Goodwin’s Model Analysis

by Jacob Enerio

So, I wanted to do a mathematical analysis of Goodwin’s Model since the sources I could find didn’t really touch too much on that. Here are the equations for Goodwin’s Model:

$$\frac{\delta\mu}{\delta t} = g_\mu * \mu$$

$$\frac{\delta u}{\delta t} = g_u * u$$

Where μ and u are the employment rate (% of people employed) and wage share of the economy (wages * amount of laborers employed/output, or the % of output going to worker wages) respectively. Since those 2 variables are percentages, they should always be in the range of 0 (exclusive) and 1 inclusive. The natural growth rate of each variable is represented with a g. Also t is time.

The derivative of u is also equivalent to the following:

$$\frac{\delta u}{\delta t} = (g_w - g_\lambda) * u = (g_w - \theta) * u$$

Where g_λ is the natural growth rate of labor productivity and will be represented with θ. g_w is the growth rate of wages and is equal to the Phillips Curve relationship. Here, I will use a linear and exponential Phillips Curve. Here are the Phillips Curve equations for the linear and exponential versions respectively:

$$g_w = -\alpha + \beta\mu$$

$$g_w = -\alpha + \beta e^{k\mu}$$

Where α, β, and k depend on the data for the Phillips Curve. They’re not really representative of a specific economic variable. Anyways, as seen in the equations above, the inflation of wages increases when employment increases (unemployment decreases), and the inflation of wages decreases when employment decreases (unemployment increases).

Now moving back to the employment rate, the equation for the employment rate can also be expressed as:

$$\frac{\delta\mu}{\delta t} = \left(\frac{1}{v} - (\theta + n) - \frac{u}{v}\right) * \mu$$

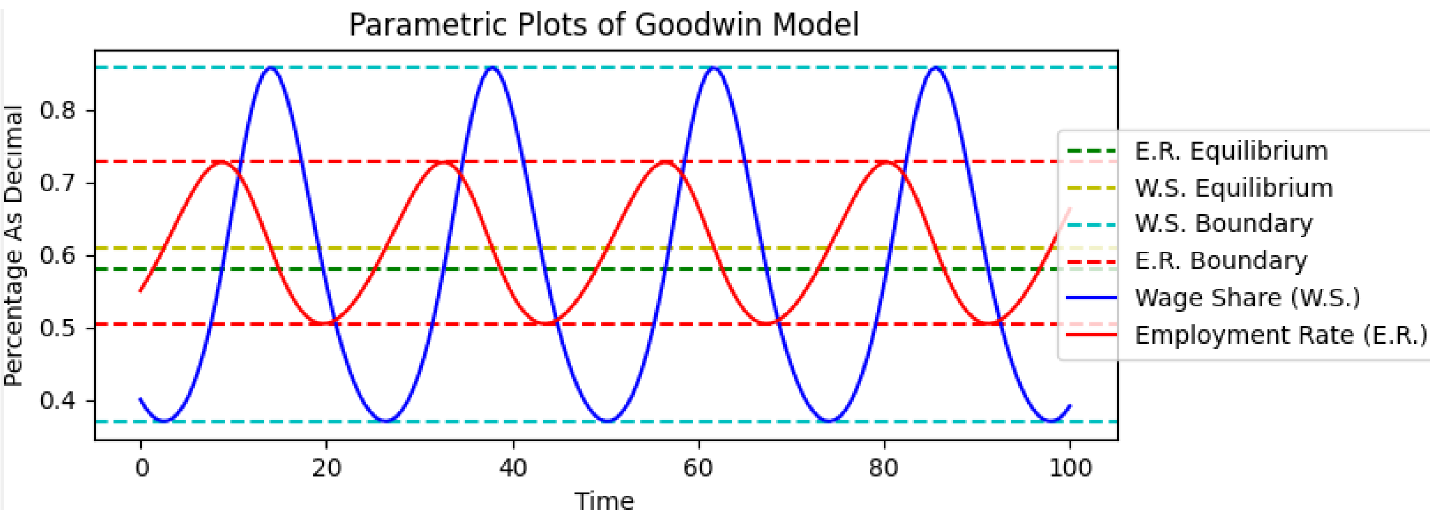
Where v is the capital-output ratio ($\frac{K}{Y}$), and n is the natural growth rate of the amount of workers.

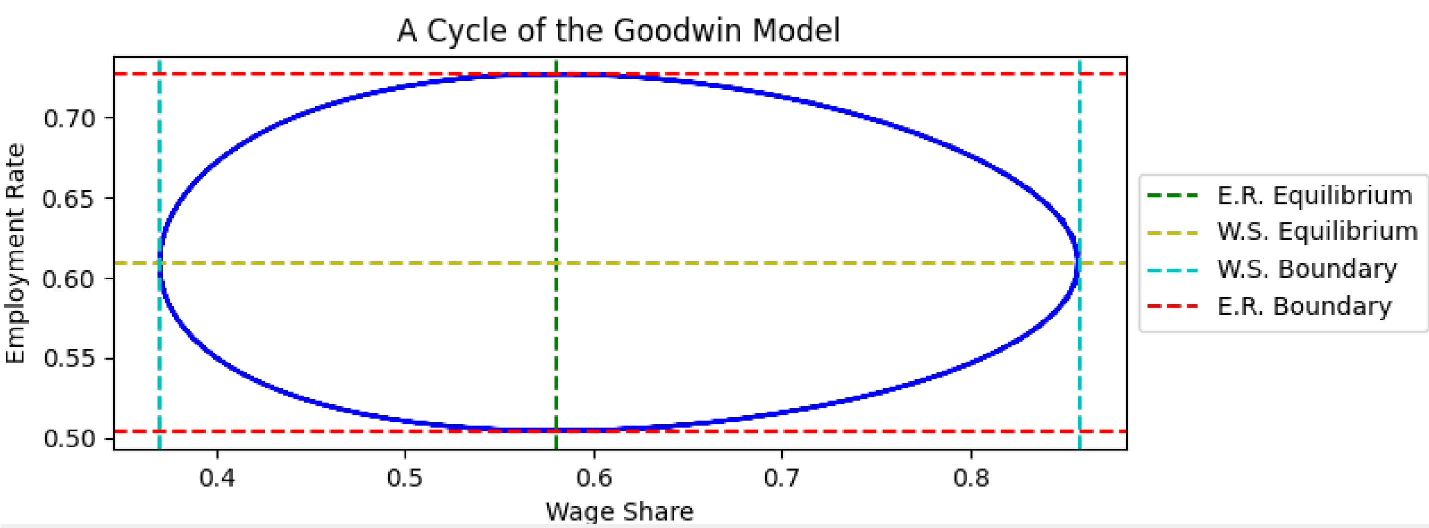
So a Goodwin’s model with a Linear Phillips Curve will have the following equations:

$$\frac{\delta\mu}{\delta t} = (\frac{1}{v} - (\theta + n) - \frac{u}{v}) * \mu$$

$$\frac{\delta u}{\delta t} = (-\alpha + \beta\mu - \theta) * u = (-(\alpha + \theta) + \beta\mu) * u$$

Here is an example of the above using python:





The variables used for the above Goodwin's Model are $v = 5, \theta = 0.009, n = 0.075, \alpha = 0.6,$ and $\beta = 1$. The initial values are $\mu = 0.55$ and $u = 0.45$.

The yellow line represents the equilibrium of wage share, and the green line represents the equilibrium of employment rate. The intersection of those 2 lines give the equilibrium point of the model. The cycle goes around the equilibrium point. Additionally, the equilibrium lines indicate the derivative is 0 when they intersect the non-parametric Goodwin's Model. To solve for these points, we will just set the derivative equal to 0 and solve. For $\frac{\delta\mu}{\delta t}$, that is:

$$\frac{\delta\mu}{\delta t} = (\frac{1}{v} - (\theta + n) - \frac{u}{v}) * \mu = 0$$

$$\frac{1}{v} - (\theta + n) - \frac{u}{v} = 0$$

$$-\frac{u}{v} = -\frac{1}{v} + (\theta + n)$$

$$u = (-\frac{1}{v} + (\theta + n)) * (-v)$$

$$u = 1 - v(\theta + n) = u^*$$

where u^* will refer to the equilibrium value of $\frac{\delta\mu}{\delta t}$. The asterisk will indicate an equilibrium value. Note: μ is presumed to be not zero, so it can't be a solution for the equilibrium. (There will always be some workers employed). Now for $\frac{\delta u}{\delta t}$:

$$\frac{\delta u}{\delta t} = (-(\alpha + \theta) + \beta\mu) * u = 0$$

$$(-(\alpha + \theta) + \beta\mu) = 0$$

$$\beta\mu = \alpha + \theta$$

$$\mu = \frac{\alpha+\theta}{\beta} = \mu^*$$

where μ^* will refer to the equilibrium value of $\frac{\delta u}{\delta t}$.

Now, notice in the Goodwin Model graph, the each of the equilibrium lines intersects the non parametric curve twice. Those are the minimum and maximum values. To calculate both of those values, let's make it so the Goodwin Model is only shown with one coefficient.

Since the Goodwin Model is based off the Lokta-Volterra Equations, we can use the substitution as stated in here.

So let's convert the Goodwin Model into the Lokta-Volterra form:

$$\frac{\delta\mu}{\delta t} = ((\frac{1}{v} - (\theta + n)) * \mu - (\frac{u}{v}) * \mu$$

$$\frac{\delta u}{\delta t} = (-(\alpha + \theta)) * u + \beta\mu u$$

Now let's substitute so there's only one parameter:

$$\mu = \frac{\alpha+\theta}{\beta}M$$

$$u = \frac{\alpha+\theta}{\frac{1}{v}}U = v(\alpha+\theta)U$$

$$t = \frac{1}{\alpha+\theta}T$$

$$A = \frac{\frac{1}{v}-(\theta+n)}{\alpha+\theta}M$$

where A is the parameter. Now let's substitute back into the Goodwin Model:

$$\frac{\delta M}{\delta T} = AM - MU$$

$$\frac{\delta U}{\delta T} = -U + MU$$

We can then combine the parametric equations above to yield:

$$\frac{\delta M}{\delta U} = \frac{AM-MU}{-U+MU} = \frac{M(A-U)}{U(-1+M)}$$

Then we can begin integrating both sides to remove the differentials. First, move all M variables to one side and U variables to the other.

$$\frac{-1+M}{M}\delta M = \frac{A-U}{U}\delta U$$

Now we can integrate both sides

$$\int \frac{-1+M}{M}\delta M \ = \int \frac{A-U}{U}\delta U$$

$$-lnM + M = AlnU - U + E$$

where E is the constant of integration. We can find E by putting in the initial conditions of M and U, which will depend on the initial conditions of u and μ.

Now to find the min/max, we will want to find the function of M in terms of U and the function of U in terms of M. We will need to use the lambert function to accomplish this:

$$W_k(z)e^{W_k(z)} = z$$

where $W_k(z)$ is the lambert function. For real numbers, k is either 0 or -1, where the k = 0 branch gives higher $W_k(z)$ values than if k is -1.

First let's solve for M:

$$-lnM + M = AlnU - U + E$$

$$-(-lnM + M) = -(AlnU - U + E)$$

$$lnM - M = U - AlnU - E$$

$$e^{lnM-M} = e^{U-E-AlnU}$$

$$e^{lnM} * e^{-M} = e^{U-E} * e^{-AlnU}$$

$$Me^{-M} = U^{-A}e^{U-E}$$

$$-Me^{-M} = -U^{-A}e^{U-E}$$

$$-M = W_k(U^{-A}e^{U-E})$$

$$M = -W_k(U^{-A}e^{U-E})$$

Next, let's solve for U:

$$-lnM + M = AlnU - U + E$$

$$\frac{-lnM+M}{A} = \frac{AlnU-U+E}{A}$$

$$\frac{-lnM}{A} + \frac{M}{A} = \frac{AlnU}{A} - \frac{U}{A} + \frac{E}{A}$$

$$\frac{-lnM}{A} + \frac{M}{A} = lnU - \frac{U}{A} + \frac{E}{A}$$

$$\frac{-lnM}{A} + \frac{M}{A} - \frac{E}{A} = lnU - \frac{U}{A}$$

$$e^{\frac{-lnM}{A}+\frac{M}{A}-\frac{E}{A}} = e^{lnU-\frac{U}{A}}$$

$$e^{\frac{-lnM}{A}} * e^{\frac{M}{A}-\frac{E}{A}} = e^{lnU} * e^{-\frac{U}{A}}$$

$$M^{\frac{-1}{A}}e^{\frac{M-E}{A}} = Ue^{\frac{-U}{A}}$$

$$Ue^{\frac{-U}{A}} = M^{\frac{-1}{A}}e^{\frac{M-E}{A}}$$

$$Ue^{\frac{-U}{A}} = M^{\frac{-1}{A}}e^{\frac{M-E}{A}}$$

$$-\frac{U}{A}e^{\frac{-U}{A}} = -\frac{1}{A}(M^{\frac{-1}{A}}e^{\frac{M-E}{A}})$$

$$-\frac{U}{A} = W_k(-\frac{1}{A}(M^{\frac{-1}{A}}e^{\frac{M-E}{A}}))$$

$$U = -AW_k(-\frac{1}{A}(M^{\frac{-1}{A}}e^{\frac{M-E}{A}}))$$

The minimum of the U and M values will be on the k = -1 branch of the lambert function, while the maximum of the U and M values will be on the k = 0 branch of the lambert function. The extrema (min and max) will occur when the other variable is at its equilibrium value (as seen in the plots above). In this case the derivative with respect to T (the substituted time variable) of the other variable is 0. Since we substituted variables, we will need to recalculate the equilibrium values. For the derivative of M with respect to T:

$$\frac{\delta M}{\delta T} = AM - MU = 0$$

$$M(A - U) = 0$$

$$A - U = 0$$

$$U = A = U^*$$

Then for the derivative of U with respect to T:

$$\frac{\delta U}{\delta T} = -U + MU = 0$$

$$-U + MU = 0$$

$$U(-1 + M) = 0$$

$$-1 + M = 0$$

$$M = 1 = M^*$$

Now we can substitute those values in to get the extrema of the M and U values:

$$M_{extrema} = -W_k(U^{*-A}e^{U^*-E}) = -W_k(A^{-A}e^{A-E})$$

$$U_{extrema} = -AW_k(-\frac{1}{A}(M^{*\frac{-1}{A}}e^{\frac{M^*-E}{A}})) = -AW_k(-\frac{1}{A}(1^{\frac{-1}{A}}e^{\frac{1-E}{A}})) = -AW_k(-\frac{1}{A}(e^{\frac{1-E}{A}}))$$

We can then substitute 0 and -1 into k to find the max and min values:

$$M_{min} = -W_{-1}(A^{-A}e^{A-E})$$

$$M_{max} = -W_0(A^{-A}e^{A-E})$$

$$U_{min} = -AW_{-1}(-\frac{1}{A}(e^{\frac{1-E}{A}}))$$

$$U_{max} = -AW_0(-\frac{1}{A}(e^{\frac{1-E}{A}}))$$

Remembering that these are substituted variables, we can get:

$$\mu_{extrema} = \frac{\alpha+\theta}{\beta}M_{extrema}$$

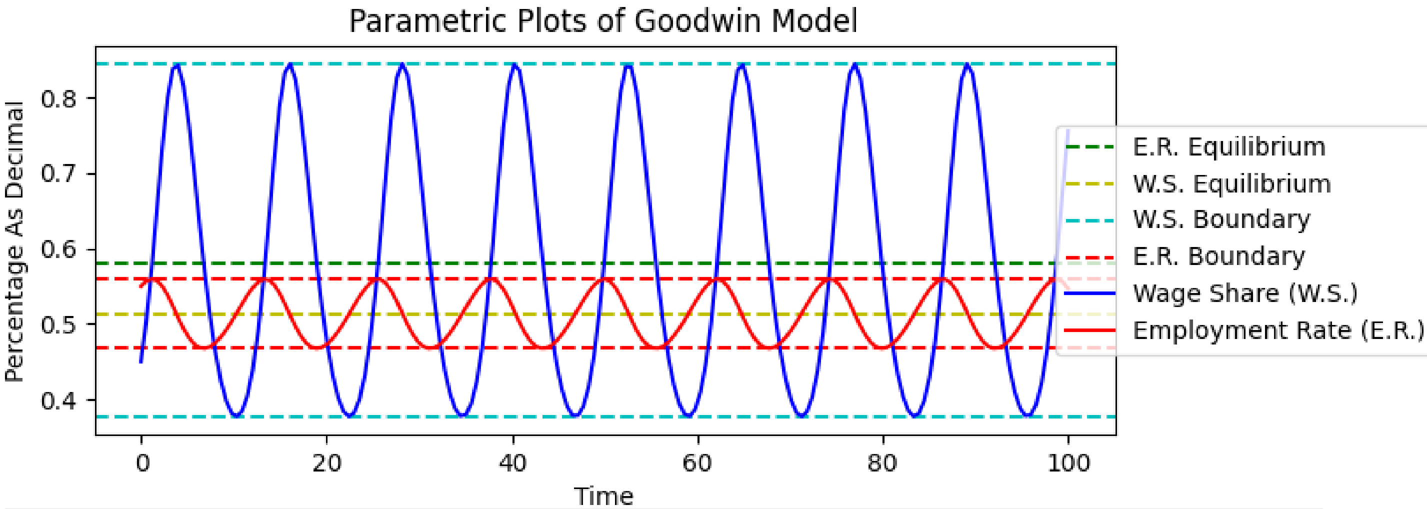
$$u_{extrema} = v(\alpha+\theta)U_{extrema}$$

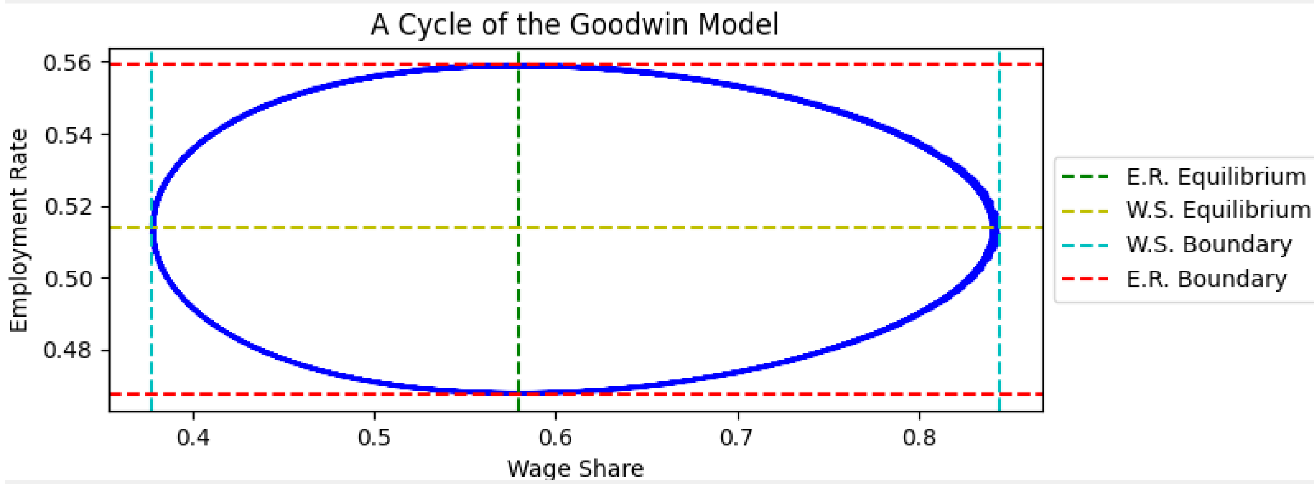
Now let’s analyze the Goodwin Model using an exponential Philips curve:

$$\frac{\delta\mu}{\delta t} = (\frac{1}{v} - (\theta+n) - \frac{u}{v}) * \mu$$

$$\frac{\delta u}{\delta t} = (-\alpha+\beta e^{k\mu}-\theta)*u = (-(\alpha+\theta)+\beta e^{k\mu})*u$$

Here is an example using python:





The variables used for the above Goodwin's Model are $v = 5, \theta = 0.009, n = 0.075, \alpha = 1, \beta = 0.1$, and $k = 4.5$. The initial values are $\mu = 0.55$ and $u = 0.45$.

Now, since the equation for $\frac{\delta\mu}{\delta t}$ is the same as the linear version, its equilibrium point will be the same as the linear version:

$$u^* = 1 - v(\theta + n)$$

Now let's solve for the equilibrium of $\frac{\delta u}{\delta t}$:

$$\frac{\delta u}{\delta t} = (-(\alpha + \theta) + \beta e^{k\mu}) * u = 0$$

$$(-(\alpha + \theta) + \beta e^{k\mu}) = 0$$

$$\beta e^{k\mu} = \alpha + \theta$$

$$e^{k\mu} = \frac{\alpha+\theta}{beta}$$

$$ln(e^{k\mu}) = ln(\frac{\alpha+\theta}{beta})$$

$$k\mu = ln(\frac{\alpha+\theta}{beta})$$

$$\mu = \frac{ln(\frac{\alpha+\theta}{beta})}{k} = \mu^*$$

Now since the exponential Goodwin model is not an expression of the Lokta-Volterra equations, substitution can't be used to make finding the extrema easier.

So, to find the extrema of the exponential model, it is necessary to first combine the parametric equations of the Goodwin model so it can be integrated:

$$\frac{\delta\mu}{\delta t} = (\frac{1}{v} - (\theta + n) - \frac{u}{v}) * \mu$$

$$\frac{\delta u}{\delta t} = (-(\alpha + \theta) + \beta e^{k\mu}) * u$$

$$\frac{\delta\mu}{\delta u} = \frac{(\frac{1}{v}-(\theta+n)-\frac{u}{v})*\mu}{(-(\alpha+\theta)+\beta e^{k\mu})*u}$$

Now move both variables and differentials to the same side so the equation can be integrated:

$$\frac{-(\alpha+\theta)+\beta e^{k\mu}}{\mu}\delta\mu = \frac{(\frac{1}{v}-(\theta+n)-\frac{u}{v})}{u}\delta u$$

Now integrate both sides:

$$\int \frac{-(\alpha+\theta)+\beta e^{k\mu}}{\mu}\delta\mu = \int \frac{(\frac{1}{v}-(\theta+n)-\frac{u}{v})}{u}\delta u$$

$$-(\alpha+\theta)ln\mu+\beta A_\mu = (\frac{1}{v}-(\theta+n))lnu-\frac{u}{v}+C$$

where C is the constant of integration and A_μ is the value of $\int \frac{\beta e^{k\mu}}{\mu}\delta\mu$. To find the value A_μ, a McLaurin series approximation will be used. Since the restriction on μ prevents it from being 0 and since its value will be close to 0, the McLaurin series approximation is okay to use. Here is the derivation of the McLaurin series of A_μ:

Each term of the McLaurin series is generated via:

$$M_{f(x)}(n) = \frac{f^{(n)}(0)}{n!}x^n$$

Where $M_{f(x)}(n)$ is the nth term of the McLaurin series of f(x). For the derivative A_μ (the derivative of $\int \frac{\beta e^{k\mu}}{\mu}\delta\mu$, which is just the original value without the integral) that is:

$$M_{A'_\mu}(n) = \frac{A_\mu'^{(n)}(0)}{n!}\mu^n = \frac{k^n e^{k*0}}{n!\mu}\mu^n = \frac{k^n}{n!\mu}\mu^n$$

Notice, only a McLaurin series was needed for the top half of the fraction in A'_μ. The bottom half remained the same. The reason this is done is because it makes A'_μ more easy to integrate. Thus, the integrated terms of the McLaurin series of A'_μ are:

$$\int M_{A'_\mu}(n)\delta\mu = \int \frac{k^n}{n!\mu}\mu^n\delta\mu$$

$$\int M_{A'_\mu}(n)\delta\mu = \begin{cases} \frac{k^n}{n!*n}\mu^n, & n > 0 \\ ln\mu, & n = 0 \end{cases}$$

Thus, the actual value of A_μ in its McLaurin series form is:

$$A_\mu(m) = \begin{cases} \sum_{n=1}^{m} \frac{k^n}{n!*n}\mu^n + ln\mu, & m > 0 \\ ln\mu, & m = 0 \end{cases}$$

Where $A_\mu(m)$ gives the McLaurin series of A_μ up to the first m+1 terms. The exact value of A_μ occurs when m is infinity, but for the point of the python code and practical uses, I used 20 terms instead.

The value of μ can't be directly solved in variable form due to the impossibility of getting an exact form of A_μ, and a root bisection method will need to be used in order to find the value of the extrema of μ. However, the value of u can still be obtained using the lambert function:

$$-(\alpha+\theta)ln\mu+\beta A_\mu=(\tfrac{1}{v}-(\theta+n))lnu-\tfrac{u}{v}+C$$

$$-(\alpha+\theta)ln\mu+\beta A_\mu-C=(\tfrac{1}{v}-(\theta+n))lnu-\tfrac{u}{v}$$

There will be a couple of substitutions to make the equation look nicer. The first one will be:

$$f_\mu=\tfrac{1}{v}-(\theta+n)$$

Thus, the integrated equation becomes:

$$-(\alpha+\theta)ln\mu+\beta A_\mu-C=f_\mu lnu-\tfrac{u}{v}$$

Divide by f_μ to get:

$$\tfrac{1}{f_\mu}(-(\alpha+\theta)ln\mu+\beta A_\mu-C)=lnu-\tfrac{u}{v*f_\mu}$$

$$\tfrac{1}{f_\mu}(-(\alpha+\theta)ln\mu+\beta A_\mu-C)=lnu-\tfrac{u}{1-v(\theta+n)}$$

Now here is the other substitution:

$$f_{\mu2}=1-v(\theta+n)$$

Thus, the integrated equation becomes:

$$\tfrac{1}{f_\mu}(-(\alpha+\theta)ln\mu+\beta A_\mu-C)=lnu-\tfrac{u}{f_{\mu2}}$$

Finishing up the derivation yields:

$$e^{\frac{1}{f_\mu}(-(\alpha+\theta)ln\mu+\beta A_\mu-C)}=e^{lnu-\frac{u}{f_{\mu2}}}$$

$$e^{\frac{-(\alpha+\theta)ln\mu}{f_\mu}*\frac{\beta A_\mu-C}{f_\mu}}=e^{lnu}*e^{-\frac{u}{f_{\mu2}}}$$

$$e^{\frac{-(\alpha+\theta)ln\mu}{f_\mu}}*e^{\frac{\beta A_\mu-C}{f_\mu}}=ue^{-\frac{u}{f_{\mu2}}}$$

$$e^{ln\mu^{\frac{-(\alpha+\theta)}{f_\mu}}}*e^{\frac{\beta A_\mu-C}{f_\mu}}=ue^{-\frac{u}{f_{\mu2}}}$$

$$\mu^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}} = ue^{-\frac{u}{f_{\mu 2}}}$$

$$ue^{-\frac{u}{f_{\mu 2}}} = \mu^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}$$

$$-\frac{u}{f_{\mu 2}} e^{-\frac{u}{f_{\mu 2}}} = -\frac{\mu^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}}$$

$$-\frac{u}{f_{\mu 2}} = W_k(-\frac{\mu^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}})$$

$$u = -f_{\mu 2} W_k(-\frac{\mu^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}})$$

Now to find the extrema of u, μ^* needs to be used:

$$u_{extrema} = -f_{\mu 2} W_k(-\frac{(\frac{ln(\frac{\alpha+\theta}{beta})}{k})^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}})$$

Then to find the min and max, insert -1 and 0 for k in the lambert function:

$$u_{min} = -f_{\mu 2} W_{-1}(-\frac{(\frac{ln(\frac{\alpha+\theta}{beta})}{k})^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}})$$

$$u_{max} = -f_{\mu 2} W_0(-\frac{(\frac{ln(\frac{\alpha+\theta}{beta})}{k})^{\frac{-(\alpha+\theta)}{f\mu}} * e^{\frac{\beta A_\mu - C}{f\mu}}}{f_{\mu 2}})$$