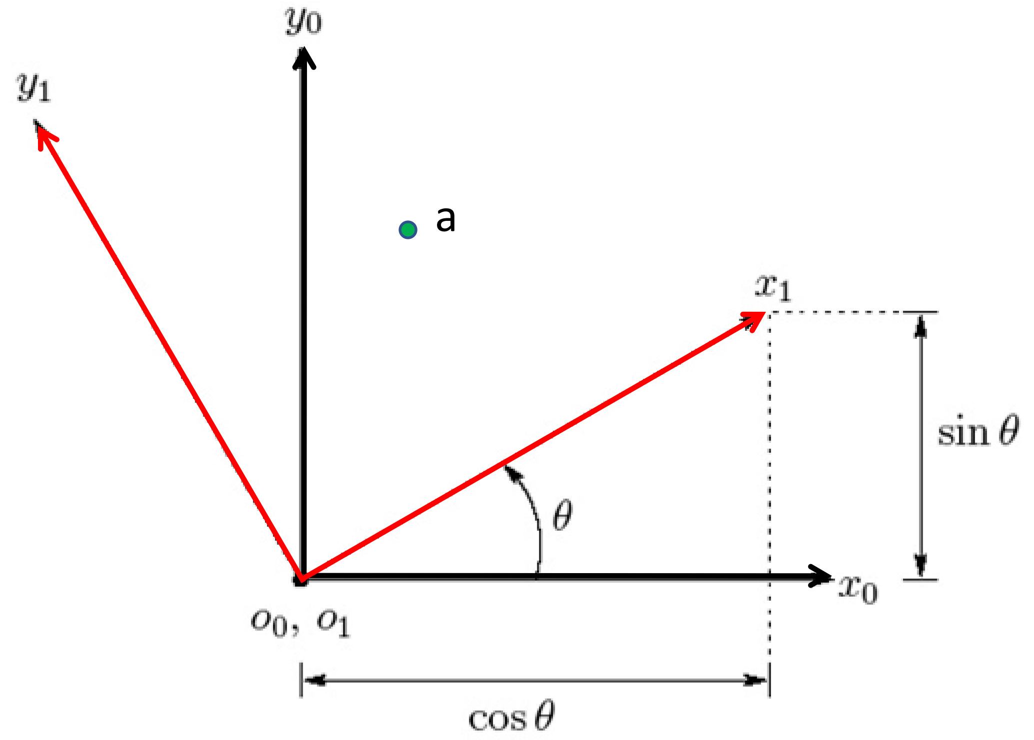
Intro to Robotics

Lecture 3

Transformation

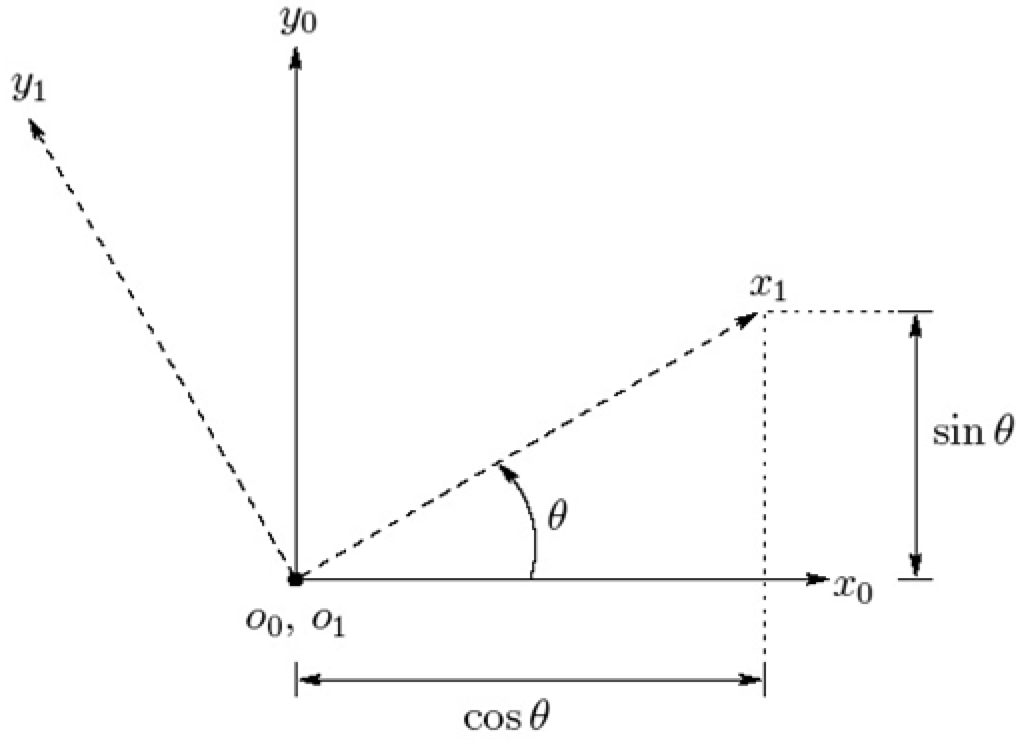


$x_1^0 \quad y_1^0$?

$a_0^1 = \left(a_{x1}, \quad a_{y1} \right)$

a_0^0 ?

Coordination Rotation in 2D

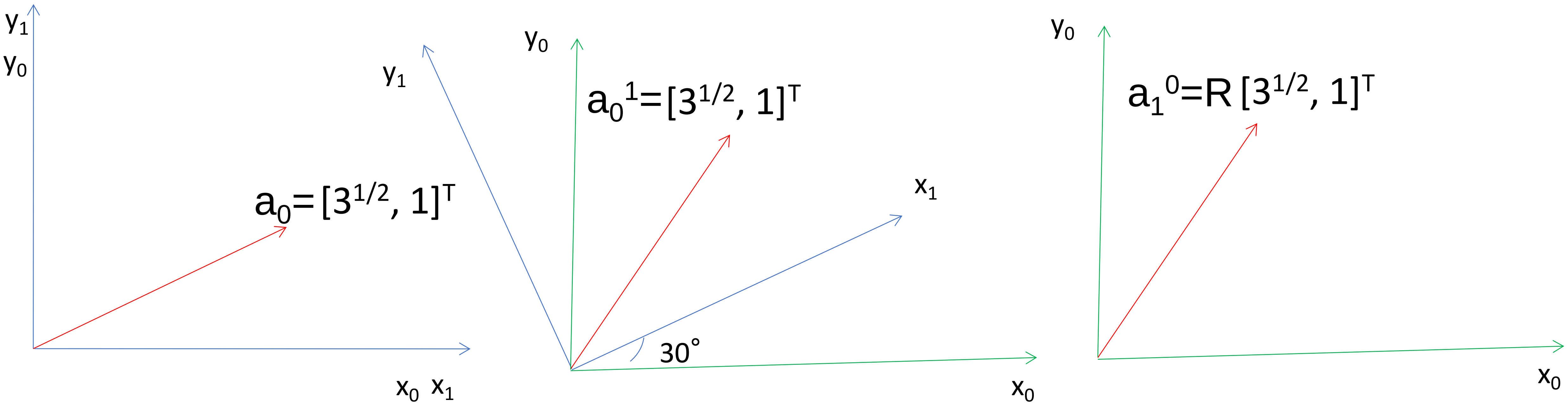


$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example



Rotation Matrix

$$R_0^1 = (R_1^0)^T$$

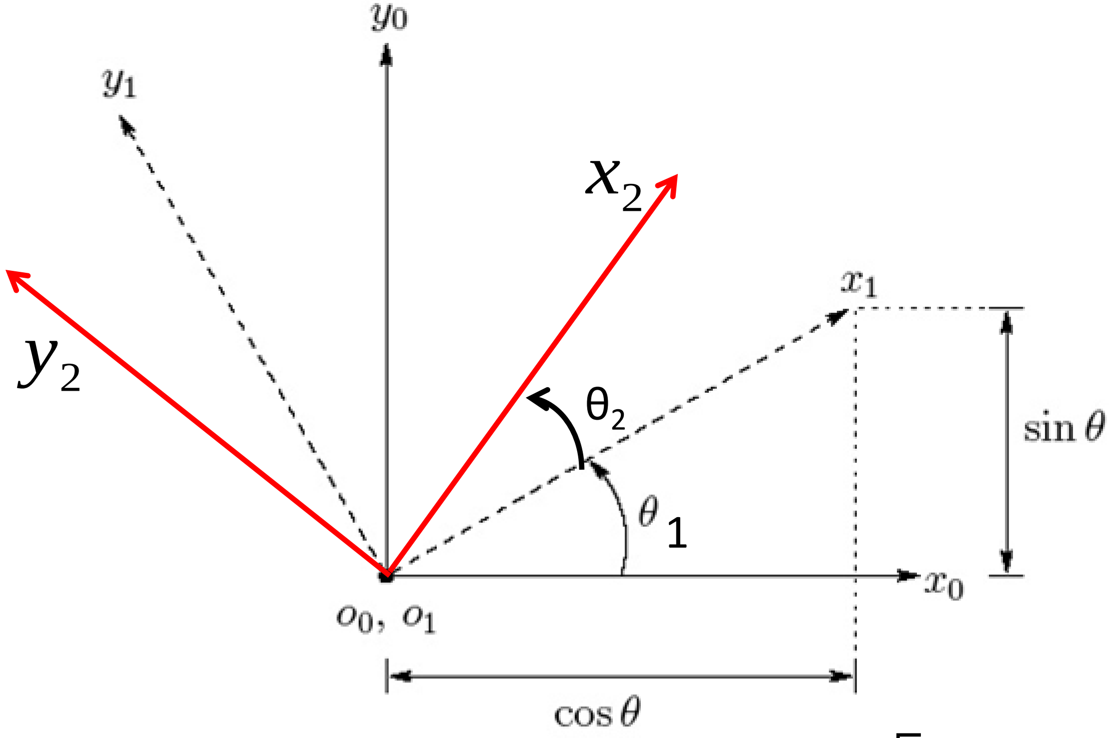
$$(R_1^0)^T = (R_1^0)^{-1}$$

$$\det(R_0^1) = 1$$

$$R_1^0 = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Orthogonal}}$$

Continue Rotation

Continue rotate θ_1, then θ_2

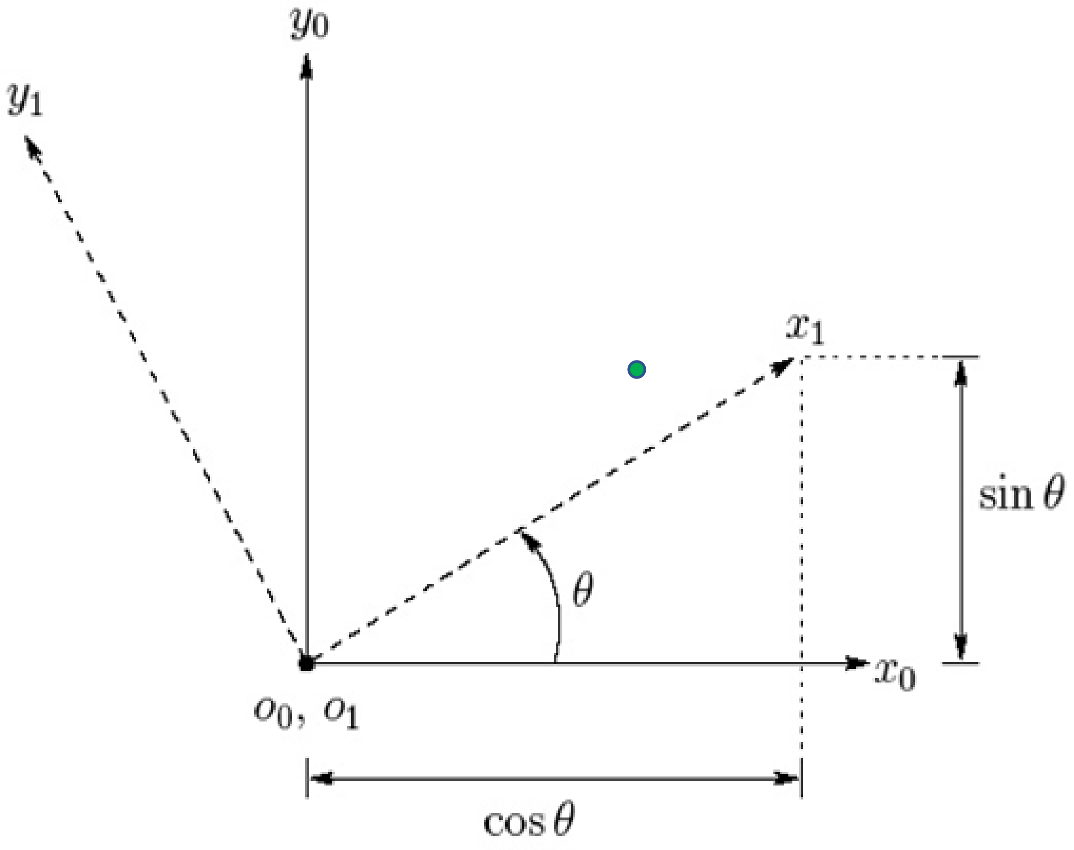


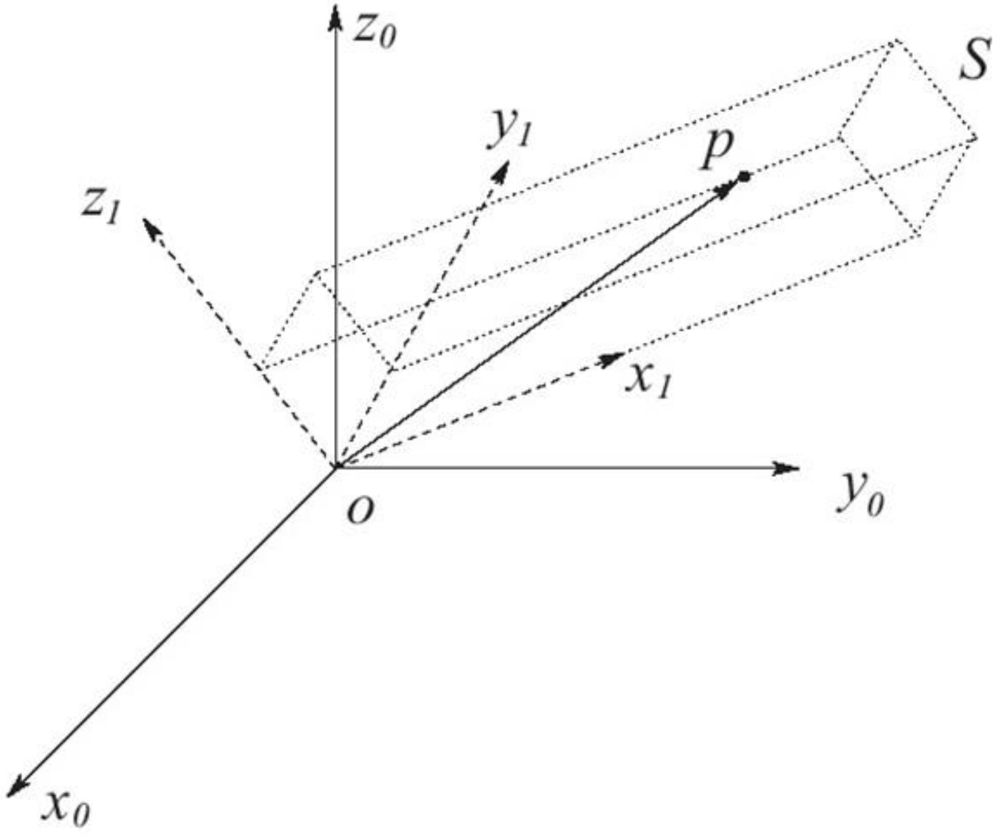
$$R_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

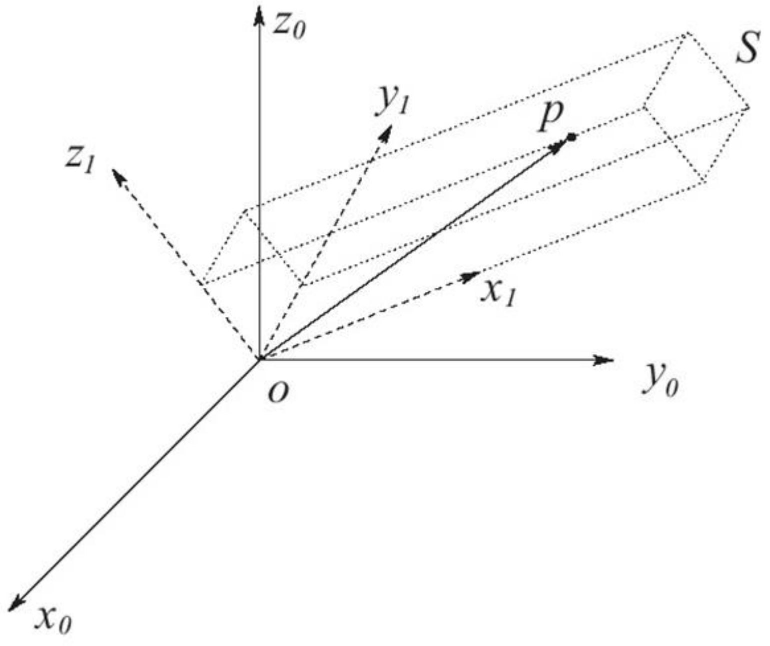
$$R_2^0 = R_1^0 R_2^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Rotation in 3D





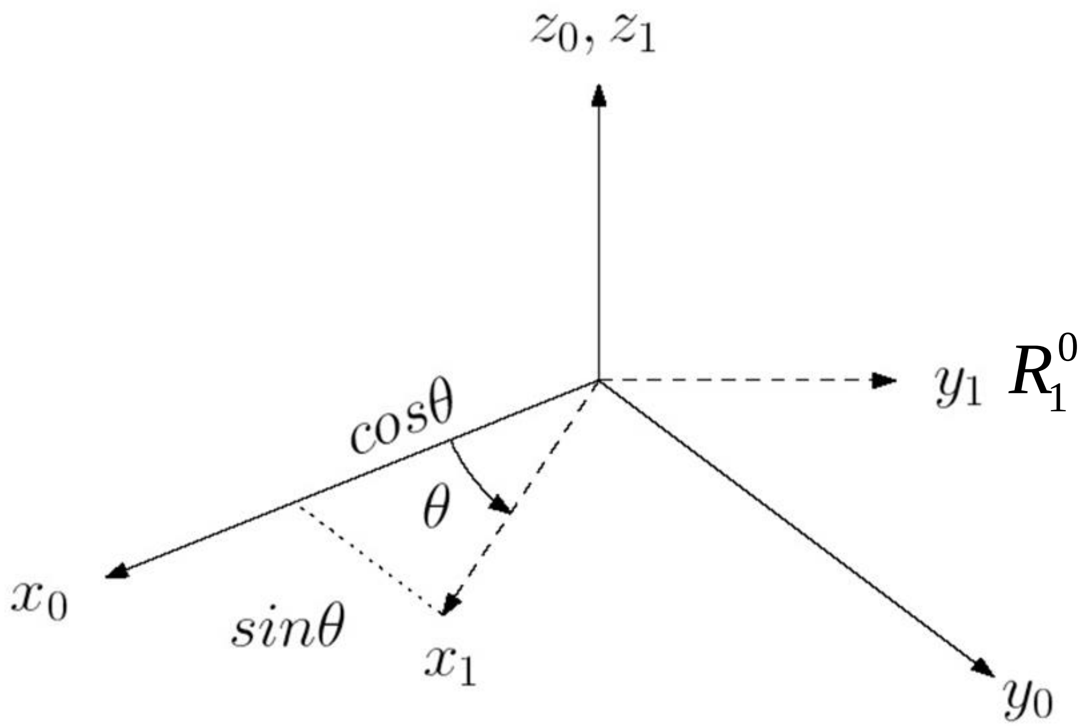
Rotation with Dot Product



$$\begin{array}{cc} & \begin{array}{ccc} x_1^0 & y_1^0 & z_1^0 \end{array} \\ R = & \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \end{array}$$

$$p^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} p^1$$

Rotation around Z



$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around X

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Rotation around Y

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

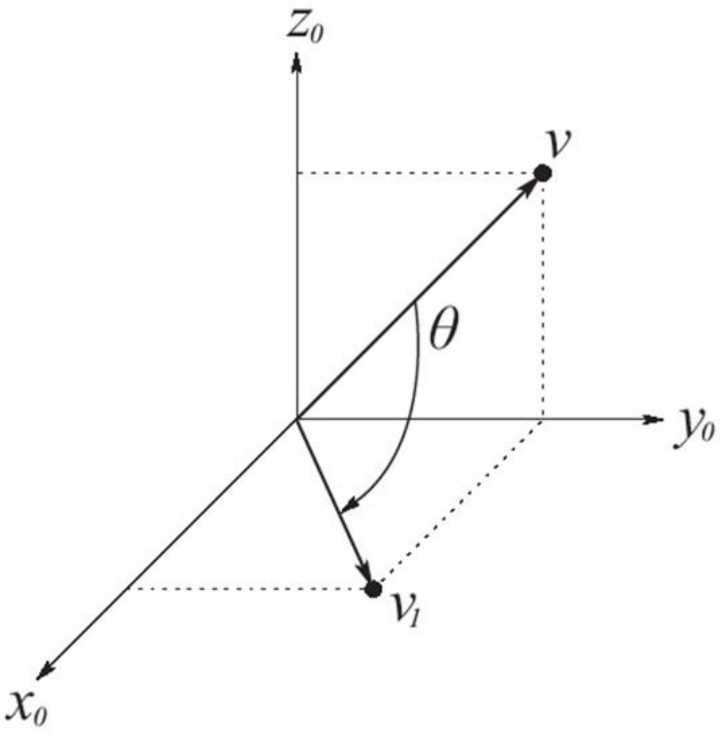
Example

A point $a = (4,3,2)$ is attached to a rotating frame 1, the frame rotates 60 degree about the OZ axis of the reference frame 0. Find the coordinates of the point relative to the reference frame 0 after the rotation.

$$a^0 = Rot(z,60)a^1$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

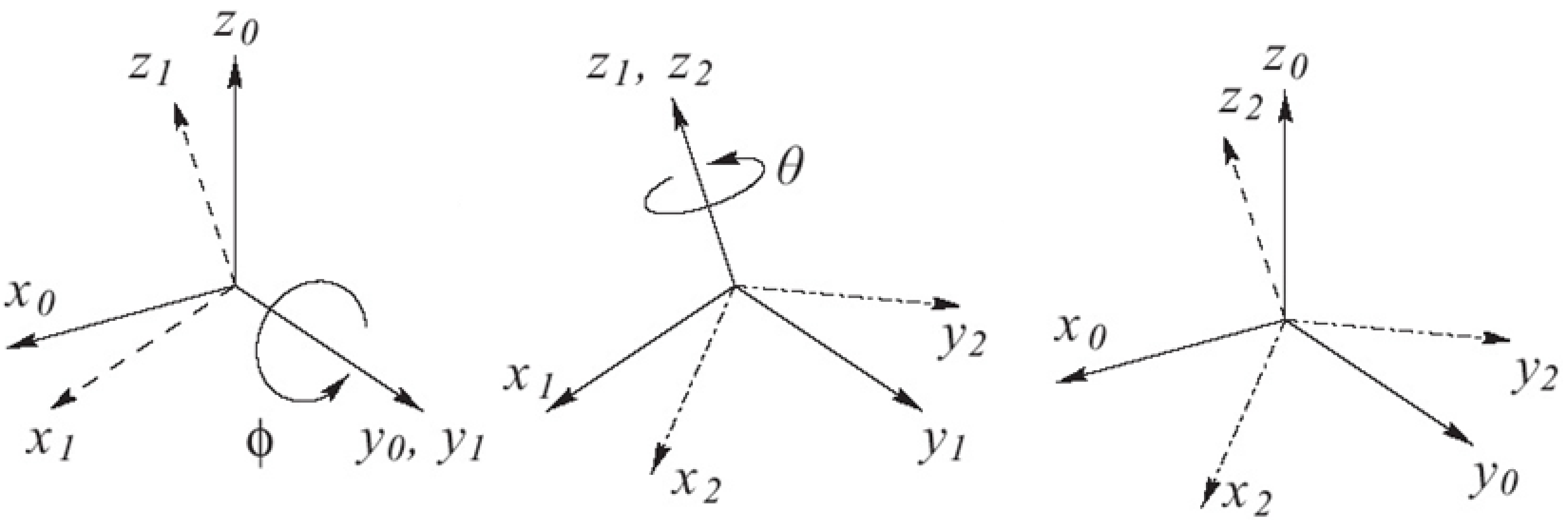
Rotate Vector



$$v_1^0 = R_{y,\frac{\pi}{2}} v^0$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Composition of Rotations

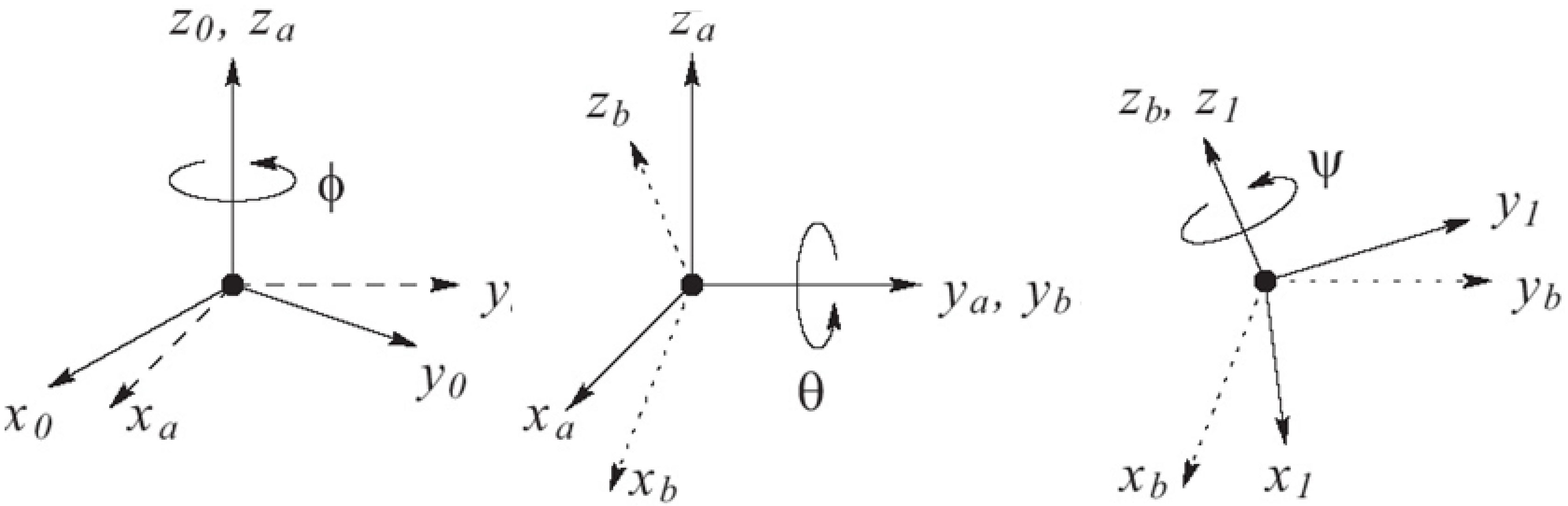


Respect to Current Frame

$$R_2^0 = R_1^0 R_2^1$$

- Any rotation can be described by three successive rotations about linearly independent axes

Example --Euler Angles



Respect to Current Frame

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{array} \right]$$

Inverse Problem

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

Inverse Problem

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

$$r_{33} = c\theta \qquad \tan\phi = \frac{r_{23}}{r_{13}} \qquad \begin{aligned} r_{13}c\phi + r_{23}s\phi &= (c\phi s\theta)c\phi + (s\phi s\theta)s\phi \\ &= (c^2\phi + s^2\phi)s\theta \\ &= s\theta \end{aligned}$$

Read textbook pp 54 - 56

Two sets of Euler angles for every R
for almost all R's
- Multiple conventions
- Singular cases

Example

$$\mathbf{R} = \mathbf{R}_z(0)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix}$$

$$\phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

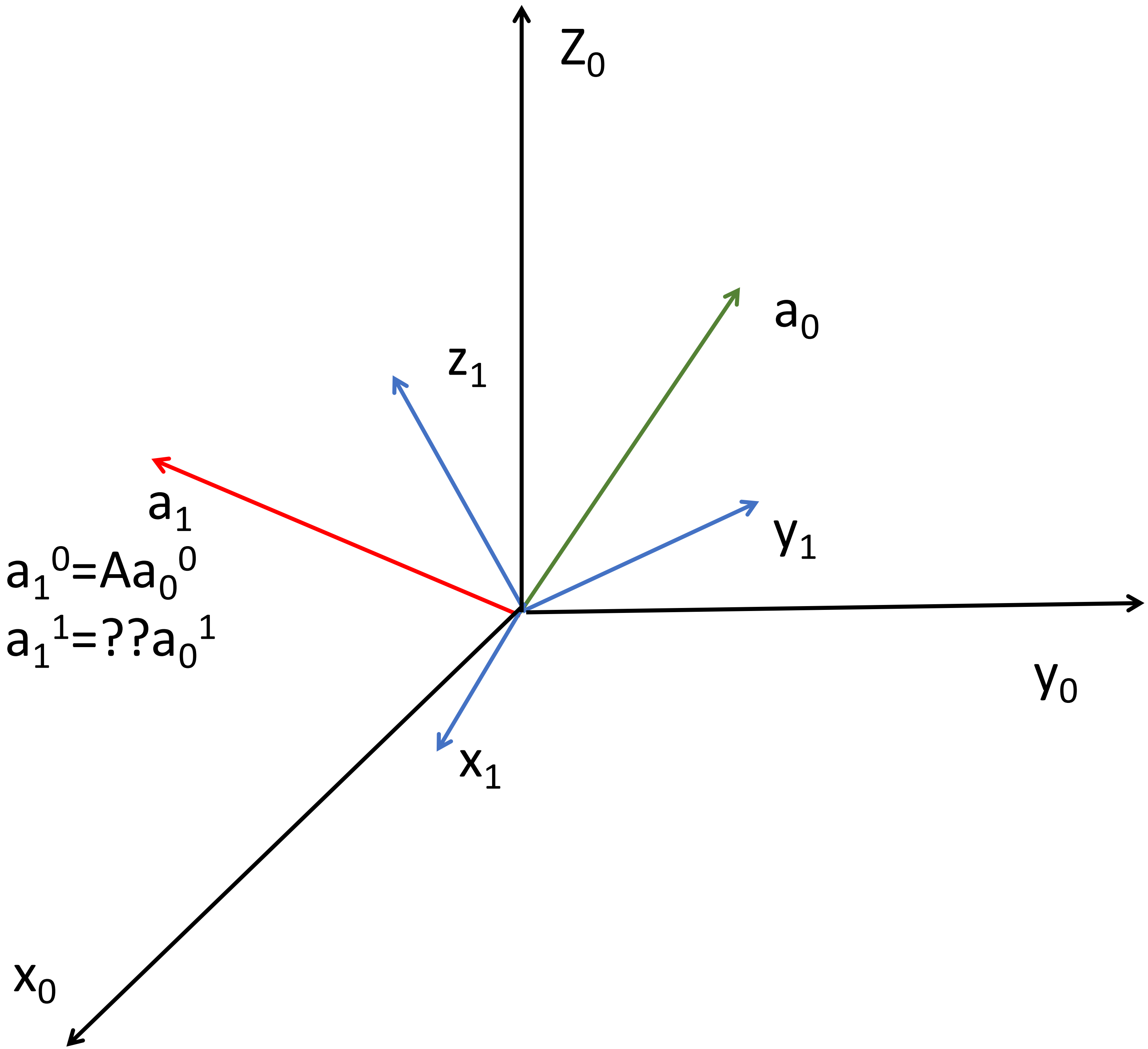
$$\theta = \mathrm{ATAN2}(1, 0) = \pi/2$$

$$\psi = \mathrm{ATAN2}(1, 0) = \pi/2$$

Similarity Transformation

- Rotation A in Frame 0
- Frame 1 to Fame 0 -- R_1^0
- What about the rotation A in Frame 0 relative to Frame 1?

$$B = (R_1^0)^{-1} A R_1^0$$



$a_1^1 = ??a_0^1$, $B = ??$

We know

$a_0^0 = R_1^0 a_0^1$

$a_1^0 = Aa_0^0$

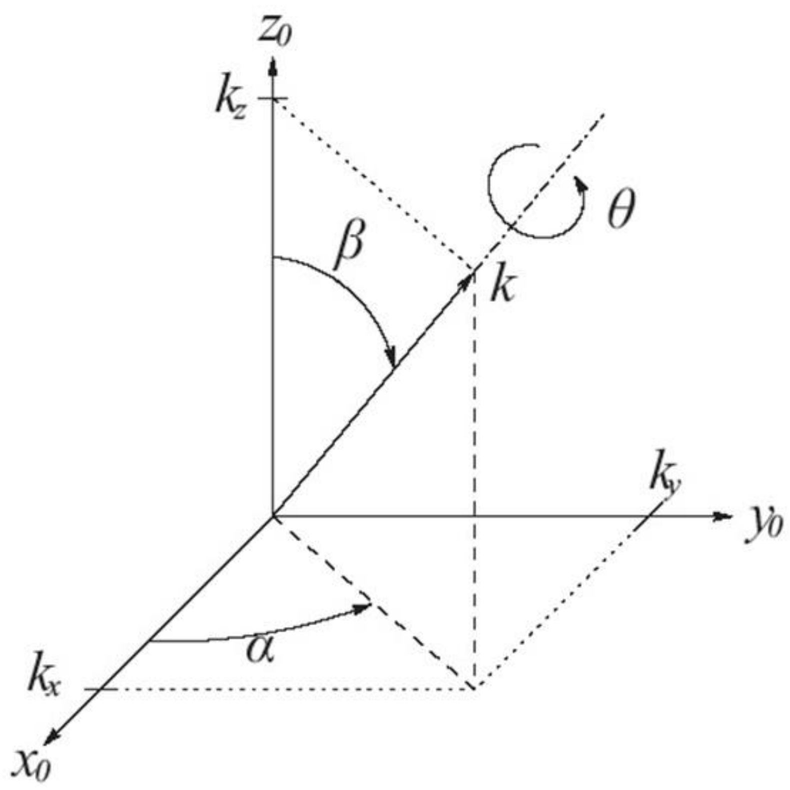
$a_1^1 = R_0^1 a_1^0$

So

$a_1^1 = R_0^1 A R_1^0 a_0^1$

$B = R_0^1 A R_1^0$

Rotate around a Vector

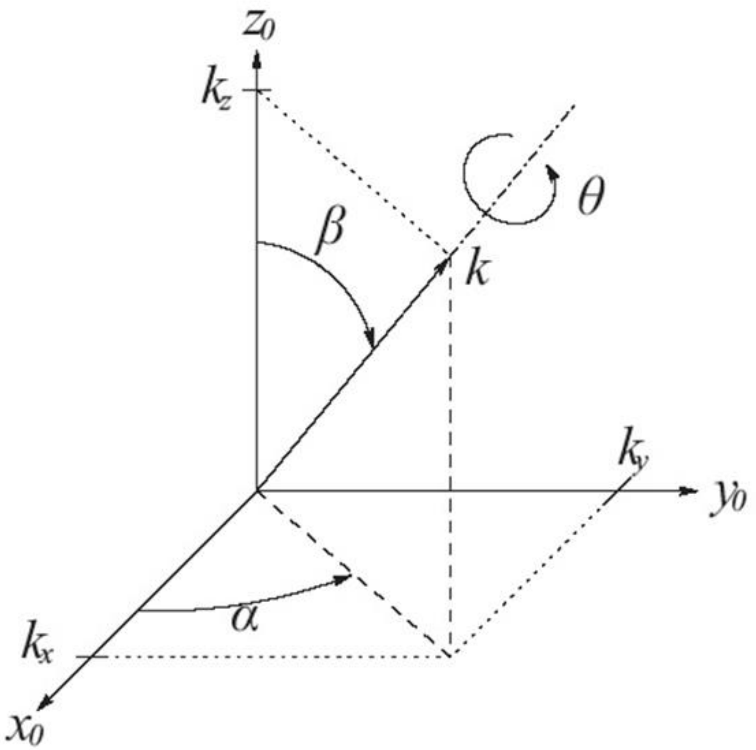


- k as z_1
- Rotate around k for θ
- R_θ is in frame 1, what is the rotation in frame 0

$$B = (R_1^0)^{-1} A R_1^0$$

$$R_{k,\theta} = R R_{z,\theta} R^{-1}$$

$$R_{k,\theta} = \underbrace{R_{z,\alpha} R_{y,\beta}}_{R} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$



$$\begin{bmatrix} k_1^2 \mathrm{v}\theta + \mathrm{c}\theta & k_1 k_2 \mathrm{v}\theta - k_3 \mathrm{s}\theta & k_1 k_3 \mathrm{v}\theta + k_2 \mathrm{s}\theta \\ k_1 k_2 \mathrm{v}\theta + k_3 \mathrm{s}\theta & k_2^2 \mathrm{v}\theta + \mathrm{c}\theta & k_2 k_3 \mathrm{v}\theta - k_1 \mathrm{s}\theta \\ k_1 k_3 \mathrm{v}\theta - k_2 \mathrm{s}\theta & k_2 k_3 \mathrm{v}\theta + k_1 \mathrm{s}\theta & k_3^2 \mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix}$$

$$\mathrm{v}\theta = 1 - \mathrm{c}\theta$$

Inverse Problem

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} k_1^2 \mathrm{v}\theta + \mathrm{c}\theta & k_1 k_2 \mathrm{v}\theta - k_3 \mathrm{s}\theta & k_1 k_3 \mathrm{v}\theta + k_2 \mathrm{s}\theta \\ k_1 k_2 \mathrm{v}\theta + k_3 \mathrm{s}\theta & k_2^2 \mathrm{v}\theta + \mathrm{c}\theta & k_2 k_3 \mathrm{v}\theta - k_1 \mathrm{s}\theta \\ k_1 k_3 \mathrm{v}\theta - k_2 \mathrm{s}\theta & k_2 k_3 \mathrm{v}\theta + k_1 \mathrm{s}\theta & k_3^2 \mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix}$$

$c\theta$

$$\mathrm{c}\theta = \frac{\mathrm{Tr}(\mathbf{R}) - 1}{2}$$

$r_{32}-r_{23}$

$r_{13}-r_{31}$

$r_{21}-r_{12}$

$$\mathrm{s}\theta = \pm\frac{1}{2}\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2}$$

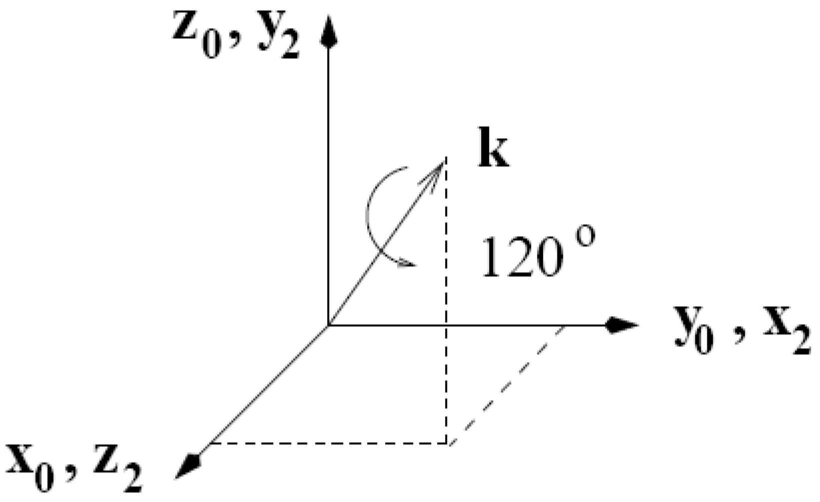
$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2\mathrm{s}\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Example

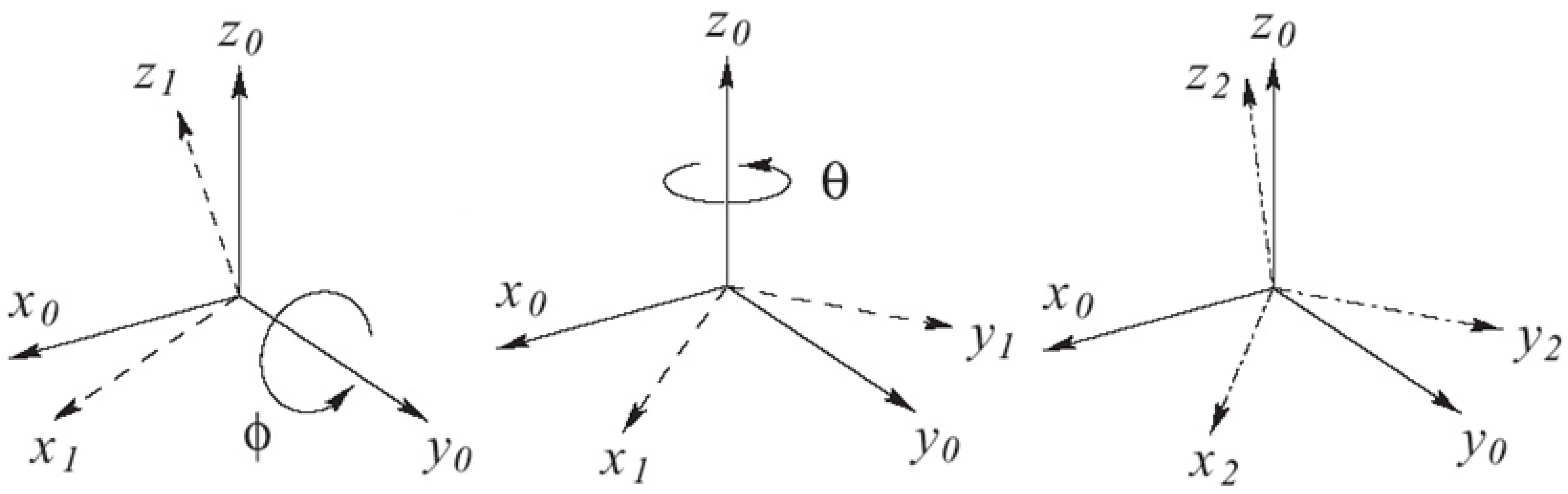
$$\mathbf{R} = \mathbf{R}_y(\pi/2)\mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$c\theta = -1/2, \quad s\theta = \sqrt{3}/2, \quad \theta = \text{atan2}(\sqrt{3}/2, -1/2) = 120^o$$

$$\mathbf{k} = \frac{1}{2\frac{\sqrt{3}}{2}} \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



Composition of Rotations

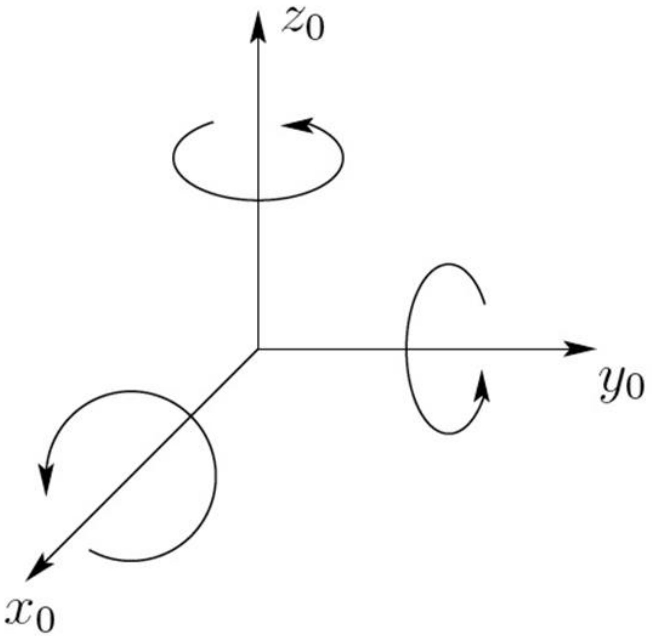


Respect to Fixed Frame

$$R_2^0 = R_1^0[(R_1^0)^{-1} R_\theta R_1^0] = R_\theta R_1^0$$

Example

-- Roll, Pitch, Yaw Angles



Respect to Fixed Frame

$$R_{XYZ} = R_{z,\phi}R_{y,\theta}R_{x,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$

What about Translation?

$$p^0 = R_1^0 p^1 + d^0$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

Homogeneous transformation

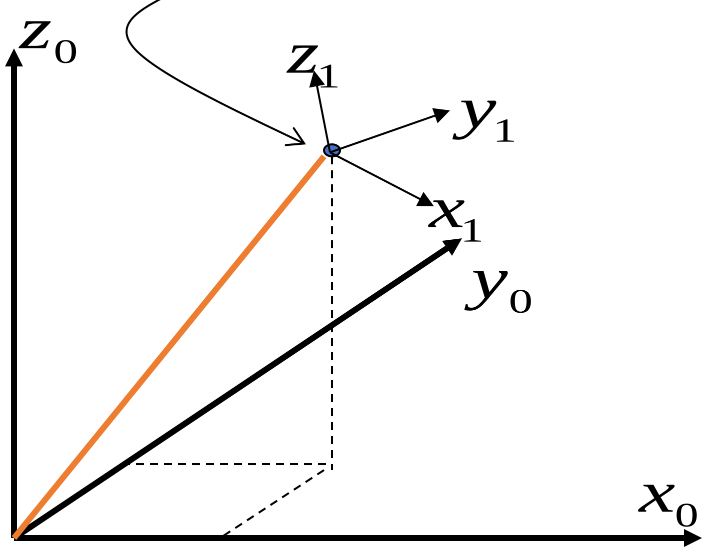
What about Translation?

$$p^0 = R_1^0 p^1 + d^0$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

$$d^0 = (d_x, d_y, d_z)^T$$



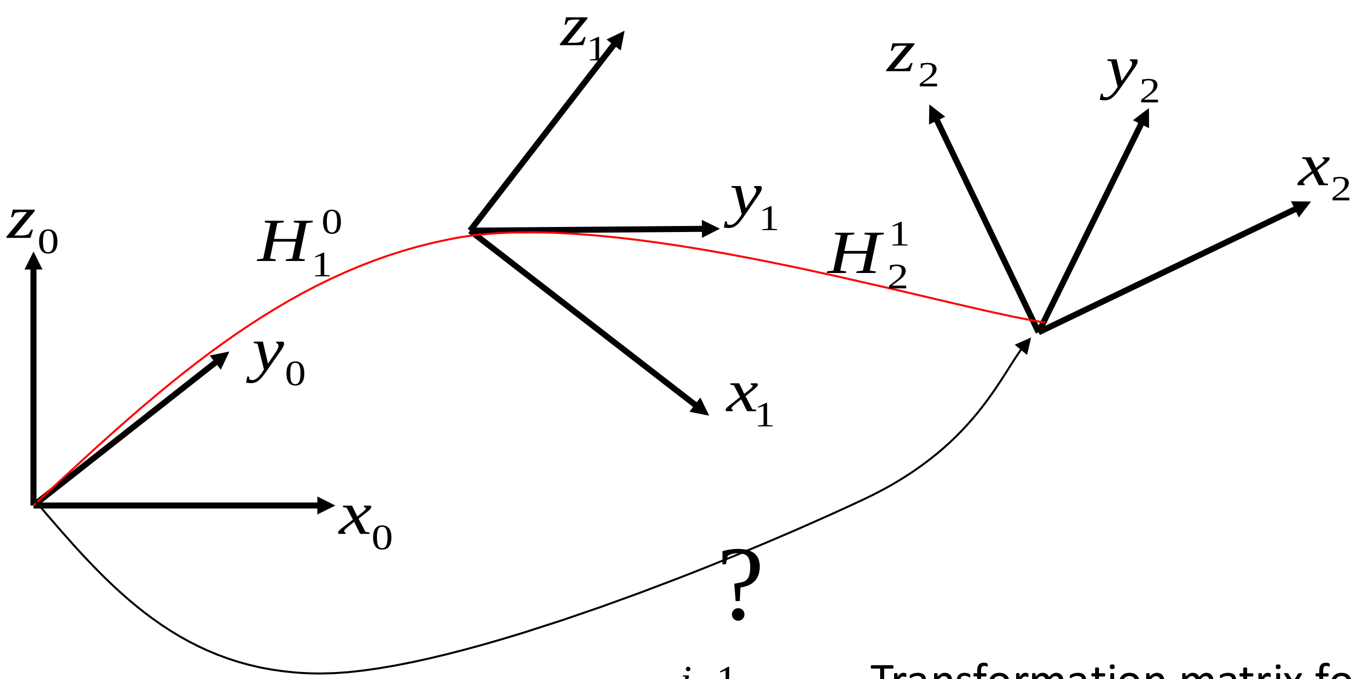
$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{R_{3\times3}} & \boxed{d_{3\times1}} \\ 0 & \boxed{1} \end{bmatrix}$$

Rotation matrix

Position vector

Homogeneous Transformation

Composite Homogeneous Transformation Matrix

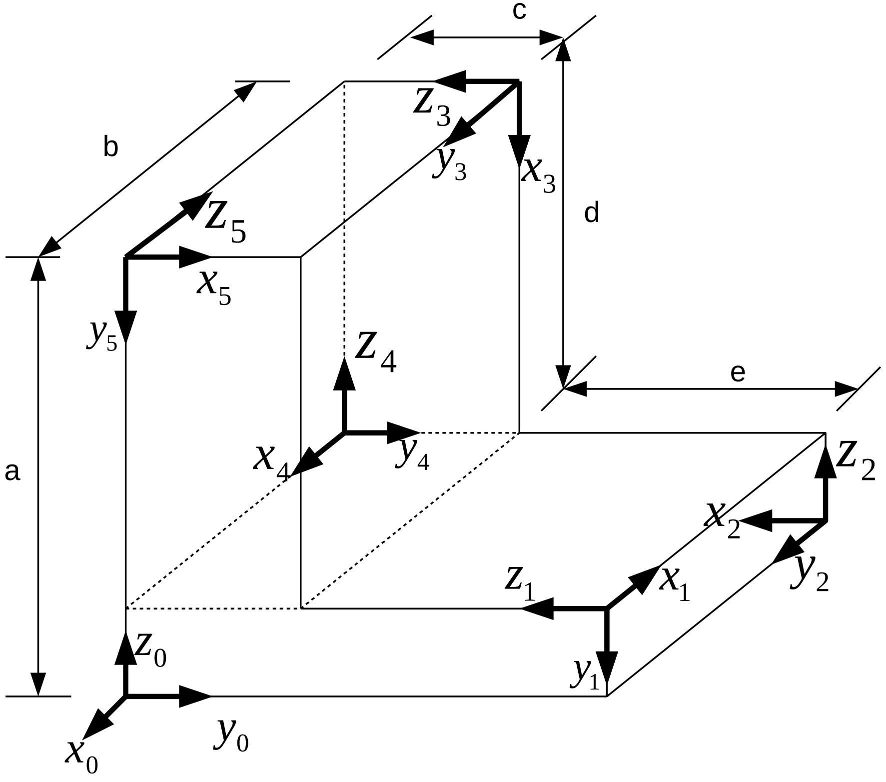


H_i^{i-1} Transformation matrix for adjacent coordinate frames

$$H_2^0 = H_1^0 H_2^1$$

Chain product of successive coordinate transformation matrices

Example



$$H_1^0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & e+c \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & e+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

$$H^{-1}H = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & 0 \\ 0 & 1 \end{bmatrix} = I_{4\times 4}$$

Coordinate System and Transformation for Robots