module 5

11.1

4) $a_n = \frac{n^2-1}{n^2+1}$ $\quad \{0, \frac{3}{5}, \frac{8}{10}, \frac{15}{17}, \frac{24}{26}, \ldots\}$

7) $a_n = \frac{1}{(n+1)!}$ $\quad \{\frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \frac{1}{6!}, \ldots\} = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots\}$

11) $a_1 = 2,\ a_{n+1} = \frac{a_n}{1+a_n}$ $\quad \{2, \frac{2}{3}, \frac{2/3}{5/3}, \frac{2/5}{7/5}, \frac{2/7}{9/7}, \ldots\} = \{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots\}$

$\uparrow \frac{2}{5} \quad \uparrow 2/7 \quad \uparrow 2/9$

13) $a_n = \boxed{\frac{1}{2}n}$ $\qquad$ 16) $a_n = \boxed{5+3(n-1)}$

24) $a_n = \frac{3+5n^2}{1+n}$ $\quad$ If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

$\therefore \lim_{x\to\infty} \frac{3+5x^2}{1+x} = \lim_{x\to\infty} \frac{10x}{1} = \infty$ so $\boxed{\text{diverges}}$

ii) also you can divide by n $\quad \lim_{n\to\infty} \frac{\frac{3}{n}+5n}{\frac{1}{n}+1} = \lim_{n\to\infty} \frac{5n}{1} = \infty$ ✓

28) $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$ $\quad$ divide by $\sqrt{n}$

$= \lim_{n\to\infty} \frac{3}{1+\frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3}{1+0} = \boxed{3}$

30) $a_n = \frac{4^n}{1+9^n}$ $\quad$ ~~use theorem 3~~

~~divide by 9~~ $\quad \cancel{\frac{4^n}{9}}$

$\cancel{\lim_{x\to\infty} \frac{4^x}{1+9^x}} =$

divide by 4^n so

$a_n = \frac{1}{\frac{1}{4^n} + \frac{9}{4}^n}$

$\lim_{n\to\infty} \frac{1}{0+\infty} = \boxed{0}$

33) $a_n = \frac{n^2}{\sqrt{n^3+4n}} = \infty$ $\quad$ ~~mult~~ divide by $n^{1.5}$ $\quad \frac{n^{0.5}}{\sqrt{1+\frac{4}{n^2}}}$

~~divide by n^2~~

~~$\lim_{n\to\infty} \frac{1}{\sqrt{n^{-1}+\frac{4}{n^3}}} = \frac{1}{\sqrt{\frac{n^2}{n^4}+\frac{4}{n}}} = \frac{1}{0+0} = \infty$~~

OR

$\frac{n^2}{\sqrt{n^3+4n}} \geq \frac{n^2}{\sqrt{n^3+4n^3}} = \frac{n^2}{\sqrt{5n^3}} = \frac{n^2}{\sqrt{5}n^{\frac{3}{2}}} = \frac{n^{3.5}}{\sqrt{5}n^3} = \frac{\sqrt{n}}{\sqrt{5}} \to \boxed{+\infty}$

$\uparrow$ divergent minorant

so $\frac{n^2}{\sqrt{n^3+4n}}$ $\boxed{\text{is also divergent}}$

36) $a_n = \frac{(-1)^{n+1} n}{n + \sqrt{n}}$ divide by n $a_n = \frac{(-1)^{n+1}}{1 + \frac{1}{\sqrt{n}}} \to \lim_{n\to\infty} \frac{(-1)^{n+1}}{1}$ $\boxed{\text{diverges}}$

42) $a_n = \ln(n+1) - \ln(n)$

using log rules

$\cancel{a_n = \frac{\ln(n+1)}{\ln(n)}}$ ~~using theorem 3 and L'Hopital:~~ $\cancel{\lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1}$

$a_n = \ln\left(\frac{n+1}{n}\right)$ so $\lim_{n\to} \frac{n+1}{n} = 1$ so $\cancel{a_n = \ln(1)}$

so $\lim_{n\to\infty} \ln(1) = \boxed{0}$

55) n! grows faster than 2^n

when $n=5$, $n! = 5\cdot4\cdot3\cdot2\cdot1 =$ $\boxed{\infty}$

$2^n = 2\cdot2\cdot2\cdot2\cdot2 =$ ~~[illegible]~~

formal proof?

$\lim_{n\to\infty} \frac{a(n+1)}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2^{(n+1)}} \cdot \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} \cdot \frac{n!}{n!} = \lim_{n\to\infty} \frac{n+1}{2} \to \boxed{\infty}$

56) $a_n = \frac{(-3)^n}{n!}$ so $\lim_{n\to\infty} \frac{(-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{-3}{n+1} \frac{n!}{n!} = \lim_{n\to\infty} \frac{-3}{n+1} \to \boxed{0}$

$\boxed{\text{Converges}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

74) $a_n = \frac{1-n}{2+n}$ inc or dec? monotonic?

is $a_{n+1} > a_n$? $n \geq 1$

$\frac{1-(n+1)}{2+(n+1)} > \frac{1-n}{2+n}$

$\frac{-n}{3+n} > \frac{1-n}{2+n} \leftrightarrow$ $-2n - n^2 > 3 - 2n - n^2$ $0 > 3$ false

so it's ~~decreasing~~

~~false when n ≥ 1~~ so

so it's ~~increasing~~ $\boxed{\text{decreasing}}$ because $a_{n+1} > a_n$ is false

$0, \frac{-1}{4}, \frac{-2}{5}, \frac{-3}{6}, \frac{-4}{7}, \frac{-5}{8}, \frac{-6}{9}, \frac{-7}{10}$

use derivative test;

$f(x) = \frac{1-x}{2+x}$ $f'(x) = -1$ ~~so decreasing as $n\to\infty$~~

so a_n is decreasing as $n \to \infty$

sequence terms $0 \geq a_n$

are bounded above by 0 ~~but not below since we can make n bigger and a_n will be more negative.~~ $\boxed{\text{Bounded above by 0 } \cancel{\text{but not below}}}$

$\boxed{\text{bounded below by } -1}$

Module 5

11.2

1) a) series is a sum of a_n infinite sequence, sequence is an ordered list of numbers.

b) if S_n is the n^{th} partial sum, a series is convergent if $\lim_{n\to\infty} S_n = L$
It's divergent if L is not a real number.

18) Convergent or divergent? find sum.

$4+3+\frac{9}{4}+\frac{27}{16}$ $\quad r=\frac{3}{4}$ $\quad a_n = 4\left(\frac{3}{4}\right)^{n-1}$ $\quad$ Sum $= \frac{4}{1-\frac{3}{4}} = \frac{4}{\frac{1}{4}} = \boxed{16}$

$\boxed{\text{Convergent}}$ $\quad \frac{2}{9}, 1, \ldots$

20) series conv or div

$2+0.5+0.125+0.03125$

$2+\frac{1}{2}+\frac{1}{128}+\frac{1}{32}$ $\quad a_n = 2\left(\frac{1}{4}\right)^{n-1}$ $\quad a_n = \frac{2}{\frac{3}{4}} = \boxed{\frac{8}{3}}$

23) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4\cdot 4^{(n-1)}} = \sum_{n=1}^{\infty} \frac{1}{4}\cdot\left(\frac{-3}{4}\right)^{n-1} = \frac{\frac{1}{4}}{\frac{7}{4}} = \boxed{\frac{1}{7}}$

24) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = \sum_{n=1}^{\infty} \frac{3^n}{(-2)^{n-1}} = \sum_{n=1}^{\infty} \frac{3\cdot 3^{n-1}}{(-2)^{n-1}}$ $\quad \sum_{n=1}^{\infty} 3\left(\frac{3}{-2}\right)^{n-1}$ $\quad |r|>1$ $\quad \boxed{\text{So diverges}}$

26) $\sum_{n=1}^{\infty} \frac{6\cdot 2^{(2n-1)}}{3^n} = \sum_{n=1}^{\infty} \frac{6\cdot 2^{(2n-1)}}{3\ 3^{n-1}} = \sum_{n=1}^{\infty} 2\,\frac{2^{2n}}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{2^n\,2^n}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{2\cdot 2^{n-1}\cdot 2^n}{3^{n-1}}$

$= \sum_{n=1}^{\infty} 2^{n+1}\,\frac{2}{3}^{n-1} = \frac{2^{n+1}}{\frac{1}{3}}$

$\boxed{\text{diverges}}$

30) $\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5} = 1$

theorem 3 + L'Hopital

$= \lim_{k\to\infty} \frac{k^2}{k^2-2k+5} = \lim_{k\to\infty} \frac{2k}{2k-2} = \lim_{k\to\infty} \frac{2}{2} = \boxed{1}$ $\quad \boxed{\text{So diverges}}$ by theorem 7

34) $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ $\quad \sum_{n=1}^{\infty} \frac{2^n}{e^n}+\frac{4^n}{e^n} = \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n + \left(\frac{4}{e}\right)^n$ $\quad e = 3.1417\ldots$

converges + diverges $\quad$ so $\boxed{\text{diverges}}$

40) $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$ Since series sums can be added let's look at these one at a time

$\sum_{n=1}^{\infty} \frac{3}{5^n} = \sum_{n=1}^{\infty} 3\cdot\frac{1}{5}^n = \sum_{n=1}^{\infty} \frac{3}{5}\cdot\frac{1}{5}^{n-1}$ $\quad$ so Sum $= \frac{a}{1-r} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

and $\sum_{n=1}^{\infty} \frac{2}{n} = \sum_{n=1}^{\infty} 2\,\frac{1}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge $\quad$ so $\frac{3}{4}$ + diverge = diverge

harmonic series

52) $a_n = \frac{46}{10^2}+\frac{46}{10^4}$ $\quad a_1 = \frac{46}{100}$ $\quad r = \frac{1}{10^2}$ $\quad a_n = \frac{\frac{46}{100}}{\frac{99}{100}} = \boxed{\frac{46}{99}}$

~~5.9) So $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ would $=1$ So $x=5$, $a_n=1$ $\quad 2<x<5$, $a_n=0$~~

59) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ we know $\sum_{n=0}^{\infty} \frac{3^n}{3^n} = 1$ so 5 is an upper bound.

$1 + \left(\frac{(x-2)}{3}\right)^1 + \left(\frac{x-2}{3}\right)^2 + \dots$

$a = 1$

$r = \frac{x-2}{3}$

to converge $\left|\frac{x-2}{3}\right| < 1 \rightarrow |x-2| < 3 \rightarrow -3 < x-2 < 3$

$\boxed{-1 < x < 5}$ converges

sum $= \frac{1}{1-\frac{(x-2)}{3}} \rightarrow \frac{3}{3-x+2} = \boxed{\frac{3}{5-x}}$