Module 2

7.4

3a) $\frac{1}{x^2+x^4} = \frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

3b) $\frac{x^3+1}{x^3-3x^2+2x} = \frac{x^3+1}{x(x^2-3x+2)} = \frac{x^3+1}{x(x-1)(x-2)}$

10) $\int \frac{y}{(y+4)(2y-1)}dy = \int \frac{A}{y+4} + \int \frac{B}{2y-1}$

eval partial frac first:

$y = A(2y-1) + B(y+4)$
$= 2Ay - A + By + 4B$
$= y(2A+B) + 4B - A$

so $2A+B=1$ weird... so $B = \frac{1}{9}$, $A = \frac{4}{9}$
$4B - A = 0$
$8B - 2A = 0$
add top and bottom so $9B = 1$

so $= \frac{4}{9}\int \frac{dy}{y+4} + \frac{1}{9}\int \frac{dy}{2y-1}$

$u = y+4$
$\frac{du}{dy} = 1$ so $dy = du$

$u = 2y-1$
$du = 2\,dy$
$\frac{du}{2} = dy$

$$\boxed{= \frac{4}{9}\ln(y+4) + \frac{1}{18}\ln(2y-1) + C}$$

12) $\int_0^1 \frac{x-4}{x^2-5x+6}dx = \int_0^1 \frac{x-4}{(x-3)(x-2)}dx$

partial fractions: $\frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$x-4 = A(x-2) + B(x-3)$
$= Ax - 2A + Bx - 3B$
$x-4 = (A+B)x - 2A - 3B$

$A+B = 1$ or $2A+2B=2$
$-2A-3B = -4$ $\quad -B = -2$
$B = 2$
$A = -1$

$= \int_0^1 \frac{-1}{x-3}dx + \int_0^1 \frac{2}{x-2}dx$

$u = x-3$
$du = dx$

$= -1 \ln|x-3|\Big]_0^1 + 2 \ln|x-2|\Big]_0^1$

$-1(\ln 2 - \ln 3) + 2(\ln 1 - \ln 2)$

$-\ln 2 + \ln 3 + 2\ln 1 - 2\ln 2 = \cancel{\ldots}$

$= -3\ln 2 + \ln 3 + 2\ln 1 = \ln 3 - 3\ln 2 = \ln 3 - \ln 2^3 = \boxed{\ln\frac{3}{8}}$

15) $\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}dx$

divide $x^2-3x+2 \overline{)x^3-4x+1}$ with quotient $x+3$

x^3-3x^2+2x

$0+3x^2-4x-2x+1$

$~~3x^2-9x+6~~$

$~~\ldots~~$

$3x^2-6x+1$
$3x^2-9x+6$
$3x-5$

$= x+3 + \cancel{\frac{-13x-2x-5}{x^2-3x+2}}$ $\frac{3x-5}{x^2-3x+2}$

$= \int_{-1}^0 x+3+\frac{3x-5}{x^2-3x+2}$

partial fractions

$\frac{3x-5}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

$3x-5 = A(x-1) + B(x-2)$

$x=1$; $-2 = -B$ so $B=2$
$x=2$; $1 = A$

so $\int_{-1}^0 x+3+\frac{1}{x-2}+\frac{2}{x-1}$

$\frac{x^2}{2}+3x+\ln|x-2|+2\ln|x-1|\Big]_{-1}^0$

(on back)

15) continued...

$\cancel{0+0+\ln 2 + 2\ln 1 - \tfrac{1}{2} + 3 - \ln 3 - 2\ln 2}$

$\cancel{\ln 2 + \ln 3} + 2.5$

$\ln 2 + 2\ln 1 - \ln 3 - 2\ln 2 + 2.5$

$\ln 2 - \ln 3 + 2(\ln 1 - \ln 2)$

$\ln \frac{2}{3} + \ln \frac{1}{4} + 2.5 = \boxed{\ln \frac{1}{6} + 2.5}$

18) $\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx$

$x^2+3x+2 \overline{)\,3x^2+6x+2}$ quotient 3

$3x^2+9x+6$

$-3x-4$

so $= 3 + \frac{-3x-4}{x^2+3x+2}$

$9-8=1$ so reducible

$= \int_1^2 3 - \frac{3x+4}{(x+2)(x+1)}$

partial fractions $\frac{3x+4}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$

$3x+4 = A(x+1) + B(x+2)$

$3x+4 = Ax + A + Bx + 2B$

$3x+4 = (A+B)x + A + 2B$

$A+B=3$

$A+2B=+4$

$\cancel{B = 7}$ $\cancel{B=-2}$

$B=1, A=2$ $\cancel{A=10}$

$= \int_1^2 3 - \frac{2}{x+2} - \frac{1}{x+1}$

$= 3x - 2\ln|x+2| - \ln|x+1| \Big]_1^2$

$\cancel{= 6 - 10\ln 4 + 7\ln 3 - (3 - 10\ln 3 + 7\ln 2)}$

$\cancel{= 6 - 10\ln 4 + 7\ln 3 - 3 + 10\ln 3 - 7\ln 2}$

$\cancel{= 3 + 10(\ln 3 - \ln 4) + 7(\ln 3 - \ln 2) = 3 + \frac{\ln 3^{10}}{\ln 4^{10}} + \frac{\ln 3^7}{\ln 2^7}}$

$\cancel{3 + 10\ln 3 - 10\ln 4 + 7\ln 3 - 7\ln 2}$

$\cancel{3 + 3\ln 3 - 10\ln 4 - 7\ln 2}$

$\cancel{3 + 3\ln 3 - 20\ln 2 - 7\ln 2}$

$\cancel{3 + 3\ln 3 - 27\ln 2}$

$= 6 - 2\ln 4 - \ln 3 - (3 - 2\ln 3 - \ln 2)$

$3 - 2\ln 4 - \ln 3 + 2\ln 3 + \ln 2$

$3 - 2\ln 4 + \ln 3 + \ln 2$

$3 - 4\ln 2 + \ln 3 + \ln 2$

$3 + \ln 3 - 3\ln 2 = \boxed{3 + \ln \frac{3}{8}}$

14) $\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}$: using partial frac : $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

$x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

$x^2+x+1 = (Ax+A)(x+2) + Bx + 2B + C(x^2+2x+1)$

$Ax^2 + 2Ax + Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$

$x^2(A+C) + x(2A+A+B+2C) + 2A + 2B + C$

$A+C=1$ $\quad$ $2A+2C=2$

$2A+A+B+2C=1$ $\quad$ $3A+B+2C=1$

$2A+2B+C=1$ $\quad$ $2A+2B+C=1$

$A+2B=0$ $\quad$ $A+B=-1$

$2B-C=-1$ $\quad$ $A+B+C=2$

$B=1$

$A=-2$

$C=3$

$\int_0^1 \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$

$u=x+1$

$du=dx$

$\int \frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u} = \frac{-1}{x+1}$

$\left[-2\ln|x+1| + \frac{-1}{x+1} + 3\ln|x+2|\right]_0^1$

$-2\ln 2 + \frac{-1}{2} + 3\ln 3 - (-1+3\ln 2)$

$-5\ln 2 + \frac{1}{2} + 3\ln 3 = \frac{1}{2} + \ln 3^3 - \ln 2^5$

$\boxed{\frac{1}{2} + \frac{\ln 27}{\ln 32}}$

21) $\int \frac{dt}{(t^2-1)^2} = \int \frac{1}{(t+1)(t-1)(t+1)(t-1)} = \int \frac{1}{(t+1)^2(t-1)^2}$

$= \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$

$A+B+C+D=1$

$1 = A(t-1)(t+1)^2 + B(t+1)^2 + C(t+1)(t-1)^2 + D(t-1)^2$

$t=-1$; $D=\frac{1}{4}$

$t=1$; $B=\frac{1}{4}$

$t=2$; $9A + 9B + 3C + D$

$1 = 9A + 3C \quad \frac{9}{4} + \frac{1}{4}$

$1 = \frac{-9}{4} + \frac{9}{4} + \frac{1}{4} + 3C$

$\frac{3}{4} = 3C$ $\quad$ $C=\frac{1}{4}$

$A=\frac{-1}{4}$

$D=\frac{1}{4}$

$B=\frac{1}{4}$

$t=3$; $32A + 16B + 16C + 4D$

$1 = 32A + 16C + 4 + 1$ $\quad$ $-4 = 32A + 16C$

$t=5$; $1 = 144A + 36B + 96C + 16D$

$1 = 144A + 9 + 96C + 4$ $\quad$ $-12 = 144A + 96C$

$64C$ $\quad$ $C=$

$12 = -48A$ $\quad$ $A=\frac{-1}{4}$

$\int \frac{dt}{(t^2-1)^2} = \int \frac{-1}{4(t-1)} + \frac{1}{4(t-1)^2} + \frac{1}{4(t+1)} + \frac{1}{4(t+1)^2}$

$\boxed{= -\frac{1}{4}\ln(t-1) - \frac{1}{4(t-1)} + \frac{1}{4}\ln(t+1) - \frac{1}{4t+1}} + C$

23) $\displaystyle\int \frac{10}{(x-1)(x^2+4)}\,dx$

partial fractions:

$$\frac{10}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$10 = A(x^2+4) + (Bx+C)(x-1)$$

$$Ax^2+4A + Bx^2 - Bx + Cx - C$$

$$10 = x^2(A+B) + x(C-B) + 4A - C$$

$A+B=0$ $\quad A+C=0$

$C-B=0$

$4A-C=10$ $\quad 5A=10$

$A=2$

$B=-2$

$-2=C$

check $\dfrac{2}{x-1} + \dfrac{-2x-2}{x^2+4}$

$2x^2+8-2x^2+2x-2x+2$
$=10$ ✓

$$= \int \frac{2}{x-1} + \frac{-2x-2}{x^2+4}$$

$$2\ln|x-1| + \int \frac{-2x-2}{x^2+4}$$

$$\int \frac{-2x}{x^2+4}\,dx + \int \frac{-2}{x^2+4}\,dx$$

$z = x^2+4$ $\qquad$ use tan sub

$\dfrac{dz}{dx} = 2x$

$dz = 2x\,dx$

$$= 2\ln|x-1| - \int \frac{dz}{z} + \tan^{-1}\frac{x}{2} + C$$

$$= \boxed{2\ln|x-1| - \ln|x^2+4| - \tan^{-1}\frac{x}{2} + C}$$

24) $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{x^2-x+6}{x(x^2+3)}$

partial frac: $\dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$

$$x^2-x+6 = A(x^2+3) + (Bx+C)x$$
$$= Ax^2 + 3A + Bx^2 + Cx$$
$$= x^2(A+B) + Cx + 3A$$

$3A=6$ $\quad A=2$

$C=-1$ $\quad C=-1$

$A+B=1$ $\quad B=-1$

$$= \int \frac{2}{x} + \int \frac{-x-1}{x^2+3}$$

$$= 2\ln|x| + \int \frac{-x}{x^2+3}\,dx + \frac{-1}{x^2+3}\,dx$$

$u = x^2+3$ $\qquad a=\sqrt{3}$

$\dfrac{du}{dx} = 2x$ $\qquad$ use $\tan^{-1}$

$du = 2x\,dx$

$$\boxed{2\ln|x| + \frac{-1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

34) $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$ $\qquad$ use $u=\sqrt{x-1}$ $\quad u^2 = x-1$ $\quad x = u^2+1$

$\dfrac{dx}{du} = 2u$ $\quad dx = 2u\,du$

$$\int \frac{2u\,du}{(u^2+1)u}$$

partial fractions $\dfrac{2u}{u^2+1(u)} = \dfrac{A}{u} + \dfrac{Bu+C}{u^2+1}$

$$2u = A(u^2+1) + (Bu+C)u$$

$$2u = Au^2 + A + Bu^2 + Cu$$

$$u^2(A+B) + Cu + A$$

$C=2$

$A=0$

$B=0$

$$= \int \frac{2}{u^2+1}$$

$\dfrac{2}{u^2+1}$

$$= 2\int \frac{1}{u^2+1}\,du$$

$$= 2\arctan(u)$$

$$\boxed{= 2\arctan(\sqrt{x-1})}$$

why does this not work?

do I have to reduce first?

42) $\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$ let $u=\sqrt[3]{x}$ $u^3=x$ $\frac{dx}{du}=3u^2$ $dx=3u^2du$

$\int_0^1 \frac{3u^2du}{1+u}$

$1+u \overline{)3u^2}$ quotient $3u-3+\frac{3}{1+u}$
$3u^2+3u$
$-3u$
$-3u-3$
3

$\int_0^1 \left(3u-3+\frac{3}{1+u}\right)du$

$\frac{3u^2}{2}-3u+3\ln|1+u|\Big]_0^1$

$\frac{3}{2}-3+3\ln 2-3\ln 1$

$\boxed{-1.5+3\ln 2}$

45) $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ $u=\sqrt[6]{x}=x^{1/6}$ $u^2=x^{\frac{1}{3}}$ $u^3=x^{\frac{1}{2}}$
$u^6=x$
$dx=6u^5du$

$u^3-u^2 \overline{)6u^5}$ quotient $6u^2+6u+6+\frac{6u^2}{u^3-u^2}$
$6u^5-6u^4$
$6u^4$
$6u^4-6u^3$
$6u^3$
$6u^3-6u^2$
$6u^2$

$=\int \frac{6u^5}{u^3-u^2}du \quad = \cancel{\int \frac{6u^3}{u-1}du}$

$=6\int u^2+u+1+\frac{u^2}{u^3-u^2}$

$=6\left(\frac{u^3}{3}+\frac{u^2}{2}+u+\int\frac{u^2}{u^3-u^2}\right) \longrightarrow \boxed{6\left(\frac{u^3}{3}+\frac{u^2}{2}+u+\ln|x-1|\right)+C}$

$\int \frac{1}{x-1}dx$ $u=x-1$ $\frac{du}{dx}=1$ $du=dx$

$\int\frac{du}{u}$

$=\ln|x-1|$

7.5 1) $\int \frac{\cos x}{1-\sin x}\,dx$ $u=1-\sin x$
$\frac{du}{dx}=-\cos x$ $du=-\cos x\,dx$

$=-\int\frac{du}{u}=-\ln|u|=\boxed{-\ln|1-\sin x|+C}$

3) $\int_1^4 \sqrt{y}\ln y\,dy$

$u=\ln y$ $dv=\sqrt{y}=y^{\frac{1}{2}}$
$du=\frac{1}{y}dy$ $v=\frac{2}{3}y^{\frac{3}{2}}$

Integral first:

$=\ln y\cdot\frac{2}{3}y^{\frac{3}{2}}-\int\frac{\frac{2}{3}y^{\frac{3}{2}}}{y}dy$

$\int\frac{2}{3}y^{\frac{1}{2}}$

$=\ln y\cdot\frac{2}{3}y^{\frac{3}{2}}-\frac{2}{3}\int y^{\frac{1}{2}}=\ln y\cdot\frac{2}{3}y^{\frac{3}{2}}-\frac{2}{3}\cdot\frac{2}{3}y^{\frac{3}{2}}$

$=\frac{2}{3}y^{\frac{3}{2}}\left(\ln y-\frac{2}{3}\right)$

$\Longleftarrow$

$\frac{2}{3}y^{\frac{3}{2}}\left(\ln y-\frac{2}{3}\right)\Big]_1^4$

$=\frac{2}{3}\cdot 8\left(\ln 4-\frac{2}{3}\right)-\frac{2}{3}\cdot\frac{-2}{3}$

$\frac{16}{3}\left(\ln 4-\frac{2}{3}\right)+\frac{4}{9}$

37) $\int_0^{\frac{\pi}{4}} \tan^3\theta \sec^2\theta \, d\theta$ $\quad u = \tan\theta$

$\frac{du}{dx} = \sec^2\theta \quad du = \sec^2\theta \, d\theta$

4) $\int \frac{\sin^3 x}{\cos x} dx = \int \frac{\sin^2 x \sin x}{\cos x} dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}$ $\quad u = \cos x \quad \frac{du}{dx} = -\sin x \quad du = -\sin x \, dx$

$= -\int \frac{(1-u^2)}{u} du = -\left(\int \frac{1}{u} du - \int u \, du\right) = -\left(\ln|u| - \frac{u^2}{2}\right) = -\left(\ln|\cos x| - \frac{\cos^2 x}{2} + C\right)$

$= -\ln|\cos x| + \frac{\cos^2 x}{2} + C$

6) $\int_0^1 \frac{x}{(2x+1)^3} dx$ partial fractions

$\frac{x}{(2x+1)^3} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$

$x = A(2x+1)^2 + B(2x+1) + C$

$x = A(4x^2+4x+1) + 2Bx + B + C$

$x = 4Ax^2 + 4Ax + A + 2Bx + B + C$

$x^2(4A) + x(4A+2B) + A + B + C$

$4A = 0 \qquad A = 0$

$4A + 2B = 1 \qquad B = \frac{1}{2}$

$A + B + C = 0 \qquad C = -\frac{1}{2}$

$\int \frac{x}{(2x+1)^3} = \int \frac{1}{2(2x+1)^2} - \int \frac{1}{2(2x+1)^3}$

$= \frac{1}{2} \int \frac{1}{(2x+1)^2} dx - \frac{1}{2} \int \frac{1}{(2x+1)^3} dx$

$u = 2x+1 \quad \frac{du}{dx} = 2 \quad du = 2dx$ $\qquad u = 2x+1 \quad \frac{du}{dx} = 2 \quad du = 2dx$

$\frac{1}{4} \int \frac{1}{u^2} - \frac{1}{4} \int \frac{1}{u^3} \Big]_1^3$

$= \frac{1}{4} - u^{-1} - \frac{1}{4} \frac{1}{-2} u^{-2} \Big]_1^3$

$= -\frac{1}{4} u^{-1} + \frac{1}{8} u^{-2} \Big]_1^3 =$ $\cancel{-\frac{1}{12} + \frac{1}{72} - \left(-\frac{1}{4} + \frac{1}{8}\right) = \ldots}$

$= \frac{1}{12}$ is this even right? let's check another way...

What if... $2x+1 = u \quad \frac{du}{dx} = 2 \quad du = 2dx$

$2x = u - 1$

$x = \frac{u-1}{2} = \frac{1}{2}u - \frac{1}{2}$

So $\int \frac{x \, dx}{(2x+1)^3} = \frac{1}{2} \int \frac{\frac{1}{2}u - \frac{1}{2}}{u^3} du = \frac{1}{2} \int \frac{(u-1)}{2u^3} = \frac{1}{4} \int \frac{u-1}{u^3}$

$= \frac{1}{4} \int \frac{1}{u^2} - \frac{1}{4} \int \frac{1}{u^3}$

$= \frac{1}{4} \cdot -u^{-1} - \frac{1}{4} \cdot \frac{1}{-2} u^{-2} = \frac{-1}{4} u^{-1} + \frac{1}{8} u^{-2} \Big]_1^3$

$\cancel{= \frac{-1}{6} + \frac{1}{36} - \left(\frac{-1}{2} + \frac{1}{4}\right)}$

$\cancel{\frac{-6}{36} + \frac{1}{36} - \left(\frac{-18}{36} + \frac{9}{36}\right)}$

$\cancel{\frac{-5}{36} - \left(\frac{-9}{36}\right) = \frac{4}{36} = \frac{1}{9}}$

$\frac{-1}{12} + \frac{1}{72} - \left(\frac{-1}{4} + \frac{1}{8}\right)$

$\frac{-6}{72} + \frac{1}{72} - \left(\frac{-18}{72} + \frac{9}{72}\right)$

$\frac{-5}{72} - \frac{-9}{72} = \frac{4}{72} = \boxed{\frac{1}{18}}$

$$\begin{array}{r} x+1 \\ x^2-x-6 \,\big)\, x^3-4x-10 \\ x^3-x^2-6x \\ \hline x^2-4x+6x-10 \\ x^2+2x-10 \\ x^2-x-6 \\ \hline 3x-4 \end{array}$$

9) $\int_2^4 \frac{x+2}{x^2+3x-4}\,dx = \int_2^4 \frac{x+2}{(x-1)(x+4)}$

partial fractions:

$$\frac{x+2}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$x+2 = A(x+4) + B(x-1)$$

$$x+2 = Ax+4A+Bx-B$$

$$x+2 = x(A+B)+4A-B$$

$A+B=1$

$4A-B=2$

$5A=3$

$A=\frac{3}{5}$ $B=\frac{2}{5}$

so check $\frac{x+2}{(x-1)(x+4)} = \frac{3}{5(x-1)} + \frac{2}{5(x+4)}$

$$\frac{3(x+4)}{5(x-1)(x+4)} + \frac{2(x-1)}{5(x+4)(x-1)}$$

$$\frac{3x+12+2x-2}{5(x-1)(x+4)}$$

$$= \frac{5x+10}{5(x-1)(x+4)} \checkmark$$

so Integral $\int_2^4 \frac{x+2}{x^2+3x-4} = \int_2^4 \frac{3}{5(x-1)}dx + \int_2^4 \frac{2}{5(x+4)}dx$

$= \frac{3}{5}\int_2^4 \frac{dx}{x-1}dx + \frac{2}{5}\int_2^4 \frac{dx}{x+4}dx$ $\quad u=x+4 \quad \frac{du}{dx}=1 \quad du=dx$

$= \frac{3}{5}\int_2^4 \ln|x-1| + \frac{2}{5}\int_2^4 \ln|x+4|$

$= \frac{3}{5}(\ln 3 - \ln 1) + \frac{2}{5}(\ln 8 - \ln 6)$

$\frac{3}{5}\ln 3 + \frac{2}{5}\ln\left(\frac{4}{3}\right)$

10) $\int \frac{\cos\frac{1}{x}}{x^3}dx = \int \cos\frac{1}{x}\, x^{-3}dx$

$u=\frac{1}{x}$ $\quad \frac{du}{dx} = -x^{-2}$

$u^3 = \frac{1}{x^3} = x^{-3}$ $\quad du = -x^{-2}dx \quad -du = x^{-2}dx$

$u^2 = \frac{1}{x^2} = x^{-2}$

$= -\int \frac{\cos(u)\, u^3}{u^2}du$

$= -\int \cos(u) u\, du$ $\quad t=u \quad dz=\cos u$

$dt=1 \quad z=\sin u$

IBP $\quad = u\sin u - \int \sin u\, du$

$\boxed{= u\sin u + \cos u\, du}$

13) $\int \sin^5 t \cos^4 t\, dt$

$= \int(\sin^2 x)^2 \cos^4 x \sin x\, dx = \int(1-\cos^2 x)^2 \cos^4 x \sin x\, dx$ $\quad u=\cos x \quad du=-\sin x\, dx$

$= -\int (1-u^2)^2 u^4\, du$

$= \int (1-2u^2+u^4)u^4\, du = \int u^4 - 2u^6 + u^8\, du = \frac{u^5}{5} - \frac{2}{7}u^7 + \frac{u^9}{9} + C$

$$= \boxed{-\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9} + C}$$

15) $\int x \sec x \tan x \, dx$ IBP $u = x$ $du = 1\,dx$ $dv = \sec x \tan x$ $v = \sec x$

$= \int x \sec x - \int \sec x$

$= \boxed{x \sec x - \ln|\sec x + \tan x| + C}$

16) $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$ $x = \sin\theta$ $\frac{dx}{d\theta} = \cos\theta$ $dx = \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta = \int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos(2x) \, dx$

$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)\Big]_0^{\frac{\pi}{4}} = \frac{1}{2}x - \frac{1}{4}\sin 2x \Big]_0^{\frac{\pi}{4}}$

$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$

25) $\int_0^1 \frac{1+12t}{1+3t} dt$ left side: $\int \frac{1}{1+3t} dt$ $u = 1+3t$ $du = 3dt$ $\frac{du}{3} = dt$

$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|1+3t| + C$

right side: $\int \frac{12t}{1+3t} = \int \frac{12t+4-4}{1+3t} = \int \frac{4(3t+1)-4}{1+3t} = \int 4 - \frac{4}{1+3t} = \int 4 - \int \frac{4}{1+3t} dt$ $u = 1+3t$ $du = 3dt$ $\frac{du}{3} = dt$

$= 4t - \frac{4}{3}\ln|1+3t| + C$

$= \frac{1}{3}\ln|1+3t| + 4t - \frac{4}{3}\ln|1+3t|\Big]_0^1$

$= \frac{1}{3}\ln|4| + 4 - \frac{4}{3}\ln|4|$

$= \boxed{-\ln|4| + 4}$

32) $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2} dx = \int_1^3 e^{\frac{3}{x}} x^{-2} dx$ $u = \frac{3}{x}$ $\frac{du}{dx} = \frac{-3}{x^2} = -3x^{-2}$

$du = -3x^{-2} dx$ so $\frac{du}{-3} = x^{-2} dx$

$= \frac{-1}{3}\int_3^1 e^u = \frac{-1}{3} e^u \Big]_3^1$

$= \frac{-1}{3}e^1 - \frac{-1}{3}e^3 = \boxed{\frac{-1}{3}e(1 - e^2)} + C$

37) $\int_0^{\frac{\pi}{4}} \tan^3\theta \sec^2\theta \, d\theta$

$u = \tan\theta$

$\frac{du}{dx} = \sec^2\theta \quad du = \sec^2\theta \, d\theta$

$\int_0^1 u^3 \, du = \frac{u^4}{4}\Big]_0^1 = \boxed{\frac{1}{4}}$

55) $\int \frac{dx}{x + x\sqrt{x}} dx$

$u = \sqrt{x}$

$u^2 = x$

$\frac{dx}{du} = 2u \quad dx = 2u\,du$

$\int \frac{2u\,du}{u^2+u^3} = \int \frac{2}{u^1+u^2} du = \int \frac{2}{u(1+u)}$

partial fractions

$\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$

$2 = A(1+u) + Bu$

$2 = A + Au + Bu = A + u(A+B)$

$A = 2 \quad A = 2$

$A + B = 0 \quad B = -2$

$= \int \frac{2}{u} + \int \frac{-2}{1+u}$ ← $z = 1+u$, $dz = du$

$= 2\ln|u| + \int \frac{-2}{z} dz$

$= 2\ln|u| + -2\ln|1+u|$

$= 2\ln|\sqrt{x}| + -2\ln|1+\sqrt{x}|$

$\boxed{= \ln|x| - 2\log|1+\sqrt{x}| + C}$