Module 11

SECTION 5.4: INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM

The symbols $\int f(x)\,dx$ are used to represent the antiderivative $F(x)$ without having to evaluate $F(b) - F(a)$. Then:

$$\int (x^5 - x^4)\,dx = \frac{x^6}{6} - \frac{x^5}{5} + C$$

while $$\int_0^1 (x^5 - x^4)\,dx = \left.\frac{x^6}{6} - \frac{x^5}{5}\right]_0^1 = \frac{1}{6} - \frac{1}{5} = -\frac{1}{30}.$$

HOW DO WE FIND ANTIDERIVATIVES?

As you have probably noticed, the Fundamental Theorem does not indicate *how* to find an antiderivative. The process of finding an antiderivative is called *integration* and the first step is shown in condensed form in the ***Table of Indefinite Integrals (1)***. The first two entries apply to all functions. The fourth entry:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

combined with the first two provide the essential tools to integrate any polynomial function. All of the remaining entries reflect derivatives of basic functions. For example the derivative of $\tan x$ is $\sec^2 x$. This can be restated as $\tan x$ is the antiderivative of $\sec^2 x$ or

$$\int \sec^2 x\,dx = \tan x + C$$

Because the derivative of $\cos x$ is $-\sin x$ the derivative of $-\cos x$ is $-(-\sin x) = \sin x$.

Therefore, $$\int \sin x\,dx = -\cos x + C.$$

One way of remembering this is to recognize that:

$$\int \sin x\,dx \neq \cos x + C$$

because the derivative of $\cos x$ is $-\sin x$ and then add the negative sign.

OVERVIEW OF INTEGRATION

Hopefully, it is evident that it is absolutely essential to know the derivatives of the twelve basic functions listed in ***Table 1***. Then the corresponding integrations will be an easy task.

The first general method of integration, the Substitution Rule, will be covered in ***Section 5.5***. This will be more demanding, but a definite pattern is involved. In ***Chapter** 7*, five more methods of integration are developed. In *each* of these more advanced methods you will encounter the *basic integrals* listed in ***Table 1***—another reminder of the importance of knowing basic facts.

Other comments on the process of integration or the process of finding an antiderivative:

1. There is no single method that covers all types of functions.

2. Each method applies to just certain types of functions. If the set of all functions is divided into subsets, then method one applies to one subset, method two to another subset, and so forth. However, there will be functions that cannot be integrated by any of these methods. $\int e^{x^2}\,dx$ is one example.

3. The *form* of the function will determine which method will work. This will be your most significant insight in covering methods of integration. Integration is performed by recognizing forms.

4. Some functions may fit more than one form. In part 2 the subsets may overlap.

5. The Product Rule for differentiating functions works for *any product.* There is no method of integration that reverses any product. The same is true for the Quotient Rule.

6. One method *relates* to reversing the Power Rule but only applies to certain types of powers.

The above comments suggest some complexity in the process of integration. Each method will be more difficult if you can't recognize the "simple" forms in ***Table 1***.

THE NET CHANGE THEOREM

After carrying out an integration, what do we have? The Net Change Theorem addresses several possibilities.

First note that the equation

$$\int_a^b F'(x)\,dx \;=\; F(b) - F(a)$$

is a rewrite of part 2 of the Fundamental Theorem with $F'(x)$ replacing $f(x)$. The derivative $F'(x)$ can be thought of as a rate of change. So in the above integral we are integrating a rate of change. In the eight examples on ***pages 406–407***, it is important to see an integral of a derivative in each case. Then the form of the right side will be easier to understand. We start with a simple example.

Suppose you drive a car for 6 hours at a constant speed, 50 mph. How far do you travel in the 6 hours? We don't need calculus to solve this but by the Net Change Theorem this distance is represented by $\int_0^6 50\ dt$. If we integrate we get $50t\big]_0^6 \;=\; 50(6) \;=\; 300$. In 6 hours, we travel 300 miles. What does the integral $\int_2^5 50\ dt$ represent? The distance traveled between the second and the fifth hour.

$$\int_2^5 50\ dt \;=\; 50t\big]_2^5 \;=\; 250 - 100 = 150$$

In five hours we traveled 250 miles and in 2 hours we traveled 100 miles. Between the second and the fifth hour, the distance traveled is 150 miles.

Each example on ***pages 406–407*** has a similar interpretation. It is important to remember that in each case we have the integral of a rate of change or a derivative.

SECTION 5.5: THE SUBSTITUTION RULE

Before proceeding, it may be a good idea to reread "Overview of Integration" at the beginning of this assignment. Integration and differentiation are two major procedures in calculus, and the learning process involves some adjustment of expectations.

The text begins with the notation,

$$\int f\big[g(x)\big]g'(x)\,dx$$

on ***page 413***. It is a common practice in integration to clarify the algebraic structure by using a single letter for a function. So we let

$$u = g(x)$$

The definition of differentials on ***page 254*** indicates that

$$du = g'(x)\,dx$$

Then $$\int f[g(x)]g'(x)\,dx$$

becomes $$\int f[u] \; du$$

But this is just a less complex form. It doesn't tell us how to evaluate the integral. More details can be provided by looking at three possibilities for $f(u)$.

TYPE I

We first consider $f(u) = u^n$ where $u = g(x)$ and show that

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

This is similar to

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

but we add a word of caution. In the integral $\int u^n \, du$ the differential du is actually $g'(x)\,dx$. The key is to make sure that du or $g'(x)\,dx$ exactly fits the given problem.

EXAMPLE 1

Evaluate $\int \left(x^2 + 3\right)^4 x \, dx$.

Let $u = x^2 + 3$ and note that $du = 2x \, dx$. Then

$$\int \left(x^2 + 3\right)^4 x \, dx = \int u^4 \, x\,dx$$

We insert 2 to get the exact du and balance this with $\frac{1}{2}$. Then

$$\int u^4 \; x\,dx \;=\; \tfrac{1}{2}\int u^4 \; 2x\,dx \;=\; \tfrac{1}{2}\int u^4 \; du$$

Now integrate by adding one to the exponent and dividing by the new exponent:

$$\tfrac{1}{2}\int u^4 \; du \;=\; \frac{1}{2}\,\frac{u^5}{5} + C \;=\; \frac{1}{10}\left(x^2 + 3\right)^5 + C$$

Check by finding the derivative of the answer:

$$\frac{d}{dx}\,\frac{1}{10}\left(x^2 + 3\right)^5 + C \;=\; \frac{1}{10}\cdot 5\left(x^2 + 3\right)^4 (2x) \;=\; \left(x^2 + 3\right)^4 x \;\; \text{OK}$$

The text uses a slightly different approach in dealing with

$$\int u^4 \; x\,dx$$

It compares $\quad du \;=\; 2x \; dx \;$ with $\; x\,dx$

and writes $\quad \dfrac{du}{2} \;=\; x \; dx$

Then $\int u^4 \; x\,dx$ becomes $\int u^4 \; \dfrac{du}{2}$, which equals $\tfrac{1}{2}\int u^4 \; du$

The result is the same. Just a different path.

EXAMPLE 2

Evaluate $\int x^2 \sqrt{x^3 + 1} \; dx$.

Let $u \;=\; x^3 + 1$ and note that $du \;=\; 3x^2 \; dx$. Then

$$\int x^2 \sqrt{x^3 + 1} \; dx \;=\; \int x^2 \sqrt{u} \; dx \;=\; \int u^{\frac{1}{2}} \; x^2 \; dx$$

We insert 3 to get the exact du and balance this with $\tfrac{1}{3}$. Then

$$\int u^{\frac{1}{2}} \; x^2 \; dx \;=\; \tfrac{1}{3}\int u^{\frac{1}{2}} \; 3x^2 \; dx \;=\; \tfrac{1}{3}\int u^{\frac{1}{2}} \; du$$

Now integrate by adding one to the exponent and dividing by the new exponent:

$$\tfrac{1}{3}\int u^{1/2}\ du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{2}{9}\left(x^3 + 1\right)^{3/2} + C$$

Check by finding the derivative of the answer:

$$\frac{d}{dx}\ \frac{2}{9}\left(x^3 + 1\right)^{3/2} + C = \frac{2}{9}\cdot\frac{3}{2}\left(x^3 + 1\right)^{1/2}(3x^2) = \left(x^3 + 1\right)^{1/2} x^2\ \ OK$$

EXAMPLE 3

Evaluate $\displaystyle\int \frac{x}{\left(3x^2 + 1\right)^5}\,dx$

To match the form

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

rewrite the integral using negative exponents:

$$\int \frac{x}{\left(3x^2 + 1\right)^5}\,dx = \int \left(3x^2 + 1\right)^{-5} x\ dx$$

Then let $u = 3x^2 + 1$ and note that $du = 6x\ dx$. Next insert 6 to get the exact du and balance this with $\tfrac{1}{6}$. Then

$$\tfrac{1}{6}\int \left(3x^2 + 1\right)^{-5} 6x\ dx = \tfrac{1}{6}\int u^{-5}\ du$$

Now integrate by adding one to the exponent and dividing by the new exponent.

$$\tfrac{1}{6}\int u^{-5}\ du = \frac{1}{6}\frac{u^{-4}}{-4} + C = -\frac{1}{24}\left(3x^2 + 1\right)^{-4} + C$$

Check by finding the derivative of the answer:

$$\frac{d}{dx}\left[-\frac{1}{24}\left(3x^2 + 1\right)^{-4}\right] + C = -\frac{1}{24}\cdot(-4)\left(3x^2 + 1\right)^{-5}(6x) = \left(3x^2 + 1\right)^{-5} x\ \ \text{OK}$$

EXAMPLE 4

Evaluate $\int \sin^3 x \cos x \, dx$.

Let $u = \sin x$ and note that $du = \cos x \, dx$. Then

$$\int \sin^3 x \cos x \, dx = \int u^3 \, du$$

Integrate by adding one to the exponent and dividing by the new exponent:

$$\int u^3 \, du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4 x + C$$

Check by finding the derivative of the answer:

$$\frac{d}{dx}\frac{1}{4}\sin^4 x + C = \frac{1}{4}\cdot 4\sin^3 x \ (\cos x) = \sin^3 x \cos x \text{ OK}$$

EXAMPLE 5

Evaluate $\int \frac{(\ln x)^3}{x} \, dx$.

Let $u = \ln x$ and note that $du = \frac{1}{x} \, dx$. Then

$$\int \frac{(\ln x)^3}{x} \, dx = \int (\ln x)^3 \, \frac{1}{x} \, dx = \int u^3 \, du$$

Integrate by adding one to the exponent and dividing by the new exponent:

$$\int u^3 \, du = \frac{u^4}{4} + C = \frac{1}{4}(\ln x)^4 + C$$

Check by finding the derivative of the answer:

$$\frac{d}{dx}\frac{1}{4}(\ln x)^4 + C = \frac{1}{4}\cdot 4\,(\ln x)^3 \,\frac{1}{x} = \frac{(\ln x)^3}{x} \text{ OK}$$

SUMMARY

The equation $\int u^n \, du = \dfrac{u^{n+1}}{n+1} + C$ where $\begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}$

can also be written as $\int [g(x)]^n \, g'(x)dx = \dfrac{[g(x)]^{n+1}}{n+1} + C$

The key to this form is:

1. the power $[g(x)]^n$ with the base $g(x)$ and
2. the derivative of the base, $g'(x)$

As shown in the above examples, a constant can be missing from the derivative of the base, $g'(x)$. If a variable is missing, that is a different matter. Consider

$$\int (x^2 + 4)^5 \, dx$$

The base is $x^2 + 4$ and its derivative is $2x$. We can insert a 2 and a balancing ½ but we can't insert the variable *x*. The missing $g'(x)$ means this integral does not fit $\int [g(x)]^n \, g'(x)dx$. Another method must be used.

This will be a pattern in integration. You will look at the form of the integral and then try to find a match with a method. Each method will have a characteristic form.

Also in the above examples, we have used three types of exponents, positive integers, fractional, and negative. If you see a radical sign, convert it to a fractional exponent, and then see if there is a fit. If a fraction is present, try using a negative exponent. After these changes are made, the exponent n usually stands out along with the base. Then ask the question, "Is the derivative of the base present except for maybe a constant"? If so, the integration is easy. Just add one to the exponent, and divide by the new exponent. The set up is the more difficult part.

TYPE 2

A distinction must be made between $\int \sin x \, dx$ and $\int \sin 2x \, dx$. The first integral $\int \sin x \, dx$ equals $-\cos x + C$. If you write

$$\int \sin 2x \, dx = -\cos 2x + C$$

the check $\frac{d}{dx}\left(-\cos 2x + C\right)$ produces $-(-\sin 2x)(2)$ or $2\sin 2x$.

Change the answer to $\frac{-\cos 2x}{2} + C$ and the check will work.

[Remember $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$. The Chain Rule requires the $\frac{du}{dx}$ or the derivative of the "angle" at the end.]

The general form is

$$\int \sin u \, du = -\cos u + C \text{ where } u = g(x).$$

This form is concise, but it hides the fact that

$$du = g'(x)\,dx.$$

We could also write

$$\int [\sin g(x)]\, g'(x)\,dx = -\cos g(x) + C$$

to emphasize the need for $g'(x)$, the derivative of the "angle" $g(x)$.

Similar forms match the other trig functions. For example,

$$\int \cos u \, du = \sin u + C \text{ where } u = g(x)$$

or $$\int [\cos g(x)] g'(x)dx = \sin g(x) + C$$

EXAMPLE 1

Find $\int \cos 3x \, dx$.

In this case, $u = 3x$ and $du = 3\,dx$. Then insert 3 and balance this with $\frac{1}{3}$:

$$\int \cos 3x \, dx = \tfrac{1}{3}\int \cos 3x \; 3\,dx = \tfrac{1}{3}\sin 3x + C$$

Check: $$\frac{d}{dx}\left(\tfrac{1}{3}\sin 3x\right) = \tfrac{1}{3}(\cos 3x)\cdot 3 = \cos 3x \text{ OK}$$

Example 2

Find $\int \sec 5x \tan 5x \; dx$.

Let $u = 5x$ and note that $du = 5\,dx$. Then insert 5 and balance this with $\frac{1}{5}$:

$$\int \sec 5x \tan 5x \; dx \;=\; \tfrac{1}{5}\int \sec 5x \tan 5x \; 5\,dx \;=\; \tfrac{1}{5}\sec 5x \;+\; C$$

Check: $$\frac{d}{dx}\left(\tfrac{1}{5}\sec 5x\right) \;=\; \tfrac{1}{5}\left(\sec 5x \tan 5x\right)\cdot 5 \;=\; \sec 5x \tan 5x \;\; \text{OK}$$

Example 3

Find $\int \left(\sec^2 e^{3x}\right) e^{3x} \, dx$.

As you look at this form, it is essential that you remember $\sec^2\theta$ is the derivative of $\tan\theta$. Then let $u = e^{3x}$ and note that $du = e^{3x}\cdot 3\;dx$. Insert 3 and balance this with $\frac{1}{3}$:

$$\int \left(\sec^2 e^{3x}\right) e^{3x}\, dx \;=\; \tfrac{1}{3}\int \left(\sec^2 e^{3x}\right) e^{3x} \; 3\,dx \;=\; \tfrac{1}{3}\tan e^{3x} \;+\; C$$

Check: $$\frac{d}{dx}\left(\tfrac{1}{3}\tan e^{3x}\right) \;=\; \tfrac{1}{3}\left(\sec^2 e^{3x}\right)\cdot e^{3x}\cdot 3 \;=\; \left(\sec^2 e^{3x}\right) e^{3x} \;\; \text{OK}$$

Summary

1. Note that in these examples the actual integration (the step where the integral sign vanishes) involved remembering that $\cos\theta$, $\sec\theta\tan\theta$, and $\sec^2\theta$ were the derivatives of $\sin\theta$, $\sec\theta$, and $\tan\theta$ respectively.

2. The set up to get an *exact* fit involved $\dfrac{d\theta}{dx}$ in a derivative like
$$\frac{d}{dx}\tan\theta \;=\; \sec^2\theta\,\frac{d\theta}{dx}$$

3. θ, u, and $g(x)$ are different representations for the angle in a trig function.

TYPE 3

The equation $\int \frac{1}{x}\,dx = \ln|x| + C$

leads to the more general form

$$\int \frac{1}{u}\,du = \ln|u| + C \qquad \text{where } u = g(x).$$

Again it is concise but hides the fact that $du = g'(x)dx$.

Possibly a better form is

$$\int \frac{1}{g(x)}\,g'(x)dx = \ln|g(x)| + C$$

and maybe the best is

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$$

Think of the form

$$\frac{g'(x)}{g(x)}$$

as asking the question, "*is* the numerator the derivative of the denominator?" If the answer is yes, then the integral equals the *natural log* of the *denominator*. Which form you prefer depends on how you see the problem. Select the one that makes the most sense to you.

EXAMPLE 1

Evaluate $\int \frac{x}{x^2 + 5}\,dx$.

The numerator will be the derivative of the denominator if we insert 2 and balance with $\frac{1}{2}$:

$$\int \frac{x}{x^2 + 5}\,dx = \frac{1}{2}\int \frac{2x}{x^2 + 5}\,dx = \tfrac{1}{2}\ln|x^2 + 5| + C$$

Note that the answer is ½ of the natural log of the denominator. A check of the answer will verify that it is correct.

Repetition: If you prefer the form $\int \frac{1}{u}\ du \ = \ \ln|u| + C$, which can be written $\int \frac{du}{u} \ = \ \ln|u| + C$, then the explanation would begin with, let $u = x^2 + 5$, $du = 2x\,dx$. Then

$$\int \frac{x}{x^2 + 5}\ dx \ = \ \frac{1}{2}\int \frac{2x}{x^2 + 5}\ dx \ = \ \tfrac{1}{2}\ln|x^2 + 5| + C$$

EXAMPLE 2

Calculate $\int \cot x\ dx$.

Rewrite in terms of $\sin x$ and $\cos x$.

$$\int \cot x\ dx \ = \ \int \frac{\cos x}{\sin x}\ dx$$

The numerator is the derivative of the denominator:

$$\int \cot x\ dx \ = \ \int \frac{\cos x}{\sin x}\ dx \ = \ \ln|\sin x| + C$$

The answer is the natural log of the denominator.

Check. $\quad \frac{d}{dx}\ln|\sin x| + C \ = \ \frac{1}{\sin x}\cos x \ = \ \cot x$ OK

We have described type 3 in terms of a numerator and a denominator. Some functions have a derivative which is expressed in a fractional form. Three examples are

$$\ln x \quad , \quad \sin^{-1} x \quad , \quad \tan^{-1} x$$

whose derivatives are

$$\frac{1}{x} \quad , \quad \frac{1}{\sqrt{1 - x^2}} \quad , \quad \frac{1}{1 + x^2}$$

So instead of seeing

$$\int \frac{1/x}{\ln x}\,dx \quad , \quad \int \frac{1/\sqrt{1 - x^2}}{\sin^{-1} x}\,dx \quad , \quad \int \frac{1/(1 + x^2)}{\tan^{-1} x}\,dx$$

you would see

$$\int \frac{1}{x \ln x}\,dx \quad , \quad \int \frac{1}{\sqrt{1 - x^2}\,\sin^{-1} x}\,dx \quad , \quad \int \frac{1}{\left(1 + x^2\right)\tan^{-1} x}\,dx$$

The answers would be

$$\ln\left|\ln x\right| + C \;, \quad \ln\left|\sin^{-1} x\right| + C \;, \qquad \ln\left|\tan^{-1} x\right| + C$$

You may feel that these last three problems involve rather complex forms but we hope you concentrate on seeing the structure rather than the complexity.

SUMMARY

1. A key aspect of this type is its fractional form. The function $\ln g(x)$ is not a fraction, but its derivative, $\dfrac{g'(x)}{g(x)}$ is a fraction. Because integration reverses this process, we start with a fractional form that is sometimes complex.

2. However, some fractional forms will fit another method. For example

$$\int \frac{(\ln x)^2}{x}\,dx \text{ and } \int \frac{x}{\sqrt[3]{x^2 + 4}}\,dx$$

match

$$\int u^n\,du = \frac{u^{n+1}}{n + 1} + C$$

Rewrite as

$$\int (\ln x)^2\,\frac{1}{x}\,dx \text{ and } \int \left(x^2 + 4\right)^{-1/3}\,x\,dx$$

and the form becomes more apparent.

3. Make a special note of Example 2 above and ***Example 6***, on ***page 415***.

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx = \ln\left|\sin x\right| + C$$

and

$$\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx = -\ln\left|\cos x\right| + C$$

$$= \ln\left|\sec x\right| + C$$

Each provides a rather surprising connection between a trig function and the log of a trig function. A similar connection exists for the integral of $\sec x$:

$$\int \sec x \; dx = \ln\left|\sec x + \tan x\right| + C$$

You can verify this by finding the derivative of $\ln\left|\sec x + \tan x\right|$ or read the derivation on ***page 483, Sec. 7.2***.

A SPECIAL SUBSTITUTION

In ***Example 2, page 414***, the text demonstrates a special substitution in the second solution, $u = \sqrt{2x + 1}$. The integral in this example also matches

$$\int u^n \; du = \frac{u^{n+1}}{n + 1} + C$$

but that is not the case for the integral here:

$$\int \frac{x^2}{\sqrt{1 - x}} \; dx$$

By letting $u = \sqrt{1 - x}$, in effect we eliminate the radical sign. A first substitution yields

$$\int \frac{x^2}{u} \; dx$$

Then square both sides of $\quad u = \sqrt{1 - x}$

to get $u^2 = 1 - x\,.$

This leads to $x = 1 - u^2$

$$dx = -2u\,du$$

and

$$\int \frac{x^2}{u}\,dx = \int \frac{\left(1 - u^2\right)^2}{u}\left(-2u\,du\right) = -2\int\left(1 - 2u^2 + u^4\right)du$$

Now we just have a polynomial to integrate:

$$-2\int\left(1 - 2u^2 + u^4\right)du = -2\left(u - 2\frac{u^3}{3} + \frac{u^5}{5}\right) + C$$

Replace u with $\sqrt{1 - x}$ and the answer is

$$-2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C$$

This is a rather complex answer, but if you find the derivative, it will simplify to $\dfrac{x^2}{\sqrt{1 - x}}$.

This type of substitution can be used for Exercises ***67*** and ***68*** on ***page 419***.

CHANGING LIMITS FOR DEFINITE INTEGRALS

When using the Substitution Rule, there will be a change of variables, frequently from a function of x to a function of u. For a definite integral there are two choices:

1. Change the limits of integration as shown in ***Examples 7, 8,*** and ***9, pages 416–417***.

2. Use a variable like u in the intermediate steps to carry out the integration but then express the answer in terms of x. Then use the original limits of integration.

Step 2 is consistent with each integration we have performed. The answer was never left in terms of u but always converted back to a function of x.

Using the procedure described in step 2, we redo ***Example 7***:

$$\int \sqrt{2x + 1}\ dx \ = \ \frac{1}{2}\int u^{1/2}\ du \ = \ \frac{1}{2}\cdot\frac{2}{3}\ u^{3/2} \ = \ \frac{1}{3}(2x + 1)^{3/2}$$

Now use the original limits of integration:

$$\int_0^4 \sqrt{2x + 1}\ dx \ = \ \frac{1}{3}(2x + 1)^{3/2}\Big]_0^4 \ = \ \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) \ = \ \frac{1}{3}\left(3^3 - 1\right) \ = \ \frac{26}{3}$$

INTEGRALS OF SYMMETRIC FUNCTIONS

Make a mental note of ***Theorem 7, page 417***. In some applications you may wish to review this theorem.

SECTION 6.1: AREAS BETWEEN CURVES

In ***Chapter 6*** we consider three applications of the definite integral. In the coverage of methods of integration some rather abstract forms appeared, so now you get a chance to look at something more concrete. The text contains many drawings of regions that have an area or a volume. These are objects that you can see in a different way from forms like $u^n\ du$ or $\frac{du}{u}$. However, between a drawing and the number that is the area or volume, there is a *framework* that provides a connection. We start with several comments that relate to this framework:

1. Every vertical line segment is represented by the algebraic form

$$y_2 - y_1$$

2. Every horizontal line segment is represented by the algebraic form

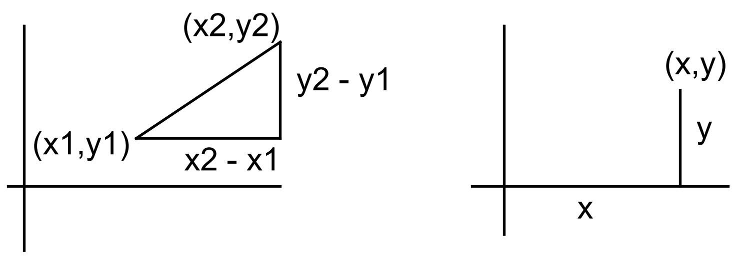
$$x_2 - x_1$$

3. Associated with a point (x, y) there is a line segment whose length is *x* and another whose length is *y*.

4. The definite integral $\int_a^b f(x)\ dx$ has two interpretations. It equals $F(b) - F(a)$ where $F(x)$ is the antiderivative of $f(x)$. Also it equals $\lim \sum f(x_i)\ \Delta x_i$, a limit of a sum of products. In setting up an integral, you

will need to remember that $\int_a^b f(x)\,dx$ somehow represents the sum, $\sum f(x_i)\,\Delta x_i$. The products must be seen in a very concrete way in a drawing. Otherwise, the process that leads to areas or volumes will not make sense.

You may feel that we are insulting your intelligence in the first three comments because they are so basic. The key is for these tools to be present in your mental background and for you to know when to use them. Some seemingly complex forms are simple when you can look at a graph and recognize what the component parts represent.



"VERTICAL" RECTANGLES

An example of a "vertical" rectangle is shown in ***Figure 2(a)***. The width of the rectangle is Δx and the height fits $y_2 - y_1$ in the form $f(x_i^*) - g(x_i^*)$. This is further clarified in the drawing by showing two line segments; one has length $f(x_i^*)$ and the other has length $-g(x_i^*)$. Remember that in the drawing, $g(x_i^*)$ is a negative *y*-value, which makes $-g(x_i^*)$ positive. Then adding the lengths of these two shorter segments gives the distance from the bottom curve (in blue) to the top curve (in red) at $x = x_i^*$.

Then ***Figure 2(b)*** shows nine rectangles that blanket the entire region between the two curves. These nine rectangles match the sum $\sum f(x_i^*)\,\Delta x_i$. It is important to visualize this sum when setting up a definite integral to verify that the boundaries of the rectangles do not change. Here the red curve is the top boundary and the blue curve is the bottom boundary for all rectangles. In ***Figure 11, page 432***, this is not the case. In S_1 and S_3 the red curve is the top boundary of imaginary vertical rectangles while in S_2 the blue curve is the top boundary. In ***Figure 12***, the bottom boundary in A_1 differs from that in A_2. The width of the rectangles does not change, but an adjustment must be made in the heights.

Returning to ***Figure 2***, the area of one rectangle is represented by

$$\left[f(x_i^*) - g(x_i^*)\right]\Delta x$$

and the area of the nine rectangles is represented by

$$\sum\left[f(x_i^*) - g(x_i^*)\right]\Delta x$$

Then *imagine* taking a limit which increases the number of rectangles, and the approximation becomes exact. The definite integral matching

$$\lim\sum\left[f(x_i^*) - g(x_i^*)\right]\Delta x$$

is

$$\int_a^b \left[f(x) - g(x)\right] dx$$

COMMENTS ON EXAMPLE 2, PAGE 429

1. Note the limits of integration, a and b, are determined by solving two equations simultaneously.

2. Only one rectangle, shown in ***Figure 6***, is needed if the boundaries do not change.

3. The height is $f(x) - g(x)$ and the width is dx. The area of the single rectangle is $\left[f(x) - g(x)\right] dx$.

4. In the notation, y_T and y_B, the subscripts T and B are used to indicate the Top curve and the Bottom curve.

5. The definite integral

$$\int_0^1 \left[\left(2x - x^2\right) - x^2\right] dx$$

represents the limit of a sum of the areas of rectangles from $x = 0$ to $x = 1$.

6. Switch your thinking to antiderivatives, evaluate and the area is $\frac{1}{3}$.

In step 5 note that we used x instead of x_i. As long as you remember that the integral sign $\int$ represents a sum *and* a limit, the intermediate step involving the notation $\sum$ can be omitted.

"HORIZONTAL" RECTANGLES

What we mean by a "horizontal" rectangle is shown in ***Figure 13***. Now the length will be represented by $x_2 - x_1$. In the text, this is written as $x_R - x_L$ where x_R is the x-value from the boundary on the right and x_L is the x-value from the boundary on the left.

EXAMPLES

	Left Boundary	**Right Boundary**	$x_R - x_L$
1.	$y = x^3$	$y = x + 10$	
Solve for x.	$x = y^{1/3}$	$x = y - 10$	$(y - 10) - y^{1/3}$
2.	$y^2 = x + 1$	$y = x + 1$	
Solve for x.	$x = y^2 - 1$	$x = y - 1$	$(y - 1) - (y^2 - 1)$

The two expressions in the third column

$$(y - 10) - y^{1/3} \qquad \text{and} \qquad (y - 1) - (y^2 - 1)$$

represent the lengths of rectangles with the given

boundaries. In the definite integral, ***page 433***,

$$f(y) - g(y)$$

provides a general description of the above two lengths. Also in ***Figure 13***, the width of the rectangle is Δy. This becomes dy in

$$\int_c^d \left[f(y) - g(y) \right] dy$$

If you clearly see this definite integral as a *sum* of horizontal rectangles, then the limits of integration, c and d, must be y-values. In ***Figure 13***, the summing starts at $y = c$ and continues until we get to $y = d$.

COMMENTS

1. How do you choose between vertical and horizontal rectangles? By reading graphs that show the region under consideration. Look at the upper and lower boundaries when using vertical rectangles to be certain that they apply throughout the region. Do the same for horizontal rectangles by looking at the boundaries on the right and on the left.

2. You need only one rectangle to determine the length and width but you need to visualize the set of rectangles covering the region to get the limits of integration. In the integral

$$\int_a^b \left[f(x) - g(x)\right] dx$$

which matches vertical rectangles, the variable is x. As we sum the rectangles covering the region x varies from a to b. In the integral

$$\int_c^d \left[f(y) - g(y)\right] dy$$

which matches horizontal rectangles, the variable is y. Now as we sum the rectangles covering the region, y varies from c to d.

3. For those regions bounded by two curves the points of intersection of the two curves will determine the limits of integration.

4. On ***page 432***, the definite integral ***(3)***

$$\int_a^b \left|f(x) - g(x)\right| dx$$

represents the *area* between two curves because of the absolute value signs. However this is just a general description of an idea not a method.

5. Comments 1 through 4 relate to the graph of a region. Accurate graphs are essential in finding the area of the region.

SUGGESTION ON INTEGRATION

A good way to become proficient in finding antiderivatives is to do as many of the odd-numbered exercises from *7* to ***45, Sec. 5.5***, as time permits. At least look at these exercises to determine if you recognize which method would work. Recognizing the form and getting the constants right are the two major demands in integration.