Module 5

SECTION 11.1: SEQUENCES

In the introduction on ***page 693***, the author indicates that sequences and series are important in calculus because of the idea of representing functions as sums of infinite series. To make this a bit more concrete, look ahead to the table on ***page 768*** and note the five examples of functions expressed as infinite series. The second example,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

indicates that the function e^x can be expressed as the sum of terms of the form, $\frac{x^n}{n!}$. Since this is a totally new idea, you may have some doubts. In fact most students probably feel very uneasy about this idea.

It may help a bit if we let $x = 1$ and check the result. Note that we have included more terms.

$$e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \ldots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots$$

$$= 1 + 1 + .5 + .166\bar{6} + .041\bar{6} + .008\bar{3} + \ldots$$

$$e \approx 2.71\bar{6}$$

We have used just 6 terms to get this approximation. If we use 4 more terms,

$$e \approx 2.71\bar{6} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!}$$

$$e \approx 2.718281526$$

The value of e to 20 decimal places is

$$e \approx 2.71828182845904523536$$

So with 10 terms we are accurate to 6 decimal places. By adding 3 more terms, we are accurate to 9 decimal places.

$$e \approx 2.718281828$$

Greater accuracy can be achieved by adding more terms.

It is important in this example to make a distinction between the set

$$1 , \frac{1}{1!} , \frac{1}{2!} , \frac{1}{3!} , \frac{1}{4!} , \ldots$$

and the sum,

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

The first is a *sequence* and the second is a *series*. The first section in this chapter deals only with sequences. Later we show how series can be related to sequences.

Also we have shown just one use of the infinite series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

representing e^x. So don't be disheartened if you have only a vague idea of its significance. Just read carefully what follows.

TYPES OF SEQUENCES

Look carefully at the sequences in ***Examples 1*** and ***2***, ***page 694***. It is important to note that a sequence follows a definite pattern.

VISUALIZING THE PATTERN

This first section covers several basic concepts involving sequences that will be important later. If you can form mental pictures related to these concepts, they will make much more sense. Look carefully at ***Figures 1*** and ***2*** and see how they match $a_n = \frac{n}{n + 1}$. Do this slowly for $n = 1$, then $n = 2$, etc., and visualize a mental picture that shows the *pattern* of the sequence. The most demanding part of this section is the **(12)** theorem on ***page 702***. The "proof" may be hard to follow, but if you can picture the two major components, increasing and bounded (both of which are shown in ***Figures 1*** and ***2***), you will get the feeling that, of course, it is true. So don't just *glance* at the figures in this section, but use them as a basis for mental pictures that you will create from *verbal* descriptions later.

DEFINITION OF A LIMIT

If you look back on Calculus I, the δ, ε definition of a limit was definitely in the background. The definition **(2)** on ***page 696*** is different. First of all, recognize that the starting point is $\lim_{n\to\infty} a_n = L$. The definition indicates what this means in symbolic language. A verbal description is

a_n gets closer and closer to L as n gets larger and larger.

But how close and how large?

In the definition, ε indicates how close and N indicates how large.

The key to understanding is deciphering the meaning of

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$

First, remember this is true for *any* $\varepsilon > 0$. By making ε smaller and smaller,

the difference, $a_n - L$, must get smaller and smaller,

or a_n gets closer and closer to L.

Next, rewrite $|a_n - L| < \varepsilon$ whenever $n > N$

as $-\varepsilon < a_n - L < \varepsilon$ whenever $n > N$

or $L - \varepsilon < a_n < L + \varepsilon$ whenever $n > N$

For *each* ε (how close) we have to find an *N* (how large) so that ALL a_n are in the interval from $L\ \varepsilon -$ to $L\ \varepsilon +$ when n is larger than N. ***Figures 3*** and ***4*** provide a visualization of the process.

It is important that you can agree with the word ALL.

1. Suppose one a_n is not in the interval, say a_K is not in the interval. Then we have selected the wrong *N*. The new *N* should be $K + 1$. Then ALL of the terms

$$a_{K+1}\ ,\ a_{K+2}\ ,\ a_{K+3}\ ,\ \ldots$$

will be in the interval.

2. Or suppose a finite number of terms are not in the interval. Pick that one with the largest subscript and label it K again. Repeat the above argument, and you will arrive at the same conclusion.

3. Now suppose there are an infinite number of terms not in the interval. Then we can't find an N AND we conclude that

$$\lim_{n\to\infty} a_n = L$$

is *not* true. The a_n's are not approaching L because some are not "close" to L as $n \to \infty$ or as n gets larger and larger.

Why are we putting so much stress on this? Because there is a mental picture involved that will support an intuitive feeling that ideas related to sequences make sense.

POINTS VERSUS A SMOOTH CURVE

A distinction must be made between the function, $y = \dfrac{1}{x}$, and the sequence, $\{a_n\}$ where $a_n = \dfrac{1}{n}$. The graph of the first is a smooth curve while the graph of the second is an isolated set of points. For $a_n = \dfrac{1}{n}$, the domain is $\{1, 2, 3, \ldots\}$. The first three points are

$$(1, 1),\quad (2, 1/2),\quad (3, 1/3)$$

Between these points nothing appears in the graph of $a_n = \dfrac{1}{n}$. Why is this important? The derivative has no meaning for a set of isolated points.
The derivative is the limit, $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

The numerator is undefined when $x + h$ and x are not integers. We can't find the limit of values that are undefined.

Theorem **(3)**, ***page 697***, fills the gap by replacing n with x, which in effect creates a smooth curve between the isolated points. If the limit exists for *f(x),* it will also exist for $a_n = f(n)$. Hence in ***Example 6***, L'Hopital's Rule is used to find the limit.

THEOREM (6) (PAGE 698)

We illustrate the theorem with an example. Suppose $a_n = \frac{(-1)^n}{n}$. The sequence is $\left\{-1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ \ldots\right\}$ Note that the signs alternate.

Then $\left|a_n\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$. Theorem **(6)** says that since

$$\lim_{n\to\infty} \left|a_n\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

we can conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n} \quad \text{also equals zero.}$$

You can gain a strong intuitive feeling for this result if you consider a graph of each sequence. For

$$\left\{\left|a_n\right|\right\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right\}$$

the terms, $\frac{1}{n}$, start at 1 and march down toward zero. All terms are to the right of zero but eventually get arbitrarily close to zero. For

$$\left\{a_n\right\} = \left\{-1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ \ldots\right\}$$

the terms, $\frac{(-1)^n}{n}$, appear on both sides of zero but also eventually get arbitrarily close to zero.

Again we are stressing a mental picture, not as a substitute for a proof but as a mental process that will enable you to gain more insight.

EXAMPLE 10

In this example, a comparison is made between $n!$ and n^n. Since

$$n! = n(n-1)(n-2)(n-3) \ \dots \ 2 \cdot 1$$

and
$$n^n = n \cdot n \cdot n \dots n \quad (n \text{ times})$$

$$a_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \quad \dots \quad \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}$$

After $\frac{n}{n}$ each fraction is less than one. Multiplying by fractions less than one results in a product that gets smaller and smaller. Consider

$$a_{10} = \frac{10}{10} \cdot \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} \quad \dots \quad \frac{3}{10} \cdot \frac{2}{10} \cdot \frac{1}{10}$$

Note that $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100}$ and that the product is smaller than either factor. In fact

$$a_{10} = (.0036288) \cdot \frac{1}{10}$$

So
$$a_{10} < \frac{1}{10}$$

which supports the claim in the text that $a_n \leq \frac{1}{n}$. Because $\frac{1}{n}$ approaches zero, a_n also approaches zero.

EXAMPLE 7

There are two ways a sequence can fail to converge. First it may diverge because a_n approaches infinity instead of a finite number. The second way is illustrated in this example. It may fail to converge to a finite number because it *oscillates* between *-1* and *+1*.

EXAMPLE 11

Remember the limit,

$$\lim_{n \to \infty} r^n = 0 \quad \text{if} \quad -1 < r < 1$$

This won't be surprising if you look at an example. Say $r = 2/3$. Then

$$a_n = \left(\frac{2}{3}\right)^n = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} \quad \cdots \quad \frac{2}{3}$$

Each time you multiply by $2/3$ you are finding two-thirds of the previous result. As n approaches infinity, the products must approach zero.

If $r = -2/3$, signs will alternate but the magnitudes will still approach zero.

Bounded and Monotonic

First note the definition ***(10)*** of a monotonic sequence on ***page 700***. It just means a sequence is either increasing or decreasing but not doing both.

In ***Example 12***, we can show the sequence is decreasing by just noting which denominator is larger. In solution 1 for ***Example 13***, it is not as simple. In comparing a_n and a_{n+1}, the fraction with the larger denominator also has a larger numerator. A more complex algebraic procedure must be used.

Also note the method used in solution 2. A negative derivative that matches a negative slope for the tangent line indicates the sequence is decreasing. All three of these techniques will be used later in this chapter.

Next carefully read the definition of a bounded sequence. An upper bound is just a number that is larger than (or equal to) *every* member of the sequence. Similarly a lower bound is a number smaller (or equal to) every member of the sequence.

For the sequence whose *n*th term is $\dfrac{(-1)^n}{n}$

$$\left\{-1,\ 1/2,\ -1/3,\ 1/4,\ \ldots\right\}$$

the number 1 is an upper bound and the number -1 is a lower bound. But we can also say that 3/4 is an upper bound and you may note that 1/2 is an upper bound. Which leads to the important idea of a *least upper bound*, the smallest member of the set of upper bounds. For the above sequence, 1/2 is the least upper bound. (Note also that this sequence is not monotonic.)

Theorem **(12)**, ***page 702***, asserts that if a sequence is bounded and monotonic, then it is convergent. We hope that you are inclined to follow the proof carefully. Also we hope that you can form a mental picture that indicates the theorem is true.

Consider a decreasing sequence that has a lower bound. Then it has a greatest lower bound by the Completeness Axiom. As n increases the terms, a_n, are marching to the left, but they can not go beyond the greatest lower bound, call it B. They will have to start "bunching" up to keep moving left but not go beyond B. Can you imagine how the sequence could diverge? The proof on ***page 702*** for an increasing sequence essentially carries forth this type of reasoning in symbolic form.

LAST COMMENT

Reviewing the sequences in this section, you will find six "components"

(A) $$\ln n \ , \ n \ , \ n^2 \ , \ r^n \ , \ n! \ , \ n^n$$

that were building blocks to form other sequences. As n approaches infinity each one, except r^n, also approaches infinity. But if we form quotients, such as $\dfrac{\ln n}{n}$ (***Example 6***) or $\dfrac{n!}{n^n}$ (***Example 10***), the sequences approached zero. These limits were zero because the denominators increased faster than the numerators.

In the above listing, for r^n, assume $r > 1$. Then the above list (A) has the "slowest" entry on the left, and each entry to the right is "faster" in terms of how fast elements of the sequence approach infinity. Sequences such as

$$\frac{n^2}{2^n} \ , \ \frac{3^n}{n!} \ , \ \frac{\ln n}{n^2}$$

where the denominator is to the right of the numerator in the above list, will approach zero. For the sequence whose nth term is $\dfrac{3^n}{n!}$, use the method shown in ***Example 10***.

For the other two, change to $\dfrac{x^2}{2^x}$ and $\dfrac{\ln x}{x^2}$ and use L'Hopital's Rule to show that the limits are zero.

The significance of these last comments will become more apparent later.

SECTION 11.2: SERIES

We now connect concepts involving *sequences* to the new topic, *series*. The first important distinction between the two is that a series is the *sum* of the members of a sequence. This may seem like a daunting task, to add an infinite number of terms. And it is. So we start with a more modest goal. We look at the sum of just the first n terms. As always, notation is important. The symbol, s_n , denotes the *sum of the first n terms*.

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n$$

where the $a_i{}'s$ are members of some sequence, $\{a_n\}$. s_n is called a *partial sum*. A possible confusion arises when we note that the partial sums, s_n, also form a sequence. If you sense this is a bit complicated, you are right. However, the good news is that this topic forms an intermediate step, which is used to develop other tools. Later we won't be too concerned with partial sums but will concentrate on using the new tools. Now, however, you need to pay attention to understand the underlying foundation.

DEFINITION (2)

A key to understanding the definition ***(2)*** on ***page 708*** is being able to form a mental picture of "If the sequence $s_n\{\ \}$ is convergent". It may help to look at the table on this page. Each entry in the right column is a partial sum of the series,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n}$$

The seventh entry, $n = 7$, matches

$$s_7 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

and a calculator verifies that this equals .9921875 as shown in the table. The last entry is for the partial sum where $n = 25$.

$$s_{25} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{25}}$$

The last term in this partial sum, $\frac{1}{2^{25}}$, is approximately .00000003. Adding this term has no effect on the first six decimal places. The fact that the amount being added is very small is shown again by

$$s_{26} = s_{25} + \frac{1}{2^{26}}$$

The extra term, $\frac{1}{2^{26}}$, is approximately .000000015, which also has no effect on the first six decimal places.

So the table suggests that the sequence of partial sums does converge and

$$\lim_{n\to\infty} s_n = 1$$

In the definition, 1 is the value for s, the sum of an infinite number of terms. The claim is that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 1$$

or
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Again we emphasize that this example illustrates the concept of partial sums and the words *the series converges*. In most cases we will not find the sum, s, but only determine whether or not the series converges.

A WORD OF CAUTION

The above discussion contains a potential implication that is important to look at. "If we add smaller and smaller numbers, the sequence of partial sums will converge". It is important to realize this is *not* necessarily true.

The series,
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$$

is going to be a very significant series. The amount being added, $\frac{1}{n}$, will certainly get smaller and smaller. But there is a huge difference between this series and the series in the last example. We noted above that the last term in the partial sum, s_{25}, was $\frac{1}{2^{25}} \approx .00000003$. For the series,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$$

the partial sum, s_{25}, is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{25}$$

Now the last term added is $\frac{1}{25} = .04$ which is a much larger number than $.00000003$. In fact we have to go to the $33{,}333{,}333$th term to get a term that is this small.

$$\frac{1}{n} = \frac{1}{33{,}333{,}333} \approx .00000003$$

But you might say we are adding an *infinite* number of terms. It takes longer for the terms being added to get small, but the effect will be the same. Wrong. This series *diverges,* as is shown in ***Example 9***. At this point we make two comments.

1. Even if a_n approaches zero, a series may not converge. (***Note 2, page 713***)

2. Using a calculator may not tell you whether a series converges or diverges. Try using a calculator to evaluate

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots$$

for larger and larger values of n and see if a pattern appears.

THE GEOMETRIC SERIES

The *geometric series* on ***pages 710–11*** is one of two basic series developed in this section. The other is the *harmonic series* on ***page 713***. Each of these series will be used many, many times in comparison tests that will be covered later in this chapter.

The geometric series has the form

$$a + ar + ar^2 + ar^3 + \ \ldots. \ = \sum_{n=1}^{\infty} ar^{n-1}$$

and converges if $|r| < 1$. If it converges, we also know its sum, which has the fairly simple form $\dfrac{a}{1-r}$. Both of these assertions are based on an algebraic trick using the partial sum,

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

Multiply both sides by r and line up similar terms.

$$rs_n = \qquad ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

Subtracting the second equation from the first eliminates all but four terms and leads to a formula for a partial sum.

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

On ***page 700***, it was established that $r^n \to 0$ if $|r| < 1$, so the limit of the partial sums is $\frac{a}{1 - r}$. This means that the sum of the entire infinite geometric series is

$$\frac{a}{1 - r}.$$

A major difficulty will be picking out the form that indicates you are dealing with a geometric series. Basically look for ar^n. In ***exercises 9*** through ***14*** on ***page 715***, ***# 9*** and #***12*** are geometric series. Is ***# 34*** a geometric series? No, but it is the difference of two geometric series.

THE HARMONIC SERIES

The second special series in this section is called the *harmonic series*. Its form is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

It will be important to convince yourself that this series diverges. Even though the amount being added, $\frac{1}{n}$, becomes smaller and smaller, the total sum of the terms in the series gets larger and larger.

Note the pattern in the solution in ***Example 9, page 713***.

1. $\frac{1}{3} + \frac{1}{4}$ is replaced with $\frac{1}{4} + \frac{1}{4}$, which equals $\frac{1}{2}$.

2. $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ is replaced with $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

3. $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$

   is replaced with

   $$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$$

Each time we need twice as many terms to get $\frac{1}{2}$, but since there are an infinite number of terms in the series, we will never run out of terms.

The second significant point is that the harmonic series

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \ldots$$

is larger than

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{16} + \ldots + \frac{1}{16}\right) + \ldots$$

This means that if the sum of the terms in the second series gets larger and larger, then the sum of the terms in the first series, the harmonic series, will be larger still.

The conclusion is that the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

is larger than

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

where the adding of another 1/2 goes on forever. The second series approaches infinity, so the harmonic series also approaches infinity.

This analysis provides a significant result which is used in the next four sections.

A Test for Divergence

A major task in working with infinite series is to determine if they converge or diverge. This is the major topic of the next five sections. A basic test for divergence ***(7)***, ***page 713***, is to show that the nth term, a_n, does not approach zero. Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{2n-1} = \frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \ldots$$

We can get a sense of the behavior of the series by letting $n = 100$ and $n = 1000$.

$$\sum_{n=1}^{\infty} \frac{n}{2n-1} = \frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \ldots + \frac{100}{199} + \ldots + \frac{1000}{1999} + \ldots$$

Now we can see that the nth term is approaching 1/2. A better approach is to find the limit of the nth term.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-1/n} = \frac{1}{2}$$

If you think that the series *approaches* the following sum,

$$\sum_{n=1}^{\infty} \frac{n}{2n-1} = \frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \ldots + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

then it cannot converge. Eventually the amount we are adding *each* time another term is included is approximately 1/2. Adding 1/2 an infinite number of times means the series must diverge.

COMMENTS

1. The series in ***Example 7***,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

is a different type of series. It contains a variable while all of the other series contain only constants. This type of series is called a *power* series and will be covered extensively in the last five sections of this chapter. Despite this difference, hopefully you recognized that ***Example 7*** is a geometric series. This example could have been solved in reverse. Start with

$$\frac{1}{1-x}$$

and perform a long division. The first few terms of the series will appear, but a question arises when we equate the fraction to an infinite number of terms.

2. ***Example 8*** contains a special type of series. By rewriting $\dfrac{1}{n(n+1)}$ as $\dfrac{1}{n} - \dfrac{1}{n+1}$, the partial sum, s_n, simplifies to $1 - \dfrac{1}{n+1}$. Then remember that if the series converges, the limit of the partial sums as n approaches infinity is the sum of the series. Because the limit is one, the sum of the series is one.

This is the second type of series where we can actually find the *sum*. The first was a convergent geometric series whose sum was $\dfrac{a}{1-r}$. In the following sections the emphasis is on whether a series converges or diverges, and in a few cases we *approximate* the sum.

3. Reread notes **2** and **3**, ***page 713-14***.

4. The summation notation in Theorem **(8)**, ***page 714***, tends to hide information. If you write out a few terms of the sum, you can readily agree that each equation is true.

Part (i): $$\sum_{n=1}^{\infty} c a_n = c a_1 + c a_2 + c a_3 + \dots$$

Now you can see that c is a common factor in all terms.

$$\sum_{n=1}^{\infty} c a_n = c(a_1 + a_2 + a_3 + \dots)$$
$$= c \sum_{n=1}^{\infty} a_n$$

Now suppose $\sum a_n$ converges to s. Then $\sum c a_n$ converges to c times s.

Part(ii):

$$\sum_{n=1}^{\infty}(a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots$$

Now rearrange the a_i's and the b_i's.

$$= (a_1 + a_2 + a_3 + \dots) + (b_1 + b_2 + b_3 + \dots)$$
$$= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Intuitively you can now see that if both series converge, then the sum also converges.

5. Read note **4**, ***page 715***. The major point here is that the "infinite tail" of a series determines whether or not it converges, not the first few terms. The geometric series, $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$, converges because $r = \frac{3}{4}$ and $\frac{3}{4}$ is less than one.

The first term, a, equals $\frac{3}{4}$, so the sum of the series is $\dfrac{3/4}{1 - 3/4}$, which equals 3.

Next consider the series, $\sum_{n=3}^{\infty}\left(\frac{3}{4}\right)^{n}$. It also converges because r is still $\frac{3}{4}$, but now the first term is $\left(\frac{3}{4}\right)^{3}$, so the sum is

$$\frac{\left(3/4\right)^{3}}{1-3/4} = \frac{27}{16}$$

We could also find this sum by noting that

$$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n} = 3 = \frac{3}{4}+\left(\frac{3}{4}\right)^{2}+\sum_{n=3}^{\infty}\left(\frac{3}{4}\right)^{n}$$

Then subtract $\frac{3}{4}+\frac{9}{16}$ from 3 to get the same result.