Module 10

SECTION 5.1: AREAS AND DISTANCES

We now begin the study of the branch of calculus called integral calculus. Read ***the first page*** carefully. In particular, the paragraph underneath Figure 2. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area." This will be the main thrust of this section and the next.

OVERVIEW

How do we find the area of the shaded region in ***Figure 3***? ***Figure 4(b)*** begins with an approximation of the shaded area by using four rectangles. The error in the approximation involves that part of each rectangle that is above the curve. This error is reduced by doubling the number of rectangles, eight in ***Figure 6(b)***, and then continuing the process in ***Figure 7*** and the table on ***page 367***. This suggests a limit by letting n, the number of rectangles, approach infinity and leads to the *definition* of area in ***(2) on page 371***. But first we look at some details.

EXAMPLE 2

It's important to understand the ideas in this example.

1. In ***Figure 7***, the width of each rectangle is $\frac{1}{n}$. The total length of the interval from $x = 0$ to $x = 1$ is one. Divide one into n equal parts and the width of each rectangle is $\frac{1}{n}$.

2. Next we need the x coordinate of each point on the interval $[0,1]$. The coordinates are $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}$. Each width is $\frac{1}{n}$. Two widths equal $2\left(\frac{1}{n}\right) = \frac{2}{n}$ up to n widths which equals $n\left(\frac{1}{n}\right) = \frac{n}{n}$ We then have a set of n subintervals:

$$\left[0,\frac{1}{n}\right], \left[\frac{1}{n},\frac{2}{n}\right], \left[\frac{3}{n},\frac{4}{n}\right], \dots, \left[\frac{n-1}{n},\frac{n}{n}\right]$$

3. Going back to ***Figure 7***, to find the area of each rectangle, we need the height. Each height is a y value or an $f(x)$ value. Because:

$$y = f(x) = x^2$$

$$f(1/n) = (1/n)^2 \ , \ f(2/n) = (2/n)^2 \text{ up to } f(n/n) = (n/n)^2$$

4. The area of the first rectangle is $\frac{1}{n}\left(\frac{1}{n}\right)^2 = \frac{1}{n^3}$.

   The second $\quad \frac{1}{n}\left(\frac{2}{n}\right)^2 = \frac{2^2}{n^3}$

   The third $\quad \frac{1}{n}\left(\frac{3}{n}\right)^2 = \frac{3^2}{n^3}$

   The last $\quad \frac{1}{n}\left(\frac{n}{n}\right)^2 = \frac{n^2}{n^3}$

   The width of each rectangle is the same, namely $\frac{1}{n}$, while the heights gradually increase.

5. Now add the areas of all of the rectangles to get an approximation to the area under the curve and above the *x-a*xis from $x = 0$ to $x = 1$. Note that $\frac{1}{n^3}$ is a common factor in all terms so the sum can be written:

$$\frac{1}{n^3}\left(1^1 + 2^2 + 3^2 + \ldots n^2\right)$$

6. (a) The equation

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

on ***page 368*** is a formula for adding the squares of integers.

Check it out for $n = 3$. $\quad 1^2 + 2^2 + 3^2 =? \frac{3(3+1)(6+1)}{6}$

Does $\quad 1 + 4 + 9 = \frac{3(4)(7)}{6} \quad ?$

Yes $\quad 14 = 14$

For $n = 5$. $\quad 1 + 4 + 9 + 16 + 25 =? \frac{5(5+1)(10+1)}{6}$

Yes $\quad 55 = 5(11) = 55$

(b) Note that as n gets larger, the number of terms on the left also gets larger but on the right side there are always just three factors. Then as shown in the text, R_n, can be expressed in a more condensed form. Also remember that R_n represents the areas of ALL of the rectangles in ***Figure 7*** and provides an approximation to the area under the curve.

7. This step is crucial. To get a better approximation, we increase the number of rectangles. As we go from 4 rectangles to 8 and then to n rectangles, note that we get a better approximation. By letting n approach infinity the number of rectangles approaches infinity, and the error approaches zero. As shown in the text

$$\lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \text{ is } \frac{1}{3}$$

It seems very reasonable then to claim the *exact* area under the curve is $1/3$. In fact note on ***page 371*** that the area is *defined* to be the limit. We don't have another way to find the area so it is defined to be the limit when the limit exists.

THE GENERAL CASE

In ***Example 2***, the procedure in the example above is extended to any function, $y = f(x)$ on the interval $[a,b]$.

The width of each rectangle is now, $\Delta x = \frac{b-a}{n}$. In ***Figure 10***, note that

$$x_1 = a + \Delta x \,,\ x_2 = a + 2\Delta x ,\ \ldots\ x_i = a + i\Delta x$$

In ***Figure 11***, the height of the first rectangle is $f(x_1)$. In this example the heights are determined using the right-hand endpoint of each sub-interval. The second and third heights, $f(x_2)$ and $f(x_3)$, follow the same pattern.

The area of the first rectangle is $f(x_1)\Delta x$. Everything will make more sense if you make sure you agree with this when looking at ***Figure 11***.

The sum of **all** rectangles, represented by R_n, is

$$f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

In ***Figure 12***, note how the difference in the area under the curve and R_n, the area of **all** rectangles, decreases as *n* increases.

The exact area is defined as a limit, $Lim_{n\to\infty} R_n$.

In ***equation 4, page 371*** you see $f(x_1^*)$ instead of $f(x_1)$ to represent a more general situation. The difference is that x_1^* can be any x value in the first subinterval, not just an endpoint. This distinction is important in mathematical theory but will not be a major factor in this course.

SIGMA NOTATION

The text introduces the **sigma notation** at the end of ***Example 2***.

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

The Greek letter $\sum$ is used to represent a sum. The variable i takes on the values 1, 2, 3, .. up to *n* changing

$$f(x_i)\Delta x \quad \text{to} \quad f(x_1)\Delta x \,,\ f(x_2)\Delta x \quad \text{up to} \quad f(x_n)\Delta x \ .$$

The sum

$$1^1 + 2^2 + 3^2 + \ldots + n^2$$

can be written, $\sum_{i=1}^{n} i^2$

Then ***formula 1*** on ***page 368*** ,

$$1^1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

matches $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

More examples are given on ***page A34*** in ***Appendix E***.

APPENDIX E: SIGMA NOTATION

Theorem 2 on ***page A35*** in ***Appendix E*** contains three properties of summations that will be used in the next section. Again patience is the key. The summation notation tends to hide information that isn't evident in the condensed form. Write out the expanded form and you will be looking at more familiar forms:

(a) $$\sum_{i=m}^{n} c\,a_i = c\sum_{i=m}^{n} a_i$$

In the expanded form for the left side, ***page A35***, you can see that c is a common factor and hence can be written outside $\sum$:

(b) $$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

As shown on ***page A35***, the terms on the left side can be rearranged to match those on the right. Part (c) is essentially the same:

(c) $$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Parts (b) and (c) indicate that the a_i and b_i terms can be summed separately.

The summation, $\sum_{i=1}^{4} 2$, is a bit different. There is no i to the right of the summation symbol. Interpret this in the same way that you treat $f(x) = 2$.
$f(1) = 2$, $f(2) = 2$, $f(3) = 2$, $f(4) = 2$. In both cases 2 is a constant that doesn't change value.

$$\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 4 \cdot 2 = 8$$

A more general form is contained in ***Theorem 3(a)*** and ***(b), page A37***:

(a) $$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{n} = n \cdot 1 = n$$

The challenge here is to count accurately. Since i varies from 1 to n the numeral 1 is written n times:

(b) $$\sum_{i=1}^{n} c = \underbrace{c + c + c + \ldots + c}_{n} = n \cdot c = nc$$

Essentially (b) is the same as (a) with c replacing 1.

SPECIAL SUMS

Theorem 3, page A37, contains three special sums that will be needed in the next section:

$$\sum_{i=1}^{n} i \quad , \quad \sum_{i=1}^{n} i^2 \quad , \quad \sum_{i=1}^{n} i^3$$

The first two are proved in ***Examples 4*** and ***5*** on ***page A35***. The last can be proved using Mathematical Induction as outlined on ***page A36***. We concentrate only on their use.

Consider the sum of the first one hundred squares of integers:

$$\sum_{i=1}^{100} i^2$$

In expanded form we would have 100 terms and quite a chore in adding them. By ***Theorem 3(d), page A37***:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

With this formula, we just insert 100 into the right side to get the sum:

$$\sum_{i=1}^{100} i^2 = \frac{100(100+1)(2 \cdot 100 + 1)}{6} = 338{,}350$$

EXAMPLE 7, PAGE A37

This form may look a bit complicated, but the limit-sum of ? is going to be very important in what follows.

By limit-sum of ?, we mean $\lim_{n\to\infty} \sum ?$ In the general form what replaces the ? will be different from what you see in this problem, but the idea will be the same. Basically, we are saying that you need to take this form seriously.

In going from

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n} \right] \text{ to } \lim_{n\to\infty} \left[\frac{3}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \right]$$

note the following:

1. The summation is applied to each term in the sum.

$$\sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n} \right] = \sum_{i=1}^{n} \frac{3}{n^3} i^2 + \sum_{i=1}^{n} \frac{3}{n}$$

2. Then in the first sum, $\frac{3}{n^3}$, does not contain the variable i.

It can be "factored out" by placing it to the left of the summation symbol. The same is true for $\frac{3}{n}$ in the second term. The right side becomes

$$\frac{3}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1$$

Then $\sum_{i=1}^{n} i^2$ is replaced by $\frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} 1$ by n.

After rewriting twice, the limit can be taken. Limits similar to this will appear in the next section along with a graphical interpretation that will give more meaning to the process.

THE DISTANCE PROBLEM

In ***Example 4, Sec. 5.1***, it is important to note that from $t = 0$ to $t = 5$ in the second table the velocity is treated as if it is a constant. Over this time interval, the velocity is 25 ft/sec. So the approximate distance traveled in the first five seconds is 125 feet. Next look at ***Figure 17***, and note that the height of the first shaded rectangle is 25 and its width is 5. So the area is 125 sq units. Although the units are different the number, 125, is the same. The general sum near the bottom of ***page 374*** has the same form as the one near the top of ***page 372***. (The second sum is a distance while the first is an area.) Also in ***(5)***, ***page 375***, a limit leads to an exact distance with the same reasoning as for areas.

SECTION 5.2: THE DEFINITE INTEGRAL

The major idea in this section is the introduction of the concept of a *definite integral* which is represented by the symbols, $\int_a^b f(x)\,dx$. Before indicating what it is, we mention that a definite integral will have two significant interpretations. One involves *antiderivatives* and the other *relates* to finding areas by the methods discussed in the last section. The fact that these are seemingly unrelated interpretations is the subject of the Fundamental Theorem of Calculus in the next section. After completing this chapter you may still feel that two ideas connect in some mysterious way even after looking at a proof.

DEFINITION OF A DEFINITE INTEGRAL

In the last section we established a procedure for finding the exact area of a region with curved sides. Now we attach a new name to this procedure. The *definite integral* represented by the symbols, $\int_a^b f(x)\,dx$ is equal to $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$. So for now, a definite integral is the limit of a sum of products of the form $f(x_i^*)\,\Delta x$.

The form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

is a rather complex set of symbols but it will be important to remember what they represent. In summary there are three parts.

(a)	(b)	(c)
$\lim_{n\to\infty}$	$\sum_{i=1}^{n}$	$f\left(x_i^*\right) \Delta x$

(a) The limit provides an exact value.

(b) Add individual components to get a total.

(c) The product $f\left(x_i^*\right)\Delta x$ represents the area of one rectangle or one distance or one ??

EXAMPLE 2(B)

As a way of stressing the above three parts we consider another example. The width of each rectangle is $\frac{3}{n}$. Then observe the pattern,

$$x_1 = \frac{3\cdot 1}{n}, \; x_2 = \frac{3\cdot 2}{n}, \; x_3 = \frac{3\cdot 3}{n} \; \ldots \; x_i = \frac{3\cdot i}{n}$$

The height of each rectangle is $f(x_i)$ which equals

$$\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right) \quad \text{or} \quad \frac{27i^3}{n^3} - \frac{18i}{n}$$

The summation, $\sum_{i=1}^{n}\left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)$ can be written

$$\sum_{i=1}^{n}\frac{27i^3}{n^3} - \sum_{i=1}^{n}\frac{18i}{n}$$

Because $\frac{27}{n^3}$ and $\frac{18}{n}$ do not contain i, they are can be treated as constants as i varies from 1 to n and placed before $\sum$.

$$\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i$$

Multiply by the width $\frac{3}{n}$ and use ***(5)*** and ***(7)*** on ***page 381*** to get line ***(6)*** in the text. The limit produces -6.75. ***Figure 5*** indicates why the answer is negative.

Is there a better way to evaluate a definite integral? Yes. Before completing this section we skip ahead to give a quick overview.

SECTION 5.3: THE FUNDAMENTAL THEOREM OF CALCULUS

For now we skip the proof of this very important theorem and discuss the end result. Part 2 is on ***page 396***:

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$

We illustrate the theorem by going back to ***Example 2(b), Sec. 5.2***.

$$\int_a^b f(x)\,dx \quad \text{matches} \quad \int_0^3 (x^3 - 6x)\,dx$$

$$f(x) = x^3 - 6x\ ,\ a = 0\ ,\ b = 3$$

Now according to the Fundamental Theorem we find any antiderivative of $x^3 - 6x$ and call it $F(x)$. Then

$$F(x) \quad \text{can be} \quad \frac{x^4}{4} - 6\frac{x^2}{2} \quad \text{or} \quad \frac{x^4}{4} - 3x^2$$

Note $\quad F'(x) = f(x)$

$$\left(\frac{x^4}{4} - 3x^2\right)' = \frac{4x^3}{4} - 3(2x) = x^3 - 6x$$

Next
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

becomes
$$\int_0^3 (x^3 - 6x)\,dx = F(3) - F(0)$$

$$= \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0^4}{4} - 3(0)\right)$$

The final result is
$$\int_0^3 (x^3 - 6x)\,dx = \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

This agrees with the result above.

NOTATION AGAIN

In ***Example 5***, ***page 392***, you will see

the notation $\quad F(x)\Big]_a^b = F(b) - F(a)$

We redo the last example using this notation.

$$\int_0^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 3x^2\Big]_0^3 = \left(\frac{3^4}{4} - 3(3^2)\right) - \left(\frac{0^4}{4} - 3(0)\right) = -6.75$$

To get the full impact of the Fundamental Theorem, compare the last line with the steps on ***page 382, Sec. 5.2***. Finding an antiderivative and making a calculation using the limits of the definite integral is somehow equivalent to adding those rectangles you see on ***page 382*** and taking a limit of sum. Later we look at the proof of the Fundamental Theorem to see if it offers any insight into this marvelous shortcut.

ONCE MORE

Example 2, Section 5.1, provided the first pattern of adding the areas of rectangles and then taking a limit. We offer one more comparison with the shorter method that comes from the Fundamental Theorem. The area of the region shown in ***Figure 3, page 360***, can be found by:

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Same answer with less work.

WARNING

This shortcut can be performed efficiently for basic functions but if you don't know the meaning it won't be of value. The meaning lies in the symbols

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

which we abbreviate to: $\quad \lim \sum f\left(x_i^*\right) \Delta x$

to emphasize: 1. a limit which gives exact values

2. $\sum$ for a sum of

3. the products, $f(x_i)\Delta x$.

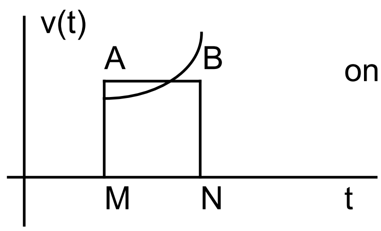
We hope the following question is burning inside of you:

How can the evaluation of an antiderivative, $F(b) - F(a)$ match $\lim \sum f(x_i)\, \Delta x$?
And even if you don't get what you consider to be an adequate answer hold onto the question. It keeps the limit of a sum of products in your mind as you are doing an algebraic procedure.

Then you can say, of course, to the assertion that:

$$\int_0^4 v(t)\; dt \quad \text{is the distance traveled in four hours}$$

if $v(t)$ is a positive velocity in $^{\text{miles}}\!/_{\text{hour}}$ and t hours represents time. It will help if you note in the diagram that the curve represents a varying velocity while on line segment AB the velocity is constant.



one of many rectangles

The length of MA is a constant velocity over a time interval, MN. The product

$$v(t_i)\Delta t \qquad \text{from} \qquad \lim \sum v(t_i)\, \Delta t$$

is now $(velocity)\cdot(time)$ which is *distance* instead of *area*. So $\sum v(t_i)\, \Delta t$ is now adding distances to get an approximation and in $\lim \sum v(t_i)\, \Delta t$ the limit makes the approximation exact.

This illustration shows how the formula:

$$(rate)\cdot(time) = \text{distance}$$

which is true only when the rate is constant, is extended via

$$\lim \sum v(t_i)\, \Delta t$$

to $\displaystyle\int_a^b f(x)\; dx$ or $\displaystyle\int_0^4 v(t)\; dt$

BACK TO SECTION 5.2

After introducing the concept of a definite integral, this section presents two types of properties of integrals. The first set on ***page 385*** will be used in evaluating a definite integral. This process is called *integration* and will rely on the Fundamental Theorem. The second set of properties on ***page 387*** is important in proving the Fundamental Theorem. We will comment on some of the properties and also bring your attention to other ideas in this section.

1. ***Pages 378-379***. If $f(x_i)$ is a negative number, then the product $f(x_i)\,\Delta x_i$ will also be negative. The sum $\sum f(x_i)\,\Delta x_i$ can contain negative terms as shown in ***Figure 3***. It will be important to keep this in mind when interpreting a definite integral.

2. ***Page 385***. If $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. If the limits are reversed, then the sign of the answer is also reversed.

3. Note the Midpoint Rule on ***page 384***. It is the simplest method for approximating a definite integral. Many calculators have a button labeled $\int$ that will provide a numerical approximation to a definite integral. The Midpoint Rule is the first of several methods of approximating definite integrals.

PROPERTIES OF THE INTEGRAL

The four properties listed at the bottom of ***page 385*** are significant. In particular note property 2, which indicates that one can focus on individual terms of a sum. Later, you will be concerned as to whether or not you can integrate a function. Property 2 says that $x^2 + 1$ can be separated into two parts. The same is not true for $\sqrt{x^2 + 1}$. It must be treated as a single entity and consequently is much harder to deal with.

Property 4 is similar to ***Property 2***.

Property 3 says a *constant* c can be placed outside the integral sign. This matches the similar property for differentiation. However, these properties apply only to constants, not variables. For example,

$$\int_0^1 x^2 \sqrt{x^2 + 1}\,dx \quad \text{is not the same as} \quad x\int_0^1 x\sqrt{x^2 + 1}\,dx .$$

You will learn how to evaluate $\int_0^1 x\sqrt{x^2+1}\,dx$ in ***Section 5.5*** but the evaluation of the first integral will require a more elaborate method involving the use of trig identities.

For ***Property 5, page 387***, look at ***Figure 15***. In terms of areas, the area from a to b, $\int_a^b f(x)\,dx$, equals the sum of the area from a to c, $\int_a^c f(x)\,dx$, and the area from c to b, $\int_c^b f(x)\,dx$.

Remember that area is only one interpretation of a definite integral. A mathematician's proof must be general enough to cover all situations. At this point, area provides a framework to give meaning to $\int_a^b f(x)\,dx$, but if this were the only interpretation, definite integrals would not be significant.

ORDER PROPERTIES

Three more properties of definite integrals are listed on ***page 387***. In particular, note the geometric interpretation of ***Property 8*** in ***Figure 16***. Remember that another drawing is required if $f(x)$ is negative for some values of x. The proof shown to the right of ***Figure 16*** covers all possibilities in one set of algebraic statements.

SECTION 5.3: THE FUNDAMENTAL THEOREM OF CALCULUS

A key part of the proof of the Fundamental Theorem is a new type of function shown on ***page 392***:

$$g(x) = \int_a^x f(t)\,dt$$

Using the area interpretation of a definite integral shown in ***Figure 1***, note that as x changes, the shaded area changes. So the shaded area depends on the value of x, or the area is a function of x. Of course, the area also depends on the particular curve, $f(t)$, but *after* we have selected $f(t)$ the area depends only on the value of x.

Think of $g(x) = \int_a^x f(t)\,dt$ as the area under the curve, $y = f(t)$, but above the x-axis from $t = a$ to $t = x$.

The first part of the Fundamental Theorem, ***page 394***, establishes that the derivative of the area function $g(x)$ is $f(x)$. This is the same as saying that $g(x)$ is the antiderivative of $f(x)$. So going back to the definition of a derivative, we must prove:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

In the paragraph to the right of ***Figure 5***, the author gives an intuitive outline of the proof for $f(t) \geq 0$. In particular, note that $g(x+h) - g(x)$ is the shaded area in ***Figure 5***. This is repeated in the more general proof on ***page 394***. It is essential that the notation makes sense.

Starting with $\quad g(x) = \int_a^x f(t)\, dt$

replace x with $x+h$ $\quad g(x+h) = \int_a^{x+h} f(t)\, dt$

Then $\quad g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

Two choices.

1. Use the graph in ***Figure 5***, to interpret the right side of the last equation.

   Subtract the area from a to x, $\int_a^x f(t)\,dt$,

   from the area from a to $x+h$, $\int_a^{x+h} f(t)\,dt$

   to get the area from x to $x+h$, $\int_x^{x+h} f(t)\,dt$. This approach depends on $f(t) \geq 0$.

2. The more general approach is to use ***Property 5, page 387*** to rewrite $\int_a^{x+h} f(t)dt$ as shown on ***page 394***.

Both choices lead to $\quad g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt$

Divide by h $\quad \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$.

The last step is to find the limit as h approaches zero.

Using ***Property 8, page 387, Sec. 5.2***, produces:

$$m\,h \;\le\; \int_x^{x+h} f(t)\,dt \;\le\; M\,h$$

In ***Figure 6, Sec. 5.3***, imagine a rectangle whose width is h and whose height is m. Then do the same for a rectangle whose width is h and height is M. Clearly, the area, $\int_x^{x+h} f(t)\,dt$, will be less than the area of the big rectangle, $M\,h$, but larger than the area of the small rectangle, $m\,h$.

Divide by h to get $\quad \frac{1}{h}\int_x^{x+h} f(t)\,dt$ in the middle,

which can be replaced with $\quad \dfrac{g(x+h) - g(x)}{h}$

This leads to $\quad \dfrac{g(x+h) - g(x)}{h}$ being placed

between $m = f(u)$ and $M = f(v)$.

The last important step is to find the limit as h approaches zero.

Peer closely into ***Figure 6***, and as $h \to 0$ both u and v must approach x. Then

$$f(u) \;\le\; \frac{g(x+h) - g(x)}{h} \;\le\; f(v)$$

tends to

$$f(x) \;\le\; \frac{g(x+h) - g(x)}{h} \;\le\; f(x)$$

So the limit of $\quad \dfrac{g(x+h) - g(x)}{h}$ must be $f(x)$ or

$$g'(x) = f(x)$$

$g(x)$ being the antiderivative of $f(x)$.

PROOFS AND ILLUSTRATIONS

Are you convinced that $\lim \sum f(x_i)\,\Delta x_i$ is connected to the antiderivative of $f(x)$? As you have probably noticed, a mathematical proof is a series of logical steps. In contrast, an illustration is designed to explain a concept or a procedure. Expecting a proof to provide a clear explanation frequently leads to disappointment. We hope that isn't the case here, but keep in mind the connection between finding areas and antiderivatives is a significant accomplishment. If it was easy to see the connection, it would have been developed much sooner. The last paragraph beginning on ***page 399*** provides an interesting perspective on its significance.

Proof of Part II

Part I establishes that $\int_a^x f(t)\,dt$ equals an antiderivative of $f(x)$. Next, the corollary on ***page 291, Sec. 4.2*** asserts that if $F(x)$ and $g(x)$ are both antiderivatives of $f(x)$, then:

(1) $\quad F(x) = g(x) + C$ for all x in (a,b).

Because $\quad g(x) = \int_a^x f(t)\,dt$ equation (1) can be rewritten as

(2) $\quad F(x) = \int_a^x f(t)\,dt + C$

Now make two observations:

1. Replace x with b to get:

$$F(b) = \int_a^b f(t)\,dt + C$$

2. Replace x with a to get:

$$F(a) = \int_a^a f(t)\,dt + C = 0 + C$$

The constant C equals $F(a)$ and the equation in part 1 becomes

$$F(b) = \int_a^b f(t)\,dt + F(a)$$

which can be written

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

It has been a long route to this result, but we now have a very powerful tool.

EQUATION

Equation 5 on ***page 395***, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, is in some sense a test of your understanding of the Fundamental Theorem. Remember Part II:

$\int_a^x f(t)\,dt = F(x) - F(a)$ where $F(x)$ is an antiderivative of $f(x)$ or $F'(x) = f(x)$.

Then the derivative,

$$\frac{d}{dx}\int_a^x f(t)\,dt \quad \text{equals} \quad \frac{d}{dx}\left[F(x) - F(a)\right]$$

$$= F'(x) - \frac{d}{dx}F(a)$$

Because a is a constant, $F(a)$ is also a constant and $\frac{d}{dx}F(a) = 0$.

$$\frac{d}{dx}\int_a^x f(t)\,dt \;\text{ equals }\; F'(x) \;\text{ which is }\; f(x)$$

Example:

$$\int_1^x (t^2 + 3t)\,dt = \left.\frac{t^3}{3} + 3\frac{t^2}{2}\right]_1^x$$

$$= \left(\frac{x^3}{3} + \frac{3x^2}{2}\right) - \left(\frac{1}{3} + \frac{3}{2}\right)$$

Now find the derivative of the result and verify that:

$$\frac{d}{dx}\int_1^x (t^2 + 3t)\,dt = x^2 + 3x$$

Also look at ***Example 3, Sec. 5.3***. For the Fresnel function:

$$S(x) = \int_0^x \sin\left(\pi t^2/2\right)\,dt$$

we can not find an antiderivative in algebraic form. However, we do know the derivative is $\sin\left(\pi t^2/2\right)$. The example illustrates a method of analyzing the function.