Module 12

SECTION 17.3: APPLICATIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

In this section an attempt is made to show how differential equations arise in science and engineering. We will concentrate on the first application, vibrating springs. Granted it may not be on the top of your list of things you want to see, but it does provide some insight into how important differential equations are in some applications.

The third paragraph on ***page 1168***, along with ***Figures 1*** and ***2***, is essential for background understanding. Note the natural or equilibrium position of the spring. It is then stretched or compressed. The distance that the mass m moves is represented by x. This means that x is *not* the length of the spring but a measure of how far it is stretched or compressed.

Next, match the words

"the force is proportional to x" with "restoring force = - $k\,x$"

where k is a positive number. At this point don't worry about the negative sign.

After the spring is stretched (or compressed) it wants to return to its equilibrium position with a restoring force that is just the constant (-k) times the distance x.

The next step requires a leap of faith, unless you have some background in physics. Newton's Second Law says that this force is the mass m times the acceleration, $\frac{d^2x}{dt^2}$, and we suddenly have the differential equation,

$$m\,\frac{d^2x}{dt^2} = -k\,x$$

Of course, you can't look at the spring and imagine it vibrating and see how this equation is a description. Newton's Second Law provides the mysterious connection. But we can connect it to ***section 17.1*** by rewriting it as

$$m\,\frac{d^2x}{dt^2} + k\,x = 0$$

and recognizing that it is a homogeneous second-order equation with constant coefficients. The solution has the now familiar form,

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

A missing step in the text is to multiply and divide by $\sqrt{c_1^2 + c_2^2}$ to get

$$x(t) = \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos \omega t + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin \omega t\right]$$

Then use **(12b)** on ***page A29*** in ***Appendix D*** to write the solution as

$$x(t) = A\cos(\omega t + \delta)$$

The details in the changes aren't as important as the fact that now you can visualize a cosine curve going up and down that matches a vibrating spring. Except why would the spring keep vibrating the same distance in step with a cosine curve whose "arches" have the same amplitude? Well, there is something we left out called a damping force.

DAMPED VIBRATIONS

Equation **(3)**, on ***page 1169***, includes a damping force, determined experimentally to be proportional to the velocity. Now there are three cases to consider. Note that graphs of the solutions for different values for c_1 and c_2 provide valuable information. It appears that the best balance occurs in case II. As noted, the damping force is just sufficient to suppress the vibrations. In case III, note that a larger c-value in the factor $e^{-(c/2m)t}$ would increase the damping effect. Also, a larger c-value in $c^2 - 4mk$ would make it less negative and move in the direction of case II.

FORCED VIBRATIONS

Equation **(5)**, ***page 1171***, is a nonhomogeneous equation. Now an external force, *F(t),* provides a term that differs from the other three. The external force is a function of time *t.* It does not involve the distance *x.*

ELECTRIC CIRCUITS

If you have the required background, you can see the basis for equation **(7)**, ***page 1172***. We mention in passing that equations **(5)** and **(7)** have the same algebraic structure. The table ***at the end of this section*** matches corresponding components. The mathematical ideas discussed in the preceding sections provide a general structure, which has two physically different manifestations. This makes it possible to analyze certain mechanical systems related to **(5)** by using electrical circuits matching **(7)**. The solutions for the electrical analogs can then be interpreted for the mechanical system.

SECTION 17.4: SERIES SOLUTIONS

Two of the three major topics in this course come together in this last section of the text. We now show how *infinite series* can be a valuable tool in solving *differential equations.* Recall that a power series has the form,

$$\sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

For those values of x for which the series converges, the sum is equal to some number, which you can call y or *f(x).*

$$y \;=\; f(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

The question we now consider is how to determine the coefficients, c_i , to get a solution to a differential equation.

As mentioned in the text, the technique is similar to the method of undetermined coefficients. However, now we *always* start with the series

$$y \;=\; f(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Find the first and second derivatives in series form and substitute into the differential equation. Then it is a matter of rearranging terms, equating coefficients, and looking for patterns.

When finding the first and second derivatives, you can concentrate on the general case involving either the summation notation or the first few terms of the series. In ***Examples 1*** and ***2*** in the text, the emphasis is on the derivatives of the general term. We show how to get the same result by looking at the derivatives of the first five or six terms and then observing the patterns that appear.

EXAMPLE 1

Use power series to solve the equation $y'' + y = 0.$

First $\quad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots$

Then $\quad y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \ldots$

and $\quad y'' = 2c_2 + 2\cdot 3c_3 x + 3\cdot 4c_4 x^2 + 4\cdot 5c_5 x^3 + \ldots$

Next substitute into the equation.

$$y'' + y = \left(2c_2 + 2\cdot 3c_3x + 3\cdot 4c_4x^2 + 4\cdot 5c_5x^3 + \ldots\right)$$

$$+ \left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots\right) = 0$$

Rewrite as

$$(2c_2 + c_0) + (2\cdot 3c_3 + c_1)x + (3\cdot 4c_4 + c_2)x^2$$

$$+ (4\cdot 5c_5 + c_3)x^3 + (5\cdot 6c_6 + c_4)x^4 + \ldots \qquad = 0$$

The constant term and all coefficients must equal zero.

$$2c_2 + c_0 = 0 \quad ; \quad 2\cdot 3c_3 + c_1 = 0 \quad ; \quad 3\cdot 4c_4 + c_2 = 0$$

$$c_2 = \frac{-c_0}{2} \qquad c_3 = \frac{-c_1}{2\cdot 3} \qquad c_4 = \frac{-c_2}{3\cdot 4}$$

$$4\cdot 5c_5 + c_3 = 0 \quad ; \quad 5\cdot 6c_6 + c_4 = 0$$

$$c_5 = \frac{-c_3}{4\cdot 5} \qquad c_6 = \frac{-c_4}{5\cdot 6}$$

By noting patterns can you predict the forms for c_7 and c_8?

Next we express c_4 through c_8 in terms of c_0 and c_1.

$$c_4 = \frac{-c_2}{3\cdot 4} = \frac{-1}{3\cdot 4}\,\frac{-c_0}{2} = \frac{c_0}{4!}$$

$$c_5 = \frac{-c_3}{4\cdot 5} = \frac{-1}{4\cdot 5}\cdot\frac{-c_1}{2\cdot 3} = \frac{c_1}{5!}$$

$$c_6 = \frac{-c_4}{5\cdot 6} = \frac{-1}{5\cdot 6}\cdot\frac{c_0}{4!} = \frac{-c_0}{6!}$$

By observing patterns we predict

$$c_7 = \frac{-c_1}{7!} \quad \text{and} \quad c_8 = \frac{c_0}{8!}$$

Of course, these are the same results as shown on ***page 1177***. There is a general form for c_{n+2}, which leads to the above results we found by looking at the first few terms. Which approach is best? Perhaps a combination of the two. You decide.

We now substitute into

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots$$

to get

$$y = c_0 + c_1x - \frac{c_0}{2}x^2 - \frac{c_1}{3!}x^3 + \frac{c_0}{4!}x^4 + \frac{c_1}{5!}x^5 - \frac{c_0}{6!}x^6 - \ldots$$

Regroup and factor.

$$y = c_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

Note that from this last equation you can write the two general terms shown in the text. The solution in condensed form is

$$y = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

COMMENT

In the solution in the text, note that for the first derivative, the summation starts with $n = 1$. For the second derivative, the summation starts with $n = 2$. Whenever there is a question on which values of n to use, write out a few terms and compare. In particular the assertion that

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

can be tricky. The text suggests replacing all n's on the left with $n + 2$. Try it to see that it works. But, then, you may wonder if this works when the exponent for x is $2n$.

If $$y = \sum_{n=0}^{\infty} c_n x^{2n}$$

does $$y'' = \sum_{n=2}^{\infty} 2n(2n-1)c_n x^{2n-2} \quad ?$$

Compare with

$$\begin{aligned} y &= c_0 + c_1x^2 + c_2x^4 + \ldots + c_nx^{2n} + \ldots = \sum_{n=0}^{\infty} c_nx^{2n} \\ y' &= \quad 2c_1x + 4c_2x^3 + \ldots + 2nc_nx^{2n-1} + \ldots \\ y'' &= \quad 2c_1 + 4\cdot 3c_2x^2 + \ldots + 2n(2n-1)c_nx^{2n-2} + \ldots \end{aligned}$$

The pattern is different here, and the summation

$$\sum_{n=2}^{\infty} 2n(2n-1)c_n x^{2n-2}$$

misses the first term, $2c_1$. Some adjustment must be made. One possibility is

$$2c_1 + \sum_{n=2}^{\infty} 2n(2n-1)c_n x^{2n-2}$$

Try to find others if you feel so inclined. In any case, note the pitfalls in working only with the *n*th term.

EXAMPLE 2

We conclude by repeating the process shown in the first example. Again

$$\begin{aligned} y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots \\ y' &= \quad c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots \\ y'' &= \quad\quad 2c_2 + 2\cdot 3c_3x + 3\cdot 4c_4x^2 + 4\cdot 5c_5x^3 + 5\cdot 6c_6x^4 + \ldots \end{aligned}$$

Substitute into the equation,

$$y'' \;-\; 2xy' \;+\; y \;=\; 0$$

$$\left(2c_2 \;+ 2\cdot 3c_3x \;+\; 3\cdot 4c_4x^2 \;+\; 4\cdot 5c_5x^3 + 5\cdot 6c_6x^4 \ldots\right)$$
$$-\left(2c_1x \;+\; 4c_2x^2 \;+\; 6c_3x^3 \;+\; 8c_4x^4 \;+\; 10c_5x^5 \;+\ldots\right)$$
$$+\left(c_0 \;+\; c_1x \;+\; c_2x^2 \;+\; c_3x^3 \;+\; c_4x^4 \;+\; c_5x^5 \;+\ldots\right)$$
$$= \quad 0$$

Rewrite to show the coefficients of like terms.

(A) $$(2c_2 \;+\; c_0) \;+\; (2\cdot 3c_3 - c_1)x + (3\cdot 4c_4 - 3c_2)x^2 + (4\cdot 5c_5 - 5c_3)x^3 + \ldots$$
$$= \quad 0$$

Before continuing, note the pattern that the coefficients follow. Try predicting the coefficient of the x^4 term, and then check from the substitution. In the text this pattern is shown in the summation above the line beginning with "This equation is true . . ." Again you may have a strong preference for one or the other approach. Use the one you prefer or a combination of the two.

On the bottom of ***page 1178*** and the top of ***page 1179***, the text goes from the general form for c_{n+2} to particular forms for $n \;=\; 0$ through $n \;=\; 7$. We will get the same results by setting all coefficients from the last equation **(A)** equal to zero. The text goes all the way to c_9 to make sure the pattern is clearly visible. The two general forms on ***page 1179*** are needed for the summations shown in equation **(8)**. For c_{2n} the increases of 4 in ($3\cdot 7\cdot 11\cdot$) relate to the 4 in the factor $(4n \;-\; 5)$. Determining the "subtract 5" will depend on the n-values used. In equation **(8)** the summation starts at $n \;=\; 2$ and $(4n \;-\; 5) \;=\; 3$, matching the first factor. This can be confusing, so remember that trial and error will also help you through the maze.

Overview

In 1665-1666, Trinity College, the school Isaac Newton attended, was closed because of the bubonic plague. While living at home, in approximately one year he created the calculus and the law of gravitation. Leibniz is also credited with independently creating calculus about ten years later. At the time Newton created calculus, he was also solving differential equations using what we now call infinite series. The last topic in this course is separated from the beginnings of calculus by some 1100+ pages, but in the historical development of calculus they were interwoven.

Now that you have some understanding of calculus, you may be interested in its historical development. The following books on the history of mathematics provide fascinating reading:

James Newman, editor, *The World of Mathematics* (4 volumes)

Howard Eves, *An Introduction to the History of Mathematics*

Morris Kline, *Mathematical Thought from Ancient to Modern Times*

Carl Boyer, *A History of Mathematics*

LAST COMMENT

The process of finding the coefficients of an infinite series that is the solution to a differential equation was the main emphasis in this section. Hopefully this doesn't overshadow the main idea that series provide a valuable tool in solving differential equations. Early on Newton recognized that in certain situations one could discard higher powers and use the first few terms of a series as an approximation. In these cases, finding the *n*th term isn't necessary, and thus the procedures covered in this section would be shortened.

A LAST NOTE

Congratulations on completing all the assignments! All that remains is the final examination. To prepare for it, review the material covered in the assignments, making sure that you have a good grasp of theory (definitions, theorems, and proofs), of algebraic procedures and geometric interpretations, and of the applications of calculus. Good luck on the examination.