5) $\int t e^{-3t}\,dt = t\dfrac{e^{-3t}}{-3} - \int \dfrac{e^{-3t}}{-3}\,dt$

$u = t \quad du = dt$

$dv = e^{-3t} \quad v = \dfrac{e^{-3t}}{-3}$

$= \dfrac{t}{-3}e^{-3t} - \dfrac{-1}{3}\dfrac{e^{-3t}}{-3} = \boxed{\left(\dfrac{t}{-3} - \dfrac{1}{9}\right)e^{-3t} + C}$

8) ~~$\int t^2 \sin Bt\,dt = t^2 \dfrac{-\cos Bt}{B} - \int \dfrac{-\cos Bt}{B} 2t\,dt$~~ $\boxed{\text{8 is on bottom}}$

~~$u = t^2 \quad du = 2t\,dt$~~

~~$dv = \sin Bt \quad v = \dfrac{-\cos Bt}{B}$~~

~~$t^2 \cdot \dfrac{\cos Bt}{B} - \left(\dfrac{1}{B}\int -\cos Bt\, 2t\,dt\right)$~~

~~$t^2 \cdot \dfrac{\cos Bt}{B} - \left(\dfrac{1}{B} \cdot -\dfrac{\sin Bt}{B} \cdot 2t\,dt\right) = t^2 \cdot \dfrac{\cos Bt}{B} + \dfrac{1}{B^2} \cdot \sin Bt \cdot 2t\,dt + C$~~

10) $\int \ln x^{\frac{1}{2}}\,dx = \ln x^{\frac{1}{2}} \cdot x - \int x \dfrac{1}{2x}\,dx$

$u = \ln x^{\frac{1}{2}} \quad du = \dfrac{1}{x^{\frac{1}{2}}} \cdot \dfrac{1}{2}\dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{2x}$

$dv = dx \quad v = x$

$= x\ln x^{\frac{1}{2}} - \dfrac{1}{2}x$

$= x\ln x^{\frac{1}{2}} - \dfrac{1}{2}x$

$= \boxed{\dfrac{1}{2}x(\ln x - 1) + C}$

12) Remember $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$

$\int \tan^{-1} 2y\,dy = y\tan^{-1}2y - 2\int y \dfrac{1}{1+4y^2}\,dy$

$u = \tan^{-1}2y \quad du = 2\dfrac{1}{1+(2y)^2}\,dy$

$dv = dy \quad v = y$

$u = y \quad du = 1$

$dv = \dfrac{1}{1+4y^2}dy \quad v = \log(1+4y^2)$

$2\int y\dfrac{1}{1+4y^2}dy = \dfrac{2}{8}\int \dfrac{1}{u}\,du \qquad u = 1+4y^2,\ du = 8y\,dy$

so ~~$y\tan^{-1}2y - \frac{1}{4}\log(4y^2+1)$~~

So $\boxed{y\tan^{-1}(2y) - \dfrac{1}{4}\log(4y^2+1) + C}$

8) $\int t^2 \sin Bt\,dt = -\dfrac{1}{B}t^2\cos Bt + \int \dfrac{1}{B}\cos Bt\, 2t\,dt$

$u = t^2 \quad du = 2t\,dt$

$dv = \sin Bt\,dt \quad v = \dfrac{-1}{B}\cos Bt$

$= -\dfrac{t^2}{B}\cos Bt + \dfrac{2}{B}\int t\cos Bt\,dt$

Integrate again: $u = t \quad dv = \cos Bt\,dt$

$du = dt \quad v = \dfrac{\sin Bt}{B}$

so $\int t\cos Bt\,dt = t\dfrac{\sin Bt}{B} - \int \dfrac{\sin Bt}{B}\,dt$

$= t\dfrac{\sin Bt}{B} - \dfrac{1}{B}\int \sin Bt\,dt = \dfrac{t}{B}\sin Bt - \dfrac{1}{B}\dfrac{-\cos Bt}{B} + C$

so $\int t^2 \sin Bt\,dt = \boxed{\dfrac{-t^2}{B}\cos Bt + \dfrac{2}{B}\left(\dfrac{t}{B}\sin Bt + \dfrac{1}{B^2}\cos Bt\right) + C}$

13) $\int t\csc^2 t\,dt$

$u = t \quad du = 1\,dt$

$dv = \csc^2 t\,dt \quad v = -\cot t$

$= t\cdot -\cot(t) - \int -\cot(t) \quad = -t\cdot\cot(t) + \int \cot(t)\,dt$

u sub $\int \frac{\cos t}{\sin t}\,dt$

$u = \sin x$ so $du = \cos t\,dt$

$\boxed{-t\cdot\cot(t) + \ln|\sin t| + C}$ so $\int \frac{\cos t}{\sin t}\,dt = \int \frac{du}{u} = \ln|u| + C = \ln|\sin t| + C$

18) $\int e^{-\theta}\cos 2\theta\,d\theta \qquad u = e^{-\theta} \quad du = -e^{-\theta}\,d\theta$

$dv = \cos 2\theta\,d\theta \quad v = \frac{\sin 2\theta}{2}$

$= e^{-\theta}\frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2} - e^{-\theta}\,d\theta$

$= \frac{e^{-\theta}}{2}\sin 2\theta + \frac{1}{2}\int \sin 2\theta\cdot e^{-\theta}\,d\theta \leftarrow$ Integrate by parts: $u = e^{-\theta} \quad du = -e^{-\theta}$

$dv = \sin 2\theta \quad v = \frac{-\cos 2\theta}{2}$

so $\int \sin 2\theta\cdot e^{-\theta}\,d\theta = e^{-\theta}\cdot\frac{-\cos 2\theta}{2} - \int -\frac{\cos 2\theta}{2}\cdot -e^{-\theta}\,d\theta$

so $\int e^{-\theta}\cos 2\theta\,d\theta = e^{-\theta}\frac{\sin 2\theta}{2} + \frac{1}{2}\left(\frac{-1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int \cos 2\theta\, e^{-\theta}\,d\theta\right)$

$= \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}\int \cos 2\theta\, e^{-\theta}\,d\theta$

$\frac{5}{4}\int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C$

$\int e^{-\theta}\cos 2\theta\,d\theta = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C\right)$

$\boxed{= \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C}$

19) $\int z^3 e^z dz$ $= z^3e^z - \int e^z 3z^2$ ~~← integrate by parts~~

$u = z^3$ $du = 3z^2$

$dv = e^z$ $v = e^z$ $= z^3e^z - \cancel{(3z^2e^z - \int e^z 6z)}$ ~~$u = 3z^2$ $du = 6z$ $dv = e^z$ $v = e^z$~~

$-3\int e^z z^2$ ← integrate by parts $u = z^2$ $du = 2z\,dx$

$dv = e^z$ $v = e^z$

$= z^3e^z - 3(z^2e^z - \int e^z 2z dz$

$= z^3e^z - 3(z^2e^z - 2\int e^z z dz)$ ← 3rd time

$u = z$ $du = dz$

$dv = e^z$ $v = e^z$

$= z^3e^z - 3(z^2e^z - 2(ze^z - \int e^z dz)$

$\boxed{= z^3e^z - 3(z^2e^z - 2ze^z + 2e^z) + C}$

23) $\int_0^{1/2} x\cos \pi x\,dx$ forgot the definite integral for now

$u = x$ $du = 1$

$dv = \cos \pi x$ $v = \dfrac{\sin \pi x}{\pi}$

so indefinite integral of $\int x\cos \pi x\,dx$

$= x \cdot \dfrac{\sin \pi x}{\pi} - \int \dfrac{\sin \pi x}{\pi}$

$= x\dfrac{\sin \pi x}{\pi} - \dfrac{1}{\pi}\int \sin \pi x$

Indef integral $= x\dfrac{\sin \pi x}{\pi} + \dfrac{1}{\pi}\dfrac{\cos \pi x}{\pi} + C$

so...

$\int_0^{1/2} x\cos \pi x\,dx = \left(x\dfrac{\sin \pi x}{\pi} + \dfrac{1}{\pi^2}\cos \pi x\right)\Big]_0^{\frac{1}{2}}$

$= \left(\dfrac{1}{2}\dfrac{\sin(\frac{1}{2}\pi)}{\pi} + \dfrac{1}{\pi^2}\cos\dfrac{\pi}{2}\right) - \left(0 + \dfrac{1}{\pi^2}\right)$

$\dfrac{1}{2\pi} + 0 - \dfrac{1}{\pi^2}$

$\dfrac{\pi}{2\pi^2} - \dfrac{2}{2\pi^2} = \boxed{\dfrac{\pi - 2}{2\pi^2}}$

54) $\int \tan^n x\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2} x\,dx \quad (n \neq 1)$

$\cancel{\text{let } u = \tan^{n-1} x \qquad dv = \tan x\,dx}$

$\cancel{du = (n-1)\tan^{n-2} x \sec^2 x\,dx \qquad v = \ln|\sec x|}$

$\cancel{= \tan^{n-1} x \cdot \ln|\sec x| - \int \ln|\sec x| \cdot (n-1)\tan^{n-2} x \sec^2 x\,dx}$

$= \int \tan^{n-2} x \cdot \tan^2 x\,dx$ so since $\tan^2 x = \sec^2 x - 1$

$\int (\sec^2 x - 1)\tan^{n-2} x\,dx = \int \sec^2 x \tan^{n-2} x\,dx - \int \tan^{n-2} x\,dx$

start w/ $\int \sec^2 x \tan^{n-2} x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

so $\int u^{n-2}\,du - \int \tan^{n-2} x\,dx$

$= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\,dx$

$\boxed{= \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x\,dx}$

$\boxed{7.2}$ 1) $\int \sin^2 x \cos^3 x\,dx$

$= \int \sin^2 x \cos^2 x \cos x\,dx$

$= \int \sin^2 x (1 - \sin^2 x)\cos x\,dx \qquad u = \sin x \quad du = \cos x\,dx$

$= \int u^2 (1 - u^2)\,du$

$= \int u^2 - u^4\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

check: $\frac{d}{dx} = \sin^2 x \cos x - \sin^4 x \cos x \qquad (1 - \cos^2 x)\cos x - \sin^2 x \sin^2 x \cos x$

$\cancel{\cos x (\sin^2 x - \sin^4 x)}$

$\cos x - \cos^3 x - (1 - \cos x)(1 - \cos x)\cos x$

$-(1 - 2\cos x + \cos^2 x)\cos x$

$-(\cos x - 2\cos^2 x + \cos^3 x)$

$-\cos x + 2\cos^2 x - \cos^3 x$

$-2\cos^3 x + 2\cos^2 x$

2) $\int \sin^3\theta \cos^4\theta \, d\theta$

$= \int \sin^2\theta \cos^4\theta \sin\theta \, d\theta \qquad u = \cos\theta \quad du = -\sin\theta \, d\theta$

$= \int (1-\cos^2\theta)\cos^4\theta \sin\theta \, d\theta$

$= -\int (1-u^2)u^4 \, du = -\int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \boxed{\frac{\cos^5\theta}{5} - \frac{\cos^7\theta}{7} + C}$

4) $\int_0^{\pi/2} \sin^5 x \, dx$ first the indef integral:

$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 \sin x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \, dx \qquad u = \cos x \quad du = -\sin x \, dx$

$= -\int (1-u^2)^2 \, du = -\int (1 - 2u^2 + u^4) \, du = -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C$

$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$

Indef integral $\rightarrow$ $\boxed{= -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + \cancel{C}}$

$-\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} \Big]_0^{\pi/2} = (0) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = 1 - \frac{2}{3} + \frac{1}{5} = \frac{1}{3} + \frac{1}{5} = \boxed{\frac{2}{5}}$

7) $\int_0^{\pi/2} \cos^2\theta \, d\theta$ find indef integral first: $= \int \frac{1+\cos 2x}{2} \, dx$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{1}{2}\int 1 + \cos 2x \, dx$

so $\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta =$

$= \frac{1}{2}\left(x + \int \cos 2x \, dx\right)$

$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)$

$\frac{1}{2}x + \frac{1}{4}\sin 2x \Big]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi}{4} + 0\right) - (0) = \frac{\pi}{4}$

16) $\int \tan^2 x \cos^3 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^3 x \, dx = \int \sin^2 x \cos x \, dx \qquad u = \sin x \quad du = \cos x \, dx$

$= \int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$

21) $\int \tan x \sec^3 x \, dx = \int \cancel{\tan x} \sec^2 x \sec x \tan x \, dx \qquad u = \sec x \quad \frac{du}{dx} = \sec x \tan x \, dx$

$= \int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{\sec^3 x}{3} + C}$

~~$\int \tan^2 x \, dx$ $\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$ $= \int \frac{1}{1+\cos(2x)} \, dx - \int \frac{\cos 2x}{1+\cos(2x)} \, dx$~~

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~~$= \frac{1}{2}\int \frac{1}{1+\cos(2x)} \frac{d}{dx}2x \, dx - \frac{1}{2}\int \frac{\cos(2x)}{1+\cos(2x)} \frac{d}{dx}2x \, dx$~~

$x^2e^x - 2xe^x - 2e^x + C$

23) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1\,dx = \boxed{\tan x - x + C}$

26) $\int_0^{\pi/4} \sec^6\theta \tan^6\theta$ find indef integral

$\int \sec^6\theta \tan^6\theta = \int \tan^6\theta (\sec^2\theta)^2 \sec^2\theta\,d\theta = \int \tan^6\theta (1+\tan^2\theta)^2 \sec^2\theta\,d\theta$

usub $u = \tan\theta$, $du = \sec^2\theta$ so... $\int u^6 (1+u^2)^2\,du = \int u^6(1 + 2u^2 + u^4)\,du$

$= \int u^6 + 2u^8 + u^{10}\,du = \frac{u^7}{7} + \frac{2}{9}u^9 + \frac{u^{11}}{11} + C$

$= \boxed{\frac{\tan^7\theta}{7} + \frac{2}{9}\tan\theta^9 + \frac{\tan\theta^{11}}{11} + C}$

28) $\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x \sec x \tan x\,dx = \int ((\sec^2 x - 1))^2 \sec^2 x \sec x \tan x\,dx$

let $u = \sec x$, $du = \sec x \tan x$ so... $\int (u^2-1)^2 u^2\,du = \int (u^4 - 2u^2 + 1)u^2\,du$

$= \int u^6 - 2u^4 + u^2\,du = \frac{u^7}{7} - \frac{2}{5}u^5 + \frac{u^3}{3} + C$

$= \boxed{\cancel{- \frac{2}{5}\sec^5 x + \frac{\sec^3 x}{3} + C}}$

$\boxed{\frac{\sec^7 x}{7} - \frac{2}{5}\sec^5 x + \frac{\sec^3 x}{3} + C}$

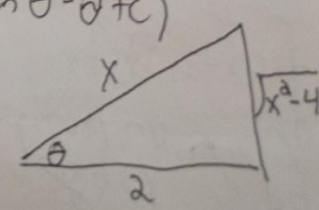
$\boxed{7.3}$ 3) $\int \frac{\sqrt{x^2-4}}{x}\,dx$ $\quad dx = 2\sec\theta\tan\theta\,d\theta$, $x = 2\sec\theta$

remember that $\tan^2\theta + 1 = \sec^2\theta$ so $\tan^2\theta = \sec^2\theta - 1$

$\int \frac{\sqrt{4\sec^2\theta - 4}}{x} = \int \frac{\sqrt{4(\sec^2\theta - 1)}}{2\sec\theta} = \int \frac{\sqrt{4\tan^2\theta}}{2\sec\theta} = \int \frac{2|\tan\theta|}{2\sec\theta}\,dx$

$= \int \frac{2\tan\theta}{2\sec\theta} 2\sec\theta\tan\theta\,d\theta = \int 2\tan^2\theta\,d\theta = 2\int \tan^2\theta\,d\theta = 2(\tan\theta - \theta + C)$

$\boxed{2\left(\frac{\sqrt{x^2-4}}{2} - \operatorname{arcsec}\left(\frac{x}{2}\right)\right) + C}$

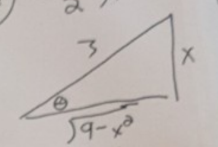


$dx \quad x^2e^x - 2xe^x - 2e^x + C$

4) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx \qquad x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta$

so $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$

so $= \frac{9\sin^2\theta}{3\cos\theta}\, 3\cos\theta\, d\theta = \int 9\sin^2\theta = 9\int \sin^2\theta = \frac{9}{2}\int 1-\cos(2\theta)\,d\theta$



$= \frac{9}{2}\int d\theta - \frac{9}{2}\int \cos 2\theta\, d\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C$

u sub

$u = 2\theta$

$du = 2\,d\theta \qquad \int \cos u\, du = \frac{1}{2}\sin u \quad \frac{1}{2}\sin 2\theta$

$\sin = \frac{x}{3}$

$\cos = \frac{\sqrt{9-x^2}}{3}$

$\sin 2\theta = \frac{9}{4}(2\sin\theta\cos\theta)$

$= \frac{9}{4}\cdot 2 \cdot \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} = \frac{1}{2}x\sqrt{9-x^2}$

so $\boxed{\frac{9}{2}\arcsin\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2}}$

9) $\int_2^3 \frac{dx}{(x^2-1)^{\frac{3}{2}}} \qquad x = \sec\theta \quad dx = \sec\theta\tan\theta\, d\theta$

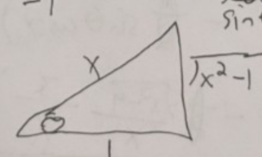
indef integral $= \int \frac{\sec\theta\tan\theta}{(\sec^2\theta - 1)^{3/2}} = \int \frac{\sec\theta\tan\theta}{(\tan^2\theta)^{3/2}}$

$= \int \frac{\sec\theta\tan\theta}{\tan^3\theta} = \int \frac{\sec\theta}{\tan^2\theta} \quad \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta} = \int \frac{\cos\theta}{\sin^2\theta}\,dx$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$\int \frac{1}{u^2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{\sin\theta} + C$



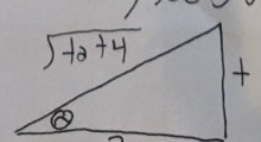
so $= -\frac{1}{\frac{\sqrt{x^2-1}}{x}} = \frac{-x}{\sqrt{x^2-1}} + C$

$\left.\frac{-x}{\sqrt{x^2-1}}\right]_2^3 = \frac{-3}{\sqrt{8}} - \frac{-2}{\sqrt{3}} = \boxed{\frac{-3}{\sqrt{8}} + \frac{2}{\sqrt{3}}}$

12) $\int_0^2 \frac{dt}{\sqrt{4+t^2}} \qquad$ Let $t = 2\tan\theta$ so $\sqrt{4+t^2} = \sqrt{4+4\tan^2\theta} = \sqrt{4(1+\tan^2\theta)}$

$dt = 2\sec^2\theta\, d\theta \qquad = \sqrt{4\sec^2\theta} = 2\sec\theta$

so $\int \frac{1}{2\sec\theta}\,dt = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$



so $\sec\theta = \frac{\sqrt{t^2+4}}{2}$

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12 cont) $\ln|\sec\theta + \tan\theta| = \ln\left|\frac{\sqrt{t^2+4}}{2} + \frac{t}{2}\right|$

so $\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \ln\left|\frac{\sqrt{t^2+4}}{2} + \frac{t}{2}\right| \Big]_0^2 = \ln\left|\frac{1}{2}(\sqrt{t^2+4} + t)\right| \Big]_0^2$

$= \ln\left|\frac{1}{2}\sqrt{8} + 1\right| - \ln|1|$

$\boxed{= \ln|1 + \sqrt{2}|}$

13) $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$ let $x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$= \int \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta}\,dx = \int \frac{\sqrt{9(\sec^2\theta - 1)}}{27\sec^3\theta} 3\sec\theta\tan\theta\,d\theta = \int \frac{3\tan\theta\, 3\sec\theta\tan\theta\,d\theta}{27\sec^3\theta}$

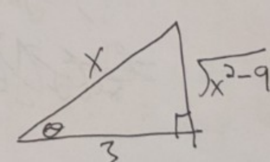
$= \int \frac{9\tan^2\theta}{27\sec^2\theta} = \int \frac{\tan^2\theta}{3\sec^2\theta} = \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\cos^2\theta}{3} = \int \frac{\sin^2\theta}{3} = \frac{1}{3}\int \sin^2\theta$

$= \frac{1}{3}\int \frac{1}{2}(1 - \cos(2\theta))\,dx = \frac{1}{6}\int 1 - \cos 2\theta\,dx = \frac{1}{6}\left(\theta - \frac{\sin 2\theta}{2}\right)$

convert to x:

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{6}\left(\sec^{-1}\frac{x}{3} - \sin\theta\cos\theta\right)$



$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$\cos\theta = \frac{3}{x}$

$= \frac{1}{6}\left(\sec^{-1}\frac{x}{3} - \frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x}\right)$

$\boxed{= \frac{1}{6}\left(\sec^{-1}\frac{x}{3} - \frac{3\sqrt{x^2-9}}{x^2}\right)}$

$x^2e^x - 2xe^x - 2e^x + C$

15) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$ let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $\theta \in [0, \frac{\pi}{2}]$ so $\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\,d\theta$

$= \int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2 - a^2\sin^2\theta}\;a\cos\theta\,d\theta = \int a^2\sin^2\theta\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta\,d\theta$

focus on indef integral!

$= \int a^2\sin^2\theta\sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta = \int a^2\sin^2\theta\cdot a\cos\theta\, a\cos\theta\,d\theta = \int a^4\sin^2\theta\cos^2\theta\,d\theta$

$= a^4\int\sin^2\theta\cos^2\theta\,d\theta$

so $\sin 2\theta = 2\sin\theta\cos\theta$

so $\frac{1}{4}\sin^2 2\theta = \sin^2\theta\cos^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

so using the trig identities on the right:

$= a^4\int\frac{1}{4}\sin^2 2\theta = a^4\int\frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right) = a^4\int\frac{1}{8}(1-\cos 4\theta)\,d\theta$

$= \frac{a^4}{8}\int 1-\cos 4\theta\,d\theta = \frac{a^4}{8}\left(\int d\theta - \int\cos 4\theta\,d\theta\right)$

$= \frac{a^4}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$

$= \boxed{\frac{a^4\pi}{16}}$

24) $\int_0^1\sqrt{x-x^2}\,dx$ ~~let $x = \sin\theta$, $dx = \cos\theta\,d\theta$~~ complete the square: $\int\sqrt{-x^2+x-\frac{1}{4}+\frac{1}{4}}\,dx$

$= \int\sqrt{-(x-\frac{1}{2})^2+\frac{1}{4}}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{-u^2+\frac{1}{4}}\,du = \int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{\frac{1}{4}-u^2}\,du$

so $u = \frac{1}{2}\sin\theta$

$du = \frac{1}{2}\cos\theta\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\frac{1}{4}(1-\sin^2\theta)}\,dx = \int\sqrt{\frac{1}{4}\cos^2\theta}\,\frac{1}{2}\cos\theta\,d\theta$

$= \int\frac{1}{2}\cos\theta\,\frac{1}{2}\cos\theta\,d\theta = \int\frac{1}{4}\cos^2\theta\,d\theta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta =$

$= \frac{1}{4}\int\frac{1}{2}(1+\cos(2\theta))\,dx = \frac{1}{8}\int(1+\cos 2\theta)\,dx =$

$\frac{1}{8}\left(\theta + \frac{1}{2}\sin(2\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{16} + \frac{\pi}{16} = \boxed{\frac{\pi}{8}}$