Module 9

Section 9.1: Modeling with Differential Equations

The third and last major topic in this course is differential equations. A differential equation is easy to define, but it can be difficult to solve. An equation that contains an *unknown* function and some of its *derivatives* is called a differential equation. Recall that derivatives relate to rates of change. When a person observes a situation where change is involved, then an appropriate description might involve derivatives, that represent this change. Accounts in the history of mathematics reveal that in the seventeenth century, differential equations like $d^2\theta/dt^2 + mg\sin\theta = 0$ for the circular pendulum existed, but analytic methods for solving them had not been developed. In some sense, you will be dealing with a reverse situation. You will be looking at methods of *solving* differential equations before you have the experience of writing one. Then when your opportunity comes, you will have some background that may provide illumination.

Models of Population Growth

It is important to note in the first paragraph of this topic that the model is based on an *assumption* that the rate of growth of a population is proportional to the size of the population. This leads to the differential equation, $\dfrac{dP}{dt} = kP$

The mathematical meaning of the word proportional is that the two quantities aren't equal but that the rate of change equals a constant, *k*, times the population, *P*.

Also note that the *model* is the *equation*. It may help to think of two things. First, over time the "real world" population is going to change. Secondly, we have created an "abstract" equation that we hope will closely match this change in population. Of course the population is not going to change by looking at this equation: rather we hope that we can get predictions from the equation of what happens, say, in ten years. If the prediction doesn't match reality, then we need to look at the basic assumption.

The intent in this first section is to give an overview of the idea of a model. Toward the bottom of ***page 586***, an exponential function, Ce^{kt}, appears out of thin air. Don't be concerned now about this magic trick. Later we will spend time finding appropriate functions using mathematical steps. Now the author needs the function to be able to create and talk about the graphs in ***Figures 1*** and ***2***. To the right of ***Figure 2*** note the reference to "ideal conditions" and a "more realistic model". The latter leads to the much more complex equation,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Keep in mind that it may have taken years for a research scientist to come up with this particular combination of symbols to describe population growth. Appropriate differential equations don't just pop into our heads. The task is to see if it is a reasonable form.

M is called the carrying capacity. Think of the fact that only so much food is available and if the population exceeds a particular number, M, there isn't enough food for more growth to occur. This is a hypothetical number but still it seems reasonable that some constant M exists.

Concentrate on the fraction, $\frac{P}{M}$, and the factor, $1 - \frac{P}{M}$. If $P = M$, then $1 - \frac{P}{M}$ equals zero and the rate of growth is zero. This is reasonable because the population has reached the carrying capacity. Also if P is larger than M, then $\frac{P}{M}$ is larger than one and $1 - \frac{P}{M}$ becomes negative. In turn this means the rate of growth is negative and the population will decrease. Again this seems reasonable, lending support to the idea that the equation, $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ seems a better description of population growth. ***Figure 3*** contains graphs of possible solutions.

GENERAL DIFFERENTIAL EQUATIONS

For a complex topic like differential equations, knowing basic definitions is important. Consider two words on ***page 588***, *order* and *solution*. We have talked about first and second derivatives extensively, but we didn't associate *first order* with the *first derivative* or *second order* with the *second derivative*. Not too complicated but fundamentally important. In ***chapters 9*** and ***17***, which cover differential equations, one topic is "first-order linear equations"; another is "second-order linear equations." What this means is that we first cover a method for solving differential equations that contain a first derivative but no higher derivatives. After working with first-order differential equations, you will recognize that the jump to second order differential equations is huge.

The second word is *solution*. A *function* is a *solution* of a different equation. The text at the bottom of ***page 586*** indicates that $P = Ce^{kt}$ is a solution of the differential equation, $\dfrac{dP}{dt} = kP$.

The function $y = \sin x$ is a solution of the differential equation

$$y'' + y = 0$$

because

$$-\sin x + \sin x = 0.$$

However, the same function will not be a solution of

$$y'' - y = 0.$$

Does a function come to mind that would be a solution? Try $y = e^x$.

Every integration that we have performed can be related to a differential equation. For example,

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

provides the solution to the differential equation,

$$y' - x^2 = 0$$

because $\left(\dfrac{x^3}{3} + C\right)' - x^2 = 0$.

However, this example is also misleading. Consider

$$x^2 y' + xy = 1 .$$

We are to find a function, $y = f(x)$, such that, after its derivative, $f'(x)$, is multiplied by x^2 and added to the product of x and the function, the resulting sum equals one. Hmm. Maybe rewriting the differential equation by solving for y' will help.

$$y' = \frac{1 - xy}{x^2}$$

Then $$y \overset{?}{=} \int \frac{1 - xy}{x^2} \, dx$$

A spreadsheet would say “cannot resolve circular references.” The function represented by y is part of the integrand and is also the answer.

The intent here is to show some of the complexity in solving a differential equation. The method of *solving* this first-*order* DE (differential equation) will be developed in ***Section 9.5. Example 2, page 622***, indicates that a solution is

$$y = \frac{\ln x + C}{x}$$

At this point, content yourself with knowing what it means to say that this function is a *solution*. Find the derivative and drop both the derivative and the function into

$$x^2 y' + xy = 1$$

to see if the sum is really one. Then don't confuse this *check* with *finding* the solution.

INITIAL-VALUE PROBLEMS

Another important concept. Solving a differential equation produces a general solution. But a person doing research needs a solution that fits a particular situation. The data that fits a particular situation goes by the name *initial conditions*.

In ***Example 1, page 589***, the function

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is shown to be a solution of the differential equation

$$y' = \frac{1}{2}\left(y^2 - 1\right)$$

Then consider replacing the constant c with different numbers. ***Figure 5*** contains seven graphs that would match different values for c.

In ***Example 2***, an initial condition, $y(0) = 2$ is included. This means that $y = 2$ when $t = 0$. Graphically we pick the the curve in Figure 5 that passes through the point (0,2). Algebraically we put this pair of numbers in the solution

$$y = \frac{1 + ce^t}{1 - ce^t}$$

to get $c = 1/3$ as shown in the text. Then

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$

is a particular solution of the differential equation.

A SECOND EXAMPLE

A car is traveling 60 mph (88 ft/sec) when the brakes are applied in a way that produces a *constant* deceleration. If the car stops in 8 seconds, how far did it travel?

The words *constant deceleration,* bring to mind the differential equation

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = k, \quad \text{where } k \text{ is a constant.}$$

For many students, the calculus part of this problem is easier than finding what we now call initial conditions. Create a structure in your mind that includes deceleration, velocity, and distance and that can be related to two values for time. Also, recognise that you have choices. When the brakes are applied, what does time t equal? The easiest choice is to let $t = 0$ at the instant when the brakes are applied *and* to start measuring distance at this time. Note that $t = 8$ when the car stops. Collecting the information we have

	deceleration	velocity	distance
time, $t = 0$	constant	88	0
$t = 8$	constant	0	?

Stopped means velocity is zero.

Hence there are three initial conditions:

$$(1)\ t = 0,\ v = 88 \quad (2)\ t = 0,\ s = 0 \quad (3)\ t = 8,\ v = 0$$

We have now completed one major task. The second major task is to know how to *use* this information. This involves solving two differential equations.

(a) $$\frac{d^2s}{dt^2} = \frac{dv}{dt} = k$$

General solution: $v = kt + C_1$

Use $t = 0,\ v = 88$: $88 = k(0) + C_1$

$C_1 = 88$

Particular solution: $v = kt + 88,$ but k is still unknown.

Use $t = 8,\ v = 0$: $0 = k(8) + 88$

$k = -11$

Really particular solution: $v = -11t + 88$

Second differential equation.

$$\text{(b)} \qquad v = \frac{ds}{dt} = -11t + 88$$

General solution: $$s = -11\frac{t^2}{2} + 88t + C_2$$

Use $t = 0,\ s = 0$ $$0 = -11\frac{0^2}{2} + 88(0) + C_2$$

$$C_2 = 0$$

Particular solution: $$s = \frac{-11}{2}t^2 + 88t$$

Now we can answer the question, "how far did the car travel?" by putting $t = 8$ into the last equation.

$$s = \frac{-11}{2}(8)^2 + 88(8) = 352$$

The car travels 352 ft in the 8 seconds that it takes the car to stop.

REVIEW

The answers in the last example are, of course, not the important point. Instead concentrate on the method with particular emphasis on finding and using the initial conditions. Solving the differential equations produces general solutions, which are then altered to fit the given information or initial conditions. Without this alteration, we would not be able to solve the problem. We would just have functions and would be wondering: how on earth do these relate to the person in a car stomping on the brakes?

SECTION 9.2: DIRECTION FIELDS AND EULER'S METHOD

DIRECTION FIELDS

This section introduces two other views of differential equations. Each example in this section contains a differential equation that is or can be written in the form

$$y' = F(x, y)$$

This just means that we can isolate the derivative symbol on one side and put all x's, y's, and constants on the other side. Now select *any* point, say, (1,2), and $F(1,2)$ is the slope of the tangent line at this point. ***Figure 3*** on ***page 592*** contains 121 short line segments indicating the slope of the solution at 121 points for the differential equation

$$y' = x + y$$

Yes, this would be a lot of tedious work but computers are very efficient in performing this type of task. For now concentrate on the creation of a background that suggests the graphs of solutions to a DE, even if we don't have the solution in algebraic form.

The first question is, where does one start? The direction field provides information for the *general* solution. Drawing a single curve matches a *particular* solution, which requires that we have some *initial* conditions. The initial condition is that $y = 1$ when $x = 0$. The red curve in ***Figure 4*** passes through the point (0,1) and then follows a path suggested by the indicated slopes of the tangent line. Note that the curve doesn't use any of the blue line segments as a tangent line but passes close to them. Remember each blue line segment is associated with a point, but we don't know that the curve passes through that point. We only know it passes through the point (0,1). If the curve actually touches a blue line segment as if it were a tangent line then it is also passing through a point on the segment. Perhaps it *should* go through that point, but the only *given* initial condition is that the curve should pass through the point (0,1).

Example 1, provides a second illustration of direction fields. ***Figure 5*** contains 81 short line segments that indicate the slopes of solutions at 81 points for the given differential equation. The number 81 increases to 195 in ***Figure*** *7*, to show more detail. The solution passing through the origin is drawn in ***Figure 6***. The five different curves drawn in ***Figure*** *7* illustrate quite different behavior. The curve passing through the origin is the only one of the five that has a relative maximum and minimum. If the differential equation is related to some physical situation, then the "swirling" behavior near the origin may have significance.

Also note that if we drew a curve on the far left or on the far right, it would be very steep. There would be no max or min points but perhaps a point of inflection.

The differential equation **(1)** of ***page 593*** describes the electric circuit in ***Figure 8***. It is important to realize that *L* and *R* are constants. The only variables are *I* and *t*. In ***Example 2***, particular values are assigned to *L* and *R* and *E(t)* is also a constant. The differential equation can then be written as $\frac{dI}{dt} = 15 - 3I$. If a value is assigned to *I*, say $I = 2$, then 9 is the slope of a solution regardless of the value of *t*. This is discussed in the paragraph below ***Figure 10***. By shifting a solution right or left we get another solution.

EULER'S METHOD

This is a numerical method that provides a set of points that can be connected with line segments to get an approximate solution of a differential equation. It does not provide an algebraic solution, just a set of points that are close to the graph of a solution. The method is described in the three equations before ***Example 3, page 595***.

SECTION 9.3: SEPARABLE EQUATIONS

We now consider a special differential equation where the variables can be separated. This means all terms containing x and dx can be placed on one side of the equation and all terms containing y and dy can be placed on the other side. Then if we can integrate each side we will have a solution to the differential equation. The answer may be in implicit form.

Example. Solve $$x^2 y' + y = 0$$

First rewrite as $$x^2 \frac{dy}{dx} = -y$$

Multiply by dx. $$x^2\,dy = -y\,dx$$

Divide by $x^2 y$. $$\frac{dy}{y} = -\frac{dx}{x^2} = -x^{-2}dx$$

Integrating we have $$\ln y = -\frac{x^{-1}}{-1} + C = x^{-1} + C$$

We demonstrate that we really have a solution to $x^2 y' + y = 0$

by finding the derivative of the answer. $\frac{1}{y}y' = -x^{-2}$

Then $$y' = -x^{-2}y$$

and substituting into $x^2 y' + y = 0$

we have $x^2\left(-x^{-2}y\right) + y = 0$

which indicates we do have a solution.

ORTHOGONAL TRAJECTORIES

A common complaint from calculus students is the lament, where am I going to use this? But then applications are shown and there are two difficulties. One, if you are entering the biology field, for example, you may not yet have the advanced knowledge needed to understand an application presented in a calculus book. The ideas from biology can't be developed extensively in a non-biology course. The second difficulty is that many applications aren't in your field. An instructor emphasizes engineering or physics problems, and you are in another field. So if we indicate that in electrostatics, thermodynamics, and hydrodynamics, finding curves that are perpendicular (orthogonal) to a family of curves is an important mathematical procedure, it is understandable that you may not be impressed. However, being able to briefly look at a procedure and pick out the significant steps, without being intimidated by the algebraic symbols, puts you in a special group. Employers may be very happy to discover that you have these skills. Potentially, if you can see mathematical structures, your mind may work differently as you look at ideas that appear to have no mathematical content. Different possibilities may emerge as you look at what is behind surface phenomena. No guarantee, but a definite possibility.

The major point in finding orthogonal trajectories is expressed in the differential equations on ***page 603***. Perpendicular tangents must have slopes that are negative reciprocals. As you skim the procedure, you may wonder why the constant k had to be eliminated. Try doing the problem without eliminating k and see what happens. If you get a different answer, can you show it is equivalent?

MIXING PROBLEMS

Example 6 on ***page 604*** illustrates the use of a new problem-solving tool. Now you can write a differential equation to describe a changing physical situation. The key is to concentrate on the rate of change of the salt in the tank, which is just the (rate in) minus the (rate out). First, each rate must be expressed in terms of kilograms of salt per minute. For the (rate in) which equals

$$\left(0.03\,\frac{kg}{L}\right)\left(25\,\frac{L}{\text{min}}\right) \quad = \quad 0.75\,\frac{kg}{\text{min}}$$

think of canceling the two L's just as if they were fractions. The combination of kg/L and L/min becomes kg/min. Every minute 0.75 kg of salt is coming into the tank. A similar procedure establishes the rate out: (y/200) kg of salt leave the tank every minute. It is essential to include the units to get these rates, but they are not included in the differential equation.

After integrating to get one form of the general solution,

$$-\ln|150 - y| = \frac{t}{200} + C$$

return to the stated problem and note the initial condition; y is 20 when t is zero. Now the particular solution,

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

fits the given problem. The first change in form

$$150 - y = e^{-t/200 + \ln 130}$$

leads to

$$150 - y = e^{-t/200} e^{\ln 130}$$

Then recall that according to the rules of exponential functions (e) and ln, the

following is true

$$e^{\ln 130} = 130$$

Finally

$$y = 150 - 130e^{-t/200}$$

As t increases, y approaches 150 as shown in the graph in ***Figure 10*** on ***page 604***.