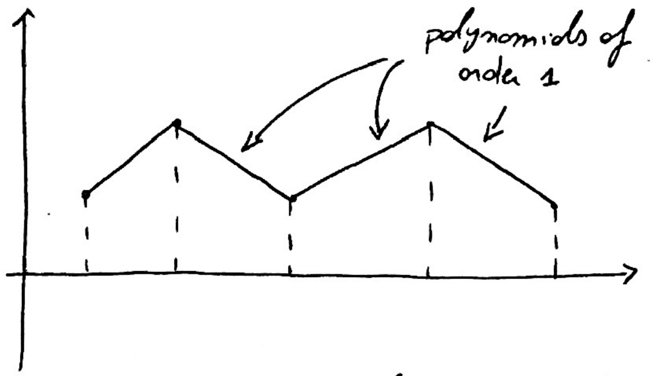
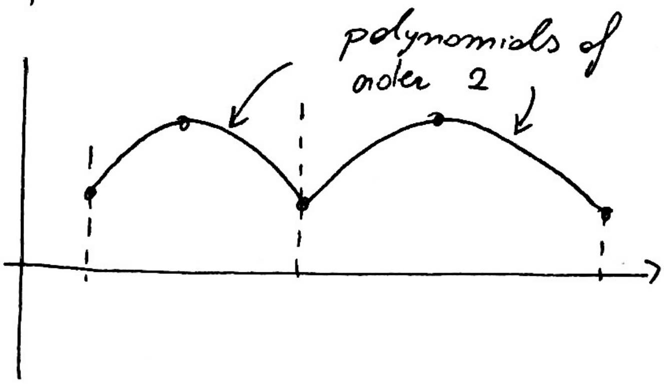
Piecewise polynomial interpolation

- Piecewise Lagrangian interpolation



piecewise linear



piecewise quadratic

Note that in both cases we obtain a continuous function that has discontinuous derivatives at some nodes. In many applications, e.g., computer graphics, it is desirable to have polynomial approximations of data and functions which are at least differentiable in x. $\Rightarrow$ SPLINES

Approximation by splines

Spline functions allow for a piecewise interpolation (OR REGRESSION) with global smoothness.

__Definition__ Let $\{x_i\}_{i=0,\cdots,n}$ $(n+1)$ distinct nodes in

$[a,b]$ $\qquad a = x_0 < x_1 < \cdots < x_n = b$

The function $S_K(x)$ $(x \in [a,b])$ is a SPLINE of degree K if

RESTRICTION OF $S_K(x)$ TO $x \in [x_J, x_{J+1}]$

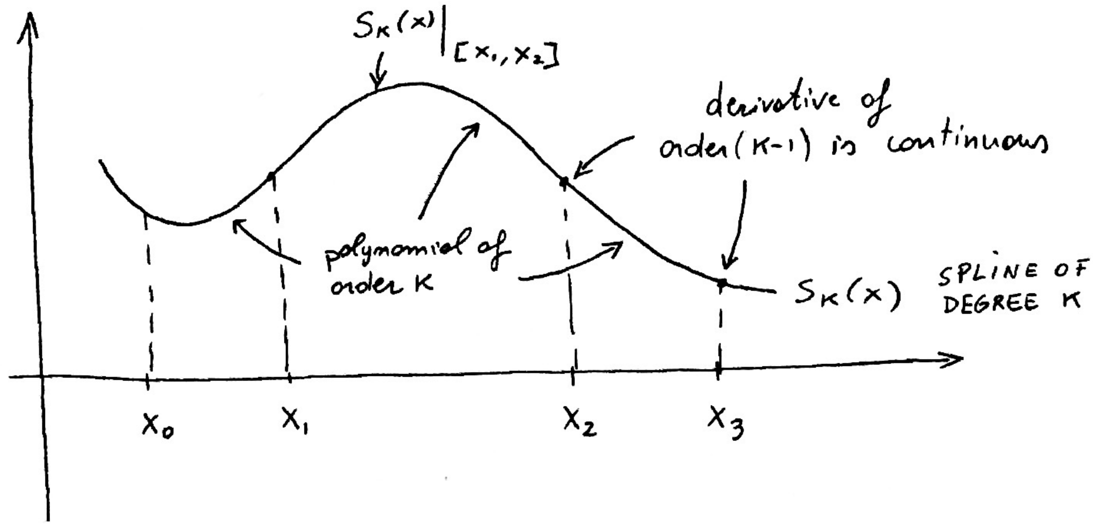
1) $S_K(x)\Big|_{[x_J, x_{J+1}]} \in \mathbb{P}_K \qquad J = 0, \cdots, n-1$

polynomial of order K

2) $S_K(x) \in C^{(K-1)}([a,b])$ (global smoothness)

From this definition it follows that the derivative of a spline of degree K is a spline of degree $K-1$, while the integral of a spline (primitive function) is a spline of degree $K+1$.

__Remark__: Piecewise Lagrangian interpolants are not splines.



Remark Any polynomial of degree K is a SPLINE of degree K. For example, a global Lagrangian interpolant is a spline.

Remark (Degrees of freedom of a spline of degree K)

Denote by $\quad S_{K,i}(x) = S_K(x)\big|_{[x_i,x_{i+1}]} \quad i=0,\dots,n-1$

From condition 1) we have that $S_{K,i}(x)$ is a polynomial of order K in $x \in [x_i, x_{i+1}]$.

Therefore:

$$S_{K,i}(x) = \sum_{q=0}^{K} \underbrace{a_{qi}}_{\text{polynomial coefficients } (K+1)} (x-x_i)^q \qquad x \in [x_i, x_{i+1}]$$

$\Rightarrow$ We need $(K+1)\cdot \overset{\text{number of elements}}{n}$ coefficients to identify the spline in $[a,b]$.

By imposing the regularity requirements

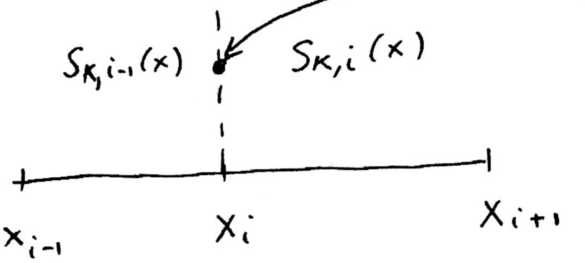
$$S_K(x) \in C^{(K-1)}([a,b])$$

at all nodes (the inner ones) $\{x_1, \dots, x_{n-1}\}$

$$S_{K,i-1}^{(m)}(x_i) = S_{K,i}^{(m)}(x_i)$$

$m = 0, \dots, K-1$

$i = 1, \dots, n-1$



We obtain $(n-1)K$ conditions. Therefore we need to set

$$\underbrace{n(K+1)}_{\text{degrees of freedom}} - \overbrace{(n-1)K}^{\text{regularity requirements}} = (n+K) \text{ degrees of freedom.}$$

<u>Remark</u> (Interpolatory cubic splines $S_3(x)$)

In this case we have $K=3$ and $n+3$ degrees of freedom, $n+1$ of which can be set through interpolation conditions at $\{x_i, y_i\}_{i=0,\dots,n}$. Thus we are left with only 2 degrees of freedom: We can set them as:

1) $S_3''(x_0) = S_3''(x_n) = 0$ (natural splines)

2) $S_3'''(x)$ is continuous at x_1 and x_{n-1} (not-a-knot splines) (IMPLEMENTED IN MATLAB AS SPLINE( ))