Module 7

SECTION 11.6: ABSOLUTE CONVERGENCE AND THE RATIO AND ROOT TESTS

ABSOLUTE CONVERGENCE

Understanding the symbols

$$\sum |a_n|$$

is a key first step to feeling comfortable with the idea of an *absolutely convergent* series. Not really that complex. The absolute value sign just removes all negative signs, if any were present. So an alternating series becomes a series with only positive terms. In fact, a series with negative signs in any pattern becomes a series with only positive terms. So the definition on ***page 737*** is just giving the name *absolutely convergent* to a series, $\sum a_n$, if it converges when all negative signs, if any, are removed.

In ***sections 11.2*** through ***11.4***, with only a couple of exceptions, we only considered series with positive terms. Now we say that each positive term series that converged was absolutely convergent because $\sum a_n$ and $\sum |a_n|$ depict the same series.

Why is this significant? Well, the punch line is theorem **(3)**, ***page 738***, which can be stated in a different way. If you take any positive term series that converges and make it into an alternating series, then that alternating series will also converge. (These statements aren't totally the same but close.) In practice what this means is that you *may* be able avoid the Alternating Series Test and test a positive term series instead.

Example: Does $\sum (-1)^n \frac{1}{n^2}$ converge?

Yes. Because $\sum \frac{1}{n^2}$ is a convergent *p*-series, with $p = 2 > 1$, the original series converges by theorem **(3)**.

Now you may still wonder, did theorem **(3)** really say that?

REWORDING THEOREM (3)

First what you read in the text.

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

The language is a bit confusing. What does "it" refer to and what does "absolutely convergent" mean in this context? The word "it" refers to the series $\sum a_n$. Not much help yet. Using the basic definition, we can write,

If the series $\sum |a_n|$ converges, then $\sum a_n$ also converges.

For the above example, $a_n = (-1)^n \frac{1}{n^2}$

and $|a_n| = \frac{1}{n^2}$

Rewriting the theorem to match the example, we have

If the series $\sum \frac{1}{n^2}$ converges, then $\sum (-1)^n \frac{1}{n^2}$ also converges.

Now we have something that makes sense as far as what theorem **(3)** claims is true. The proof of the theorem, which is a bit complex, indicates *why* this statement is correct.

However, there is also an intuitive approach that makes sense. Look at ***Figure 1, page 695,*** and change a_1, a_2, a_3, a_4, ... to partial sums, s_1, s_2, s_3, s_4, ... and also change 1 to S.

Then the diagram illustrates a positive term series which converges to S. Each time we add one more (positive) term, s_n moves to the right and gets closer to S. This illustrates the convergence of $\sum |a_n|$.

Now imagine what happens when we change the positive term series to an alternating series. It will look like the diagram in ***Figure 1, page 733***. Instead of the s_n's marching to the right, they bounce back and forth but still approach some number *s.* (S and *s*, of course, are different numbers.)

Remember that if $\sum |a_n|$ converges, then $|a_n|$ approaches zero and in turn a_n approaches zero. The bounces get smaller and smaller.

Example 3, contains a series that includes negative signs that do not alternate. Theorem (3) must also cover this situation: hence, the need for the proof in the text instead of the above intuitive argument.

CONDITIONALLY CONVERGENT SERIES

Suppose we try to use theorem **(3)** on the series $\sum (-1)^n \dfrac{1}{\sqrt{n}}$.

$$a_n = (-1)^n \frac{1}{\sqrt{n}} \quad \text{and} \quad |a_n| = \frac{1}{\sqrt{n}}$$

Theorem **(3)** says,

> If the series $\sum \dfrac{1}{\sqrt{n}}$ converges, then $\sum (-1)^n \dfrac{1}{\sqrt{n}}$ also converges.

But now $\sum \dfrac{1}{\sqrt{n}}$ is a divergent p-series, because $p = 1/2 < 1$. We cannot use the theorem. To show that $\sum (-1)^n \dfrac{1}{\sqrt{n}}$ converges, we must use the Alternating Series Test.

We also call this type of series a *conditionally convergent* series.

SUMMARY

1. For an absolutely convergent series, *both* $\sum |a_n|$ and $\sum a_n$ converge.

Example: The series $\sum (-1)^n \dfrac{1}{n^3}$ is absolutely convergent.

This means both $\sum \dfrac{1}{n^3}$ and $\sum (-1)^n \dfrac{1}{n^3}$ converge.

2. For a conditionally convergent series, $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example: The series $\sum (-1)^n \dfrac{1}{\ln n}$ is conditionally convergent because

$\sum (-1)^n \dfrac{1}{\ln n}$ converges but $\sum \dfrac{1}{\ln n}$ diverges.

3. Also remember that you can now claim that $\sum (-1)^n \dfrac{n+5}{n^3 - n}$ converges without using the Alternating Series Test. Instead use the Limit Comparison Test on $\sum \dfrac{n+5}{n^3 - n}$ with $b_n = \dfrac{1}{n^2}$. This would establish absolute convergence, and by theorem **(3)** the alternating series converges.

4. This procedure doesn't work on a conditionally convergent series. In this situation, you must use the Alternating Series Test.

GENERAL LANGUAGE

At some point you may repeat definition **(1)**, ***page 737***, with a series like $\sum \frac{1}{n^4}$ where $a_n = \frac{1}{n^4}$ and $|a_n|$ also equals $\frac{1}{n^4}$. The statement

$$\text{“}\sum \frac{1}{n^4} \text{ is absolutely convergent if } \sum \frac{1}{n^4} \text{ is convergent”}$$

doesn't sound right, but it is still true. Note that the text asserts that, for a *positive* term series, *absolute convergence* is the same as *convergence*.

A more edifying statement is

$$\text{“}\sum (-1)^n \frac{1}{n^4} \text{ is absolutely convergent if } \sum \frac{1}{n^4} \text{ is convergent.”}$$

This more closely matches the following wording for the definition.

> "An alternating series is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent."

However, this doesn't cover the series in ***Example 3,*** so more general language was chosen. This also explains why math texts are sometimes hard to read.

THE RATIO TEST

In the rest of this chapter, the Ratio Test will be used more than any other test. It is based on finding the limit of the ratio of consecutive terms, excluding any negative signs. If the limit L is *less* than one, including $L = 0$, then the series not only converges but converges absolutely. If L is *greater* than 1, or the limit approaches infinity, the series diverges.

In the ratio of consecutive terms, note the absolute value signs. This is why any negative signs are excluded, which in turn means the limit cannot be negative.

Also because of the absolute value signs, the test is the same for an alternating series and its positive term counterpart. Use the Ratio Test on $\sum (-1)^n \frac{1}{n2^n}$, and then use it on $\sum \frac{1}{n2^n}$, and all steps of the procedure will be the same. So the Ratio Test confirms that

$\sum \frac{1}{n2^n}$ or $\sum |a_n|$ converges, which by definition means that we have absolute convergence.

The proof of part (i) is based on inequality **(4)**, which is used to create a "larger" convergent geometric series. Then, by the Comparison Test, the original series converges absolutely.

Read the Note on ***page 740*** carefully. When the limit L is equal to one, the test fails. *No conclusion can be drawn*. Experience in using the test will give you some indication beforehand which types of series have a limit equal to one. For now observe that both examples in the Note are rational functions, the quotient of two polynomials. In ***Example 4***, the limit is not one because of the presence of the exponential, 3^n. More on this later.

THE ROOT TEST

In the last test that we will consider, we find the limit of the nth root of the absolute value of the nth term, $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. As in the Ratio Test, call the limit L. Then the conclusions are the same as for the Ratio Test.

We illustrate the test with two examples.

1. Consider the series $\sum \frac{2^n}{n^n}$. By the Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{2^{n/n}}{n^{n/n}} = \lim_{n\to\infty} \frac{2}{n} = 0$$

Because $L = 0 < 1$, the series converges absolutely. This means

$$\sum (-1)^n \frac{2^n}{n^n} \text{ converges also.}$$

2. Now consider the series $\sum \frac{n}{3^n}$. By the Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{n^{1/n}}{3^{n/n}} = \frac{\lim_{n\to\infty} n^{1/n}}{3} = \frac{\infty^0}{3}$$

We have a bit of a chore here in finding the limit of the nth root of n. Does ∞^0 have meaning? The answer is back on ***page 310 (Sec. 4.4)***.

Consider $y = x^{1/x}$. Then $\ln y = \frac{1}{x} \ln x = \frac{\ln x}{x}$.

Now take the limit of $\ln y$ as $x \to \infty$.

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

Use L'Hopital's Rule. $= \lim_{x\to\infty} \frac{1/x}{1} = 0$

Now remember that we have shown that $\ln y \to 0$. It follows that

$$y \to e^0 = 1.$$

Then $\lim_{n\to\infty} \sqrt[n]{\frac{n}{3^n}} = \frac{1}{3}$ and we conclude that the series converges absolutely. Whew!!!

The last example was included to show that the Root Test is not a very good choice. Compare the following solution using the Ratio Test.

3. Given $\sum \frac{n}{3^n}$, $a_n = \frac{n}{3^n}$ and $a_{n+1} = \frac{n+1}{3^{n+1}} = \frac{n+1}{3 \cdot 3^n}$

In the Ratio Test, write $\frac{a_{n+1}}{a_n} = (a_{n+1}) \cdot \frac{1}{a_n}$.

Then instead of dividing by a_n, we multiply by the reciprocal of a_n.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{3 \cdot 3^n} \cdot \frac{3^n}{n} = \lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$$

Because $L = 1/3 < 1$, we conclude that the series converges absolutely.

CONCLUSION

Only use the Root Test if you have exponential forms where the exponent is n. If some other component, like a polynomial in n, $\ln n$, $n!$, $\sqrt{n}$ or $\sqrt[3]{n}$ is present, don't use the Root Test.

Look at all exercises 2 through 30, pages 742–43, and try to decipher why it is better to use the root test in those problems.

WHEN DOES THE RATIO TEST FAIL?

Now we are concerned with situations where $L = 1$. Consider the following examples.

EXAMPLE 1

$$\sum \frac{n+3}{n^3+n^2} \qquad a_{n+1} = \frac{n+4}{(n+1)^3+(n+1)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+4}{(n+1)^3+(n+1)^2} \cdot \frac{n^3+n^2}{n+3}$$

$$= \frac{n^3+n^2}{(n+1)^3+(n+1)^2} \cdot \frac{n+4}{n+3}$$

$$= \frac{1+1/n}{\frac{(n+1)^3}{n^3}+\frac{(n+1)^2}{n^3}} \cdot \frac{1+4/n}{1+3/n}$$

Now take the limit as n approaches infinity.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1+0}{1+0} \cdot \frac{1+0}{1+0} = 1$$

Sad news. After all this work the Ratio Test *fails*. Hopefully you noted at the start that the Limit Comparison Test would work with $b_n = \frac{1}{n^2}$. The limit of $\frac{a_n}{b_n}$ would again equal one, but now it is OK.

Conclusion: If we have *only* polynomials in n in the numerator and denominator, the Ratio Test will fail.

EXAMPLE 2

$$\sum \frac{\ln n}{\sqrt{n}} \qquad a_{n+1} = \frac{\ln (n+1)}{\sqrt{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln (n+1)}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\ln n}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{\ln (n+1)}{\ln n}$$

Then note that $\dfrac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\dfrac{n}{n+1}}$, and its limit is $\sqrt{1} = 1$.

Next consider $\lim_{x\to\infty} \dfrac{\ln (x+1)}{\ln x} = \lim_{x\to\infty} \dfrac{1/(x+1)}{1/x} = \lim_{x\to\infty} \dfrac{x}{x+1} = 1$

So the final limit is

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1 = 1$$

More sad news. The Ratio Test fails. So to the list of polynomials in n, add $\sqrt{n}$ and $\ln n$. Would $\sqrt[3]{n}$ be any different?

EXAMPLE 3

$$\sum \frac{2^n}{n!} \qquad\qquad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Now we get into the good stuff.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\ &= \frac{2^n\,2}{(n+1)\,n!} \cdot \frac{n!}{2^n} \\ &= \frac{2}{(n+1)}\end{aligned}$$

Now the limit is zero.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2}{(n+1)} = 0$$

Good news. The series converges absolutely by the Ratio Test.

Conclusion: Exponential forms like 2^n and 3^n along with factorials are Ratio Test friendly.

REARRANGEMENTS

The last part of this section may be a bit puzzling. The emphasis is twofold.

1. To make a distinction between a finite sum and an infinite sum.

2. To also make a distinction between an absolutely convergent series and a conditionally convergent series.

Also, you can read about Bernhard Riemann on ***page 379***. This doesn't remove the puzzling aspect of rearrangements, but if he proved the statement, we can rest assured it is true.

SECTION 11.7: STRATEGY FOR TESTING SERIES

There is actually nothing new in this section. You will encounter a more diverse group of exercises, which will provide a test of how well you can detect patterns that have been stressed.

By all means read ***pages 744-45*** carefully. Some parts are just review, but others relate to insights we hope you are developing. In particular pay attention to the following:

1. #2. Some algebraic manipulation may be required to see the form, $a\,r^n$, which is characteristic of a geometric series.

   Example: $$\frac{5^{n+2}}{3^{2n+1}} = \frac{5^n\,5^2}{3^{2n}\,3} = \frac{25}{3}\cdot\left(\frac{5}{9}\right)^n$$

2. #3. A rational function of n is of the form, $\dfrac{\text{polynomial in } n}{\text{polynomial in } n}$. There are no square roots or cube roots in this type of function, but an algebraic function can contain roots.

   Examples: $\dfrac{x-3}{x^4+x^2}$ is a rational function, while

   $\dfrac{1}{\sqrt[3]{x}}$ and $\dfrac{\sqrt{x}}{x+1}$ are algebraic functions. See ***page 30***.

The corresponding nth terms of an infinite series would be,

$$\frac{n-3}{n^4+n^2}\,,\quad \frac{1}{\sqrt[3]{n}}\,,\quad \text{and}\quad \frac{\sqrt{n}}{n+1}.$$

(Remember x can equal any real number in the domain of the function, but n is restricted to the set of positive integers.)

Note the suggestion to use a comparison test. Remember that there may be difficulties in establishing the required inequality for the Comparison Test. For this reason we suggested using the Limit Comparison Test. However, remember there are exceptions to any general rule.

Also note the comment in #6 that the Ratio Test fails (limit equals 1) for rational and algebraic functions of n. Do you remember hearing that before?

3. #4. Don't forget the Test for Divergence. It should be easier now to recognize series where the *n*th term does *not* approach zero.

4. #5. Always look for the possibility of showing absolute convergence (drop the negative signs, and show that the positive term series converges) instead of using the Alternating Series Test.

5. #6. The Ratio Test likes factorials, *n*th powers of constants, and *n*th powers of *n*. In other words, it likes $n!$, $(n+2)!$, 2^n , and n^n. Or, maybe you prefer the Ratio Test when using these components because they don't produce limits equal to one. Remember,

$$(n+2)! \quad = \quad (n+2)(n+1)\,n!$$

and

$$\frac{(2n)!}{[2(n+1)]!} = \frac{(2n)!}{(2n+2)(2n+1)\,(2n)!} = \frac{1}{(2n+2)(2n+1)}$$

COMPONENTS

If you look at the exercises on ***page 746***, you will see what can be called components or factors in the *n*th terms. Examples include,

$$\sqrt{n} \;,\; n^2+1 \;,\; 4^n \;,\; \ln n \;,\; e^n \;,\; n! \;,\; n^n$$

Quotients are then formed using components to form *n*th terms. As you gain experience, you will note a difference between

$$\frac{4^n}{n^2+1} \qquad \text{and} \qquad \frac{n^2+1}{4^n}.$$

4^n is the more dominant component of the two. Compare

$$4^2 \;,\; 4^3 \;,\; 4^4 \;,\; 4^5 \qquad \text{or} \qquad 16\,,\;\; 64\,,\;\; 256\,,\;\; 1024$$

with

$$2^2+1\,,\, 3^2+1\,,\, 4^2+1\,,\, 5^2+1 \qquad \text{or} \qquad 5\,,\;\; 10\,,\;\; 17\,,\;\; 26$$

This observation leads to an intuitive sense as to whether the series will converge or diverge.

The most powerful or most dominant are n^n and $n!$. Have you formed an opinion as to which will win in this quotient, $\frac{n!}{n^n}$? Have you seen this before? Good. ***Example 5, page 741***, contains the reciprocal. Following the suggestion in the Note below the example, write

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \quad \cdots \quad \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}$$

This indicates that n^n dominates and $a_n \to 0$. It leads to the intuitive feeling that the series will converge. Remember, you have to prove this. Which test will you use? If you said the Ratio Test, you are ready for the following exercises.

SECTION 11.8: POWER SERIES

The concept of a power series will present a new challenge. In one sense we just have a new component, x^n, that is combined with other familiar components. The major test to determine convergence will be the Ratio Test, but now the variable x will always be present *after* taking a limit. This will require some adjustments, but the major difference is that a power series is a *function* whose domain is the set of all values of x for which the series converges. Consider the power series,

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots$$

For $x = 1$, we have the harmonic series

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

which we know diverges. So $x = 1$ is not in the domain of the function. Compare this situation with the function, $y = \frac{1}{x^2 - 1}$. When $x = 1$, $y = \infty$, or y is undefined and 1 is not in the domain of the function.

When $x = 1$ in the power series, the harmonic series is defined, but since it approaches infinity as we consider more and more terms, there is no finite value associated with $x =$ 1.

Next let $x = -1$. Then the power series becomes an alternating series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We have shown that this series converges by the Alternating Series Test. So we can put $x = -1$ in the domain of the power series. In fact on ***page 742 (Sec. 11.6)***, equation **(6)**, the alternating series equals $\ln 2$. So think of the point $(1, \ln 2)$ as one point on the graph of the function

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots$$

Wow, this is a lot of work. But before we consider an easier method, suppose $x = 1/2$. Then

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n+1} = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \ldots + \frac{1}{(n+1) \cdot 2^n} + \ldots$$

The Ratio Test will show that this series converges, so we can put $x = 1/2$ in the domain of the power series.

Obviously, this method of determining the domain of the power series is very time-consuming. However, it does illustrate that the power series is a function. A more efficient approach in finding the domain is to use the Ratio Test on the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \quad \text{with} \quad a_n = \frac{x^n}{n+1}.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \cdot \frac{n+1}{n+2}\right|$$

$$= \lim_{n\to\infty} \left|x \cdot \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}\right| = |x|$$

In this case, $L = |x|$ and by the Ratio Test for the series to converge, L must be less than one. So the series converges if

$$|x| < 1$$

which is equivalent to $\quad -1 < x < 1$.

The Ratio Test fails when $L = |x| = 1$, but we have shown above that the power series diverges at $x = 1$ and converges at $x = -1$. So the series converges when $-1 \le x < 1$. This interval is the domain of the function defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$. Select any value of x in the interval, $-1 \le x < 1$, substitute it in the power series (as we did above for $x = 1/2$) and the result will be a series that converges to some number C. This pairing of x's with C's defines the function.

FORMS OF A POWER SERIES

The text uses two forms to represent a power series.

$$(1) \quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$(2) \quad \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

The second is the more general form. If $a = 0$ in the second form, the result is the same as the first. Each of these forms will be used extensively later.

A MAJOR CHANGE IN THINKING

Because a different perspective will be important in the rest of this chapter, we again emphasize a major change in thinking about infinite series. In the last few sections we were just concerned about the convergence of a series involving constants (which we called components). Now, in a power series, there is a variable x involved in addition to the variable n. A very important point is that a *single* x value leads to one infinite series. As we consider this series, n varies from 0 to infinity. Then we consider another value of x, which results in another infinite series, and again n varies from 0 to infinity. Theoretically we continue this process until we have found all values of x for which the power series converges and call this the domain of the function. On ***page 749***, the domain is called the *interval of convergence,* and theorem **(3)** indicates that we get clearly defined boundaries on this set of x values.

It is also worth noting that when using the Ratio Test, or the Root Test, we find a limit as n approaches infinity. In this process, n is a variable but x is a constant. Think of having selected a single value of x. As we find a limit, the value of x does not change. So when looking at the symbols,

$$\lim_{n\to\infty} \frac{x^2}{4(n+1)^2} \qquad\qquad \textbf{\textit{Example 3}}$$

n is changing but x is not changing. If n becomes very large, then the fraction will be small even if x also is a large number. Suppose x is one million, 10^6. Then $x^2 = 10^{12}$. Because n is approaching infinity, eventually $n + 1$ can equal 10^{12} times 10^{12}, and then

$$\frac{x^2}{4(n+1)^2} = \frac{10^{12}}{4\left(10^{12}\right)\left(10^{12}\right)} = \frac{1}{4\left(10^{12}\right)} = \frac{1}{4{,}000{,}000{,}000{,}000}$$

As the text indicates, the limit is zero.

Theorem (3)

Think of the first two parts of theorem **(4)**, ***page 749***, as special cases. The third part is illustrated in ***Figure 3***. For $R > 0$, the given power series converges for all x in the interval defined by $\left|x \quad a\right| \;<-R$. This inequality is equivalent to

$$-R < x - a < R$$

or $$a - R < x < a + R$$

So in ***Figure 3***, the series converges for all x in the interval marked by the red line segment AND diverges for all x outside this interval, except possibly at the endpoints. Theorem (3) says nothing about the behavior at the endpoints of the interval. The series can converge or diverge at each endpoint.

This discussion relates to a power series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$. If $a = 0$, then the series has the form, $\sum_{n=0}^{\infty} c_n x^n$. In the above inequalities, replace a with zero; $\quad |x| < R \quad$ and $\quad -R < x < R$

TERMINOLOGY

Why is R called the radius of convergence? Now the domain has just one variable x, which is graphed on a single number line. In higher mathematics, the domain can have two variables, which are graphed in an x-y plane. Then the convergence occurs in a circle which has the radius R. Hence the terminology *R is the radius of convergence*. In our situation, we consider only a diameter of this circle, which is called the interval of convergence.

In some power series, instead of x^n, you will see x^{2n} , x^{2n-1} or x^{2n+1}. Consider the following series.

$$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \ldots$$

The exponent $2n$ produces only even exponents.

$$\sum_{n=1}^{\infty} x^{2n-1} = x + x^3 + x^5 + \ldots$$

Now the exponent $2n$ - 1 produces only odd exponents.

$$\sum_{n=0}^{\infty} x^{2n+1} = x + x^3 + x^5 + \ldots$$

The exponent $2n$ + 1 also produces only odd exponents. The last two series are the same because we fudged on the beginning values of n. More on this later.

ROOT TEST

Earlier we mentioned that in most cases the Ratio Test is used to determine the interval of convergence. But the new components x^n and x^{2n} would also fit the Root Test.

$$\sqrt[n]{|x^n|} = |x| \quad \text{and} \quad \sqrt[n]{|x^{2n}|} = x^2$$

So the Root Test would be a good choice if *all* other components are powers of n.
Look at all ***exercises 3*** through ***28*** on ***page 751***. How many contain *only* powers of n?
Two is the correct answer.

ENDPOINTS

Every finite interval has two endpoints. For a finite interval of convergence, both the Ratio Test and the Root Test fail to give any information at the endpoints.

In ***Example 4,*** the Ratio Test produces $L = 3|x|$, which leads to the endpoints, $\pm \frac{1}{3}$. Substitute either of these values in $L = 3|x|$ and L $=$ 1. You can check that the same thing happens in ***Example 5***.

So the three-step process is: use the Ratio (or Root) Test, find the interval of convergence, *and* check the endpoints.