Lesson 13

1) $\begin{bmatrix} -10 & 12 \\ -8 & 10 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda+10 & -12 \\ 8 & \lambda-10 \end{vmatrix} = \lambda^2 - 100 + 96 = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

$(\lambda I - A)x = 0$ for $\lambda = 2$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -10 & 12 \\ -8 & 10 \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ 8 & -8 \end{bmatrix}$

eigenvector for $\lambda = 2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $t \neq 0$

$(\lambda I - A)x = 0$ for $\lambda = -2$ $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} -10 & 12 \\ -8 & 10 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$

so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $x_2 = 2t$, $x_1 = 3t$

$\boxed{\text{eigenvalues} = \{-2, 2\} \quad \text{eigenvectors} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}}$

2) $|\lambda I - A| = 0$

$\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} -1 & 0 & 0 \\ 3 & -1 & -3 \\ -3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 0 & 0 \\ -3 & \lambda+1 & 3 \\ 3 & 0 & \lambda-2 \end{vmatrix}$

-1 is an eigenvalue

$-1(I) - A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 3 \\ 3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $x_3 = t$
$x_2 = s$
$x_1 = t$

$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{matrix} t \\ s \\ t \end{matrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\text{basis} = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)}$

3) eigenvalues along diagonal are $2, -1, 1$

for eigenvalue $\lambda = 2$; $2I - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $x_3 = t$
$x_2 = -\frac{1}{3}t$
$x_1 = -t$

eigenvector for $\lambda = 2 = t\begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ -\frac{1}{3} \\ -1 \end{bmatrix}$

for eigenvalue $\lambda = -1$ $-I - A = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -2 \\ -3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

eigenvector in form $t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Let $x_3 = t$
$x_2 = 0$
$x_1 = 2t$

$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -2 + -9 & -1.5 + -7.5 \\ 2 + -3 & 1.5 + 2.5 \end{bmatrix} = \begin{bmatrix} -11 & -9 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$-11 + -9$

eigenvalue $= 1$

eigenvector $= t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$I - A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = 0$

$x_1 = 0$

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

4) $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ ← Not diagonalizable because it's a 2×2 matrix w/ 1 eigenvector

~~if invertible → unique solution to $(\lambda I - A)x = 0$ → linearly independent vectors~~

~~If not invertible → non linearly independent vectors → not diagonalizable.~~

$$\cancel{\begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 4 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}}$$

eigenvalue $\lambda = 2$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

~~it is not invertible so A is not diagonalizable~~

5) $A = \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix}$

eigenvalues $= \{1, -2\}$

find characteristic polynomial:

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 4 & -3 \\ 6 & \lambda + 5 \end{vmatrix} = \lambda^2 + \lambda - 20 + 18 = \lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 1)(\lambda + 2)$$

eigenvalue $\lambda = 1$

$$I - A = \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix} \to \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$

$x_1 = -t$

eigenvector $= t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvalue $\lambda = -2$

$$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$x_2 = t$

$x_1 = 3t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -2+-9 & -1.5+-7.5 \\ 2+-3 & 1.5+2.5 \end{bmatrix} = \begin{bmatrix} -11 & -9 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$-11 + -9$

eigenvalue $\lambda = -2$ $\quad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 6 & 3 \end{bmatrix} \to \begin{bmatrix} -6 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$x_2 = 2t$
$x_1 = -t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

check $= \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -8+6 & -6+5 \\ 4-6 & 3-5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}$

diagonal

$$P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

6) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \qquad \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$

eigenvalues $= 1, -1$ $\qquad$ eigenvector $= t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvalue: $1I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} -t \\ t \end{matrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

basis $= \{(1,1), (-1,1)\}$

$P^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$