Module 8) Find Power Series and Interval convergence

11.9) 3) $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$ we know $\frac{1}{1-x} = 1 + x + x^2 + x^3$ = geometric series w $a=1$, $r=x$

$= \sum_{n=0}^{\infty} 1x^n$ (a)

so just replace x with $-x$

so $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$ $\boxed{\text{Interval of convergence: } -1 < x < 1}$

$= 1 - x + x^2 - x^3 + x^4$

Since it's a geometric series where $r = -x$, $|-x| < 1$ = $|x| < 1$ so interval is $-1 < x < 1$

4) $f(x) = \frac{5}{1-4x^2} = 5 \cdot \frac{1}{1-4x^2}$ let's evaluate $\frac{1}{1-4x^2}$ first

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n \quad \text{provided } |4x^2| < 1 \text{ or } |x^2| < \frac{1}{4}$$

$$|x| < \frac{1}{2}$$

$$= \sum_{n=0}^{\infty} (4^n) x^{2n}$$

so $5 \cdot \frac{1}{1-4x^2} = 5 \sum_{n=0}^{\infty} (4^n) x^{2n} = \boxed{\sum_{n=0}^{\infty} 5(4^n) x^{2n} = \sum_{n=0}^{\infty} 5(4x^2)^n}$

$\boxed{\text{Interval of conv: } \frac{-1}{2} < x < \frac{1}{2}}$

5) $\frac{2}{3-x} = 2 \cdot \frac{1}{3-x} = 2 \cdot \frac{1}{3(1-\frac{x}{3})} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n} = \boxed{\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n}$

geometric series w $\frac{-x}{3} = r$

so $\left|\frac{-x}{3}\right| < 1 \rightarrow |-x| < 3$

Interval is $\boxed{-3 < x < 3}$

6) $\frac{4}{2x+3} = 4 \frac{1}{2x+3} = 4 \cdot \frac{1}{3(1+\frac{2}{3}x} = \frac{4}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n = \frac{4}{3} \sum_{n=0}^{\infty} \frac{(-2)^n}{3^n} x^n = \boxed{\sum_{n=0}^{\infty} \frac{4(-2)^n}{3^{n+1}} x^n}$

$\left|\frac{-2}{3} x\right| < 1$ so $|x| < \frac{3}{2}$ $\boxed{\frac{-3}{2} < x < \frac{3}{2} \text{ Interval of Convergence}}$

8) $f(x) = \frac{x}{2x^2+1} = x \cdot \frac{1}{2x^2+1} = x \cdot \frac{1}{1-(-2x^2)} = x \sum_{n=0}^{\infty} (-2x^2)^n = \boxed{\sum_{n=0}^{\infty} (-2)^n x^{2n+1}}$

geo w: $r = -2x^2$ so

$|-2x^2| < 1$ to converge

$|2x^2| < 1$ $|x^2| < \frac{1}{2}$

$\boxed{\text{Interval of conv: } \frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}}$

10) $f(x) = \dfrac{x+a}{x^2+a^2}$ $a>0$ $= (x+a)\cdot\dfrac{1}{a^2+x^2} = \dfrac{(x+a)}{a^2}\cdot\dfrac{1}{1+\left(\frac{x}{a}\right)^2} = \dfrac{(x+a)}{a^2}\cdot\dfrac{1}{1-\left(\frac{x^2}{-a^2}\right)}$

$= \dfrac{(x+a)}{a^2}\cdot\sum_{n=0}^{\infty}\left(\dfrac{-x^2}{-a^2}\right)^n = \dfrac{(x+a)}{a^2}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x^2}{a^2}\right)^n = \sum_{n=0}^{\infty}\dfrac{(x+a)\left((-1)^n x^{2n}\right)}{a^{2n+2}}$

(done elsewhere)

$-a < x < a$

$|x| < a$

radius of conv $= a$

11. $f(x) = \dfrac{2x-4}{x^2-4x+3}$

$x^2-4x+3 = (x-3)(x-1)$ $\qquad \dfrac{A}{x-3} + \dfrac{B}{x-1}$

$A(x-1) + B(x-3) = 2x-4$

$Ax - A + Bx - 3B = 2x-4$

$Ax + Bx - A - 3B = 2x-4$

$(A+B)x + (-A-3B) = 2x-4$

so $A+B = 2$

$-A-3B = -4$

$A = 2-B$

$-2+B-3B = -4$

$-2B = -2$

$B = 1$

$A = 1$

$\dfrac{2x-4}{x^2-4x+3} = \dfrac{1}{x-3} + \dfrac{1}{x-1} =$

quick check:

$\dfrac{x-1}{(x-3)(x-1)} + \dfrac{x-3}{(x-1)(x-3)}$

$\dfrac{2x-4}{x^2-4x+3}$ ✓

Lets find power series using $\dfrac{1}{x-3} + \dfrac{1}{x-1}$

right side: $\dfrac{1}{x-1} = \dfrac{1}{-1(1-x)} = -1\cdot\dfrac{1}{1-x} = -1\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}-x^n$

left side: $\dfrac{1}{x-3} = \dfrac{1}{-3+x} = \dfrac{1}{-3(1-x)} = \dfrac{1}{-3}\cdot\dfrac{1}{1-x} = \dfrac{1}{-3}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\dfrac{x^n}{3}$

combine: $\sum_{n=0}^{\infty}-x^n + \sum_{n=0}^{\infty}\dfrac{x^n}{3} = \boxed{\sum_{n=0}^{\infty}\dfrac{x^n}{3} - x^n}$

interval is the overlapping interval of convergence

interval of conv for $\frac{1}{x-1}$: $-1 < x < 1$

interval of conv for $\frac{1}{x-3}$: $-1 < x < 1$

overlapping interval of convergence: $\boxed{-1 < x < 1}$

28) $\int \frac{t}{1+t^8}$ Eval Integral as power series. Find radius of convergence.

10) $\frac{x+a}{x^2+a^2}$ (from 11.9) split into 2 fractions

$, a>0$

a) $\frac{x}{x^2+a^2} = x \cdot \frac{1}{x^2+a^2} = \frac{x}{a^2} \cdot \frac{1}{1+\frac{x^2}{a^2}} = \frac{x}{a^2} \cdot \frac{1}{1-\left(\frac{-x^2}{-a^2}\right)} = \frac{x}{a^2} \cdot \frac{1}{1-\left(\frac{-x}{-a}\right)^2} = \frac{x}{a^2}\sum_{n=0}^{\infty}\left(\frac{-x^2}{-a^2}\right)^n$

b) $\frac{a}{x^2+a^2} = a \cdot \frac{1}{x^2+a^2} = \frac{a}{a^2} \cdot \frac{1}{1+\left(\frac{x^2}{a^2}\right)} = \frac{a}{a^2} \cdot \frac{1}{1-\left(\frac{-x^2}{-a^2}\right)} = \frac{1}{a} \cdot \frac{1}{1-\left(\frac{-x}{-a}\right)^2}$

combine

$a = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{a}\right)^{2n} \cdot \frac{x}{a^2}$ $\qquad b = \frac{1}{a} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{a}\right)^{2n}$ $\qquad \left|-\left(\frac{x}{a}\right)^2\right| < 1 \to \left|\frac{x^2}{a^2}\right| < 1$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{a^{2n+2}}\right) + \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n}}{a^{2n+1}}\right)$

$|x| < a$

so $\boxed{\text{interval: } -a < x < a \quad \text{radius: } a}$

25) Eval indef integral as power series

$\int \frac{t}{1-t^8}\,dt$ Step 1: Express integrand as power series:

$$\frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t \cdot \sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} = 1 + t^9 + t^{17} + t^{25}$$

take integral directly: $\int \sum_{n=0}^{\infty} t^{8n+1} = \boxed{\sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} \qquad \text{radius} = 1}$

29) Approx to 6 decimal places

$\int_0^{0.3} \frac{x}{1+x^3}\,dx$ — first express integrand as power series, then integrate, then use Fundamental Theory of calc, then use AST estimation, Find gen term for AST first

$$\frac{x}{1+x^3} = x \cdot \frac{1}{1+x^3} = x \cdot \frac{1}{1-(-x^3)} = x \cdot \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n+1}$$

So $\int \cancel{\sum_{n=0}^{\infty} (-x^{3n+1})} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + C$ if $x=0, C=0$

$\int \sum_{n=0}^{\infty} (-1)^n x^{3n+1}$

first few terms: $\frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} \dots$

So $\int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n x^{3n+1} = \left. \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} \dots \right]_0^{0.3}$

$$= \frac{0.3^2}{2} - \frac{0.3^5}{5} + \frac{0.3^8}{8} - \frac{0.3^{11}}{11} + \dots + \frac{(-1)^n \cdot 0.3^{(3n+2)}}{3n+2}$$

$$= \frac{(-1)^n\, 3^{(3n+2)}}{10^{3n+2} \cdot 3n+2}$$

So ~~k=2~~

$\int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n x^{3n+1} \approx$

$$\boxed{\frac{0.3^2}{2} - \frac{0.3^5}{5} + \frac{0.3^8}{8}}$$

when $n = 3$, $a_n = \frac{0.3^{11}}{11} \approx$ 0.000000161042 which does not affect 6th decimal place

12) continued)

module 11.10 | 8) Find first four non-zero terms centered at a

Taylor Series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) = \ln x,\ a = 1$

$f(x) = \ln x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2x}{x^4} = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f''''(x) = \frac{-2(3)x^2}{x^6} = \frac{-2(3)}{x^4}$ $\qquad f^{IV}(1) = -6$

$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$

at $x=1$

$\ln(x) = 1(x-a) + \frac{-1}{2}(x-a)^2 + \frac{2}{6}(x-a)^3 + \frac{-6}{24}(x-a)^4$

$$\boxed{= (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{-1}{4}(x-1)^4}$$

9) $f(x) = \sin x;\ a = \frac{\pi}{6}$

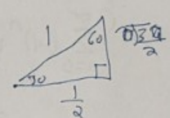
$f(x) = \sin x$ $\qquad f(x) = \frac{1}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{6}\right) = \frac{-1}{2}$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

$f^{IV}(x) = \sin x$ $\qquad f^{IV}\left(\frac{\pi}{6}\right) = \frac{1}{2}$



$f(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{\frac{1}{2}\left(x-\frac{\pi}{6}\right)^2}{2} - \frac{\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^3}{6}$

$$= \boxed{\frac{1}{2} + \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)}{2} - \frac{\left(x-\frac{\pi}{6}\right)^2}{4} - \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)^3}{12}}$$

12) Find Maclaurin Series, and radius of conv

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{IV}(x) = \frac{-2(3(1+x)^2)}{(1+x)^6} = \frac{-6}{(1+x)^4}$ $\qquad f^{IV}(0) = -6$

few terms $= \frac{1x^1}{1} - \frac{-1x^2}{2} + \frac{2x^3}{3 \cdot 2} - \frac{6x^4}{4 \cdot 3 \cdot 2}$

Series $= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(n-1)!\,x^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

14) a) find power rep of $\ln(1-x)$ so... $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x} = -1 - x - x^2 - x^3 \dots$

$\cancel{\sum_{n=0}^{\infty} -x^n}$ $\cancel{|x|<1}$

so $\int \frac{d}{dx}\ln(1-x) = \int -1 - x - x^2 - x^3 dx$

with C= because $\ln(1) = 0$

~~b) find power series f(x) = x ln(1-x)~~ $\cancel{= x\sum_{n=0}^{\infty} -x^n = \sum_{n=0}^{\infty} -x^{n+1}}$

$\cancel{|x|<1}$ ~~radius of conv = 1~~

so $\ln(1-x) = \int(-1 - x - x^2 - x^3, \dots)\,dx$

$= C - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$

$= \sum_{n=1}^{\infty} \frac{-x^n}{n} = \sum_{n=1}^{\infty} (-1)\frac{x^n}{n} = \boxed{\sum_{n=1}^{\infty} \frac{-1}{n} x^n}$ so $|x|<1$

but at -1, this is just an alternating harmonic series so

interval of con: $\boxed{-1 \le x < 1}$

b) find power series: $f(x) = x\ln(1-x)$

$x \cdot \sum_{n=1}^{\infty} \frac{-1}{n} x^n = \boxed{\sum_{n=1}^{\infty} \frac{-1}{n} x^{n+1}}$

c) Find ln2 using a) using $x = \frac{1}{2}$ in result from a)

$\ln(1 - \frac{1}{2}) = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$

So $\ln 2 = -\ln(1 - \frac{1}{2}) = -\sum_{n=1}^{\infty} \frac{-1}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n = \boxed{\sum_{n=1}^{\infty} \frac{1}{n 2^n}}$

15) $f(x) = \ln(5-x)$ so $\frac{d}{dx}\ln(5-x) = \frac{-1}{5-x} = -1 \cdot \frac{1}{5(1-\frac{x}{5})} = \frac{-1}{5} \cdot \frac{1}{1-\frac{x}{5}}$

$= \frac{-1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \frac{-1}{5} + \frac{-x}{25} + \frac{-x^2}{125} + \frac{-x^3}{625} \dots$

$= -\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$

so $\int -\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}(n+1)}$

$\cancel{\text{so} \int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}}$

when $x = 0$, the sum is 0 so

$f(0) = C = \ln 5$ so

$= \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}(n+1)}$

~~radius =~~ $\cancel{-5 < x < 5}$

radius of convergence is the same as $\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ so $|x| < 5$ $\boxed{\text{radius} = 5}$

12) continued)

Find radius of $\ln(1+x)$

so $a_n = \frac{(-1)^{n-1} x^n}{n}$ $\quad a_{n+1} = \frac{(-1)^n x^{n+1}}{n+1}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} x^n}\right|$

$= \left|\frac{x^{n+1} n}{(n+1) x^n}\right| = \left|\frac{xn}{n+1}\right| = |x|$ converges when

$|x| < 1$

$-1 < x < 1$

$\boxed{\text{radius} = 1}$

16) $f(x) = x\cos x$ $\quad$ From ex 6 we know $x\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$

$f(x) = x\cos x$

$f'(x) = x - \sin x + \cos x$

$f''(x) = x - \cos x - \sin x - \sin x$

$= x - \cos x - 2\sin x$

$f'''(x) = x\sin x - \cos x - \cos x - \cos x$

$= x\sin x - 3\cos x$

$f^{iv}(x) = x\cos x + 4\sin x$

$f^{v}(x) = x - \sin x + \cos x + 4\cos x$

$= -\sin x \cdot x + 5\cos x$

$f^{vi}(x) = x - \cos x - \sin x - \sin x + 4 - \sin x$

$-\cos x \cdot x - 6\sin$

$f^{vii}(x) = -\cos x + x - \sin x - 6\cos x$

$= x - \sin x - 7\cos x$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -3$

$f^{iv}(0) = 0$

$f^{v}(0) = 5$

$f^{vi}(0) = 0$

$f^{vii}(0) = -7$

Maclaurin $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

values $= 0 + \frac{1}{1!}x^1 + 0 - \frac{3}{3!}x^3 + 0 + \frac{5x^5}{5!} + 0 - \frac{7x^7}{7!} \cdots$

$= \frac{1}{1!}x^1 - \frac{3}{3!}x^3 + \frac{5x^5}{5!} - \frac{7x^7}{7!} \cdots$

$\boxed{a_n = (-1)^n \frac{(2n+1)\, x^{(2n+1)}}{(2n+1)!}}$

$\boxed{= \frac{(-1)^n x^{(2n+1)}}{(2n)!}}$

Use ratio test to find radius

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{(2n+2)}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{(2n+1)}}\right| = \left|\frac{x}{(2n+2)(2n+1)}\right|$

R.

x can be anything

$\boxed{\begin{array}{c} -\infty < x < \infty \\ \text{Radius} = \infty \end{array}}$

18) $\cosh x = \frac{e^x + e^{-x}}{2}$ $\sinh x = \frac{e^x - e^{-x}}{2}$

$f(x) = \cosh x$ $f(0) = 1$

$f'(x) = \sinh x$ $f'(0) = 0$

$f''(x) = \cosh x$ $f''(0) = 1$

$f'''(x) = \sinh x$ $f'''(0) = 0$

$f^{IV}(x) - \cosh x$ $f^{IV}(0) = 1$

$\frac{d}{dx} \cosh x = \frac{2(e^x - e^{-x})}{2^2} = \frac{e^x - e^{-x}}{2} = \sinh x$

values $= 1 + 0 + \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \frac{1}{6!}x^6 \ldots$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots$

$= \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$

Find radius using ratio

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)}$

< 1 for all x

$\boxed{\text{Radius} = \infty}$

20) Find Taylor series centered at a

$f(x) = x^6 - x^4 + 2$, $a = -2$ again, Taylor Series formula:

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$f'(x) = 6x^5 - 4x^3$ $f(-2) = 50$

$f''(x) = 30x^4 - 12x^2$ $f'(-2) = (6 \cdot -32) + 32$

$f'''(x) = 120x^3 - 24x$ $f''(-2) = (30 \cdot 16) - (12 \cdot 4)$

$f^{IV}(x) = 360x^2 - 24$ $f'''(-2) = (120 \cdot -8) + 48$

$f^{V}(x) = 720x$ $f^{IV}(-2) = (360 \cdot 4) - 24$

$f^{VI}(x) = 720$

$f(-2) = 50$

$f'(-2) = -192 + 32 = -160$

$f''(-2) = 480 - 48 = 432$

$f'''(-2) = -960 + 48 = -912$

$f^{IV}(-2) = 1440 - 24 = 1416$

$\frac{6!\,x^{6-n}}{(6-n)!}$

$f(x) = \sum_{n=0}^{\infty} \frac{6!(-2^{6-n})}{(6-n)!\,n!}(x+2)^n$

$\boxed{\text{Radius} = \infty}$ because polynomial?

30) $\cosh x = \frac{1}{2}(e^{x}+e^{-x}) = \frac{e^x}{2} + \frac{e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

remember $f'(x) = e^x$
$f''(x) = e^x$
etc...
and $f(0)=1$
$f'(0)=1$

$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

when n is odd, this $=0$

n is even, this is

$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\boxed{\text{valid for all } x}$

34) use binomial series to expand to power series, state radius

$f(x) = (1-x)^{3/4} = (1-(-x))^{3/4} = \sum_{n=0}^{\infty} \binom{3/4}{n}(-x)^n$ $\quad k=\frac{3}{4} \quad x=-x$

so radius $|-x|<1 \to |x|<1$ $\boxed{\text{radius} = 1}$

$= 1 - \frac{\frac{3}{4}x}{1!} + \frac{\frac{3}{4}\left(\frac{-1}{4}\right)(-x)^2}{2!} + \frac{\frac{3}{4}\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)(-x)^3}{3!} + \frac{\frac{3}{4}\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)\left(\frac{-9}{4}\right)(-x)^4}{4!} \ldots$

$\frac{\cancel{3 \cdot -1 \cdot -5} - 4(n-1)}{n!\,4^n}(-x)^n$ $\quad \boxed{\frac{3 \cdot -1 \cdot -5 \ldots \overset{-4n+7}{\cancel{3-(4(n-1))}}}{n!\,4^n}(-x)^n}$

$-4(n-2)-1$
$-4n+7$

$\boxed{\text{radius} = 1}$

47) Find Maclaurin and radius; Graph, What do you notice b/w poly and f?

$f(x) = xe^{-x}$ we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ so $xe^{-x} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

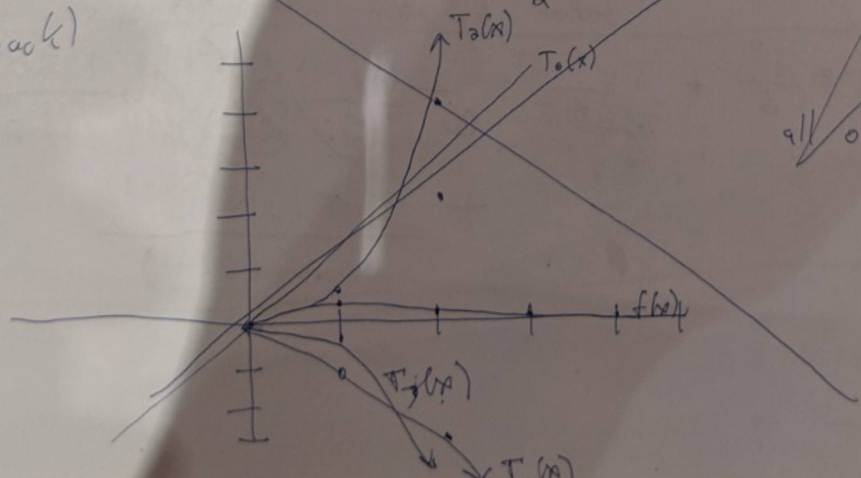
$\boxed{\text{radius} = \infty}$

$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}}$

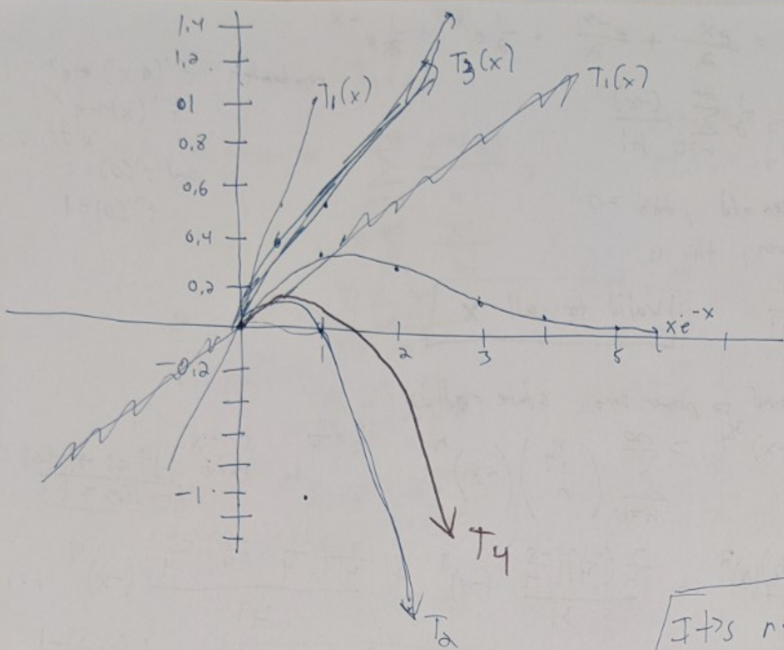
$T_0(x) = 0$ $\quad T_1(x) = -1 \cdot x^2 = -x^2$

$T_2(x) = \frac{x^3}{2}$ $\quad T_3(x) = \frac{-x^4}{6}$

(on back)



all over the place?



$T_0(x) = x$

~~$T_1(x) = x - x^2$~~

~~$T_0(x) = x - x^2 + \frac{x^3}{2}$~~

$T_0(x) = 0$

$T_1(x) = x$

$T_2(x) = 0 + x + 0 - x^2$

$T_3(x) = x - x^2 + \frac{x^3}{2}$

$T_5(x) = x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24}$

$\boxed{\text{It's not close?}}$

51) a) Binomial expand $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$ $\quad k = \frac{-1}{2} \quad x = -x^2$

$$\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{-x}{2}\right)^n = 1 + \underbrace{-\frac{1}{2}(-x^2)}_{kx} + \underbrace{\frac{-\frac{1}{2} \cdot \frac{-3}{2}}{2}}_{\frac{k(k-1)}{2!}} (-x^2)^2 + \underbrace{\frac{-\frac{1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2}}{6}}_{\frac{k(k-1)(k-2)}{3!}} (-x^2)^3$$

$$\boxed{\sum_{n=0}^{\infty} = \frac{1.3.5 \ldots 2n-1}{2^n n!} x^{2n}} \boxed{= 1 + \frac{x^2}{2} + \frac{1.3\, x^4}{2 \cdot 2^2} + \frac{1.3.5\, x^6}{6 \cdot 2^3} \ldots \frac{1.3.5(2n-1)}{2^n n!} x^{2n}}$$

b) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ $\quad$ so $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x$

term by term integration

$$\boxed{= C + x + \frac{x^3}{6} + \frac{3x^5}{2 \cdot 2 \cdot 5} + \frac{1.3.5\, x^7}{3!\, 2^3\, 7} + \frac{1.3.5 \ldots x^{(2n+1)}}{n!\, 2^n (2n+1)}}$$

Module 8 1111)

4) $f(x) = xe^{-2x}$ find $T_3(x)$

so e^x we know is $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ $\quad f'(n) = e^x = e^x \quad f'(0) = 1$

$f''(n) = e^x = e^x \quad f''(0) = 1$ etc

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

so $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$ so $e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

$xe^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1} x^n}{(n-1)!}$

$T_0(x) = 0 \quad T_1(x) = x$

$T_2 = x - 2x^2$ $\quad \boxed{T_3 = x - 2x^2 + \frac{4x^3}{2}}$

16) $f(x) = \sin x$, at $a = \frac{\pi}{6}$, $n = 4$ degree, $0 \le x \le \frac{\pi}{3}$

a) $f'(x) = \cos x \quad f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \quad f''(\frac{\pi}{6}) = \frac{-1}{2}$

$f'''(x) = -\cos x \quad f'''(\frac{\pi}{6}) = \frac{-\sqrt{3}}{2}$

$f^{iv}(x) = \sin x \quad f^{iv}(\frac{\pi}{6}) = \frac{1}{2}$

$f^{v}(x) = \cos x \quad f^{v}(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

T Series at $\frac{\pi}{6}$ of 4th degree:

$$\boxed{T_4(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) + \frac{\frac{-1}{2} \cdot (x - \frac{\pi}{6})^2}{2!} + \frac{\frac{-\sqrt{3}}{2} \cdot (x - \frac{\pi}{6})^3}{3!} + \frac{\frac{1}{2} \cdot (x - \frac{\pi}{6})^4}{4!}}$$

b) Taylor inequality to estimate $T_4(x)$ above.

Error in approx $\sin x$ at $\frac{\pi}{6}$ by the first 4 terms is at most

so If $0 \le x \le \frac{\pi}{3}$ ~~error is smaller than~~

so M = 1

≈ 0.0000247850

2.47850×10^{-5}

$R4(x) = $ $f^{v}(x) = \cos x$

$f^{v}(x)$ is largest at 0, $\cos 0 = 1$

$\boxed{3.28 \times 10^{-4}} = \boxed{R_4(x) \le \frac{1}{5!} (\frac{\pi}{6})^5}$

24) estimate $\sin 38°$ - in radians is $\sin \frac{38}{180}\pi = \sin(\frac{19}{90}\pi)$ which uses $\frac{\pi}{6}$ as the estimation point

plugging into desmos $\approx \boxed{0.61563 \cdot 140736}$

(Error on back)

24) continued) $38° = 30° + 8° = \frac{\pi}{6} + \frac{4\pi}{90} = \frac{19\pi}{90}$ radians $\quad \frac{\pi}{6} = \frac{15}{90}\pi$

$$R_4\left(\frac{\pi}{6} + \frac{2\pi}{45}\right) \le \frac{M}{5!}\left(\frac{19\pi}{90} - \frac{15\pi}{90}\right)^5 =$$

$x - a$

$f^v(x) = \cos x \qquad 0 \le x \le \pi/3$

so $M = 1$

$$R_4\left(\frac{19\pi}{90}\right) = \frac{1}{5!}\left(\frac{4\pi}{90}\right)^5 = 4.4 \times 10^{-7}$$

so the estimate to 5 decimal places is:

$\boxed{0.61566}$

26) $\ln(1+x)$

$\ln(1.4) = \ln(1+0.4)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3}$

$f^{iv}(x) = \frac{-2(3(1+x)^2)}{(1+x)^6} = \frac{-6}{(1+x)^4}$

$\cancel{f'(0.4)}$

$\cancel{\sum_{n=0}^{\infty} \frac{(n-1)!\,(-1)^{n-1}}{n!\,(1+x)^n}(x)^n}$

$f'(0) = 1$

$f''(0) = -1$

$f'''(0) = 2$

$f^{iv}(0) = -6$

$f^n(0) = (-1)^{n-1}(n-1)!$

Maclaurin so $a = 0$

so $\ln(1+x) = \sum_{n=1} \frac{(-1)^{n-1}(n-1)!}{n!} x^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

so $\sum_{n=0}^{\infty} = \frac{(-1)^n}{n+1} x^{n+1}$

$= x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$

Alternating series so use the estimation theorem

so $|R_n| = |s - s_n| \le b_{n+1}$

$$\ln(1.4) = \ln(1+0.4) = 0.4 + \frac{0.4^2}{2} + \frac{0.4^3}{3} + \frac{0.4^4}{4}$$

$\frac{0.4^5}{5} = 0.002048$

$\frac{0.4^6}{6} = 0.0006 < 0.001$

So $R_5 \le \frac{0.4^6}{6} < 0.001$

needs $\boxed{5 \text{ terms}}$