Module 4

Section 8.1: Arc Length

In ***chapter 5***, the concept of a definite integral was introduced as the limit of a sum of products.

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Taking away some of the details, the general idea in this definition can be conveyed by

$$\int_a^b f(x)\,dx \;=\; \lim \sum \; f(x_i)\,\Delta x$$

to emphasize

1. a limit that gives exact values
2. $\sum$ for a sum of
3. the products, $f(x_i)\,\Delta x$.

We now use these ideas to show that the length of a curve can be expressed as an integral.

Overview

A visual picture of the method used to find the length of a curve is shown in ***Figure 4, page 544***. In the top drawing, the length of the blue curve is approximated by the length of the red line segment. Then in the second drawing the *two* red line segments give a better approximation. Continuing the process, *three* and then *four* red line segments give better and better approximations to the length of the blue curve. It seems reasonable then to *define* the length of the curve as the end result of this process, which is a particular limit.

The Details

The harder part is to follow the algebraic description of this process. First we use the distance formula

$$\sqrt{(x_i \;-\; x_{i-1})^2 \;+\; (y_i \;-\; y_{i-1})^2}$$

to represent one line segment. For convenience this is written

$$\sqrt{(\Delta x)^2 \;+\; (\Delta y)^2}$$

Then the key algebraic step is to replace Δy_i with $f'(x_i{}^*)\Delta x_i$. The justification for this is the Mean Value Theorem way back on ***page 288***. In this case,

$$f(b) - f(a) = f'(c)\,(b - a)$$

becomes $$y_i \;-\; y_{i-1} = f'(x_i^*)(x_i \;-\; x_{i-1})$$

which can be written $$\Delta y \;=\; f'(x_i^*)\Delta x.$$

After the substitution, $(\Delta x)^2$ is a common factor.

$$\sqrt{(\Delta x)^2 \;+\; \left(f'(x_i^*)\right)^2 (\Delta x)^2}$$

$$= \sqrt{(\Delta x)^2\left(1 \;+\; \left[f'(x_i^*)\right]^2\right)}$$

$$= \Delta x \sqrt{\left(1 \;+\; \left[f'(x_i^*)\right]^2\right)}$$

After all of this algebraic effort, it is important to remember that the last line represents just one red line segment. We then sum *n* of these line segments,

$$\sum_{i=1}^{n} \sqrt{\left(1 \;+\; \left[f'(x_i^*)\right]^2\right)}\;\Delta x$$

and think of taking a limit by letting n approach infinity. The result,

$$\lim \sum_{i=1}^{n} \sqrt{\left(1 \;+\; \left[f'(x_i^*)\right]^2\right)}\;\Delta x$$

is a limit of the sums of products and can be represented by the integral

$$\int_a^{-b} \sqrt{1 + [f'(x)]}\;^2\;\, dx$$

Yes, this formula for the length of a curve is a bit complex, but it can be an important tool. Because of the square root, expect some difficulty in carrying out the integration.

THE ARC LENGTH FUNCTION

We now switch gears a bit and consider a special function, *s(x),* developed on ***page 547***.
First note that *s(x)* , defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^{\,2}}\; dt$$

represents the length of a curve from $t = a$ to $t = x$. Starting with the fixed point, $(a, f(a))$, on the curve, x varies as the length of the curve varies, so that *s(x)* represents this varying length. Hold this thought in your mind and refer to it when confusion arises.

As a review of *derivatives of integrals,* consider

$$F(x) = \int_1^x t^3\, dt\,,$$

Integrating, we have $$F(x) = \left.\frac{t^4}{4}\right|_1^x = \frac{x^4}{4} - \frac{1}{4}\,.$$

Hopefully you aren't surprised by the fact that the derivative of $F(x)$ is

$$F'(x) = \frac{4x^3}{4} = x^3$$

Differentiation reverses the process of integration. In other words the derivative of the integral is the integrand with a change of variable.

For the arc length function, *s(x)* , the derivative is

$$\frac{ds}{dx} = \sqrt{1 + \,[f'(x)\,]^{\,2}}$$

The right side is just a formula, which can be fairly easily calculated. If

$$f(x) = \sin x$$

then

$$\sqrt{1 + \,[f'(x)\,]^{\,2}} = \sqrt{1 + \,cos^2 x}$$

But what is the meaning of writing

$$\frac{ds}{dx} = \sqrt{1 + \cos^2 x}\ ?$$

Recall that

$$\frac{ds}{dx} \approx \frac{\Delta s}{\Delta x}$$

where Δs is the change in the arc length that corresponds to Δx , the change in x. Then $\frac{\Delta s}{\Delta x}$ is the average rate of change of arc length with respect to x, and $\frac{ds}{dx}$ is the instantaneous rate of change. More on this later.

THE DIFFERENTIAL OF ARC LENGTH

The text includes an interesting (and significant) relationship between the differentials *ds, dx,* and *dy*.

$$(ds)^2 = (dx)^2 + (dy)^2$$

This follows from

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}$$

through the following steps:

$$\frac{ds}{dx} = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}$$

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx$$

$$ds = \sqrt{\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}}\ dx$$

Now bring the rightmost dx under the square root by multiplying both terms under the radical sign by $(dx)^2$. After multiplying

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

and $$(ds)^2 = (dx)^2 + (dy)^2 .$$

With the Pythagorean Theorem in mind, the right triangle in ***Figure 7*** provides an easy way to remember the relationship.

Why is this significant? Formulas **(3)** and **(4)** on ***pages 545*** and ***546*** can be written in the simpler form,

$$s = \int ds$$

with $$ds = \sqrt{(dx)^2 + (dy)^2} \ .$$

The suggestion is to think in terms of differentials to find arc

length. ***Example 2,***

Given $\quad y^2 = x, \quad$ find ds.

First $\quad 2\,y\,dy = dx$

Then
$$\begin{aligned} ds &= \sqrt{(2\,ydy)^2 + (dy)^2} \\ &= \sqrt{\left[4y^2 + 1\right](dy)^2} \\ &= \sqrt{\left[4y^2 + 1\right]}\; dy \end{aligned}$$

Finally $\quad s = \int ds$

becomes $\quad s = \int \sqrt{\left[4y^2 + 1\right]}\; dy$

This the same integral found in ***Example 2***.

At this point you may not be convinced that the formula for ds is significant, but in the next section formulas become more complicated. Then the value of the differential ds will become more apparent.

THE RATE OF CHANGE $\frac{ds}{dt}$

Earlier, we indicated that $\frac{d\,s}{d\,x}$ is the rate of change of arc length with respect to x. In a similar manner, $\frac{ds}{dt}$ is the rate of change of arc length with respect to time t. Now we are comparing a change in the length of a curve to the corresponding change in time. Under certain conditions this differential is the speed of an object as it moves along a curve. Before we could only consider motion along a line. Now we have a tool that indicates how fast we are moving on a curve.

We go back to the equation,

$$(ds)^2 = (dx)^2 + (dy)^2$$

and divide by $(dt)^2$.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

The algebraic change is easy enough, but for dx/dt and dy/dt to have meaning, both x and y must be functions of time *t*. This is equivalent to saying that we must be able to indicate the time t that the point *(x,y)* will be at a particular location. The types of equations involved here are called *parametric* equations. (See ***pages 640***, ***824***, and ***849***). These equations are a separate topic, but knowing that $\frac{ds}{dt}$ is the speed on a curve hopefully provides a ground for solving equations that sometimes seem just abstract entries.

SECTION 8.2: AREA OF A SURFACE OF REVOLUTION

It is important to note that in this section we are concerned with outer surface *area* of a solid, not its volume. The volume is the number of cubic units that would fit inside, whereas the area is the number of square units required to cover the outer surface.

Figure 1, page 551, clearly illustrates the rectangle that results from making the indicated cut and flattening the outer layer. The length of the rectangle is $2\pi r$, the circumference of the circular base and the area is $2\pi r h$.

Next, without going into the details of the derivation, we note that the lateral area of the frustum of the cone shown in ***Figure 3,*** has a similar algebraic form. The area $2\pi r l$ is shown as **(2)** on ***page 552***. The cylinder's height is *h*, while for the frustum, *l* is the slant height. Treat this similarity as a rather surprising result that makes it easier to remember.

An even more surprising result is the formula ***(7)*** for the surface area on ***page 547***.

$$S = \int 2\pi y \, ds$$

Remember that the integral sign represents a limit of a summation. In this case, we are summing frustums of a cone, shaded red in ***Figure 4(b)***. After taking a limit (theoretically), the form, $2\pi y\, ds$, appears. So you can look at the following as an aid in remembering the form for the area of a surface of revolution.

Cylinder	$2\pi r h$	
Frustum of a cone	$2\pi r l$	
Surface Area	$\int 2\pi y\ ds$	Rotation about x-axis
	$\int 2\pi x\ ds$	Rotation about y-axis

$$(ds)^2 = (dx)^2 + (dy)^2$$

For the rotation about the x-axis, in ***Figure 5(a)***, note that y is the radius of a circle. For a rotation about the y-axis, in ***Figure 5(b)***, note that x is the radius of a circle. This type of visualization is an important aid in understanding why x or y should be used.