Module 10

SECTION 9.4: MODELS FOR POPULATION GROWTH

We return to the logistic model that was discussed in ***section 9.1*** and ***Module 9***. The exponential model was based on the assumption that the growth rate is proportion to the population.

$$\frac{dP}{dt} = kP$$

The logistic model introduced the carrying capacity M . This is a maximal population, perhaps because of a limited food supply. P may go above M but will gradually come back down to M. The logistic model is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

The growth rate will be zero for $P = 0$ and for $P = M$. These are the equilibrium solutions where no growth would occur. If P is less than M, then $1 - \frac{P}{M}$ will be positive and in turn the growth rate is positive. P will increase toward M. If P is larger than M, then $1 - \frac{P}{M}$ will be negative and the growth rate will be negative. The population P will decrease to M.

The direction field for a particular situation is shown in ***Figure 1, page 612***. Then in ***Figure 2***, three solutions are drawn. The solutions agree with the above comments.
The text uses the method for separable equations to provide an algebraic solution to the logistic equation. We comment on several steps of the process ***after Figure 2***.

First, equation **(5)** is the result of dividing the preceding equation by $P\left(1 - \frac{P}{M}\right)$ and multiplying by dt. Partial fractions can be used in the integration to produce the logarithmic form,

$$\ln|P| - \ln|M - P| = kt + C$$

Multiply both sides by –1 and use the log property; the difference of two logs is the log of a quotient. Remember that a natural log is an exponent with base e and e^{-C} is a constant so rename it with another letter A. Then

$$\frac{M - P}{P} = Ae^{-kt}$$

P and *t* are the variables, *M, A*, and *k* are constants. The text uses a different route but note that if we put $t = 0$ into the last equation and remember that P_0 is the population when $t = 0$, we have

$$\frac{M - P_0}{P_0} = A$$

This is part of the final answer **(7)** on ***page 614***.

Below **(6)**,

$$\frac{M}{P} - 1 = Ae^{-kt}$$

can be written

$$\frac{M}{P} = 1 + Ae^{-kt} = \frac{1 + Ae^{-kt}}{1}$$

Then use the reciprocals of each side to get

$$\frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

Multiply by *M* and the algebraic solution is

$$P = \frac{M}{1 + Ae^{-kt}}$$

with

$$A = \frac{M - P_0}{P_0}$$

The derivation and solution are a bit complex but consider the following.

Note the differential equation in ***Example 2***. Should we find the solution by separating variables or should we use a matching process with the above derivation. Finding a solution by separating variables would be a *duplication* of the 12 step derivation we just completed. We don't need to do that again. Match the given differential equation with the general logistic differential equation,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

to determine the values for *k* and *M*. Then go to the solution,

$$P = \frac{M}{1 + Ae^{-kt}}$$

with

$$A = \frac{M - P_0}{P_0}$$

to determine the value for A and the algebraic solution is,

$$P = \frac{1000}{1 + 9e^{-0.08t}}$$

Now to find *P(40)*, we need only evaluate $\frac{1000}{1 + 9e^{-0.08(40)}}$.

COMPARISON OF THE NATURAL GROWTH AND LOGISTIC MODELS

Keep in mind that these models are based on assumptions that seem reasonable. But how do they compare with actual observations? The table on ***page 615*** indicates that in the long range the logistic model is much better. Two other models are shown at the end of this section as an attempt to improve predictions.

SECTION 9.5: LINEAR EQUATIONS

We now consider a second method that produces an algebraic solution to a differential equation. Recall that first order means that only a first derivative is present in the equation. Linear refers to the fact that y is to the first power. If you look at the left side of the equation,

$$\frac{dy}{dx} + P(x)y = Q(x),$$

use your imagination and compare it to the Product Rule; $\frac{dy}{dx}$ and y fit, but also something is missing. The missing part is called an *integrating* factor and is represented by *I(x)* . The derivation claims that

$$I(x) = e^{\int P(x)\,dx}.$$

Yes, this is a rather strange looking function, but the following examples show that it does produce an integrating factor. Remember that the method is based on the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and *P(x)* is the coefficient in the *y* term. When integrating *P(x),* a constant term need not be included.

EXAMPLE 1

Solve $y' - 2y = e^x$.

Here, $P(x) = -2$, and $I(x) = e^{\int(-2)\,dx}$.

Then $I(x) = e^{-2x}$. Multiply both sides of the DE by e^{-2x} to get

$$e^{-2x}y' - 2ye^{-2x} = e^{-2x}e^x = e^{-x}$$

This method always produces the derivative of the product *I(x) y* on the left side. Check $\left(e^{-2x}y\right)' = e^{-2x}y' + ye^{-2x}(-2) =$ the left side. Then,

$$\left(e^{-2x}y\right)' = e^{-x}$$

and we can integrate each side.

$$e^{-2x}y = -e^{-x} + C$$

and $$y = e^{2x}\left(-e^{-x} + C\right) = -e^x + Ce^{2x}.$$

We verify that this is a solution by substituting y and $y' = -e^x + 2Ce^{2x}$ into the equation, $y' - 2y = e^x$.

$$\left(-e^x + 2Ce^{2x}\right) - 2\left(-e^x + Ce^{2x}\right) \stackrel{?}{=} e^x$$

Simplify the left side and indeed we do have a solution.

EXAMPLE 2

Solve $\dfrac{dy}{dx} + (\cot x)y = 4x^2\csc x$.

The integrating factor is $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$. Multiply both sides by this integrating factor to get

$$\sin x\,\frac{dy}{dx} + \sin x(\cot x)y = 4x^2\csc x\sin x$$

which equals

$$\sin x \frac{dy}{dx} + \cos x \; y = 4x^2 .$$

Then

$$(\sin x \; y)' = 4x^2$$

and

$$\sin x \; y = \frac{4x^3}{3} + C$$

The solution is

$$y = \frac{4x^3 \csc x}{3} + C \csc x .$$

You can show that this is indeed a solution by substituting y and y' into the original DE.

EXAMPLE 3

Solve $\quad (2y - x^2 + 1)\,dx + (x^2 - 1)\,dy = 0 .$

First we must determine if this equation fits the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Possibly, since y is to the first power. Divide by dx and rearrange terms.

$$(x^2 - 1)\frac{dy}{dx} + 2y = x^2 - 1$$

Next divide by $(x^2 - 1)$.

(A) $$\frac{dy}{dx} + \frac{2}{x^2 - 1}\,y = 1$$

It is a first-order linear equation with

$$P(x) = \frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1} .$$

Then

$$e^{\int P(x)dx} = e^{\ln|x-1| - \ln|x+1|} = \frac{e^{\ln|x-1|}}{e^{\ln|x+1|}} = \frac{x-1}{x+1}.$$

Multiply (A) by the integrating factor, $\dfrac{x-1}{x+1}$.

$$\left(\frac{x-1}{x+1}\right)\frac{dy}{dx} + \left(\frac{x-1}{x+1}\right)\frac{2}{x^2-1}y = 1\left(\frac{x-1}{x+1}\right)$$

Simplify.

$$\left(\frac{x-1}{x+1}\right)\frac{dy}{dx} + \frac{2}{(x+1)^2}y = \left(\frac{x-1}{x+1}\right)$$

The left side is the derivative of a product.

$$\left[\left(\frac{x-1}{x+1}\right)y\right]' = \left(\frac{x-1}{x+1}\right) = 1 - \frac{2}{x+1}$$

Then, integrating both sides,

$$\left(\frac{x-1}{x+1}\right)y = x - 2\ln|\,x+1\,| + C$$

and solving for y, we have

$$y = \left(\frac{x+1}{x-1}\right)\left(x - 2\ln|\,x+1\,| + C\right)$$

The solution is a bit complicated, but surprisingly one can show that it satisfies the original DE fairly easily. Just leave the answer in the product form when finding the derivative.

Electric Circuits

Examples 4 and ***5, page 624***, show how the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

provides information about an electric circuit. If you are familiar with Kirchhoff's laws you can understand the basis for the equation. The letter I in the differential equation represents the current in amperes, not an integrating factor. In ***Example 4***, the initial conditions change the general solution

$$I(t) = 5 + Ce^{-3t}$$

to the particular solution,

$$I(t) = 5 - 5e^{-3t}$$

As you read this example, note that almost all of the information relates to the initial conditions. At this point we hope you understand that you need an equation to drop in the initial conditions. The differential equation springs from other knowledge about circuits that is not included in the problem. A good understanding of what is happening also relates to the other knowledge about circuits. These examples are included to show how a differential equation is essential in creating a mathematical context within a particular field, but as long as you are outside the field, these equations lack vitality.

You may know that a battery provides direct current while a generator can provide either direct or alternating current. The word *alternating* relates to a sine curve that matches the voltage, 60 *sin* 30t, in ***Example 5***. This leads to different types of functions in the solution shown at the end of the section. In a general way you can see some of the inner workings, even if they don't make complete sense.

SECTION 9.6: PREDATOR-PREY SYSTEMS

The discussion of models involving differential equations continues for a more complex but real situation. Two populations are involved. Wolves are the predators and rabbits are the prey. This requires two differential equations instead of one. And, of course, the model is based on *assumptions*. The central question should always be, does the model fit real life situations. This requires data accumulated over a period of time and the text is able to supply this.

The beginning is to use the natural growth model twice. The rate of growth or decline is proportional to the size of the population. Wolves, W, are the predator and rabbits, R, are the prey. For rabbits, $\frac{dR}{dt} = kR$, where k is a *positive* constant. For wolves, $\frac{dW}{dt} = -rW$, where r is again a *positive* constant. Additional assumptions are that there is an adequate food supply for the rabbits and no predators $\left(\frac{dR}{dt}\right.$ is positive and the rabbit population is increasing) and there is an absence of prey for the wolves $\left(\frac{dW}{dt}\right.$ is negative and the wolf population is decreasing). This is a somewhat arbitrary starting point.

The mixing of the wolves and rabbits together is represented by the product of R and W. Since the predator feeds on the prey, the number of wolves should increase,

$$\frac{dW}{dt} = -rW + bRW$$

and the number of rabbits should decrease,

$$\frac{dR}{dt} = pR - aRW$$

Again these are intended as approximations and need to be checked with actual data.

In ***Example 1***, the equilibrium solution is 1000 rabbits and 80 wolves. For these numbers, both $\frac{dR}{dt}$ and $\frac{dW}{dt}$ are equal to zero so there is a tendency for neither population to change. But suppose outside forces not included in the equations causes a decrease in the number of wolves to say 75 wolves. In $\frac{dR}{dt}$, the factor $(0.08 - 0.001W)$ is zero for 80 wolves and will be positive for 75. In turn $\frac{dR}{dt}$ will be positive and as expected the number of rabbits would increase. Or suppose outside forces cause the number of rabbits to decrease to say 900. Now look at the factor $(-0.02 + 0.00002R)$ in $\frac{dW}{dt}$. This factor is zero for 1000 rabbits and will be negative for 900. This will make $\frac{dW}{dt}$ negative and the number of wolves to decrease. Again this is what would be expected suggesting that the equations provide some accurate information.

Two equations in two unknowns, like $2x + y = 7$ and $x - y = 5$ can be solved by adding to get $3x = 12$ and $x = 4$. The value for y would be –1. But the two differential equations we are considering present different challenges. On ***page 624***, the Chain Rule is used to combine $\frac{dR}{dt}$ and $\frac{dW}{dt}$ into another rate of change $\frac{dW}{dR}$. The expression,

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

can be used to create the direction field in ***Figure 1***. Then in ***Figure 2***, several solution curves are drawn. Now we have a visual form that provides more information. The curve drawn in ***Figure 3*** passes through 1000 rabbits and 40 wolves. Woops, it passes through the point whose coordinates are (1000, 40). Because these numbers make $\frac{dR}{dt}$ positive, R is increasing and movement along the curve is counter-clockwise. By reading the paragraph below ***Figure 3***, you can gain more conviction that the model is reasonable.

Further visual information is presented in ***Figures 4*** and ***5***. In ***Figure 5***, the first peak for the blue curve matches P_1 in ***Figure 3*** while the first peak in the red curve matches P_2 .

In ***Figure 6***, records over a 90-year period display coupled oscillations at least somewhat similar to those in ***Figure 5***. The differential equations below ***Figure 6*** are an attempt to get better predictions.

Footnote: Why is RW used and not $(R + W)$ on ***page 627***? Consider

$$\frac{dR}{dt} = kR - a(R + W) = (k - a)R - aW$$

Then $(k - a)$ is some constant, call it p, and $\frac{dR}{dt} = pR - aW$

The first term, pR, matches the natural growth model but the second term, $-aW$, does not contain the variable R. The second term does not reflect any interaction between rabbits R and wolves W. The RW term seems to be the only algebraic way to represent the desired interaction.