Module 6

SECTION 1.5: INVERSE FUNCTIONS AND LOGARITHMS

The first part of this section can be skimmed with just a brief look at the different ways of seeing the relationships between inverse functions. A major point is the algebraic method for finding an inverse function on ***page 58***.

1. Solve for x in terms of y.
2. Interchange x and y.

This sounds simple enough but frequently it is *impossible* to carry out. If:

$$y = x^5 + 3x^4 + 2x^3 + x^2 + 5x + 3,$$

then it is not possible to solve for x in terms of y. See ***Note 2*** on ***page 211***.

However, when it is possible, the method will produce an inverse function.

Example. Given: $y = 3x - 4$

Solve for x: $x = \dfrac{y + 4}{3}$

Interchange x and y: $y = \dfrac{x + 4}{3}$

Then $\dfrac{x + 4}{3}$ is the inverse of $3x - 4$.

But what does this mean? A good summary is given in the top paragraph on ***page 58***, "start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Thus f^{-1} undoes what f does." f represents the function and f^{-1} represents the inverse. For the above linear function, $y = 3x - 4$, and its inverse we can describe the relationship in words:

First apply f to x. Multiply x by 3 and then subtract 4.

Then apply f^{-1} Add 4 (to the first result) and then divide by 3.

If $x = 7$, then $3x - 4 = 3(7) - 4 = 17$

Put this result, 17, into the inverse and $\dfrac{x + 4}{3} = \dfrac{17 + 4}{3} = \dfrac{21}{3} = 7$

The second result, 7, is the same as the first input, $x = 7$.

Now we encounter a familiar problem, saying the same thing with more general notation.

NOTATION

The procedure in the last five lines is described in ***[4]*** on ***page 57***.

$$f^{-1}\left[f(x)\right] = x$$

Using function notation for the above linear function, we have:

$$f(x) = 3x - 4$$

and

$$f^{-1}(x) = \frac{x + 4}{3}$$

Then

$$f^{-1}\left[f(x)\right] = f^{-1}\left[3x - 4\right] = \frac{(3x - 4) + 4}{3} = \frac{3x}{3} = x.$$

This relationship will hold for any function and its inverse. The reverse will also be true:

$$f\left[f^{-1}(x)\right] = x$$

$$f\left[f^{-1}(x)\right] = f\left[\frac{x + 4}{3}\right] = 3\left[\frac{x + 4}{3}\right] - 4 = x + 4 - 4 = x$$

This process shows the relationship between a function and its inverse.

GRAPHING AN INVERSE FUNCTION

The algebraic procedure for finding an inverse involves interchanging x and y. This means that the graph of an inverse can be found from the graph of the function by reflecting about the line $y = x$.

Consider $y = x^3$ and the matching table:

x	-2	-1	0	1	2
y	-8	-1	0	1	8

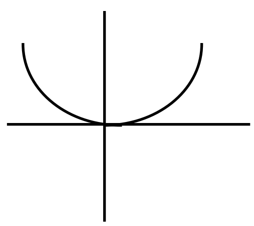
The table indicates that the points $(-2,-8)$, $(-1,-1)$, $(0,0)$, $(1,1)$, $(2,8)$ are on the curve, $y = x^3$.

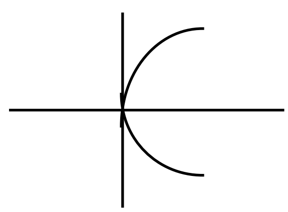
Now interchange the x and y coordinates and the resulting points $(-8,-2)$, $(-1,-1)$, $(0,0)$, $(1,1)$, $(8,2)$ are on the inverse.

If you plot these points on a graph you will discover the reflection about the line $y = x$.

WHEN IS THE INVERSE A FUNCTION?

If the graph of $y = x^2$ is rotated about the line $y = x$ we get another parabola shown in the graph below:

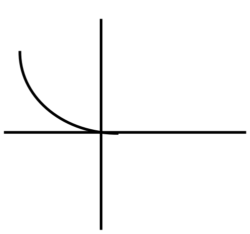


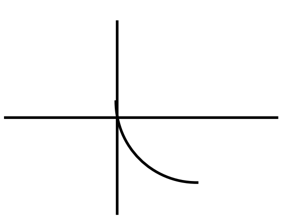


But in the graph on the right we do not have a function. Using the Vertical Line Test, a vertical line crosses the graph in two points.

The concept of a *one-to-one* function is introduced on ***page 56***. If a *horizontal* line intersects the graph only once then the function is one-to-one and the inverse will be a *function*.

In the graph below on the left, we now have a one-to-one function and the graph on the right is the inverse *function:*





$y = x^2 \ , \ x \leq 0$ $\qquad\qquad$ $x = y^2 \ , \ y \leq 0$

DOMAINS AND RANGES

Going from a one-to-one function f to its inverse f^{-1}, we interchange x and y. Then the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}.

DEFINITION OF AN INVERSE FUNCTION

First note that we are not discussing the definition in ***Box 2*** before looking at other interpretations of an inverse function. Why? Because the definition is correct but after reading it you may not have a feeling that it *tells* you what an inverse is. There are three parts in the definition.

1. f must be a one-to-one function for the inverse to be a *function.*

2. The domains and ranges are interchanged.

3. The assertion that $f^{-1}(y) = x$ if and only if $f(x) = y$.

   In statement 3 it is important to recognize that x and y have not been interchanged. The symbols $f^{-1}(y) = x$ just represent the process of solving for x. If you can do this then $f^{-1}(y)$ or $f^{-1}(x)$ will show the algebraic *form* of the inverse.

Statement 3 also implies the *cancellation equation:*

$$f^{-1}\left[f(x)\right] = x.$$

Replace y with $f(x)$ in the equation, $f^{-1}(y) = x$.

This definition is abstract and indirect. Some prior knowledge of an inverse function makes it easier to understand.

LOGARITHMIC FUNCTIONS

Sec. 1.5 contains a review of basic properties of logarithms covered in a Precalculus or Intermediate Algebra course. Bear in mind, however, that knowing a basic fact and knowing when to use it are two different things. Hence, we will stress basic properties and also suggest how they become tools. The tricky part is not knowing which tool to use beforehand. So a willingness to try an approach even when you can't see far enough ahead to know if you are on the right path will be important.

EVERY LOGARITHM IS AN EXPONENT

The basic definition of a logarithm

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x$$

matches a log form with an exponential form. In the exponential form, $a^y = x$, y is an exponent. In the log form, $\log_a x = y$, y represents the logarithm. Hence every logarithm is an exponent.

However, that is only part of the story. The exponent must be associated with some base, a. Consider two situations:

1. What is the log of 25, base 5, or $\log_5 25 = ?$

   Easy. Look at $5^? = 25$ and wait till the answer 2 pops into your head.

2. What is the log of 7, base 3, or $\log_3 7 = ?$

   Not easy. Look at $3^? = 7$ and the answer is not apparent. If you note that $3^1 = 3$ and $3^2 = 9$ then ? is between one and two but that only gives us a rough approximation. We will indicate two methods for finding a solution later.

THREE BASIC PROPERTIES OF LOGS

You need to know these three properties like the back of your hand.

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a\left(x^y\right) = y\log_a x$

In words:

1. The log of a product equals the sum of two logs.

2. The log of a quotient equals the difference of two logs.

3. In the log of a power the exponent becomes a coefficient.

Each of these properties can be used in two ways. To use the first property look for the log of a product and change to the sum of two logs *or* look for the sum of two logs and change to the log of a product. Similar comments apply to the other two properties.

The third property will be especially important. To solve any equation in which the variable is an exponent, use the third property.

1. The equation, $3^{?} = 7$, states that the left side equals the right side. It follows that the log of the left side will equal the log of the right side. Why do this? Look ahead and see what happens.

$$3^{?} = 7$$

$$\log 3^{?} = \log 7$$

Now use the third property $\quad ?\log 3 = \log 7$

Divide by $\log 3$ $\quad ? = \dfrac{\log 7}{\log 3}$

Use a calculator $\quad ? = 1.77$

When writing $\log 3$ the base is understood to be 10; $\log 3 = \log_{10} 3$ The third line is the crucial step where the exponent ? changes position and becomes a coefficient.

2. Later you will need to solve the following type of equation:

$$1000 = 10\left(2^{0.1x}\right)$$

First divide by 10 $\quad 100 = 2^{0.1x}$

Then take the log of both sides $\quad \log 100 = \log 2^{0.1x}$

Use property 3 on the right side $\quad \log 100 = (0.1x)\log 2$

Then solve for *x*. $\quad x = \dfrac{\log 100}{(0.1)\log 2}$

Use a calculator and $\quad x = 66.44$

Comment: If you don’t divide by 10 to simplify, line 2 would be

$$\log 1000 = \log 10\left(2^{0.1x}\right)$$

The right side is the log of a product. The next line would be

$$\log 1000 = \log 10 + \log\left(2^{0.1x}\right)$$

Then

$$\log 1000 - \log 10 = \log\left(2^{0.1x}\right)$$

The left side is now a difference of two logs and can be rewritten as

$$\log 1000 - \log 10 = \log\frac{1000}{10} = \log 100$$

Substitution produces line 2 in the first solution. The second method requires three extra steps to get to the same equation.

THE NUMBER *e* AGAIN

Earlier we noted that the most frequently used exponential function in calculus is e^x. The most frequently used logarithmic function in calculus also uses the base e with the special notation:

$$\ln x \quad for \quad \log_e x$$

The log form $y = \ln x$ matches the exponential form, $x = e^y$

Interchange x and y and $y = e^x$.

So the inverse of the function, $\ln x$, is e^x.

It follows that the *cancellation equations* on ***page 61*** apply to $\ln x$ and e^x:

$$f^{-1}\left[f(x)\right] = x \qquad \text{and} \qquad f\left[f^{-1}(x)\right] = x$$

Then $\qquad e^{\ln x} = x \qquad$ and $\qquad \ln(e^x) = x$

The second equation can be justified by using property 3:

$$\ln(e^x) = x \ln e$$

Then note that $\ln e = 1$ $= x$

[The log form $\ln e = 1$ matches the exponential form $e^1 = e$ because $\ln e = 1$ is the same as $\log_e e = 1$]

The first equation is a bit trickier and has some important uses. We give several examples of the form with different bases.

1. $2^{\log_2 4} = 4$ Verify this by noting that $\log_2 4 = 2$, $2^2 = 4$.

2. $5^{\log_5 125} = 125$ Because $5^3 = 125$ and $\log_5 125 = 3$.

3. $e^{\ln e^2} = e^2$ Note $\ln e^2 = 2$.

In these examples it's as if you are doing the same thing twice. In #1 to find $\log_2 4$ you need to use $2^2 = 4$. And then when you replace $\log_2 4$ with 2 you are looking at $2^2 = 4$ again. This repetition illustrates the fact that we are using a function 2^x and its inverse $\log_2 x$ and to decipher $\log_2 x$ we use 2^x. Yes, it is a bit convoluted.

WHERE IS $e^{\ln x}$ USED?

One use of $e^{\ln x} = x$ allows us to write $2^x = e^{\ln 2^x}$

Then in the exponent, $\ln 2^x = x\ln 2$, so we have $2^x = e^{x\ln 2}$

This last result can be used to find the derivative of 2^x:

$$y = 2^x = e^{x\ln 2}$$

$$y' = e^{x\ln 2}(x\ln 2)' \qquad \text{Use } \frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$y' = e^{x\ln 2}(\ln 2) \qquad \text{Remember } \ln 2 \text{ is a constant.}$$

or $$y' = 2^x \ln 2$$

Now compare $\frac{d}{dx}e^x = e^x$ with $\frac{d}{dx}2^x = 2^x \ln 2$ and you can see why the preference for the simpler derivative exists. Even though the number e seems to be a strange number as an exponential base it offers significant advantages.

CHANGE OF BASE FORMULA

In ***Box 10***, you will see the formula

$$\log_a x \;=\; \frac{\ln x}{\ln a}$$

It is worth reading the proof carefully to note the tools used.

1. Start with the left side, $\log_a x$, and try to find a logical path to the right side.

2. Go from the log form, $y = \log_a x$, to the equivalent exponential form,
$$a^y = x.$$

3. Then introduce a new base, e, by taking the natural log of both sides.
$$y \ln a = \ln x$$

4. Solve for *y*. $$y = \frac{\ln x}{\ln a}$$

5. Remember that *y* equals $\log_a x$. $$\log_a x = \frac{\ln x}{\ln a}$$

The proof provides an example where you probably can not see the end result directly from the starting point. The intermediate steps provide the path. But when working with similar applications you will need to *create* the intermediate steps. *Be willing to try something and then look at the resulting next step.* One possibility is changing a log form to an exponential form. You will see things in the exponential form that are not apparent in the log form. (Eventually you may be able to see both simultaneously.) Also consider taking the log of both sides of an equation. In the result you may be able to see ahead to the final goal.

We are suggesting a *dynamic* mental process that you create. Many students get stuck and then keep looking at a *static* situation. That, by definition doesn't change.

The above formula is a special case of the more general formula,

$$\log_a x \;=\; \frac{\log_b x}{\log_b a}$$

The logarithmic base changes from a to b.

Try constructing a proof by following the five steps listed above.

USES OF THE CHANGE OF BASE FORMULA

Calculators have just two log buttons, LOG and LN. The first is base 10 and the second is base e. The equation $3^x = 7$ is equivalent to $x = \log_3 7$ but there is no log, base 3, button on any calculator. Use the change of base formula:

$$\log_3 7 = \frac{\ln 7}{\ln 3}$$

and a calculator can then be used:

$$\log_3 7 = \frac{1.9459}{1.0986} = 1.7712$$

Earlier we solved the same equation using logs, base 10. Then

$$x = \frac{\log 7}{\log 3}$$

Using a calculator $x = \dfrac{0.8451}{0.4471} = 1.7712$ and we get the same answer.

A second use of the change of base formula:

$$\log_2 x = \frac{\ln x}{\ln 2} = \frac{1}{\ln 2}\ln x$$

Here we have converted the log function, base 2, to a natural log function. Remember that $\ln 2$ is a constant, $\ln 2 = 0.6931$. The right side is just a constant times $\ln x$. We can write:

$$\log_2 x = 1.4427 \ln x$$

to better illustrate the change.

Why do we want to do this? The answer lies in the next section. By knowing the derivative of $\ln x$ we can find the derivative of *any* log function. Just use the above conversion. For $\log_2 x$ multiply the derivative of $\ln x$ by the constant $\dfrac{1}{\ln 2}$.

GRAPHS OF LOG FUNCTIONS

Study the graphs in the ***Graph and Growth of the Natural Logarithm*** *subsection* carefully. In particular, in the graph of $y = \ln x$, note the following:

1. The domain is $(0,\infty)$.

2. The point $(1,0)$ matches $\ln 1 = 0$ because $e^0 = 1$

3. The natural log is negative between 0 and 1.

4. As x approaches 0 from the right, $\ln x$ approaches $-\infty$.

5. $y = \ln x$ is a very slowly increasing function.

To understand the third line, remember that:

$$y = \ln x \text{ is the same as } e^y = x .$$

Then think of inserting negative values for y in $e^y = x$:

y	-4	-3	-2	-1
x	$\frac{1}{e^4}$	$\frac{1}{e^3}$	$\frac{1}{e^2}$	$\frac{1}{e}$

The negative exponent leads to a power of e in the denominator and a fraction between 0 and 1.

The fact that $y = \ln x$ is a very slowly increasing function is emphasized in the table and graphs on ***page 63***. For x = 100,000, $\ln x$ has only reached 11.5. For x = 1,000,000, $\ln x$ increases to just 13.8. There is also a comparison with $\sqrt{x}$. The bigger context here to look at:

$$\ln x \,,\ \sqrt{x} \,,\ x \,,\ x^2 \,,\ x^3 \,,\ x^4 \ \ldots$$

and note that $\sqrt{x}$ increases slowly compared to those on it's right but $\ln x$ moves at a slow snails pace. These comparisons will be significant in the study of infinite series in Chapter 11.

SECTION 3.6: DERIVATIVES OF LOGARITHMIC FUNCTIONS

The derivative of the log function with base ***a*** is shown *in **Box 1***. To understand why the derivative is a quotient with *x ln a* in the denominator, check the proof. Yes it is just a set of algebraic steps but still the form of the derivative depends on these steps.

However the more significant result is (2). The derivative of the natural log function, *ln x*, is the quotient, *1/x*. Remember that the log of x, base *a*, becomes *ln x* when *a* is equal to *e*. Also *ln a* vanishes because *ln e* equals one.

It may appear to be a strange result in that it connects a log function with a rational function. To make the result more intuitive we look at two intervals and one point. Look at the graph in ***Figure 13, page 62*** and remember that the derivative $\frac{1}{x}$ is the slope of the tangent line:

1. As $x \to 0^+$, $\frac{1}{x} \to \infty$.
2. At $x = 1$, $\frac{1}{x} = 1$.
3. As $x \to \infty$, $\frac{1}{x} \to 0$.

Each of these statements match the graph of $y = \ln x$.

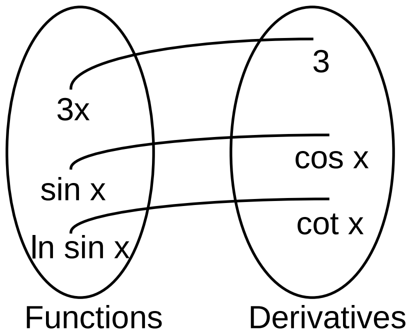
Example 1, Sec. 3.6 illustrates the procedure for finding a derivative of
ln u when *u* is a function of x, say *u = g(x).*
The procedure is summarized in ***Box 3*** in terms of *u* and also *g(x)* . For now concentrate on the first form.

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

The derivative of *ln u* is "(1 over *u)* times the derivative of *u*"
This will be a significant formula in the remainder of the course.

STRANGE HAPPENINGS

In ***Example 2***, the derivative of $\ln(\sin x)$ is $\cot x$. Think of the process of finding a derivative as a connection between a set of functions and the corresponding derivatives as suggested below:



In addition to the pairings shown we could have:

$$5x^2 + 8x + 3 \text{ and } 10x + 8$$

or $$\tan x \quad \text{and } \sec^2 x$$

The first connects a polynomial to another polynomial and the second connects a trig function to another trig function. In the connection of $\ln(\sin x)$ with $\cot x$, however, we have a log-trig function mating with a trig function. And somehow this strange number e is involved as the base of a logarithm. These connections are very important. Later a process called integration will reverse the above connection — given $10x + 8$ you will produce $5x^2 + 8x + a$ constant or given $\cot x$ you will produce $\ln(\sin x) + a$ constant.

Why say all of this now? Well, many students ask "Where am I going to use this?" and answers exist but you have to be willing to listen a while, perhaps develop a new point of view and remember formulas like

$$\frac{d}{dx} e^u = e^u \frac{du}{dx} \qquad\qquad \frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

Even if a result looks strange it can be important in the broader view of things.

THE DERIVATIVE OF ln $|x|$

In ***Example 6***, the definition of absolute value:

$$\begin{aligned} |x| &= x \quad \text{if} \quad x > 0 \\ &= -x \quad \text{if} \quad x < 0 \end{aligned}$$

is used to show that the derivative of $\ln|x|$ is the same as the derivative of $\ln x$:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Using the Chain Rule:

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}$$

What is the difference between $\ln|x|$ and $\ln x$?

The domain of $y = \ln x$ is $(0,\infty)$. We can find the natural log of positive numbers only. For $y = \ln|x|$, the absolute value produces only positive numbers and hence its domain is $(-\infty,0) \cup (0,\infty)$. In a similar manner, the domain of $\ln(\sin x)$ must exclude all negative values of $\sin x$ and values where $\sin x$ equals zero. However, for the domain of $\ln|\sin x|$ we need only exclude values where $\sin x$ equals zero. For *both* functions the derivative is $\cot x$.

LOGARITHMIC DIFFERENTIATION

Example 7, illustrates a significant use of the three properties of logs. The original function is a quotient with both products and powers present which would require the use of the Quotient Rule, the Product Rule and the Power Rule. By taking the log of both sides, the quotient becomes a difference, the product a sum, and the exponents in the powers become coefficients. Each of these changes simplifies the process of finding the derivative. This method is used only when the function is complex. Remember that $\ln(A \pm B)$ is not equal to $\ln A \pm \ln B$. The method does not work for sums and differences.

A NEW TYPE OF FUNCTION

On ***page 221***, the text lists four types of functions involving exponents. Simple forms of each are:

$$3^2 \quad , \quad x^2 \quad , \quad 3^x \quad , \quad x^x$$

The Power Rule applies to only one of these, x^2. The last form, x^x, is a totally different type of function. *Both* the base and the exponent are variables. To find the derivative, use logarithmic differentiation:

$$y = x^x$$

$$\ln y = \ln x^x = x\ln x$$

$$\frac{1}{y}y' = x\frac{1}{x} + \ln x$$

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

THE NUMBER *e* AS A LIMIT

As mentioned earlier the special number e can be expressed as a limit:

$$\lim_{x \to 0}(1 + x)^{1/x} = e$$

By replacing x with $\frac{1}{n}$ which leads to $n = \frac{1}{x}$ the limit can be expressed as a limit with $n \to \infty$:

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$$

Try several large values for n and watch the powers approach 2.718281828.

SUMMARY

We conclude with a listing of the exponential and logarithmic derivatives.

Basic	**General** $u = f(x)$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^u = e^u\frac{du}{dx}$
$\frac{d}{dx}a^x = a^x \ln a$	$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}\ln\lvert x \rvert = \frac{1}{x}$	$\frac{d}{dx}\ln\lvert u \rvert = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$	$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$

Three comments:

1. The right column can be created from the left column by replacing x with u and multiplying by $\frac{du}{dx}$.

2. In every case the effect of the Chain Rule appears as a multiplication by $\frac{du}{dx}$.

3. Note that the derivatives of e^x and a^x follow the same pattern except for the need for $\ln a$ in the derivative of a^x. The same is true for $\ln x$ and $\log_a x$.

To Memorize or Not to Memorize

Many students say they don't want to rely on rote memorization but rather on understanding of concepts. Keep in mind that the derivation of the derivative of e^x was rather complex. In contrast the derivation of the derivative of $\ln x$ was very concise. However, mentally going through that derivation each time you use the formula:

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

is not an efficient mental process.

A second point relates to the fact that there are two major procedures in basic calculus. The first is finding a derivative. The second will involve a reversal of this process. A function is treated as a derivative and you have to find a starting function that produces this particular derivative. If the above derivatives exist only as a list on paper instead of in your head you will be at a disadvantage.

Section 3.7: Rates of Change in the Natural and Social Sciences (Optional)

This is a very challenging section. The author has selected problems from different fields. However, if you have no knowledge in a particular field, it is more difficult to understand the use of these new calculus concepts. Therefore, we will make very brief comments on six of the eight examples and go in detail on the other two, ***Examples 1*** and ***8***. If some examples relate to your field, by all means look at them carefully. Read the first two paragraphs carefully since they provide a good review.

EXAMPLE 2

Note that density ρ is mass m per unit length l, $\rho = \frac{m}{l}$. Without this definition, your mind can't function. Then note Δx is a change in *x*, which makes it a length l. Δm is a change in mass. From the definition, $\frac{\Delta m}{\Delta x}$, is density, and because we haven't taken the limit, it is the average density. $\frac{dm}{dx}$ is the limit of $\frac{\Delta m}{\Delta x}$, and its meaning is described in the last two lines. It is the density right at a point, and an evaluation is shown at $x = 1$.

EXAMPLE 3

Average current is defined as $\frac{\Delta Q}{\Delta t}$, which is $\frac{\text{net charge}}{\text{change in time}}$, where the net charge is for the time interval Δt. The limit of $\frac{\Delta Q}{\Delta t}$ is the current at a particular time or the rate at which charge flows through a surface.

Note in these two examples, definitions of terms are basic to a first understanding. Further activity is required before one can gain a broader understanding.

EXAMPLE 4

Here, $\frac{\Delta[C]}{\Delta t}$ is the average rate of reaction of the product C over the time interval, Δt. The limit $\frac{d[C]}{dt}$ is the instantaneous rate of reaction. Read the rest if you've had some chemistry.

EXAMPLE 5

This example is a bit different. Start with the "volume V depends on the pressure P."
This implies V is a function of P, and hence the derivative of V with respect to P, $\frac{dV}{dP}$, is a meaningful concept even though we don't specifically know the form of the function. Then we encounter another definition.

$$\text{Isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Then a verbal interpretation is given. Remember, $\frac{dV}{dP}$ and $\frac{\Delta V}{\Delta P}$ are connected by the limit process. And $\frac{\Delta V}{\Delta P} = \frac{\text{change in volume}}{\text{change in pressure}}$. If the above minus sign is put in front of this and applied to the numerator, we have a decrease in volume and an increase in pressure. The words, "per unit volume," relate to $\frac{1}{V}$ in the definition. (Not only is the approach different but the concepts are also difficult to grasp.) The last line of the example follows the pattern in which $-\frac{1}{V}\frac{dV}{dP}$ is rewritten as $\frac{-\frac{dV}{dP}}{V}$ which roughly translates to a negative rate of change per unit volume.

EXAMPLE 6

The function $n = f(t)$ matches n animals or plants whose number depends on time t. Then fill in the ?? for

$$\frac{\Delta n}{\Delta t} = \frac{change\ in\ ??}{change\ in\ ??}$$

and the limit is the instantaneous rate of growth of the number of animals or plants. However, because n must be a whole number, $n = f(t)$ is really a step function, and the graph has discontinuous jumps which mess up the limit. The red smooth curve in ***Figure 5*** provides an approximation that removes these difficulties.

EXAMPLE 7

We start with the law of laminar flow,

$$v = \frac{P}{4h\ l}\left(R^2 - r^2\right).$$

A formula like this can not be understood without a legend explaining what each letter represents. On the right side, which letters are constants and which are variable? Velocity v and distance r are the variables, so $\frac{dv}{dr}$ is the rate of change of velocity v with respect to r. To find this derivative from the above formula, remember $\frac{P}{4\eta\ l}$ is a constant; none of the letters is a variable. Then $\frac{d}{dr}\left[\frac{P}{4\eta\ l}\left(R^2 - r^2\right)\right] = \frac{P}{4\eta\ l}\ \frac{d}{dr}\left(R^2 - r^2\right)$ by $(cf)' = c\,f'$. In finding the derivative of $\left(R^2 - r^2\right)$, treat R^2 as a constant. Its derivative is zero, and the derivative of $-r^2$ is $-2r$. Finally, $\frac{dv}{dr} = \frac{P}{4\eta\, l}(-2r) = -\frac{P\,r}{2\eta\, l}$ Granted it is difficult to make this meaningful without more knowledge in the field, but the process of finding the derivative is significant. All you have to agree to is that all letters are constants except r and then use $\frac{d}{dr} constant = 0$ and $\frac{d}{dr}(c\,f) = c\frac{df}{dr}$. Of course, you also have to be able to read all of the notation. The challenge will become easier with more experience.

Hopefully, the above comments help you to at least vaguely see some uses of derivatives as rates of change without going into all of the details. A common thread is to determine what each letter represents and to look at

$$\frac{\text{change of the dependent variable}}{\text{change of the independent variable}}$$

Coming back to this section later when you have more experience is also a good idea.

Now examples 1 and 8 in more detail.

EXAMPLE 1

Above ***Example 1*** is the statement that the particle is moving (back and forth) on a straight line. The particle does *not* move along the graph of

$$s = t^3 - 6t^2 + 9t$$

This equation just gives the position of the particle on the line. In the schematic in ***Figure 2***, the particle moves back and forth on the line, with the red path suggesting what happens *on* the line.

1. In ***part (b)***, $v(2) = -3\ m/s$. The negative sign means that the distance s is decreasing, or at $t = 2$, the particle is moving to the left. Why? $v = \frac{ds}{dt}$, and $\frac{ds}{dt}$ is approximately $\frac{\Delta s}{\Delta t}$ and this is negative. Starting with $t = 0$, then $t = 1$, $t = 2$, time t is increasing, and hence Δt is positive. So $\frac{\Delta s}{\Delta t}$, is negative only when Δs is negative. This means that s is decreasing (say $s = 2,$ then $s = 1$) or the particle is moving to the left.

2. ***Part (c)*** should possibly be reworded as, "when is velocity v equal to zero?" At least that is what is meant by "at rest." The answer, $v = 0$ if

$t = 1$ or $t = 3$, helps to explain the graph in ***Figure 2***. The t values where velocity is zero separate the negative velocities and the positive velocities and hence determines the direction of movement on the line.

3. Note that when $t = 5$, $f(5) = 20$, which is its position or location on the line. When $t = 0$, $f(0) = 0$. It is important to see why $20 - 0 = 20m$ is not the distance traveled in the first five seconds. The drawing in ***Figure 2*** indicates why the distance traveled must be greater than $20\,m$. Another plug for a visual interpretation.

EXAMPLE 8

This is the type of problem that is significant for business and economics majors. As a reminder,

$$\frac{\Delta C}{\Delta x} = \frac{\text{change in total cost}}{\text{change in number of items produced}}$$

The *number* of items and the *change* in that number are not the same. The key phrase in this problem is *marginal cost*. Note that the marginal cost is the limit of $\frac{\Delta C}{\Delta x}$. This means if a cost function is known, as in the middle of ***page 232***, then differentiation formulas can be used to find the derivative or marginal cost, $\frac{dC}{dx}$.

1. In the approximation involving n units, n represents a particular value of x. Letting $\Delta x = 1$ is a common technique in interpreting marginal cost. It is the cost of producing one more unit. $\frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \frac{\text{change in cost}}{\text{one unit}}$.

2. The approximation $C'(n) \approx C(n+1) - C(n)$ comes from the fact that the derivative, $C'(n)$, can be approximated by $\frac{\Delta C}{\Delta x}$ when Δx is small and

   $$\frac{\Delta C}{\Delta x} = \frac{C\big(n + \Delta x\big) - C(n)}{\Delta x} = \frac{C(n+1) - C(n)}{1} \text{ when } \Delta x = 1.$$

3. Because $C(n+1)$ is the cost of producing $n+1$ units and $C(n)$ is the cost of producing n units, then $C(n+1) - C(n)$ is the cost of producing the $(n+1)st$ unit.

4. In the middle of ***Example 8***, a particular cost function is given. *How* this type of function is derived is a topic in an economics or business course. Going from

$C(x) = 10{,}000 + 5x + 0.01x^2$ to the derivative, $C'(x) = 5 + 0.02x$ is easy once you are comfortable with the differentiation formulas. But the *meaning* doesn't come from the formulas, as we have shown above. Recognition of this fact provides a reason why you must know certain definitions and the meaning of the symbols used.

SECTION 3.8: EXPONENTIAL GROWTH AND DECAY

SOMETHING NEW: A DIFFERENTIAL EQUATION

The following is the key idea in this section.

The rate of change with respect to time *t* $\rightarrow\rightarrow\rightarrow$ $\dfrac{dy}{dt}$

is proportional to *y*. $\rightarrow\rightarrow\rightarrow$ equals a constant times *y*.

Thus

$$\frac{dy}{dt} = k\,y \text{ where } k \text{ is some constant.}$$

The last equation is called a *differential equation* which by definition is an equation that contains a derivative. To solve a differential equation one must find a *function* that satisfies the equation. We can readily verify that:

$$y = C e^{kt} \text{ where } C \text{ and k are constants}$$

is a solution to the equation. First find $\dfrac{dy}{dt}$. Remember to treat C and k as constants:

$$\frac{dy}{dt} = C\,\frac{d}{dt}e^{kt} = C e^{kt}\,k$$

Substitute both $\dfrac{dy}{dt}$ and y into the differential equation:

$$\frac{dy}{dt} = k\,y$$

becomes $$C e^{kt}\,k = C\left(k\,e^{kt}\right)$$

The two sides are equivalent to each other which demonstrates that $y = C e^{kt}$ is a solution of the equation.

WHAT HAPPENS WHEN *t* EQUALS ZERO?

Setting *t* equal to zero is a significant step in this type of problem.

$$e^{kt} \text{ becomes } e^{k\cdot 0} = e^{0} = 1$$

Then

$$C e^{kt} \text{ becomes } C e^{k\cdot 0} = C\cdot 1 = C.$$

Because we are multiplying by one when $t = 0$, in effect, e^{kt} vanishes.

The full story is $\qquad y(t) = C e^{kt}$

becomes $\qquad y(0) = C e^{k\cdot 0} = C$

The constant C is the value of the function when $t = 0$ and

$$y(t) = C e^{kt}$$

becomes $\qquad y(t) = y(0) e^{kt}$

The symbols $y(0)$ represent both a number and a process which is illustrated in the examples.

RELATIVE GROWTH RATE

Suppose that we discover that a population is growing at the rate of 50,000 per year. Next compare three situations.

1. The population is 50,000 and the growth rate is 50,000 per year. In one year, the population doubles or *increases* by 100%.

2. The population is 500,000 and the growth rate is 50,000 per year. In one year, the population increases by 10%.

3. The population is 5,000,000 and the growth rate is 50,000 per year. In one year, the population increases by 1%.

In each of the above situations, the growth rate is 50,000 per year; $\dfrac{dP}{dt} = 50{,}000$

The text defines the *relative* growth rate as the growth rate divided by the population size or $\frac{1}{P}\frac{dP}{dt}$. Remember that dividing by P is the same as multiplying by the reciprocal, $\frac{1}{P}$. In the second case above,

$$\frac{1}{500{,}000}\cdot 50{,}000 \;=\; .1 \;=\; 10\%$$

CHOICES YOU GET TO MAKE

In each of the examples in this section, time t is the independent variable. Imagine that time t is measured by a stopwatch and you decide when to push the button that starts the stopwatch. Pushing the button matches $t = 0$. But there are two variables, P and t. So look at the problem and ask, "What is the value of P when $t = 0$?" In Example 1, $t = 0$ matches 1950 when P = 2560. This choice makes the following algebra easier.

POINTS ON A GRAPH

A crucial factor in problem solving is organizing the given information. The functions used for population growth, $P(t) = P_0 e^{kt}$, connect a value of t to a value of P. Each pair is a point (t, P) on a graph. In Example 1, the pair or point (0,2560) was used to determine the value for P_0 and the pair (10,3040) was used to determine the value for k. In addition the exercise asked for estimates in 1993 and 2020. The points, (43,?) and (70,?) would match this situation. Now the function is used by dropping in 43 (and 70) for t to determine the corresponding values for P.

RADIOACTIVE DECAY

First note that the algebraic form of the function is the same except for using m instead of P. From a graphical point of view, the new words, half-life, refer to 2 points. Half-life is the time required for half of a given quantity to decay. Read time as a time interval which has two endpoints. In ***Example 2***, the half life is 1590 years. This matches the interval from $t = 0$ to $t = 1590$. In this length of time a sample having a 100 mg mass decreases to 50 mg. So the two points are (0,100) and (1590, 50). The first pair determines m_0 and the second pair determines the value for k. In effect, the half-life determines both constants in

$$m(t) \;=\; m_0 e^{kt}$$

The result, $$m(t) \;=\; 100\, e^{-(\ln 2/1590)t}$$

can now be used in two ways. Put in a t value to determine the corresponding mass as in part (b). The second way is to put in an m value to determine the corresponding time as in part (c).

One last comment. In ***Example 2***, the exponential base is changed from e to 2 by using the fact that $e^{\ln 2} = 2$. When using this form, $e^{\ln 2}$, there can not be anything between e and $\ln 2$. The form, $e^{3\ln 2}$, must be changed to $e^{\ln 2^3}$ or $\left(e^{\ln 2}\right)^3$.
Either form would lead to the answer 8. In ***Example 2***, rewrite $e^{-(\ln 2/1590)t}$ as $e^{\ln 2(-1/1590)t} = 2^{(-1/1590)t}$ which in turn equals the form in the text.

Continuously Compounded Interest

In the formula, $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ in ***Example 4***, A_0 is the amount of an investment at the annual interest rate r which is compounded n times per year. Then A is the value of the investment t years later. When $n \to \infty$ the number of times interest is paid each year gets larger and larger and we say that interest is *compounded continuously*. The derivation on ***page 242*** shows how that strange number e is involved. The limit, $\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m$ which equals e and substitution produces

$$A(t) = A_0 e^{rt}$$

where A_0 is the initial investment (matches $t = 0$), r is the annual interest rate, and t is the time in years.

It may seem strange that the number e is involved in computing the value of an investment after t years, but the key is to see how the exponents can be rewritten in line one of the derivation.

$$\lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt}$$

In the second form, multiply the exponents; $\frac{n}{r}\cdot rt$ equals nt, so the two forms are the same. Then the exponent, n/r is replaced by m. Note that $\frac{1}{m} = \frac{1}{n/r} = \frac{r}{n}$. The reason for all of this symbol manipulation is to get the form, $\left(1 + \frac{1}{m}\right)^m$, whose limit is e. On ***page 242***, note that the value of a $1,000 investment is $1,197.20 if interest is compounded daily and for continuous compounding the value is $1,197.22. There is just a 2 cent difference after a 3 year period. However, the formula, $A(t) = A_0 e^{rt}$ is more concise and easier to use.

Suppose we ask how long it takes for an investment A_0 to double in value if the interest rate is 8%. When it doubles, the value is $2A_0$.

Then $$A = A_0 e^{rt}$$

becomes $$2A_0 = A_0 e^{.08t}$$

Divide by A_0 $$2 = e^{.08t}$$

$$\ln 2 = \ln e^{.08t} = .08t$$

Solve for t $$t = \frac{\ln 2}{.08} = 8.66$$

The amount A_0 doubles in 8.66 years if the interest rate is 8%.

SUMMARY ON USING PAIRS OF VALUES OR THE COORDINATES OF POINTS

Because this is a significant idea in problem solving, we provide this summary.

1. We start with a function or an equation that contains constants.

   Consider the function, $y(t) = Ae^{kt}$.

2. Focus your attention on finding the constants, in this case A and k.

3. Put a pair of values into the function. The result is an equation in one or more unknowns. If there is only one unknown then solve the equation for that unknown.

4. Repeat step 3. Each pair of values leads to an equation.

5. If there is more than one unknown, shift your thinking to solving a system of equations; two equations in two unknowns, three equations in three unknowns, etc.

6. After finding a solution, the general function $y(t) = Ae^{kt}$ becomes a particular function, maybe $y(t) = 20e^{0.018t}$

7. Then use the function by putting in a given t value or by putting in a given y value.

When you read step 5 and have a sinking feeling that this is getting complicated you are understanding the process better. In most cases you will only have to deal with two equations in two unknowns. The topic of solving systems of equations is very broad and leads to a course called Linear Algebra.

SECTION 3.11: HYPERBOLIC FUNCTIONS

If you browse through this section and see all of the formulas you may feel overwhelmed. It may help to develop a general perspective before plunging in.

1. Note the definitions of $\sinh x$ and $\cosh x$ on ***page 259***. Why are trig forms used and why the "h" which is pronounced "hyperbolic"?

2. In ***Figure 6***, the point $P(\cos t, \sin t)$ is on the circle because of the basic trig identity, $\cos^2 t + \sin^2 t = 1$.

   This matches $x^2 + y^2 = 1$

3. In ***Figure 7***, the point $P(\cosh t, \sinh t)$ has a similar relationship to the hyperbola, $x^2 - y^2 = 1$.
   This accounts for the "h" in $\sinh t$ and $\cosh t$.

4. Many trig identities have a similar form using $\sinh t$ and $\cosh t$.

   Trig form: $\sin 2x = 2 \sin x \cos x$

   Hyperbolic form: $\sinh 2x = 2 \sinh x \cosh x$

5. The hyperbolic functions have "real life" applications as mentioned on ***page 259***.

Parts 3 and 4 have no meaning if you don't look at the algebraic justifications:

Drop $u = \sinh t = \dfrac{e^t - e^{-t}}{2}$ and $v = \cosh t = \dfrac{e^t + e^{-t}}{2}$

into $v^2 - u^2 = 1$.

The result on the left side is $\left(\dfrac{e^t + e^{-t}}{2}\right)^2 - \left(\dfrac{e^t - e^{-t}}{2}\right)^2$

Complete the algebraic steps and you will see that this simplifies to 1. The trig identity $\cos^2 x + \sin^2 x = 1$ corresponds to the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$.

This correspondence with trig identities can be extended by looking at ***3.11 Exercises, 11–16***. The forms are either exactly the same or they differ by a negative sign, a rather strange coincidence.

GRAPHS

The graphs of $\sinh x$, $\cosh x$, and $\tanh x$ are shown in ***Figures 1, 2,*** and ***3*** on ***page 259***. The negative sign in the definition of $\sinh x$ gives a shape similar to $y = x^3$ while the plus sign in the definition of $\cosh x$ keeps the graph above or at $y = 1$. The graph for $\tanh x$ stays between $y = -1$ and $y = 1$.

DERIVATIVES

In keeping with the similarities to trig functions, the derivatives on ***page 261*** are the same except for sign differences. So this part does not create any overload.

INVERSE HYPERBOLIC FUNCTIONS

However, now you may be back on overload. We concentrate on the derivatives in ***Box 6*** and compare with the inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1 + x^2}}$$

(These are the same except for the difference in one sign.)

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}}$$

(Under the radical sign $1 - x^2$ switches to $x^2 - 1$ and the quotient is not negative.)

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}$$

(These have the same form except one denominator has a plus sign while the other has a negative sign.)

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}} \qquad \frac{d}{dx}\left(\text{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1 - x^2}}$$

Under the radical sign $x^2 - 1$ switches to $1 - x^2$ and the quotient is negative.

Why the Comparison?

At this point you may feel that the multitude of forms before your eyes are a result of someone playing games. Later, however, in *integral* calculus, you will be performing an operation on some of these forms (which are derivatives) and will be wondering what the answers are. Then when you revisit these tables of derivatives they will make more sense.

Inverses and Logs

The relationship, $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

on ***page 262*** may seem like a very strange combination of symbols. We start with the unusual definition of $\sinh x$ and then consider an abstract idea, $\sinh^{-1} x$. Now a claim is made that this is a logarithm. Strange indeed. We just make a few comments on the proof in ***Example 3***.

After using the definition of $\sinh y$ the equation

$$e^y - 2x - e^{-y} = 0$$

appears. This is the same as:

$$e^y - 2x - \frac{1}{e^y} = 0$$

Multiply by e^y:

$$e^{2y} - 2x e^y - 1 = 0$$

The key is to recognize this equation as a quadratic, $e^{2y} = \left(e^y\right)^2$, and use the quadratic formula with $a = 1$, $b = -2x$, $c = -1$. The result

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

simplifies to: $e^y = x \pm \sqrt{x^2 + 1}$

Because e^y is always positive we can drop the negative sign. Then take the natural log of each side and we get the result:

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Finally, remember that y equals $\sinh^{-1} x$ and:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

In ***Example 4***, this relationship is used as an alternative way of showing that the derivative of $\sinh^{-1} x$ is $\dfrac{1}{\sqrt{x^2 + 1}}$. So even though the relationship may look strange, everything fits together.

ONE LAST PLUG ON HOW THINGS FIT TOGETHER

We conclude this section by finding the derivative of $\ln(\sec x + \tan x)$:

If $y = \ln(\sec x + \tan x)$

then $\dfrac{dy}{dx} = \dfrac{1}{\sec x + \tan x} \sec x \tan x + \sec^2 x$

Note that $\sec x$ is a common factor: $\dfrac{dy}{dx} = \dfrac{1}{\sec x + \tan x} \sec x(\tan x + \sec x)$

and then $\dfrac{dy}{dx} = \sec x$

Two comments:

1. Later you will want to memorize this result.

$$\frac{d}{dx} \ln(\sec x + \tan x) = \sec x$$

2. Compare this derivative with the derivative in ***Example 5***. Because $\ln(\sec x + \tan x)$ and $\tanh^{-1}(\sin x)$ have the same derivative they must be equal? You can show that this is the case by using (from ***Box 5***)

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Replace x with $\sin x$, multiply by $\dfrac{1+\sin x}{1+\sin x}$ and simplify.