Lesson 1

Reading: Larson, Section 1.1, Introduction to systems of linear equations.

Suggested exercises: Larson, Section 1.1: 11, 13, 27, 37, 47, 49, 51, 53, 61, 63, 71, 77, 79

Submit: Lesson 1: Solution sets of linear systems

Free variables and parameters

This section of the text introduces terminology that will be used throughout the course. First, a *free variable* is a variable whose value you are allowed to set, and which then determines the value of the other variable or variables. For example, in the equation $x + 2y = 2$, either variable could be considered as a free variable. Once a value for that variable has been set, the other variable must assume a specific value. For example, if we decide to take x as our free variable, then choosing $x = 1$ forces y to be equal to $1/2$. If we choose y to be the free variable, then choosing $y = 1$ forces $x = 0$. For this situation, there is no preference for which variable is free.

If we have two equations in three unknowns, then there may be a clear choice of free variable. For example, if we have

$$\begin{aligned} x + z &= 1 \\ y - 2z &= 3, \end{aligned}$$

then the variable z is the only variable appearing in both equations, so it is the natural choice for the free variable. That means that we can set any value we wish for z, and once that choice is made, we have determined the values of the other variables: we must have $x = 1 - z$ and $y = 2z + 3$.

It is not absolutely necessary to take z as our free variable in this example. If we really wanted to, we could, for example, choose x to be the free variable. Assigning a value to x would then force a value of z in the first equation, and then that forced value of z would force a value of y in the second equation. For example, choosing $x = 2$ in the first equation forces $z = -1$. Plugging this value into the second equation gives us $y = 1$. It works, but it is not the obvious choice to make for this set of equations.

The term *parameter* is introduced also. In what we are going to do, there is really not a substantial difference between a parameter and a free variable. In the example given above, we could represent the solutions of the two given equations in three variables by stating $x = 1 - z$, $y = 2z + 3$, and z acts as the parameter, meaning that a value must be provided to it in order to determine the values of the other variables. Or, we could explicitly introduce another symbol to represent our parameter, by setting $z = s$, for example. This would produce a solution that looks like this:

$$\begin{aligned} x &= 1 - s \\ y &= 2s + 3 \\ z &= s \end{aligned}$$

All that has really happened is that we have given a new name to z, but at least now we have a slot in which to insert a number for each of the three variables in order to generate a solution, and as will be seen later, this can be useful.

Row-echelon form and Gaussian elimination

Two important terms, row-echelon form and Gaussian elimination, deserve special attention. The term "row-echelon form" is casually introduced in the section "Solving a system of linear equations." This idea will be explored further later, but for now, it is enough to think of it in the simple terms given in the text: a system in row-echelon form has the stair-step, diagonal form as shown in the text, in which the first variable in each equation is farther to the right than the first variable in the equation above it (assuming, of course, that the variables appear in a consistent order). To be in row-echelon form, we also require that the coefficient of the first variable appearing in each row be a 1, but for Gaussian elimination with back-substitution, we will not always insist on this: the stair-step structure will be enough. For example,

$$\begin{aligned} x - y + 4z &= 1 \\ y - 2z &= 3 \\ 6z &= 1 \end{aligned}$$

is in the stair-step form that is ready for back-substitution. (If we wrote the final equation as $z = 1/6$, then it would be in row-echelon form.) A linear system written in this form is easy to solve, if you work from the bottom up: solve for z in the bottom equation, then plug that value into the second equation and solve it for y, and finally plug the values found for y and z into the top equation and solve that for x.

The linear system

$$\begin{aligned} x - y + 4z &= 1 \\ y - 2z &= 3 \\ 5y + 6z &= 1 \end{aligned}$$

is *not* in row-echelon form because of the presence of the $5y$ term in the third equation. This form does not deserve a special name: there is a bit more work to be done before it is easy to read off the solutions.

Gaussian elimination is the name given to the process of eliminating terms from equations in a linear system with the goal of arriving at row-echelon form (although it is sufficient to get the stair-step form for Gaussian elimination with back-substitution). The elimination of terms is done systematically from left to right, and from top to bottom within each column. Following this pattern of elimination of terms will prevent you from going around in circles, and eliminating a term only to see a term reappear in that spot later on.

For example, in the system

$$\begin{aligned} x - y + z &= 0 \\ 2x + y - z &= 3 \\ 3x + y + z &= 8 \end{aligned}$$

we first eliminate the $2x$ and $3x$ terms in the first column by adding -2 times the first equation to the second, and then by adding -3 times the first equation to the third:

$$\begin{aligned} x - y + z &= 0 \\ 3y - 3z &= 3 \\ 4y - 2z &= 8 \end{aligned}$$

We are done dealing with the x column now; none of the terms in that column will change in what follows. This is because we will no longer work with the first equation; we will simply leave it in place, and as a result, we have no opportunity to change any of the x terms.

We have only one term left to eliminate: the $4y$ term in the third equation. To prepare to eliminate this term, we can divide the second equation by 3 to get the system

$$\begin{aligned} x - y + z &= 0 \\ y - z &= 1 \\ 4y - 2z &= 8 \end{aligned}$$

Now, we add -4 times the second equation to the third equation, resulting in

$$\begin{aligned} x - y + z &= 0 \\ y - z &= 1 \\ 2z &= 4, \end{aligned}$$

and finally, dividing the third equation by 2,

$$\begin{aligned} x - y + z &= 0 \\ y - z &= 1 \\ z &= 2. \end{aligned}$$

The system is now in row-echelon form, so we can work from the bottom up, solving for one variable at a time. This process is called *back-substitution.* In the third equation, we have $z = 2$. We plug this into the second equation, which gives us $y - 2 = 1$, or $y = 3$. Plug both values into the first equation, getting $x - 3 + 2 = 0$, or $x = 1$. (Since the system was reduced to row-echelon form, we did not have to divide when performing back-substitution.)

When putting a system in row-echelon form, it is helpful to avoid fractions, if possible. You may be able to avoid them by interchanging equations. For example, given

$$\begin{aligned} x - 2y + z &= 1 \\ 5y + 2z &= 3 \\ y - z &= 2 \end{aligned}$$

don't assume that you *must* eliminate the y term in the third equation, just because it appears in that spot. To do so would require dividing the second equation by 5 before subtracting it from the third equation, and you would have to deal with fractions (not that there's anything wrong with that—it's just less convenient and more likely to introduce an arithmetical error). Instead, switch the second and third equations:

$$\begin{aligned} x - 2y + z &= 1 \\ y - z &= 2 \\ 5y + 2z &= 3 \end{aligned}$$

Now, you can subtract 5 times the second equation from the third, obtaining

$$\begin{aligned} x - 2y + z &= 1 \\ y - 3z &= 2 \\ 7z &= -7 \end{aligned}$$

which leads to

$$\begin{aligned} x - 2y + z &= 1 \\ y - 3z &= 2 \\ z &= -1 \end{aligned}$$

and the system is in row-echelon form with no fractions.

You can also avoid fractions (for at least one more step) as follows. Suppose you have

$$\begin{aligned} x + y + z &= 1 \\ 3y + 2z &= 4 \\ 2y + 5z &= 1 \end{aligned}$$

You can eliminate the y term in the third equation without using fractions if you multiply the second and third equations by factors that produce compatible coefficients. For example, if we multiply the second equation by 2 and the third equation by 3, we get

$$\begin{aligned} x + y + z &= 1 \\ 6y + 4z &= 8 \\ 6y + 15z &= 3 \end{aligned}$$

We can subtract the second equation from the third to eliminate the $6y$ term, producing

$$\begin{aligned} x + y + z &= 1 \\ 6y + 4z &= 8 \\ 11z &= -5 \end{aligned}$$

We're going to have to deal with fractions once we solve for z, but at least we postponed it until this point. Note that this system is not in row-echelon form, but it is ready for back-substitution, nonetheless.

Types of solution sets

The case of two equations in two variables, in which we are dealing with lines in the plane, illustrates the types of solution sets we can expect from linear systems. Two lines in the plane can intersect in only three ways. If they have different slopes, then they intersect in exactly one point. That unique intersection point is the unique solution in the solution set of the associated linear system. If they are parallel but not the same line, then they do not intersect at all, and the associated linear system has no solutions. If the two equations represent the same line (which happens precisely when one equation is a scalar multiple of the other), then the lines coincide and every point on the common line is a solution. In this case, there are infinitely many solutions.

This type of thing happens no matter how many equations or variables we have. The geometry may be much more complicated, but the types of intersection are the same: either there is no intersection, a unique point of intersection, or infinitely many points of intersection. There is no such thing as a linear system with exactly three solutions! If it has more than one solution, then it has to have infinitely many.