Module 8

SECTION 11.9: REPRESENTATION OF FUNCTIONS AS POWER SERIES

As background for this section, review ***Example 6, page 707***. In this example, we considered the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

Because it fits the pattern of a geometric series with $r = x$, it converges if $|x| < 1$. Then the formula for the sum, $\dfrac{a}{1-r}$, became $\dfrac{1}{1-x}$, and we could write

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

ANOTHER MAJOR CHANGE IN THINKING

When this equation is written

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

we have the first example of a function, $\dfrac{1}{1-x}$, being expressed as a power series. This totally new idea will be the major focus in the rest of this chapter. The idea is to start with the function,

$$\frac{1}{1-x}$$

We then claim that it can be expressed as a power series. In one sense we are getting ahead of ourselves. On ***page 760***, ***Sec. 11.10***, formula **(6)** tells us how to find the power series for a particular function. This is a much more general approach than the one used in this section. Why the switch? The intent of ***section 11.9*** is to show how several functions can be expressed as power series. It just happens that the basis for doing this is the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Several surprising results will follow from this equation.

BUT WHY DO THIS?

In the above example, working with the function $\frac{1}{1-x}$ seems much simpler than working with the power series. Good point, but the text includes two reasons concerning the importance of this process.

1. ***Note 3***, ***page 755***, refers to a very important topic, differential equations, which is covered later in this course. Not very convincing now, but it will be later.

2. ***Example 11, page 768***, suggests a method for evaluating the integral $\int e^{-x^2}\, dx$ by using power series. Earlier we noted that it was not possible to find an elementary function whose derivative was e^{-x^2}.

It is always difficult when considering basic ideas to look ahead and see how they are used. Look back to a trig course and imagine how meaningful a discussion of trig substitutions in integration would have been.

NEW RULES

1. The first new rule is that one can make a *substitution,* such as that shown in ***Example 1, page 753***. It is mathematically OK to replace all x's with $-x^2$ to get a valid power series for $\frac{1}{1+{}^{2}x}$. The result can be *justified* by noting that the power series

$$1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

is a geometric series, with $r = -x^2$. It converges if $|x| < 1$, and its sum is

$$\frac{a}{1-r} = \frac{1}{1-\left(-x^2\right)} = \frac{1}{1+x^2}$$

Using the same procedure without any justification,

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \ldots$$

2. In $\frac{1}{1-x}$, we can replace x with another expression, but we can't do the same for the two ones. ***Example 2*** illustrates a more complex procedure. On ***page 753***, note that

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

Now compare the second fraction with $\frac{1}{1-x}$ and make a substitution. Note that the single 2 in $\frac{1}{2+x}$ impacts the series by introducing the exponential 2^{n+1}.

Using the same technique,

$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{\left[1-\frac{x}{3}\right]} = \frac{1}{3}\left(1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$$

The interval of convergence is (- 3 , 3).

3. ***Example 3*** illustrates an easy rule. Just write

$$\frac{x^3}{x+2} = x^3 \frac{1}{x+2}$$

and we only need to multiply the series $\frac{1}{x+2}$ by x^3.

Theorem (2) contains two significant results that are easy to apply.

4. Given a function expressed as a power series, to find the derivative of the function, it is OK to differentiate the series term-by-term. The interval of convergence for both series is the same, except possibly at the endpoints.

5. Same statement for integration. It is OK to integrate the series term-by-term. Again, interval of convergence is the same, except possibly at the endpoints.

USING THE NEW RULES

With the function $\frac{1}{1-x}$ as a starting point and by using the five rules listed above, we can express three related functions as power series.

1. The derivative of $\dfrac{1}{1 - x}$ is $\dfrac{1}{(1 - x)^2}$. Differentiating term-by-term produces the power series in ***Example 5***.

2. The derivative of $\ln(1 - x)$ is $\dfrac{-1}{1 - x}$. Start with

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$$

multiply by -1,

$$\frac{-1}{1 - x} = -1 - x - x^2 - x^3 - \ldots$$

and then integrate term-by-term to get the power series in ***Example 6***.

3. The derivative of the function $\tan^{-1} x$ is $\dfrac{1}{1 + x^2}$, the power series which was developed in ***Example 1***. Integrating term-by-term produces the power series in ***Example 7***. The last example in this section involves the function $\dfrac{1}{1 + x^7}$, which is an arbitrary variation of the basic function in this section. ***Example 8*** illustrates that it is easy to integrate term-by-term to get a power series that equals $\displaystyle\int \frac{1}{1 + x^7}\, dx$. Then we can approximate a definite integral with great accuracy.

COMMENTS

1. On one hand it is impressive to see how a number of functions can be expressed as power series using one basic series, but the method does have limitations. It was fairly easy to go from $\dfrac{1}{1 - x}$ to $\dfrac{1}{1 + x^2}$ and then to $\tan^{-1} x$, but we can't do the same for $\sin^{-1} x$, whose derivative is $\dfrac{1}{\sqrt{1 - x^2}}$. The square root is the problem. The first step is easy.

$$\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \ldots$$

but

$$\sqrt{\frac{1}{1 - x^2}} = \sqrt{1 + x^2 + x^4 + x^6 + \ldots}$$

doesn't lead anywhere. Remember $\sqrt{9 + 16} \neq 3 + 4$.

2. When differentiating term-by-term, the question arises whether to use a summation notation or the expanded form. Consider

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Suppose we find the derivatives as follows.

$$\sum_{n=0}^{\infty} (-1)^n (2n) x^{2n-1} = -2x + 4x^3 - 6x^5 + 8x^7 - \ldots$$

Are the two sides the same? The answer is yes. On the left, $n = 0$ produces zero as the first term, so the equation can also be written with n starting at one.

$$\sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1} = -2x + 4x^3 - 6x^5 + 8x^7 - \ldots$$

In ***Example 5, page 749***, the assertion is made that

$$\sum_{n=1}^{\infty} n x^{n-1} \quad \text{is equivalent to} \quad \sum_{n=0}^{\infty} (n+1) x^n .$$

This can be verified by finding the first two or three terms of each summation. We can even add a third equivalent summation, $\sum_{n=0}^{\infty} n x^{n-1}$.

There is no general rule as to which is best.

Suggestion: When differentiating or integrating term-by-term, use both the summation form and the expanded form to make sure that the two match.

SECTION 11.10: TAYLOR AND MACLAURIN SERIES

In this section a more general procedure is developed for expressing a function as a power series. The procedure is straightforward, but there are exceptions which complicate the presentation. In particular, note the question in the first paragraph: "Which functions have power series representations?" This is answered later in the section, but before the answer is given, *assume* that $f(x)$ is in the good category, i.e., that it does have a power series representation.

OK, now you can forget about the exceptions and just concentrate on a rather clever development. In

$$\textbf{(1)} \qquad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

we want a way of determining c_0 , c_1 , c_2 , *etc*. The first step is to let $x = a$. The result is

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \ldots$$

or $\qquad f(a) = c_0$

To determine the first constant, we just drop *a* into the function, *f(x),* and the result is the first constant c_0.

Then take a derivative of both sides of equation **(1)**.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$$

and again let $x = a$. Now the result is

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + 4c_4(a-a)^3 + \ldots$$

or $\qquad f'(a) = c_1$

To determine the second constant c_1, drop *a* into the first derivative, $f'(x)$.

Find the second derivative.

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \ldots$$

and repeat the process to get $f''(a) = 2c_2$.

The third derivative leads to

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \ldots$$

and $\qquad f'''(a) = 3!\, c_3$.

It is important to see how the factorials appear as we continue to take more derivatives. The general conclusion is

$$c_n = \frac{f^{(n)}(a)}{n!}$$

We can find the constant c_n by dropping a into the nth derivative and dividing the result by n factorial. Equation **(1)** becomes

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

and is called the *Taylor series of* f(x)*, about* a. It is important to recognize that if you change the value of a, the Taylor series will be different. More on this later.

Frequently $a = 0$, and this form is called a *Maclaurin series*.

Example

Find the Taylor series for $f(x) = \dfrac{1}{x} = x^{-1}$ at $a = 1$.

Consider

$$\begin{aligned} f(x) &= x^{-1} & f(1) &= 1 \\ f'(x) &= (-1)x^{-2} & f'(1) &= -1 \\ f''(x) &= 2x^{-3} & f''(1) &= 2 \\ f'''(x) &= -6x^{-4} & f'''(1) &= -6 \end{aligned}$$

Then

$$\frac{1}{x} = 1 + \frac{-1}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \ldots$$

or

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$$

Granted, it is a lot of work to find the coefficients, but the patterns are straightforward. By recognizing that the power series is a geometric series, $r = -(x-1)$, we can show the interval of convergence is (0,2). We can also show that the sum, $\dfrac{a}{1-r}$, is $\dfrac{1}{1+(x-1)} = \dfrac{1}{x}$. The method does work.

EXAMPLE 1

The result in this example, on ***page 760***, is that *IF* e^x has a Maclaurin series, it is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Now we don't have a geometric series, so we proceed in a different direction. In many applications involving infinite series only the first few terms are used.

Maybe the first five terms are used as an approximation to e^x .

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

But how good is the approximation? The answer to this question involves a special polynomial.

TAYLOR POLYNOMIALS

An *n*th-degree Taylor polynomial, T_n , is defined on ***page 761***. It is just the first (*n* + 1) terms of a Taylor series. R_n is the remainder of the series, or the error in the approximation.

$$R_n = f(x) - T_n(x)$$

This equation can be rewritten as

$$f(x) = T_n(x) + R_n$$

Two comments.

1. If we can show that $R_n \to 0$, then *f* can be represented as a power series. This is the mechanism to avoid mistakes.

2. R_n looks a lot like the $(n+1)$ term of a Taylor series.

   $$R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}, \text{ where } z \text{ is between } x \text{ and a.}$$

Several examples in the text find R_n and show that it approaches zero, but in this course this process remains in the background. The greater emphasis is on the creation of power series for basic functions and the significance of Taylor polynomials.

When using a Taylor polynomial,

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

as an approximation to $f(x)$, the following is true:

1. Both $T_n(x)$ and $f(x)$ pass through the point $(a, f(a))$.

Replace x with a in the above equation.

2. Both $T_n'(x)$ and $f'(x)$ are the same at $x = a$.

$$T_n'(a) = \frac{f'(a)}{1!} + \frac{f''(a)}{2!}2(a-a) + \ldots + \frac{f^{(n)}(a)}{n!}n(a-a)^{n-1}$$

Didn't we do this before? Yes, this is a review but from a different perspective. In any case, this means that both curves have the same tangent line but *only* at $x = a$.

3. We can repeat #2 and conclude that *all* derivatives of $T_n(x)$ and $f(x)$, up to the nth derivative, will be the same at $x = a$.

Conclusion: Near $x = a$ the graph of $T_n(x)$ will be "close" to the graph of $f(x)$. Further away we can't be sure what happens.

These ideas are illustrated in ***Figure 1, page 761***, which contains the graph of $y = e^x$ (in red) and the first three Taylor or Maclaurin polynomials. (Remember a = *0* for a Maclaurin series.) Note that the approximations are better near x= 0. All curves go through the point (0,1) and have the same tangent line at (0,1). The second derivatives are also the same at (0,1) for T_2, T_3, and f. This means the rate of change of the slopes of the tangent lines are the same at (0,1) which possibly smoothes out the curves at this point. For T_3 and f, the third derivatives are the same, but there is no clear graphical interpretation for this phenomenon.

Perhaps you have noticed that, in the third and fourth quadrants, the approximations aren't very good, to say the least. Keep this in mind as we proceed.

THE SINE CURVE

Example 4, develops the Maclaurin series (a = 0) for *sin x*.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Figure 2 shows the graph of the sine curve along with the graphs of the first three Taylor or Maclaurin polynomials. Near $x = 0$, all three polynomials are close to the sine curve. T_5 continues to be close to the sine curve, but at $x = \pi$ it heads up while the sine curve decreases.

To get better approximations near $x = \pi$, find the Taylor series with $a = \pi$. The form would be

$$\sin x = \sin \pi + \frac{\cos \pi}{1!}(x-\pi) + \frac{-\sin \pi}{2!}(x-\pi)^2 + \frac{-\cos \pi}{3!}(x-\pi)^3 + \ldots$$

$$= 0 - (x-\pi) + 0(x-\pi)^2 + \frac{1}{3!}(x-\pi)^3 + \ldots$$

$$= - (x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \ldots$$

The graph of the Taylor polynomial

$$T_3 = - (x-\pi) + \frac{1}{3!}(x-\pi)^3$$

would be close to the sine curve near $x = \pi$. So T_3 would be a better tool to approximate, say, $\sin 178^\circ$ than a corresponding polynomial based on the Maclaurin series for *sin* x.

REPETITION

The Taylor series for e^x about -2 is

$$e^x = e^{-2} + \frac{e^{-2}}{1!}(x+2) + \frac{e^{-2}}{2!}(x+2)^2 + \frac{e^{-2}}{3!}(x+2)^3 + \ldots$$

All derivatives of e^x are the same, e^x. Evaluate at $x = -2$, and we have the above series.

The Maclaurin series for e^x , from ***Example 1***, is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Which series would you use to approximate $e^{-0.1}$? Same question for $e^{-2.1}$. You are correct if you said the second, or the Maclaurin series for $e^{-0.1}$ and the first, the Taylor series, for $e^{-2.1}$. Each series will be an alternating series, so we can approximate the answers by looking at the first term *not* used.

$$e^{-0.1} = 1 + \frac{-0.1}{1!} + \frac{(-0.1)^2}{2!} + \frac{(-0.1)^3}{3!} + \ldots$$

If we use the first three terms as an approximation, the error is R_3 and

$$|R_3| \le \left|\frac{(-0.1)^3}{3!}\right| = \frac{1}{6{,}000} \approx .00017$$

$$e^{-2.1} = e^{-2} + \frac{e^{-2}}{1!}(-2.1+2) + \frac{e^{-2}}{2!}(-2.1+2)^2 + \frac{e^{-2}}{3!}(-2.1+2)^3 + \ldots$$

$$= e^{-2} + \frac{e^{-2}}{1!}(-0.1) + \frac{e^{-2}}{2!}(-0.1)^2 + \frac{e^{-2}}{3!}(-0.1)^3 + \ldots$$

If we use the first three terms as an approximation, the error is again R_3

and

$$|R_3| \le \left|\frac{(-0.1)^3}{e^2(3!)}\right| = \frac{1}{6{,}000e^2} \approx .000023$$

If we had used the Maclaurin series to approximate $e^{-2.1}$,

$$e^{-2.1} = 1 + \frac{-2.1}{1!} + \frac{(-2.1)^2}{2!} + \frac{(-2.1)^3}{3!} + \ldots$$

by using the first three terms, the error R_3 would have been much larger.

$$|R_3| \le \left|\frac{(-2.1)^3}{3!}\right| = \frac{9.261}{6} \approx 1.54$$

Back to Rules

Finding a Taylor series for a function like $x \cos x$ involves using the Product Rule to find the first derivative, which is $-x \sin x + \cos x$. Then finding the second and higher derivatives becomes more complicated. Earlier in this lesson, five rules for finding power series from other series were discussed. Frequently these rules can be used instead of the Taylor series pattern. ***Example 6, page 765***, illustrates the process for $x \cos x$. Just multiply the series for $\cos x$ by x.

In a similar manner, in ***Example 5***, the Maclaurin series for *cos x* is found by term-by-term differentiation of the series for *sin x*.

In ***exercises 35*** through ***46***, all series can be found by using one or more of the five rules.

Also, we found the Taylor series for e^x about -2 to be,

$$e^x = e^{-2} + \frac{e^{-2}}{1!}(x+2) + \frac{e^{-2}}{2!}(x+2)^2 + \frac{e^{-2}}{3!}(x+2)^3 + \ldots$$

Note that e^{-2} is a common factor of all terms of the series. Hence,

$$e^x = e^{-2}\left[1 + \frac{1}{1!}(x+2) + \frac{1}{2!}(x+2)^2 + \frac{1}{3!}(x+2)^3 + \ldots\right]$$

Multiply by e^2.

$$e^x e^2 = 1 + \frac{1}{1!}(x+2) + \frac{1}{2!}(x+2)^2 + \frac{1}{3!}(x+2)^3 + \ldots$$

Then

$$e^{x+2} = 1 + \frac{1}{1!}(x+2) + \frac{1}{2!}(x+2)^2 + \frac{1}{3!}(x+2)^3 + \ldots$$

Compare this result with the Maclaurin series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

We could have used one of the five rules, substitution, to get the same result, but we would have had to develop a new perspective first.

EXAMPLE 11

This example provides a significant insight into the use of a power series representation of a function. Before we could not perform an integration, but now with power series, integration is a straightforward task. The error can easily be reduced by using more terms of the series.

MULTIPLICATION AND DIVISION

This section concludes with the possibility of multiplying or dividing two series to get a third series. Yes, it is a complicated process, but the Taylor series pattern can also be complicated. The intent here is to present a choice. By all means select the simpler method.

SECTION 11.11: APPLICATIONS OF TAYLOR POLYNOMIALS

Parts of this section were emphasized earlier. In particular, read the last paragraph on ***page 774***, concerning the graphical interpretation of using Taylor polynomials to approximate a function. This type of graph can provide a background setting for the three methods of estimating errors listed on ***page 775***. We will concentrate on the third method, using the remainder

$$R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

where z is between x and a.

The idea is not to find an exact value for R_n. Instead find an upper bound B and then indicate that $R_n \le B$. Two steps are commonly used in finding an upper bound.

1. Make the numerator as large as possible.

2. Make the denominator as small as possible. For example, 1/2 is larger than 1/5.

In ***Example 1***, $\qquad R_2 = \dfrac{5(x-8)^3}{81z^{8/3}}$, and z is between 8 and x.

Part **(b)** also restricts x to $7 \le x \le 9$. There are two possibilities; $7 \le x \le 8 \le 9$ or $7 \le 8 \le x \le 9$. In either case, z is greater than or equal to 7 because z is between 8 and x.

Now replace x with its largest possible value, 9, and replace z with its smallest possible value, 7. This gives one estimate of the error.

In ***Example 2***, we have an alternating series. An upper bound on the error is the first term not used. Put the largest possible x-value in the numerator, and we have the required estimate of the error. If we used only two terms to approximate sin x, an upper bound on the error would be

$$\frac{(0.3)^5}{5!} = 2.025 \; X \; 10^{-5}$$

which doesn't provide accuracy to six decimal places as requested.

Last Comment

If you have some knowledge of physics, ***Example 3*** on ***page 778*** may be interesting.
In any case, note the power series

$$\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \ldots$$

along with the comment that if v is much smaller than c, the fraction v/c is very small. In turn v^4/c^4 , v^6/c^6 , and higher powers are so small that just using the first term provides a good approximation.

Power series are frequently used in this manner.