Module 5

Section 3.4: The Chain Rule

We have noted that the Power Rule can be applied to x^4 , x^{-2}, and $x^{3/4}$, where each base is *x*. But if the base is, say, $x^3 + 1$ as in $\left(x^3 + 1\right)^2$, the same pattern will not give a correct answer. The derivative of $\left(x^3 + 1\right)^2$, is not $2\left(x^3 + 1\right)^1$. If we expand $\left(x^3 + 1\right)^2$ to $x^6 + 2x^3 + 1$ and then find the derivative, the result is $6x^5 + 6x^2$. The results are not the same. The tool that we need is called the Chain Rule.

Composite Functions

If $y = f(u) = u^2$ and $u = g(x) = x^3 + 1$, then we have a situation where the value of *y* depends on the value of *u* and in turn the value of *u*, depends on the value of *x*. This can be thought of as a *chain* reaction.

First consider $y = f(u) = u^2$, and take a derivative with respect to *u*. We might write $y' = f'(u) = 2u$, but this doesn't give as much information as

$$\frac{dy}{du} = \frac{df}{du} = 2u.$$

(Why we are using both the y and $f(u)$ notation will be explained below.) Next consider $u = g(x) = x^3 + 1$, and now take a derivative with respect to *x*.

$$\frac{du}{dx} = \frac{dg}{dx} = 3x^2$$

Since the *y* value depends on the *u* value and the *u* value depends on the *x* value, it is also true that the *y* value depends on the *x* value. (The *u* variable is just an intermediary.) Then, $\frac{dy}{dx}$, the rate of change of *y* with respect to *x*, also has meaning. But how do we determine this derivative? The answer is provided by the Chain Rule in a fairly straightforward manner.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

In other words, we just multiply the two derivatives, $\frac{dy}{du}$ and $\frac{du}{dx}$. So in the above situation, $\frac{dy}{dx} = 2u \cdot 3x^2$. This doesn't seem clear because we have two variables, so replace u with $x^3 + 1$

and $\dfrac{dy}{dx} = 2(x^3 + 1) \cdot 3x^2 = 6x^5 + 6x^2.$

Going back to $y = f(u) = u^2$ and $u = g(x) = x^3 + 1$, we can find the composite function, $f[g(x)]$, which is $(x^3 + 1)^2$. Multiplying this out as we did above, we get $x^6 + 2x^3 + 1$, and taking a derivative produces the same result, $6x^5 + 6x^2$, as for $\dfrac{dy}{dx}$ above.

PROOF OF THE CHAIN RULE

The last example seems to confirm that the Chain Rule is true, but one example does not constitute a proof. The proof on is straightforward except for one flaw mentioned at the end. We suggest that you read it, but we will direct our comments to notation and the use of the Chain Rule.

NOTATION AGAIN

The Chain Rule is expressed in two forms in the text.

1. If $y = f(u)$ and $u = g(x)$,
   then $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

2. If $F(x)$ is a composite function defined by $F(x) = f[g(x)]$,
   then $$F'(x) = f'[g(x)]g'(x).$$

The second form is harder to read, but it more closely matches the problems you will encounter. To show that the two forms are the same, we rewrite the first statement.

Since $y = f(u)$ and $u = g(x)$,

$\dfrac{dy}{du} = f'(u) = f'[g(x)]$ We have replaced u with $g(x)$.

Also $$\frac{du}{dx} = g'(x)$$

Then $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

becomes $\quad \dfrac{dy}{dx} = f'[g(x)]g'(x)$, which matches the second form of $y = F(x)$.

THE POWER RULE: LAST VERSION

In the preceding sections, the Power Rule could be used for different types of exponents, but the base had to be a single letter. We can now extend the Power Rule so that the base can be a *function* instead of a single letter.

If $\quad y = u^n \quad$ and $\quad u = g(x) \quad$ then $\quad y = [g(x)]^n$

and $\quad \dfrac{dy}{du} = nu^{n-1} \qquad \dfrac{du}{dx} = g'(x)$

By the Chain Rule,

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= nu^{n-1}\, g'(x)$$

Replace u with $g(x)$

$$= n[g(x)]^{n-1}\, g'(x)$$

In summary,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\, g'(x)$$

On the left side, $g(x)$ is the *base* and n is the exponent.
On the right side, note that there are three steps.

1. The exponent n becomes a coefficient.
2. Reduce the exponent by one, DO NOT change the base.
3. Multiply by the derivative of the base.

We state these three steps in verbal form because of the importance of this last version of the Power Rule. Since the base can now be a function, you will encounter more complex functions. It is very important to include all three steps, especially the third one.

USING THE CHAIN RULE

It is important to recognize that the Chain Rule is a general rule and the Power Rule is a particular application of the Chain Rule. There will be five or six other significant applications of the Chain Rule in this course. A second application is discussed in *The Chain Rule and Trig Functions* below.

We now look at six powers that illustrate the use of the Power Rule.

1. $$\frac{d}{dx}\left(x^3 + 5x^2\right)^4 = 4\cdot\left(x^3 + 5x^2\right)^3\cdot\left(3x^2 + 10x\right)$$

↓ ↓ ↓

Note the three steps: 1 2 3

2. $$\frac{d}{dx}\,\frac{1}{(x^2+1)^3} = \frac{d}{dx}\,(x^2+1)^{-3} = -3\,(x^2+1)^{-4}\,(2x) = \frac{-6x}{(x^2+1)^4}$$

↓ ↓ ↓

The three steps: 1 2 3

You could use the Quotient Rule on the fraction, $\frac{1}{(x^2+1)^3}$, but the Power Rule is more efficient.

3. $$\frac{d}{dx}\left(\frac{x^2+3}{x^2-4}\right)^5 = 5\left(\frac{x^2+3}{x^2-4}\right)^4\,\frac{(x^2-4)(2x) - (x^2+3)(2x)}{(x^2-4)^2}$$

↓ ↓ ↓

Note the three steps: 1 2 3

In the third step, we used the Quotient Rule on the base, $\frac{x^2+3}{x^2-4}$.

Continuing, $$5\left(\frac{x^2+3}{x^2-4}\right)^4\,\frac{-14x}{(x^2-4)^2} = 5\,\frac{(x^2+3)^4}{(x^2-4)^4}\,\frac{-14x}{(x^2-4)^2}$$

The final result is $$\frac{-70x\,(x^2+3)^4}{(x^2-4)^6}$$

4. $$\frac{d}{dx}\sqrt{\frac{3x+1}{2x-5}} = \frac{d}{dx}\left(\frac{3x+1}{2x-5}\right)^{1/2} = \frac{1}{2}\left(\frac{3x+1}{2x-5}\right)^{-1/2}\,\frac{(2x-5)3 - (3x+1)2}{(2x-5)^2}$$

↓ ↓ ↓

Before we simplify, note the 3 steps: 1 2 3

In the third step, we used the Quotient Rule on the base, $\dfrac{3x+1}{2x-5}$.

Continuing, $\dfrac{1}{2}\left(\dfrac{3x+1}{2x-5}\right)^{-1/2}\dfrac{-17}{(2x-5)^2} = \dfrac{1}{2}\,\dfrac{(3x+1)^{-1/2}}{(2x-5)^{-1/2}}\,\dfrac{-17}{(2x-5)^2}$

The final result is $\dfrac{-17}{2(3x+1)^{1/2}\,(2x-5)^{3/2}}$

Many students look at this type of problem and think there is an easier way. There are different ways but *not* easier ways.

a. You can write the original problem as a quotient and then use the Quotient Rule. $\left(\dfrac{3x+1}{2x-5}\right)^{1/2} = \dfrac{(3x+1)^{1/2}}{(2x-5)^{1/2}}$ = a quotient

It is true that the *first* step is easier but negative fractional exponents will appear and the *simplification* will be more complex. Try it and see. Remember you should get the same result as above.

b. The original problem can even be written as a product.

$$\left(\frac{3x+1}{2x-5}\right)^{1/2} = \frac{(3x+1)^{1/2}}{(2x-5)^{1/2}} = (3x+1)^{1/2}\,(2x-5)^{-1/2} = \text{a product}$$

Now the Product Rule can be used but again negative fractional exponents will be involved. The simplification requires factoring out negative fractional exponents which is tricky. The simplification should provide the above answer.

c. Review the use of the Power Rule above. It does provide the best method.

5. $$\frac{d}{dx}\cos^3 x = \underset{1}{\underset{\downarrow}{3}}\,\underset{2}{\underset{\downarrow}{\cos^2 x}}\,\underset{3}{\underset{\downarrow}{(-\sin x)}} = -3\sin x\,\cos^2 x$$

The 3 steps: 1 2 3

Look at this example carefully, especially step 2. Reduce the exponent by one, but don't change the base which is $\cos x$. If needed write $\cos^3 x$ as $(\cos x)^3$

6. $\dfrac{d}{dx}\sec^3 x = 3\sec^2 x\,(\sec x\,\tan x) = 3\sec^3 x\,\tan x$

$$\begin{array}{lccc} & \downarrow & \downarrow & \downarrow \\ \text{The 3 steps:} & 1 & 2 & 3 \end{array}$$

The behavior here is different. We start with a cubic, $(\sec x)^3$, the exponent goes down one but then back up to 3.

If you understand these six examples, then you understand how to use the Power Rule. Note the Chain Rule was not used in any of these examples. It was used to derive the Power Rule.

THE CHAIN RULE AND TRIG FUNCTIONS

The Chain Rule leads to a second extension of basic differentiation procedures where a *function* replaces the *angle* in a trig form.

Consider $y = \sin u$ where u is a function of *x*, $u = f(x)$.

Note $\dfrac{dy}{du} = \cos u$

By the Chain Rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

which becomes $\dfrac{dy}{dx} = \cos u\,\dfrac{du}{dx}$

The same pattern applies to the five other trig functions. We give a complete list.

$$\frac{d}{dx}\sin u = \cos u\,\frac{du}{dx} \qquad \frac{d}{dx}\csc u = -\csc u\,\cot u\,\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\,\frac{du}{dx} \qquad \frac{d}{dx}\sec u = \sec u\,\tan u\,\frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u\,\frac{du}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 u\,\frac{du}{dx}$$

These are very important formulas and must be memorized. If you note the patterns, this isn't too hard to do. In each case *x* is replaced by *u* in the basic derivative *and* we multiply by the derivative of the *angle.*

Here are several examples.

1. $\dfrac{d}{dx}\sin 2x \;=\; \cos 2x\cdot 2 \;=\; 2\cos 2x$

2. $\dfrac{d}{dx}\cos 3x \;=\; -\sin 3x\cdot 3 \;=\; -3\sin 3x$

3. $\dfrac{d}{dx}\sec(5x^2) \;=\; \sec(5x^2)\,\tan(5x^2)\cdot 10x \;=\; 10x\,\sec(5x^2)\,\tan(5x^2)$

A form like $\cot^4(5x)$ is a power, so use the Power Rule first.

4. $\dfrac{d}{dx}\cot^4(5x) \;=\; 4\cot^3(5x)\;\left[-\csc^2(5x)\right]5 \;=\; -20\cot^3(5x)\,\csc^2(5x)$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 + \dfrac{du}{dx} \end{array}$$

Note the 3 steps: 1 2 $3 + \dfrac{du}{dx}$

THE CHAIN RULE AND EXPONENTIAL FUNCTIONS

Recall that for $y = e^x$

$$\frac{dy}{dx} = e^x$$

Now consider $y = e^u$ where u is a function of x.

$$\frac{dy}{du} = e^u$$

Using the Chain Rule $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

We have $\dfrac{d}{dx}\left[e^u\right] = e^u\,\dfrac{du}{dx}$

In effect the derivative of e^u is the *same* e^u times the derivative of the exponent *u*.

Examples. $\dfrac{d}{dx}\left[e^{3x}\right] = e^{3x}(3) = 3e^{3x}$

$$\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2}(2x) = 2x\,e^{x^2}$$

$$\frac{d}{dx}\left[e^{\tan x}\right] = e^{\tan x}(\sec^2 x) = \sec^2 x\, e^{\tan x}$$

ANOTHER DIFFERENTIATION FORMULA

Recall that the derivative of e^x is e^x. However the derivative of 2^x follows a slightly different pattern. On ***page 202***, the author shows the derivation of the derivative of the power function a^x where a is a constant other than *e*. Understanding the derivation requires that one understand that $e^{\ln a} = a$. Consider the following:

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

Then $$2^{\log_2 8} = 2^3 = 8$$

For *ln a* the base is the number *e*. We can write *ln a* = $\log_e a$.

Then if we follow the above pattern

$$e^{\ln a} = e^{\log_e a} = a$$

You can also verify this equation by using a calculator and assigning values to *a*.

Then if $$y = a^x$$

We can write $$y = a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

Then $$\frac{d}{dx}\left[e^{x\ln a}\right] = e^{x\ln a}(\ln a) = a^x \ln a$$

or $$\frac{d}{dx}\left[a^x\right] = a^x \ln a$$

Examples. $$\frac{d}{dx}\left[3^x\right] = 3^x \ln 3$$

$$\frac{d}{dx}\left[5^x\right] = 5^x \ln 5$$

Finally using the Chain Rule for the situation where *u* is a function of *x*,

$$\frac{d}{dx}\left[a^u\right] = a^u \ln a \frac{du}{dx}$$

Then $\qquad \dfrac{d}{dx}\left[2^{3x}\right] = 2^{3x}(\ln 2)(3)$

ONE LAST COMMENT ON FINDING DERIVATIVES

You have a number of rules now to find derivatives. To use the rules correctly, *first* look for powers, products, and quotients.

1. $(3x+2)^2(5x+1)^4$ is a product. Each factor is a power, but the overall form is a product. Use the Product Rule first.

2. $\sqrt[3]{(7x-4)^4}$ is a power because $\sqrt[3]{(7x-4)^4} = (7x-4)^{4/3}$
   Use the Power Rule first.

3. $\dfrac{\sin x}{1+\cos x}$ is a quotient. Use the Quotient Rule first.

SIMPLIFYING BY REMOVING COMMON FACTORS

Note that $(3x+2)^2(5x+1)^4$ is a product, and we must use the Product Rule first.

$$\frac{d}{dx}(3x+2)^2(5x+1)^4 = (3x+2)^2\,4(5x+1)^3\,5 + (5x+1)^4\,2(3x+2)\,3$$

The *common* factors are $\quad 2(3x+2)(5x+1)^3$

Continuing = $\qquad 2(3x+2)(5x+1)^3\left[10(3x+2) + 3(5x+1)\right]$

which equals $\qquad 2(3x+2)(5x+1)^3\left[30x+20 + 15x+3\right]$

The final answer is $\qquad 2(3x+2)(5x+1)^3\left[45x+23\right]$

The sections covered in the last two assignments contain exercises that need to be *mastered*. Think of the Product, Quotient, and Power Rules as types of algebraic procedures that are as fundamental to calculus as solving equations is to basic algebra. You will have a serious handicap in the rest of this course if you aren't proficient in using these procedures. For this reason I advise you to do a large number of practice exercises before completing the submitted assignments. You don't have to attempt all of them, but try honestly to determine if you can do them. This includes the simplification that may be more than one half of the effort. I especially recommend the following odd numbered exercises, whose answers appear at the end of your text: ***3.2 Exercises, 3–25; 3.3 Exercises, 1–15; 3.4 Exercises, 7–33.***

Section 3.5: IMPLICIT DIFFERENTIATION

Understanding the words, *implicit equation*, is important in this section. The equation of a circle with center at the origin and radius 5 can be written in two forms:

Implicit form	Explicit form
$x^2 + y^2 = 25$	$y = \pm\sqrt{25 - x^2}$

The explicit form matches, $y = f(x)$, where *y* is isolated on one side of the equation. The equation, $x^2 + y^2 = 25$, *implies* that *y* is a function of *x*. In this case it is not too difficult to find the explicit form shown above. But in some cases it will be impossible to solve for y. *Implicit differentiation* is a procedure for finding a derivative using the implicit form of an equation.

THE TECHNIQUE

Step 1. Earlier we emphasized that the Power Rule requires *three* steps. One of the big adjustments in implicit differentiation is finding the derivative of y^2 *or* y^3:

$$\frac{d}{dx}y^2 = 2 \cdot y \cdot y' \text{ where } y' = \frac{dy}{dx}.$$

In particular, note the third step, the derivative of the base, y'. A significant adjustment in implicit differentiation is making a distinction between $\frac{d}{dx}x^3 = 3x^2 \cdot 1$ and $\frac{d}{dx}y^3 = 3y^2 y'$.

In the first form, 1 is the derivative of the base *x* and can be left out. But in the second form, the derivative of the base *y* is $\frac{dy}{dx}$ or y' and must be included. Getting in the habit of including y' or $\frac{dy}{dx}$ is the needed adjustment. This isn't a big deal, but you will have to work on it.

Step 2. The second adjustment is to recognize the term, *xy*, as a product and use the Product Rule.

$$\frac{d}{dx}(xy) = xy' + y \cdot 1 = xy' + y$$

Step 3. If you see a form (such as $\frac{x}{y}$) then use the Quotient Rule.

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y \cdot 1 - xy'}{y^2} = \frac{y - xy'}{y^2}$$

Step 4. For trig functions, we first developed $\frac{d}{dx}\sin x \ = \ \cos x$

and then the more general case, $\frac{d}{dx}\sin u \ = \ \cos u \frac{du}{dx}$

The second case applies when we have $\sin y$ instead of $\sin x$.

$$\frac{d}{dx}\sin y = \cos y \frac{dy}{dx} = \cos y\, y'$$

In summary, the two big adjustments in using implicit differentiation are remembering to include:

1. The derivative of the base when using the Power Rule

2. The derivative of the "angle" for trig functions.

SOLVING FOR y'

In general, expect the forms that come from implicit differentiation to be a bit complicated. Look for two types of terms. One type contains y' as a factor and the rest do not. Place all terms containing y' as a factor on one side of the equation and all other terms on the other side. Then y' will be a common factor on one side and can be isolated by a division.

Example: Given $x^2y \ + \ y^3 \ = \ 5x$ find the derivative. y'.

The derivative of the left side of the equation must equal the derivative of the right side.

$$\frac{d}{dx}\left(x^2y \ + \ y^3\right) \ = \ \frac{d}{dx}5x \qquad \text{Treat } x^2y \text{ as a product.}$$

$$x^2y' + 2xy + 3y^2y' = 5 \qquad \text{Treat } y^3 \text{ as a power.}$$

$$x^2 y' + 3y^2 y' = 5 - 2xy \qquad \text{Place both } y' \text{ terms on the left.}$$

$$y'\left(x^2 + 3y^2\right) = 5 - 2xy \qquad \text{Factor out } y'.$$

$$y' = \frac{5 - 2xy}{x^2 + 3y^2} \qquad \text{Isolate } y'.$$

A NEW WORD

In the last example, implicit differentiation produced the equation,

$$x^2 y' + 2xy + 3y^2 y' = 5$$

which has a derivative y' in two terms. This is an example of a *differential equation.* A *differential equation* is an equation that contains derivatives and usually *x* and *y* terms. This is a major topic in calculus, but for now we only introduce the word to distinguish between an equation and the result after differentiating implicitly.

EXAMPLE 4, PAGE 212

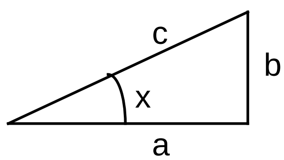
In ***Figure 10***, the slope of the tangent line is given by $y' = -\frac{x^3}{y^3}$. To understand why the graph has the flattened appearance, note that *x* varies from $-2 \;\; to \;\; +2$. Looking at $x^4 + y^4 = 16$ as *x* approaches -2, the *y* value must get smaller and smaller. Put an *x* value close to -2 and a small *y* value in $y' = -\frac{x^3}{y^3}$ and the quotient will be a large number. Consider $\frac{15.99}{.01} = 1599$ as the slope of a steep line. When *x* is close to zero, *y* is close to ± 2. Then $-\frac{x^3}{y^3}$ produces a slope near zero at the flat top and bottom part of the graph. Also in this example, y' appears in the second derivative and is replaced by $-\frac{x^3}{y^3}$. The result is still complicated, but this is to be expected when implicit differentiation is used.

REVIEW OF INVERSE TRIGONOMETRIC FUNCTIONS, PAGES 67–69

Just as the number e has a special significance in calculus, the same is true for inverse trig functions. ***Section 1.5*** contains a review of **Inverse Trig Functions**. Their derivatives are derived on ***pages 213–214***. Later in finding an area or in some other application, you will be surprised to find the appearance of an inverse trig function for reasons that will unfold slowly.

NOTATION AGAIN

In a basic trig form such as $y = \sin x$, x can represent the angle shown in the diagram and $\sin x = \dfrac{b}{c}$



Now suppose we want to solve for x in the equation $y = \sin x$. Is it possible? Yes, we just introduce new notation, $x = \arcsin y$. A first step in understanding is to recognize that

$$x = \arcsin y \text{ and } y = \sin x$$

are two ways of presenting the same information. To understand

$$\theta = \arcsin 3z$$

write $\quad 3z = \sin\theta$

or $\quad z = \dfrac{1}{3}\sin\theta$

Next consider $\quad y = \arcsin x$

which has the same meaning as $\quad x = \sin y$

Interchanging x and y we have $\quad y = \sin x$

The last three steps match the process of finding an inverse function. This justifies writing $\arcsin x$ as $\sin^{-1} x$. Both of these notations are used for inverse sine x.

PRINCIPAL VALUE RANGES

Look at ***Figure 17, Sec. 1.5,*** and imagine the interchange of x and y.

A partial graph of the result is shown in Figure A:

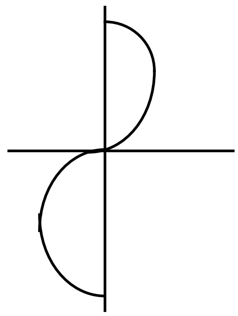


Figure A

Note that we do not have the graph of a function. A vertical line will cut the graph in more than one place. In Figure B we do have a graph of a function:

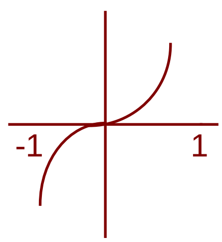


Figure B

Because we have interchanged x and y the angles $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ now appear on the y-axis as shown in ***Figure 20***. This is *very important*. For the inverse sine to be a function the range is limited to:

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

This means $\sin^{-1}\left(-\frac{1}{2}\right)$

is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ as shown in Figure C:

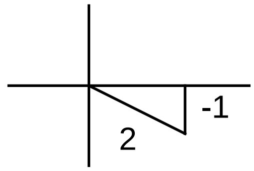


Figure C

The angle is $-\frac{\pi}{6}$.

The graph of the inverse cosine function is shown in ***Figure 22*** and the inverse tangent function is shown in ***Figure 25***. We summarize the results here:

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \qquad 0 \le \cos^{-1} x \le \pi \qquad -\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$$

Note that the principal value ranges are the same for $\sin^{-1} x$ and $\tan^{-1} x$.

VERBAL DESCRIPTIONS

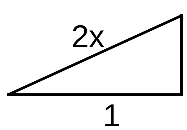
When you see a form such as $\tan^{-1}(1)$ you can read this as

the inverse tangent of 1

or as the angle whose tangent is 1.

Remember that $\theta = \tan^{-1}(1)$ is the same as $\tan\theta = 1$.

It may be a good reference point to identify *any* inverse function as an angle. Then $\sec^{-1}(2x)$ is the angle whose secant is 2*x* which matches the diagram:



Next use the Pythagorean Theorem to find the third side which is $\sqrt{4x^2 - 1}$

These results can be used to find: $\tan\left(\sec^{-1}(2x)\right)$

The symbols $\sec^{-1}(2x)$ represent the angle shown in the above diagram.

Find the tangent of this angle. The answer is: $\dfrac{\sqrt{4x^2-1}}{1}$ or $\sqrt{4x^2-1}$.

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS, Sec. 3.5

The derivation of the derivatives of the inverse trig functions is a straightforward process. The results will probably seem a little strange because none of them involve trig functions. The key in each derivation is the use of *implicit* differentiation.

Given $\quad y = \sin^{-1} x \quad$ we want to find $\dfrac{dy}{dx}$.

Rewrite as $\quad \sin y = x$

Then $\quad \cos y \dfrac{dy}{dx} = 1 \quad$ by implicit differentiation.

Solve for $\dfrac{dy}{dx}$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

We can determine $\cos y$ from the trig identity $\cos y = \pm\sqrt{1 - \sin^2 y}$

Because $\quad \sin y = x \quad \cos y = \pm\sqrt{1 - x^2}$

We can drop the $\pm$ because $\cos y$ is positive in the first and fourth quadrants. Remember

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

matches $\quad -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

The last line indicates that y is an angle in the first or fourth quadrant.

So $\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

becomes $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

Don’t be too concerned if you find the result,

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

to be a bit strange. In Module 1, we gave a list of different types of functions. The derivative of the inverse sine function turns out to be an algebraic function. Why? The answer lies in the derivation, which means it is hidden in the algebraic forms instead of being *intuitively* evident. The derivatives of the six inverse trig functions are shown *at the end of this section*. There are, however, just three basic derivatives and three others that are the same, except each is negative.

For example $$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$$

Compare with $$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

CHAIN RULE

Remember that the form of the Chain Rule is:

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Instead of $y = \sin^{-1} x$

consider $y = \sin^{-1} u$ where $u = f(x)$.

The derivative, $\frac{dy}{du}$, has the form of the basic derivative of $\sin^{-1} x$. We just replace *x* with *u:*

$$\frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$$

The Chain Rule then says, multiply this basic derivative by $\frac{du}{dx}$. (This same pattern has been used in *each* application of the Chain Rule.)

Concentrate on knowing the derivatives that follow the above pattern:

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

Then for each co-function of the above, insert the negative sign.

Principal Value Ranges

In the derivation of the derivative of $\sin^{-1} x$ we used the principal value range,

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

to conclude that we didn't need the symbols $\pm$ in front of the square root sign. This simplified the derivative of $\sin^{-1} x$. On ***page 66*** in ***box 11***, two of the principal value ranges are a bit complex. These were selected so the inverse would be a *function* and to *simplify* the derivative. You can investigate the details in selected exercises or just note the end results which we listed above.