Lesson 6

Reading: Larson, Section 3.1, The Determinant of a Matrix; Section 3.2, Determinants and Elementary Operations

Suggested exercises: Larson, Section 3.1: 3, 9, 15, 19, 31 (choose row or column carefully!), 33, 39, 49, ; Section 3.2: 1, 4, 7, 9, 11, 13, 21, 25, 33

Submit: Lesson 6: Computing determinants

Section 3.1: The Determinant of a Matrix

The name *determinant* comes from systems of linear equations. See the first example in this section, a linear system of two equations in two variables. The denominators of the solutions are the same, $a_{11}a_{22} - a_{21}a_{12}$, and consequently, if $a_{11}a_{22} - a_{21}a_{12}$ is nonzero, then the two solutions shown are the unique solutions of the system. If $a_{11}a_{22} - a_{21}a_{12}$ is zero, then the formula presented is not valid, and there are either no solutions or infinitely many solutions. Hence, the number $a_{11}a_{22} - a_{21}a_{12}$ determines what type of solution a system has, in other words, it is a determinant of this property.

The term *determinant* has come to be applied to matrices, since linear systems can be expressed in the matrix form $A\mathbf{x} = \mathbf{b}$. The determinant of the coefficient matrix A performs the same role: it tells us whether to expect a unique solution of $A\mathbf{x} = \mathbf{b}$ or not.

You should know the formula for the determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The general definition given of determinant in this section is recursive (the text says inductive–it means the same thing): the determinant of an $n \times n$ matrix is defined in terms of determinants of $(n-1) \times (n-1)$ matrices, with a starting point consisting of the given formula that the determinant of a 1×1 matrix $[a]$ is a.

This method of computing determinants relies on the ideas of *minors* and *cofactors*. Given a square matrix A, the *minor* of an entry a_{ij} is the determinant of the matrix found by eliminating the row and column containing a_{ij} (that is, row i and column j). The *cofactor* of a_{ij} is the minor multiplied by $(-1)^{i+j}$. Note the pattern of signs given in the text that shows you where $(-1)^{i+j}$ is $+1$ and where it is -1. You start with a $+1$ in the upper-left corner, and then change sign with each move by one place horizontally or vertically, which gives you the checkerboard pattern shown. This is the easiest way to remember how to apply the signs in the cofactors.

Once you are proficient in finding cofactors of entries in a square matrix, then you can compute the determinant by cofactor expansion: pick any row or column in the matrix, and for each entry in that row or column, compute the product of that entry and its cofactor. Add these up for all entries in the given row or column, and the result is the determinant. It is by no means intuitive that you will get the same result no matter which row or column you choose to expand on, but it is true, and you should probably test it out and verify it for yourself at least once, using several different choices or row and column.

The formula for 2×2 determinants results from this method. Let's expand on the top row, remembering that a 1×1 determinant is just the value of the entry:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot (+1) \cdot \begin{vmatrix} d \end{vmatrix} + b \cdot (-1) \cdot \begin{vmatrix} c \end{vmatrix} = ad - bc.$$

Since you get the same result no matter what row or column you expand on, it makes sense to choose the one that will make your task as easy as possible. For this reason, you usually want to expand on the row or column containing the most zero entries. You do this because the product of the entry and its cofactor will be zero no matter what the value of the cofactor is, so there is no need to compute the value of the cofactor, saving you a bit of work. Therefore, if you have to compute

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix},$$

then the best choice is to expand on column two, since two of the three entries are zero. Starting with a $+1$ in the upper-left corner, moving down to the 1 at the bottom of column two gives us a sign factor of -1, and therefore the determinant will be

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1 \cdot (-1) \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -(4+1) = -5.$$

We didn't have to bother with the cofactors of the two zeros in this column: it doesn't matter what they are equal to, since they will be zeroed out by the entries. This choice made the computation of the determinant relatively quick and painless.

Finally, the determinant of a triangular matrix is seen to be simply the product of the entries on its main diagonal. This is because of all of the zeros in a triangular matrix: you start by expanding on the row with only one nonzero entry (which is on the main diagonal), and the minor of that nonzero element will be the determinant of a triangular matrix, so it will also have a row with only one nonzero entry (again, on the main diagonal). This continues all the way down to the 1×1 case at the end, and all that survives is the product of the entries on the main diagonal. Try it, and see.

These determinants are especially easy to compute: you don't have to bother with cofactor expansions at all. Just multiply the entries on the main diagonal, and you're done. Like this:

$$\begin{vmatrix} -2 & 1 & 3 \\ 0 & 3 & 11 \\ 0 & 0 & 1 \end{vmatrix} = (-2)(3)(1) = -6.$$

Section 3.2: Determinants and Elementary Operations

In this section, a different and often very useful method of computing determinants is shown. The method relies on three basic properties of determinants: (1) if you interchange the rows of a matrix, you change the sign of the determinant, (2) if every entry in a row of a matrix is a multiple of a scalar k, then the determinant of that matrix is k times the determinant of the matrix without the factor k in that row, and (3) if you add a scalar multiple of any row to another row, you do not change the value of the determinant. We can demonstrate these properties for 2×2 determinants without too much trouble, and you can probably see how the ideas would extend to larger determinants.

First, let's interchange the rows of a 2×2 matrix and see what happens to the determinant. We know that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and interchanging rows gives us

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Next, let's consider a 2×2 determinant with a row containing entries multiplied by a scalar k:

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = (ka)d - (kb)c = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The same thing happens if we multiply the second row by k; check it. Note what this is showing us: you can factor out a common factor of k from a row, and the k comes out as a multiplier of the determinant. For example,

$$\begin{vmatrix} 4 & 8 \\ 1 & -2 \end{vmatrix} = \begin{vmatrix} 4 \cdot 1 & 4 \cdot 2 \\ 1 & -2 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}.$$

(Check that the values of the two sides of this equation agree.) This is how this property will typically be used.

Finally, let's start with

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and add k times row one to row two:

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d+kb) - b(c+ka) = ad + abk - bc - abk = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Again, you can check that something similar happens if you add k times row two to row one.

We can exploit these properties to compute determinants: perform elementary row operations on the matrix to reduce it to row-echelon form, which is an upper-triangular matrix. Keep track of the effect of each row operation as you do it by introducing the correct factor in front of the changed determinant. When you have reduced to row-echelon form, then you can finish by using the fact that the determinant of that matrix is the product of the elements on the main diagonal. For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix},$$

since we performed the row operation of adding -3 times row one to row two to produce the determinant on the right, and this does not change the value of the determinant. Therefore,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(-2) = -2.$$

Maybe not so impressive, since this was just a 2×2 determinant, so let's try it on a larger matrix. Let's compute

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 4 \end{vmatrix}.$$

We can reduce this to row-echelon form by adding row one to row three, so we'll get

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{vmatrix} = (1)(1)(6) = 6.$$

Nice!

Now, let's try one that requires us to use all of the properties. We will compute

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix}.$$

We will first interchange rows one and two, changing the sign of the determinant:

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 4 \end{vmatrix}.$$

Now, eliminate the 2 at the bottom of the first column by adding 2 times row one to row three. This will not change the value of the determinant, and we get:

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 10 \end{vmatrix}.$$

Next, let's change the 2 in row two to a 1, so we can use it to eliminate the 1 at the bottom of column two. We are factoring out the 2 from row two, so this multiplies the determinant by 2:

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 10 \end{vmatrix} = -2\begin{vmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 10 \end{vmatrix}.$$

Finally, we'll add -1 times row two to row three (not changing the value of the determinant) to achieve row-echelon form:

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 10 \end{vmatrix} = -2\begin{vmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 10 \end{vmatrix} = -2\begin{vmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{vmatrix}.$$

Now that the determinant is in upper-triangular form, we compute:

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = (-2)(-1)(1)(10) = 20.$$

One of the consequences of the property that factoring out a k from a row gives k as a multiplier of the determinant is that a row of zeros causes the determinant to be zero. This is because the row of zeros can be considered as $k = 0$ times that row, and therefore, you can pull out the k to get a multiplier of zero, zeroing out the entire determinant. (Of course, having a row of all zeros can be seen to cause the determinant to be zero without this property: just do a cofactor expansion on this row, and everything is zero!)

You should note that everything we say and do with rows in a determinant will also apply to columns, so all of these properties can be applied to columns as well as to rows. If a matrix has a form in which it makes sense to look at columns instead of rows, then apply these properties to the columns. Typically, though, you will work with rows.