Module 3

SECTION 7.6: INTEGRATION USING TABLES AND COMPUTER ALGEBRA SYSTEMS

In the two preceding modules, we covered six major procedures for evaluating integrals. These procedures will enable you to follow discussions in any text that involve the process of integration. If you had to evaluate a large number of integrals requiring, say, trig substitutions, the work would be time-consuming and repetitious. This section discusses two other possibilities — the use of tables of integrals and computer algebra systems. If you have access to a CAS, by all means investigate its capabilities. The ones mentioned in the text — Derive, Mathematica, and Maple — are very powerful tools and will have an impact on the study of calculus that is not yet clearly defined. However, a CAS is not required for this course, so we limit our comments to the use of tables of integrals.

TABLES OF INTEGRALS (*Reference Pages 5-10*)

First, note the ten headings in the Reference Pages at the back of the book that identify forms like $\sqrt{a^2 + u^2}$ or $a + bu$. You may need to make a substitution to find a matching form. For example,

$$\int e^x \ \sec h(e^x)\ dx$$

will match entry ***107*** if you let $u = e^x$. Because $du = e^x\ dx$,

$$\int e^x \ \sec h(e^x)\ dx \ = \ \int \sec h(u)\ du$$

and the table indicates that the answer is $\tan^{-1}\left|\sin h\ e^x\right| + C$.

A second major point is to make changes when necessary to get an *exact* match. Consider,

$$\int \frac{\sqrt{9x^2 - 1}}{x^2}\ dx$$

It appears to be close to entry ***42*** in the table. Let $u = 3x$ and $a = 1$. The denominator x^2 must be changed to $9x^2$ to match u^2. Also note that $du = 3\ dx$. Then

$$\frac{9}{3}\int \frac{\sqrt{9x^2 - 1}}{9x^2}\, 3dx \;=\; 3\int \frac{\sqrt{u^2 - 1}}{u^2}\, du$$

and the rest is easy. Multiply the answer in entry ***42*** by 3, and replace u with $3x$ and a with 1.

1. A pattern that is sometimes used in the last problem involves substituting $u/3$ for x and $du/3$ for dx. The *9* and the *3* appear in different places but the result will be the same.

$$\int \frac{\sqrt{9x^2 - 1}}{x^2}\, dx \;=\; \int \frac{\sqrt{u^2 - 1}}{u^2/9}\, du/3 \;=\; 3\int \frac{\sqrt{u^2 - 1}}{u^2}\, du$$

Two other observations:

2. ***Example 3, page 510***, uses the reduction formula in entry ***84***. A similar procedure should be used to evaluate

$$\int \sec^5 x\; dx$$

The reduction formula is entry *77* in the table.

$$\int \sec^n u\; du \;=\; \frac{1}{n - 1}\tan u\, \sec^{n-2} u \;+\; \frac{n - 2}{n - 1}\int \sec^{n-2} u\; du$$

Then, with $n = 5$,

$$\int \sec^5 x\; dx \;=\; \frac{1}{4}\tan x\, \sec^3 x \;+\; \frac{3}{4}\int \sec^3 x\; dx$$

Then use the reduction formula a second time, with $n = 3$,

$$\int \sec^3 x\; dx \;=\; \frac{1}{2}\tan x\, \sec x \;+\; \frac{1}{2}\int \sec x\; dx$$

or use entry ***71***.

Then

$$\int \sec^5 x\; dx \;=\; \frac{1}{4}\tan x\, \sec^3 x \;+\; \frac{3}{4}\left(\frac{1}{2}\tan x\, \sec x \;+\; \frac{1}{2}\int \sec x\; dx\right)$$

The final answer would be

$$\frac{1}{4}\tan x\,\sec^3 x \;+\; \frac{3}{8}\tan x\,\sec x \;+\; \frac{3}{8}\ln\left|\sec x \;+\; \tan x\right| + C$$

3. In ***Example 4, page 502***, note that the trinomial, $x^2 + 2x + 4$, was changed to a binomial form, $(x + 1)^2 + 3 = u^2 + 3$, by completing the square. A more extensive table would include entries for trinomials.

SECTION 7.7: APPROXIMATE INTEGRATION

In this section we cover five numerical methods of approximating integrals. Why are they needed? As pointed out in the first three paragraphs of this section, some integrals do not have an antiderivative. The integral $\int e^{-x^2}\,dx$ is important in statistics, but no function exists whose derivative is e^{-x^2}. We can, however, approximate its value by one of five methods listed in this section.

THE FIVE METHODS

The first method appears on ***page 515***.

$$\text{(1)} \qquad \int_a^b f(x)\,dx \;\approx\; L_n \;=\; \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

It is essential that you match the summation

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x \;=\; f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

with ***Figure 1(a)***. The first term, $f(x_0)\Delta x$, is the area of the leftmost rectangle in ***Figure 1(a)***. The height, $f(x_0)$, involves the left endpoint x_0. The same is true for each of the other rectangles. Look at the base of each rectangle and note that the height measures from the left endpoint. The symbol L_n, representing the sum,

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

indicates the *L*eft endpoint is used.

The second method, also on the same page,

$$(2) \qquad \int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

contains just one change in the summation: $f(x_i)$ replaces $f(x_{i-i})$. Now when $i = 1$, we have $f(x_1)\Delta x$ instead of $f(x_0)\Delta x$. ***Figure 1(b)*** shows that we are now using the *R*ight endpoint of the base of each rectangle, matching the symbol R_n.

The third method

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

is similar except that we now use the midpoint, $\bar{x}_i$, of each base. The formula $\Delta x = \dfrac{b - a}{n}$ divides the interval [*a,b*] into *n* equal parts. This formula applies to the first two methods also.

The fourth method

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

is described as an average and as the Trapezoidal Rule. ***Figure 2*** shows how all heights, $f(x_i)$, are used twice except for the first and last. Another way of describing this is that the middle heights are "weighted" more heavily.

The last method is Simpson's Rule on ***page 520***.

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[\begin{array}{r} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3)\ldots + 2f(x_{n-2}) \\ + 4f(x_{n-1}) + f(x_n) \end{array}\right]$$

This formula is based on using the area under parabolas instead of rectangles or trapezoids. The result of the derivation is that $f(x_1)$, $f(x_3)$ etc. are weighted more heavily than $f(x_2)$, $f(x_4)$, etc., with $f(x_0)$ and $f(x_n)$ carrying the least weight. Note also that Δx is divided by 3 instead of 2.

ESTIMATING ERRORS

It is important to recognize that the Trapezoidal Rule or Simpson's Rule shouldn't be used if one can find an antiderivative. So when a number comes out of either of these methods, we don't know by staring at the number how accurate it is. It is only an approximation, and knowing its accuracy can be essential. Hence, the error bounds on ***page 518*** and on ***page 522*** are significant.

The intent in this section is not mastery of numerical methods but an exposure to the concepts of approximation and estimation of errors. Courses in numerical analysis go into much greater detail. Here we just point out that it is possible to estimate the error when using the Midpoint, Trapezoidal or Simpson's rules.

For the Midpoint and Trapezoidal rules one must find an upper bound K for the second derivative over the interval $[a,b]$. There are two things to keep in mind when finding K.

1. Make the numerator as *large* as possible

and 2. make the denominator as *small* as possible.

EXAMPLE

Find an upper bound for $\dfrac{(x-2)^2}{5x}$ on the interval [1,5].

The largest value of $(x-2)^2$ on [1,5] occurs at $x = 5$.

The smallest value of $5x$ on [1,5] occurs at $x = 1$.

Hence $\left|\dfrac{(x-2)^2}{5x}\right| \le \dfrac{(5-2)^2}{5\cdot 1} = \dfrac{9}{5}$

It is OK to use different values for x in the numerator and the denominator. We can assert that $\dfrac{(x-2)^2}{5x}$ will not be larger than *9/5* on [1,5] . Finding the upper bound is not the same as finding the maximum value of $\dfrac{(x-2)^2}{5x}$ on the interval. The maximum value is the "best" value, which is also called the least upper bound. If we use calculus methods to find the maximum value (setting the first derivative equal to zero, etc.) it is more time-consuming than the method in the last example. Of course, you could also use a graphics calculator to find an upper bound or K value. Just remember to graph the second derivative for the Trapezoidal Rule or the fourth derivative for Simpson's Rule.

SECTION 7.8: IMPROPER INTEGRALS

This section serves a dual purpose. First, the last two types of integrals are introduced. But instead of serving as an ending, this section provides a tie-in with another major topic in this course, namely infinite series. The ideas in this section provide an important base for future work.

TYPE 1 INFINITE INTERVALS

Pay special attention to

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = 1 - \frac{1}{t}$$

on ***page 527. Figure 1*** illustrates how $\left(1 - \frac{1}{t}\right)$ matches the area under the curve $y = \frac{1}{x^2}$ from $x = 1$ to $x = t$. It is important that this concept remain crystal clear in your mind. Then below ***Figure 1***, you can visualize the limit

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

The area approaches 1 as shown in ***Figure 2***.

The hope is that this seems like a totally logical process to evaluate the integral $\int_1^\infty \frac{1}{x}\,dx$ and that the definition on ***page 528*** merely describes this process. We first integrate from a to t (or from t to b) and then take a limit of the result by letting t approach infinity.

This type of integral is called a Type 1 *improper* integral to distinguish it from other integrals. Also the word *convergent* is significant. If an improper integral approaches a *finite* number, then we say it *converges* to this finite number. If it does not converge, then we say it *diverges*. *Divergent* is the opposite of *convergent,* and *converge* means to approach a finite limit.

Example 4, will have significance later. We illustrate the result, summarized as **(2)** on ***page 531***.

(2) $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

First consider the integral,

$$\int_1^\infty \frac{1}{x^{1.1}}\,dx$$

Using negative exponents we have,

$$\int_1^\infty x^{-1.1}\,dx = \lim_{t\to\infty}\left(\left.\frac{x^{-0.1}}{-0.1}\right]_1^t\right) = \lim_{t\to\infty}\left(\frac{-10}{t^{0.1}} + 10\right)$$

The first term will approach zero because the denominator will get larger and larger. The denominator, $t^{0.1} = t^{1/10} = \sqrt[10]{t}$, but the tenth root of larger and larger numbers will also get larger and larger, and the fraction will get smaller and smaller. So the integral converges to 10. Note that p is equal to 1.1, which is greater than one. This agrees with **(2)**.

Next consider,

$$\int_1^\infty \frac{1}{x^{0.9}}\,dx$$

Again using negative exponents, we have

$$\int_1^\infty x^{-0.9}\,dx = \lim_{t\to\infty}\left(\left.\frac{x^{0.1}}{0.1}\right]_1^t\right) = \lim_{t\to\infty}\left(10t^{0.1} - 10\right)$$

Now the first term approaches infinity because $t^{0.1} = \sqrt[10]{t}$ is in the numerator. Now we say the integral is divergent because it does not approach a finite number. In this case, p is equal to 0.9, which is less than one. This agrees with **(2)**.

Summary. The key in the last two integrals is whether p is larger than one or is smaller than one. If p is larger than one, then after adding one to negative p, $-p + 1$ is still a negative exponent which pushes t into the denominator. When we let t approach infinity, the denominator approaches infinity and the fraction approaches 0. Otherwise, if p is less than one, the limit is infinity and the integral does not converge.

TYPE 2 DISCONTINUOUS INTEGRANDS

A Type 2 *improper* integral

$$\int_a^b f(x)\, dx$$

looks like an ordinary definite integral except that the integrand $f(x)$ has a point of *discontinuity*

	1.	at $x = b$, the upper limit
or	2.	at $x = a$, the lower limit
or	3.	at $x = c$, where c is some value between a and b.

Look carefully at ***Figures 7, 8***, and ***9*** on ***page 531*** as you read the definition on the same page. The graphs will indicate the similarity between the three types of integrals. The words *convergent* and *divergent* have the same meaning as discussed above. Again you can check your understanding of improper integrals by seeing in the graphs the behavior described. When things aren't making sense, it may be because you aren't matching words with a mental picture suggested by the graphs.

COMMENTS

1. ***Example 6***. The limit of *sec x* as *x* approaches $\frac{\pi}{2}$ from the left will be more meaningful if you look at the graph of $y = sec\, x$ in the appendix on ***page A31***. On the same page check the graph of $y = tan\, x$ to confirm that *tan x* also approaches infinity.

2. ***Example 8***. Again a graph indicates a vertical asymptote at $x = 0$. For $\lim_{t\to 0+} (t \ln t)$ the graph indicates that the limit is of the form $0\ (\infty -\cdot)$. You may wish to review indeterminate products on ***page 308***.

3. Comparison Theorem, ***page 533***. At first glance, this theorem may seem a bit complex. But remember that the two integrals

   $$\int_a^\infty f(x)\, dx \text{ and } \int_a^\infty g(x)\, dx$$

   can be thought of as representing areas because both *f(x)* and *g(x)* are positive or zero. In this context, a *convergent* integral represents a *finite* area, while a *divergent* integral represents an area that is infinite. The fact that *f(x)* is larger than or equal to *g(x)* leads to the possible graph shown in ***Figure 12***. The paragraph ***below the Comparison Theorem box*** describes parts (a) and (b) of the Comparison Theorem. Just substitute *finite area* for *convergent* and *infinite area* for *divergent.*

It is well worthwhile to study comment 3 carefully. This type of reasoning will appear later in the study of infinite series.

4. ***Example 9***. This example shows how the Comparison Theorem can be used to show that the integral $\int_0^\infty e^{-x^2}\,dx$ is convergent. A quick reading of this example is sufficient to get a general idea of the method. A crucial part of the discussion is the inequality

$$e^{-x^2} \leq e^{-x}$$

Rewrite it as

$$\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$$

and then remember that the larger denominator produces the smaller fraction. This holds only for $x \geq 1$, which is why the original integral was written as the sum of two integrals.