Theorem (Convergence order of fixed-point Iterations)

Let I_α^δ be a neighborhood of α and $\phi \in C^{p+1}(I_\alpha^\delta)$ $p \geq 1$ (integer).

If $\dfrac{d^i\phi(\alpha)}{dx^i} = 0$ $i = 1, \dots, p$ and $\dfrac{d^{p+1}\phi(\alpha)}{dx^{p+1}} \neq 0$ (FINITE)

then the sequence $x^{(k+1)} = \phi(x^{(k)})$ converges to α with order $p+1$ for any $x^{(0)} \in I_\alpha^\delta$

$$\lim_{k \to \infty} \frac{(x^{(k+1)} - \alpha)}{(x^{(k)} - \alpha)^{p+1}} = \frac{1}{(p+1)!} \frac{d^{p+1}\phi(\alpha)}{dx^{p+1}}$$

Proof Consider the Taylor series

$$x^{(k+1)} - \alpha = \phi(x^{(k)}) - \phi(\alpha) = \sum_{i=1}^{p} \frac{1}{i!} \frac{d^i\phi(\alpha)}{dx^i} (x^{(k)} - \alpha)^i +$$

$$+ \frac{1}{(p+1)!} \frac{d^{p+1}\phi}{dx^{p+1}}(\eta^{(k)}) (x^{(k)} - \alpha)^{p+1}$$

$\eta^{(k)} \in [x^{(k)}, \alpha]$ or $[\alpha, x^{(k)}]$

By continuity:

$$\lim_{k \to \infty} \frac{(x^{(k+1)} - \alpha)}{(x^{(k)} - \alpha)^{p+1}} = \frac{1}{(p+1)!} \frac{d^{p+1}\phi}{dx^{p+1}}(\alpha)$$

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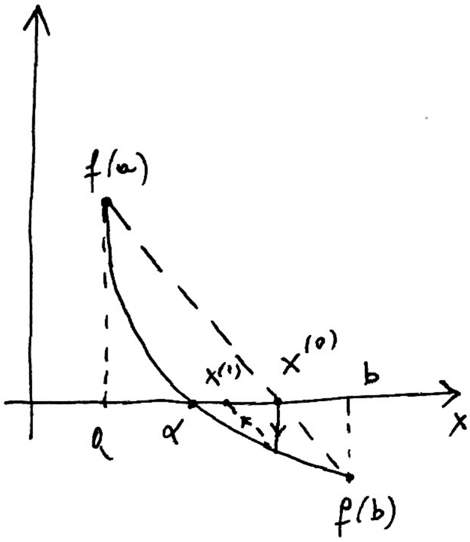
Remark: We do not need to assume that $\left|\frac{d^{p+1}\phi(\alpha)}{dx^{p+1}}\right| < 1$. Remember that this is necessary for convergence only if $p=0$ (Ostrowsky theorem).

Convergence Results for some fixed point methods

The chord method



$$x^{(k+1)} = x^{(k)} - \frac{(b-a)}{f(b)-f(a)} f(x^{(k)})$$

$$\Rightarrow x^{(k+1)} = \phi(x^{(k)})$$

$$\phi(x) = x - \frac{(b-a)}{f(b)-f(a)} f(x)$$

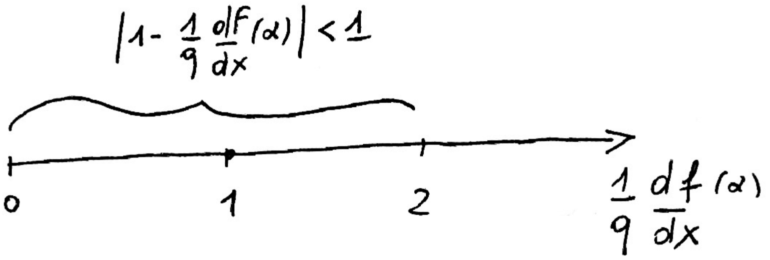
(AUXILIARY FUNCTION)

$$\frac{d\phi}{dx} = 1 - \frac{(b-a)}{f(b)-f(a)} \frac{df}{dx}$$

If $\frac{df}{dx}(\alpha) = 0$ (multiple zero) we have $\frac{d\phi}{dx}(\alpha) = 1$ and the chord method is not granted to converge.

Let $q = \frac{f(b)-f(a)}{b-a}$. In order to have $\left|\frac{d\phi}{dx}(\alpha)\right| < 1$

we must have $\left| 1 - \frac{1}{q}\frac{df}{dx}(\alpha)\right| < 1$



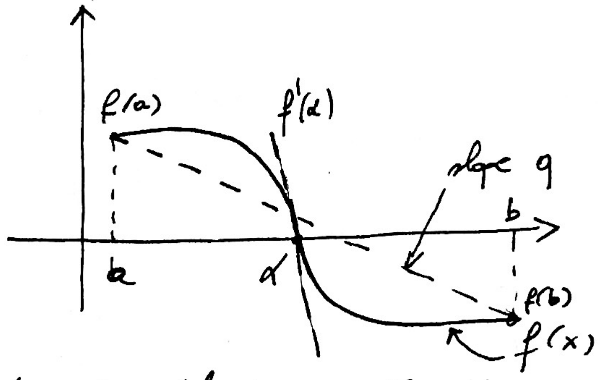
This means that

$$0 < \frac{df(\alpha)}{dx}\cdot\frac{1}{q} < 2$$

(1) $\Rightarrow$ the slope of the chord q must be of the same sign as $\frac{df(\alpha)}{dx}$ and must be such that:

$$|q| > \frac{1}{2}\left|\frac{df(\alpha)}{dx}\right|$$

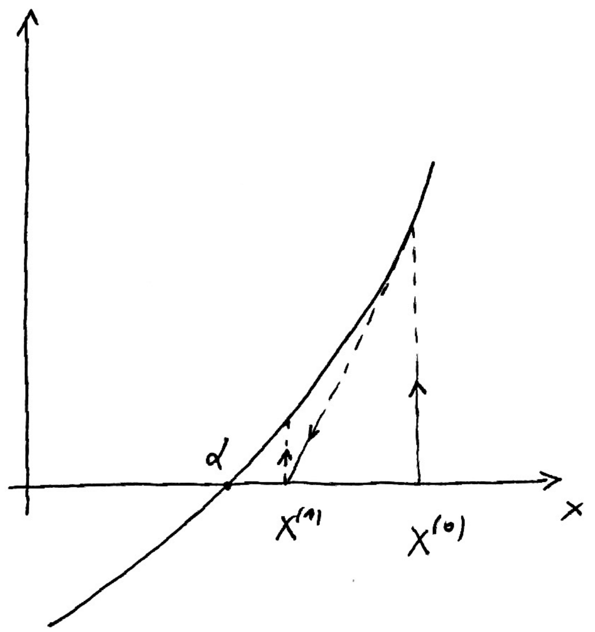
Example:



The chord method will not converge here as $|q| < \frac{1}{2}|f'(\alpha)|$

(2) $\Rightarrow$ The chord method converges with ORDER 1 since $\phi'(\alpha) \neq 0$

Newton's method



$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$\Rightarrow \quad x^{(k+1)} = \phi(x^{(k)})$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

(auxiliary function)

$$\phi'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)}{f'(x)^2} f''(x)$$

$$= \frac{f(x)}{f'(x)^2} f''(x)$$

$$\phi''(x) = \frac{f'(x)\, f''(x)}{f'(x)^2} + \frac{f(x)\, f'''(x)}{f'(x)^2} - 2\,\frac{f(x)\, f''(x)^2}{f'(x)^3}$$

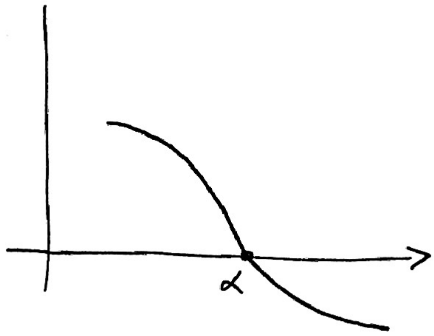
$$\Rightarrow \quad \phi'(\alpha) = 0$$

$$\phi''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$$

$\Rightarrow$ if $f''(\alpha) \neq 0$ then the Newton's method converges with ORDER 2.

$$\lim_{k \to \infty} \frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|^2} = \frac{1}{2} \frac{|f''(\alpha)|}{|f'(\alpha)|}$$

Question: Can the Newton's method converge with order 3? Yes, if $f'(\alpha) \neq 0$ and $f''(\alpha) = 0$ (simple zero at inflection point of $f(x)$):



How to stop fixed point iterations

- Control on the residual $\quad |f(x^{(k)})| < \varepsilon$

  could be too restrictive

  could be excessively optimistic

- Control on the increment

$$x^{(k+1)} - x^{(k)} = (x^{(k+1)} - \alpha) + (\alpha - x^{(k)})$$

$$= (\alpha - x^{(k)})\left(1 - \phi'(\eta^{(k)})\right)$$

In fact $\quad x^{(k+1)} - \alpha = \phi(x^{(k)}) - \phi(\alpha) = \phi'(\eta^{(k)})(x^{(k)} - \alpha)$

(mean value theorem)

This implies that:

$$\alpha - x^{(k)} = \frac{1}{1-\phi'(\eta^{(k)})}\left(x^{(k+1)} - x^{(k)}\right)$$

($\eta^{(k)}$ between α and $x^{(k)}$)

$$\simeq \frac{1}{1-\phi'(\alpha)}\left(x^{(k+1)} - x^{(k)}\right)$$

If $\phi'(\alpha) = 0$ (e.g. Newton's method) then the control on the increment gives us a very good estimate of the distance to the zero. On the other hand, if $\phi'(\alpha) \simeq 1$ then the test on the increment has issues.

~~Ex~~ <u>Example</u>: the chord method with a double root ($f'(\alpha) = 0$) has $\phi'(\alpha) \simeq 1$

Exercise: Show that the sequence

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} - \frac{f\left(x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}\right)}{f'(x^{(k)})}$$

converges with order 3 to any simple zero of $f(x)$ - provided we select $x^{(0)}$ close enough to such zero. (assume $f \in C^{(4)}$)