Lesson 7

Reading: Larson, Section 3.3, Properties of Determinants

Suggested exercises: Larson, Section 3.3: 1, 3, 9, 13, 17, 23, 25, 27, 33, 37, 53, 71

Submit: Lesson 7: Properties of determinants

In this section, we are given several properties of determinants. First, that the determinant of a product of square matrices is the product of their determinants: $\det(AB) = \det A \det B$, or $|AB| = |A||B|$. Do you recall in the section on LU-factorization the comment that it was useful in computing determinants of large matrices? This is due to this property of determinants. If A has an LU-factorization, then $A = LU$, which means that $|A| = |LU| = |L||U|$. Go back to that section and check the last example of an LU factorization that was computed. Note that L is a lower-triangular matrix with only ones on the main diagonal, which means that its determinant is 1. This must happen! Therefore, we have $|A| = |U|$, and since U is upper-triangular, its determinant is just the product of the entries on its main diagonal. For that example, then, the determinant of A is 13. It may not look like there is a huge advantage to computing a determinant like this, but that's because the matrix is so small. For a 600×600 matrix, the speed advantage of computing a determinant this way as opposed to cofactor expansion is ENORMOUS. (Yes, that needed to be capitalized; the speed advantage is that big.)

From the result that a scalar can be factored out of a row and become a multiplier of the determinant, we get the result on the determinant of a scalar multiple of a matrix. Given an $n \times n$ matrix A, the matrix cA is the matrix obtained by multiplying each entry of A by c. Factor out c one row at a time; each time you do it, you introduce a multiplicative factor of c. Since there are n rows, you get an overall multiplicative factor of c^n, so that

$$|cA| = c^n|A|,$$

for an $n \times n$ matrix A. Thus:

$$\begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}.$$

(Check the numbers!)

We see the important result that a matrix is invertible if, and only if, its determinant is nonzero. This is key: you test a matrix for invertibility by computing its determinant.

If A is invertible, then $AA^{-1} = I$, and the determinant of an identity matrix is 1, so we get $|AA^{-1}| = |I| = 1$, which gives us $|A||A^{-1}| = 1$, so

$$|A^{-1}| = \frac{1}{|A|}.$$

This is a nice result; you can get the determinant of an inverse matrix without knowing it, as long as you know the determinant of the original matrix.

The determinant of the transpose of a matrix is the same as the determinant of the original matrix: $|A^T| = |A|$. I think you can visualize a proof of this if you think about cofactor expansions. Rows of A correspond to columns of A^T, so a cofactor expansion about a row of A will give exactly the same numbers in the same places as a cofactor expansion about the corresponding column in A^T. Same numbers, same determinants.