Module 9

SECTION 4.7: OPTIMIZATION PROBLEMS

If you have been wondering where you might use calculus other than in graphing functions, the answer is near at hand. We now look at some practical applications of calculus, but first a word of warning. Most students find the translation of words into functions and equations, along with the required algebraic manipulations one of the greatest challenges in Calculus I. We don't wish to frighten you, but if you have some difficulty, don't despair. You are not alone. Perseverance will help you to see patterns that are not apparent at first glance.

The ***Steps in Solving Optimization Problems*** are an excellent summary of the steps involved in solving applied maximum and minimum word problems. Expect that the first time you read this summary some steps will be vague. Steps 4 and 5 are the most significant but may not make sense until you match them with particular examples.

One trick to learn in solving word problems is to break the overall problem into parts, select key words like area or volume, and translate a part into a symbol form. Frequently, an equation or a function comes from a single word with a drawing providing an intermediate step. In the comments that follow step 4 and step 5, refer to the previous steps. Once you see these two steps clearly these problems will be much easier.

EXAMPLE 1

"Area" is a key word leading to the equation $A = xy$

Step 4 also matches the equation $A = xy$

Step 5: In $A = xy$, A is the dependent variable and x and y are the independent variables. We want only one independent variable, or we want A to be a function of only one variable. This is accomplished by replacing y with ??

The equation $2x + y = 2400$ is used to find ?? which is the replacement for y. Solving for y we have:

$$y = 2400 - 2x$$

and $A = xy$

becomes $A = x(2400 - 2x) = 2400x - 2x^2$

Finally, find the derivative, set it equal to zero, and solve for x.

EXAMPLE 2

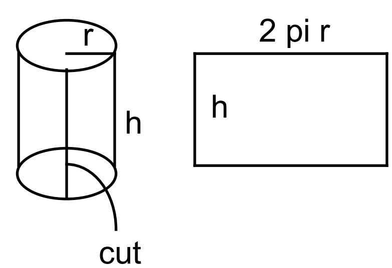
"Can" is a key word matching the drawing in ***Figure 3***. Inside the front cover, in the "Geometry" column, there are formulas related to spheres, cylinders, and cones. "1 L of oil" (one liter of oil) relates to the volume of the can or cylinder. The formula for the volume is:

$$V = \pi r^2 h$$

Replace V with one liter in the form 1000 cubic centimeters.

$$1000 = \pi r^2 h$$

Next, the "cost of the metal" must be equated with the total surface area. The top is a circle as is the bottom, so their combined area is $2\pi r^2$. We also need the area of the "side" of the can. Imagine there is no top or bottom and we make a cut in the can as shown below. It will the flatten out to the rectangle shown:



2 pi r in the diagram is $2\pi r$ which is the circumference of a circle. The top and bottom circles unravel into line segments whose length is the circumference of the circle. The lateral (side) area is the area of the rectangle, $2\pi r h$. The total area is:

$$A = 2\pi r^2 + 2\pi r h$$

All this to execute step 4. Note that we have two independent variables, r and h. Very important. How do we get rid of one variable? Go back to:

$$1000 = \pi r^2 h$$

and solve for h.

$$h = \frac{1000}{\pi r^2}$$

Use this to remove h from $A = 2\pi r^2 + 2\pi r h$.

$$A = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

Simplify: $$A = 2\pi r^2 + \frac{2000}{r}$$

We have completed step 5. A is now a function of one variable, r. Finally, find the derivative, set it equal to zero, and solve for r.

EXAMPLE 3

Here the word "closest" refers to a distance. Then remember to use the distance formula between two points. (x, y) is *any* point on the parabola. The translation for this type of problem is easier than most, once the idea of distance is introduced.

On ***page 333***:

$$d^2 = f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

The text says "Instead of minimizing d, we minimize its square". To justify this step consider:

$$d^2 = f(y)$$

Then $$2dd' = f'(y)$$

and $$d' = \frac{f'(y)}{2d}$$

The minimum of d occurs when $d' = 0$

or when $$\frac{f'(y)}{2d} = 0$$

Multiply both sides by $2d$. The result is $f'(y) = 0$ which justifies the claim in the text.

EXAMPLE 4

The drawing is crucial here. You perhaps possess exceptional mathematical abilities if you can think this problem through without referring to the drawing or a similar one in your head. A key here is to remember that distance = rate (times) time:

$$D = RT$$

If you travel for 3 hours at 50 mile per hour the distance traveled is 50 X 3 = 150 miles.

Solve for time T in $D = RT$:

$$T = \frac{D}{R}$$

Time T equals distance D divided by rate R.

The words "reach B" "as soon as possible" must be translated to *minimize time* and time is $\frac{\text{distance}}{\text{rate}}$.

The rates are given: row $6\ km/h$ and run $8\ km/h$

So look for the corresponding distances which are:

$$\sqrt{x^2 + 9} \qquad \text{and} \qquad 8 - x$$

The total time is: $\dfrac{\sqrt{x^2 + 9}}{6} + \dfrac{8 - x}{8}$

Find the derivative, set it equal to zero, and solve for *x*.

In the discussion of the above examples, comments were limited to the translation to a symbol form. This translation can present special difficulties that we have hopefully addressed. The process of solving for *x* after setting the derivative equal to zero can also present challenges. Don't despair if this happens. The difficulties are dealt with elsewhere.

APPLICATIONS TO BUSINESS AND ECONOMICS

At first glance, you might feel that this section will illustrate a more practical use of calculus. Eventually this will be the case but first it is important to become familiar with terms like marginal cost and marginal profit. The words, marginal cost, were first introduced in ***Section 3.7***. We start with a cost function *C(x)* which "is the total cost that a company incurs in producing *x* units of a certain commodity." The marginal cost is $C'(x)$, the derivative of $C(x)$. In the form, $\frac{dC}{dx}$, think of dx as one. Then dC is an approximation to the change in cost for one additional unit produced. This is one way of thinking when seeing the words, marginal cost. It is the cost of producing one more unit. In a similar manner, marginal profit is the change in profit when one more unit is produced. Also marginal revenue is the change in revenue when one more unit is produced. Attaching these words to the appropriate derivatives will clarify some ideas in this section.

SECTION 4.8: NEWTON'S METHOD

Newton's method provides a formula for approximating roots of equations.

1. The first step is to make a guess of a root. For example, a graph may show that the curve crosses the x-axis between one and two, slightly closer to one than to two. So the guess might be 1.4 or $x_1 = 1.4$.

2. Then find the corresponding y_1 value, $y_1 = f(x_1)$, which leads to the point $\left(x_1, f(x_{1)}\right)$ on the curve.

3. Find the equation of the tangent line at the point $\left(x_1, f(x_{1)}\right)$.

4. This tangent line passes through a point on the *x*-axis. Let x_2 be the *x* value of this point.

5. This number, x_2, becomes the second guess, and the entire process is repeated.

To translate the above words into an algebraic form we need the equation of the tangent line at $\left(x_1, f(x_{1)}\right)$, slope is $f'(x_1)$:

$$y - y_1 = m(x - x_1)$$

becomes

$$y - f(x_1) = f'(x_1)(x - x_1).$$

The tangent line crosses the *x*-axis when $y = 0$, $x = x_2$. Put this into the equation and we get:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1).$$

Solve for x_2 which is the second guess:

$$-\frac{f(x_1)}{f'(x_1)} = (x_2 - x_1)$$

and

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x_2$$

or

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} .$$

Repeating the process means that the third guess, x_3, is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} .$$

The general form is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

Figure 3, gives a graphical interpretation of how we hope the method works with x_1, x_2, x_3, x_4 approaching the root labeled *r*. However, ***Figure 4*** shows that the second guess, x_2, may be worse than the first. Note in ***Figure 4***, however, if x_1, the first guess, was closer to the root *r,* then the results would lead to a good approximation.

SECTION 4.9: ANTIDERIVATIVES

Antiderivative? Sounds like a subversive term rather than a reversal of the process of finding a derivative. In the rest of this course you will spend much energy in finding these elusive objects. Given $f'(x) = x^3$ the task is to find $f(x)$, the function whose derivative is x^3. $f(x)$ is called the antiderivative of x^3. In this case, one antiderivative is $\frac{x^4}{4}$, because $\left(\frac{x^4}{4}\right)' = \frac{4x^3}{4} = x^3$. So $\frac{x^4}{4}$ is a an antiderivative of x^3.

TERMINOLOGY

It becomes a bit more confusing when we note that:

$$\left(\frac{x^4}{4}+1\right)' = x^3 \text{ and also } \left(\frac{x^4}{4}-7\right)' = x^3$$

because the derivative of any constant is zero. Extending this one step further

$$\left(\frac{x^4}{4}+C\right)' = x^3 \text{ where } C \text{ is any constant.}$$

$\frac{x^4}{4} + C$ is called the *general* antiderivative of x^3 and

$\frac{x^4}{4}$ is a *particular* antiderivative of x^3 with $C = 0$

Then $\frac{x^4}{4} - 7$ is also a *particular* antiderivative of x^3 with $C = -7$.

The constant C will have significant interpretations in applications which we cover later.

FORMULAS FOR FINDING ANTIDERIVATIVES

The first step in developing an organized approach to finding antiderivatives is shown in the ***Table of Antidifferentiation Formulas (2)***. In the last eight entries, the column on the left contains the basic derivatives of the functions listed in the right column. Suppose you need the antiderivative of $\frac{1}{1+x^2}$. Simple. Just remember that $\frac{1}{1+x^2}$ is the derivative of $\tan^{-1} x$ and you have the answer. This may seem like a game, but keep in mind this is just the beginning. Greater complexity will unfold in the next chapter and then you will have your hands full. Now concentrate on basics.

The first three entries in the table are more general. The paragraph above the table describes the first two. Then the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.

This is a general formula that you will use many times. It is easy to prove:

$$\left(\frac{x^{n+1}}{n+1}\right)' = \frac{(n+1)x^n}{n+1} = x^n$$

To find the antiderivative of x^7 add one to the exponent and divide by the new exponent to get, $\frac{x^8}{8}$. The general antiderivative of x^7 is $\frac{x^8}{8} + C$.

The form $\frac{x^{n+1}}{n+1}$ is undefined if $n = -1$. So the antiderivative of x^{-1} will not be $\frac{x^{-1+1}}{-1+1}$. It just happens that the natural log function fills the gap. The antiderivative of $x^{-1} = \frac{1}{x}$ is $\ln x$.

CHECKING YOUR ANSWER

Initially you may encounter some difficulty in finding an antiderivative by adding one to the exponent instead of subtracting one.

The derivative of $\sin x$ is $\cos x$ but the antiderivative of $\sin x$ is $-\cos x$.
$(-\cos x)' = -(-\sin x) = \sin x$. This check verifies the answer.

You can save yourself some grief by checking all answers that are supposed to be antiderivatives. This will also reinforce the new process of reversing old patterns. For example, suppose you write $\sec^2 x$ as the antiderivative of $\tan x$. Then a check, using the Power Rule, would produce:

$$\left(\sec^2 x\right)' = 2\left(\sec x\right)^1\left(\sec x \tan x\right) \neq \tan x$$

and you would realize your mistake. We can supply the answer which is $-\ln\cos x$ because:

$$\left(-\ln\cos x\right)' = -\left(\frac{1}{\cos x}\right)\left(-\sin x\right) = \tan x$$

but keep in mind there are procedures covered in ***Chapter 5*** that produce this answer. For now just check your answers to avoid foolish assertions and also to better understand a new process.

DIFFERENTIAL EQUATIONS

Note the definition of a differential equation on ***page 353***. A *differential equation* is an equation that contains derivatives of a function. Then $y' = x^2$ is a simple differential equation and its *solution* is the function $y = \frac{x^3}{3} + C$. The graphs in ***Figure 1*** show six solutions corresponding to six different values for C.

INITIAL CONDITIONS

In ***Figure 1***, suppose we need the curve passing through the point (3, 11). Starting with

$$y = \frac{1}{3}x^3 + C$$

drop in $x = 3$ and $y = 11$ to get $$11 = \frac{1}{3}\cdot 3^3 + C$$

$$11 = 9 + C$$

and $$C = 2$$

So the graph of the equation $y = \frac{1}{3}\cdot 3^3 + 2$ will be a curve passing through (3, 11).

The point (3, 11) can also be expressed as *f(3)* = 11. This pair of values is called an "initial condition." It is an additional requirement that allows us to find the constant that appears when finding an antiderivative.

In ***Example 3***, the initial condition is given in the form, *f(0) = -2*. The graph of the solution is to pass through the point (0, -2). The significant idea here is that the initial condition determines the value of C in the general solution.

In ***Example 4***, the two initial conditions can be thought of as two points, namely, (0,4) and (1,1). Two points are needed because we start with the second derivative. Finding one antiderivative produces the first derivative and one constant *C*. Finding a second antiderivative produces the solution function along with another constant *D*. Then putting $x = 0$, $y = 4$ into $f(x)$ determines the value of *D*, $D = 4$. Repeat the process for $x = 1$, $y = 1$ and $C = -3$.

This procedure is shown again in ***Example 6***. Here the initial conditions are given as $v(0) = -6\ {}^{cm}/_{sec}$ and $s(0) = 9\ cm$. To make sense of this, remember that v represents velocity and s represents distance. Think of time *t* equals zero as the starting time. We begin timing the movement at $t = 0$. At this starting time, the velocity is $-6\ {}^{cm}\!/_{\sec}$ and the distance is $9\ cm$ from some reference point. $v(0) = -6\ {}^{cm}\!/_{\sec}$ is used to determine the value of C, and $s(0) = 9\ cm$ is used to determine the value of D.

SUMMARY

Each time an antiderivative is found, a constant C (or D or whatever) appears. To determine the constant, some extra condition (called an initial condition) is given (directly or indirectly). This changes a general antiderivative into a particular antiderivative. Graphically, you can think of this as selecting one function in ***Figure 1***, by specifying a point the curve must pass through.

EXAMPLE 7

In the first equation, $$a(t) = \frac{dv}{dt} = -32$$

the number -32 indicates the rate of change of velocity v.

Recall that $\dfrac{dv}{dt} \approx \dfrac{\Delta v}{\Delta t} = \dfrac{\text{change in velocity}}{\text{change in time}}$, which is the *definition* of the word

acceleration. So $-32\ {}^{ft}\!/_{\sec^2}$ is actually $-32\ {}^{{}^{ft}\!/_{\sec}}\!/_{\sec}$. Physicists make measurements and tell us that the effect of the force of gravity on a free falling object is a constant change in velocity of $-32\ {}^{{}^{ft}\!/_{\sec}}\!/_{\sec}$. Velocity decreases $32\ {}^{ft}\!/_{\sec}$ in one second.

The next second the same decrease occurs and this continues. So the starting point is the differential equation:

$$a(t) = \frac{dv}{dt} = -32$$

Find an antiderivative $\quad v(t) = -32t + C$

Look at the first line "A ball is thrown upward with a speed of $48\ {}^{ft}\!/_{\sec}$ from the edge of a cliff 432 ft above the ground." Nothing is said about time and the easiest approach is to start measuring time $(t = 0)$ when $v = 48$ and $s = 432$.

Then $\quad v(t) = -32t + C$

becomes $\quad v(0) = 48 = -32 \cdot 0 + C$

Result: $\quad C = 48$ and $v(t) = -32t + 48$

Because $\quad v = \frac{ds}{dt} \qquad \frac{ds}{dt} = -32t + 48$

and we find another antiderivative. $\quad s(t) = -32\frac{t^2}{2} + 48t + D$

Now remember $s = 432$ when $t = 0$. $\quad 432 = -16 \cdot 0^2 + 48 \cdot 0 + D$

Result: $D = 432$ and $\quad s(t) = -16t^2 + 48t + 432$

Conclusion: The function $s(t) = -16t^2 + 48t + 432$ is the solution of the differential equation $a(t) = \frac{dv}{dt} = -32$ subject to the initial conditions, $v = 48$, $s = 432$, when $t = 0$.

LAST COMMENT

The functions $\quad s(t) = -16t^2 + 48t + 432$

and $\quad v(t) = -32t + 48$

can be used in two ways.

1. Put a time t into either function to determine the corresponding distance or velocity.

2. Put a distance s into $s(t) = -16t^2 + 48t + 432$ or a velocity v into $v(t) = -32t + 48$ and solve the resulting equation to determine the corresponding time t.

At the maximum height, velocity (or slope) is zero.

When the ball hits the ground, distance is zero.

Use the pattern in step 2 to find the corresponding time.