Module 7

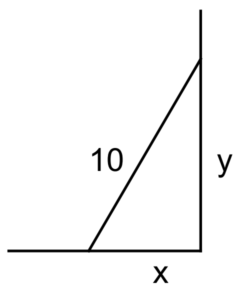
SECTION 3.9: RELATED RATES

In an earlier module, we noted that implicit differentiation required an adjustment in using the Power Rule.

$$\frac{d}{dx}x^3 = 3x^2 \cdot 1 \text{ but } \frac{d}{dx}y^3 = 3y^2 y'$$

Related rate problems require a similar kind of adjustment. In this type of problem you will find derivatives with respect to time t.

Consider ***Example 2***. A 10 ft ladder is leaning against a wall and the base of the ladder slides along the ground at a rate of one foot per second.



Applying the Pythagorean Theorem we have:

$$x^2 + y^2 = 100$$

Now comes the new adjustment. Find a derivative with respect to time t.

$$\frac{d}{dt}x^2 = 2x\frac{dx}{dt}$$

The third step of the Power Rule, the derivative of the base, produces $\frac{dx}{dt}$.

What is $\frac{dx}{dt}$?

1. It is the rate of change of distance x with respect to time t.

2. It tells us how *fast* the base of the ladder is moving.

3. It is given as one foot per second.
4. It is approximately $\dfrac{\Delta x}{\Delta t}$ which is $\dfrac{\text{change in distance } x}{\text{change in time } t} \approx \dfrac{1\,\text{ft}}{1\text{ sec}}$

In a similar manner, the derivative of y^2 with respect to time is

$$\frac{d}{dt}\,y^2 \;=\; 2y\frac{dy}{dt}$$

Again we have used the three step Power Rule. $\dfrac{dy}{dt}$ is the derivative of the base y with respect to t and indicates how fast the top of the ladder is moving. Going back to $x^2 + y^2 = 100$ differentiation with respect to time produces $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.

Note the following:

1. The number of unknowns has increased from two to four. The new unknowns are $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

2. $\dfrac{dx}{dt}$ is given as $1\,\dfrac{\text{ft}}{\text{sec}}$ and to answer the question "how fast is the top of the ladder moving?" we must find $\dfrac{dy}{dt}$.

3. In the differential equation, to find $\dfrac{dy}{dt}$, we must insert values for the other three unknowns, x, y, and $\dfrac{dx}{dt}$. This is a common step in all related rate problems.

4. The value of x is given as 6. Use $x^2 + y^2 = 100$ to find y.

   Putting in $x = 6$, $36 + y^2 = 100$

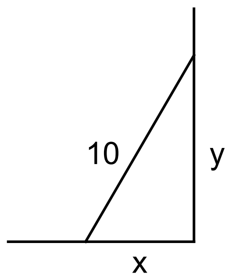
$$y^2 = 64$$

$$y = 8$$

Put $x = 6$, $y = 8$, and $\dfrac{dx}{dt} = 1$ into the second equation and out pops $\dfrac{dy}{dt} = -\dfrac{3}{4}$. The negative sign means y is decreasing.

A significant aspect of this type of problem is the passage from one type of equation, $x^2 + y^2 = 100$, to a differential equation, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, that is completely different.

How do we explain the connection? There is nothing to say beyond "find derivatives with respect to *t*." The connection is an algebraic procedure. One doesn't look at $x^2 + y^2 = 100$ and the diagram



and have an insight to how *x, y,* $\frac{dx}{dt}$ and $\frac{dy}{dt}$ relate to each other. The relationship is determined by the algebraic process of finding derivatives. In some cases one needs insight as an aid in the thinking process but in other situations algebraic procedures need to be used in a somewhat mechanical manner. Seeing a distinction between performing algebraic procedures to connect parts of a problem and the need to do more creative thinking can reduce frustration. In related rate problems, after writing an equation, remember the next step is to take a derivative with respect to time *t*.

WARNING

The warning on ***page 247*** is worth understanding. In the above example suppose we had substituted $x = 6$ into $x^2 + y^2 = 100$ to get $36 + y^2 = 100$ and then taken the derivative. The result would have been $2y\frac{dy}{dt} = 0$ and since $y \neq 0$, $\frac{dy}{dt}$ would have to equal zero. This means that *y* is not changing.

The equation $x^2 + y^2 = 100$ represents a dynamic situation where *x* and *y* can take on different values.

The equation $36 + y^2 = 100$ represents a static situation where there is no movement. *x* is fixed at 6 and *y* must equal 8.

EXAMPLE 3

An important procedure is shown in this problem. In the formula for the volume of a cone, $V = \frac{1}{3} \pi r^2 h$

the volume V depends on two variables, *r* and *h*.**.** If this formula is used to find derivatives, $r^2 h$ must be treated as a product. The result,

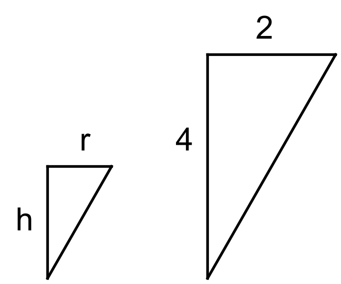
$$\frac{dV}{dt} = \frac{1}{3} \pi \left(r^2 \frac{dh}{dt} + 2r \frac{dr}{dt} h \right)$$

is complicated. It contains five variables, $\frac{dV}{dt}$, r, $\frac{dh}{dt}$, $\frac{dr}{dt}$, and h. In the text, the formula,

$$V = \frac{1}{3} \pi r^2 h \text{ is changed to:}$$

$$V = \frac{\pi}{12} h^3$$

Now the volume *V* depends on only *one* variable *h*. The variable *r* has been eliminated. Only *h* and constants are left in the formula. (This will also be a significant step in max-min word problems later.) How was *r* eliminated? The similar triangles in ***Figure 3***



are shown here as two separate triangles. Both are right triangles and the bottom angles are the same. (Similar triangles require just two congruent angles.) Then the ratios of radius to height are equal: $\frac{r}{h} = \frac{2}{4}$ Multiply by *h* and reduce $\frac{2}{4}$ to $\frac{1}{2}$ and $r = \frac{h}{2}$
Now replace *r* with $\frac{h}{2}$ in $V = \frac{1}{3} \pi r^2 h$ as shown in the text and the variable *r* is eliminated.

STRATEGY

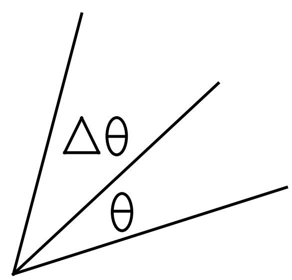
Step 5 of the Problem Solving Strategy is very important. Write an equation. In ***Example 4***, the equation is:

$$z^2 = x^2 + y^2$$

The tool used to find this equation was the Pythagorean Theorem. This tool is used in many problems, so always look for right triangles. Note, however, that the value of z depends on two variables, x and y, but this is not changed as in the last example. Here the Power Rule can be used three times. There is no *product* of variables that complicates the derivative.

EXAMPLE 5

In this example the question is "At what rate is the searchlight rotating?" The key here is to introduce an angle θ and note that as the searchlight rotates there is a change in the angle, $\Delta\theta$.



This change in the angle, $\Delta\theta$, takes place in a time interval, Δt .

Then $\dfrac{\Delta\theta}{\Delta t}$ which is $\dfrac{\text{a change in angle}}{\text{a change in time}}$ is an approximation to $\dfrac{d\theta}{dt}$.

The units for $\dfrac{d\theta}{dt}$ will be $\dfrac{\text{radians}}{\text{second}}$ because the angle must be in radians.

But how do we get the derivative, $\dfrac{d\theta}{dt}$? The first step is to use a trig function. From ***Figure 5***, $\tan\theta = \dfrac{x}{20}$ which changes into:

$$x = 20\tan\theta$$

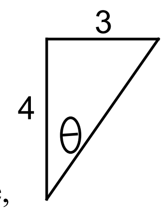
Then find derivatives with respect to *t:*

$$\frac{dx}{dt} = 20\,\frac{d(\tan\theta)}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

Remember that the derivative of $\tan\theta$ is $\sec^2\theta$ times the derivative of the angle with respect to time, $\frac{d\theta}{dt}$. So $\frac{d\theta}{dt}$ appears as part of the derivative of a trig function.

The problem is finished by putting in given values. When $x = 15$,

$\tan\theta = \frac{x}{20}$ becomes $\tan\theta = \frac{15}{20} = \frac{3}{4}$. This matches the right triangle,



The hypotenuse must be 5 and then $\cos\theta = \frac{4}{5}$.

This result could also be found by using the identity,

$$1 + \tan^2\theta = \sec^2\theta$$

Put in $\tan\theta = \frac{3}{4}$ and $1 + \frac{9}{16} = \sec^2\theta$

$$\frac{25}{16} = \sec^2\theta$$

Since $\cos\theta$ is the reciprocal of $\sec\theta$, $\cos^2\theta = \frac{16}{25}$ or $\cos\theta = \frac{4}{5}$.

The final answer is 0.128 rad/sec. Since $\pi\,radians = 180^\circ$, $1\ radian = \frac{180^\circ}{\pi}$ and

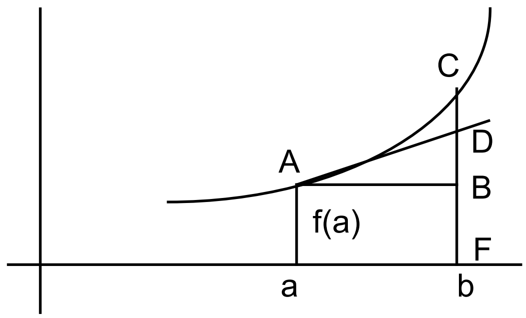
$$0.128 \text{ rad} = \frac{0.128 \cdot 180^\circ}{\pi} \cong 7.33^\circ$$

The searchlight rotates approximately 7.33° every second.

SECTION 3.10: LINEAR APPROXIMATIONS AND DIFFERENTIALS

LINEAR APPROXIMATIONS

Consider the following diagram.



Suppose the graph of $y = f(x)$ is the curve passing through A and C. The tangent line at x = a is the line through A and D. *The major new idea is to use the tangent line as an approximation to the curve.* This last sentence is emphasized because this involves quite a shift in thinking. You may ask "why on earth do we want to do this?" A partial answer is that some functions can be quite complicated while polynomial functions are basic, well understood, and easy to use. Therefore, it is natural to use a linear or quadratic function to approximate a more complicated function.

First, we find the equation of the tangent line at A. The tool we use is

$$y - y_1 = m(x - x_1)$$

where point A (x_1, y_1) is $(a, f(a))$ and the slope is the derivative evaluated at $x = a$, $m = f'(a)$. Substituting, we get:

$$y - f(a) = f'(a)(x - a).$$

Solving for y, we have:

$$y = f(a) + f'(a)(x - a)$$

(This is the equation at the top of ***page 252***).

Now it is important to remember that

$$y = f(x)$$

matches the curve through A and C, while

$$y = f(a) + f'(a)(x - a)$$

is the equation of the tangent line through A and D. We are going to use the tangent line as an approximation to the curve. Therefore,

$$f(x) \quad \text{is approximated by } f(a) + f'(a)(x - a).$$

At $x = b$ in the above diagram we use FD to approximate FC. The length of FD will come from $f(a) + f'(a)(x - a)$ by letting $x = b$. The length of FC will come from $f(x)$ by letting $x = b$. For all of this to be meaningful, it is crucial to associate $y = f(x)$ with the curve and $f(a) + f'(a)(x - a)$ with the tangent line.

EXAMPLE 1

In this example the curve is associated with $\sqrt{x + 3}$ and the tangent line with $\frac{7}{4} + \frac{x}{4}$. This is shown in ***Figure 2***.

EXAMPLE 2

This example requires a graphing calculator. We comment only on the inequalities. Starting with:

$$\left| \sqrt{x + 3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

the intermediate steps that are omitted are as follows. Rewrite as

$$-0.5 < \sqrt{x + 3} - \left(\frac{7}{4} + \frac{x}{4} \right) < 0.5$$

Then multiply by -1. Reverse the inequality signs because of the multiplication by a negative number:

$$0.5 > -\sqrt{x + 3} + \left(\frac{7}{4} + \frac{x}{4} \right) > -0.5$$

Next add $\sqrt{x + 3}$ three times:

$$\sqrt{x+3} + 0.5 > \left(\frac{7}{4} + \frac{x}{4}\right) > \sqrt{x+3} - 0.5$$

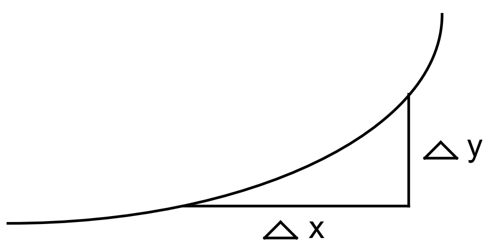
After rewriting this last inequality, it is the same as the one in the text. Remember that the middle part of these inequalities matches the tangent line.

DIFFERENTIALS

Some concepts in calculus are more challenging because they have different interpretations. Such is the case for differentials.

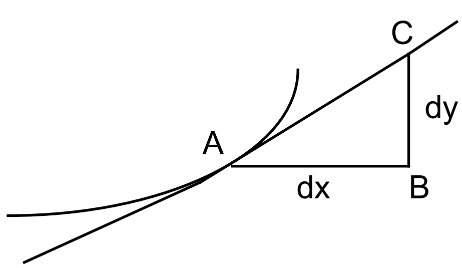
THE DEFINITION

To this point, the symbols $\frac{\Delta y}{\Delta x}$ represent a fraction that has a clear geometric interpretation:



We also have $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\frac{dy}{dx}$ represents the derivative of a function, but can it also have meaning as a fraction? The definition on ***page 254*** is designed to separate dy and dx into separate parts so indeed $\frac{dy}{dx}$ can be considered as a fraction. As background to the definition recall that $\frac{dy}{dx}$ is the slope of the tangent line to a curve. It then makes sense to define dy as the length of BC shown below and dx as the length of AB.



This is accomplished by the definition,

$$dy = f'(x)\,dx$$

where dy is called the *differential* of y and dx the *differential* of x. How does this match the drawing? From the drawing, the slope of the line through A and C is $\dfrac{dy}{dx}$. Using the above definition,

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x) = \text{slope of the tangent line.}$$

In other words, the definition of dy as $dy = f'(x)\,dx$ agrees with $\dfrac{dy}{dx} = f'(x)$

Now is this important? Well in the second part of calculus, called integral calculus, you will perform an operation called integration and every time a differential will be involved. So it will be important to know the above definition. But this isn't hard. The differential dy is just the derivative $f'(x)$ times the differential dx.

FINDING THE DIFFERENTIAL *dy*

1. Given $y = 3x^2 + 4x$

   $dy = (6x + 4)\,dx$

2. Given $y = \sqrt{3x}$

   $dy = \dfrac{3}{2\sqrt{3x}}\,dx$

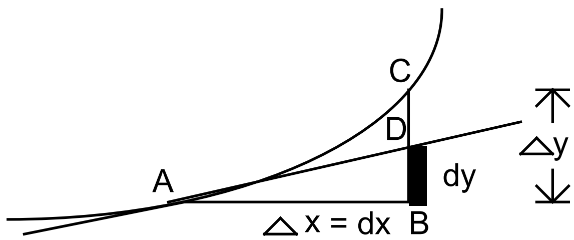
3. Given $y = \sec^3 2x$

   $dy = 6\sec^3 2x \tan 2x\, dx$

Perhaps you are wondering, what is the meaning of all of this algebraic manipulation?

THE GEOMETRIC MEANING OF DIFFERENTIALS

In the drawing below, we have included the tangent line through A and D.



We want to compare $\frac{dy}{dx}$ and $\frac{\Delta y}{\Delta x}$ so we make Δx and dx the same. dx is an independent variable so we can assign a value to it like $dx = .01$. Then we can assign the same value to Δx. In the above diagram, Δx and dx both equal the length of AB. What is very important to see in the diagram is the comparison of Δy and dy.

1. Δy is the length of BC and dy is the length of BD.
2. Δy takes us back to the curve but dy just goes to the tangent line.
3. Δy is the *exact* change in y while dy is an *approximation* to Δy.
4. Δy equals $f(x + \Delta x) - f(x)$.
5. dy equals $f'(x)dx$.
6. It is easier to find dy than Δy.

Now we can show the meaning of the algebraic manipulations in *Finding the Differential dy* above. From the first example

$$y = 3x^2 + 4x$$

$$dy = (6x + 4)dx$$

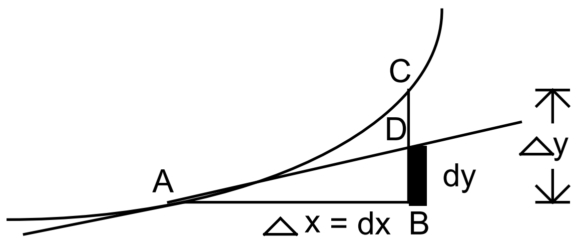
and

$$\Delta y = 3(x + \Delta x)^2 + 4(x + \Delta x) - (3x^2 + 4x)$$

$$= 3x^2 + 6x\Delta x + 3(\Delta x)^2 + 4x + 4\Delta x - 3x^2 - 4x$$

$$= 6x\Delta x + 3(\Delta x)^2 + 4\Delta x$$

Clearly, it is easier to find dy than Δy. In the diagram, if the curve through A and C was the graph of $y = 3x^2 + 4x$ then



the length of BC is $6x\,\Delta x + 3(\Delta x)^2 + 4\Delta x = \Delta y$ and the length of BD is $(6x + 4)\,dx = dy$ If we use dy as an approximation to Δy then the difference,

$$\Delta y - dy = 6x\,\Delta x + 3(\Delta x)^2 + 4\,\Delta x - 6x\,dx - 4d\,x$$

is the error in the approximation. Remember $\Delta x = dx$ so the last line is

$$\Delta y - dy = 6x\,\Delta x + 3(\Delta x)^2 + 4\,\Delta x - 6x\,\Delta x - 4\Delta x = 3(\Delta x)^2$$

and the error is $3(\Delta x)^2$. Then in the above diagram, the length of DC is $3(\Delta x)^2$, the error in using dy as an approximation to Δy. So by making Δx smaller we get a better approximation.

APPROXIMATING VALUES OF A FUNCTION

In the above discussion, we were approximating Δy, the change in y that results from increasing x by Δx. Now note that

$$\Delta y = f(x + \Delta x) - f(x)$$

can be rewritten as

$$f(x + \Delta x) = f(x) + \Delta y$$

Replace Δx with dx and let $x = a$.

$$f(a + dx) = f(a) + \Delta y$$

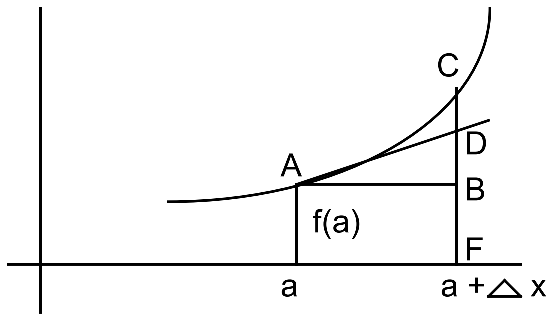
This form contains Δy which is hard to calculate so we replace Δy with dy.

$$f(a + dx) \approx f(a) + dy$$

This approximation appears on ***page 255***. We revisit an earlier diagram to illustrate the meaning.

In the diagram below $f(a) + \Delta y = \text{FB} + \text{BC} = f(a + \Delta x)$

and $f(a) + dy = \text{FB} + \text{BD}$:



Note that the approximation, $f(a) + dy$, goes up to the tangent line, not to the curve. The error in the approximation is DC but this may be difficult to determine.

Also note the definition of *relative error* in ***Example 4***. The change or error is divided by a total amount to provide a comparison. The relative change or relative error will have a form such as $\dfrac{dV}{V}$ *or* $\dfrac{dA}{A}$.

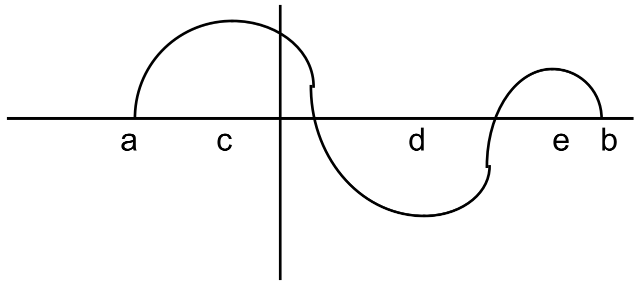
SECTION 4.1: MAXIMUM AND MINIMUM VALUES

The emphasis in preceding lessons has been on the algebraic structure of various derivatives and related limits. Attention now shifts to the use of derivatives in graphing functions and in applications. Most calculus books become rather verbose in their discussion of related background material. One reason for this is the process of describing in words and symbols what can be seen easily in a graph. There is a need for the symbolic (algebraic) description in proving theorems, but if we concentrate on the *use* of the theorems, another point of view emerges. We will concentrate on an overview and let some details stay in the background.

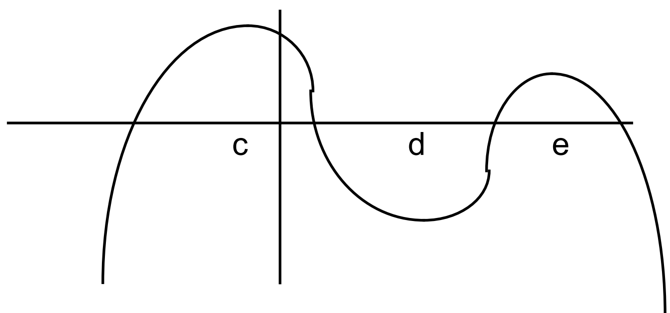
ABSOLUTE AND LOCAL MAXIMUM VALUES

First note that when discussing a maximum (or minimum) this relates to the maximum y value or the maximum value for the *dependent* variable. (If $s = f(t)$ then a max value would mean a maximum value for *s.)*There will *always* be a corresponding x value where the max occurs but we are not talking about a maximum x value.

The next major question is, “Are we finding maximum *y* values in a *limited* domain or over the *entire* domain of the function.”



Consider the graph of a fourth degree polynomial function shown above with a limited domain, $a \le x \le b$. In this domain there is a maximum value at $x = c$ and also at $x = e$. The minimum value occurs at $x = d$.



If the domain, $-\infty < x < \infty$, is used then at $x = c$, we have an *absolute* maximum. The largest *y* value over the *entire* domain is at $x = c$. The maximum at $x = e$ is a local maximum. At $x = d$ we have a local minimum. There is no absolute minimum in the entire domain. On the left and on the right the y values approach negative infinity so there is no smallest *y* value over the entire domain.

THE EXTREME VALUE THEOREM

The key word in this theorem is “attains.” Do the *y* values actually *equal* a maximum or minimum value? In ***Figure 10***, consider the interval from $x = 0$ to $x = 2$.
There is no largest *y* value on this interval. *y* gets close to 3 but never reaches 3. If you think 2.999 is the largest *y* value then note that $y = 2.9999$ is larger and $y = 2.99999$ is larger still. This process continues and as long as 3 is not included we can not find a largest number. Same comments for a smallest number on the interval (0,2) where the endpoints are not included. (The endpoints are included in the interval [*a,b*]. This is called a *closed* interval. The interval (*a,b*) does not include the endpoints and is called an *open* interval.)

FERMAT'S THEOREM

Remember that $f'(x)$ is the slope of the tangent line and $f'(c)$ is the slope of the tangent line at the point where $x = c$. Then in ***Figure 11***, the slope of the tangent line is zero at the maximum and minimum points; therefore, the theorem intuitively makes sense. Note, however, that in ***Example 6***, a minimum exists at $x = 0$, but the derivative is not defined and hence not equal to zero. Also, in ***Figure 12***, the derivative is zero at $x = 0$ but there is no maximum or minimum at (0,0). Fermat's Theorem is true only when the two conditions:

There is a maximum or minimum at $x = c$ and $f'(c)$ exists

$\left[f'(x) \text{ equals a finite number at } x = c\right]$

are satisfied.

However, a major activity in using calculus to graph functions will be to find the derivative, set it equal to 0, and solve the resulting equation. The graph will indicate whether or not there is a special case.

CRITICAL NUMBERS

This will be an important concept. The definition states that $x = c$ is a critical number if:

1. $f'(c) = 0$ [the derivative is zero at $x = c$]

or 2. $f'(c)$ does not exist [the derivative is undefined at $x = c$]

Recall that when $f'(c)$ does not exist we say $f(x)$ is not differentiable at $x = c$.

Case 1. Look for the *x* values where the slope of the tangent line is zero. Find the corresponding *y* value and the resulting point *may* match a maximum or minimum *y* value.

Case 2. This type of critical number is not as significant because it applies to special situations. The most common situation is when the tangent to the curve is a vertical line. A vertical line is the only line for which the slope is undefined.

Case 2 will appear for a function like, $y = \sqrt[3]{x}$

The derivative will contain a negative exponent $y' = \frac{1}{3}x^{-2/3}$

which in turn pushes x into the denominator. $y' = \frac{1}{3x^{2/3}}$

Now if $x = 0$ the derivative is undefined. $f'(0)$ doesn't exist but a vertical tangent will exist at the point $(0,0)$.

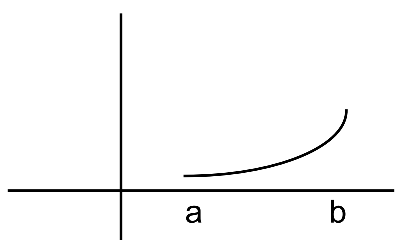
Also note that the negative exponent appeared because the original exponent $1/3$ was less than one. Every time the Power Rule is used on an exponent less than one, the result will contain a negative exponent and the potential that a denominator can equal zero. Match this with a vertical tangent line and case 2 will be less confusing.

The derivative also fails to exist when the curve has a "corner" or is "pointed". The graph of $y = |x|$ in ***Figure 13***, illustrates this characteristic. Also check the graphs in ***4.1 Exercises, 3-6***. Each has a "pointed" feature where the function is not differentiable.

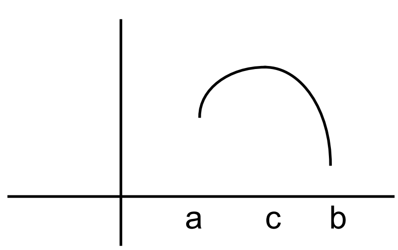
MAX-MIN ON A CLOSED INTERVAL

In applications, a function may match a physical situation for a *limited* domain. In the box on ***page 281*** a three-step procedure is given to find the absolute max and min values on a closed interval.

In the graph below the maximum and minimum *on a closed interval* occur at the endpoints where the derivative is not zero.



For the graph shown below, the derivative is zero at the maximum $x = c$, and the minimum is at an endpoint where the derivative is not zero. The three-step procedure suggests looking at all possibilities.

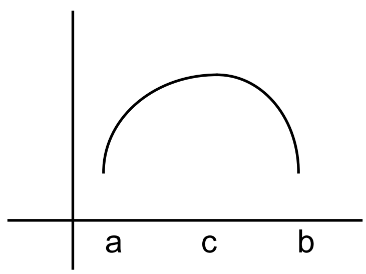


SECTION 4.2: THE MEAN VALUE THEOREM

As mentioned in the first paragraph, the Mean Value Theorem is an important theoretical theorem in calculus. Rolle's Theorem is introduced at this point as a tool to prove the Mean Value Theorem. We concentrate on the content of each theorem instead of the proofs.

First note that conditions 1 and 2 for each theorem are the same. The function must be continuous on a closed interval $\left[a,b\right]$ and differentiable on the open interval $\left(a,b\right)$. At the endpoints of the interval it is *ok* if the derivative of the function doesn't exist but it must be continuous there.

Rolle's Theorem also requires that $f(a) = f(b)$. The *y* values at $x = a$ and at $x = b$ must be the same. One possibility is shown below:



Rolle's Theorem asserts that at least one c exists between a and b where $f'(c) = 0$. So imagine a horizontal tangent line to the curve at the point where $x = c$ and the theorem seems quite reasonable.

THE MEAN VALUE THEOREM

This theorem is also intuitively clear. The key is to see that the slope of the secant line in ***Figure 3*** is:

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

Then the Mean Value Theorem asserts that at least one c exists between a and b where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

At some point c in the interval (a,b) , the slope of the tangent line is equal to the slope of the secant line or the two lines are parallel. ***Figures 3*** through ***6*** show that this is a reasonable expectation.

The form: $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

can also be written as: $$f(b) - f(a) = f'(c)(b-a)$$

This can also be written as: $$\Delta y = f'(c)\Delta x$$

A change in y can be replaced by the derivative times a change in x.

The second form is the one that is used in many applications of the Mean Value Theorem.

A THEOREM AND A COROLLARY

Theorem 5 on ***page 290*** is another reasonable assertion. If the slope of the tangent line is zero for all values of x on some interval (a,b) then the graph must be a horizontal line whose equation is $y = a$ constant .

The corollary in ***Box 7*** will have an important application in integral calculus. The condition that $f'(x) = g'(x)$ for all x in an interval (a,b) asserts that we have two functions which have the same slope at every point in an interval. The conclusion is $f(x) = g(x) + c$ where c is a constant. If c is positive, the graph of $f(x)$ is c units above the graph of $g(x)$.

Suppose $f'(x) = 2x$ and $g'(x)$ also equals $2x$. Then $f(x)$ could be $x^2 + 3$ and $g(x)$ could be $x^2 - 2$. The derivative of each function is $2x$ as required.

Then: $$f(x) = g(x) + 5$$

because $$x^2 + 3 = x^2 - 2 + 5$$

The significance of this corollary will be more evident later.