Lesson 8

Reading: Larson, Section 4.1, Vectors in R^n; Section 4.2, Vector Spaces; Section 4.3, Subspaces of Vector Spaces

Suggested exercises: Larson, Section 4.1: 7, 13, 21, 23, 33, 37, 39, 45 ; Section 4.2: 1, 21, 23; Section 4.3: 1, 7, 13, 37, 39, 41

Submit: Lesson 8: Vector operations, vector spaces, and vector subspaces

Section 4.1: Vectors in R^n

In this chapter, the idea of a vector space is introduced. You should read through the examples given in the text, so you have an idea that the idea of vector space is quite general, but we will focus on R^n in this course.

There are two arithmetic operations introduced in this section: vector addition and scalar multiplication. The rules for these operations are essentially the same as for addition and scalar multiplication of matrices. (In fact, you can consider a vector in R^n as a matrix: either a row matrix or a column matrix, depending on how you write it.)

The rules obeyed by these operations are listed in this section, and you should go over them, but they will all seem familiar to you, since they are not significantly different from the rules of arithmetic with real numbers. Note that there is a zero vector in every vector space, and we denote it in boldface (like this: $\mathbf{0}$) to distinguish it from the scalar zero, which is denoted by 0. The zero vector in R^n is an n-tuple of zeros.

The term *linear combination* is introduced in this section (and will be re-introduced in section 4.4). A linear combination of vectors is a sum of scalar multiples of those vectors. For example, $2\mathbf{v}_1 - \mathbf{v}_2 + 5\mathbf{v}_3$ is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Writing a vector as a linear combination of other vectors is a matter of solving a linear system. For example, the vector $\mathbf{x} = (2, 3)$ in R^2 can be expressed as a linear combination of $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (3, 2)$, as follows: first, set $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$, or

$$(2, 3) = a(1, 1) + b(3, 2),$$

and then write the equation for each component of the vectors:

$$\begin{aligned} a + 3b &= 2 \\ a + 2b &= 3. \end{aligned}$$

Proceed as usual to solve this system: subtract the first equation from the second to get

$$\begin{aligned} a + 3b &= 2 \\ -b &= 1 \end{aligned}$$

and back-substitution gives us $b = -1$ and $a = 5$. Thus, the linear combination is

$$(2, 3) = 5(1, 1) + (-1)(3, 2),$$

or

$$\mathsf{x} = 5\mathsf{u} - \mathsf{v}.$$

Section 4.2: Vector Spaces

We will focus on R^n on this section, but you should read the other examples as well, if only to shake off any bad habits you might have of thinking of vectors in terms of arrows. The arrow representation can be helpful in R^2 and R^3 in some situations, but in most vector spaces, an arrow just doesn't make sense. For example, the set of all continuous functions on the real line can be made into a vector space (and is one of the examples in this section), so what arrow would you draw to represent x^3 or $\sin x$?

There are examples given of sets which do not qualify as vector spaces. You should pay attention to these, because understanding why a definition fails to apply can be as valuable as understanding why it does.

Section 4.3: Subspaces of Vector Spaces

If V is a vector space, then a subset W of V is called a *subspace* of V if W is itself a vector space, if we use the same operations used in V. For example, in the vector space R^3, the subset W consisting of all vectors of the form $(x, y, 0)$ is a subspace of R^3. (This subset is essentially just R^2 embedded in R^3.) So, do we have to go through all of the axioms of vector spaces to verify that this is a subspace? Fortunately, no, thanks to a theorem appearing in this section. There is a test that can be applied to determine if a nonempty subset W of a vector space V is a subspace of V: (1) W must be closed under vector addition, and (2) W must be closed under scalar multiplication. This means that (1) the vector sum of any two elements in W must also be in W, and (2) if v is in W and c is a scalar, then $c \cdot \mathsf{v}$ must also be in W.

Therefore, in R^3, the nonempty set W consisting of vectors of the form $(x, y, 0)$ can be seen to be a subspace as follows: first, if $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ are vectors in W, then their sum is $(x_1 + x_2, y_1 + y_2, 0)$, which is an element of W. Next, if $(x, y, 0)$ is in W and c is any real number, then $c \cdot (x, y, 0) = (cx, cy, 0)$, which is in W. And that is all that needs to be done: W is a subspace of R^3.

Here's an example of a subset that fails to be a subspace: In R^2, let W be the line with equation $y = x + 2$. A typical element is $(x, y) = (x, x + 2)$. Suppose (x_1, y_1) and (x_2, y_2) are elements of W. Then $(x_1, y_1) = (x_1, x_1 + 2)$ and $(x_2, y_2) = (x_2, x_2 + 2)$. If we add these vectors together, we get

$$(x_1, y_1) + (x_2, y_2) = (x_1, x_1 + 2) + (x_2, x_2 + 2) = (x_1 + x_2, x_1 + x_2 + 4).$$

This vector is *not* a member of W: its y-component is four more than its x-component, not two more, so it does not lie on the line $y = x + 2$. Therefore, W is *not* closed under vector addition, and is therefore not a subspace of R^2.

This is enough to show that W is not a subspace of R^2, but we'll also show that it is not closed under scalar multiplication. Let's choose the scalar $c = 0$. For a typical member $(x, x + 2)$ of W, we have

$$c \cdot (x, x + 2) = 0 \cdot (x, x + 2) = (0, 0),$$

and the y-component is *not* two more than the x-component, so $(0, 0)$ does not lie on the line $y = x + 2$, and therefore W is not closed under scalar multiplication, either.