Module 12

SECTION 6.2: VOLUMES

The concept of volume provides the second application of a definite integral. Now

$$\lim \sum f(x_i)\,\Delta x_i$$

is replaced by

$$\lim \sum A(x_i)\,\Delta x_i$$

where $A(x_i)$ is the *area* of a cross-section of a solid. Examples of cross-sections are shown in red in almost every drawing in this section. Artists have talents most of us do not have, so these drawings present clearer images than most of us can draw. However, it is possible and necessary to visualize a solid and the perpendicular cut that we call a cross-section. Areas require just two dimensions, length and width, but volumes relate to three dimensional solids. Seeing things in three dimensions or 3-space is a greater challenge. More on this later.

CYLINDERS

It is also important to understand the concept of a cylinder and the volume of a cylinder. In ***Figure 1***, note the congruent regions in parallel planes and the perpendicular line segments connecting these two regions. Because the line segments are perpendicular, each solid is called a *right* cylinder. The formula for the volume of each cylinder, $V = Ah,$ indicates that volume is just the area A of the base times the height. It will help in what follows if this is crystal clear. Starting with A , the area of the base, mentally create a solid one unit high ($h = 1$). Volume is the number of cubic units in the solid, which in this case is A . Each square in the base becomes a cube (or each part of a square becomes part of a cube). Since there are A square units in the base, there are A cubic units in our solid, which is one unit high. Think of this as one layer. If $h = 2,$ we have two layers and $2A$ cubic units and for h layers we have Ah cubic units. Hence the formula,

$$V = Ah$$

Why is this so important? Because we are going to approximate the volume of a solid by creating a series of cylinders as shown in ***Figure 3*** and ***Figure 5***. The volume of the ith cylinder is $A(x_i)\,\Delta x_i,$ where $A(x_i)$ is the area of the base of a cylinder and Δx is the height or thickness of a cylinder. If you think the base should be on the bottom, just imagine tilting the cylinders into an upright position. Or think of slicing a loaf of bread and each slice is one cylinder. Then the sum of all cylinders, $\sum A(x_i)\,\Delta x_i$, is an approximation to the volume of the solid and a limit, $\lim \sum A(x_i)\,\Delta x_i$, makes it exact.

By the Fundamental Theorem, $\int_a^b A(x)\,dx$ will give the exact volume.

The key is going to be the visualization of a cross-section of the solid. The cross-section is the base of a cylinder and $A(x)$ is the area of the cross-section. Multiplying by the height Δx or dx changes area into volume.

SOLIDS OF REVOLUTION

The discussion on ***page 439*** is more general than on ***page 440***. For a solid of revolution, $A(x)$ will be the area of a circle, ***Figure 6***, or the difference of the areas of two concentric circles, ***Figure 8(c)***. Because the area of a circle is πr^2 we need only look for the radius of one or two circles. Visualizing the radius and finding an algebraic representation for it depend on first visualizing the solid itself. A solid of revolution is formed by rotating a two dimensional region about an axis. In the rotation, a single point in the region follows a circular path. The solid in some sense is a set of densely packed circles. To get a cross-section the solid is sliced *perpendicular* to the axis of revolution. The cross-section will be either a circle, ***Figure 4***, or a washer, as shown in ***Figure 8(c)***. In the second case there is a hollow part in the solid.

ALGEBRAIC REPRESENTATIONS OF RADII

In ***Figure 6(b)***, the radius of the circular base is the distance from the x-axis to the curve. This is just the *y* coordinate of a point $(x\,,y)$ on the curve $y = f(x)$. The area of this circle is $\pi\, y^2$ or $\pi\,[f(x)]^2$ and the thickness of the cylinder is Δx or dx. Then

$$V = \int_a^b \pi\left[f(x)\right]^2\,dx$$

will determine the indicated volume.

In ***Figure 7(b)***, the radius of the circular base is the *x* coordinate of a point $(x\,,y)$ on the curve. The area of this circle is πx^2 and the thickness of the cylinder is Δy or dy. In this particular example, start with $y = x^3$ and solve for *x*. The result $x = \sqrt[3]{y}$ has the general form, $x = g(y)$. The area of the circular base is $\pi\left[g(y)\right]^2$ and the volume is

$$V = \int_c^d \pi\left[g(y)\right]^2 \, dy$$

Taking a second look at the integrals,

$$\int_a^b \pi\left[f(x)\right]^2 \, dx \quad \text{and} \quad \int_c^d \pi\left[g(y)\right]^2 \, dy$$

note that x is the only variable in the first and y is the only variable in the second. How do we know which to use? The answer lies in the thickness of the cross-sectional cylinder. Is it Δx or Δy ? Just determine if the thickness involves a change in x or in y. If it is Δx (which becomes dx in the integral) then express the radius in terms of x : if Δy then the radius must be expressed in terms of y.

EXAMPLE 1

In ***Figure 4***, the sphere intersects the xy plane in a circle with radius r . The equation of the circle is

$$x^2 + y^2 = r^2$$

Next imagine a point (x, y) at the top of the red circle. This leads to the x and y coordinates shown in the drawing. In the rotation about the x-axis, the point (x, y) traces out the circular boundary of the red circle with y as the radius. Because the thickness is Δx express the radius y in terms of x.

Also note, on ***page 440***, that

$$\int_{-r}^{r} \pi(r^2 - x^2) \, dx = 2\pi \int_0^r (r^2 - x^2) \, dx$$

This can be justified in two ways:

1. Note that the integrand is even and use ***Theorem (7a), page 417, Sec. 5.5***.

2. Use ***Figure 4, page 440***, and symmetry, and reason that the volume from $x = -r$ to $x = r$ is twice the volume from $x = 0$ to $x = r$. This requires that you visualize the summation process over these two intervals.

This change in the limits of integration simplifies the computation. Compare the following with the shorter form in the text:

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left(r^3 - \frac{r^3}{3}\right) - \pi\left(-r^3 - \frac{-r^3}{3}\right)$$

$$= \pi\left(r^3 - \frac{r^3}{3}\right) + \pi\left(+r^3 - \frac{+r^3}{3}\right)$$

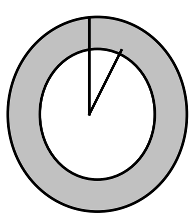
$$= 2\pi\left(r^3 - \frac{r^3}{3}\right)$$

This longer path produces the same result as in the text.

EXAMPLE 4

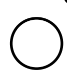
In ***Figure 8(a)***, note the vertical red line segment between $y = x$ and $y = x^2$. As this segment is rotated about the x-axis, it forms the washer shown in ***8(b)*** and ***8(c)***. The radius of the large circle is the y coordinate on the line, which equals x. The radius of the inside smaller circle is the y coordinate on the parabola, which equals x^2. The area of the red band in 8(c) is found by subtracting the area of the small circle from the area of the large circle as shown in the text. As we carry out the summation to get the total volume, x must vary from 0 to 1. Visualize the summation process as you look at Figure 8(b) and 8(a) to get these limits.

If you are tempted to subtract the small radius from the large radius and then square consider the following.



large radius = 3
small radius = 2

large radius - small radius =
3 - 2 = 1



The area of the washer is $\pi 3^2 - \pi 2^2 = 9\pi - 4\pi = 5\pi$.

Subtracting radii first produces a circle with radius 1.

Its area is $\pi 1^2 = \pi$, which is only $1/5$ as large.

A general formula for this type of volume is:

$$V = \pi \int_a^b \left\{[\text{outer radius}]^2 - [\text{inner radius}]^2\right\} dx$$

However, remember that the rotation must be around the x-axis for this formula to be valid.

LIMITATIONS OF FORMULAS

From the disk method we have two general formulas:

1. $\int_a^b \pi[f(x)]^2 \, dx$ Rotation about the x-axis.

2. $\int_c^d \pi[g(y)]^2 \, dy$ Rotation about the y-axis.

The washer method led to:

3. $V = \pi \int_a^b \left\{[\text{outer radius}]^2 - [\text{inner radius}]^2\right\} dx$ Rotation about the x-axis.

However, if you try to use these formulas in a rote way, you won't be able to adjust to variations. In formulas 1 and 2, we are summing cylinders whose circular bases have the area πr^2. If you see the radius r in a drawing, you don't really need a formula. The height or thickness of a cylinder is either Δx or Δy and this can also be seen in a drawing.

The form for formula 3 is $\pi \int_a^b \left[r_L^2 - r_S^2\right] dx$ where the outer or large radius is r_L and the inner or small radius is r_S and $\pi \left[r_L^2 - r_S^2\right]$ is the area of a washer.

EXAMPLE 5

This problem presents greater challenges. First, the rotation is not about the x-axis or the y-axis. From ***Figure 9***, the thickness of a "washer" is Δx and the form

$$\pi \int_a^b \left[r_L^2 - r_S^2\right] dx$$

can be used. We need only find r_L and r_S, which are shown in ***Figure 9*** on the right. The key is that each radius is a vertical line segment and hence of the form $y_2 - y_1$. Each radius starts at the axis of the rotation, which is $y = 2$. The large radius ends on the parabola where $y = x^2$. The small radius ends on the line where $y = x$. Then $r_L = 2 - x^2$, and $r_S = 2 - x$. It is important to see both of these as differences in *y*-values.

EXAMPLE 6

Figure 11 contains an excellent drawing of the red washer that is a cross-section. First note that the thickness of the washer is Δy or dy and the radii must be expressed in terms of *y*.

The two radii will be of the form, $x_2 - x_1$. For the large or outer radius, use the x-value on the parabola but remember this is the horizontal distance from the y-axis to the parabola. Add one to this x-value to go from the parabola to the vertical line, $x = -1$. Or you can also note that $x_2 - x_1$ is $x_P - (-1) = x_P + 1$ where x_P is the x-value on the parabola. From the parabola, $y = x^2$, solving for *x*, we have $x = \sqrt{y}$ and $x_P + 1$ becomes $\sqrt{y} + 1$ as the larger radius.

For the smaller radius, use $x_L + 1$, where x_L is the x-value on the line, $y = x$. The smaller or inner radius is $y + 1$ and the form

$$\pi \int_a^b \left[r_L^2 - r_S^2\right] dy$$

becomes

$$\pi \int_0^1 \left[\left(\sqrt{y} + 1\right)^2 - (y + 1)^2\right] dy$$

When evaluating this integral, you must expand $\left(\sqrt{y} + 1\right)^2$ before integrating. The text also expands the second square of a binomial and combines like terms before showing the integration.

BACK TO BASICS

The volume of each solid is this section was based on the integrals

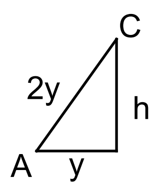
$$\int_a^b A(x)\, dx \quad \text{or} \quad \int_c^d A(y)\, dy$$

where $A(x)$ or $A(y)$ was the area of a cross-section and dx or dy was the height of a cylinder. For solids of revolution $A(x)$ or $A(y)$ was the area of a circle or the area of a washer. The last three examples in this section cover other types of solids (optional).

EXAMPLE 7

First the circular base is described by the equation $x^2 + y^2 = 1$. Also note in ***Figure 13(b)*** the variable y is used in three different ways. As part of the equation of a circle in $y = \sqrt{1 - x^2}$, as a coordinate in the point (x, y), and as the length of a line segment. The length of the line segment is an interpretation of y as a coordinate but still important to remember.

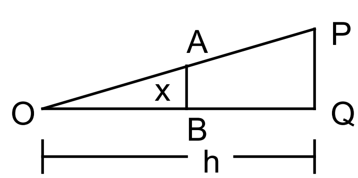
In the equilateral triangle in ***Figure13(c)***, $AC = 2y$



and $(2y)^2 = y^2 + h^2$ leads to $h^2 = 3y^2$ and $h = \sqrt{3}\, y = \sqrt{3}\,\sqrt{1 - x^2}$. $A(x)$ is the area of an equilateral triangle and the summation must extend from $x = -1$ to $x = 1$. We hope that when the word *summation* is used, both $\sum A(x)\,\Delta x$ and $\int_a^b A(x)\,dx$ come to mind.

EXAMPLE 8

In ***Figure 14***, imagine the red square moving from $x = 0$ to $x = h$ and filling up or creating the pyramid. If s is the length of one side of a red square, then s^2, the area of the red square, is $A(x)$. Similar triangles provide a connection between the variables x and s and the constants h and L as shown in ***Figure 15***. As further clarification, consider



where AB is one-half of s and PQ is one-half of L.

Triangle OAB is similar to triangle OPQ; three pairs of angles are congruent. Then corresponding sides are proportional.

OB is to OQ as AB is to PQ

which becomes x is to h as $\frac{s}{2}$ is to $\frac{L}{2}$

or x is to h as s is to L

$$\frac{x}{h} = \frac{s}{L}$$

Solve for s. $\quad \frac{x\,L}{h} = s$

and $\quad A(x) = s^2 = \frac{x^2 L^2}{h^2}$

In the integration, remember $\frac{L^2}{h^2}$ is a constant.

EXAMPLE 9

The key in this example is to see that a cross-section perpendicular to the x-axis is the triangle shown in ***Figure 17***. Because $\tan 30 = \frac{BC}{y}$, $y \tan 30 = BC$. The area of the triangle is

$$\frac{1}{2}\,y\cdot BC = \frac{1}{2}\,y\cdot y\tan 30 = \frac{1}{2}\,y^2\,\frac{1}{\sqrt{3}} = \frac{16 - x^2}{2\sqrt{3}}$$

The summation starts at $x = -4$ and continues to $x = 4$ but

$$\int_{-4}^{4} A(x)\,dx = 2\int_{0}^{4} A(x)\,dx$$

because of symmetry. This change simplifies the computation.

SECTION 6.3: VOLUMES BY CYLINDRICAL SHELLS

A second method for finding volume involves cylindrical shells. A single shell is shown in ***Figure 2***. We are not concerned with the volume inside the cylinder but the amount of "material" in the wall of the shell. Volumes will be approximated by stacking a series of these shells next to each other with their walls touching. ***Figure 4***, attempts to show five shells nestled together with an open center. Unfortunately, the outer shells block the view of the inner shells, so dotted lines are included to suggest the walls of individual shells. The heights of the shells vary to give an approximation of the solid shown in ***Figure 3***.

AN IMPORTANT FORMULA

On ***page 450***, the text develops formula **(1)**

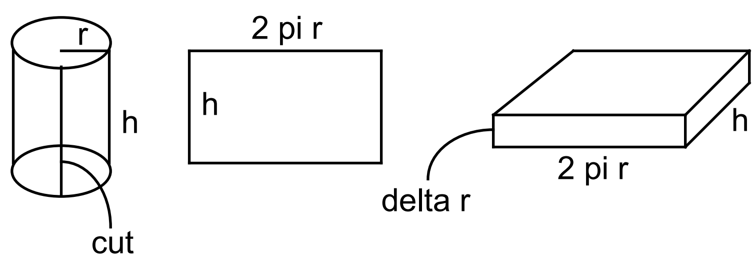
$$V = 2\pi r h \, \Delta r$$

along with

$$V = [\text{circumference}] \cdot [\textit{height}] \cdot [\textit{thickness}]$$

Using the average radius r for the cylinder in ***Figure 2***, imagine making a vertical cut so the cylinder can be flatten into a rectangle. The length of the rectangle will be the circumference of the circular top and the width will be the height of the cylinder.

2 pi r = $2\pi r$



The area of the rectangle is $2\pi r h$. Next let Δr be the thickness of the wall of the shell and the volume is $2\pi r h \Delta r$. Again, this is the volume of the wall of the shell, not the volume inside the shell. The change in the radius, Δr, will be either dx or dy in the integral form.

The form, $2\pi r h \Delta r$, represents the volume of one shell and

the sum $\sum 2\pi r h \Delta r$ is an approximation to the volume of a solid.

The limit $\lim \sum 2\pi r h \Delta r$ gives the exact volume. In integral form, we have

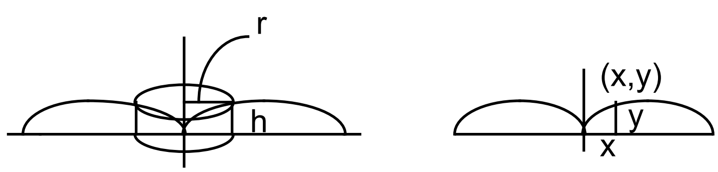
$$\int_a^b 2\pi r h\, dx \text{ with } r \text{ and } h \text{ expressed in terms of } x$$

or

$$\int_c^d 2\pi r h\, dy \text{ with } r \text{ and } h \text{ expressed in terms of } y.$$

SEEING CYLINDRICAL SHELLS

Using the method of cylindrical shells will be straightforward if a representative shell is clearly seen. The wall of a shell is formed by rotating a line segment or thin rectangle about the axis of rotation. The following drawing relates to ***Example 1*** and ***Figure 6***:



Note the radius r and the height h of a shell on the left and the point (x, y) on the right. A comparison indicates that $r = x$ and $h = y$. Because $y = 2x^2 - x^3$ and $\Delta r = dx$

the integral $\int_a^b 2\pi r h\, dx$

becomes $\int_0^2 2\pi x\left(2x^2 - x^3\right) dx$

To agree that the limits are 0 and 2, imagine drawing a shell for a particular value of x, say, $x = \frac{1}{4}$. Then note that the shell extends around the y-axis and passes through $x = -\frac{1}{4}$. This is true for all shells, so summing from 0 to 2 also covers the interval from 0 to -2.

EXAMPLE 2

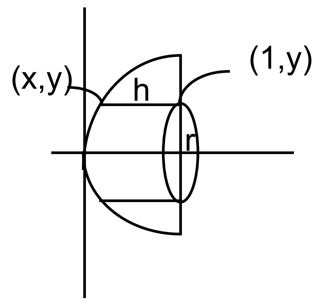
The shell shown in ***Figure 8*** is generated by a thin rectangle extending from the parabola $y = x^2$ to the line $y = x$. The height of the shell is $y_2 - y_1$ or $x - x^2$. The thickness of the wall of the shell is Δx and the radius is the common x-value on the line and the parabola. The region is covered if x varies from 0 to 1. For this situation the integral

$$\int_a^b 2\pi r h\, dx$$

becomes $\int_0^1 2\pi x(x - x^2)\,dx$.

EXAMPLE 3

One shell is shown in the drawing below:



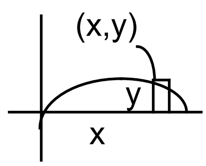
The radius r is the common y-value of the two points shown, and h is a horizontal line segment, $x_2 - x_1$ or $1 - x$. In this case the thickness of the wall is Δy which becomes dy.. Then the volume is

$$\int_0^1 2\pi y(1 - x)\; dy$$

Replace x with y^2 and $\int_0^1 2\pi y(1 - y^2)\; dy$ will determine the volume.

EXAMPLE 4

This problem is a bit more complex, but r and h can be found by the same methods. In ***Figure 10*** put in the point (x, y) as shown in the drawing:



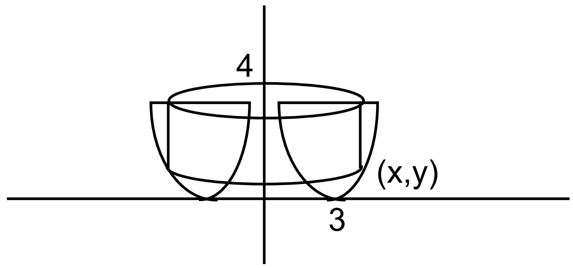
The radius is a horizontal line segment and $x_2 - x_1$ is $2 - x$. The height of a shell is y, which equals $x - x^2$ and the thickness is Δx which becomes dx. Then

$$V = \int_a^b 2\pi r h\, dx$$

becomes $$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx$$

YOUR LAST EXAMPLE IN THIS COURSE

Find the volume of the solid formed by rotating about the y-axis the region bounded by $y = (x-3)^2$ and $y = 4$. By now, do you know that you can't do these problems without a drawing?



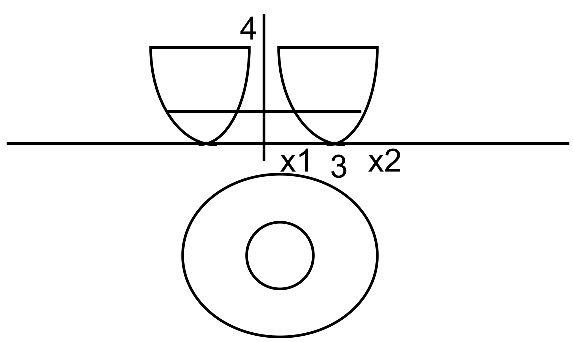
The radius of a shell is the x coordinate of the point shown. The height is a vertical line segment; $y_2 - y_1$ equals $4 - y$. To find the limits of integration, put $y = 4$ into $y = (x-3)^2$, and solve for x. The parabola intersects the line $y = 4$ at $x = 1$ and $x = 5$. Then the volume is

$$\int_1^5 2\pi x(4-y)\,dx.$$

Replace y with $(x-3)^2$ and

$$V = \int_1^5 2\pi x\left[4-(x-3)^2\right]dx = 2\pi\int_1^5 \left(-x^3+6x^2-5x\right)dx = 64\pi$$

Can this be done using the washer method? Yes. It turns out the integration is easier but the setup is more difficult. Another drawing:

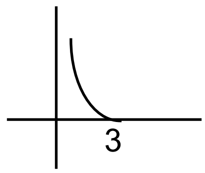


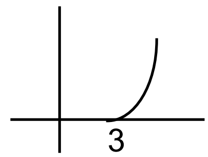
The slice is shown as a horizontal line segment and the bottom pair of circles is a view from below. The radii of the two circles are labeled $x1$ and $x2$ for x_1 and x_2. The area of the band between the two circles is $\pi(x_2)^2 - \pi(x_1)^2$. But how do we distinguish between x_1 and x_2?

Start with $y = (x-3)^2$ and solve for x.

$$\pm y^{1/2} = (x-3)$$

and $x = 3 \pm y^{1/2}$ Then separate the parabola into two parts:





$x = 3 - y^{1/2}$ $\qquad$ $x = 3 + y^{1/2}$

On the left, $x_1 = 3 - y^{1/2}$ and on the right $x_2 = 3 + y^{1/2}$. Then

$$\int_a^b \left[\pi(x_2)^2 - \pi(x_1)^2\right] dy$$

becomes

$$\int_0^4 \left[\pi\left(3 + y^{1/2}\right)^2 - \pi\left(3 - y^{1/2}\right)^2\right] dy$$

The thickness of the washer is Δy, and the summation is from $y = 0$ to $y = 4$. Surprisingly the above integral simplifies to

$$\pi\int_0^4 12 y^{1/2}\, dy = 12\pi \frac{2}{3} y^{3/2} \Big]_0^4 = 8\pi \cdot 8 = 64\pi$$

The first method seems easier, but by now you have developed an abundance of good judgment and can decide for yourself.

Volume problems tend to be challenging, so concentrate on the assigned problems below. Remember to include in your sketch a disk, a washer or a shell, instead of relying on formulas which may need to be adjusted.