Lesson 9

1) $S = \{(1,0,-1),(2,1,0),(0,1,1)\}$ spans R^3

Let (v_1, v_2, v_3) be a random vector.

$$(v_1, v_2, v_3) = c_1(1,0,-1) + c_2(2,1,0) + c_3(0,1,1)$$
$$= (c_1 + 2c_2,\ c_2 + c_3,\ -c_1 + c_3)$$

$$\begin{aligned} c_1 + 2c_2 \quad &= v_1 \\ c_2 + c_3 &= v_2 \\ -c_1 \quad + c_3 &= v_3 \end{aligned} \rightarrow \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 1 \times 1 \times -1 = -1$$

so a unique solution exists so any vector in R^3 can be written with those 3 vectors so these vectors span R^3

2) $S = \{(2,-1,2),(1,0,3),(3,-2,1)\}$ is linearly dep or not?

$$c_1(2,-1,2) + c_2(1,0,3) + c_3(3,-2,1) = (0,0,0)$$

$$\begin{aligned} 2c_1 + c_2 + 3c_3 &= 0 \\ -1c_1 \quad - 2c_3 &= 0 \\ 2c_1 + 3c_2 + c_3 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 2 & 1 & 3 & 0 \\ -1 & 0 & -2 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0.5 & 1.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $c_3 = t$
$c_2 = t$
$c_1 = -2t$

So there are infinite nontrivial solutions. $\boxed{\text{Linearly dependent}}$

$$v_1 = \frac{v_2 + v_3}{2}$$

3) $S = \{(1,0,-1),(2,1,1),(-3,0,2)\}$ is a basis?

Let v_1, v_2, v_3 be a random vector

$$(v_1, v_2, v_3) = c_1(1,0,-1) + c_2(2,1,1) + c_3(-3,0,2)$$
$$= (c_1 + 2c_2 - 3c_3,\ c_2,\ -c_1 + c_2 + 2c_3)$$

$$\begin{aligned} c_1 + 2c_2 \ \cancel{c_1} - 3c_3 &= v_1 \\ c_2 &= v_2 \\ -c_1 + c_2 + 2c_3 &= v_3 \end{aligned} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

determinant is nonzero so there is a unique solution so $\underline{S \text{ spans } R^3}$.

Linear dependent?

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

only solution is trivial so <u>linearly dependent</u>

<u>S is a basis for R^3</u>

4) There are 4 vectors so 1 of them does not add additional directionality we already know the standard basis of R^3 has 3 vectors.

$c_1(1,0,-1) + c_2(2,1,1) + c_3(-3,0,2) + c_4(1,0,5) = 0$

$$\begin{aligned} c_1 + 2c_2 - 3c_3 + c_4 &= 0 \\ c_2 \quad + \quad &= 0 \\ -c_1 + c_2 + 2c_3 + 5c_4 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 6 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & 0 \end{array}\right]$$

$c_4 = t$
$c_3 = 6t$
$c_2 = s$
$c_1 = -s + 17t$

So 1 vector can be expressed as the other vector so not linearly dependent

there are non-trivial solutions

5) $$\begin{bmatrix} -2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ -1 & 4 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 3 & 2 & 1 & 0 \\ -1 & 4 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 14 & 4 & 18 \\ 0 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{9}{7} \end{bmatrix}$$

a) $v_1 = (1,3,1,3)$ $v_2 = (0,1,\frac{2}{7},\frac{9}{7})$

b) rank is 2

6) $$A = \begin{bmatrix} 3 & -9 & 6 \\ -2 & 6 & -4 \\ 1 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \\ 3 & -9 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = t$
$x_2 = s$
$x_1 = 3s - 2t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

a) $\boxed{\text{nullity} = 2}$ $\boxed{\text{basis}: \left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)}$

b) a basis for column space: $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ rank = 1

Check using transpose method:

$$A^T = \begin{bmatrix} 3 & -2 & 1 \\ -9 & 6 & -3 \\ 6 & -4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ is a basis

or $[3,-2,1]^T$ is a basis

7) $$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & -4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

~~equations using rref:~~

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & -4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = t$
$y = 4/3\,t$
$x = -2t + y$

4) There are 4 vectors so 1 of them does not ...
we already know the standard ...
$c_1(1,0,-1)$ ...

7) continued

Let $z = t$

$y = \frac{4}{3}t$

$z = -2t + \frac{4}{3}t = -\frac{2}{3}t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2/3\,t \\ 4/3\,t \\ t \end{bmatrix} = t\begin{bmatrix} -2/3 \\ 4/3 \\ 1 \end{bmatrix}$$

$$\boxed{\text{basis} = \begin{bmatrix} -2/3 \\ 4/3 \\ 1 \end{bmatrix}}$$

8)

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & -3 & 2 \\ 4 & 7 & -3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -2 \\ 2 & 1 & 1 & 1 \\ 4 & 7 & -3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 5 & -5 & 5 \\ 0 & 15 & -15 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 + 3x_3 = -2$

$x_2 - x_3 = 1$

Let $x_3 = t$

$x_2 = 1 + t$

$x_1 = -2 - 3t + 2(1+t)$

$= -2 - 3t + 2 + 2t$

$= -t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 1 + t \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_p &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ x_h &= t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}}$$