Module 6

SECTION 11.3: THE INTEGRAL TEST AND ESTIMATES OF SUMS

Begin this module by reading the first paragraph of this section. In particular, note that we will be developing several tests to determine if a series converges or diverges. No test will work for all types of series. Rather, with experience, you will be able to match a test with particular types of series.

THE INTEGRAL TEST

In this first test we determine whether a series converges or diverges by matching it with an improper integral. If the improper integral converges, the series converges, and if the improper integral diverges, the series diverges. To understand the connection between an improper integral and a series, we need to look at the proof.

A key step is to match inequality **(4)** on ***page 725***,

$$a_2 + a_3 + \ldots + a_n \le \int_1^n f(x)\,dx$$

with ***Figure 5***. First note that a_2 is both a y-value and the area of the first shaded rectangle. The function connects to the series by

$$f(1) = a_1 , f(2) = a_2 , \ldots , f(n) = a_n$$

In ***Figure 5***, the *height* of the first rectangle is $a_2 = f(2)$. The width of each rectangle is one. So the area of each rectangle equals the height.

The total area extends from $x = 1$ to $x = n$, and the sum of the areas of the rectangles is less than the area under the curve, which is represented by

$$\int_1^n f(x)\,dx$$

We repeat that a_1 does not appear because it is not the *height* of one of the rectangles, but 1 does appear as the lower limit for the integral because the area starts at $x = 1$.

In a similar manner, inequality **(5)**

$$\int_1^n f(x)\,dx \le a_1 + a_2 + \ldots + a_{n-1}$$

matches ***Figure 6***. Now the height of the first rectangle is $a_1 = f(1)$ and the area of the first rectangle is a_1. The height of the last rectangle is $f(n-1)$ which equals a_{n-1}.

Next recall that if we are talking about the convergence of a series, we need to look at the partial sums. (At least until we establish several tests.) In

$$a_2 + a_3 + \ldots + a_n \leq \int_1^n f(x)\,dx$$

we need to add a_1 to get the partial sum

$$s_n = a_1 + a_2 + \ldots + a_n$$

Then

$$\begin{aligned} s_n &\leq a_1 + \int_1^n f(x)\,dx \\ &\leq a_1 + \int_1^\infty f(x)\,dx \end{aligned}$$

The Integral Test requires that $f(x)$ be a positive function so the area from 1 to n is smaller than the area from 1 to infinity.

Now the punch line: if $\int_1^\infty f(x)\,dx$ converges, then the improper integral equals some finite number, say *F*. Then

$$s_n \leq a_1 + F \qquad \text{for all } n.$$

This means that the sequence of partial sums, $\{s_n\}$, is bounded. Also, the sequence is increasing. Tack on another rectangle, and the area gets larger. By theorem ***(12), page 702,*** a bounded, increasing sequence must converge. By definition, if the sequence of partial sums converges, then the series converges. This establishes part (a) of the Integral Test.

To establish part (b), go back to the second inequality,

$$\int_1^n f(x)\,dx \leq a_1 + a_2 + \ldots + a_{n-1}$$

and rewrite in terms of the partial sum, s_{n-1}.

$$\int_1^n f(x)\,dx \leq s_{n-1}$$

Now let n approach infinity. If the improper integral $\int_1^\infty f(x)\,dx$ diverges, it approaches infinity, forcing s_{n-1} to also approach infinity.

If the partial sums get larger and larger, the series diverges.

USING THE INTEGRAL TEST

There is a limitation on using the Integral Test. We have to be able to carry out the integration. When the *n*th term, a_n , doesn't match a function that can be readily integrated, you should use another test. However, there will be a few situations where all other tests fail and the Integral Test comes to the rescue.

THE *P*-SERIES TEST

One important use of the Integral Test is to establish what is called the *p*-series Test. A *p*-series is of the form $\sum_1^\infty \frac{1}{n^p}$. Examples include

$$\sum_1^\infty \frac{1}{n^2} \quad , \quad \sum_1^\infty \frac{1}{n^5} \quad , \quad \sum_1^\infty \frac{1}{\sqrt[3]{n}} \; , \quad \text{and} \quad \sum_1^\infty \frac{1}{n^{1.001}}$$

where $\quad p = 2 , \quad p = 5 , \quad p = 1/3 , \quad$ *and* $\quad p = 1.001.$

Once you can recognize the form, the test is easy to apply.

A p-series converges if $p > 1$ *and diverges if* $p \leq 1$.

This can be proved by the Integral Test. We *illustrate* the *proof* by two examples.

1. The *p*-series $\sum_1^\infty \frac{1}{\sqrt[3]{n}}$ diverges because $p = 1/3 < 1$.

Supporting evidence: $$\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/3}\,dx$$

$$= \lim_{t\to\infty} \frac{3}{2}\left(t^{2/3} - 1\right) = \infty$$

By the Integral Test, because the improper integral diverges, the series diverges.

2. The p-series $\sum_{1}^{\infty} \frac{1}{n^{1.001}}$ converges because $p = 1.001 > 1$.

Supporting evidence:

$$\int_1^{\infty} \frac{1}{x^{1.001}}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-1.001}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{t^{-.001}}{-.001} + \frac{1}{.001} \right)$$

$$= 1{,}000$$

By the Integral Test, because the improper integral converges, the series converges.

In the second example, note that the exponent, -1.001, is sufficiently negative so that, after adding one, the exponent, - .001 , is still negative. This negative exponent pushes t into the denominator, and when t approaches infinity, this factor approaches zero. In the first example, the exponent, $-\frac{1}{3} + 1$, is positive, which in the limit produces infinity.

A WORD OF CAUTION

In the last two examples, you need only use the p-series Test to show divergence and convergence. The integrations were included to illustrate the *proof* of the p-series Test using the Integral Test.

DECREASING IN THE INTEGRAL TEST

In the description of the Integral Test, ***page 721***, the function, *f(x)* , must be a decreasing function. In practice the function must also approach zero. Why? Because the function will be defined by the nth term of a particular sequence. If a_n doesn't approach zero, then the series will diverge (by the Test of Divergence, ***page 713***), and you will not need to use the Integral Test.

HARMONIC SERIES AGAIN

If you had doubts about the divergence of the harmonic series, $\sum \frac{1}{n}$, note that it is a p-series, where $p = 1$. This is the borderline case but falls on the divergent side. Or use the Integral Test.

$$\int_1^{\infty} \frac{1}{x}\,dx \;=\; \lim_{t\to\infty} \left(\ln t \;-\; \ln 1\right) \;=\; \infty$$

Earlier we noted that $(\ln t)$ approaches infinity *slowly*. See the discussion on ***pp. 62-3***. For $(\ln t)$ to reach just 11.5, t must be 100,000. This in turn implies that the harmonic series increases slowly.

TWO TESTS

At this point we have developed two tests, the Integral Test and the *p*-series Test, to determine whether a series converges or diverges. We now shift gears and consider approximations.

ESTIMATING THE SUM OF A SERIES

Using the *p*-series Test we can conclude that the series $\sum \frac{1}{n^2}$ converges; $p = 2$ and $2 > 1$. There is some number s that is the sum of *all* terms in the series. But we don't know the value of s. The sum can be easily determined for a convergent *geometric* series but not for $\sum \frac{1}{n^2}$.

A logical first step is to use, say, the first ten terms as an approximation to the sum s.

$$s_{10} \;=\; 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{100} \;\approx\; 1.54977$$

If we use 15 terms, then

$$s_{15} \;=\; 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{15^2} \;\approx\; 1.58044$$

But since we don't know the total sum of the series, we don't know how accurate these estimates are. Hence, we estimate the remainder or error using **(2)** on ***page 723***.

The first point is to clearly understand the symbols,

$$R_n \;=\; s \;-\; s_n \;=\; a_{n+1} \;+\; a_{n+2} \;+ \ldots$$

We are using s_n as an estimate the total sum of the series which is s.

Then the error is the difference, $s - s_n$, which is just the sum of the remaining terms in the series, starting with a_{n+1}. The word *remainder* is used for the remaining terms in the series. In the above example,

$$R_{10} = s - 1.54977 = \frac{1}{11^2} + \frac{1}{12^2} + \ldots$$

and
$$R_{15} = s - 1.58044 = \frac{1}{16^2} + \frac{1}{17^2} + \ldots$$

All well and good, but we still don't know *s*.

The remainder estimate,

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

follows in a clear way from ***Figures 3*** and ***4***. In ***Figure 3***, the area of the rectangles is clearly less than the area under the curve. The area of the rectangles is $a_{n+1} + a_{n+2} + \ldots$, which is R_n. This establishes the right two-thirds of the above inequality.

Then, in ***Figure 4***, the rectangles are an overestimate, but we start at $n + 1$. Remember, $a_{n+1} + a_{n+2} + \ldots = R_n$ is the form being described in the figures.

For the above example,

$$\int_{11}^{\infty} \frac{1}{x^2}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx$$

The error in using 1.54977 as an estimate of the total sum of the series is between the integrals. Evaluating the integrals we have

$$\frac{1}{11} \le R_{10} \le \frac{1}{10}$$

or
$$.09 \le R_{10} \le .10$$

The error is less than 1/10 but greater than 9/100.

For the error to be less than, say, 0.001, consider

$$R_n \leq \int_n^{\infty} \frac{1}{x^2}\, dx = \frac{1}{n}$$

We need to pick n such that $\frac{1}{n} < 0.001$

Then $\frac{1}{0.001} < n$

or $n > 1{,}000$

The claim is that if we use the sum of 1,001 terms as an approximation to the total sum of the series, the error is less than 0.001 .

A Second Estimate

The inequalities **(3)** on ***page 724***,

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_n^{\infty} f(x)\, dx$$

provide an approximation for s instead of R_n. For the above example with $n = 10$, we have

$$1.54977 + \frac{1}{11} \leq s \leq 1.54977 + \frac{1}{10}$$

or $$1.64068 \leq s \leq 1.64977$$

Last Comment

The series $\sum \frac{1}{n^2 - 1}$ *resembles* a p-series, but it is not in that category. To be a p-series, the form must be $\sum \frac{1}{n^p}$ with the only possible variation being in the value of p. However, in the next section tests will be developed that allow one to take advantage of the similarity to a p-series.

SECTION 11.4: THE COMPARISON TESTS

This section covers two important comparison tests. To determine if $\sum a_n$, converges or diverges, we compare it with another series, $\sum b_n$ in one of two ways. First, however, note that a_n will always refer to the given series and b_n to a known comparison series, which we must drag from our memory banks. In both cases we will be dealing with series that have positive terms only.

THE COMPARISON TEST

Given a series, $\sum a_n$, we try to find another series, $\sum b_n$, such that one of two inequalities is true. Either

1. $a_n \leq b_n$ for all n

or 2. $a_n \geq b_n$ for all n.

In the first case, if $\sum b_n$ converges, then we can conclude that $\sum a_n$ converges. If the "larger" series converges, then the "smaller" series must also converge.

In the second case, if $\sum b_n$ diverges, then we can conclude that $\sum a_n$ diverges. If the "smaller" series diverges, then the "larger" series must also diverge.

An intuitive proof is based on comparing

$$\sum a_n \qquad \text{and} \qquad \sum b_n$$

1. If $a_1 + a_2 + \ldots + a_n + \ldots \leq b_1 + b_2 + \ldots + b_n + \ldots$

and $\sum b_n$ converges to s, then we have an upper bound for $\sum a_n$.

Because all a_i terms are positive, the sum $(a_1 + a_2 + \ldots + a_n + \ldots)$ increases as n increases. Hence, the sequence of partial sums is increasing and bounded, and by theorem **(12)** on ***page 702***, the sequence must converge. This means the series $\sum a_n$ converges.

Hopefully, you will spend some time thinking about the gist of this proof and have a sense that, of course, this is the way it must be. Why? Because you can't reverse the inequality and get the same result. If the "larger" series diverges, there is no conclusion. Right?

2. If $a_1 + a_2 + \ldots + a_n + \ldots \geq b_1 + b_2 + \ldots + b_n + \ldots$

and $\sum b_n$ diverges, then the right side gets bigger and bigger. Now won't this make the left side (it *is* larger) bigger also? Yes. So

$\sum a_n$ must diverge also.

This test should perhaps be called the Inequality Comparison Test, but it is abbreviated to just the Comparison Test.

THE LIMIT COMPARISON TEST

In this test $\sum a_n$ again refers to the given series we are to test for convergence, and $\sum b_n$ refers to the comparison series we drag from memory. Now we form the quotient, $\frac{a_n}{b_n}$, and find the limit as n approaches infinity.

The most general result is on ***page 729***, but note there are special cases in ***exercises 40*** and ***41*** on ***page 732***. First, the limit is a positive *finite* constant *not* equal to zero. The constant can be 1 or 300 or whatever; it makes no difference. Then both series converge or both diverge.

The following is not a proof but a view that supports an intuitive understanding.

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$. This means that the ratio, $\frac{a_n}{b_n}$, eventually gets very close to 2 or $a_n \approx 2b_n$. Then

$$\sum a_n \approx \sum 2b_n$$

which supports the idea that both converge or both diverge. a_n approaches being twice as large as b_n, but this fact does not affect whether either series converges or diverges.

You can think of ***exercises 40*** and ***41*** as special cases. The inequalities in a proof support only one result in each case. If the limit is zero both converge, and if the limit is infinity both diverge. You may also use these extensions, if you wish.

USING THE COMPARISON TESTS

For series with positive terms, you will find that the Limit Comparison Test is the most useful. It has fewer limitations than the Integral, p-series, or Comparison Test. For example, the series, $\sum \frac{n+3}{n^3-4}$, would present quite a challenge if you used the Integral Test. For *both* the Comparison and Limit Comparison Tests we note that

$$\frac{n+3}{n^3-4} \quad \text{is similar to} \quad \frac{n}{n^3}\text{, which equals } \frac{1}{n^2}.$$

The series $\sum \frac{1}{n^2}$ is a convergent p-series, $p = 2 > 1$, so it *appears* that the original series also converges. However, we need to prove this by using a test.

For the Limit Comparison Test we have $a_n = \frac{n+3}{n^3-4}$ and $b_n = \frac{1}{n^2}$.

Next note that $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$. Instead of dividing by b_n, think of multiplying by the reciprocal of b_n ; in this case multiply by $\frac{n^2}{1}$.

$$\frac{a_n}{b_n} = \frac{n+3}{n^3-4} \cdot \frac{n^2}{1} = \frac{n^3+3n^2}{n^3-4}$$

Before finding the limit, divide all terms by n^3. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+3/n}{1-4/n^3} = 1$$

We conclude that the series converges by the Limit Comparison Test.

If you try to use the Comparison Test, you will encounter difficulties.

The desired inequality

$$\frac{n+3}{n^3-4} \le \frac{n}{n^3}$$

is not true for two reasons. First the numerator on the left is larger, making the left fraction larger. Also the denominator on the left is smaller, which also makes the left fraction larger. The correct inequality is

$$\frac{n+3}{n^3-4} \geq \frac{n}{n^3} = \frac{1}{n^2}$$

It is very important to realize that *no conclusion* can be drawn from this inequality. Being larger than a convergent series is not what we want.

Why? Consider two series.

$$\frac{1}{n} \geq \frac{1}{n^2} \qquad \text{(larger denominator — smaller fraction)}$$

and $\sum \frac{1}{n}$ diverges. But

$$\frac{1}{n^{1.5}} \geq \frac{1}{n^2} \qquad \text{(larger denominator — smaller fraction)}$$

and $\sum \frac{1}{n^{1.5}}$ is a convergent p-series, $p = 1.5 > 1$.

Conclusion: we can find *both* convergent and divergent series "larger" than a convergent series. We must find a series "smaller" than a convergent series to know for sure that it is convergent.

FINDING b_n

For both comparison tests, we need to compare the given series to another series, which we label $\sum b_n$. A general method for determining b_n is shown in the first part of ***Example 4, page 730***. To find what the text calls the *dominant* part, focus on the term with the highest exponent. In this example, we ignore the terms $3n$ and 5. When n becomes large these terms have less effect. If n is one million, squaring n means we have one million millions. In the term $3n$ we have just 3 million. Adding 5 to one million to the fifth power is negligible.

We list possible b_n's for selected odd exercises on ***page 731***. Try using the above method before looking at the answers.

3. $\frac{1}{n^2}$ **5.** $\frac{1}{n^{0.5}}$ **7.** $\left(\frac{9}{10}\right)^n$ **15.** $4(\frac{4}{3})^n$ or just $(\frac{4}{3})^n$ **17.** $\frac{1}{n}$ **23.** $\frac{1}{n^3}$

The factorials in ***exercises 29*** and ***30*** are best handled by the Ratio Test, which is covered later. Remember $\sum b_n$ should be a simple series that you KNOW converges or diverges. The best candidates are the p-series and the geometric series.

MAKING COMPARISONS

We have suggested that you may want to try the Limit Comparison Test first. Should it fail, however, be careful in establishing an inequality involving fractions.

1. If the numerators are the same, then the *larger* denominator produces the *smaller* fraction. Just look at $\frac{1}{2}$ and $\frac{1}{10}$.

Examples: $\frac{1}{n^3} < \frac{1}{n}$, $\frac{1}{\ln n} > \frac{1}{n}$

2. If the denominators are the same, then the *larger* numerator produces the *larger* fraction.

Examples: $\frac{1}{n^2} < \frac{n+1}{n^2}$, $\frac{\sin n}{n^3} \le \frac{1}{n^3}$

3. You may have the good fortune to have a larger numerator and a smaller denominator to determine the larger fraction.

Example: $\frac{n^2+1}{n} > \frac{n^2}{3n+5}$

4. But the difficulty that frequently arises is the conflict of a larger numerator *and* a larger denominator.

$$\frac{n+2}{n^3} \quad ?? \quad \frac{n-1}{n^2}$$

You cannot draw a quick conclusion as to which is larger. A possible method is shown on ***page 701, solution 1, Example 13***.

Answers

It is important to distinguish between an intuitive sense that a series converges or diverges and a proof of this. The intuitive feeling is absolutely essential in picking a $\sum b_n$ as a comparison series. This means that early on you will have a sense whether a given series converges or diverges. And this is an important part of the learning process. However, there are going to be borderline cases where your intuition may be wrong. So we want you to look carefully at how you present your case.

We ask that you use a clear three-step process to support your conclusion.

1. Indicate your choice of a b_n and why it converges or diverges. At this point, b_n will usually be a *p*-series or a geometric series.

2. Include your work showing (a) an inequality for the Comparison Test and why it is true or (b) the limit of $\dfrac{a_n}{b_n}$ for the Limit Comparison Test. Later more tests will be available.

3. Indicate your conclusion and the test that justifies your answer.

Example

Determine whether the series, $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 + n}{n^4 - n^2 + 1}$ converges or diverges.

1. Let $b_n = \dfrac{1}{n}$. $\quad \sum \dfrac{1}{n}$ diverges, p-series, p = 1

   or $\sum \dfrac{1}{n}$ diverges, harmonic series.

2. $$\frac{a_n}{b_n} = \frac{n^3 + n}{n^4 - n^2 + 1} \cdot \frac{n}{1}$$

   $$\lim_{n\to\infty} \frac{n^4 + n^2}{n^4 - n^2 + 1} = \lim_{n\to\infty} \frac{1 + 1/n^2}{1 - 1/n^2 + 1/n^4} = 1$$

3. The series diverges by the Limit Comparison Test.

Would the Comparison Test work for this series? Yes. Part 2 could be as follows.

2. Is $$\frac{n^3 + n}{n^4 - n^2 + 1} \geq \frac{1}{n} \ ?$$

Cross multiply. $n^4 + n^2 \geq n^4 - n^2 + 1$?

$$2n^2 \geq 1$$

The last inequality is true for all n so $a_n \geq 1/n$.

Part (3) would then be: The series diverges by the Comparison Test.

REVIEW

One more plug for bounded, increasing sequences. This idea keeps reappearing because partial sums form an increasing sequence if a_n is positive. Add another positive term and the sum gets larger. Then we only need to find an upper bound to show a series converges. More on this later.

SECTION 11.5: ALTERNATING SERIES

The Alternating Series Test is a straightforward test. There are just two parts, which are the same for all alternating series. You need to show that

(a) $b_{n+1} \leq b_n$ for all n

(b) $\lim_{n\to\infty} b_n = 0$.

The second part is easy, but the first can be tricky.

Also, it will be easy to recognize an alternating series. In the sigma notation you will see $(-1)^n$, $(-1)^{n-1}$, $(-1)^{n+1}$ or another variation of these forms. In passing, note that $(-1)^{2n}$ will always be positive. The symbol $2n$ represents an *even* number. In a similar vein, $(-1)^{2n-1}$ or $(-1)^{2n+1}$ will always be negative. The symbols, $2n - 1$ and $2n + 1$, represent an *odd* number.

It is also important to note that the negative sign is included in a_n while b_n represents a positive term. Hence parts (a) and (b) of the Alternating Series Test deal only with positive terms.

On ***page 733***, there is an excellent drawing in ***Figure 1***, which illustrates the nature of a convergent alternating series. Study this figure carefully to gain an intuitive sense of what happens when an alternating series converges. Because $b_{n+1} \leq b_n$, the terms are decreasing. We continually add a term, then subtract a *smaller* term, add a *still smaller* term, and so forth without end. As long as the nth term is approaching zero, the series must converge to some number s. Make sure you see this clearly in the diagram.
We also suggest that you look at the proof because of its cleverness. The partial sums of an alternating series will not form an increasing sequence.

For a positive term series we had

$$s_1 \leq s_2 \leq s_3 \leq \ldots$$

because we were continually adding one more positive term. In ***Figure 1***, the partial sums bounce back and forth around s. However, by grouping terms in pairs,

$$(b_1 - b_2) + (b_3 - b_4) + (b_5 - b_6) + \ldots$$

we are adding positive terms and creating an increasing sequence. (Each pair is positive because the terms are decreasing; b_2 is smaller than b_1, b_4 is smaller than b_3, etc.) If we can show there is an upper bound, then the sequence must converge. The easy way to see that b_1 or s_1 is an upper bound is to look at Figure 1. Algebraically the grouping

$$b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots$$

gives the same result. We start with b_1 and continually subtract positive numbers. So, b_1 or s_1 is an upper bound, which means the sequence converges.

A BOOST TOWARD CONVERGENCE

Example 1 illustrates that some divergent series will converge when changed to an alternating series. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges. Interpret this by looking at ***Figure 1*** and the result is quite reasonable. However, if the nth term does *not* approach zero, a series will not converge as shown in ***Example 2***. The Test for Divergence,

$$\lim_{n\to\infty} a_n \neq 0$$

applies to *all* series.

DECREASING TERMS

Example 3 indicates how the derivative of a function matching the nth term of a series can be used to show that terms are decreasing. If the derivative is negative, the slope of the tangent line is negative and the function is decreasing.

Consider the series, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$.

The inequality, $\quad b_{n+1} \le b_n \quad$?

becomes $\quad \dfrac{\sqrt{n+1}}{n+5} \le \dfrac{\sqrt{n}}{n+4} \quad$?

On the left, both numerator *and* denominator are larger, which doesn't support any conclusion. Instead let

$$f(x) = \frac{\sqrt{x}}{x+4}$$

Then

$$f'(x) = \frac{(x+4)\,\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+4)^2} = \frac{4-x}{2\sqrt{x}\,(x+4)^2}$$

Because $\sqrt{x}$ and $(x+4)^2$ are both positive, the derivative is negative if $x > 4$. Then, $b_{n+1} \le b_n$, if $n > 4$. Show that the nth term approaches zero, and we can conclude that the series converges by the Alternating Series Test.

A SIMPLE BONUS

If we use the sum of the first n terms of a convergent alternating series to approximate the total sum, the error is less than b_{n+1}, the first term not used. This means it is an easy process to estimate the error.

Consider the convergent series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \ldots$$

If we use the first 10 terms to approximate the series, the error will be less than the absolute value of the 11th term, $\dfrac{1}{11^2}$ or .0083. This provides a simple procedure to estimate the error of an approximation if we are dealing with a convergent alternating series.

As mentioned in the text, one can see this result in ***Figure 1***. If s_5, the sum of the first 5 terms, is used as an approximation then the error is less than the length of the red line segment labeled, $-b_6$. Note that the error is the distance from s_5 to s.

LIMITATIONS

Can the comparison tests be used to determine whether an alternating series converges or diverges? A resounding NO. The comparison tests, the Integral Test, and the *p*-series test apply only to series with positive terms. At this point the ONLY test for an alternating series is the Alternating Series Test. This will change shortly.

It is well worth reading the note on ***page 736***.