Module 11

SECTION 17.1: SECOND-ORDER LINEAR EQUATIONS

Some problems in physics lead to first-order differential equations, while others require the inclusion of a second derivative. The general form for a second-order *linear* (*y* is to the first power) equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

However, in this section and the next, we consider only those equations where *P(x), Q(x),* and *R(x)* are constants. None of these three coefficients will contain a single *x*. Briefly look at the problems on ***pages 1160*** and ***1167***, and note that all coefficients are constants. In this section the equation has an even simpler form because $G(x) = 0$. We will consider the equation

$$a\,y'' + b\,y' + c\,y = 0$$

where *a, b,* and *c* are constants. This is called a *homogeneous* equation, while in the next section, the form

$$a\,y'' + b\,y' + c\,y = G(x)$$

is called a *nonhomogeneous* equation.

LINEAR COMBINATIONS

This is an important concept. If y_1 and y_2 are solutions, then the sum, $y_1 + y_2$, is also a solution. Also, $3y_1 + 5y_2$ is a solution. In fact, we can replace 3 and 5 with any constant and the resulting sum is a solution. In the general form, if c_1 *and* c_2 are any real numbers, then

$$c_1 y_1 + c_2 y_2$$

is a solution. This form is called a *linear combination* of y_1 and y_2. This form is important to keep in mind during what follows.

The proof on ***page 1154*** is easy to follow and indicates why the sum is a solution.

LINEAR INDEPENDENCE

Another important concept. Above we indicated that if y_1 is a solution, then $c_1 y_1$, where c_1 is any real number, is also a solution. Now we introduce terminology that indicates y_1

and $c_1 y_1$ are related to each other by saying that they are *linearly dependent.* The functions $4x^3$, and x^3 are linearly dependent, because one is just 4 times the other. However, if you start with x^3, you can't get to e^x by multiplying by constants. These two functions are said to be *linearly independent*.

THE GENERAL SOLUTION

Theorem **4**, ***page 1155***, says that not only is

$$c_1 y_1 + c_2 y_2$$

a solution, but that *all* solutions will have this form. Shortly we will show that e^{2x} and e^{-3x} are solutions of the equation,

$$y'' + y' - 6y = 0$$

Because these are linearly independent solutions, *every* solution will have the form

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

This Theorem is proved in more advanced texts, but a check gives some insight.

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

$$y'' = 4c_1 e^{2x} + 9c_2 e^{-3x}$$

Then

$$y'' + y' - 6y =$$

$$4c_1e^{2x} + 9c_2e^{-3x} + 2c_1e^{2x} - 3c_2e^{-3x} - 6c_1e^{2x} - 6c_2e^{-3x}$$

Combine like terms and the result is zero.

We also show that e^{3x} is *not* a solution because

$$y'' + y' - 6y =$$

$$9e^{3x} + 3e^{3x} - 6e^{3x} \neq 0$$

Now the coefficients do not produce 0.

THE METHOD

After looking at the combinations of derivatives in the last paragraph, it isn't too much of a jump to expect that $y = e^{rx}$ for some r is a solution of the equation

$$a y'' + b y' + c y = 0$$

Because $$y' = r e^{rx}$$

and $$y'' = r^2 e^{rx}$$

the substitution for y, y', and y'' produces

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

The common factor, e^{rx}, never equals zero. Hence, we can divide both sides by e^{rx}. The resulting quadratic equation (called the auxiliary equation)

$$ar^2 + br + c = 0$$

determines the r values that make $y = e^{rx}$ a solution to the equation. By the quadratic formula, the two roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

When possible, factoring is always the best way to solve a quadratic equation, but we can classify the roots by looking at the discriminant, $b^2 - 4ac$. It can be positive, zero, or negative. These three situations are covered as ***case I***, ***case II***, and ***case III*** in the text.

CASE I

If $b^2 - 4ac$ is positive, then there are two *distinct* real roots. We also note that $b^2 - 4ac$ may be a perfect square; 1, 4, 9, 16, 25, 36, etc. Then the roots will not contain a square root, or the roots will be rational numbers. This also means that the quadratic equation is factorable.

The two *distinct* roots r_1 *and* r_2 lead to the two particular solutions, $e^{r_1 x}$ and $e^{r_2 x}$.

The general solution is $$y = c_1e^{r_1 x} + c_2e^{r_2 x}.$$

Example. Solve the equation $y'' - 2y' - 15y = 0$

The auxiliary equation is

$$r^2 - 2r - 15 = 0$$
$$(r + 3)(r - 5) = 0$$

The two roots, -3 and 5, lead to the general solution

$$y = c_1 e^{-3x} + c_2 e^{5x}$$

CASE II

If $b^2 - 4ac$ is zero, then the roots are

$$r_1 = \frac{-b + \sqrt{0}}{2a} \quad \text{and} \quad r_2 = \frac{-b + \sqrt{0}}{2a}$$

Now we have the single root, $\frac{-b}{2a}$, repeated twice. Then $e^{r_1 x}$ and $e^{r_2 x}$ are the same, and we have only one solution. But, lo and behold, someone discovered that in this situation, $x e^{r_1 x}$ will also be a solution. In the check at the bottom of ***page 1156***, note how the factor $(2ar + b)$ appears. This factor is zero because $\frac{-b}{2a}$ is a root of the auxiliary equation: $r = \frac{-b}{2a}$ leads to $2ar + b = 0$.

So when $b^2 - 4ac$ is zero, both e^{rx} and $x e^{rx}$ are particular solutions. The general solution is $y = c_1 e^{rx} + c_2 x e^{rx}$.

The auxiliary equation will be factorable in this case. The two factors will be the same.

Example. Solve the equation $y'' + 6y' + 9y = 0$

The auxiliary equation is

$$r^2 + 6r + 9 = 0$$
$$(r + 3)(r + 3) = 0$$

The general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

CASE III

When $b^2 - 4ac$ is negative, the roots are complex numbers. Now a little magic is involved. ***Appendix H*** indicates how infinite series can be used to relate the exponential form, $e^{i\theta}$, where $i = \sqrt{-1}$, to trigonometric functions. The surprising result (see ***page A63***) is

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For the two complex roots in this third case,

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

$$e^{r_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

and

$$e^{r_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos(-\beta x) + i\sin(-\beta x))$$

$$= e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

The general solution,

$$y = C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

can be simplified to

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

by combining coefficients.

Now, if you have some doubts about this rather strange looking form, that is a healthy response. Perhaps a check of the answer in ***Example 4*** will remove a *bit* of the doubt.

For the equation,

$$y'' - 6y' + 13y = 0$$

the claim is that the general solution is

$$y = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

To make the check less complicated, we arbitrarily let $c_1 = 4$ *and* $c_2 = 5$

Then

$$y \;=\; e^{3x}(4\cos 2x \;+\; 5\sin 2x)$$

Check:

$$y' \;=\; e^{3x}(-8\sin 2x \;+\; 10\cos 2x) \;+\; 3e^{3x}(4\cos 2x \;+\; 5\sin 2x)$$

$$=\; e^{3x}(7\sin 2x \;+\; 22\cos 2x)$$

$$y'' \;=\; e^{3x}(14\cos 2x \;-\; 44\sin 2x) \;+\; 3e^{3x}(7\sin 2x \;+\; 22\cos 2x)$$

$$=\; e^{3x}(-23\sin 2x \;+\; 80\cos 2x)$$

Then

$$y'' \quad - \quad 6y' \quad + \quad 13y \;=$$

$$e^{3x}(-23\sin 2x + 80\cos 2x) - 6\left[e^{3x}(7\sin 2x + 22\cos 2x)\right] + 13\left[e^{3x}(4\cos 2x + 5\sin 2x)\right]$$

This does indeed equal zero, which verifies that $y \;=\; e^{3x}(4\cos 2x \;+\; 5\sin 2x)$ is a solution of the differential equation.

BEHAVIOR OF FUNCTIONS

Some insight into the nature of differential equations can be gained by looking at the check in the last paragraph.

The exponential function e^{3x} occurs in y, y', and y''. It never vanishes. In contrast the sine function changes to cosine and then back to negative sine. Similar changes occur for the cosine function.

Also, all three functions, e^{3x}, $\sin 2x$, and $\cos 2x$, produce coefficients 3 and 2 in the first derivative and 9 and 4 in the second derivative.

The trick is to get the right combination of coefficients for sin 2*x* and cos 2*x* so that the balance is zero. Because e^{3x} never vanishes, it is a common factor in *all* terms and can be factored out. It contributes coefficients to the zero balance but is never zero itself.

In the next section, instead of producing zero, we will need combinations that produce a particular function, and the task becomes more complex, as you will see in the next module.

INITIAL AND BOUNDARY VALUES

In the above check of ***Example 4*** from the text, we arbitrarily let $c_1 = 4$ and $c_2 = 5$. We could also specify *initial* conditions that would produce the same result. Suppose we require that

$$y(0) = 4 \qquad \text{and} \qquad y'(0) = 22$$

Then $$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

and $$y' = e^{3x}(-2c_1 \sin 2x + 2c_2 \cos 2x) + 3e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

become $$4 = e^{0}(c_1 \cos 0 + c_2 \sin 0) = c_1$$

and $$22 = e^{0}(-2c_1 \sin 0 + 2c_2 \cos 0) + 3e^{0}(c_1 \cos 0 + c_2 \sin 0)$$

or $$22 = 2c_2 + 3c_1$$

Insert $c_1 = 4$ in the last equation to determine that $c_2 = 5$.

The words *initial conditions* refer to the starting conditions. Frequently $t = 0$ at this time. Note that this is the case in ***Examples 5*** and ***6***, where initial-values are given for y and y'.

In contrast, in a boundary-value problem, two y-values are given for two different points. The first derivative is not involved.

SECTION 17.2: NONHOMOGENEOUS LINEAR EQUATIONS

In the last section, we developed a method for finding the general solution of the *homogeneous* differential equation

$$a y'' + b y' + c y = 0 \qquad \textbf{(2)}$$

where *a, b,* and *c* are constants. Now we consider the *nonhomogeneous* equation,

$$a y'' + b y' + c y = G(x) \qquad \textbf{(1)}$$

These equations are labeled to match the text. Equation **(2)** is now called the *complementary* equation.

Theorem **(3)** on ***page 1161*** indicates that the *general* solution of (1) is

$$y(x) \;=\; y_p(x) \;+\; y_c(x)$$

where $y_p(x)$ is a *particular* solution of (1) and $y_c(x)$ is the *general* solution of (2). This may seem like a surprising result, but by calculating $y''(x)$ and $y'(x)$ we can show that the sum is a solution.

First $\quad y'(x) \;=\; y_p'(x) \;+\; y_c'(x)$

and $\quad y''(x) \;=\; y_p''(x) \;+\; y_c''(x)$

Substitute these three expressions into

$$a\,y'' \;+\; b\,y' \;+\; c\,y \;=\; G(x)$$

On the left side we have,

$$a\left[y_p''(x) \;+\; y_c''(x)\right] + b\left[y_p'(x) \;+\; y_c'(x)\right] + c\left[y_p(x) \;+\; y_c(x)\right]$$

This can be rewritten as

$$\left[ay_p''(x) \;+ by_p'(x) \;+ cy_p(x)\right] \;+\; \left[ay_c''(x) \;+\; by_c'(x) \;+\; cy_c(x)\right]$$

In turn this equals

$$G(x) \qquad\qquad + \qquad\qquad 0$$

which indicates that the sum is a solution of **(1)**. Remember that $y_p(x)$ is a solution of **(1)** that produces G(x), and $y_c(x)$ is a solution of **(2)** that produces 0.

The text has a slightly different proof, but both proofs use the same ideas. Looking carefully at one or both proofs is time well spent. The essence of mathematics is to generalize from a few examples. Trying to understanding the more general language throws most human beings for a loop. Here is your chance to rise to a new level and appreciate the view from a broader perspective. It is not easy to absorb the meaning of all the symbols but once you understand what is being said you have conquered one aspect of *intimidation caused by math symbols* that seems to plague most of mankind.

In the remainder of this section, we consider the two methods for finding a *particular* solution of the nonhomogeneous equation **(1)**.

THE METHOD OF UNDETERMINED COEFFICIENTS

Earlier in this module, under the heading "Behavior of Functions," we noted the behavior of e^{3x}, sin 2x, and cos 2x as they were substituted into a second-order linear equation. Think of the expression $a y'' + b y' + c y$ as a machine that processes a combination of the second derivative, the first derivative, and the function itself. Now we want that result to be another function, *G(x)*. In ***Example 1,*** *G(x)* equals x^2. A moment's reflection will indicate that e^{3x}, sin 2x, or cos 2x will not produce x^2 after being processed by $y'' + y'$ - 2*y*. A more likely candidate is $Ax^2 + Bx + C$ for certain values of A, B, and C.

This type of reasoning is the *basis* for the method of undetermined coefficients. Maybe you thought that we should include a Dx^4 term because the second derivative would produce an x^2 term. But thinking through the pattern of the substitution, we would have only one Dx^4 term (from -2y). Since there is no x^4 term on the right side, *D* must be zero, which eliminates any fourth degree term. The same reasoning would apply to the inclusion of an x^3 term.

Once you see the rationale for the method, the actual steps are straightforward. Find the first and second derivatives and substitute into the equation. Set the resulting quadratic equal to x^2 and solve for A, B, and C. Because

$$x^2 = x^2 + 0x + 0$$

A, B, and C must be selected to eliminate the x term and the constant.

Remember that the final answer,

$$y = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

has two parts. If you substitute the *first two* terms, along with their derivatives, into the original differential equation

$$y'' + y' - 2y = x^2$$

The result would be $\qquad 0 = x^2$

By themselves, these two terms are not solutions to the nonhomogeneous equation. They are solutions to the complementary equation,

$$y'' + y' - 2y = 0$$

for *any* values of c_1 *and* c_2. The last three terms, $-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$, provide a *particular* solution to $y'' + y' - 2y = x^2$.

Check.

$$y'' + y' - 2y = (-1) + \left(-x - \frac{1}{2}\right) - 2\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}\right)$$

$$= -1 - x - \frac{1}{2} + x^2 + x + \frac{3}{2}$$

The last line simplifies to x^2.

When the claim is made that

$$y = c_1e^x + c_2e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

is the *general* solution of

$$y'' + y' - 2y = x^2$$

we mean that *any* solution must fit this form. The constants c_1 and c_2 can have any value (general solution), but the coefficients for the last three terms are fixed (particular solution).

In the following we consider examples in the text.

EXAMPLE 2

Hopefully, you can agree that the only candidate for the particular solution of

$$y'' + 4y = e^{3x}$$

must have the form, Ae^{3x}, where A is a constant. After this selection, substitute the function and its second derivative into the equation to determine the value of A. Then combine with the solutions to the complementary equation to get the general solution.

EXAMPLE 3

Is it clear why we must use a combination of sin x and cos x even though the equation,

$$y'' + y' - 2y = \sin x$$

contains only sin x? If you thought just $y_p(x) = A\sin x$ would produce a solution, you are forgetting that the first derivative is $A \cos x$. Then, after substituting,

$$-A \sin x + A\cos x - 2A\sin x = \sin x$$

we would not have a solution for any value of A.

EXAMPLE 4

Now the right side contains a combination of several types, x, e^x, and $\cos 2x$. Note that the trial solution is broken up into two parts, $(Ax + B)e^x$ and $C \cos 2x + D \sin 2x$. Also, the original differential equation is separated into two matching parts. This is only a matter of convenience. You will get the same result if you use

$$y_p(x) = (Ax + B)e^x + C\cos 2x + D\sin 2x$$

as the trial solution. It is just a bit unwieldy.

Also note that you couldn't use the abbreviated form $Ax\,e^x$ in place of $(Ax + B)e^x$. B does have a non-zero value, and without it we could not get a solution.

EXAMPLE 5

Suppose you didn't notice anything special and used $y = A\cos x + B\sin x$
as the trial solution. First, $y' = -A\sin x + B\cos x$
and $y'' = -A\cos x - B\sin x$.

Then $y'' + y = -A\cos x - B\sin x + A\cos x + B\sin x$.

When you discover that the right side is zero, you have the facts that indicate the general solution of the *complementary* equation only. Some adjustment must be made. The adjustment is shown as a *modification* in the middle of ***page 1165***. We multiply by the factor x and use $Ax\cos x + Bx\sin x$ as a trial solution. You can gain some insight into why the inclusion of the extra factor works by noting the line at the bottom of ***page 1164***.

$$y_p'' + y_p = -2A\sin x + 2B\cos x = \sin x$$

After the substitution in $y_p'' + y_p$, we have no coefficient that contains the factor *x*. Would this happen in all cases? Instead of an answer, we just say that this is how mathematicians develop ideas. Find a good question and look at all possibilities to see if you can draw clear conclusions.

THE METHOD OF VARIATION OF PARAMETERS

First a historical note. In the late 1600s, when Isaac Newton was developing calculus, he was also solving what we now call differential equations. For three hundred years, mathematicians have sought methods to solve various equations. The next method that we consider, called *variation of parameters*, was developed in the late 1700s by Lagrange, one of the greatest mathematicians of the eighteenth century. It is a more complex method, which demands some perseverance in carrying out all of the steps.

We start with the homogeneous equation and its general solution,

$$y = c_1 y_1 + c_2 y_2$$

The constant coefficients, c_1 *and* c_2, are sometimes called parameters. We replace these constants with two functions, u_1 *and* u_2. The result,

$$y = u_1 y_1 + u_2 y_2$$

is the sum of two products of functions. One goal is to select u_1 *and* u_2 in a way that would produce a new solution of the differential equation. Because we have *two* "unknown" functions, we can also impose another condition, which is labeled **(7)** in the text. The reason for choosing the condition expressed by **(7)** is to simplify the first derivative. Starting with

$$y = u_1 y_1 + u_2 y_2$$

and using the Product Rule twice, we have

$$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

Then rewrite as

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

and equation **(7)** eliminates the first two terms by setting them equal to zero. Yes, this is legal.

$$\textbf{(7)} \qquad u_1' y_1 + u_2' y_2 = 0$$

To keep track of terms, note that we have the derivatives of the "u" functions but not the "*y*" functions. Don't spend time now trying to figure out *why*. There isn't an easy answer.

The first derivative is now just two terms, but as shown in the text the second derivative has four terms. Equation **(8)** is the result of substituting in the differential equation and rearranging terms. Before you let the complexity overwhelm you, note that equation **(8)** simplifies to

$$\textbf{(9)} \qquad a(u_1' \, y_1' \; + \; u_2' \, y_2') \;=\; G(x)$$

The final step in the method will be to solve equations **(7)** and **(9)** as two equations in two unknowns.

Comment 1. As you read the derivation, keep in mind that this is a general description that will have a different appearance when you do a particular problem.

Comment 2. What do I have to remember? When using this method you will need to write equation **(7)**. But this will only require that you select the two *terms* involving $u_1' \; and \; u_2'$ and set them equal to zero.

Comment 3. After you substitute y, y', and y'' in the differential equation, pairs of terms cancel naturally, and the result is equivalent to equation **(9)**. You won't have to remember the form of this equation, but you may want to check to see that it really does match.

Comment 4. Solve the two equations in two unknowns by eliminating terms after multiplying to make two terms the opposites of each other.

Comment 5. Unfortunately, the "solutions" are u_1' and u_2'. You will have to perform two integrations to find u_1 and u_2.

Comment 6. Then remember to substitute the result in

$$y \;=\; u_1 y_1 \;+\; u_2 y_2$$

to get *one* particular solution of the nonhomogeneous equation.

Comment 7. The general solution is as before,

$$y(x) \;=\; y_p(x) \;+\; y_c(x)$$

UNDERSTANDING DERIVATIONS

You may have to read the derivation on ***page 1166*** more than once to fully comprehend it. Even then you may get a different understanding after doing several problems. Unfortunately, many students want to shortcut the process and get to the "heart" of the matter — which problems must I do and what is the minimum I need to know. Well, the answer is to write equations **(7)** and **(9)** and solve them.

You first have to find $y_1 \textit{ and } y_2$ as solutions to $a\,y'' + b\,y' + c\,y = 0$, but then you can just treat **(7)** and **(9)** as formulas by dropping in the right components. You still have to follow comments 4 through 7, though. In fact, the only missing step is substituting y, y', and y'' in the differential equation to get to **(9)**. However, keep in mind that some understanding is lost when formulas are treated in a mechanical manner. But don't feel guilty. Most people have no concept of a differential equation. Your understanding may be incomplete, but Lagrange probably felt the same way.