Lesson 13

Reading: Larson, Section 7.1, Eigenvalues and Eigenvectors; Section 7.2, Diagonalization

Suggested exercises: Larson, Section 7.1: 1, 5, 11, 15, 17, 21, 29 (you can use wolframalpha.com, and enter (without the quotes) "eigenvalues of {{ -4, 5 }, { -2, 3 }}"), 33 (use wolframalpha.com again, and enter the matrix similarly), 39, 43; Section 7.2: 1, 5, 7, 9, 15, 23, 27

Submit: Lesson 13: Eigenvalues, eigenvectors, and diagonalizable matrices

Section 7.1: Eigenvalues and Eigenvectors

Typically, when A is a square matrix, and you compute $A\mathbf{v}$ for some vector $\mathbf{v}$, you get a vector that lies along a different line through the origin than $\mathbf{v}$. For example, if

$$A = \begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix},$$

then

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix},$$

which does not lie along the line through the origin in the direction of $(1, 0)$. However,

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

which *is* contained in the line through the origin in the direction of $(2, 1)$, and in fact,

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

In this case, $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. Due to linearity, this property also holds for any scalar multiple of the vector $(2, 1)$:

$$A \begin{bmatrix} 2t \\ t \end{bmatrix} = 2 \cdot \begin{bmatrix} 2t \\ t \end{bmatrix}.$$

There is also a vector for which multiplication by A is the same as scalar multiplication by -3:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and similarly for any scalar multiple of this vector.

In general, what we are seeing here are nonzero vectors $\mathbf{v}$ and scalars λ for which

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Such a nonzero vector $\mathbf{v}$ is called an *eigenvector*, and the scalar λ is called an *eigenvalue.*

In the example above, if we were to use these vectors as a basis of R^2, then the linear transformation

$$T(\mathbf{v}) = A\mathbf{v}$$

would have an especially nice form: the matrix of T in the basis

$$\mathscr{B} = \{(2,1), (1,1)\}$$

is

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

This is because, in this basis, the first column of the matrix of T is

$$T(2,1) = (4,2) = 2 \cdot (2,1) + 0 \cdot (1,1)$$

and the second column is

$$T(1,1) = (-3,-3) = 0 \cdot (2,1) + (-3) \cdot (1,1).$$

This simplification is an important reason for finding eigenvalues and eigenvectors.

Corresponding to an eigenvalue λ, there is a subspace called an *eigenspace*, consisting of all eigenvectors of λ, together with the zero vector. For the example above, the eigenspace of $\lambda = 2$ is the span of the vector $(2,1)$, and the eigenspace of $\lambda = -3$ is the span of $(1,1)$.

To compute eigenvalues of a square matrix, we consider the equation

$$A\mathbf{v} = \lambda\mathbf{v},$$

which we rewrite as

$$\lambda\mathbf{v} - A\mathbf{v} = \mathbf{0},$$

or

$$(\lambda I - A)\,\mathbf{v} = \mathbf{0}.$$

(Recall that $I\mathbf{v} = \mathbf{v}$ for any vector $\mathbf{v}$, so we think of the $\lambda\mathbf{v}$ term above as $\lambda I\mathbf{v}$, and therefore factoring out the $\mathbf{v}$ leaves us with a difference of two matrices, $\lambda I - A$, which makes sense, instead of a difference of a scalar and a matrix, $\lambda - A$, which does not.)

We now have a matrix equation

$$(\lambda I - A)\mathbf{v} = \mathbf{0}$$

which is supposed to have nonzero solutions. According to results we've seen previously, this means that the determinant of the matrix $\lambda I - A$ must be zero. The equation

$$\det(\lambda I - A) = 0$$

is a polynomial equation whose solutions are the eigenvalues of A. This equation is called the *characteristic equation* of the matrix.

For example, for the matrix

$$A = \begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix}$$

in our example above, we have

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} \lambda - 7 & 10 \\ -5 & \lambda + 8 \end{bmatrix},$$

and

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 7 & 10 \\ -5 & \lambda + 8 \end{vmatrix} = (\lambda - 7)(\lambda + 8) + 50 = 0,$$

or

$$\lambda^2 + \lambda - 6 = 0.$$

Solving this equation gives us our two eigenvalues, $\lambda = 2$ and $\lambda = -3$.

Once we have the eigenvalues, then finding eigenvectors is a matter of solving the corresponding linear system $(\lambda I - A)\mathbf{v} = \mathbf{0}$. For example, with $\lambda = 2$, we get the system

$$\begin{bmatrix} -5 & 10 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which reduces to the single equation

$$-v_1 + 2v_2 = 0.$$

With v_2 as a free variable, and letting $v_2 = t$, we get solutions $(2t, t)$, in other words, scalar multiples of $(2, 1)$ as our eigenvectors. (Technically, we get eigenvectors for *nonzero* scalar values of t.)

A nice case to deal with is the case of a triangular matrix. If A is triangular, then the eigenvalues are simply the numbers on the main diagonal of A. This is yet another reason that triangular matrices are good to work with. Thus, the matrix

$$A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

has eigenvalues -2, 1, and -3.

Section 7.2: Diagonalization

In the previous section, our first example was a matrix A for which we found a basis in which the matrix became diagonal. The basis consisted of eigenvectors of A. This is the topic of this section.

We say that two matrices A and B are *similar* if there is an invertible matrix P for which

$$B = P^{-1}AP.$$

It can be shown that similar matrices always have the same eigenvalues.

The case we are interested in is the case where B is a diagonal matrix, in which case the matrix A is said to be *diagonalizable*. Not every square matrix can be diagonalized, but for those that can, the eigenvalues and eigenvectors give us a method of doing it. Suppose A is $n \times n$. We find eigenvalues and eigenvectors, and if we have n linearly independent eigenvectors, then we can form the matrix P using the eigenvectors as columns. Then $P^{-1}AP$ will be diagonal, and the diagonal matrix will have the eigenvalues of A on its main diagonal, in positions corresponding to their eigenvectors.

For the example leading off the previous section, that means we can let

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

giving

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

from which we compute

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

It is necessary that an $n \times n$ matrix A have n linearly independent eigenvectors in order to be diagonalizable. One outcome that guarantees n linearly independent eigenvectors is for A to have n distinct eigenvalues. If it does, then it can be shown that the eigenvectors *must* be linearly independent.

For example, the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

has characteristic equation

$$\begin{vmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 1 \end{vmatrix} = 0,$$

or

$$\lambda^2 - 3\lambda = 0,$$

which has solutions $\lambda = 3$ and $\lambda = 0$. Since we have two distinct eigenvalues, we know that we will be able to find two linearly independent eigenvectors. We find $(1, -1)$ as an eigenvector of $\lambda = 0$ and $(2, 1)$ as an eigenvector of $\lambda = 3$. (Check!)

Therefore, we have

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix},$$

giving us

$$P^{-1}AP = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}.$$

By the way, we could have loaded our eigenvectors into the matrix P in the opposite order, giving us

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

This works equally well, and gives us diagonal form

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

In other words, if you change the order in which the eigenvectors go into P, you change the order in which the eigenvalues appear in the diagonal matrix.