module 9

9.1 2) $y = -t\cos t - t$ is a solution of $t\frac{dy}{dt} = y + t^2\sin t$ $y(\pi) = 0$

$y' = t\sin t - \cos t - 1$

$t\frac{dy}{dt} = t^2\sin t - t\cos t - t$

$= t^2\sin t + y$ ✓

check initial condition!

$y(\pi) = 0$

$y = \pi - \pi = 0$ ✓

3) $e^{rx} = f(x)$

a)

$f'(x) = re^{rx}$

$f''(x) = r^2e^{rx}$

$2r^2e^{rx} + re^{rx} - e^{rx} = 0$

$e^{rx}(2r^2 + r - 1) = 0$

e^{rx} doesn't ever $= 0$ so the differential is satisfied when $2r^2 + r - 1 = 0$

$(2r - 1)(r + 1) = 0$

$\boxed{r = \{\frac{1}{2}, -1\}}$

check for -1!

$2e^{-x} - e^{-x} - e^{-x} = 0$ ✓

check for $\frac{1}{2}$!

$\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{\frac{1}{2}x} - e^{\frac{1}{2}x} = 0$ ✓

b) show $y = ae^{\frac{1}{2}x} + be^{-x}$ also satisfies

$2y'' + y' - y = 0$

$y' = \frac{1}{2}ae^{\frac{1}{2}x} + -be^{-x}$

$y'' = \frac{1}{4}ae^{\frac{1}{2}x} + be^{-x}$

$= 2\left(\frac{1}{4}ae^{\frac{1}{2}x} + be^{-x}\right) + \frac{1}{2}ae^{\frac{1}{2}x} - be^{-x} - \left(ae^{\frac{1}{2}x} + be^{-x}\right) = 0$

$= \frac{1}{2}ae^{\frac{1}{2}x} + \cancel{2be^{-x}} + \frac{1}{2}ae^{\frac{1}{2}x} - \cancel{be^{-x}} - ae^{\frac{1}{2}x} - \cancel{be^{-x}} = 0$

it does ✓

6a) $y = \frac{\ln x + C}{x}$ is a solution of $x^2y' + xy = 1$

$y' = \frac{x \cdot \frac{1}{x} - \ln x + C}{x^2} = \frac{1 - \ln x + C}{x^2}$

$1 - \ln x + C + \ln x + C = 1$ ✓

6b) if $y(2) = 1$ so $1 = \frac{\ln 2 + C}{2}$ so $2 = \ln 2 + C$ $C = \frac{2}{\ln 2}$

so the solution is $\boxed{y = \frac{\ln x + \frac{2}{\ln 2}}{x}}$

6d) check:

$y = \dfrac{\ln x + \frac{2}{\ln 2}}{x}$ so $x^2 (y') + \ln x + \frac{2}{\ln 2} = 1$

$y' = \dfrac{x \frac{1}{x} - \ln x + \frac{2}{\ln 2}}{x^2}$ so $1 - \ln x - \frac{2}{\ln 2} + \ln x + \frac{2}{\ln 2} = 1$ so still good.

7b) $y = \dfrac{1}{x+C}$

$y' = \dfrac{-1}{(x+C)^2}$

so $-y^2 = \dfrac{-1}{(x+C)^2}$ so

y is a solution because $-y^2 = y'$

7d) $y(0) = 0.5$

$0.5 = \dfrac{1}{0+C}$ $\rightarrow 0.5C = 1$

$C = \frac{1}{0.5} = 2$

so a solution to $y' = -y^2$ and $y(0) = 0.5$ is

$y = \dfrac{1}{x+2}$

8b) $y = (c-x^2)^{-1/2}$ are solutions of $y' = xy^3$

$y' = -\frac{1}{2}(c-x^2)^{\frac{-3}{2}} \cdot -2x$

$y' = x(c-x^2)^{\frac{-3}{2}}$ $\longrightarrow$ $y' = x(c-x^2)^{\frac{-3}{2}}$ ✓

check: $xy^3 = x\left((c-x^2)^{-\frac{1}{2}}\right)^3 =$

8d) $y(0) = 2$

so $y = (C-0)^{-\frac{1}{2}} = 2$

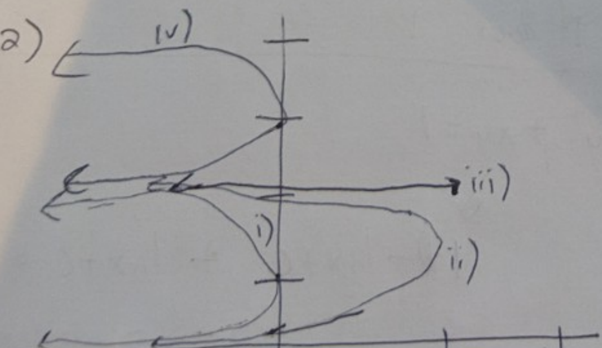
$C^{-\frac{1}{2}} = 2$

$C = 2^{-2} = \frac{1}{4}$

so a solution to $y' = xy^3$ and $y(0) = 2$ is

$\boxed{y = \left(\frac{1}{4} - x^2\right)^{-\frac{1}{2}}}$

9.2 a)



b) $y = 2$ $y = 0$

$y = 4$

$y = 2x$ generally

Solutions approach two lines as $x \to -\infty$

3) III

Slope $=0$ at $y=2$

slope positive when $x=0$

slope negative when $x=4$

4) slope $=0$ at $x=0$ and $y=2$ so I or II

at $(2,2)$ slope is 0 so I

5) at $(\frac{1}{2},\frac{1}{2})$ slope is 0 so IV or II

at $(4,-4)$ slope is 1

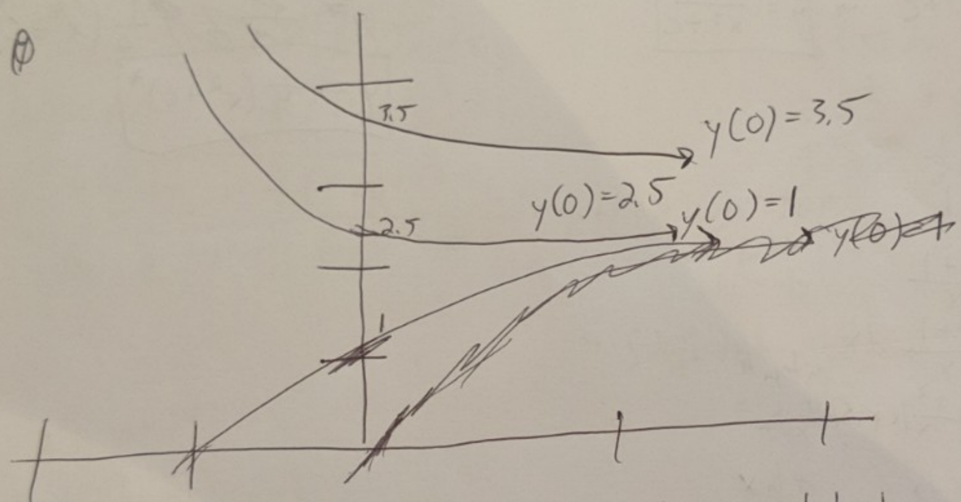
at $(-4,4)$ slope is 1 so IV

6) at $(\frac{\pi}{2},\frac{\pi}{2})$ slope 1

$(\frac{\pi}{2},-\frac{\pi}{2})$ slope -1 II

$(\frac{-\pi}{2},\frac{\pi}{2})$ slope -1

8)



24) a) $h=0.2$ to estimate $y(0.6)$ where $y(x)$ is solution to $y'=\cos(x+y), y(0)=0$

$y_0=0$

$y_{n+1}=y_n+h(F(x_n;y_n))$

$y_1=0+0.2(\cos(0))=0.2$

$y_2=0.2+0.2(\cos(0.2+0.2))=0.2+0.184=0.384$

$y_3=0.384+0.2(\cos(0.4+0.384))=0.384+0.141=\boxed{0.526}$

b) $h=0.1$ estimate $y(0.6)$ where $y(x)$ is a solution to $y'=\cos(x+y), y(0)=0$

$y_0=0$ so

$y_1=0+0.1(1)=0.1$ $\quad y_2=0.1+0.1(\cos(0.1+0.1))=.198$

$y_3=0.198+0.1(\cos(0.2+0.198))=0.290$ $\quad y_4=0.290+0.1(\cos(0.3+0.290))=0.373$

$y_5=0.373+0.1(\cos(0.4+0.373))=0.446$ $\quad y_6=0.446+0.1(\cos(0.5+0.446))=\boxed{0.504}$

1) $\frac{dy}{dx} = 3x^2 y^2$

$\frac{dy}{y^2} = 3x^2\,dx$ separate

$y^{-2}\,dy = 3x^2\,dx$ integrate

$-y^{-1} = x^3 + C$

$\frac{-1}{y} = x^3 + C \rightarrow \boxed{y = \frac{-1}{x^3+C}}$

2) $\frac{dy}{dx} = x\sqrt{y} \rightarrow \frac{dy}{\sqrt{y}} = x\,dx$

$\rightarrow \int y^{-\frac{1}{2}}\,dy = \int x\,dx \rightarrow 2y^{\frac{1}{2}} = \frac{x^2}{2} + C$

$\rightarrow y^{\frac{1}{2}} = \frac{x^2}{2} + \frac{C}{2}$

$y^{\frac{1}{2}} = \frac{1}{2}(x^2 + C)$

$\boxed{y = \frac{1}{4}(x^2+C)^2}$

3) $xyy' = x^2 + 1$

$y\frac{dy}{dx} = \frac{x^2+1}{x}$

$y\,dy = \frac{x^2+1}{x}\,dx$ integrate

$\frac{y^2}{2} = \int \frac{x^2}{x}dx + \int \frac{1}{x}dx$

$= \int x\,dx + \ln x$

$\frac{y^2}{2} = \frac{x^2}{2} + \ln x + C$

$y^2 = x^2 + 2\ln x \rightarrow \boxed{y = \sqrt{x^2 + 2\ln x + C}}$

4) $y' + xe^y = 0 \rightarrow \frac{dy}{dx} + xe^y = 0$

$\frac{dy}{dx} = -xe^y$

$\frac{dy}{e^y} = -x\,dx \rightarrow \frac{1}{e^y}dy = -x\,dx$

$\int \frac{1}{e^y}dy = \int -x\,dx \rightarrow$

$u = -y$
$du = -1\,dy$

$\int e^{-y}\,dy = \int -x\,dx$

$-e^{-y} = -\frac{x^2}{2} + C$

$e^{-y} = \frac{x^2}{2} - C \rightarrow e^y = \frac{1}{\frac{x^2}{2} - C} = \frac{2}{x^2 - C} \rightarrow \boxed{y = \ln\left(\frac{2}{x^2 - C}\right)}$

5) $(e^y - 1)y' = 2 + \cos x$ $\rightarrow (e^y - 1)\frac{dy}{dx} = 2 + \cos x$

$e^y - 1\, dy = (2 + \cos x)\, dx$ integrate

$e^y - \cancel{e}y = 2x + \sin x + C$

$\cancel{e}y - \ln y = \ln(2x + \sin x \cancel{} + c)$

8) $\frac{dH}{dR} = \frac{RH^2\sqrt{1+R^2}}{\ln H}$

$\ln H\, dH \cancel{} = RH^2\sqrt{1+R^2}\, dR$

$\frac{\ln H}{H^2}\, dH = R\sqrt{1+R^2}\, dR$ integrate

$= \cancel{\ldots}\ \frac{1}{2}\int \sqrt{u}\, du$

$\frac{-\ln H - 1}{H} + C = \frac{1}{3}(1+R^2)^{\frac{3}{2}} + C$

$$\boxed{-\frac{\ln H + 1}{H} = \frac{1}{3}(1+R^2)^{\frac{3}{2}} + C}$$

H is an implicit function of R

right side: $u = 1 + R^2$

$\frac{du}{dR} = 2R \rightarrow du = 2R\,dR$

$\frac{du}{2} = R\,dR$

$\frac{1}{2}\int \sqrt{u}\, du$

$\frac{1}{3}u^{\frac{3}{2}} + C$

$\cancel{\frac{1}{3}(\ldots)}\ \frac{1}{3}(1+R^2)^{\frac{3}{2}} + C$

left side use IBP:

$u = \ln H \quad dv = H^{-2} \quad v = -H^{-1}$

$du = \frac{1}{H}$

so $\int \frac{\ln H}{H^2}\, dH = \ln H \cdot -H^{-1} - \int -H^{-1}\frac{1}{H}$

$= \frac{-\ln H}{H} - \int \frac{-1}{H^2}$

$= \frac{-\ln H}{H} - \int -H^{-2}$

$= \frac{-\ln H}{H} - H^{-1} = \frac{-\ln H}{H} - \frac{1}{H} =$

$-\frac{\ln H - 1}{H} + C$

9) $\frac{dp}{dt} = t^2 p - p + t^2 - 1$

$= t^2(p+1) - p - 1$

$t^2(p+1) \cancel{+} - (p+1)$

$\frac{dp}{dt} = (p+1)(t^2 - 1)$

$\frac{dp}{p+1} = (t^2 - 1)\, dt$ integrate

$\ln|p+1| = \frac{t^3}{3} - t + C$

$|p+1| = e^{\frac{t^3}{3} - t + C}$

$\cancel{p = e^{\frac{t^3}{3} - t + C} - 1}$

$p + 1 = \pm e^{\frac{t^3}{3} - t + C}$

$= \pm e^C e^{\frac{t^3}{3} - t}$

$= \boxed{K e^{\frac{t^3}{3} - t} \quad \text{where } K = \pm e^C}$

10) $\frac{dz}{dt} + e^{t+z} = 0$

$\frac{dz}{dt} = -e^t e^z \rightarrow \frac{dz}{e^z} = -e^t dt$

integrate

$-e^{-z} = -e^t + c$

$e^{-z} = e^t - c$

$-z = \ln(e^t - c)$

$\boxed{z = -\ln(e^t - c)}$

11) $\frac{dy}{dx} = xe^y$, $y(0) = 0$

$\frac{dy}{e^y} = x\,dx$ integrate next

$-e^{-y} = \frac{x^2}{2} + c \rightarrow e^{-y} = \frac{-x^2}{2} - c$

$-y = \ln\left(\frac{-x^2}{2} \mp c\right)$

$y = -\ln\left(\frac{-x^2}{2} \mp c\right)$

$y =$

$\ln \frac{1}{\frac{-x^2}{2} - c}$

$y = \ln \frac{2}{-x^2 - 2c}$

so $\boxed{y = \ln \frac{2}{-x^2 + 2}}$

so if $y(0) = 0$ then

$\frac{x^2}{2} = c + 1$

$-2c = 2$

$c = -1$

12) $\frac{dy}{dx} = \frac{x \sin x}{y}$, $y(0) = -1$

$y\,dy = x \sin x\,dx$

integrate

$\frac{y^2}{2} + c = -x\cos x + \sin x + c$

$y^2 = 2(-x\cos x + \sin x + c_1)$ $\quad 2c = c_1$

$y = \pm\sqrt{-2x\cos x + 2\sin x + 2c_1}$

right side use IBP

$\int x \sin x\,dx$

$u = x \quad du = 1$

$dv = \sin x \quad v = -\cos x$

$= -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \sin x$

when $y(0)$

$-1 = \pm\sqrt{c_1}$

$c_1 = 1$

14) $x + 3y^2\sqrt{x^2+1}\ \frac{dy}{dx} = 0 \qquad y(0) = 1$

$3y^2\sqrt{x^2+1}\ \frac{dy}{\cancel{dx}} = -x\,dx$

$3y^2\,dy = \frac{-x}{\sqrt{x^2+1}}\,dx$ integrate this

right side: $u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$y^3 = \frac{1}{2}\int -\frac{1}{\sqrt{u}}\,du$

$= -\frac{2}{2}\sqrt{u} + C$

$y^3 = -\sqrt{x^2+1} + C$

$y = \cancel{\ldots}$ if $y(0) = 1$

$(-\sqrt{x^2+1} + C)^{\frac{1}{3}}$

$1 = (-\sqrt{1} + C)^{\frac{1}{3}}$ so $C = 2$

so a solution of the initial value problem is

$y = (-\sqrt{x^2+1} + 2)^{\frac{1}{3}}$

31) $y = \frac{k}{x}$ Find orthogonal trajectories

take derivative

$\frac{dy}{dx} = -k\frac{1}{x^2} = \frac{-k}{x^2}$

this is the slope of the family of curves for $y = \frac{k}{x}$

but, in order to find orthogonal trajectories, we need to find perpendicular curves. the slopes of those are $\frac{x^2}{k}$.

orthogonal trajectory slope:

$\frac{dy}{dx} = \frac{x^2}{k}$ and $k = yx$

$= \frac{x^2}{xy}$

$xy\,dy = x^2\,dx$

$y\,dy = x\,dx$ now integrate $\rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \rightarrow \boxed{y = x} + C$

45) $\frac{dy}{dt}$ = rate in − rate out

we can find y by starting with the derivative of y

rate in $= \left(10\frac{L}{min}\right)\left(0\frac{kg}{L}\right) = 0$

rate out $= \left(\frac{y(t)}{1000L}\right)\left(10\frac{L}{min}\right) = \frac{y(t)\,kg}{100 min}$

$\frac{dy}{dt} = \frac{-y(t)}{100}$

$100dy = -y(t)dt \rightarrow 100dy = -y\,dt \rightarrow \frac{dy}{y} = \frac{-dt}{100}$ now integrate

$\ln y = \frac{-1}{100}t + C \rightarrow y = e^{-1/100 t + C}$

$= e^{\frac{-1}{100}t}e^{C}$

$= Ce^{\frac{-1}{100}t}$

(on back)

initial condition: $y(0) = 15$

so $y(0) = 15 = Ce^{\frac{-1}{100}0} \rightarrow C = 15$ so the solution given the initial condition is $\boxed{y = 15e^{\frac{-1}{100}t}}$

b) so $y(20) = 15e^{\frac{-1}{100} \cdot 20} = 15e^{\frac{-1}{5}} \boxed{\approx 12.28k}$