Module 8

SECTION 4.3: HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH

We start with a brief summary of this section.

1. A function is increasing when the derivative is positive. In a graph the curve is rising.

2. A function is decreasing when the derivative is negative. In a graph the curve is falling.

3. Use the First Derivative Test to find a local maximum and local minimum.

The algebraic definition of an increasing function on an interval I (***page 19***):

$$f(x_1) \; < \; f(x_2) \text{ whenever } x_1 \; < \; x_2 \text{ in I}$$

is needed to prove algebraically that:

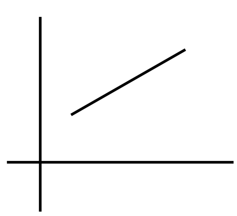
$$\text{if } \; f'(x) > 0 \; \text{ for all } x \text{ in I, then } \; f \; \text{ is increasing on I.}$$

The interval I is also represented by (a,b).

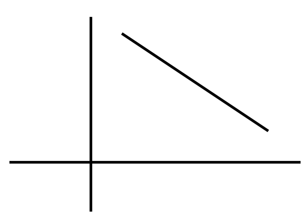
Same comments for f is decreasing and the derivative is negative.

GRAPHICAL INTERPRETATION

We now look at a graphical interpretation but use an approach that differs from the text. When the slope of a line was first introduced, a distinction was made between lines with a positive slope and ones with a negative slope:

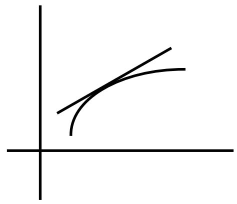


Positive slope

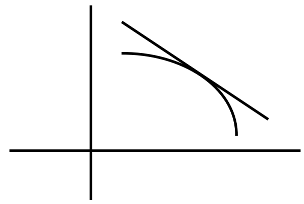


Negative slope

Next consider the tangent line to a curve:



Positive slope

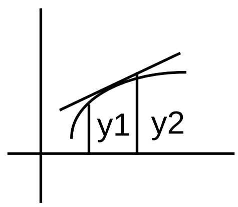


Negative slope

↑ ↑

Then note the curve is rising and is falling.

Finally, associate the curve is rising with *y* is increasing.



Positive slope

The *y* values are increasing because $y_2 > y_1$. [y2 = y_2 and $y1 = y_1$]

The same argument can be made for *y* is decreasing.

THE FIRST DERIVATIVE TEST

The important point here is to look at ***Figure 3, Sec. 4.3***, as you read the First Derivative Test. By looking at Figure 3(a) you will find that part (a) of the First Derivative Test is a description of the graph. The same is true in Figure 3(b) and part (b). Match Figure 3(c) and 3(d) with part (c) of the First Derivative Test. This is very important in seeing the First Derivative Test as describing a graphical situation and to make it more intuitive.

HOW TO USE THE FIRST DERIVATIVE TEST

The text follows the pattern of most calculus books by constructing elaborate tables, which you can find ***in this section***. The reason for a table is to illustrate the three parts of the First Derivative Test. However, this is a cumbersome procedure. Instead, consider the following.

1. Find the domain of the function.

2. When possible find the x intercepts. Put $y = 0$ into the function and try to solve for x.

3. Find the y intercept. Put $x = 0$ into the function and solve for y.

4. Find the critical numbers.

    a. Set the first derivative equal to zero and solve for x.

    b. Find any x values where the first derivative is undefined.

5. Each critical number is an x value. For each critical number find the corresponding y value. Then you will have points that you can plot on a graph. This is a crucial step in constructing a graph using ideas from calculus.

6. Go back to step 4 to check when the slope of the tangent line is zero (possible max and min points) and when the slope is undefined.

7. If parts of the graph are not clear, plot s*elected* points to gain clarification.

After completing step 5, connect the points where the slope is zero and the x and y intercepts. Frequently this will give a good sketch of the graph. From the graph you will be able to pick out the maximum and minimum values and the intervals where the function is increasing or decreasing. Contrast this approach with the use of tables shown in the text and decide which you prefer.

DOMAINS AND TYPES OF FUNCTIONS

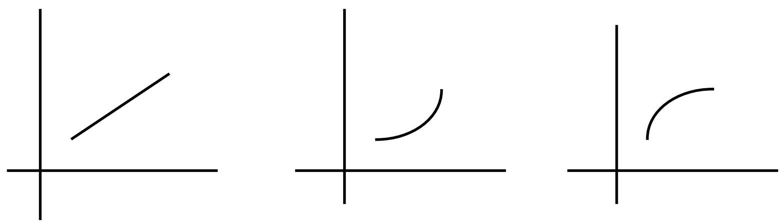
As an aid in graphing we include a brief review of the domains of selected types of functions:

Type	**Domain**
1. Polynomial	1. All real numbers — no limitation
2. Rational	2. Denominators can not equal zero
3. Algebraic	3. If a square root is present the radicand must be positive or zero. #2 may also apply. For cube roots there is no limitation except for #2.
4. Exponential	4. All real numbers — no limitation
5. Logarithmic	5. Only positive real numbers

In 5, the given domain is for the basic logarithmic functions. A translation would change the domain.

CONCAVITY AND POINTS OF INFLECTION

We now look at more subtle distinctions in the graphs of various functions. In the graphs below we start with a rising (increasing) line. Once the direction is set the path doesn't waver to the right or to the left. Straight ahead but stay on the line. Then in the middle graph we veer left and in the third graph we veer right.



Is there a tool in calculus that makes a distinction between these three situations? The answer is yes and the tool is the second derivative. To explain the significance of the second derivative we use a different approach than the text. Not better just different:

First remember, that if $\frac{dy}{dx} > 0$ then y is increasing (A)

and if $\frac{dy}{dx} < 0$ y is decreasing. (B)

Now think of the first derivative as the slope of the tangent line and let:

$$\frac{dy}{dx} = m$$

Because the second derivative is the derivative of y' or $\frac{dy}{dx}$

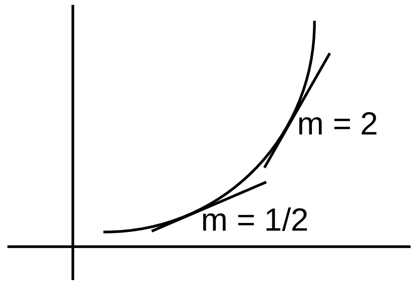
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dm}{dx}$$

The second derivative indicates the rate of change of the slope of the tangent line. Using (A) and (B) above and replacing y with m we have:

if $\frac{dm}{dx} > 0$ then m is increasing (C)

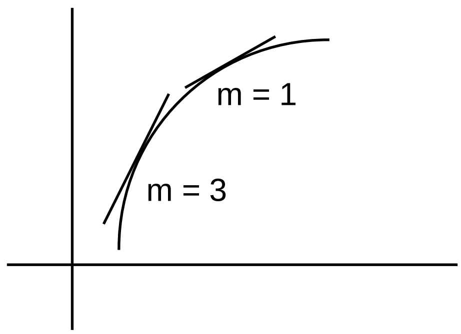
and if $\frac{dm}{dx} < 0$ m is decreasing (D)

To interpret (C) consider the following graph.

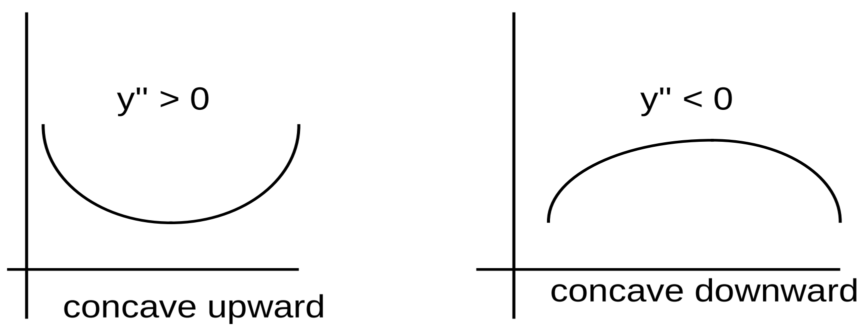


Note that the *slopes* are increasing. The curve is *concave upward* and y'' is positive.

To interpret (D) consider a similar graph:



Now the *slopes* are decreasing. The curve is *concave downward* and y'' is negative. The above graphs can be extended and the above reasoning is still valid.



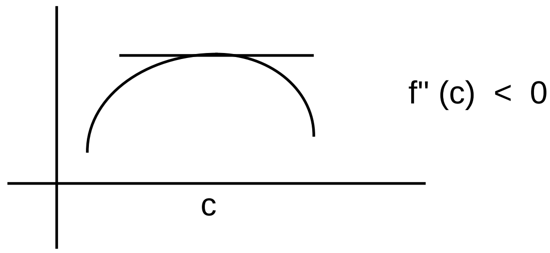
POINTS OF INFLECTION

A point of inflection separates a concave upward part of a graph from a concave downward part. Note also that it must be a point on the curve. In many cases the second derivative will be zero at a point of inflection, but not always. The second derivative can be also zero at $x = c$ and the point $(c, f(c))$ is not a point of inflection.

Having pointed out the exceptions, however, a reasonable procedure is to set the second derivative equal to zero, solve the equation, and look at a graph for guidance.

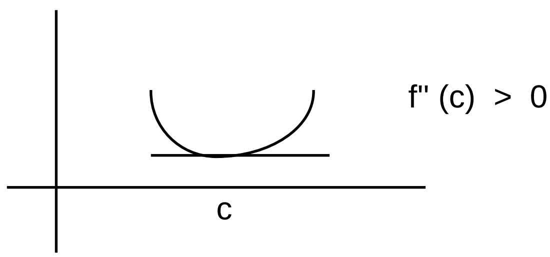
THE SECOND DERIVATIVE TEST

The second derivative can also be used to determine if a critical number *c*, where $f'(c) = 0$, produces a maximum or a minimum. For a maximum:



the curve is concave downward and $y'' < 0$. After finding the critical number *c*, put it in the second derivative. If $f''(c)$ is negative then a local maximum exists at $x = c$.

For a minimum the curve is concave upward and $y'' > 0$:



Again, after finding the critical number *c* put it in the second derivative. If $f''(c)$ is positive then a local minimum exists at $x = c$.

The Second Derivative Test is presented as a possibility. In most cases the points where a maximum or a minimum occurs can be read from a graph.

Because of the emphasis on graphing in this assignment, answers on ***page A82*** almost read like a story. Each description of an accompanying graph is a concise lesson on increasing, decreasing, maximum, minimum, and concavity which we hope is clear. To determine the numbers in the graph check the critical numbers of the given function where y' equals zero or is undefined and the *x*-values where y'' equals zero. Five graphs display *x*-values where a function is *not* differentiable.

SECTION 4.4: INDETERMINATE FORMS AND L'HOSPITAL'S RULE

We now consider some limits that will be important in later work. A new tool for finding limits is presented that simplifies the process. It is not a completely general tool but does apply to limits that are troublesome.

L'HOSPITAL'S RULE

L'Hospital's Rule states that under certain conditions one quotient can be traded for another and the limits will be the same:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

The conditions are:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are called *indeterminate* forms. By just looking at $\frac{0}{0}$ one can not determine what it equals. The same is true for $\frac{\infty}{\infty}$. Especially in the second case, don't conclude that $\frac{\infty}{\infty} = 1$. It may be true, but in most cases the quotient will not be one. The paragraph below 2 on ***page 305*** contains a good description of the interaction between the numerator and the denominator.

USING L'HOSPITAL'S RULE

First read Notes 1 and 2. L'Hospital's Rule also applies when $x \to a$ is replaced by $x \to \infty$.

$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = \lim_{x\to \infty} \frac{f'(x)}{g'(x)}$$

However, make sure that the limit is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before using L'Hospital's Rule. Also note that $\frac{f(x)}{g(x)}$ is *not* treated as a quotient when finding derivatives. Find the derivatives of the numerator and denominator as separate functions.

TYPES OF FUNCTIONS

Each time L'Hospital's Rule is used a quotient will be involved. As noted on ***page 305***, there is a struggle between the numerator and the denominator. Who wins the contest depends on the types of derivatives.

For a polynomial function like $x^3 - 4x^2$ the derivative is $3x^2 - 8x$. A cubic becomes a quadratic; the degree is reduced by one. Contrast that with e^x whose derivative is e^x. The function e^x will win any contest with a polynomial function.

EXAMPLE 1

$$\lim_{x\to\infty} \frac{x^3 - 4x^2}{e^x} = \lim_{x\to\infty} \frac{3x^2 - 8x}{e^x} = \frac{\infty}{\infty}$$

Use L'Hospital's Rule a second time: $= \lim_{x\to\infty} \frac{6x - 8}{e^x} = \frac{\infty}{\infty}$

Once more: $= \lim_{x\to\infty} \frac{6}{e^x} = 0$

The denominator approaches infinity so the fraction approaches 0.

EXAMPLE 2

$$\lim_{x\to\infty} \frac{e^x}{x^3 - 4x^2} = \lim_{x\to\infty} \frac{e^x}{3x^2 - 8x} = \frac{\infty}{\infty}$$

Use L'Hospital's Rule a second time. $= \lim_{x\to\infty} \frac{e^x}{6x - 8} = \frac{\infty}{\infty}$

Once more: $= \lim_{x\to\infty} \frac{e^x}{6} = \infty$

The numerator approaches infinity so the fraction also approaches infinity.

In both examples, the polynomial progresses from a cubic to a quadratic to a linear form to a constant, while e^x does not change. Keep this pattern in mind as you look at other functions.

In ***Example 3***, the function $\sqrt[3]{x}$ behaves differently than a polynomial function. One derivative produces a negative exponent and then the base appears in the other part (numerator or denominator) of the fraction:

EXAMPLE 3

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^2}}{x^2 + 3x} = \lim_{x\to\infty} \frac{\frac{2}{3}\, x^{-\frac{1}{3}}}{2x + 3}$$

Rewrite: $$\lim_{x\to\infty} \frac{2}{x^{\frac{1}{3}}(2x + 3)} = 0$$

The limit is zero because the denominator approaches infinity.

Also, the log function, $\ln x$, is in ***Example 3***. This function will lose most contests with the functions covered above because its derivative is $\frac{1}{x}$ or x^{-1} which causes a shift.

EXAMPLE 4

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

EXAMPLE 5

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

In both cases, the simple polynomial, x, determines the outcome. A partial list of "weakest" to "strongest" functions is:

$$\ln x \;,\; \sqrt[3]{x} \;,\; x \;,\; x^2 \;,\; x^3 \;, \;\; , e^x$$

Any function in this list will dominate a function to its left.

Indeterminate Products

If a limit has the form, $0 \cdot \infty$, no conclusion can be drawn. One factor is getting smaller and smaller while the other factor is getting larger and larger. While we know that zero times any finite number is zero, this may NOT be the case for $0 \cdot \infty$. To determine the limit convert to a quotient whose form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example: $\lim_{x\to-\infty} x e^x = -\infty \cdot 0$

Rewrite as: $\lim_{x\to-\infty} x e^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \frac{-\infty}{\infty}$

Use L'Hospital's Rule: $= \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0$

Indeterminate Powers

The types here can be 0^0, ∞^0, or 1^∞. ***Examples 8*** and ***9*** illustrate the method. A power A^B is changed to $\ln A^B = B \ln A$ which in turn is changed to a quotient whose form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then use L'Hospital's Rule. If this limit is L then the answer is e^L.

Example 9, page 307

$\lim_{x\to0^+} \left(x^x\right) = 1$. The following example illustrates a variation of this form.

Example: Find $\lim_{x\to0^+} (3x)^{2x}$.

If $y = (3x)^{2x}$ then $\ln y = 2x \ln(3x)$ and

$$\lim_{x\to0^+} \ln y = \lim_{x\to0^+} 2x \ln(3x) = 0\cdot(-\infty)$$

Rewrite as a quotient: $= \lim_{x\to0^+} \frac{2\ln 3x}{x^{-1}} = \frac{-\infty}{\infty}$

By L'Hospital's Rule: $= \lim_{x\to0^+} \frac{2\,\frac{3}{3x}}{(-1)x^{-2}} = \lim_{x\to0^+} (-2x) = 0$

we have shown that $\lim_{x\to 0^+} \ln y = 0$.

Then $\lim_{x\to 0^+} y = e^0 = 1$

comparing this result $\lim_{x\to 0^+} (3x)^{2x} = 1$

with ***Example 9*** $\lim_{x\to 0^+} x^x = 1$

the 3 in the base and the 2 in the exponent do not change the limit.

SECTION 4.5: SUMMARY OF CURVE SKETCHING

To a great extent, we have addressed general curve sketching techniques in ***Section 4.3*** and now present a summary.

SYMMETRY

In ***Guidelines for Sketching a Curve***, the text lists steps A through H that provide useful information for constructing a graph. In step C, the idea of symmetry is discussed. Note the examples of *even* functions given in ***Step C.*** Then do the same for *odd* functions on the same page. All are simple functions that can be checked easily by looking at $f(-x)$. For more complicated functions, the check is time-consuming. A shortcut based on understanding the significance of odd and even as it applies to exponents can be useful. Then a glance at $x^6 + x^4 + x^2 + 7$ indicates that it is even while $x^5 + x^3 + x$ is odd. Also $x^5 - x^4$ and $x^3 - 1$ are neither odd nor even. This shortcut can be extended to $\dfrac{x^2}{1 + x^2}$ and $\dfrac{x}{x^5 - x^3}$.

Symmetry will also appear when finding critical numbers and the corresponding *y* values. So we didn't completely ignore this concept in the method outlined under the heading "How to Use the First Derivative Test" early in this module's lecture (above).

The periodic nature of trig functions is significant and worth reviewing on ***pages A30-A31*** in ***Appendix D***.

ASYMPTOTES

Asymptotes are a significant part of a graph.

1. Horizontal Asymptotes. These are found by finding a limit as *x* approaches $\pm\infty$. L'Hopital's Rule makes this task much easier.

2. Vertical Asymptotes. These are tied in with the domain of a function. Most vertical asymptotes will be found by determining when a denominator is zero.

3. Slant Asymptotes. Note on ***page 320*** that a slant asymptote exists for a rational function when the degree of the numerator is more than or equal to the degree of the denominator. Then a long division can be performed as shown in ***Example 6***.

Another example: $f(x) = \dfrac{x^2 - 1}{x + 2}$

After a long division, $f(x) = x - 2 + \dfrac{3}{x+2}$.

As x approaches $\pm\infty$, $\dfrac{3}{x+2}$ approaches zero. The slant asymptote is $y = x - 2$

because the function $f(x) = x - 2 + \dfrac{3}{x+2}$

gets closer and closer to $f(x) = x - 2$

as x approaches $\pm\infty$.

A similar situation occurred in ***Sec. 3.11*** in the graphs of $y = \sinh x$ and $y = \cosh x$. As

x approaches ∞,

$\dfrac{e^x \pm e^{-x}}{2}$ approaches $\dfrac{e^x}{2}$, as shown in ***Figures 1*** and ***2*** of that section.

In this case a curve is the asymptote instead of a line. The term, *slant asymptote* applies only to lines, but the idea is the same.

A SECOND APPROACH

At this point, we again list steps for sketching a graph from ***Section 4.3***. We have added two steps to include asymptotes and the second derivative.

1. Find the domain of the function.

2. When possible find the x intercepts. Put $y = 0$ into the function and try to solve for x.

3. Find the y intercept. Put $x = 0$ into the function and solve for y.

4. Find vertical and horizontal asymptotes. When appropriate, find a slant asymptote.

5. Find the critical numbers.

    a. Set the first derivative equal to zero and solve for *x*.

    b. Find any *x* values where the first derivative is undefined.

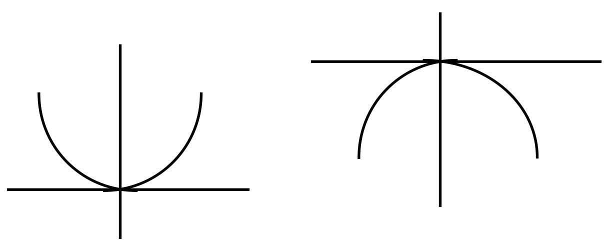
6. Each critical number is an *x* value. For each critical number find the corresponding *y* value. Then you will have points that you can plot on a graph. This is a crucial step in constructing a graph using ideas from calculus.

7. Go back to step 5 to check when the slope of the tangent line is zero (possible max and min points) and when the slope is undefined.

8. Look for points of inflection by setting y'' equal to zero and solving for *x*. All claims must agree with your graph.

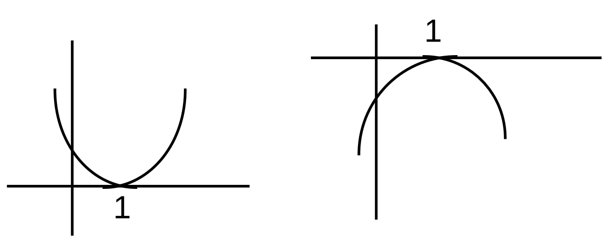
9. If parts of the graph are not clear, plot s*elected* points to gain clarification.

We add one last tidbit that is sometimes useful. The graphs of $y = x^2$ and $y = -x^2$ are shown below:



Note that both are tangent to the *x*-axis; x^2 is always positive or zero. The same is true for:

$$y = (x-1)^2 \text{ and } y = -(x-1)^2$$



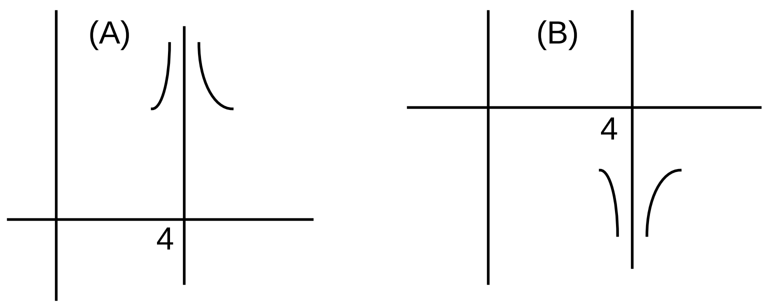
Because the factor $(x - 1)$ is squared, the curves are tangent to the *x*-axis. The reason for making this observation is that the same is true for:

$$y = x(x - 4)^2(x + 3)$$

and $$y = \frac{x(x - 4)^2}{x + 3}$$

The graph of each of these functions will be tangent to the *x*- axis at $x = 4$. Why? Consider the interval $[3.9,\ 4.1]$ and movement from left to right in this interval. The factor $(x - 4)^2$ will be positive or zero throughout the interval. When $x = 4$, the curve will *touch* the *x*-axis but will not cross it.

If the factor $(x - 4)^2$ is present in the denominator, there will be a vertical asymptote at $x = 4$, and the graph will behave like (A) or (B) below:



If we stay close to $x = 4$ then either *y* is positive on both sides of $x = 4$ or *y* is negative on both sides of $x = 4$. The sign depends on the other parts of the function.

SECTION 4.6: GRAPHING WITH CALCULUS AND CALCULATORS

This is an optional section depending on whether or not you have a graphing calculator. The functions in this section are selected to demonstrate the capabilities of a graphing calculator. To get the most benefit from this section, duplicate the steps in the text, especially those involving the boundaries on the viewing window.

GRAPHING DERIVATIVES

In ***Example 1***, it is difficult to locate the local max and min values from the graph in ***Figure 1***. With a graphing calculator it is easy to graph the first derivative as shown in ***Figure 3***. Then recall that the first derivative is zero at a local max or min. So in ***Figure 3***, look for the *x* values where the graph crosses the *x*-axis. These are estimated to be at $x = -1.62,\ 0$ and 0.35 and then checked by using an appropriate viewing window. The major point here is that ***Figure 3*** contains the graph of $y = f'(x)$, not the function $y = f(x)$, and one must adjust one's thinking accordingly.

To find points of inflection, the graph of the second derivative is shown in ***Figure 5***. Now the two crossings of the *x*-axis produce values of *x* where $f''(x) = 0$ and *possibly* lead to points of inflection.

It may be of value to duplicate these steps for a function like:

$$f(x) = 2x^3 - 3x^2 - 12x$$

and verify that the graph of $y = f'(x)$ crosses the *x* axis at $x = -1$ and at $x = 2$ and that the graph of $y = f''(x)$ crosses the *x* axis at $x = \frac{1}{2}$.

Because of the ease of graphing with a graphing calculator, information can be found in different ways.