Lesson 2

Reading: Larson, Section 1.2, Gaussian Elimination and Gauss-Jordan Elimination

Suggested exercises: Larson, Section 1.2: 3, 7, 9, 11, 17, 21, 25, 31, 35, 41

Submit: Lesson 2: Gaussian elimination

Augmented matrices

This section introduces the three types of elementary row operation, which are used to solve linear systems by elimination. These operate on the *augmented matrix* representing the system: a matrix containing the essential information of the linear system, namely the coefficients of the variables, and the numbers on the right-hand sides of the linear equations. For example, the augmented matrix of the linear system

$$\begin{aligned} x + 2y &= 3 \\ 4x - 5y &= 6 \end{aligned}$$

is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}.$$

As long as make sure to line up the variables correctly, and understand that this is an augmented matrix (that is, that the rightmost column represents the right-hand sides of the equations) then this matrix is all we need to represent the system.

We can now take the operations that were performed on equations in linear systems to eliminate terms, and replace them with elementary row operations, which achieve the same things, but with rows in a matrix rather than equations. Eliminating a term in an equation now corresponds to producing a zero in the appropriate entry of the augmented matrix. The stair-step pattern we sought in solving linear systems in the last section is replaced with a stair-step pattern of leading 1's in the rows of the augmented matrix, for example

$$\begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

which represents the system

$$\begin{aligned} x \quad\;\; - 2z &= 2 \\ y - z &= 3 \\ z &= -1, \end{aligned}$$

or perhaps

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{bmatrix},$$

representing

$$\begin{aligned} x_1 - 2x_2 + x_3 \qquad + 2x_5 &= -3 \\ x_4 - x_5 &= 2. \end{aligned}$$

Elementary row operations

The elementary row operations have the important property that, while they may change the appearance of the augmented matrix of the linear system, they do not change the solution set, because each operation maintains the balance in the affected equation. The goal of applying these operations is to put the system into row-echelon form, and possibly the more restrictive *reduced row-echelon form.*

The process of putting the system into row-echelon form and solving for the values of the variables from the bottom up is Gaussian elimination with back-substitution, which has already been discussed. This is generally sufficient for a small system with a unique solution. (You can recognize that the system has a unique solution if, for each row in the augmented matrix, its row-echelon form contains a leading 1 in a column corresponding to one of the variables.)

You can go further and put the system into reduced row-echelon form, which has the additional requirement that, in each column containing a leading 1, the leading 1 is the only nonzero number appearing in that column. Adding this one requirement has an important effect: reduced row-echelon form is unique for a given matrix! It represents, in the context of solving linear systems, the simplest form that the matrix can be put into by elementary row operations—simplest, because it is the best form for reading out the solutions of the linear system.

For example, the linear system

$$\begin{aligned} x + 2y &= 5 \\ 2x - y &= 5 \end{aligned}$$

has augmented matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & -1 & 5 \end{bmatrix},$$

which has reduced row-echelon form (check!)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix},$$

representing the equivalent linear system

$$\begin{aligned} x \qquad &= 3 \\ y &= 1. \end{aligned}$$

You can't get any simpler than that!

Here's an example with infinitely many solutions. The linear system

$$\begin{aligned} x + y - z &= 3 \\ 2x + y - 3z &= 4 \end{aligned}$$

has augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & -3 & 4 \end{bmatrix},$$

which has reduced row-echelon form (again, check!)

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

representing the equivalent linear system

$$\begin{aligned} x \qquad - 2z &= 1 \\ y + z &= 2. \end{aligned}$$

We see that z is a free variable, and introducing the parameter t by setting $z = t$ allows us to write the solutions as

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 - t \\ z &= t. \end{aligned}$$

The point of reducing all the way to reduced row-echelon form is that we completely separated the dependent variables x and y. If we had been satisfied with row-echelon form, then there would have been a y term in the equation for x, and getting the solutions would have required the additional step of computing y first, and then plugging this value into the x equation. With reduced row-echelon form, we are guaranteed that x and y each appear only once, which allows us to compute x and y independently.

This process of reducing the augmented matrix all the way to reduced row-echelon form is called *Gauss-Jordan elimination*. It is not really necessary for a small system with a unique solution, but if there are infinitely many solutions, as in the preceding example, then Gauss-Jordan elimination is the way to go: it produces a clean representation of the solutions.