1) a) $\begin{bmatrix} 1 & -2 & 1 & 0 \\ -2 & 0 & 2 & 1 \\ 1 & 1 & -1 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ not symmetric so $a_{32} \neq a_{23}$ and $a_{42} \neq a_{24}$ and $a_{43} \neq a_{34}$

b) totally symmetric

2) a) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$ $P^T =$ (same matrix) $PP^T = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 & \frac{1}{2}+\frac{-1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2}+\frac{-1}{2} & 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$

$= I$

b) are columns orthonormal?

$P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $P_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ $P_1 \cdot P_2 = 0$, $P_1 \cdot P_3 = 0$, $P_2 \cdot P_3 = 0$ so orthogonal ✓

$\|P_1\| = \sqrt{\frac{1}{2}+0+\frac{1}{2}} = 1 = \|P_2\| = \|P_3\|$ so orthonormal too

3) $A = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 2 \\ 2 & \lambda-2 \end{vmatrix} = \lambda^2 - \lambda - 2 - 4 = (\lambda - 3)(\lambda + 2)$

$\lambda = \{3, -2\}$

for eigenvector 3:

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $\begin{matrix} t_1 \\ t_2 \end{matrix} = \begin{matrix} -t \\ 2t \end{matrix} = t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

eigenvector $= (-1, 2)$

for eigenvector -2

$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{matrix} 2t \\ t \end{matrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

eigenvector: $(-2, 1)$

$(-1, 2) \cdot (2, 1) = 0$ so orthogonal expand w/ row 1 $\lambda(\lambda-1)(\lambda+1)$ $|\lambda I - A| = 0$

4) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ symmetric ✓

$|\lambda I - A| = \cancel{\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix}} = \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{vmatrix} = \lambda^2 - 4\lambda + 4 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda-4)$

$$\boxed{\begin{array}{l} \lambda = \{0, 4\} \\ \text{eigenvector} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \text{eigenvector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}}$$

dot product $= 0$ ✓

theorem 7.9

eigen 0:

$\lambda I - A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} -t \\ t \end{matrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvector 4:

$4I - A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{matrix} x_1 & t \\ x_2 & t \end{matrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

make orthonormal: $\left\{ \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$

$$\boxed{P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

find $P^{-1} \to \left[\begin{array}{cc|cc} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \end{array}\right]$

$\to \begin{bmatrix} 1 & -1 & -\sqrt{2} & 0 \\ 1 & 1 & 0 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -\sqrt{2} & 0 \\ 0 & 2 & \sqrt{2} & \sqrt{2} \end{bmatrix}$

$\to \begin{bmatrix} 1 & -1 & -\sqrt{2} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$P^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = P^T$

so $D = P^T A P$
$= P^{-1} A P =$

$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$ = temp variable

$\begin{bmatrix} 0 & 0 \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & \frac{8}{\sqrt{2}} \end{bmatrix} = \text{Diagonal}}$

5) matrix is symmetric ✓

$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix}$ expand $\overline{w}$ row 1 $\lambda \cdot (\lambda^2 - 1) = \lambda(\lambda-1)(\lambda+1)$

$\lambda = \{0, 1, -1\}$ is when $|\lambda I - A| = 0$

for eigenvalue = 0

$-A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for eigenvalue = 1

$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} \to t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

for eigenvalue = -1

$-I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

- all 3 eigenvectors are orthogonal because eigenvector 1 · eigenvector 2 = 0
  2 · 3 = 0
  1 · 3 = 0

now divide each eigenvectors by its length

$(1,0,0) \to (1,0,0)$

$(0,1,1) \to (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$(0,-1,1) \to (0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

so $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

columns are orthonormal so $P^{-1} = P^T$
theorem 7.8

$P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$P^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = Diagonal

1) $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$|\lambda I - A| = 0$ for nontrivial solutions

$$\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} = 0 \rightarrow \lambda \cdot (C_{11}) + (-1)(C_{13})$$

use first row

use $C_{11} = -1^2 \cdot \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda & 0 \\ -1 & 0 & \lambda \end{vmatrix}$

use 2nd row

$= -1^2 \cdot \lambda \cdot (-1)^4 \cdot (\lambda^2 - 1)$

$= \lambda \cdot (\lambda^2 - 1) = \lambda \cdot (\lambda - 1)(\lambda + 1)$

$C_{13} = (-1)^4 \cdot \begin{vmatrix} 0 & \lambda & -1 \\ -1 & 0 & 0 \\ 0 & -1 & \lambda \end{vmatrix}$

cofactor use 2nd row

$= -1^4 \cdot -1 \cdot (-1)^3 \cdot (\lambda^2 - 1) = (\lambda + 1)(\lambda - 1)$

so $|\lambda I - A| = \lambda^2 \cdot (\lambda - 1)(\lambda + 1) + (-1)(\lambda - 1)(\lambda + 1)$

$= (\lambda - 1)(\lambda + 1)(\lambda^2 - 1)$

$= (\lambda - 1)(\lambda + 1)(\lambda + 1)(\lambda - 1)$

$= \boxed{(\lambda - 1)^2 (\lambda + 1)^2}$

a) eigenvalues $= \{-1, 1\}$

b) dimension is 2 of each eigenspace multiplicity of 2