module 7 | absolutely or conditionally convergent

11.6 | 2) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ def conditionally convergent by Alternating Series Test

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \ldots$ divergent p series because $p = \frac{1}{2}$ so it is $\boxed{\text{conditionally convergent}}$

also by comparison $\frac{1}{\sqrt{n}} > \frac{1}{n}$ harmonic so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent too

3) $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{5n+1}$ find if Absolutely Convergent by Limit Comparison, can't use standard comparison test because $\frac{1}{5n+1} < \frac{1}{n}$

$\frac{1}{1} - \frac{1}{6} + \frac{1}{11} + \frac{1}{16}$ $\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} = \frac{1}{5}$ so also divergent

so Not absolutely convergent

but it does pass Alternating series test so it is $\boxed{\text{conditionally convergent}}$

5) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ $|\sin n| \leq 1$ Test for absolute convergence: $\sum_{n=1}^{\infty} \left| \frac{\sin n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{2^n} < \frac{1}{2^n}$

Is $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent? $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$= \sum_{n=1}^{\infty} \frac{1}{2}^n$ $r = \frac{1}{2}$ so convergent

so $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ is $\boxed{\text{absolutely convergent}}$ by comparison test

8) use ratio test just use general term formula

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-2)^n} \right| = \cancel{\frac{2}{n+1}} = \cancel{0}$ ~~so absolutely convergent~~ $\cancel{\boxed{\text{convergent}}}$

$2 \cdot \frac{n^2}{(n+1)^2} = 2 \cdot \frac{1}{(1+\frac{1}{n})^2} = 2 > 1$ $\boxed{\text{diverges}}$

$\sum_{k=1}^{\infty} k e^{-k}$ $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(k+1)e^{-(k+1)}}{k\, e^{-k}} \right| = \frac{1}{e} < 1$ $\boxed{\text{converges}}$

almost screwed up the negative sign in numerator, oops

17) $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cos((n+1)\pi/3)}{(n+1)!} \cdot \frac{n}{\cos n\pi/3} \right| = \frac{\cos((n+1)\pi/3)}{(n+1)\cos(n\pi/3)}$?

so $\cos(n\pi/3) \in \{-1, 1, \frac{-1}{2}, \frac{1}{2}\}$

so $\left| \frac{\cos((n+1)\pi/3)}{\cos(n\pi/3)} \right| \leq 2$ so $= \frac{2}{n+1} = 0$

so $\boxed{\text{absolutely converges}}$

using comparison!

$\frac{\cos(n\pi/3)}{n!} \leq \frac{1}{n!}$ so absolutely converges too

Is this right?? help!!

18) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|\frac{(n+1)\, n^n}{(n+1)^{n+1}}\right| = \frac{\cancel{n^{n+1} + n^n}\ \cancel{(n+1)}(n^n)}{\cancel{(n+1)^{n+1}}\ (n+1)^n \cdot \cancel{n+1}}$

get rid of $| \, |$

$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$

~~as $n \to \infty = 1 = 1$ so absolutely converges~~

neat move

$\left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1+\frac{1}{n}\right)^{-n}$

$= \frac{1}{e} < 1$

so $\boxed{\text{absolutely converges}}$

20) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2(n+1))!}{(n+1)!^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} = 4$ $\boxed{\text{diverges by ratio test}}$

$(n+1)^2 (n)!^2$

$= \frac{2\cancel{(n+1)}(2n+1)}{\cancel{(n+1)}(n+1)} = \frac{4n+2}{n+1} = 4$

$a_{n+1} = a_n \cdot \frac{2+3(n)}{3+2(n)}$

22) $\sum_{n=1}^{\infty}$ ~~[scribbled out]~~

nth term $= \frac{2+3(n-1)}{3+2(n-1)}$

so $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2+3n)}{3+2(n)} = \frac{3}{2}$ so $\boxed{\text{diverges}}$

because $a_{n+1} = a_n \cdot \frac{2+3(n)}{3+2(n)}$

30) root test

$\sum_{n=0}^{\infty} (\arctan n)^n$ $\lim_{n\to\infty} \sqrt[n]{(\arctan n)^n} = \lim_{n\to\infty} |\arctan n| = \frac{\pi}{2} > 1$ so $\boxed{\text{diverges}}$

11.7 converge or diverge?

2) $\lim_{n\to\infty} \frac{n-1}{n^3+1}$ compare to $\frac{1}{n^2}$ which converges using Limit Comparison

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n-1}{n^3+1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n^3}} = 1$ So $\boxed{\text{Converges}}$

8) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ $b = \frac{n^4}{4^n}$ alternating series

eventually $b_{n+1} < b_n$

~~$\lim_{n\to\infty} \frac{n^4}{4^n}$ using L'Hopital $\frac{4n}{}$~~

$\boxed{\text{Converges by AST}}$

first few terms $n=6$ $\frac{6\,6\,6\,6}{4\,4\,4\,4\,4\,4}$ seems as if $\to 0$

~~9) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$~~

~~can also use absolute convergence test: $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n^4}{4^n}\right| = \sum_{n=1}^{\infty} \frac{n^4}{4^n}$ use Limit Comparison to $\frac{1}{2^n}$ which is convergent~~

~~$\lim_{n\to\infty} \frac{\frac{n^4}{4^n}}{\frac{1}{2^n}} = \frac{n^4}{2^n} = 0$~~ (next page)

5

11) $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty} \cancel{\left(\frac{3^n}{n^3 3^n} + \frac{n^3}{3^n n^3}\right)} = \cancel{\sum_{n=1}^{\infty} \left(\frac{3^n + n^3}{n^3 3^n}\right)}$

8 continued) use ratio test

$\lim_{n\to\infty} (-1)^n \frac{n^4}{4^n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{n^4}\right| = \frac{(n+1)^4}{4n^4} = \frac{\left(1+\frac{1}{n}\right)^4}{4} = \frac{1}{4} < 1$

$\boxed{\text{So convergent by ratio test}}$

9) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ $\quad$ using ratio test $\left|\frac{\pi^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{\pi^{2n}}\right|$

$= \frac{\pi^2}{2n+2 \cdot 2n+1} = 0 < 1$ so $\boxed{\text{converges by ratio test}}$

11) $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$ maybe compare to $\frac{1}{n^3}$ which converges. p series where $p > 3$ so converges

Limit Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\frac{1}{n^3} + \frac{1}{3^n}\right) \cdot \frac{n^3}{1} = \frac{n^3}{n^3} + \frac{n^3}{3^n} =$

$1 + \frac{n^3}{3^n} = 1$ so

$\lim \frac{n^3}{3^n}$ using L'Hopital $= \frac{3n}{3^n \ln 3} = \frac{3}{\ln 3^2 3^n} = 0$ $\quad \boxed{\text{converges too}}$

12) $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ $\quad$ Integral test? $\quad$ comparison? to $\frac{1}{k^2}$ which

p series $p=2$ converges

$u = k^2+1 \qquad du = 2k\,dk$

$\int_1^{\infty} \frac{1}{k}\left(k^2+1\right)^{-1/2} dk = \int_1^{\infty}$ cont. ugh

$\frac{1}{k\sqrt{k^2+1}} < \frac{1}{k\sqrt{k^2}} = \frac{1}{k^2}$ so $\boxed{\text{converges}}$ by comparison test

13) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ $\quad$ factorial present so use ratio test?

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}\right| = \left|\frac{3 \cdot (n+1)^2}{(n+1) \cdot n^2}\right| = \frac{3}{n+1} = 0$ so < 1

converges by ratio test

15) $\sum_{k=1}^{\infty} \frac{2^{k-1} 3^{k+1}}{k^k}$ $\quad$ root test?

$\lim_{n\to\infty} \sqrt[k]{a_k} = \frac{2^{\frac{k-1}{k}} 3^{\frac{k+1}{k}}}{k} = \frac{6}{k} = 0$

16) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ Limit comparison? it behaves like $\frac{(n^4+1)^{\frac{1}{2}}}{n^3+n} \sim \frac{1}{n}$ which diverges

$$\frac{(n^4+1)^{\frac{1}{2}}}{n^3+n} \cdot n = \frac{(n^6+n^2)^{\frac{1}{2}}}{n^3+n} = \frac{n^3+n}{n^3+n} = 1 \text{ so } \boxed{\text{diverges too}}$$

$\boxed{11.8}$ 5) $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ use ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{2(n+1)-1} \cdot \frac{2n-1}{x^n}\right| = \left|\frac{x \cdot 2n-1}{2n+1}\right| = \left|\frac{x \cdot (1-\frac{1}{2n})}{1+\frac{1}{2n}}\right|$

$= |x|$

so $|x| < 1$ so $-1 < x < 1$

check endpoints: when $x = 1$, then $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ which by Limit Comparison test w/ $\frac{1}{n}$ is divergent

Limit Comparison: $\lim_{n\to\infty} \frac{1}{2n-1} \cdot \frac{n}{1} =$

$\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$ so diverges

when $x=-1$, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ use AST $b = \frac{1}{2n-1}$ is decreasing and

$\lim \frac{1}{2n-1} \to 0$

so $\boxed{\begin{array}{l}\text{radius of convergence: } 1 \\ \text{interval of convergence: } -1 \leq x < 1\end{array}}$

6) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ ratio test? $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n}\right| = \left|\frac{(-1) \cdot x \cdot n^2}{(n+1)^2}\right|$

$= |x|$ So at least $-1 < x < 1$ but check endpts

when $x=-1$, then $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ p series w/ $p=2$ converges

when $x=1$, then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ AST decreasing and $b_{n+1} < b_n$

So radius of convergence:

Interval of convergence: $-1 \leq x \leq 1$

9) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$ ratio test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^4 4^{n+1}} \cdot \frac{n^4 4^n}{x^n}\right| = \left|\frac{x\, n^4}{(n+1)^4 4}\right| = \left|\frac{x}{4}\right|$

converges when $\left|\frac{x}{4}\right| < 1$ $\quad |x| < 4$ $\quad -4 < x < 4$ but check endpoints

when $x = -4$, then $\sum_{n=1}^{\infty} \frac{-4^n}{n^4 4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^4 4^n}$ decreasing and $\lim_{n\to\infty} b_n = 0$

So converges by AST

when $x = 4$, then $\sum_{n=1}^{\infty} \frac{4^n}{n^4 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ $\quad$ p series $p = 4$, so converges

$$\boxed{\begin{array}{l}\text{~~Radius~~ Interval of convergence} = -4 \le x \le 4 \\ \text{Radius of convergence} = 4\end{array}}$$

10) $\sum_{n=1}^{\infty} 2^n n^2 x^n$ $\quad$ try ratio test?

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}(n+1)^2 x^{n+1}}{2^n n^n x^n} = \left|\frac{2(n+1)^2 x}{n^2}\right| = |2x|$

converges when $|2x| < 1$ so $-\frac{1}{2} < x < \frac{1}{2}$

check endpoints!

when $x = \frac{-1}{2}$, then $\sum_{n=1}^{\infty} 2^n n^2 \frac{-1}{2}^n = \sum_{n=1}^{\infty} -1^n n^2$ $\quad$ n^2 does not converge $\lim_{n\to\infty} n^2 \neq 0$

when $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} 1^n n^2$ $\quad$ Test for Divergence $\lim_{n\to\infty} 1^n n^2 \neq 0$ so diverges

$$\boxed{\begin{array}{l}\text{interval of convergence}: \frac{-1}{2} < x < \frac{1}{2} \\ \text{radius of convergence}: \frac{1}{2}\end{array}}$$

12) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n$ $\quad$ ratio test? $\left|\frac{a_{n+1}}{a_n}\right| = \left|\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n \cdot 5^n}{(-1)^{n-1} x^n}\right| = \left|\frac{x \cdot n}{(n+1)5}\right| = \left|\frac{x}{5}\right|$

So converges if $\left|\frac{x}{5}\right| < 1$ so $-5 < x < 5$ but let's check endpoints.

when $x = -5$, then $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} -5^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} (-1)^n 5^n = \frac{(-1)^{2n-1}}{n} = \frac{-1}{n}$ $\quad$ So -5 not in interval. negative harmonic series

12 continued) when $x=5$, then $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} 5^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ AST does converge

so $\boxed{\text{interval of convergence: } -5 < x \leq 5 \quad \text{radius of convergence: } 5}$

15) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ try ratio test: $\left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| = |x-2|$

so $|x-2| < 1$

$1 < x < 3$

before endpoint checks

$\lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = \frac{2n}{2(n+1)} = \frac{2n}{2n+2} = 1$

when $x=1$, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ use AST where $b = \frac{1}{n^2+1}$ this is decreasing and $\lim_{n\to\infty} b_n = 0$ so converges at $x=1$

when $x=3$, then $\sum_{n=1}^{\infty} \frac{1^n}{n^2+1}$, use comparison test. $\frac{1^n}{n^2+1} < \frac{1}{n^2}$ so converges

interval of convergence: $1 \leq x \leq 3$

radius of convergence: 1

24) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\cdot 4\cdot 6 \cdots (2n)} = \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n n!}$ use ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2 x^{n+1}}{2(n+1)} \cdot \frac{2n}{n^2 x^2} \right| =$

$\left| \frac{x \cdot 2n}{2(n+1)} \right| = |x|$ $|x| < 1$ for root test $-1 < x < 1$ before checking endpts.

when $x=-1$, then $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n}$ fails AST because $b = \frac{n^2}{2n}$ is $b_{n+1} > b_n$

when $x=1$, then $\sum_{n=1}^{\infty} \frac{n^2}{2n}$ fails test for divergence

$\boxed{\text{radius of convergence: } 1 \quad \text{interval of convergence: } -1 < x < 1}$

$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n n!}$ use ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2 x^{n+1}}{2^{n+1} (n+1)!} \cdot \frac{2^n n!}{n^2 x^n} \right| = \left| \frac{x}{2(n+1)} \right| \to 0$

$\boxed{-\infty < x < \infty \quad \text{Radius} = \infty}$