4a) carrying capacity 6000, $k=0.0015$, write logistic diff equation

a) $\boxed{\frac{dP}{dt} = 0.0015\,P\left(1-\frac{P}{6000}\right)}$

e) initial pop = 1000, write formula for pop after t years.

so find $P(t)$

$$P(t) = \frac{6000}{1+Ae^{-kt}} \quad \text{w/ } A = \frac{6000-1000}{1000} = 5$$

$$\boxed{= \frac{6000}{1+5e^{-kt}}}$$

5) like example 2

$$\frac{dy}{dt} = ky\left(1-\frac{y}{M}\right) \qquad M = 8\times10^7\,kg \qquad k = 0.71$$

a) ~~for~~ $y(0) = 2\times10^7\,kg$, find $y(1)$

$$y(t) = \frac{8\times10^7\,kg}{1+3\,\cancel{\times10^7\,kg}\,e^{-kt}} \qquad A = \frac{8\times10^7\,kg - 2\times10^7\,kg}{2\times10^7\,kg} = 3\ \cancel{\times10^7\,kg}$$

so $y(0) = 2\times10^7\,kg$ ✓

$$y(1) = \frac{8\times10^7\,kg}{1+3e^{-0.71}} \approx 3.23\cdot10^7$$

b) how long to reach $4\times10^7\,kg$

$$4\times10^7\,kg = \frac{8\times10^7\,kg}{1+3e^{-.71t}}$$

$$1+3e^{-.71t} = 2$$

$$3e^{-.71t} = 1 \rightarrow e^{-.71t} = \frac{1}{3} \rightarrow -.71t = \ln\frac{1}{3}$$

$$.71t = \ln 3$$

$$t = \boxed{\frac{\ln 3}{.71} \approx 1.547}$$

6) Suppose $P(t)$ satisfies

a) $\frac{dP}{dt} = 0.4P - 0.001P^2 \quad P(0) = 50$

$= 0.4P\left(1-\frac{P}{400}\right)$ so $\boxed{M=400}$

b) $A = \cancel{400-50}\ \frac{M-P_0}{P_0} = \frac{400-50}{50} = 7$

$$P(t) = \frac{400}{1+7e^{-kt}} \qquad k = 0.4$$

so $P(0) = \frac{400}{8} = 50$

so $P'(0) = 17.5$

c) $\frac{M}{2} = 200$

$200 = \frac{400}{1+7e^{-kt}} \rightarrow 200 + 1400e^{-kt} = 400$

$1400e^{-kt} = 200$

$e^{-kt} = \frac{1}{7}$

$-kt = \ln(\frac{1}{7}) \quad k = 0.4$

$t = \frac{-1}{0.4}\ln(\frac{1}{7}) \approx 4.86$

$P(t) = \frac{M}{1+Ae^{-kt}}$

k = constant

t = time

7) $P_0 = 1000$

$M = 10{,}000$

$P_1 = 2500$ find P_4

$A = \frac{M - P_0}{P_0} = \frac{10000 - 1000}{1000} = 9$

$P(t) = \frac{10000}{1+9e^{-kt}}$

$P(1) = 2500 = \frac{10000}{1+9e^{-k}} \rightarrow 2500 + 22500e^{-k} = 10000$

$e^{-k} = \frac{1}{3}$

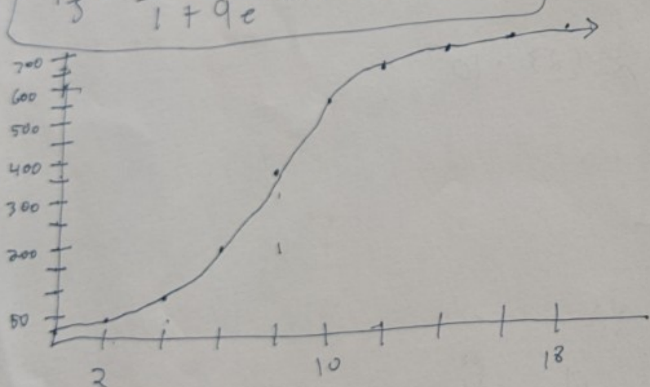
$-k = \ln(\frac{1}{3})$

$k = -\ln(\frac{1}{3})$

$= \ln(3)$

$$\boxed{\begin{aligned} P_4 &= \frac{10000}{1+Ae^{-\ln(3)4}} \\ &= \frac{10000}{1+9e^{-\ln(3)4}} \approx 10000 \\ P_3 &= \frac{10000}{1+9e^{-\ln(3)\cdot 3}} \approx 10000 \end{aligned}}$$

8)



a) $\boxed{M \approx 700}$

$e^{kt} = 2 \quad kt = \ln(2)$ doubles every 2 hrs so $t = 2$ $\quad k = \frac{\ln(2)}{2}$

b) ~~$k \approx 2$ at the beginning~~

c) exponential model $= kP = 2 \cdot 18$ $\frac{dP}{dt}$

$= 18 \cdot e^{\frac{\ln 2 t}{2}} = \boxed{18 \cdot 2^{\frac{t}{2}}}$

logistical model $= P(t) = \frac{700}{1+37.88e^{\frac{\ln 2 t}{-2}}}$

$A = \frac{700-18}{18} = 37.88$

$= \frac{700}{1+37.88 \cdot 2^{-\frac{t}{2}}}$

d) exponential does well at the beginning but overestimates at around $t = 10$. $P(t) = 509$

but exponential model $= 576$

log model $= \frac{700}{1+37.88 \cdot 2^{\frac{-t}{2}}}$

where $t = 10$

$= 320.54$

which undershoots a little at $t = 10$

— log model underestimates at $t = 10$ but estimates accurately at $t = 18$

e) $P(7) = \frac{700}{1+37.88 \cdot 2^{\frac{-7}{2}}} = 160.987$

but the

(a) $M = 800{,}000{,}000$ $y(0) = 282{,}000{,}000$ $A = \frac{M - P_0}{P_0} = \frac{800-282}{282} \approx 1.836$

Logistical model: $y(t) = \frac{M}{1+Ae^{-kt}} = \boxed{\frac{800{,}000{,}000}{1 + 1.836e^{-kt}}}$

can't get k out yet

b) $y(10) = 309{,}000{,}000$ so $309 = \frac{800}{1+1.83e^{-k(10)}}$

$$309 + 309(1.83e^{-k10}) = 800$$
$$309(1.83e^{-k10}) = 491$$
$$1.83e^{-k10} = 1.589$$
$$e^{-k10} = 0.8683$$
$$-k \cdot 10 = \ln(0.8683)$$
$$k = \frac{\ln(0.8683)}{-10} \approx \boxed{0.01412}$$

new model w/ k: $\frac{M}{1+Ae^{-kt}} = \frac{800}{1+1.83e^{-k10}}$

c) find $y(100)$ and $y(200)$

$P(100) = \frac{800}{1+1.83e^{-k\cdot 100}}$ where $k \approx 0.01412$ $\approx \boxed{553.3}$ in 2100

$P(200) = \frac{800}{1+1.83e^{-k\cdot 200}}$ where $k \approx 0.01412$ $\approx \boxed{721.6}$ in 2200

d) we find to find t so that > 500

$$500 > \frac{800}{1+1.83e^{-kt}} \quad \text{where } k \approx 0.01412$$
$$1 + 1.83e^{-kt} > \frac{800}{500}$$
$$1.83e^{-kt} > \frac{300}{500}$$
$$e^{-kt} > \frac{300}{500 \cdot 1.83}$$
$$-0.01412\,t = \ln\left(\frac{300}{500 \cdot 1.83}\right)$$
$$t = \frac{\ln\left(\frac{300}{500 \cdot 1.83}\right)}{-0.01412} = \boxed{78.97}$$

t has to be > 78.97 for the pop to exceed 500 mil

$\boxed{\text{Module 9.5}}$

5) solve diff equations: $y' + y = 1 \rightarrow \frac{dy}{dx} + y = 1$ not separable so use integrating factor

$dy + y\,dx = dx$

(on back)

5) (continued) $y' + y = 1$ $I(x) = e^{\int 1} = e^{x}$

$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}$

$\frac{d}{dx}(e^{x}y) = e^{x}$ integrate

$e^{x}y = e^{x}$

$\boxed{y = 1}$

6) $y' - y = e^{x} = y' + -y = e^{x}$ $I(x) = e^{\int -1\,dx} = e^{-x}$

$e^{-x}\frac{dy}{dx} + e^{-x}-y = e^{x}e^{-x}$

$(e^{-x}y)' = 1$

integrate $e^{-x}y = x$

$y = \frac{x}{e^{-x}} = \boxed{e^{x}x}$

check $y' = e^{x} + xe^{x}$

$y' - y = e^{x}$

$e^{x} + xe^{x} - xe^{x} = e^{x}$ ✓

7) $y' = x - y$

rewrite to $y' + P(x)y = Q(x)$

$y' + y = x$ $I(x) = e^{\int 1 dx} = e^{x}$

multiply both sides by $I(x)$

$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}x$

$(e^{x}y)' = e^{x}x$

$e^{x}y = \int e^{x}x\,dx$ IBP $u = x$ $du = 1$ $v = e^{x}$ $dv = e^{x}$

$= xe^{x} - \int e^{x}\,dx$

$= xe^{x} - e^{x}$

$y = \frac{xe^{x} - e^{x}}{e^{x}} = \boxed{x - 1}$

9) $xy' + y = \sqrt{x}$ divide by x to get $\frac{dy}{dx} + P(x)y = Q(x)$

$y' + \frac{1}{x}y = \frac{x^{\frac{1}{2}}}{x} = x^{-\frac{1}{2}}$ $I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$xy' + y = x^{\frac{1}{2}}$

$(xy)' = x^{\frac{1}{2}}$

$xy = \frac{2}{3}x^{\frac{3}{2}} + C \to y = \boxed{\frac{2}{3}x^{\frac{1}{2}}} \boxed{+ \frac{C}{x}}$

check: $y' = \frac{1}{3}x^{-\frac{1}{2}}$

$\frac{1}{3}x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{1}{2}} = x^{\frac{1}{2}}$ ✓

get into form $\frac{dy}{dx} + P(x)y = Q(x)$

10) $2xy' + y = 2\sqrt{x}$

$y' + \frac{1}{2x}y = \frac{2x^{\frac{1}{2}}}{2x} = x^{-\frac{1}{2}}$

$I(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln(2x)} = \sqrt{2x}$

$\sqrt{2x}\frac{dy}{dx} + \frac{1}{\sqrt{2x}}y = \sqrt{2x}\,x^{-\frac{1}{2}} = \sqrt{2}$

$(\sqrt{2x}\,y)' = \sqrt{2}$ integrate

$\sqrt{2x}\,y = \sqrt{2}x + C$

$y = \frac{\sqrt{2}x + C}{\sqrt{2x}} = \boxed{x^{\frac{1}{2}} + \frac{C}{\sqrt{2x}}}$

check: $y' = \frac{1}{2}x^{-\frac{1}{2}} + -C\sqrt{2x}^{-\frac{1}{2}}$

so. $2x\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{-C\sqrt{2x}^{-\frac{1}{2}}}{2x}\right) +$

$x^{\frac{1}{2}} + \frac{C}{\sqrt{2x}} = 2\sqrt{x}$?

$x^{\frac{1}{2}} + x^{\frac{1}{2}} = 2\sqrt{x}$ ✓

13) $t^2\frac{dy}{dt} + 3ty = \sqrt{1+t^2}$, $t > 0$

$I(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$

$\frac{dy}{dt} + \frac{3}{t}y = \frac{\sqrt{1+t^2}}{t^2}$

$t^3\frac{dy}{dt} + t^3\frac{3}{t}y = t\sqrt{1+t^2}$

$t^3\frac{dy}{dt} + 3t^2y = t\sqrt{1+t^2}$

$(t^3y)' = t\sqrt{1+t^2} \rightarrow t^3y = \int t\sqrt{1+t^2}\,dt$ $\quad u = 1+t^2$, $du = 2t\,dt$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}$

$t^3y = \frac{1}{3}(1+t^2)^{3/2} + C$

$y = \boxed{\frac{(1+t^2)^{\frac{3}{2}}}{3t^3} + \frac{C}{t^3}}$

15) Solve initial value problem

$x^2y' + 2xy = \ln x$, $y(1) = 2$

divide by x^2

$y' + \frac{2y}{x} = \frac{\ln x}{x^2}$

$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$x^2y' + 2xy = \ln x$

$(x^2y)' = \int \ln x\,dx$

$x^2y = \frac{1}{x} + C$

$y = \frac{1}{x^3} + \frac{C}{x^3}$

if $y(1) = 2$ then $2 = 1 + \frac{C}{1}$ so $C = 1$ so given initial condition:

$\boxed{y = \frac{1}{x^3} + \frac{1}{x^3}}$

10) $2xy' + y = 2\sqrt{x}$ get into form $\frac{dy}{dx} + P(x)y = Q(x)$

$$y' + \frac{1}{2x}y = \frac{2x^{\frac{1}{2}}}{2x} = x^{-\frac{1}{2}}$$

$$I(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln(2x)} = \sqrt{2x}$$

$$\sqrt{2x}\frac{dy}{dx} + \frac{1}{\sqrt{2x}}y = \sqrt{2x}\,x^{-\frac{1}{2}} = \sqrt{2}$$

$$(\sqrt{2x}\,y)' = \sqrt{2} \quad \text{integrate}$$

$$\sqrt{2x}\,y = \sqrt{2}x + C$$

$$y = \frac{\sqrt{2}x + C}{\sqrt{2x}} = \boxed{x^{\frac{1}{2}} + \frac{C}{\sqrt{2x}}}$$

check: $y' = \frac{1}{2}x^{-\frac{1}{2}} + -C\sqrt{2x}^{-\frac{3}{2}}$

so. $2x\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{-C\sqrt{2x}^{-\frac{1}{2}}}{2x}\right) + x^{\frac{1}{2}} + \frac{C}{\sqrt{2x}} = 2\sqrt{x}$?

$$x^{\frac{1}{2}} + x^{\frac{1}{2}} = 2\sqrt{x} \checkmark$$

13) $t^2\frac{dy}{dt} + 3ty = \sqrt{1+t^2}$, $t>0$

$$\frac{dy}{dt} + \frac{3}{t}y = \frac{\sqrt{1+t^2}}{t^2}$$

$$I(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$t^3\frac{dy}{dt} + t^3\frac{3}{t}y = t\sqrt{1+t^2}$$

$$t^3\frac{dy}{dt} + 3t^2y = t\sqrt{1+t^2}$$

$$(t^3y)' = t\sqrt{1+t^2} \rightarrow t^3y = \int t\sqrt{1+t^2}\,dt \qquad u = 1+t^2, \quad du = 2t\,dt$$

$$= \frac{1}{2}\int\sqrt{u}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}$$

$$t^3y = \frac{1}{3}(1+t^2)^{3/2} + C$$

$$y = \boxed{\frac{(1+t^2)^{\frac{3}{2}}}{3t^3} + \frac{C}{t^3}}$$

15) Solve initial value problem

$x^2y' + 2xy = \ln x$, $y(1) = 2$ divide by x^2

$$y' + \frac{2y}{x} = \frac{\ln x}{x^2}$$

$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$x^2y' + 2xy = \ln x$$

$$(x^2y)' = \int \ln x\,dx$$

$$x^2y = \frac{1}{x} + C \qquad \text{if } y(1) = 2$$

$$y = \frac{1}{x^3} + \frac{C}{x^3} \qquad \text{then } 2 = 1 + \frac{C}{1} \text{ so } C = 1 \text{ so}$$

given initial condition:

$$\boxed{y = \frac{1}{x^3} + \frac{1}{x^3}}$$

16) $t^3 \frac{dy}{dt} + 3t^2 y = \cos t, \; y(\pi) = 0$

divide by t^3

$\frac{dy}{dt} + \frac{3}{t} y = \frac{\cos t}{t^3}$ $\quad I(x) = e^{\int \frac{3}{t} dx} = e^{3 \ln t} = e^{\ln t^3} = t^3$

$t^3 \frac{dy}{dt} + 3t^2 y = \cos t$

$(t^3 y)' = \cos t$

$t^3 y = \sin t + C$ $\quad$ since $y(\pi) = 0$, $C = 0$

$y = \frac{\sin t}{t^3} + \frac{C}{t^3}$ $\quad$ so $\boxed{y = \frac{\sin t}{t^3}}$

18) $xy' + y = x \ln x$, $y(1) = 0$

div by x

$y' + \frac{1}{x} y = \ln x$ $\quad$ so $\quad I(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

mult both sides by $I(x)$

$xy' + y = x \ln x$

Rewrite

$(xy)' = x \ln x$

integrate

$xy = \int x \ln x \, dx$ $\quad u = \ln x$ $\quad du = \frac{1}{x} dx$ $\quad$ doesn't work, What about IBP?

$u = \ln x \quad du = \frac{1}{x}$

$dv = x \quad v = \frac{x^2}{2}$

$= \frac{1}{2} x^2 \ln x - \int \frac{x^2}{2x} dx$

$= \frac{1}{2} x^2 \ln x - \int \frac{x}{2} dx$

$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx$

$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \cdot \frac{x^2}{2}$

$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$

$y = \frac{1}{2} x \ln x - \frac{1}{4} x + \frac{C}{x}$ $\quad$ since $y(1) = 0$

$0 = \frac{1}{2}(1) \ln(1) - \frac{1}{4} + C$

$0 - \frac{1}{4} + C$ $\quad$ so $C = \frac{1}{4}$

so solution given the C is $\boxed{y = \frac{1}{2} x \ln x - \frac{1}{4} x + \frac{1}{4x}}$

20) $(x^2+1)\frac{dy}{dx} + 3x(y-1)=0$, $y(0)=2$

$(x^2+1)\frac{dy}{dx} + 3xy = 3x$

$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1}$

$\int \frac{3x}{x^2+1}dx$ $u=x^2+1$ $\frac{du}{dx}=2x$ $du=2x\,dx$

$I(x) = e^{\int \frac{3x}{x^2+1}dx}$ $\frac{3}{2}\int \frac{1}{u}du$ $\frac{3}{2}du = 3x\,dx$

$\frac{3}{2}\ln(u) = \frac{3}{2}\ln(x^2+1)$

$= e^{\frac{3}{2}\ln(x^2+1)}$

$((x^2+1)^{\frac{3}{2}}y)' = (x^2+1)^{\frac{3}{2}}\frac{3x}{x^2+1}$ $= (x^2+1)^{\frac{3}{2}}$

$(x^2+1)^{\frac{3}{2}}y = \int 3x(x^2+1)^{\frac{1}{2}}dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{3}{2}du = 3x\,dx$

$= \frac{3}{2}\int u^{\frac{1}{2}}du$

$= u^{\frac{3}{2}}$

$I(x)y = (x^2+1)^{\frac{3}{2}} + C$

$y = \frac{(x^2+1)^{\frac{3}{2}}}{(x^2+1)^{\frac{3}{2}}} + \frac{C}{(x^2+1)^{\frac{3}{2}}}$

$= 1 + \frac{C}{(x^2+1)^{\frac{3}{2}}}$ so since $y(0)=2$ $C = (x^2+1)^{\frac{3}{2}}$

So the solution given the initial condition is:

$\boxed{y = 2}$

27) $V=40V$, $I=2H$ $R=10$ ohms $I(0)=0$

a) $2\frac{dI}{dt} + 10I = 40$ find $I(t)$

$I(x) = e^{\int 5dx} = e^{5x}$

$\frac{dI}{dt} + 5I = 20$

$e^{5x}\frac{dI}{dt} + e^{5x}5I = e^{5x}20$

$(e^{5x}I)' = e^{5x}\cdot 20$

$e^{5x}I = 20\int e^{5x}dx$ $u=5x$ $du=5\,dx$ $4du = 20\,dx$

$= 4\int e^{u}du$

$e^{5x}I = 4e^{5x} + C$ If $I(0)=0$, then $C = -4$

$I = 4 + \frac{C}{e^{5x}}$

so $\boxed{I = 4 - \frac{4}{e^{5x}}}$

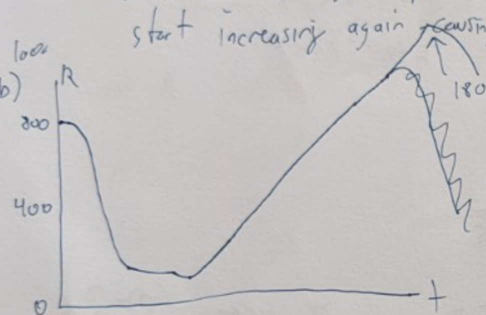
b) $I(0.1) = 4 - \frac{4}{e^{.5}} \approx 1.573$

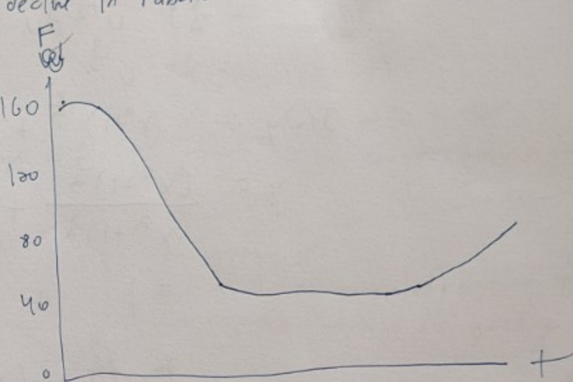
2) a) looks like cooperation because of the positive xy term in both. As the population of both gets bigger, both populations increase at a faster rate

b) looks like competition because of the negative xy term, as both populations increase the growth rate of both decline. Inc in one population → dec in the other

6) a) $t=0$ R and F both decrease from $t=0$
R eventually hit equilibrium but F keeps rapidly declining until $F \approx 38$ when rabbits start increasing again up until $R = 1800$. Then wolves, perhaps old enough to hunt, start increasing again causing rapid decline in rabbits

b)





10) $\frac{dA}{dt} = 2A - 0.01AL$

$\frac{dL}{dt} = -0.5L + 0.0001AL$

a) Find equil solutions. Aphids will be constant when derivative is 0

Aphids: $0 = A(2 - 0.01L)$ so when $\boxed{A=0}$ that is 1 equilibrium solution

so when $\boxed{L=200}$ that is another equilibrium

when $0.01L = 2$

$L = \frac{2}{0.01} = 200$

Ladybugs: $0 = -0.5L + 0.0001AL$ when $\boxed{L=0}$ that is 1 equilibrium solution

$= L(-0.5 + 0.0001A)$ when $\boxed{A=5000}$ another equilibrium

$A = \frac{0.5}{0.0001} = 5000$

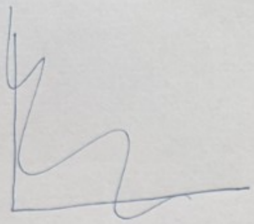
a) one pair is $A=0, L=0$ ← if no aphids or Ladybugs, none forever

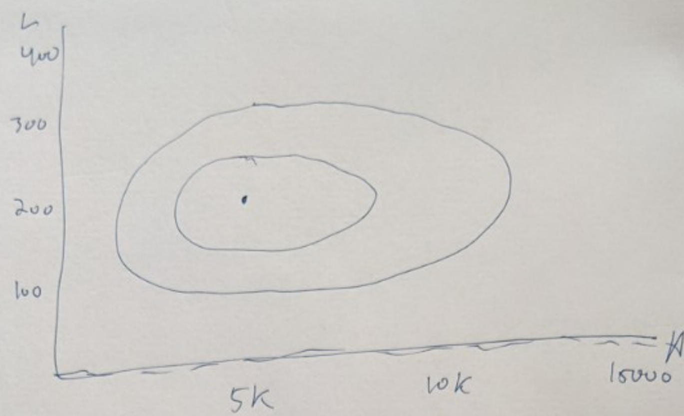
another is $L=200, A=5000$ ← populations are perfect and no changes from here

b) Find $\frac{DL}{DA}$

$\frac{dL}{dt} = \frac{dL}{dA} \cdot \frac{dA}{dt}$ so $\frac{dL}{dA} = \frac{\frac{dL}{dt}}{\frac{dA}{dt}} = \boxed{\frac{-0.5L + 0.0001AL}{2A - 0.01AL}}$

c)





- common equilibrium ~~po~~ ~~pt~~ point @ 5000 A and 200 L

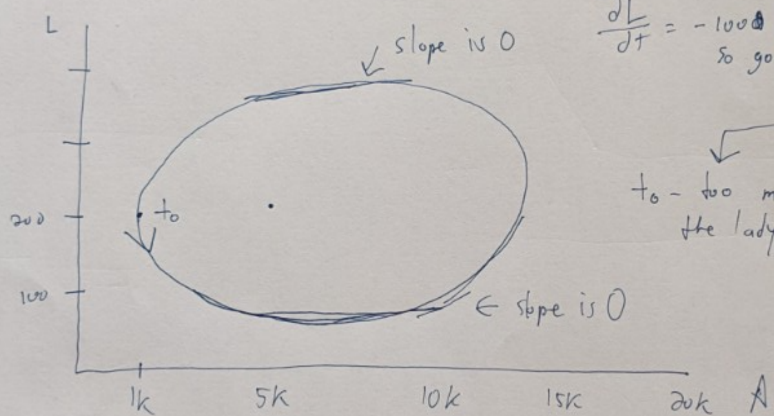
d) $t=0$, 1000 A, 200 L

$\frac{da}{dt}$ $2000 - 2000 = 0$

$\frac{dL}{dt} = -100 + 20 = -80$

so go counterclockwise first

t_0 - too many lady bugs vs aphids so the ladybugs die quickly

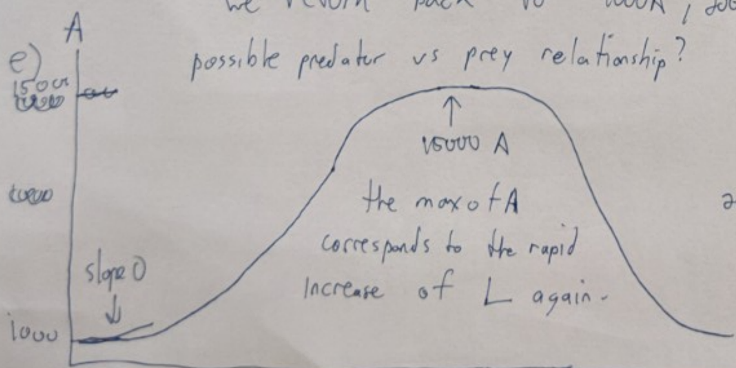


at $t=0$, ~~dA~~ Aphids are not changing since $\frac{dA}{dt}=0$, and $\frac{dL}{dt}$ is negative so we go counterclockwise first, eventually $\frac{dL}{dt}$ ~~is at equilibrium so~~ $=0$, bottoms out. then $\frac{dL}{dA}$ starts to increase again as A stops increasing, eventually, A starts decling which causes L to stop increasing until L starts decreasing again and we return back to 1000A, 200L.

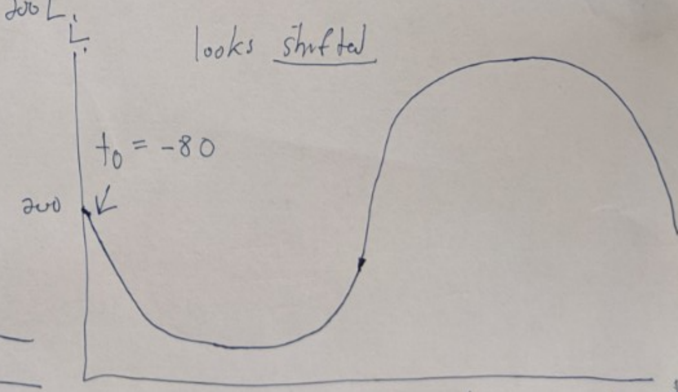
(too many ladybugs relative to aphids)

e)

possible predator vs prey relationship?



looks shifted



- one looks like a shift of ~~t~~ the other. As one ↑, the other stops or platers.
- ex, the peak of A is the decline of L. the inc of A is the ~~plateau~~ low of L