Lesson 12

Reading: Larson, Section 6.3, Matrices for Linear Transformations

Suggested exercises: Larson, Section 6.3: 1, 3, 7, 11, 15, 19, 27, 29, 31, 33, 35, 37, 41

Submit: Lesson 12: The matrix of a linear transformation

As mentioned in the last lesson, we are going to look at representing a linear transformation T in the form $T(\mathbf{v}) = A\mathbf{v}$. The simplest way to do this is when $T : R^n \to R^m$ and R^n and R^m are equipped with their standard bases. This leads us to the *standard matrix* for T. If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the standard basis vectors

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \ldots, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \ldots, 0) \\
&\vdots \\
\mathbf{e}_n &= (0, 0, 0, \ldots, 1),
\end{aligned}$$

then the standard matrix for T is the matrix A containing $T(\mathbf{e}_1)$ in its first column, $T(\mathbf{e}_2)$ in its second column, and so on, ending with $T(\mathbf{e}_n)$ in its nth and final column.

For example, for the linear transformation $T : R^3 \to R^4$ given by

$$T(x_1, x_2, x_3) = (-2x_1 + x_3, x_3, x_1 + 4x_2 - x_3, x_2 + x_3),$$

we have

$$\begin{aligned}
T(1, 0, 0) &= (-2, 0, 1, 0) \\
T(0, 1, 0) &= (0, 0, 4, 1) \\
T(0, 0, 1) &= (1, 1, -1, 1),
\end{aligned}$$

and therefore the standard matrix is

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Compare this matrix to the transformation. Do you see that each row can be found by simply listing the coefficients of the variables in their correct order for each component of $T(x_1, x_2, x_3)$? This is as it must be, because we want the following to hold:

$$T(x_1, x_2, x_3) = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Note that the matrix multiplication insists that we represent the image as a column vector, to match the shape of the matrix product. Usually we will, but as discussed before, there is no reason to panic if you see the image represented in the text as a 4-tuple.

This task becomes a little more complicated if, for some reason, the standard bases are not being used. Suppose we have a linear transformation $T : R^n \to R^m$, and suppose further that R^n is equipped with a basis $\mathscr{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and R^m is equipped with a basis $\mathscr{B}' = \{\mathbf{w}_1 \ldots, \mathbf{w}_m\}$.

In this case, the matrix A still contains the images of the basis vectors of R^n, but–and this is the complicating factor–they must be expressed in terms of the basis $\mathscr{B}'$. In practice, this means we have to solve a linear system to find the correct coefficients of these vectors in the basis $\mathscr{B}'$, and these coefficients are what we load into the columns of A.

For example, suppose $T : R^2 \to R^3$ is given by

$$T(x, y) = (x + 2y, x - y, y).$$

Suppose R^2 has basis

$$\mathscr{B} = \{(1, 1), (1, -1)\},$$

and R^3 has basis

$$\mathscr{B}' = \{(0, 1, 1), (1, 0, 1), (1, 1, 1)\}.$$

(Do you remember that you can check whether a set of n vectors acts as a basis of R^n by loading them into a determinant and checking whether the determinant is zero or not? Check these!)

We have

$$T(1, 1) = (3, 0, 1),$$

but remember that $(3,0,1)$ is *not* how this vector is represented in the basis $\mathscr{B}'$. We have to find constants a, b, and c such that

$$(3,0,1) = a(0,1,1) + b(1,0,1) + c(1,1,1),$$

which means we have to solve the system given by equating components:

$$\begin{aligned} b + c &= 3 \\ a \quad + c &= 0 \\ a + b + c &= 1. \end{aligned}$$

This system has augmented matrix

$$\begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

giving us $a = -2$, $b = 1$, and $c = 2$. Therefore,

$$(3,0,1) = (-2)(0,1,1) + (1)(1,0,1) + (2)(1,1,1),$$

so the first column of our matrix is

$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

By the way, you do not have to worry about getting more than one solution to the system set up in this way, because we are guaranteed unique representation in a basis.

Before we repeat this process with the other basis vector of R^2, we should think about another way to solve this system. We could have written the system for a, b, and c in matrix form:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix},$$

found the inverse of our coefficient matrix to be

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

and then computed

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

The advantage of this method is that, now that we have the inverse matrix, to find the other column of our matrix, we just have to replace $(3, 0, 1)$ with the image of $T(1, -1) = (-1, 2, -1)$ and compute

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}.$$

Our matrix A representing this transformation in these bases is therefore

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \\ 2 & 2 \end{bmatrix}.$$

In this section, we also see that if a linear transformation T_1 has standard matrix A_1 and a linear transformation T_2 has standard matrix A_2, then the composition $T_2 \circ T_1$ has standard matrix $A_2 A_1$.

For example, if $T_1(x, y) = (x+2y, 3x+y)$ and $T_2(x, y) = (x+y, x-y, 2x)$, then T_1 has standard matrix

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

and T_2 has standard matrix

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

Therefore, $T_2 \circ T_1$ has standard matrix

$$A_2 A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 2 & 4 \end{bmatrix},$$

so $T_2 \circ T_1$ is the transformation

$$T_2 \circ T_1(x, y) = (4x + 3y, -2x + y, 2x + 4y).$$

Finally, if T is an invertible transformation with standard matrix A, then the inverse transformation T^{-1} has standard matrix A^{-1}.

Thus, $T(x, y) = (2x + y, x + y)$ has standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

which has inverse

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

so the inverse transformation is

$$T^{-1}(x, y) = (x - y, -x + 2y).$$