Module 11 Solve.

Steps:
- find auxiliary equation
- find roots
- plug into $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ if $b^2 - 4ac > 0$.
- if $b^2 - 4ac = 0$, plug into $y = c_1 e^{rx} + c_2 x e^{rx}$
- if $b^2 - 4ac < 0$, plug into $y = e^{\alpha x}(c_1 \cos Bx + c_2 \sin Bx)$ when $r_1 = \alpha + iB$ and $r_2 = \alpha - iB$ when α is a real number

11.1) 2) $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3)$ so $r = 3$

so $\boxed{y = c_1 e^{3x} + c_2 x e^{3x}}$

3) $y'' + 2y = 0$

$r^2 + 2 = 0$

$\cancel{(r+\sqrt{2})(r-\sqrt{2})}$

$r^2 = -2 \Rightarrow r \pm \sqrt{-1 \cdot 2}$

$r = \sqrt{-1 \cdot 2} = i \cdot \sqrt{2}$

$r \pm \sqrt{2} i$

$\cancel{r = \frac{-2 \pm \sqrt{4-4}}{2} = -1 \pm 2}$

$r = \frac{0 \pm \sqrt{0 - 4(1)(2)}}{2}$

$= \frac{0 \pm \sqrt{-8}}{2}$

$= 0 \pm \sqrt{4 \cdot 2 \cdot -1} = \pm \sqrt{2} i$

$r = 0 + \sqrt{2} i$ and $r = 0 - \sqrt{2} i$ $\alpha = 0$ $B = \sqrt{2}$ and $e^{\alpha x} = 1$ so $\boxed{y = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x}$

Check: $y' = -c_1 \sin \sqrt{2} x \cdot \sqrt{2} + \sqrt{2} c_2 \cos \sqrt{2} x$

$y'' = -2 c_1 \cos \sqrt{2} x - 2 c_2 \sin \sqrt{2} x$ and $y'' + 2y = 0$ ✓

4) $y'' + y' - 12y = 0$

auxiliary equation: $r^2 + r - 12 = 0$

$(r+4)(r-3) \rightarrow r = -4, 3$ so $\boxed{y = c_1 e^{3x} + c_2 e^{-4x}}$

5) $4y'' + 4y' + y = 0$

auxiliary equation: $4r^2 + 4y + 1 = 0$

$(2r+1)(2r+1) = 0$

$r = \frac{-1}{2}$

$y = c_1 e^{\frac{-x}{2}} + c_2 x e^{\frac{-x}{2}}$

6) $\overset{a}{9}y'' + 4y = 0$

auxiliary equation: $\overset{a}{9}r^2 + \overset{c}{4} = 0$ $\rightarrow 9r^2 = -4$

$\cancel{(3r\quad)(3r\quad)}$

$r^2 = \frac{-4}{9} \rightarrow r = \pm\sqrt{\frac{-4}{9}}$

$= \pm \frac{2}{3} i$

using quadratic:

$r = \frac{-0 \pm \sqrt{0 - 4(9)(4)}}{18}$ 36 4 144

$= \frac{\pm\sqrt{-144}}{18}$

$= \pm \frac{-12}{18} = \pm \frac{2}{3} i$

$\boxed{y = c_1 \cos \frac{2}{3} x + c_2 \sin \frac{2}{3} x}$

$r = 0 + \frac{2}{3} i$ and $r = 0 - \frac{2}{3} i$

so $\alpha = 0$

$B = \frac{2}{3}$

8) $y = y'' \rightarrow y'' - y = 0$

auxiliary equation $\Rightarrow r^2 - 1 = 0$

$r^2 = 1 \rightarrow r = \pm 1$ two roots

$y = c_1 e^{x} + c_2 \cancel{x} e^{-x}$

$y' = c_1 e^{x} - c_2 e^{-x}$

$y'' = c_1 e^{x} + c_2 e^{-x}$

so check $y'' - y = 0$ ✓

substitute: $(-4A \cos 2x - 4B \sin 2x)$ ... $\sin 2x$ (on back)

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10) $3y'' + 4y' - 3y = 0$ auxiliary equation: $\underset{a}{3r^2} + \underset{b}{4r} \underset{c}{-3} = 0 \longrightarrow (3r \quad)(r \quad 1) = 0$

doesn't work yet

$r = \dfrac{-4 \pm \sqrt{16 - 4(3)(-3)}}{6} = \dfrac{-4 \pm \sqrt{16+36}}{6} = \dfrac{-4}{6} \pm \dfrac{\sqrt{52}}{6}$

$= \dfrac{-2}{3} \pm \dfrac{2\sqrt{13}}{3}$

two roots

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \quad \text{where } r_1 = \frac{-2}{3} + \frac{2\sqrt{13}}{3},\ r_2 = \frac{-2}{3} - \frac{2\sqrt{13}}{3}}$$

12) $r'' + 6r' + 34R = 0$

auxiliary: $r^2 + 6r + 34 = 0$

$(r \quad)(r \quad)$ no easy roots

$a=1 \quad b=6 \quad c=34$

$r = \dfrac{-6 \pm \sqrt{36 - 4(1)(34)}}{2}$

$= -3 \pm \dfrac{\sqrt{100}}{2} = -3 \pm 5$

$r = 2, -8$ two real roots

$$\boxed{y = c_1 e^{2x} + c_2 e^{-8x}}$$

18) Solve initial problem

$y'' - 2y' - 3y = 0 \qquad y(0) = 2 \qquad y'(0) = 2$

auxiliary equation $r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$ so $r = \{1, 3\}$

so $y = C_1 e^{-x} + C_2 e^{3x}$

and $y' = -C_1 e^{-x} + 3C_2 e^{3x}$

so $y(0) = 2 = C_1 + C_2$

$y'(0) = 2 = -C_1 + 3C_2$

Solve: $4 = 4C_2$ so $C_2 = 1$

$C_1 = 1$

so ~~initial~~ solution to initial value problem is:

$$\boxed{y = e^{-x} + e^{3x}}$$

19) $9y'' + 12y' + 4y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$9r^2 + 12r + 4 = 0$

$(3r + 2)(3r + 2) = 0$

$r = \frac{-2}{3}$ one root

general equation: $y = C_1 e^{\frac{-2x}{3}} + C_2 x e^{\frac{-2x}{3}}$

$C_1 = \frac{3}{2}$

$C_2 = \frac{-1}{2}$

$y(0) = 1 = C_1 + C_2$

$y'(0) = 0 = \frac{-2}{3} C_1 + \frac{-2}{3} C_2 \cdot 0 + 1$

$y' = \frac{-2}{3} C_1 e^{\frac{-2x}{3}} + \frac{-2}{3} C_2 x e^{\frac{-2x}{3}} + e^{\frac{-2x}{3}}$

specific solution given the initial conditions: $\boxed{y = \frac{3}{2} e^{\frac{-2}{3}x} + \frac{-1}{2} x e^{\frac{-2x}{3}}}$

27) $y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y(1) = 0$

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$r = -2$ two real but equal roots

general: $y = C_1 e^{-2x} + C_2 x e^{-2x}$

if $y(0) = 2 = C_1$ ~~+C_2~~

if $y(1) = 0 = C_1 e^{-2} + C_2 e^{-2} = e^{-2}(C_1 + C_2)$

solve: ~~add $2 = C_1 + C_2 + e^{-2}(C_1 + C_2)$~~

~~$2 - (C_1 + C_2) = e^{-2}(C_1 + C_2)$~~

~~$\frac{2}{C_1 + C_2} - 1 = e^{-2}$~~

$0 = e^{-2}(2 + C_2) \qquad C_2 = -2$

solution given initial conditions

$$\boxed{y = 2e^{-2x} - 2xe^{-2x}}$$

Module 17.2

1) Use undetermined coefficients

$y'' + 2y' - 8y = 1 - 2x^2$

$y = y_c + y_p$

Find complimentary first: so find auxiliary now:

$r^2 + 2r - 8 = (r-2)(r+4) = 0 \qquad$ roots $= \{2, -4\}$

$y_c = C_1 e^{2x} + C_2 e^{-4x}$

Lets find particular: in form of

$y_p(x) = Ax^2 + Bx + C \qquad$ find derivatives to plug in.

$y'_p = 2Ax + B$ and $y''_p = 2A$

$(2A) + 2(2Ax + B) - 8(Ax^2 + Bx + C) = 1 - 2x^2$

$-8Ax^2 + (4A - 8B)x + (2A + 2B - 8C) = 1 - 2x^2$

$-8A = -2 \qquad A = \frac{1}{4}$

$4A - 8B = 0 \qquad B = \frac{1}{8}$

$2A + 2B - 8C = 1$

$\frac{1}{2} + \frac{1}{4} - 8C = 1$

$-8C = \frac{1}{4}$

$C = \frac{-1}{32}$

so $y_p = \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}$

so general is add them together

$$y = \text{general solution} = \boxed{C_1 e^{2x} + C_2 e^{-4x} + \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}}$$

2) $y'' - 3y' = \sin 2x$

$y = y_c + y_p$

Find complimentary equation first by finding auxiliary: $r^2 - 3r = 0$

$r(r-3) = 0$

roots $= \{0, 3\}$

$y_c = C_1 e^{0x} + C_2 e^{3x}$

Find particular: $y_p = A\cos kx + B\sin kx$

$y'_p = -kA\sin kx + kB\cos kx$

$y''_p = -k^2A\cos kx - k^2B\sin kx$

~~substitute: $(-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) = \sin 2x$~~

~~$(-B + 3A)\sin x + (-A - 3B)\cos x = \sin 2x$~~

~~substitute: $(-4A\cos x - 4B\sin x) - 3(-2A\sin x + 2B\cos 2x)$~~

substitute: $(-4A\cos 2x - 4B\sin 2x) - 3(-2A\sin 2x + 2B\cos 2x) = \sin 2x$ (on back)

5 2) Continued

$(-4B+6A)\sin 2x + (-4A-6B)\cos 2x = \sin 2x$

$-4B+6A=1 \qquad -8B+\overset{12}{\cancel{8}}A=2$

$-6B-4A=0 \rightarrow -6B=4A \rightarrow B=\frac{-2}{3}A$

$-10B+2A=1$

so $-\frac{8}{3}A+6A=1$

$\frac{26}{3}A=1$

$A=\frac{3}{26}$

$-6B=\frac{12}{26} \qquad B=\frac{-12}{\cancel{216}\,156}$

$y=\text{general}=c_1+c_2e^{3x}+\frac{3}{26}\cos 2x + \frac{-12}{\cancel{208}\,156}\sin 2x$

4) $y''-\overset{2}{\cancel{4}}y' \cancel{\;} +2y = x+e^x$

Find complimentary first via auxiliary equation

$r^2-\overset{2}{\cancel{3}}r+2=0 \rightarrow \cancel{(r\quad)(r-2)=0}$

$r=\frac{2\pm\sqrt{4-4(1)(2)}}{2}=1\pm\frac{\sqrt{-4}}{2}$

$=1\pm\frac{\cancel{\sqrt{\;}}\,2i}{2}$

$=1\pm i$ so $\alpha=1$, $\beta=1$

homo has general solution

$\boxed{y=e^x(c_1\cos x + c_2\sin x)}$

split this y:

$y_{p_1}(x)=y''-2y'+2y=x$ we try

$y_{p_1}(x)=Ax+B$

$y'_{p_1}=A$

$y''_{p_1}=0$

so $0-2A+(2Ax+2B)=x$

so $2Ax+(-2A+2B)=x$

$2A=1$ so $A=\frac{1}{2}$

$-2A+2B=0$ so $B=\frac{1}{2}$

so $\boxed{y_{p1}=\frac{1}{2}x+\frac{1}{2}}$

for y_{p2}: $y''-2y'+2y=e^x$

$y_{p2}=Ae^x$ so $y'_{p2}=Ae^x$ and $y''_{p2}=Ae^x$

$Ae^x-2Ae^x+2Ae^x=e^x$

$A=1$ so $y_{p2}=e^x$

$\boxed{y=\text{general}=e^x(c_1\cos x+c_2\sin x)+\frac{1}{2}x+\frac{1}{2}+e^x}$

5) $y'' - 4y' + 5y = e^{-x}$

general solution = $y_c + y_p$

find complimentary solution first:

$r^2 - 4r + 5 = 0$ $a = 1, b = -4, c = 5$

~~$(r+1)(r-5)$~~ ~~roots~~

roots = $\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \frac{\sqrt{-4}}{2}$

$= 2 \pm 2i$

so complimentary: $y =$ $\alpha = 2$ $\beta = 2$

$\boxed{e^{2x}(A\cos 2x + B\cos 2x)}$

particular

$y'' - 4y' + 5y = e^{-x}$

try $y_p = Ae^{-x}$

$y'_p = -Ae^{-x}$

$y''_p = Ae^{-x}$

sub

$Ae^{-x} + 4Ae^{-x} + 5Ae^{-x} = e^{-x}$

$+4Ae^{-x} + 6Ae^{-x} = e^{-x}$

$10Ae^{-x} = e^{-x}$

$10A = 1$

$A = \frac{1}{10}$

$\boxed{y_p = \frac{1}{10}e^{-x}}$

general: $\boxed{y = e^{2x}(A\cos 2x + B\cos 2x) + \frac{1}{10}e^{-x}}$

6) $y'' - 4y' + 4y = x - \sin x$

aux: $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$ roots = 2

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

$y_{p1} = y'' - 4y' + 4y = x$

in the form $Ax + B$.

so $y'_p = A$

$y''_p = 0$

subbing:

$0 - 4A + 4Ax + 4B = x$

$4Ax + (-4A + 4B) = x$

$4A = 1$

$A = 1/4$ $B = \frac{1}{4}$

so $y_p = \frac{1}{4}x + \frac{1}{4}$

other y_{p2} in form: $A\sin x + B\cos x$

$y'_{p2} = A\cos x - B\sin x$

$y''_{p2} = -A\sin x - B\cos x$

$-A\sin x - B\cos x - 4A\cos x + 4B\sin x + 4A\sin x + 4B\cos x = -\sin x$

$3A + 4B = -1$

$3B - 4A = 0$

$3B = 4A$

$\frac{3}{4}B = A$

$\frac{9}{4}B + 4B = -1$

$\frac{25}{4}B = -1$

$B = \frac{-4}{25}$

$A = \frac{-3}{25}$

$4B = \frac{-16}{25}$

$3A = \frac{-9}{25}$

$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{4}x + \frac{1}{4} + \frac{-3}{25}\sin x + \frac{-4}{25}\cos x}$

10) $y'' + y' - 2y = x + \sin 2x$ $\quad y(0) = 1, \quad y'(0) = 0$

complementary

$r^2 + r - 2 = 0$

$(r-1)(r+2) \to$ roots $= \{1, -2\}$

$y_c = c_1 e^x + c_2 e^{-2x}$

$y_{p_1} = Ax + B$

$y'_{p_1} = A$

$y''_{p_1} = 0$

$0 + A - 2Ax - 2B = x$

$-2Ax + (A - 2B) = x$

$-2A = 1$ and $A - 2B = 0$

$A = \frac{-1}{2}$ $\quad -2B = \frac{1}{2}$

$B = -1/4$

$y_{p_1} = -\frac{1}{2}x - 1/4$

$y_{p_2} = y'' - y' - 2y = \sin 2x$

$y_{p_2} = A \sin 2x + B \cos 2x$

$y'_{p_2} = 2A \cos 2x - 2B \sin 2x$

$y''_{p_2} = -4A \sin 2x - 4B \cos 2x$

~~$-4A = 1$ $\quad -4B = 0$~~

~~$A = -\frac{1}{4}$ $\quad B = 0$~~

~~$y_{p_2} = -\frac{1}{4} \sin 2x$~~

~~$\sin 2x$~~ $- \frac{1}{2}\cos 2x + \frac{1}{2}\sin 2x$

$-4A \sin 2x - 4B \cos 2x - 2A \cos 2x + 2B \sin 2x$

$-2A \sin 2x - 2B \cos 2x = \sin 2x$

sines: $-4A + 2B - 2A = 1 \to -6A + 2B = 1$

cosines: $-4B - 2A - 2B = 0 \to -6B - 2A = 0$

$-18B - 6A = 0$

$-6B = 2A$

$B = \frac{-1}{3}A$

$-6A + \frac{-2}{3}A = 1$

$-\frac{20}{3}A = 1$

$-20A = 3$

$A = \frac{3}{-20}$

$20B = 1$

$B = \frac{1}{20}$

$-6A = \frac{18}{20}$

$A = \frac{18}{-120}$

$A = \frac{3}{-20}$

$$\boxed{y = c_1 e^x + c_2 e^{-2x} + \frac{-1}{2}x - \frac{1}{4} + \frac{-3}{20}\sin 2x + \frac{1}{20}\cos 2x}$$

21) $y'' - 2y' + y = e^{2x}$

auxiliary equation:

$r^2 - 2r + 1 = 0 \to (r-1)(r-1) = 0$

root $= 1$

$y_c = c_1 e^x + c_2 x e^x$

$y_p = Ae^{2x}$

$y'_p = 2Ae^{2x}$

$y''_p = 4Ae^{2x}$

so $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x}$

$A = 1$

$y_p = e^{2x}$

general by undetermined coefficients $=$ $\boxed{y = c_1 e^x + c_2 x e^x + e^{2x}}$

21) (continued) using variation of parameters

so $y_c = c_1 e^x + C_2 x e^x$

find particular in form: $y_p = u_1(x)e^x + u_2(x)xe^x$

- $y_p' = u_1(x)e^x + u_1'e^x + u_2(x)(xe^x + e^x) + u_2'xe^x$

- set $u_1'e^x + u_2'xe^x = 0$

so $y_p' = u_1(x)e^x + u_2(x)xe^x + u_2(x)e^x$

$y_p'' = u_1(x)e^x + u_1'e^x + u_2(x)(xe^x + e^x) + u_2'xe^x + u_2(x)e^x + u_2'e^x$

so combining: $u_1(x)e^x + u_1'e^x + u_2(x)xe^x + u_2(x)e^x + u_2(x)e^x + u_2'e^x + u_2'xe^x$

$-2u_1(x)e^x - 2u_2(x)xe^x - 2u_2(x)e^x + u_1(x)e^x + u_2(x)xe^x = e^{2x}$

$u_1(x)(\underbrace{e^x - 2e^x + e^x}_{0}) + u_2(x)(\underbrace{xe^x + e^x + e^x - 2xe^x - 2e^x + xe^x}_{0}) + u_1'(e^x) + u_2'(e^x) + u_2'xe^x = e^{2x}$

so equations: $u_1'(e^x) + u_2'(e^x) + u_2'xe^x = e^{2x}$

$u_1'e^x + u_2'xe^x = 0$

$u_2'(e^x) = e^{2x}$

so $u_2' = e^x$

$u_2 = e^x + C$

so $u_1'e^x + e^x xe^x = 0$

$u_1'e^x = -e^x xe^x$

$u_1'e^x = -e^{2x}x$

$u_1' = -e^x x$

so $\int u_1' = \int -e^x x\,dx$

$= -\int e^x x\,dx$

IBP: $u = x$ $du = 1$

$dv = e^x$ $v = e^x$

so $= -xe^x + \int e^x dx$

$\boxed{\begin{aligned} &= -xe^x + e^x = u_1 \\ &e^x = u_2 \end{aligned}}$

so $y_p = (-xe^x + e^x)e^x + (e^x)xe^x$

$-xe^{2x} + e^{2x} + xe^{2x} = e^{2x} = y_p$

$\boxed{y = \text{general} = C_1e^x + C_2xe^x + e^{2x}}$

22) $y'' - y' = e^x$ via undetermined coefficients!

$~~y_p = Ae^x~~$

$~~y_p' = Ae^x~~$

$~~y_p'' = Ae~~$ ← But! this is a solution of complimentary

so use $~~y_p = Ae^x~~$

$y_p = Axe^x$

$y'_p = Axe^x + e^x A$

$= A(xe^x + e^x)$

$y''_p = Axe^x + e^x A + e^x A$

$= Axe^x + 2Ae^x$

auxiliary equation is:

$r^2 - r = 0$

$r(r-1) = 0$ roots: $\{0, 1\}$

$y_c = C_1 e^0 + c_2 e^x = C_1 + C_2 e^x$

so $Axe^x + 2Ae^x - Axe^x - e^x A = e^x$

$Ae^x = e^x$

$A = 1$

so $y_p = xe^x$

general: $\boxed{y = C_1 + C_2 e^x + xe^x}$

check: $y = xe^x$

$y' = xe^x + e^x$

$y'' = xe^x + e^x + e^x = xe^x + 2e^x$ ✓

$y'' - y' = e^x$

via variation in parameters:

Complimentary: $C_1 e^0 + C_2 e^x = y_c$

sub w/ functions: $u_1 + u_2 e^x = y_p$

$y_p' = u_1' + u_2 e^x + u_2' e^x$

set $u_1' + u_2' e^x = 0$

$y_p' = u_2 e^x$

$y_p'' = u_2 e^x + u_2' e^x$

$u_2 e^x + u_2' e^x - u_2 e^x = e^x$

$u_2' e^x = e^x$

$u_2' = 1$

$\int u_2' = x = u_2$

so $u_1 = -u_2' e^x$

$u_1 = -e^x$

so $y_p = -e^x + xe^x$

$y = y_c + y_p = -e^x + xe^x + C_1 + C_2 e^x$

$C_2 e^x - e^x = ~~C_2(-1+e^x)~~$

$e^x(C_2 - 1) = C_2 e^x$

where $C_2 = C_2 - 1$

So $\boxed{y = xe^x + C_1 + C_2 e^x}$

23) $y'' + y = \sec^2 x$, $0 \leq x < \pi/2$

variation of params. first find complimentary equation:

$r^2 + \cancel{0r} + 1 = 0$ $\quad r^2 = -1$

$r(r+1) = 0$ $\quad r = i$

so $\cancel{y_c = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}}$

$y_c = e^0 (A \cos x + B \sin x)$

$y_p(x) = A \cos x + B \sin x$

sub constants w functions: $u_1 \cos x + u_2 \sin x$

$y'_p = -u_1 \sin x + u_1' \cos x + u_2 \cos x + u_2' \sin x$

set $u_1' \cos x + u_2' \sin x = 0$ (A)

$y'_p = -u_1 \sin x + u_2 \cos x$

$y''_p = -u_1 \cos x - u_1' \sin x - u_2 \sin x + u_2' \cos x$

$y'' + y = -u_1' \sin x + u_2' \cos x = \sec^2 x$ (B) after combining

so solving (A) and (B)

(A) $= u_2' \sin x = -u_1' \cos x$

$\cancel{u_2' = -u_1' \frac{\cos x}{\sin x}}$

$u_1' = -u_2' \sin x \cdot \frac{1}{\cos x}$

$u_2' = -u_1' \cos x \cdot \frac{1}{\sin x}$

$u_2' \frac{\sin^2 x}{\cos x} + u_2' \cos x = \sec^2 x$

$u_2' \sin^2 x + u_2' \cos^2 x = \sec x$

$u_2' = \sec x$

$u_2 = \ln(\sec x + \tan x)$

$u_1' = -\frac{\sin}{\cos} \cdot \frac{1}{\cos} = -\tan x \sec x$

$u_1 = \sec x$

$y_p = -\sec x \cdot \cos x + \ln(\sec x + \tan x) \sin x$

$y_p = -1 + \ln(\sec x + \tan x) \sin x$

$\boxed{\text{general} = c_1 \cos x + c_2 \sin x - 1 + \ln(\sec x + \tan x) \sin x}$