Module 3

SECTION 2.6: LIMITS AT INFINITY; HORIZONTAL ASYMPTOTES

The distinction between $y \to \infty$ (or $f(x) \to \infty$) and $x \to \infty$ is significant. The infinite limits considered earlier (***pages 89*** and ***112***) involved $y \to \infty$ and vertical asymptotes because a denominator was approaching zero. Now $x \to \infty$ and we consider what happens to *y*.

There are three possibilities.

1. *y* or *f(x)* approaches some constant. Then we have a horizontal asymptote. Three types are illustrated in ***Figure 2*** and are discussed further below.

2. *y* or *f(x)* gets larger and larger. $y \to \infty$ as $x \to \infty$ as shown in ***Figure 3***. Note we do *not* have a vertical asymptote here even though y approaches infinity.

3. y or $f(x)$ goes downward. $y \to -\infty$ as $x \to \infty$, also shown in ***Figure 3***.

The same three possibilities exist if $x \to -\infty$. These can be visualized by drawing similar graphs but looking at the left side as x gets smaller and smaller.

Study ***Example 2*** carefully. For $\lim_{x \to \infty} \frac{1}{x} = 0$ imagine a sequence like

$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$ The denominators are getting larger and larger, and the fractions are getting smaller and smaller. This concept will make the procedures for evaluating limits as $x \to \infty$ more reasonable

The rational function on ***page 126***, $f(x) = \frac{x^2 - 1}{x^2 + 1}$, has a horizontal asymptote on the right and on the left. Also, in both cases, the curve approaches the asymptote from below. This is easy enough to see in the graph, but how is this seen in the algebraic form?

1. Note that the numerator is smaller than the denominator for all values of x. The fraction will be less than 1 for all values of x.

2. Use a long-division process to divide the denominator into the numerator.

$$
\begin{array}{r}
1 \\
x^2+1 \,\overline{\big)\, x^2-1} \\
\underline{x^2+1} \\
-2
\end{array}
$$

Then $f(x) = \dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

Now note that as $x \to \infty$, $\dfrac{2}{x^2+1}$ approaches 0 and y approaches one from below, because we are subtracting a smaller and smaller number from one. The situation is the same when $x \to -\infty$ because x^2 is positive in both cases.

Next consider, $f(x) = \dfrac{x-1}{x+1}$. A long-division produces

$$f(x) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}.$$

As $x \to \infty$, $\dfrac{2}{x+1} \to 0$ and *f(x)* approaches 1 from below.

As $x \to -\infty$, $\dfrac{2}{x+1} \to 0$ again but this time through negative numbers.

So in $1 - \dfrac{2}{x+1}$ we are actually subtracting a negative number or adding a smaller and smaller number to 1. The result is that y approaches 1 from above.

A MAJOR ALGEBRAIC PROCEDURE FOR FINDING LIMITS AS $x \to \infty$

Most of the above comments relate to graphical interpretations of limits as $x \to \infty$. The major algebraic procedure uses the fact that $\dfrac{1}{x}, \dfrac{1}{x^2}, \dfrac{1}{x^3}$ all approach zero as *x* gets larger and larger. The algebraic trick is to change the original problem to get the above forms.

Examples:

1. $$\lim_{x\to\infty} \frac{2x - 4}{3x + 6} = \lim_{x\to\infty} \frac{2 - \frac{4}{x}}{3 + \frac{6}{x}} = \frac{2}{3}$$

   The key is to divide each term in the quotient by x. Then $\frac{4}{x}$ *and* $\frac{6}{x} \to 0$.

2. $$\lim_{x\to\infty} \frac{x^3 + 4x}{x^3 - 1} = \lim_{x\to\infty} \frac{1 + \frac{4}{x^2}}{1 - \frac{1}{x^3}} = \frac{1}{1} = 1$$

   Here each term in the quotient is divided by x^3. $\frac{4}{x^2}$ *and* $\frac{1}{x^3} \to 0$.

In the above two examples, the degree of the numerator and the denominator were the same. The next two examples show what happens when the degrees are not the same.

3. $$\lim_{x\to\infty} \frac{5x - 6}{2x^2 + 3x} = \lim_{x\to\infty} \frac{\frac{5}{x} - \frac{6}{x^2}}{2 + \frac{3}{x}} = \frac{0}{2} = 0$$

   Note that each term was divided by x^2, and each of the three fractions approaches 0. This limit can also be found by dividing each term by x, but the process is a bit more complex.

$$\lim_{x\to\infty} \frac{5x - 6}{2x^2 + 3x} = \lim_{x\to\infty} \frac{5 - \frac{6}{x}}{2x + 3} = \lim_{x\to\infty} \frac{5}{2x + 3} = 0$$

To make sense out of this, in the middle term, note $\frac{6}{x}$ approaches zero.

Then $\frac{5}{2x + 3}$ approaches zero because as the denominator gets larger and larger, the fraction gets smaller and smaller.

4. $$\lim_{x\to\infty} \frac{2x^3 + 5x - 4}{3x^2 - 7} = \lim_{x\to\infty} \frac{2x + \frac{5}{x} - \frac{4}{x^2}}{3 - \frac{7}{x^2}} = \infty.$$

First each term is divided by x^2. Then $\frac{5}{x}$, $\frac{4}{x^2}$ *and* $\frac{7}{x^2}$ each approaches zero. In effect, this leaves $\lim_{x\to\infty} \frac{2x}{3}$. The limit is ∞ because the fraction gets larger and larger as $x \to \infty$.

PATTERNS

The last four examples illustrate the following patterns for rational functions.

1. If the degree of the numerator and the denominator are the same, there will be a horizontal asymptote determined by the coefficients of the terms with the highest degrees.

2. If the degree of the numerator is smaller than the degree of the denominator, then the limit is zero and the x-axis (or $y = 0$) is an asymptote.

3. If the degree of the numerator is larger than the degree of the denominator, then the limit is ∞ . This means that the y-values increase as the x-values increase. The graph will have the behavior shown in ***Figure 3***.

The long-division process used above is a second technique that can be used on those rational functions where the degree of the numerator is larger than or equal to the degree of the denominator. In other words, it could be used in 1 and 3 in the last list but not in 2. So which procedure should you use? Think of long division as a special procedure that is used infrequently. In the above examples, the result after division gave some information that is hidden in the quotient form. Consider

$$y = \frac{x^3 + 3x}{x^2 + 1} = x + \frac{2x}{x^2 + 1}$$

The limit, $\lim_{x\to\infty} \dfrac{x^3 + 3x}{x^2 + 1} = \lim_{x\to\infty} \dfrac{x + \frac{3}{x}}{1 + \frac{1}{x^2}} = \infty$, indicates that y gets large as x gets large. If we use the form resulting from the long division, $y = x + \dfrac{2x}{x^2 + 1}$, and the fact that $\dfrac{2x}{x^2 + 1} \to 0$ as x increases, we conclude that the line, $y = x$, is what is called a slant asymptote. For large values of x, the curve will approach the line, $y = x$, from above. In this case, the long-division process produces additional information.

BEYOND RATIONAL FUNCTIONS

In ***Example 4***, the quotient form includes a square root and is not in the category of rational functions. However, we can still change the algebraic form by dividing by x if we note that $x^2 = x$. As $x \to \infty \sqrt{x}$ is a positive number matching the fact that the square root is positive. In the numerator, dividing by x^2 under the square root sign matches dividing the denominator by *x*. If we were taking the limit as $x \to -\infty$, then we would use the fact that $\sqrt{x^2} = -x$ because *x* is a negative number. Remember that $-x$ is positive when x is negative.

Example 5 contains a special limit. The answer is *not* zero because ($\infty - \infty$) equals zero. Both terms are getting larger and larger but not necessarily at the same rate. Consider the following limit.

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 3x} - x\right)$$

Multiply the numerator and the denominator by the conjugate radical.

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 3x} - x\right) \frac{\sqrt{x^2 + 3x} + x}{\sqrt{x^2 + 3x} + x} = \lim_{x \to \infty} \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x}$$

Simplifying the numerator produces $\lim_{x \to \infty} \dfrac{3x}{\sqrt{x^2 + 3x} + x}$

Then divide by *x*. (Under the square root sign, divide by x^2).

$$\lim_{x \to \infty} \frac{3}{\sqrt{1 + \dfrac{3}{x}} + 1} = \frac{3}{1+1} = \frac{3}{2}.$$

So in this case, $(\infty - \infty)$ equals $\dfrac{3}{2}$. The coefficient 3 under the radical sign has an effect on the answer.

SECTION 2.7: DERIVATIVES AND RATES OF CHANGE

One of the problems that students encounter when studying calculus is deciphering all of the symbols that appear. In the hopes that an overview might help, we return to the form, $\dfrac{y_2 - y_1}{x_2 - x_1}$, the slope of a line given two points. We list the variations of this form that occur in ***Section 2.7***.

Page141: $$\frac{f(x) - f(a)}{x - a}$$

Page 142: $\dfrac{f(a+h) - f(a)}{h}$

Page 145: $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Page 145: $\dfrac{\Delta y}{\Delta x} = \dfrac{\text{the change in } y}{\text{the change in } x}$

The common thread in all of these is best captured by $\dfrac{\Delta y}{\Delta x}$. This form is relating the *change* in one variable to the corresponding *change* in another variable. The form $\dfrac{\Delta y}{\Delta x}$ doesn't tell us how to find the changes, while the form $\dfrac{f(x) - f(a)}{x - a}$ does if we are given a function. Think of the last form as a set of algebraic instructions. However, when doing the algebra, don't lose sight of the meaning of the form.

POINTS ON A CURVE

Why do we have the variations in the form, $\dfrac{y_2 - y_1}{x_2 - x_1}$? A key is to understand the notation for a point on a curve. The first step is to use *(x,y)* to indicate the coordinates of a point. If $y = f(x) = x^2$, then *(x,y)* represents *any* point on the parabola. If we select $x = 2$, then substitution produces $y = f(2) = 2^2 = 4$. We then indicate that $(2,4)$ is a point on the curve. Generally, we don't write $(2,2^2)$ or $(2, f(2))$. The simplest notation is preferred. But when we select an *x*- value like *x* = *a*, things change. The corresponding *y*-value can only be represented by $f(a)$ or a^2, and the first form is the one that is commonly used. The point is represented by $(a, f(a))$.

So in ***Figure 3,*** note that the coordinates of P are $(a, f(a))$, and the coordinates of Q are $(a+h,\ f(a+h))$. The change in *x* from P to Q is represented by h, so note the a and $a + h$ coordinates on the x-axis. When $x = a + h$, $y = f(a + h)$, the same pattern as above and very important in understanding the notation for points.

GEOMETRIC MEANING

Next consider the graphical interpretations. ***Figure 8*** contains information that is essential to carry in your mind to give meaning to algebraic manipulations and the taking of limits. Note that Δy is the *change* in *y* as we move from P to Q on the curve. Δy is the length of a vertical line segment. In the above forms, the length is first represented by $f(x) - f(a)$, which is shown in ***Figure 1(a)***. Then, Δy is $f(a + h) - f(a)$ in ***Figure 3.*** Finally, it is $f(x\ ) - f(x\ )$, in ***Figure 8,*** although it is not explicitly shown. It is important to note that Δy ,

$f(x) - f(a)$, $f(a + h) - f(a)$, and $f(x_2) - f(x_1)$ all represent the length of the same line segment. The coordinates of the points involved are different, but the length isn't.

In the same three graphs, Δx is the *change* in x as we move from P to Q on the curve. Δx is the length of a horizontal line segment, represented by $x - a$, h, and $x_2 - x_1$, the last not shown explicitly. As above, Δx, $x - a$, h, and $x_2 - x_1$ are different representations for the length of the same line segment.

TAKING THE LIMIT

We have noted earlier that the process of taking the limit produces results that have different names. The *derivative* of a function at $x = a$ is the general name for $\lim_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$, ***page 144***. Chapter 3 is devoted to procedures for finding derivatives of functions. Now we consider different interpretations of the derivative. It can be the slope of the tangent line, velocity, a rate of change or marginal cost, depending on the situation. In the two final paragraphs of ***sec. 2.7***, each rate of change mentioned is an interpretation of a derivative.

It is also important to recognize that the limit, $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, is equivalent to $\lim_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ as a way of defining the derivative—different symbols but the same meaning. The same is true for $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. This last form will be used to stress the meaning of the derivative while the first two forms contain algebraic instructions.

SLOPE OF THE TANGENT LINE: ONE INTERPRETATION OF A DERIVATIVE

The idea of a tangent line first arises as a line that touches a curve in only one point. In most situations this will provide a good geometric picture. But this idea won't produce the tangent line to a line, and tangent lines do cross a curve in more than one point (see ***Figure 1***). A more precise description of a tangent line is given on ***page 141***. First, select a point P on the curve, say, $x = a$, $y = f(a)$. Then the *slope* of the tangent line is *defined* to be the limit, $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$. If you follow the limit process in ***Figure 1(b)***, the definition will seem reasonable. The *equation* of the tangent line is found by dropping a point and the slope into $y - y \pm m(x - x)$. 1

VELOCITY: ANOTHER INTERPRETATION OF A DERIVATIVE

In the second paragraph of the Velocities section (***Sec. 2.7)***, it is important to understand what the equation of motion, $s = f(t)$, means. Suppose you are driving a car on a straight road and you are going from town B to town C. In your mind, however, you are thinking how far you are from a major city A (also on the same road). Let s represent this distance from city A. As time t changes, the distance s also changes. This is what $s = f(t)$ means. For each value of time, t the distance from city A is s.

When we plot the points, (t,s), we do not get a line. The road was straight, but (t,s) are not the coordinates of the roadbed. The values of time t and distance s will be influenced by traffic conditions and the speed of the car.

Because s is a distance and $s = f(t)$, the symbols $f(t)$ also represent a distance. Then

$f(a + h) - f(a)$ is a change in distance, and the form $\dfrac{f(a+h) - f(a)}{h}$ is

$\dfrac{\text{a change in distance}}{\text{a change in time}}$. Remember from $t = a$ to $t = a + h$ is a time interval and hence a change in time. The form $\dfrac{f(a+h) - f(a)}{h}$ is the *average velocity* and what

the text calls $\dfrac{\text{displacement}}{\text{time}}$ is $\dfrac{\text{a change in distance}}{\text{a change in time}}$ or $\dfrac{\Delta s}{\Delta t}$. The *instantaneous velocity,*

then is the limit in (3). As h *or* Δt approaches 0, the time intervals approach zero and the average velocities approach what seems reasonable to call the instantaneous velocity.

ALGEBRAIC CHANGES WHEN FINDING LIMITS

We now comment on some of the algebraic changes that are required in finding limits.

Example 1:

After factoring and removing the common factor, $x - 1$, the limit can be found by direct substitution.

Example 2:

The numerator, $\frac{3}{3+h} - 1$ is simplified by using the common denominator. The result, $\frac{\frac{-h}{3+h}}{h}$, can be simplified by multiplying by $\frac{\frac{1}{h}}{\frac{1}{h}}$.

$$\frac{\frac{-h}{3+h}}{h}\,\frac{\frac{1}{h}}{\frac{1}{h}} = \frac{-1}{3+h}.$$

The limit can then be found by direct substitution. Note that we changed the denominator, h, into a 1 by multiplying by its reciprocal, $\frac{1}{h}$. Then $\frac{NUM}{1} = NUM$. This is equivalent to writing $\frac{f(3+h) - f(3)}{h}$ in the equivalent form, $\frac{1}{h}\left[f(3+h) - f(3)\right]$, which will simplify the algebraic manipulations.

MORE ON DERIVATIVES

We have shown several interpretations of a derivative, but how does one find the derivative? There are three answers.

1. Use ***Definitions 4*** and ***5, page 144***, as is the case in examples in ***Section 2.7***. If you feel disheartened at this prospect, read on.

2. In ***Section 3.1***, instead of doing several particular problems, we concentrate on the general case and develop differentiation formulas.

3. Some problems don't fit either of the above categories and require special analysis. ***Example 7, Sec. 2.7***, illustrates the type. The starting point is a set of data instead of a function or a formula. With more experience, you will see the given data as a set of points, and your analysis can be guided by a geometric interpretation.

With this in mind, don't short-change the definition. Using differentiation formulas will sometimes involve complex algebraic manipulations that hide the basic meaning of a derivative, essentially, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, but the variables may change. When we come to applications, the basic meaning of a derivative provides guidance, not the algebraic manipulations. Now the task is not to get lost in the algebraic changes required to be able to take a limit when using the definition.

Example 4:

The common a in $f(a+h)$ and $f(a)$ leads to like terms that can be combined. Each of the remaining terms contains h as a factor. This allows for the cancellation of the h in the denominator. Before the cancellation, the limit is $\frac{0}{0}$ or undefined. After the cancellation, direct substitution can be used.

The quadratic function in this example is in the category of polynomial functions. In the next section, the above process is repeated in a more general setting to develop one of the differentiation formulas.

SECTION 2.8: THE DERIVATIVE AS A FUNCTION

Definition 2, page 152, presents a subtle but important change. If this definition had been used in ***Example 4*** in the last section, the result would have been $f'(x) = 2x - 8$ instead of $f'(a) = 2a - 8$. In the second form a is a constant while in the first x is a variable. This means that $2x - 8$ is a variable (no problem here), but $2a - 8$ is to be treated as a constant. We haven't indicated *which* constant a represents, but once selected, it doesn't change. Having said this, you may still not be convinced, but store away the idea, and later it will appear in a clearer context.

Example 3:

The major algebraic change is to rationalize the numerator by multiplying by the conjugate. After simplification, the $h's$ can be canceled, and direct substitution leads to the answer.

Example 4:

Here the key is the combining of the two fractions by using a common denominator. Granted the multiplications are a bit complex, but once accomplished, simplification leads to a single term. And again, $h's$ can be canceled, which changes the limit from $\frac{0}{0}$ to one where direct substitution can be used.

The goal here is not to master the technique of using the definition to find a derivative, but to develop an algebraic sense of the overall problem instead of focusing on details. This general view will help in many types of problems.

THE CALCULUS REFORM MOVEMENT

In recent years, there has been much discussion on the traditional way calculus has been taught. New ideas have been presented as an attempt to make calculus more student-friendly. The author addresses these concerns in the preface on ***page xi***. One of the ideas in the reform movement is a greater emphasis on reading graphs to get information. This approach appears twice in ***Sections 2.7*** and ***2.8***.

First, to the right of ***Figure 9, Sec. 2.7***, is a discussion of how to read the graph and to get information related to the idea of a rate of change. Where the graph is steeper (at P), the rate of change is larger. The second case appears in ***Figure 2, Sec. 2.8***. This type of problem would not be found in a calculus book twenty years ago. The task is to “read” the top graph and get information about the derivative (slope in this case) and then graph this information. Points A, B, and C are significant because the slope of the tangent line is zero at these three points. Recall that any horizontal line has a slope of zero. This leads to A′, B′, and C′ in the bottom graph.

The second significant step is to distinguish between positive and negative slopes for the tangent line. Between B and C in the top graph, the slope is negative. This matches the negative *y′-values* between B′ and C′ on the bottom graph. The slope of the tangent line is also negative in the top graph to the left of point A. Note the corresponding negative *y′-values* in the bottom graph.

If you found the interpretation of the graphs in ***Figure 2*** challenging, don’t be too concerned. Much new information is used. Just visualizing the tangent line is a new activity, which will become second nature with more experience. And there is also the notation.

OTHER NOTATIONS

Near the top of this subsection, one line contains no less than *seven* notations for the derivative of $y = f(x)$. At this point, there is no way you can understand the significance of these notations. The author has placed these in one place for convenient reference. The two "primes" y' and $f'(x)$, have been used in the preceding pages. The next one $\frac{dy}{dx}$, has significance when read “the derivative of y with respect to x.” This will be *very* important later.

To illustrate, consider the following: If $y = f(x)$, then y', $f'(x)$, and $\frac{dy}{dx}$ all clearly represent the derivative, and (x, y) represents a point.

Next suppose (x, y) represents a point as above, but the values of x and y depend on time t. This can be expressed by $x = f(t)$ and $y = g(t)$. Then $\frac{dy}{dt}$ indicates a derivative of y with respect to t while y' doesn't display the same information. In a similar manner, x' doesn't convey the same information as $\frac{dx}{dt}$.

The notations $\frac{df}{dx}$, $\frac{d}{dx} f(x)$, and $D_x f(x)$ also indicate derivatives with respect to x. The middle notation allows us to rewrite the result in ***Example 3***, as $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Sometimes this is a more convenient form. The same is true for the other notations.

IMPORTANT WORDS

The word, *differentiable* is defined *in* ***Definition 3***. Basically, it just means that the derivative exists as a finite number, not ∞. In ***Example 5***, note two things. First, the definition of $|x|$ has to be used to be able to investigate the limit. Second, the limit does not exist because $-1 \neq 1$—the left-hand limit does not equal the right-hand limit. ***Figure 5(a)*** shows the "corner" at $x = 0$ where one can't draw the usual tangent line. The function $f(x) = |x|$ is not differentiable at $x = 0$, but it is continuous there. There are no breaks in the graph. The derivative whose graph is shown in ***Figure 5(b)*** is not continuous at $x = 0$.

Theorem 4 shows a connection between continuity and differentiability. (The proof involves what may appear to be a trick when $f(x) - f(a)$ is expressed as a special product that just happens to lead to the desired result. It may have taken some time before someone thought of doing this, but once done it provides a concise proof. Study it if you feel so inclined, but it will not be emphasized.) Basically the theorem says that if you can draw a tangent line at $x = a$ there won't be a break in the curve there. The theorem does *not* say that if a function is continuous, it is differentiable. ***Figure 5(a)*** shows a function that is continuous but *not* differentiable at $x = 0$.

The discussion on ***pages 157-158*** contains a good summary of how a function can fail to be differentiable. The three graphs in ***Figure 7*** illustrate the different possibilities. All of these will be more meaningful later when you have more experience and a better context.