# Module 4

## SECTION 1.4: EXPONENTIAL FUNCTIONS

We begin with an outline of the review section for exponential functions.

1. How do we interpret an irrational exponent? The power $3^2$ means $3$ times $3$ and $3^{1/2}$ is $\sqrt{3}$ but what does $3^{\sqrt{2}}$ mean?

2. When does the graph of an exponential function rise and when does it fall?

3. How do quadratic and exponential functions compare?

4. The special number $e$.

### MEANING OF IRRATIONAL EXPONENTS

We know that:

$$2^3 = 2 \cdot 2 \cdot 2 \quad , \quad 2^0 = 1 \quad , \quad 2^{-3} = \frac{1}{2^3} \quad , \quad 2^{2/3} = \sqrt[3]{2^2} \, ,$$

but if we use only these types of rational exponents in graphing $y = 2^x$ there will be gaps in the graph as shown in ***Figure 1***. To illustrate the gaps in the graph, we show that there are an *infinite* number of *irrational* numbers between 0 and $\frac{2}{3}$. Start with

$$\sqrt{3} \cong 1.732050808$$

Then:

$$\frac{\sqrt{3}}{3} \cong .57735$$

and $\frac{\sqrt{3}}{3}$ is an irrational number between 0 and $\frac{2}{3}$. The same is true for *each* of the following numbers:

$$\frac{\sqrt{3}}{4} \; , \; \frac{\sqrt{3}}{5} \; , \; \frac{\sqrt{3}}{6} \; , \; \frac{\sqrt{3}}{7} \; , \; \ldots \; , \; \frac{\sqrt{3}}{n} \; , \ldots$$

where $n > 3$ and $n$ is an integer. This set will contain an infinite number of irrational numbers and 2 raised to each one of these powers is missing in the graph of $y = 2^x$, $x$ rational, in ***Figure 1***.

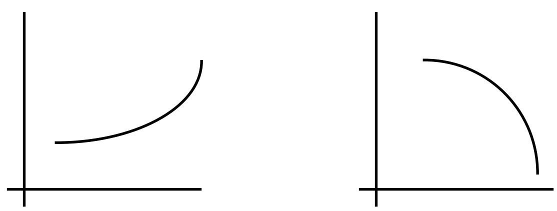
The next step is to decide how to assign meaning to a power like $2^{\sqrt{3}}$ If you say a calculator will tell us, realize that any calculator will use an approximation to $\sqrt{3}$ like $\sqrt{3} \cong 1.732050808$ and in $2^{1.732050808}$ we have a rational exponent. The exponent, 1.732050808 can be written as $\dfrac{1{,}732{,}050{,}808}{1{,}000{,}000{,}000}$ and hence by definition is a rational number. If you say, $2^{1.732050808}$ is close to $2^{\sqrt{3}}$ we agree, but there are still gaps in the graph in ***Figure 1***. $2^{1.732050808}$ is there (as a $y$ value) but $2^{\sqrt{3}}$ is not.

On ***page 46***, the set of inequalities indicates a pattern that can be followed to define the meaning of $2^{\sqrt{3}}$. We use only rational exponents to establish boundaries like $2^{1.732}$ and $2^{1.733}$ but in such a way that we get closer and closer to $2^{\sqrt{3}}$.

Having said all of this, if you have followed the gist of the discussion, you can now go back to using your calculator to get approximations. Just remember there is quite a bit behind the approximations (like how does the *circuitry* of a calculator go about its task).

## RISING AND FALLING GRAPHS

When talking about a curve rising or falling, this always assumes that $x$ is increasing, or that our attention passes from left to right.



Rising Falling

When the curve is rising the $y$ values are increasing and when the curve is falling the $y$ values are decreasing.

In ***Figure 3***, the graph of $y = \left(\frac{1}{4}\right)^x$ is shown as falling and in ***Figure 4 (a)*** the base $a = \frac{1}{4}$ is between 0 and 1. However, the equation

$$y = \left(\frac{1}{4}\right)^x$$

is usually written as $y = 4^{-x}$ $\qquad \left(\frac{1}{4}\right)^x = \left(4^{-1}\right)^x = 4^{-x}$

For equations like

$$y = 3^{-x} \ , \ y = 4^{-x} \ , \ y = 10^{-x}$$

the curve is falling (*y* is decreasing), and for equations like

$$y = 3^{x} \ , \ y = 4^{x} \ , \ y = 10^{x}$$

the curve is rising (*y* is increasing).

These categories are important in applications. Rising graphs match exponential growth and falling graphs match exponential decay. Both are important uses of the exponential function.

### COMPARING QUADRATIC AND EXPONENTIAL FUNCTIONS: EXAMPLE 2

In ***Figures 6*** and ***7***, a graphical comparison is made of

$$y = x^2 \text{ and } y = 2^x .$$

For $x > 4$, $2^x$ is larger than $x^2$. But the graphs still do not indicate the magnitude of the difference. Consider $x = 10$:

$$x^2 = 10^2 = 100 \text{ and } 2^x = 2^{10} = 1024$$

$2^x$ is 10.2 times as large as $x^2$. But note what happens in the following table where ratio $= \dfrac{2^x}{x^2}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | 10 | 11 | 12 | 13 | 14 | 15 | 20 | 30 |
| $x^2$ | 100 | 121 | 144 | 169 | 196 | 225 | 400 | 900 |
| $2^x$ | 1024 | 2048 | 4096 | 8192 | 16384 | 32768 | 1048576 | 1073741824 |
| ratio | 10.2 | 16.9 | 28.44 | 48.47 | 83.59 | 145.64 | 2621.44 | 1193046.47 |

From $x = 10$ to $x = 30$ $2^x$ jumps from being 10.2 times as large to over 1 million times as large. Keep this in mind when you hear the words *exponential growth*.

### THE NUMBER *e*

The significance of the special number *e* is indicated in ***Figure 12***. It is the base for an exponential function that produces a slope of one for the tangent line at x = 0. The number *e* is approximately 2.71828. However, at this point you will have difficulty seeing why this is important. For now just note these two facts: later you will have a better appreciation of the significance of this number.

## SECTION 3.1: DERIVATIVES OF POLYNOMIALS AND EXPONENTIAL FUNCTIONS

We now apply the definition of the derivative to selected algebraic forms to create differentiation formulas that will simplify the process of finding derivatives. These formulas are used extensively in all aspects of calculus. It is very important that you master these formulas and the algebraic procedures related to their use.

### COMMENTS ON THE DERIVATION OF THE DIFFERENTIATION FORMULAS

First we will make comments on the derivation (or proofs) of the formulas and then discuss their use. By looking carefully at the derivations, you will have a better understanding of why the formulas have a particular structure.

### DERIVATIVE OF A CONSTANT FUNCTION

In the derivation, $\lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h}$ . If you now take the limit, the result is $\frac{0}{0}$ which is undefined. So, divide *h* into 0, the result is 0 and then take the limit. Remember the limit of a constant is the constant.

### THE POWER RULE: THE DERIVATIVE OF $x^n$ IS $n x^{n-1}$

This is version I of the Power Rule. The exponent, *n*, must be a *positive* integer because the proof depends on that fact. The formula,

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$$

is true for $n = 2$, or $n = 3$, or $n = 15$ but not for $n = 1/2$ . You can agree or disagree by putting each of these *n* values into the formula and multiply out the right-hand side.

A key step in the proof is the cancellation of the factor, $x$ - $a$. The limit,

$\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$ without change is $\dfrac{0}{0}$ because $\dfrac{x-a}{x-a}$ is $\dfrac{0}{0}$ when $x = a$. After removing the common factor, the limit can be evaluated by direct substitution which produces the term, $a^{n-1}$, $n$ times. The counting procedure to get " $n$ times " can be accomplished by looking at the exponents of the $x$ terms and the extra term. The exponents are $n-1$, $n-2$, down to $1$. Counting from $1$ to $n-1$ is, of course, $n-1$, and adding 1 for the extra term produces $n$ terms.

A second proof is given but if you follow the above that is great.

### *THE POWER RULE (GENERAL VERSION)*

The only difference here is that $n$ can be any real number, not just a positive Integer. In ***Example 2***, note that $n$ is a negative integer and then a fractional exponent. Fortunately the form of the derivative stays the same.

### THE CONSTANT MULTIPLE RULE: $\dfrac{d}{dx}(cf) = c\,\dfrac{df}{dx}$

The derivative of a constant times a function is the constant times the

derivative of the function.

**Example.** The derivative of $x^4$ is $4x^3$ by the Power Rule.

Then the derivative of $5x^4$ is $5(4x^3)$ or $20x^3$. In the proof, the constant $c$

is a common factor in the numerator. It then jumps in front of LIM by a Limit

Law from ***section 2.3*** and the desired result follows.

### THE SUM RULE

The derivative of a sum is the sum of the derivatives. A large part of the proof involves using the definition of a derivative three times and also in the middle line rearranging the terms.

### THE DIFFERENCE RULE

The derivative of a difference is the difference of the derivatives.

Same comments as for the Sum Rule.

The last three rules indicate there are no twists in taking derivatives of

$$c\,f(x)\ ,\ \ f(x)+g(x)\,,\ \text{and}\ \ f(x)-g(x)\ .$$

Very straightforward. The same is *not* true for a product of two functions.

Also there is another notation introduced on ***page 176*** *under* ***The Sum Rule***. $(f+g+h)'$ represents the derivative of the sum, *f(x) + g(x) + h(x)*. It allows for concise descriptions and will be used frequently.

## DERIVATIVES OF POLYNOMIAL FUNCTIONS

It now becomes quite easy to find the derivative of *any* polynomial. ***Example 5, Sec. 3.1*** illustrates the process. The two middle lines provide descriptions of the use of the above differentiation formulas. With practice, you will go directly from line 1 to line 4.

## EXPONENTIAL FUNCTIONS

This derivation requires careful reading and lots of thinking. The starting point is the definition of a derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)\ -f(x)}{h}\,.$$

As shown in the text this leads to:

$$f'(x)=a^{x}\lim_{h\to 0}\frac{a^{h}-1}{h}\,.$$

Because $a^x$ is fixed as *h* gets smaller, it is treated as a constant. Unfortunately, none of the limits we just discussed are of any help in finding the limit. If we plug in numbers will we see anything? Not if we leave the base as the constant $a$.

$$\frac{a^{.01}\ -\ 1}{h}\ \text{and}\ \frac{a^{.001}\ -\ 1}{h}\ \text{reveal nothing.}$$

The equation:

$$f'(x)=a^{x}\lim_{h\to 0}\frac{a^{h}-1}{h}$$

can be changed to:

$$f'(x)=f'(0)a^{x}$$

but this doesn't solve the problem of finding the limit. To see why the last equation is correct, replace *x* with 0 in the definition:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}$$

The next step is to look at the limit for particular values of $a$. Your understanding will be greater if you use your calculator to verify the numbers in the table on ***page 178***. Then go beyond the entries in the table and let *h* get closer to 0. Try *h* = 0.00001 up to *h* = 0.0000000001 (nine zeros), you will get a better feeling for the limit and recognize the limitations of calculators. The limit of $\frac{2^h - 1}{h}$ is not 0 as your calculator suggests for very small values of *h*.

## A VERY SPECIAL NUMBER

$$\text{For } a = 2, \lim_{h\to 0} \frac{a^h - 1}{h} \cong 0.69 = f'(0)$$

$$\text{For } a = 3, \lim_{h\to 0} \frac{a^h - 1}{h} \cong 1.10 = f'(0)$$

Is there a value for $a$ between 2 and 3 that would produce a limit equal to 1? If so, then

$$f'(x) = f'(0)a^x$$

would become

$$f'(x) = 1 \cdot a^x = a^x$$

and we would have the simplest derivative possible.

How would we find the number that produces a limit equal to 1? At this time the only method available is trial and error. In the following table, a calculation is made only for *h* = 0.00001 and the indicated values for $a$:

| h | $\frac{2.5^h - 1}{h}$ | $\frac{2.6^h - 1}{h}$ | $\frac{2.7^h - 1}{h}$ | $\frac{2.8^h - 1}{h}$ | $\frac{2.9^h - 1}{h}$ |
|---|---|---|---|---|---|
| 0.00001 | 0.9163 | 0.9555 | 0.9933 | 1.0296 | 1.0647 |

The table indicates that the value we are looking for is between 2.7 and 2.8 and closer to 2.7. We try again:

| h | $\dfrac{2.71^h - 1}{h}$ | $\dfrac{2.72^h - 1}{h}$ | $\dfrac{2.73^h - 1}{h}$ | $\dfrac{2.74^h - 1}{h}$ | $\dfrac{2.75^h - 1}{h}$ |
|---|---|---|---|---|---|
| 0.00001 | 0.997 | 1.0006 | 1.0043 | | |

We see the value is between 2.71 and 2.72. Despite the fact that the method is cumbersome we could continue and get greater accuracy. As mentioned in the text this very special number is denoted by $e$. So for this new function we have a simple derivative.

$$\frac{d}{dx} e^x = e^x .$$

The function, $e^x$, is the only function you will encounter whose derivative is the same as the function. A lot of work produces a simple result.

If you check other calculus texts you may find that a different approach is used in the derivation of the above derivative. We say that the approach may be different but it will also be very challenging. There isn't an easy way to get the above derivative.

We add that the number $e$ is also expressed as a limit on ***page 222***:

$$e = \lim_{x\to 0} (1 + x)^{1/x}$$

The table on this page leads to an approximation to 8 decimal places in just 8 steps. How this is related to the above will be covered later. A first glance at the above limit suggests that the limit is $1^\infty = 1\cdot 1\cdot 1\cdot \;\; \cdot 1 = 1$ because $1 + x \to 1 + 0$ and $\dfrac{1}{x} \to \infty$ as $x$ gets smaller and smaller. But this is *not* the case:

$$\text{for } x = 0.1, \quad (1 + \text{x})^{1/\text{x}} = (1.1)^{1/0.1} = (1.1)^{10} = 2.5937.$$

The values of $(1 + x)^{1/x}$ with $x$ varying from 0.01 to 0.00001 are shown in the following table:

| $x$ | 0.01 | 0.001 | 0.0001 | 0.00001 |
|---|---|---|---|---|
| $(1 + x)^{1/x}$ | 2.7048 | 2.7169 | 2.7181 | 2.7183 |

In just five steps we see that the value is 2.718 to three decimal places.

### SUMMARY

Because of the complexity of the above discussion, we summarize the result. There is a mysterious number called $e$ such that an exponential function with this number as the base has a derivative equal to the function:

$$\frac{d}{dx} e^x = e^x$$

We can say that the exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications. This does not mean that the frequent appearance of this function will require the above complex reasoning. The focus will be on the resulting derivative.

## SECTION 3.2: THE PRODUCT AND QUOTIENT RULES

### THE PRODUCT RULE

Finding the derivative of a product is more complicated than finding the derivative of a sum. The rule on ***page 183*** can also be written as

$$(f g)' = f g' + g f'$$

or

$$\left[ f(x) g(x) \right]' = f(x) g'(x) + g(x) f'(x)$$

One way of remembering the rule is to repeat the words in italics below **The Product Rule box**. *The derivative of a product is the first function times the derivative of the second function plus the second function times the derivative of the first function*. This rule will be used many times in this course so committing it to memory now is a good step.

But you might say I want to understand it, not just memorize it. This, then, requires that you look carefully at the derivation. In particular, the line labeled 1 on ***page 183*** indicates there are three terms in the algebraic form for the change in the product, *uv*. Then divide by delta x and find the limit as delta x approaches zero. The third term approaches zero and the first two terms lead to what we call the Product Rule.

Now realize that *using* the rule is different than *deriving* it. Following the above verbal pattern is one approach or you can devise your own scheme. In any case, the Product Rule provides a method that is much simpler than using the definition of a derivative.

### THE QUOTIENT RULE

For the quotient of two functions the pattern for finding the derivative can be written

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\,f'(x) - f(x)g'(x)}{[g(x)]^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{g\,f' - f\,g'}{g^2}$$

In verbal form, we have *the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.* The verbal form may seem like a mouthful, but the words do create the *pattern* you must follow when using the Quotient Rule. Regardless of the mental process you use, pay particular attention to the first term, $g\,f'$. You must start with the denominator times the derivative of the numerator, not $f\,g'$. If carelessness leads you to reverse these terms, you will get the opposite of the correct answer. The simple form,

$$4 - 7 \text{ is the opposite of } 7 - 4$$

is algebraically equivalent to

$$gf' - fg' = -\left(fg' - gf'\right).$$

The reversal will produce a frustrating *sign* mistake.

## SECTION 3.3: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

***Appendix D*** contains a brief review of trigonometry. All material on these pages needs to be understood well. Although the first semester of calculus does not require extensive trigonometry, confidence that you know basic ideas is important. In particular, on ***page A24***, the formula for the length of an arc of a circle is developed. Using ***Figure 1***, the ratio of the central angle $\theta$ to $2\pi$ is the same as the ratio of the arc length $a$ to the circumference of the circle $2\pi r$. $\frac{\theta}{2\pi} = \frac{a}{2\pi r}$ Solving for $\theta$ produces $\theta = \frac{a}{r}$, which in turn leads to arc length $a = \theta r$, where $\theta$ is measured in radians. This formula will be needed in ***section 3.3***.

On ***page A26***, the definitions of the six trigonometric functions in terms of the *adjacent* and *opposite* sides and the *hypotenuse* are very useful. Also be sure that the identities ***(6)*** through ***(9)***, ***15(a)***, and ***15(b)*** are in your head. The others aren't used as extensively and can be looked up when needed. The graphs of the six basic trigonometric functions are essential, especially the $\sin x, \cos x,$ and $\tan x$ graphs.

### COFUNCTIONS

In this section, the concept of cofunction will be significant.

Cosine x is the cofunction of sine x.
Cotangent x is the cofunction of tangent x.
Cosecant x is the cofunction of secant x.

### OVERVIEW

In ***section 3.3***, the derivatives of the six trigonometric functions are derived. The greatest challenge is to derive the derivatives for $\sin x$ and $\cos x$. Then the derivatives of the other four functions can be found by using the Quotient Rule and basic identities.

### TWO IMPORTANT LIMITS

The most significant statement in this section is ***(4)*** on ***page 192***,

$$\frac{d}{dx}\sin x = \cos x$$

The pages preceding this statement lay the foundation for the derivation of this derivative. The first step is to derive the limits shown in ***(2)*** and ***(3)***. These limits are needed to prove that the derivative of $\sin x$ is $\cos x$.

Using ***Figure 2(a), Sec. 3.3,*** note that $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{OB} = \frac{BC}{1}$. Hence, $BC = \sin\theta$. Next in triangle CAB, the leg BC is smaller than the hypotenuse AB. And hypotenuse AB is smaller than arc AB. So leg BC is the smallest and arc AB is the largest.

$$BC < \text{arc } AB$$

The length of arc AB is $\theta\, r$ or arc AB = $\theta$ because the radius $r$ is equal to 1. Then $\sin\theta < \theta$, which can be rewritten as $\frac{\sin\theta}{\theta} < 1$. Call this (A).

We next use ***Figure 2(a)***, to establish $\theta < \tan\theta$. In ***Figure 2(a)***, note that arc AB is smaller than AD.

$$\text{arc AB} < \text{AD}$$

We know that arc AB = $\theta$, and from ***Figure 2(a)***,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AD}{OA} = \frac{AD}{1},$$

which leads to $\theta < \tan\theta$.

This is the same as $\theta < \frac{\sin\theta}{\cos\theta}$ or $\cos\theta < \frac{\sin\theta}{\theta}$.

Now combine $\cos\theta < \frac{\sin\theta}{\theta}$ with (A) above $\frac{\sin\theta}{\theta} < 1$,

and we have $\cos\theta < \frac{\sin\theta}{\theta} < 1$. We now conclude that $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$ because

$\cos\theta \to 1$ as $\theta \to 0^+$ and $\frac{\sin\theta}{\theta}$ is sandwiched between 1 and $\cos\theta$ which is

approaching 1. The text shows that the limit is the same when $\theta \to 0^-$ to establish the two sided limit,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad\qquad \text{(B)}.$$

The second important limit is $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$.

The proof on ***pages 191-192*** uses the identity, $\sin^2\theta + \cos^2\theta = 1$, rewritten in the form $\cos^2\theta - 1 = -\sin^2\theta$. The goal is to introduce $\sin\theta$ in the numerator and be able to use (B) above. The factored form of $\cos^2\theta - 1$ is $(\cos\theta - 1)(\cos\theta + 1)$. The key step is multiplying by $\frac{\cos\theta + 1}{\cos\theta + 1}$ to get the desired forms. Pay particular attention to the two limits, $-\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = -1$ and $\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1} = \frac{0}{2} = 0$.

In the second limit you can use direct substitution, but the first required the above lengthy derivation. Also in the second limit we need, $\sin 0 = 0$ and $\cos 0 = 1$. The easiest way to remember these results without a calculator is from the *graphs* of $y = \sin x$ and $y = \cos x$. This method works well for $\sin x$ or $\cos x$ equaling zero, $+1$, or $-1$. More on this later.

### DERIVATIVE OF SIN *X*

We can now complete the derivation on ***page 191***.

If $f(x) = \sin x$, then $\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$.

To simplify the numerator, we need identity ***12a*** from ***page A29***,

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

Then

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Note that $\sin x$ is a common factor in the first and third terms:

$$\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} = \frac{\sin x(\cos h - 1) + \cos x\sin h}{h},$$

which in turn equals, $\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$.

Now we can take the limit which is $\sin x\cdot 0 + \cos x\cdot 1 = \cos x$

So after much work we have $\frac{d}{dx}\sin x = \cos x$.

## DERIVATIVE OF COS *X*

The second basic derivative is $\frac{d}{dx}\cos x = -\sin x$ ***(5) page 193***. There is symmetry in the two forms but in the second a negative sign appears. The two proofs are very similar except for:

1. In $(\sin x)'$ we used $\sin(x+h) = \sin x\cos h + \cos x\sin h$
2. In $(\cos x)'$ we use $\cos(x+h) = \cos x\cos h - \sin x\sin h$

Without going into details on the derivation of $(\cos x)' = -\sin x$, the negative sign appears because of the negative sign in $\cos(x+h)$

## THE DERIVATIVES OF TAN *X* AND COT *X*

Because of the identities, $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, the Quotient Rule can be used to find the derivatives of $\tan x$ and $\cot x$.

1. $\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \sec^2 x$ as shown in the text, ***page 193***.
2. $\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x.$$

Comparing the two results, one is positive, the other is negative; both involve squares; and the bases are cofunctions. This suggests memorizing $\sec^2 x$ as the derivative of $\tan x$, and then substitute the cofunction $\csc x$ for $\sec x$ and insert a negative sign.

## THE DERIVATIVES OF SEC *X* AND CSC *X*

Because of the identities, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$

the Quotient Rule can also be used to find the derivatives of $\sec x$ and $\csc x$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)\cdot 0 - 1\cdot(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$$

$$= \sec x\tan x.$$

So $$\frac{d}{dx}\sec x \;=\; \sec x\,\tan x\,.$$

Does the negative cofunction rule hold? Yes, it does.

$$\frac{d}{dx}\csc\,x \;=\; -\csc x\,\cot x\,.$$

This can be verified by using the Quotient Rule on $\dfrac{1}{\sin x}$

## FINDING DERIVATIVES

1. Always look for *products* and *quotients* first and use the Product and Quotient Rules when they are found. Remember, $(f\,g)' \neq f'\cdot g'$.

2. Simplify your result. When using derivatives, the simplest form will be essential.

3. If trigonometric functions are involved, use basic trigonometric identities to simplify. This is much easier if the basic identities are in your head and not just a list on a sheet of paper.

## SPECIAL LIMIT PROBLEMS

The fact that the limit, $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, is so significant in this section leads to problems like those in ***Examples 5*** and ***6, Sec. 3.3***. The intent is to emphasize the limit by requiring that you use it in finding limits. First read the form carefully. The angle $\theta$ in the denominator must be the *same* as the $\theta$ in the numerator $\sin\theta$. Change either one and the limit is *not* one. You can use a calculator, in radian mode, and show that $\lim_{x\to 0}\dfrac{\sin 2x}{x}$ is two instead of one. To do this algebraically, rewrite $\dfrac{\sin 2x}{x}$ as $\dfrac{2\,\sin 2x}{2x}$. In the second form, the angles $2x$ are the same and

$$\lim_{x\to 0}\frac{2\,\sin 2x}{2x} \;=\; 2\lim_{x\to 0}\frac{\sin 2x}{2x} \;=\; 2\cdot 1 \;=\; 2\,.$$

In ***Example 5***, note that the rewrite, $\frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$ has the same angle, $7x$, on top and bottom. The limit then is $\frac{7}{4}\cdot 1$ or $\frac{7}{4}$.

In ***Example 6***, the first goal is to rewrite to get the form $\frac{\sin x}{x}$ and then take the limit.