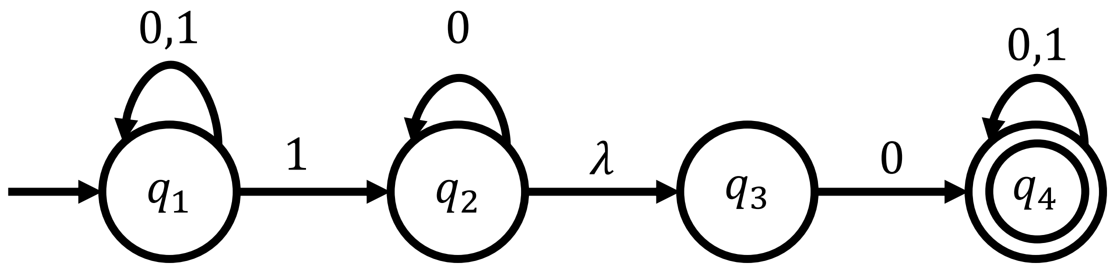
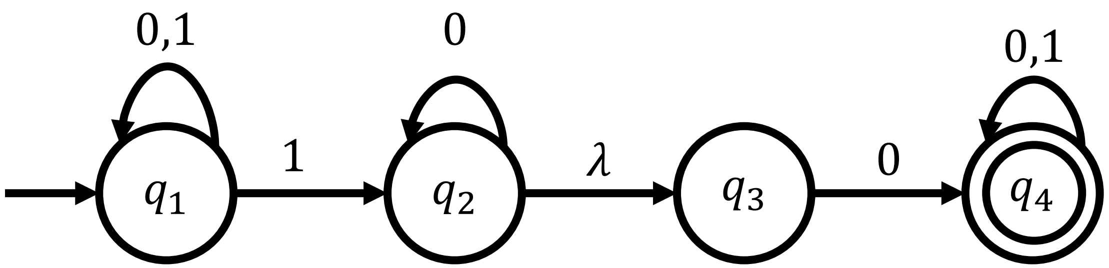
Nondetermism (1.2)

COT 4210 Discrete Structures II
Summer 2025
Department of Computer Science
Dr. Steinberg



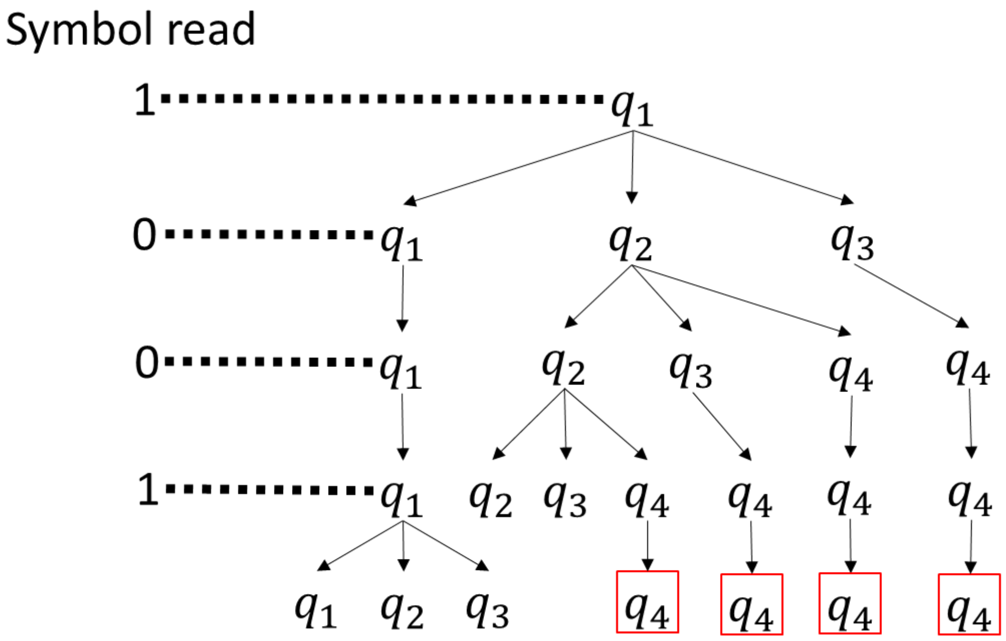
Nondeterminism

- An *Nondeterministic Finite Automata* looks like a DFA, *except* an NFA:
  - Can have more than one possible transition per state per input symbol
  - Doesn't have to have a transition for every state for every input symbol
  - Can transition on the empty string
- It also *works* like a DFA, except acceptance is nondeterministic
  - A DFA accepts if *the* path for the input string ends on an accept state
  - An NFA accepts if *any* path for the input string ends on an accept state

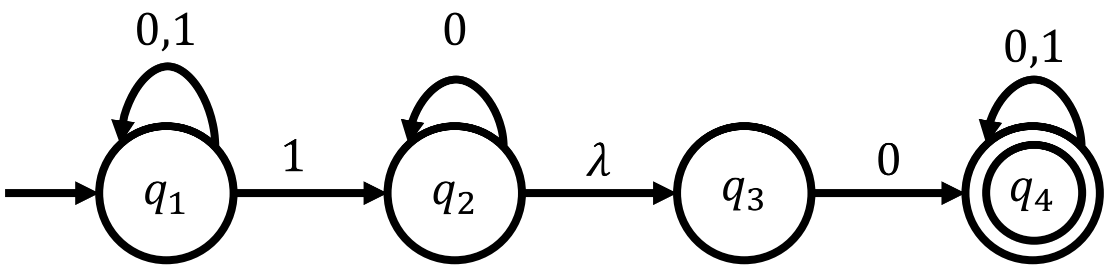


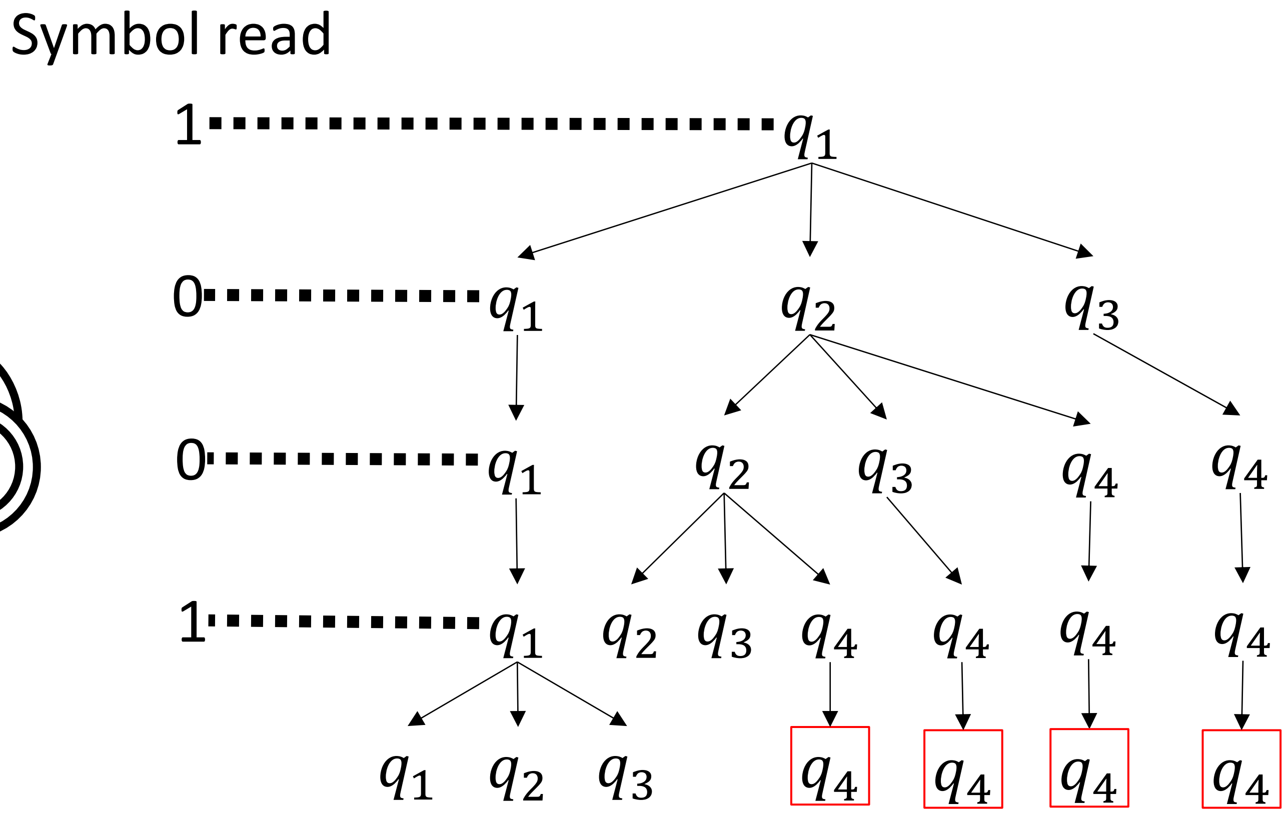
Computation in an NFA

- The easiest way to think of how an NFA works is to think of a *threaded DFA*
  - Every time there is a choice of more than one path, the NFA splits off a copy of itself to follow each path
  - The copies conceptually run in parallel
  - A copy that reaches the end of input either accepts or rejects normally
  - A copy that reaches a symbol it cannot transition on stops and rejects
  - The NFA itself accepts if *any* copy accepts

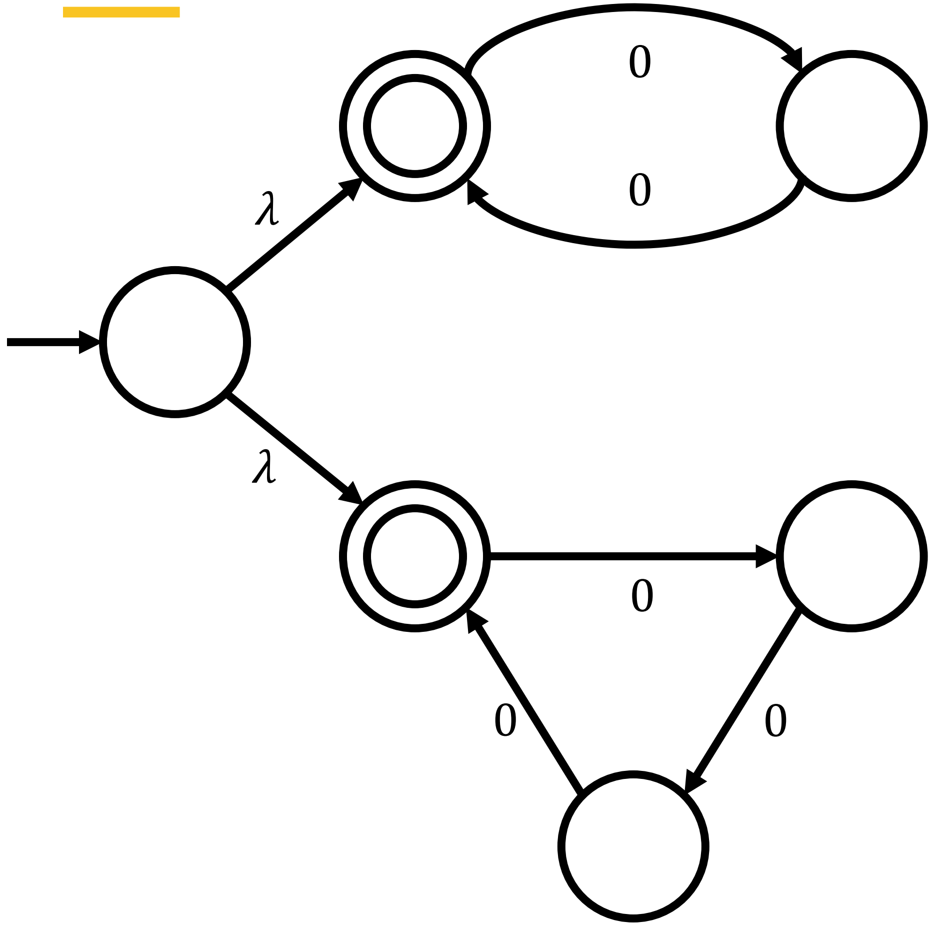


Input 1001 on our NFA



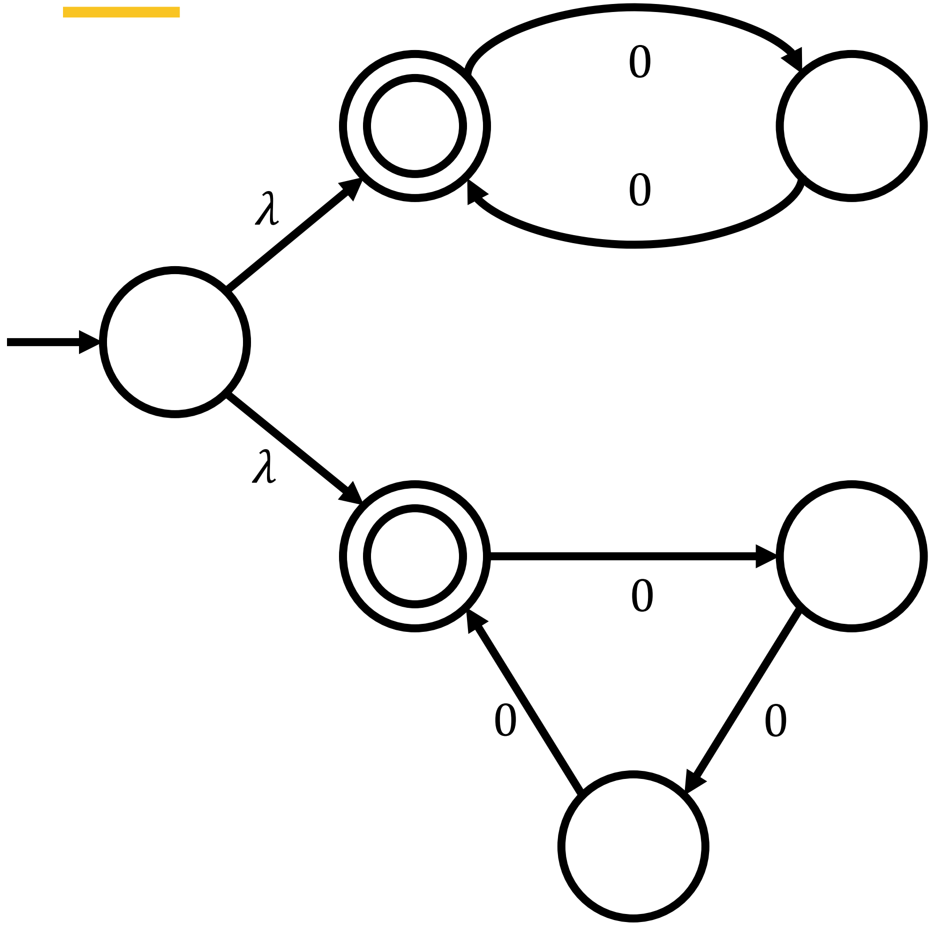


NFA Example



What language does this machine recognize?

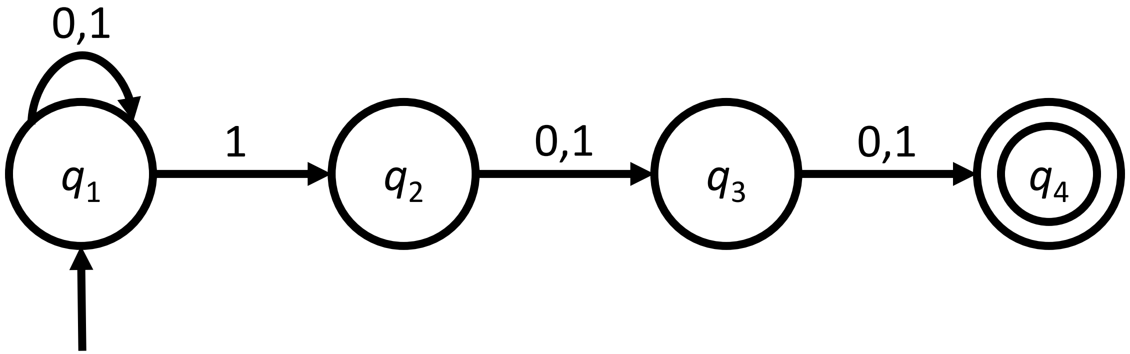
NFA Example

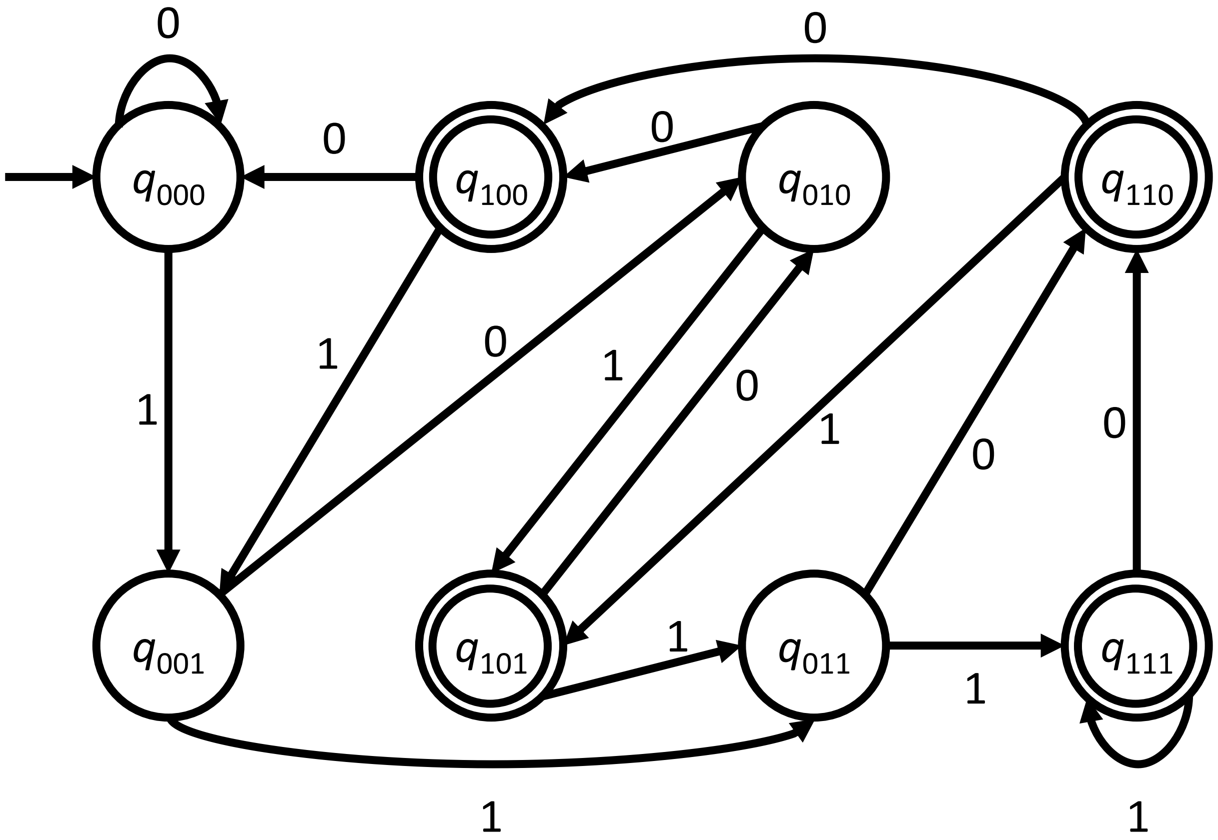


What language does this machine recognize?

$L = \{0^n : n \bmod 2 = 0 \text{ or } n \bmod 3 = 0\}$

An NFA and its DFA
(Yes, this is, in fact, the best we can do.)





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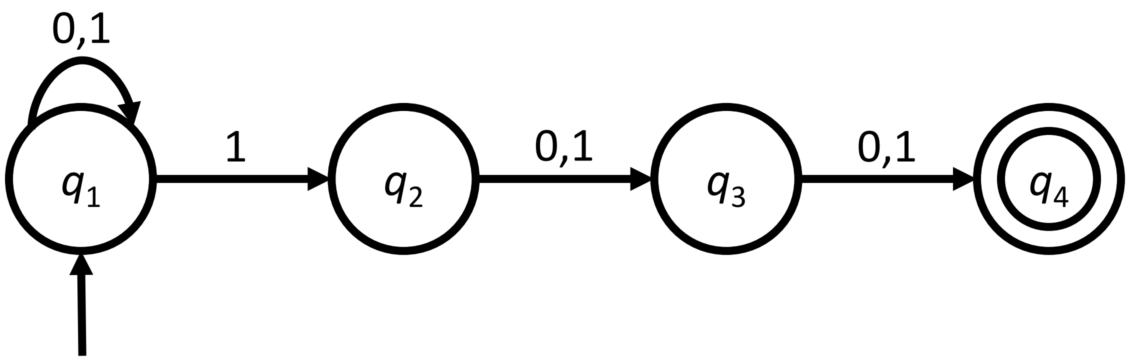
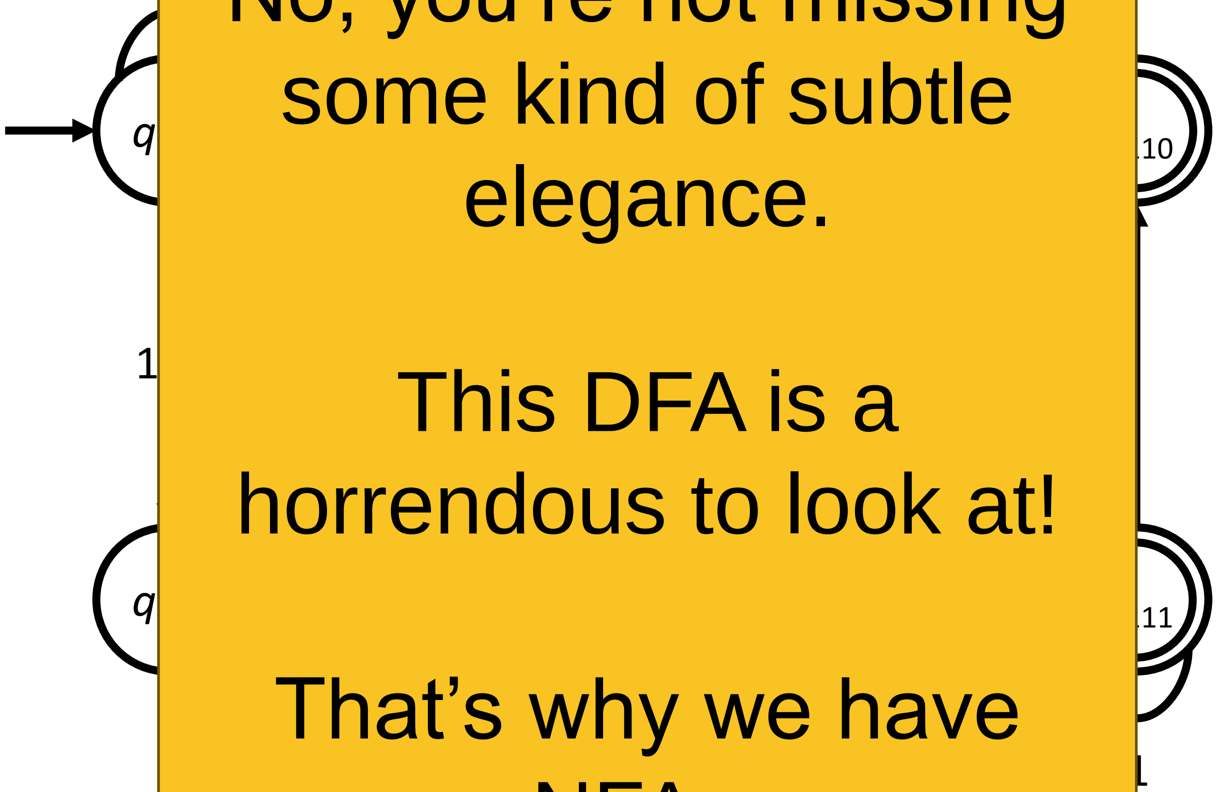




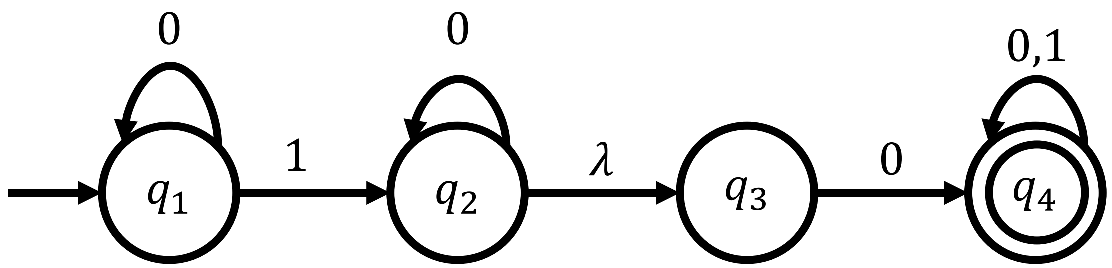
No, you’re not missing some kind of subtle elegance.

This DFA is a horrendous to look at!

That’s why we have NFAs.

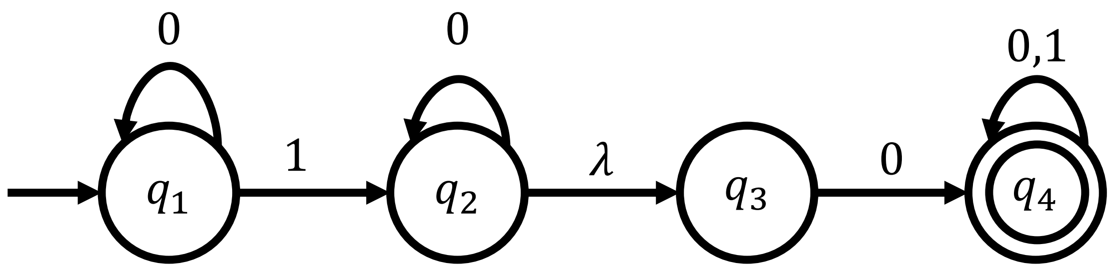
Definition: Nondeterministic Finite Automaton

- A **nondeterministic finite automaton** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ that consists of:
  - Q — A finite set of *states*
  - Σ — An *alphabet*
  - $\delta: Q \times \Sigma \rightarrow P(Q)$ — A *transition function*
  - $q_0 \in Q$ — A *start state*
  - $F \subseteq Q$ — A set of *accept* (or *final*) states



Question

You might be wondering what would happen if a string was fed to the NFA and a state does not have a definition for it. For example, what happens if the string “110” was fed to the above NFA?



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You might be wondering what would happen if a string was fed to the NFA and a state does not have a definition for it. For example, what happens if the string “110” was fed to the above NFA?

Since $\delta(q_2, 1)$ and $\delta(q_3, 1)$ is undefined, we have a situation called **dead configuration**. That means the NFA will just automatically stop without further action.

Question

What can an NFA do that a DFA can't?

Question

What can an NFA do that a DFA can’t?

Absolutely Nothing!

Let’s do some problems with NFAs!

To the board!

The Equivalence of NFAs and DFAs

Equivalence

- To prove NFAs and DFAs equivalent, it suffices to show that:
  - For every DFA, an NFA can be made that recognizes exactly the same language.
    - **This is easy.**
      - The capabilities of NFAs are a strict superset of those of DFAs, so every DFA is already an NFA. There's no proof to write.
  - For every NFA, a DFA can be made that recognizes exactly the same language.
    - **This isn't.** So let's do it.

NFA-to-DFA Conversion: Simulating the NFA

- Keep in mind our three observations about the computation process of an NFA:
  1. **Multiple transitions imply that at any given time, an NFA is in a *set of states***
  2. **Empty transitions are handled in computation by *including their possibilities* in the set of states**
  3. **Missing transitions simply *don't add anything* to the next set of states**
- Now take this a bit further:
  - The power set P(Q) of an NFA's states is the set of all possible subsets of its states
  - So at any given time, the set of states an NFA is in is an element in P(Q)
  - P(Q) is *itself* a set
- So **on a given transition, an NFA is simply transitioning from one element of P(Q) to another**

NFA-to-DFA Conversion: Simulating the NFA

- We have also observed that:

4. **On a given transition, an NFA is simply transitioning from one element of P(*Q*) to another**

- Now consider empty and missing transitions:
  - The possibilities of empty transitions are included in the set of states by look-ahead
  - Missing transitions are handled by simply not adding anything to the set of states
  - So given the next input symbol, we account for them completely in the next set of states
- This means that **an NFA transitions from one element of P(*Q*) to another element of P(*Q*) based only on the next input symbol**

NFA-to-DFA Conversion: Simulating the NFA

- Keep in mind our three observations about the computation process of an NFA:

  1. **Multiple transitions imply that at any given time, an NFA is in a *set of states***
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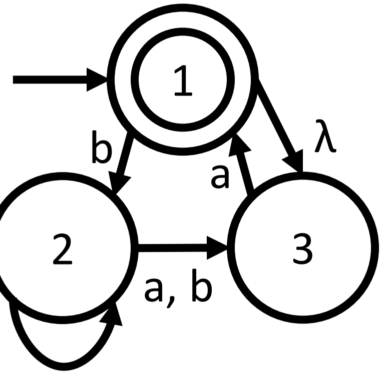
- We have also observed that:

  4. **On a given transition, an NFA is simply transitioning from one element of P(*Q*) to another**
  5. **An NFA transitions from one element of P(*Q*) to another element of P(*Q*) based only on the next input symbol**

- So while an NFA is computing, we have:
  - A finite set of states it can be in, and
  - A way to know which state it will be in next, given only its current state and the input symbol

Building the DFA

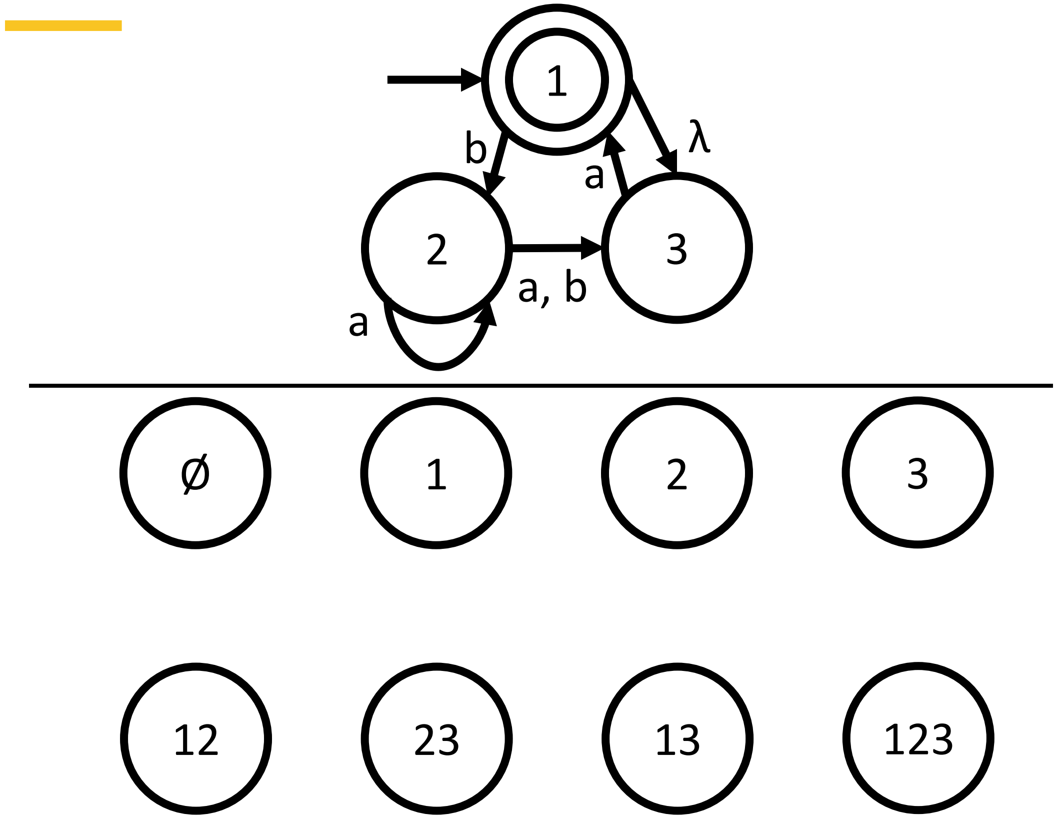


We want to build a DFA $D = \{ Q_D, \Sigma, \delta_D, q_{0D}, F_D \}$ that simulates NFA $N = \{ Q_N, \Sigma, \delta_N, q_{0N}, F_N \}$

We need to figure out:

- The state set
- The transition function
- The start state
- The final states

Building the DFA: States



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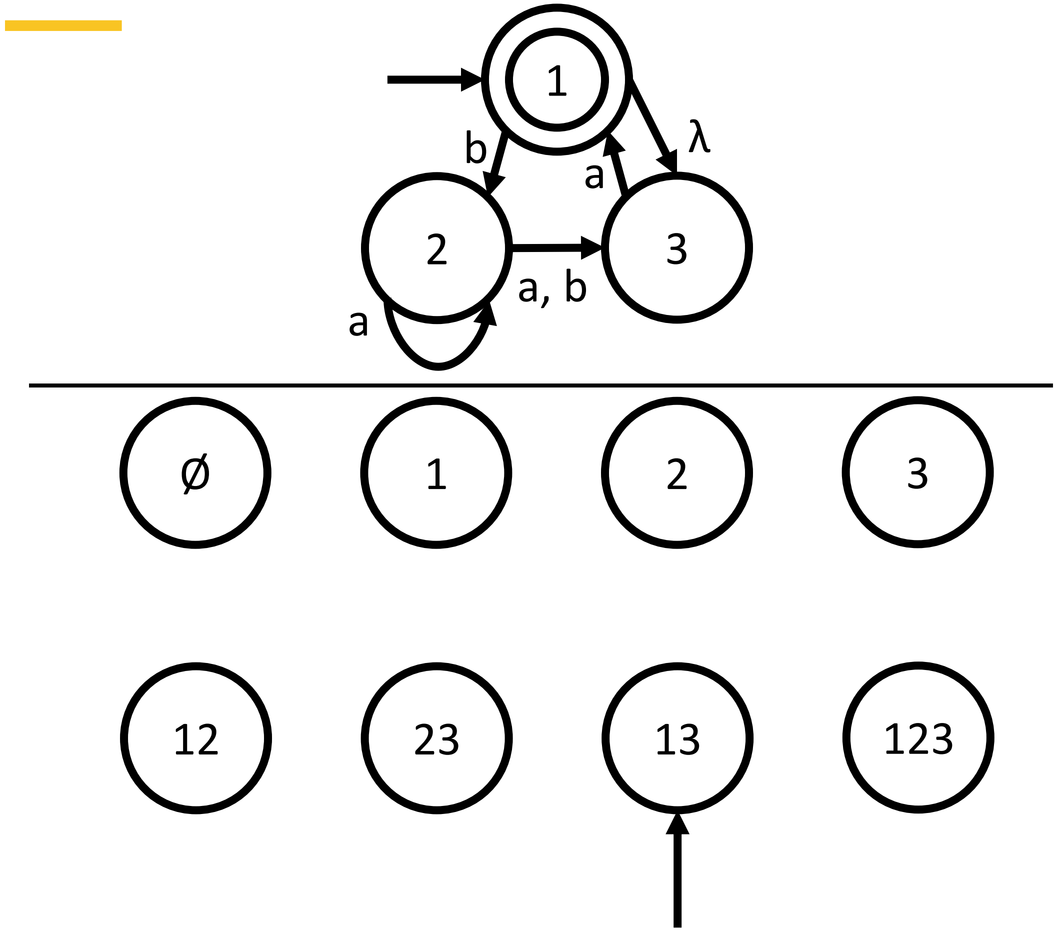
The state set is the easiest part: just remember that we need to simulate being in some subset of the states of N, and say:

- **Q_D = P(Q_N), the power set of Q_N**

Building the DFA: Starting



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- **Q_D = P(Q_N), the power set of Q_N**

Next let's look at the start state

- Easy answer: the state corresponding to being in, and only in, the start state of the NFA
- …with one wrinkle: empty-string transitions

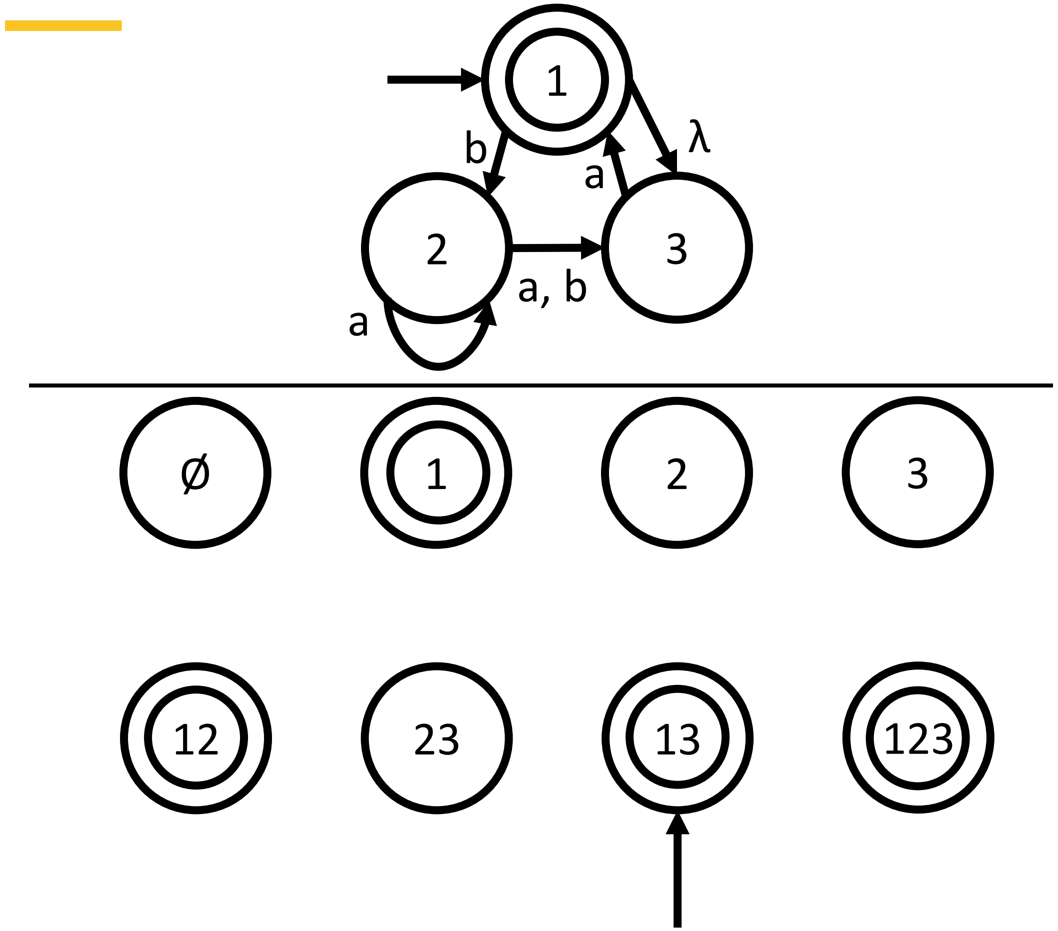
So we need the set-state containing:

- q_{0N}
- The states you can reach from q_{0N} with only empty-string transitions

Let's call that set-state E({q_{0N}}), and say:

- **q_{0D} = E({q_{0N}})**

Building the DFA: Acceptance



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- **Q_D = P(Q_N), the power set of Q_N**
- **q_{0D} = E({q_{0N}})**

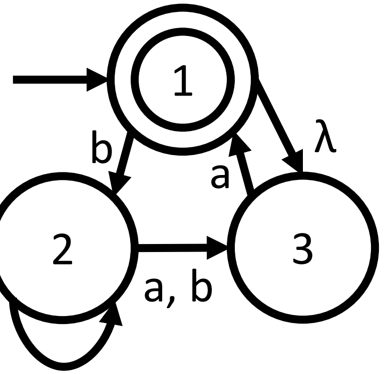
Next the accept states – which actually *are* easy

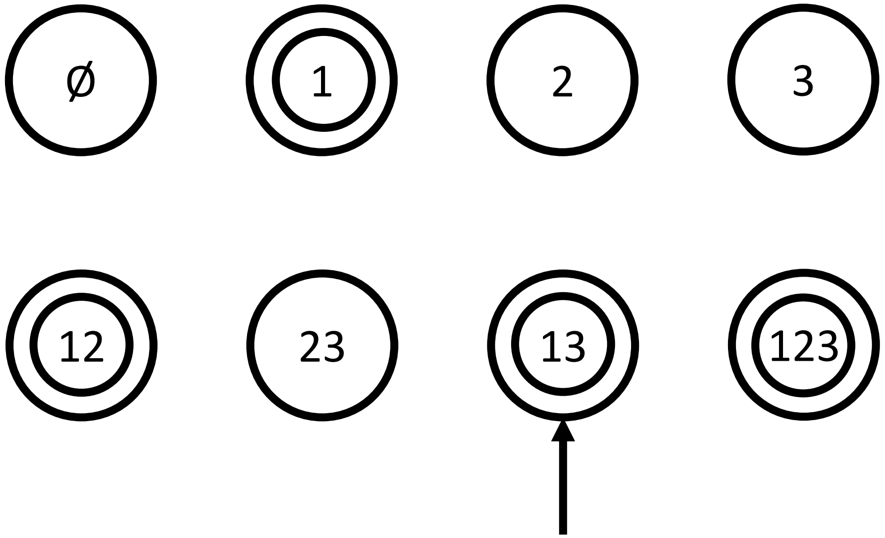
- Recall that the NFA accepts if it has *any* computation path to an accept state
- This means that in our computation, if there is any state we could be in that is an NFA accept state, we accept

So a state-set accepts if it *contains any accept state* from the NFA

- $F_D = \{R \in Q_D \mid R$ and F_N have a common member$\}$, or
- $\boldsymbol{F_D = \{R \in Q_D \mid R \cap F_N \neq \emptyset\}}$

Building the DFA: Transition





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- **$F_D = \{R \in Q_D \mid R \cap F_N \neq \emptyset\}$**

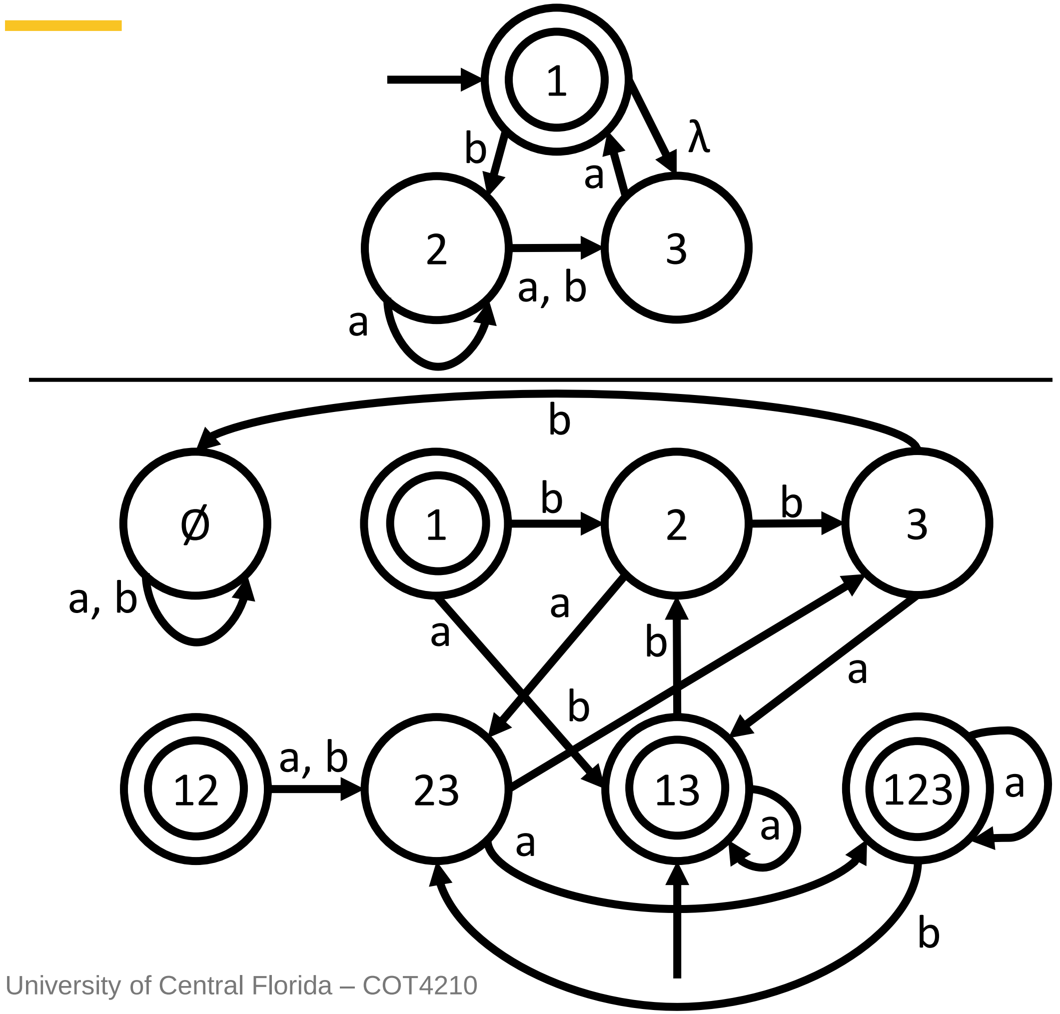
Now the transition function. Remember:

- The NFA transitions between *sets of states*
- We simulate that by having a state for each possible set

So to transition on a given symbol *a*, we:

- Look at *all* the NFA states we are currently simulating
- Look at all the states *they* can possibly transition to on *a*
- Transition to the set of all of those states

Building the DFA: Transition



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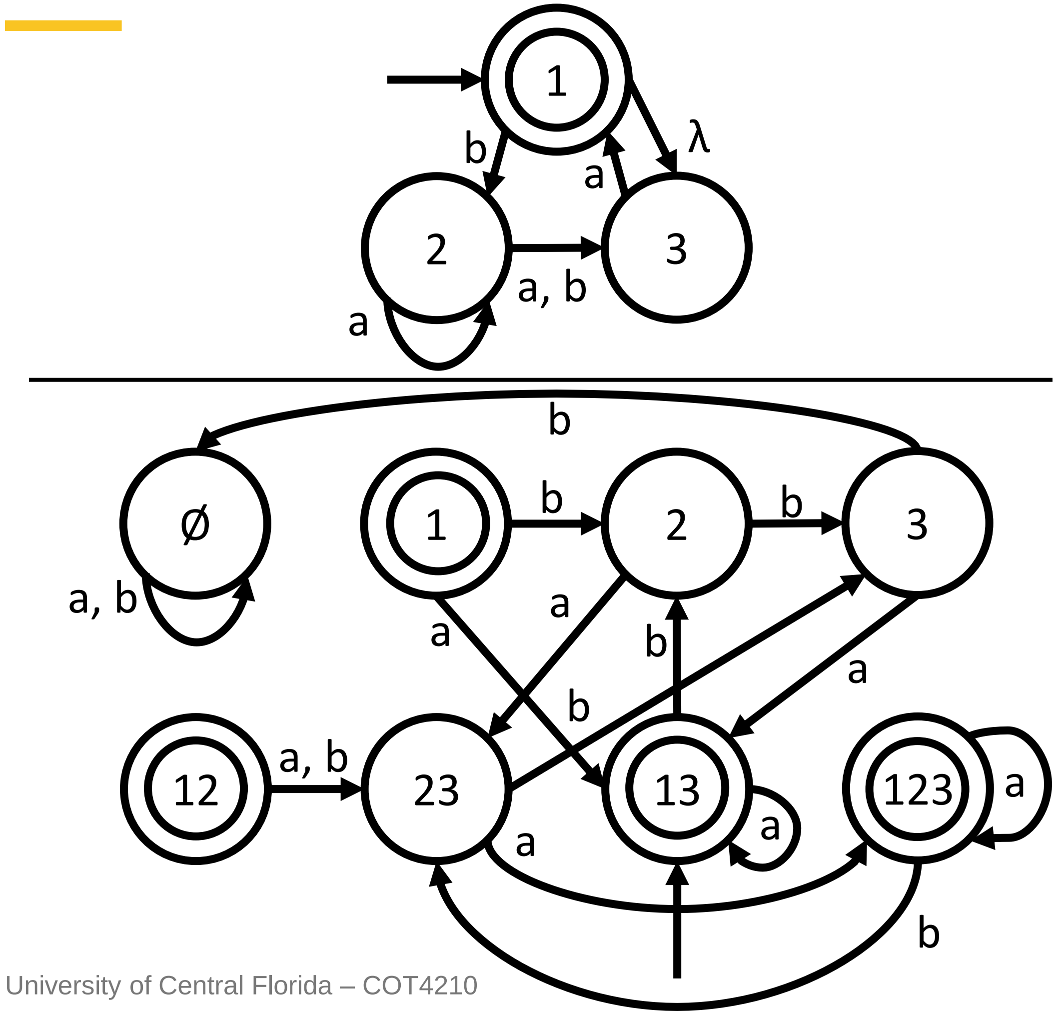
So we define $\delta_D\colon Q_D \times \Sigma \to Q_D$ as:

- $\delta_D(R, a) = \{q \in Q_N \mid q \in \delta_N(r, a) \text{ for some } r \in R\}$
- …almost

Building the DFA: Transition



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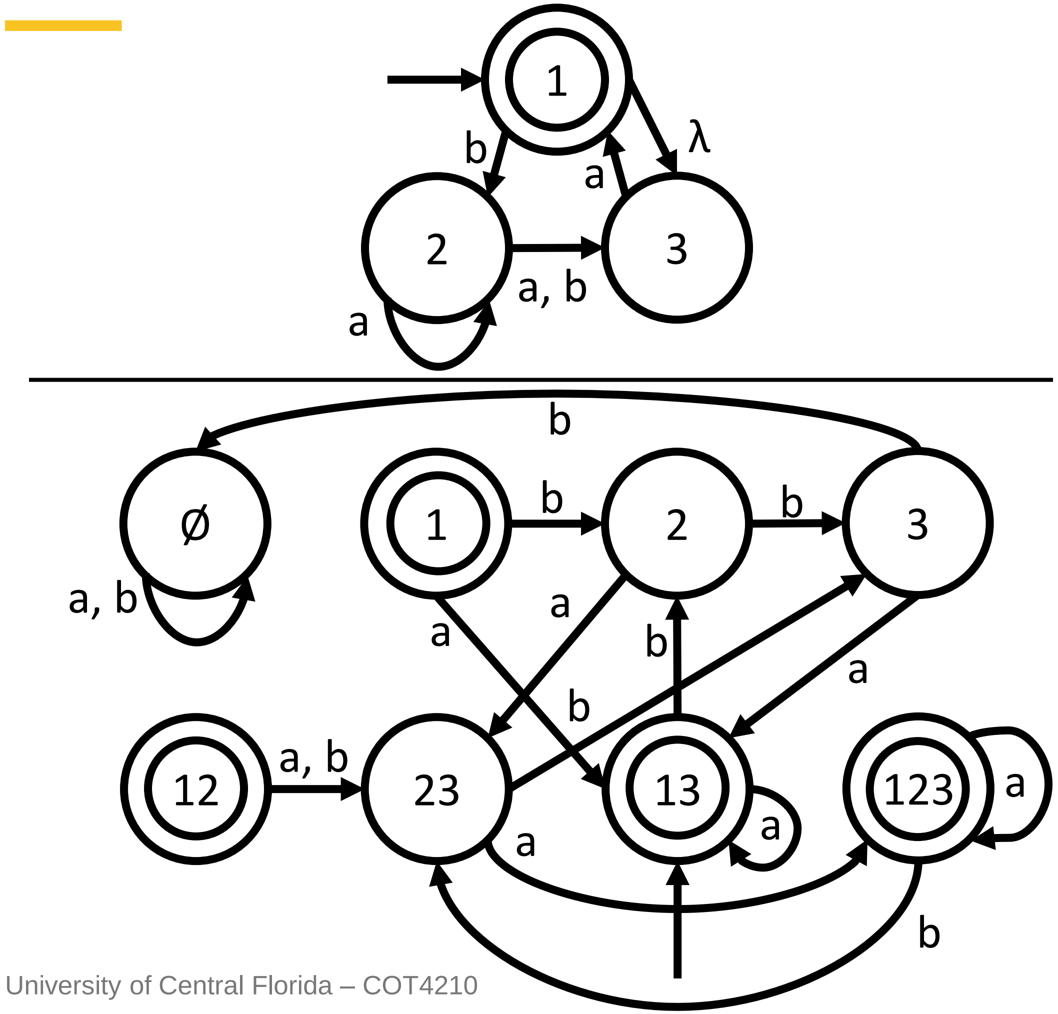
But we need to consider empty string transitions

- We can just do this the same way we did with the start state

So we finally say:

- $\boldsymbol{\delta_D(R, a) = \{q \in Q_N \mid q \in E(\delta_N(r, a)) \text{ for some } r \in R\}}$

Building the DFA: Transition



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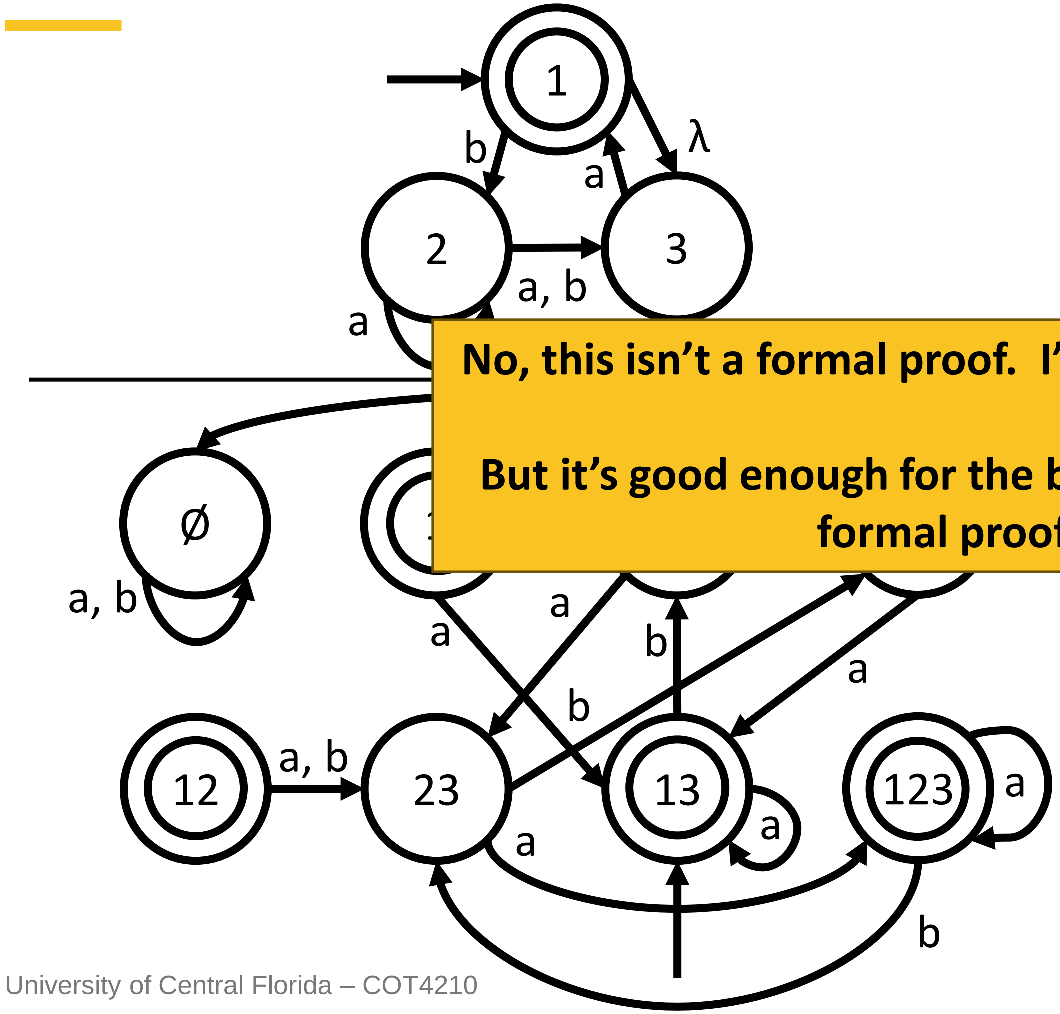
From our observations during construction, *D* is always in a state corresponding to the subset of states *N* could be in on the same input

We have observed that for any NFA *N*, a corresponding DFA *D* exists that recognizes the same language as *N*

Hence, by definition of regular languages, any language recognized by an NFA is regular.

Building the DFA: Transition



No, this isn't a formal proof. I'm not going to draw the little square.

But it's good enough for the book, let alone this class, and doing a formal proof would take weeks.

We **have built** a DFA $D = \{ Q_D, \Sigma, \delta_D, q_{0D}, F_D \}$ that simulates NFA $N = \{ Q_N, \Sigma, \delta_N, q_{0N}, F_N \}$

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- }

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The Consequences of Equivalence

Consequences 1

If every NFA has a DFA, then every language that can be recognized with an NFA can be recognized with a DFA.

But that means…

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If every NFA has a DFA, then every language that can be recognized with an NFA can be recognized with a DFA.

But that means…

Corollary to NFA/DFA Equivalence: *A language is regular if and only if it can be recognized by an NFA.*

Consequences 2

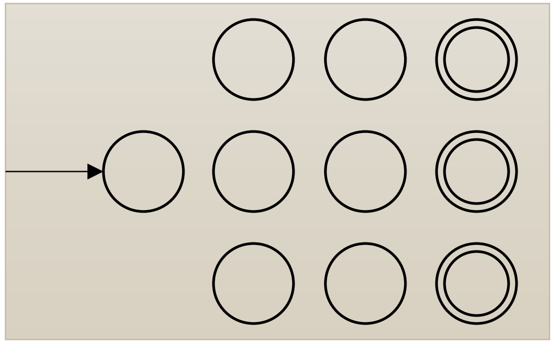
Not too long ago we proved that the class of regular languages was closed under union.

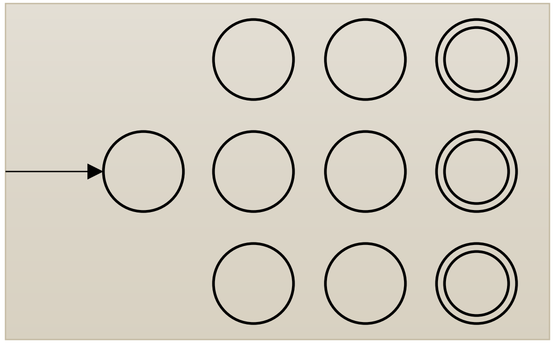
It hurt. A lot.

Let's do that again, a lot faster…

- ..and after that, we'll prove that it's closed under concatenation and star closure

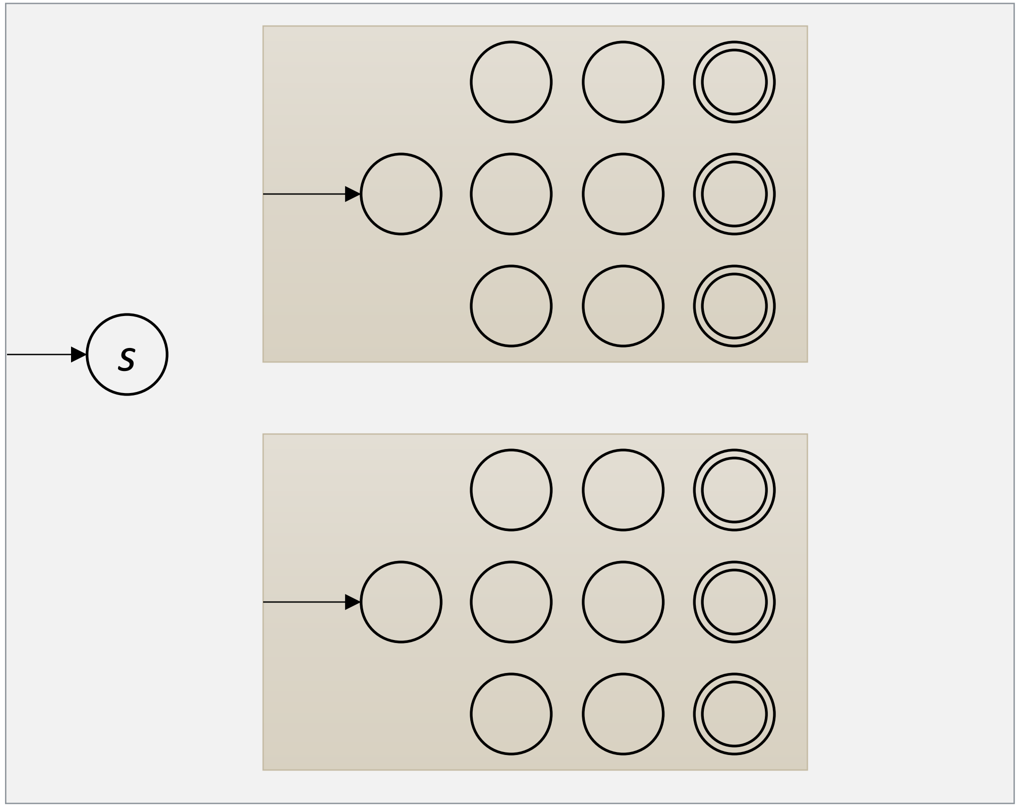
Proof: Closure of Regular Languages – Union





Let $N_A = \{ Q_A, \Sigma, \delta_A, q_{0A}, F_A \}$ and $N_B = \{ Q_B, \Sigma, \delta_B, q_{0B}, F_B \}$ be NFAs recognizing regular languages A and B.

Proof: Closure of Regular Languages – Union



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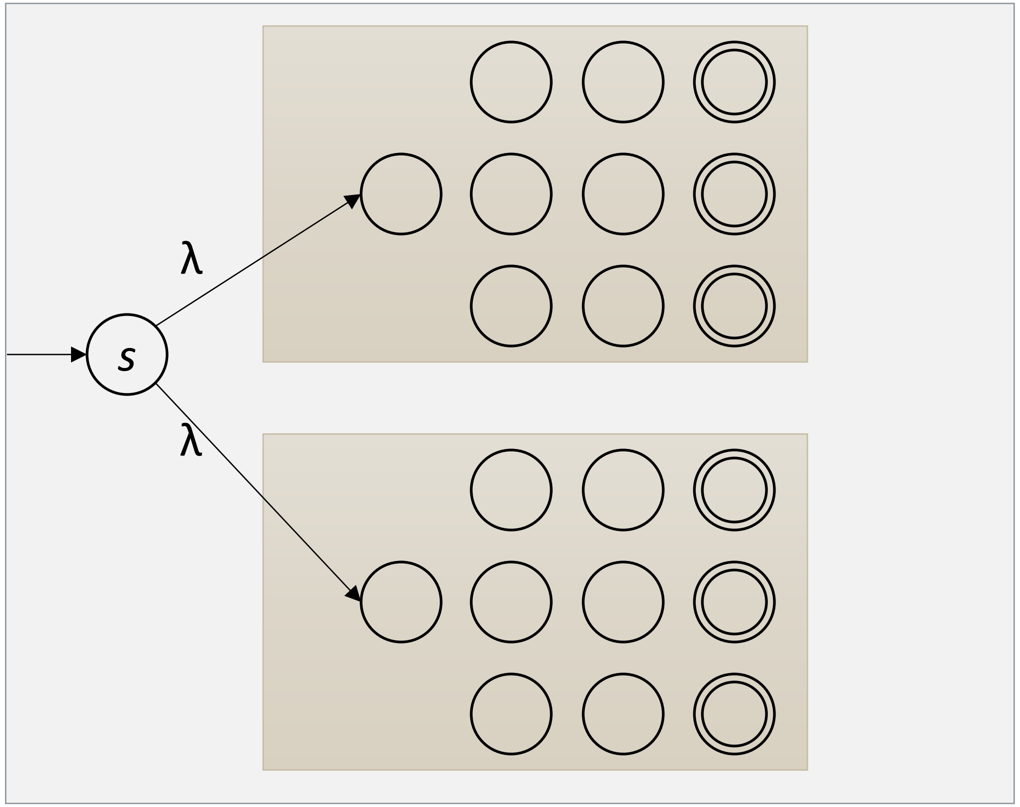
Construct a new NFA $N = \{ Q, \Sigma, \delta, s, F \}$ with:

- $Q = Q_A \cup Q_B \cup \{ s \}$
- Start state s
- $F = F_A \cup F_B$

Proof: Closure of Regular Languages – Union



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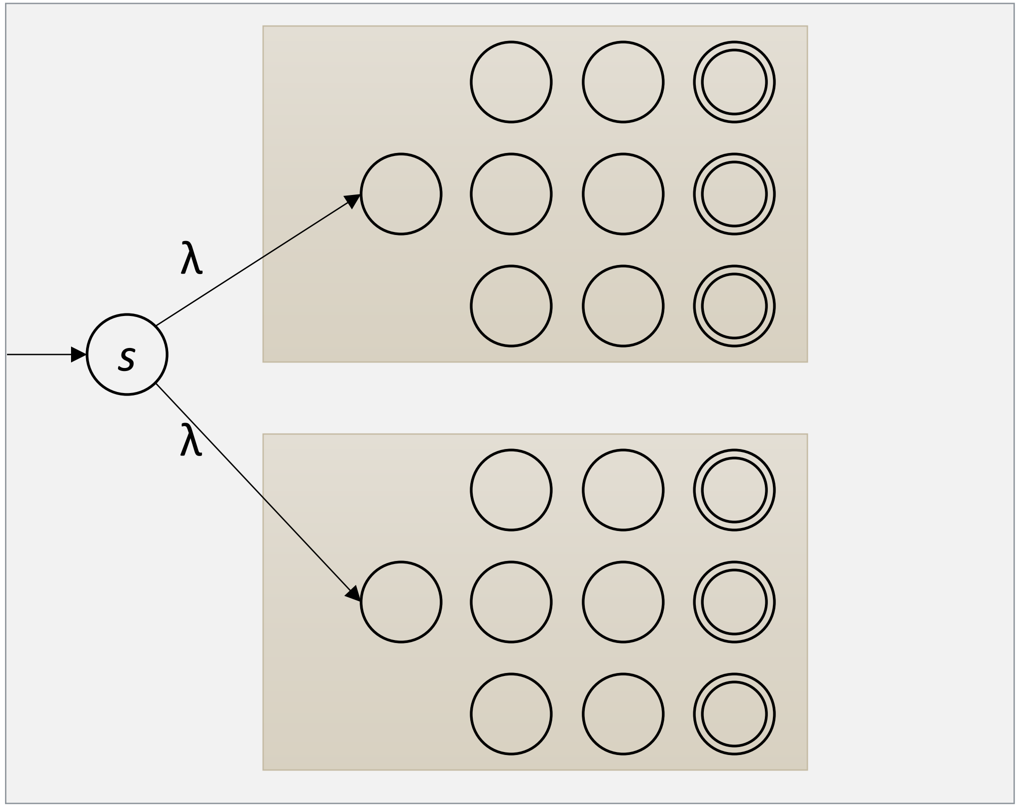
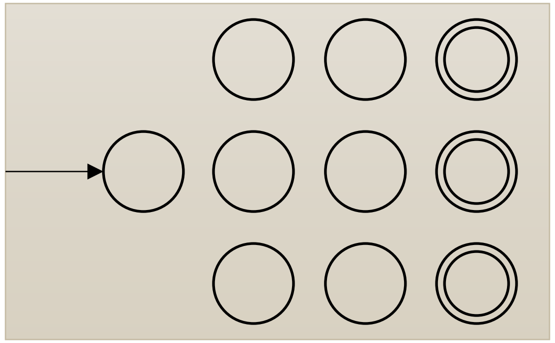
- $Q = Q_A \cup Q_B \cup \{\, s \,\}$
- Start state s
- $F = F_A \cup F_B$
- $\delta(q,a) = \begin{cases} \delta_A(q,a) & q \in Q_A \\ \delta_B(q,a) & q \in Q_B \\ \{q_{0A}, q_{0B}\} & q = s \text{ and } a = \lambda \\ \emptyset & \text{otherwise} \end{cases}$

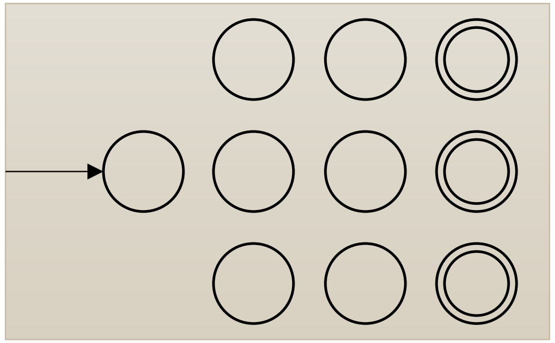
Proof: Closure of Regular Languages – Union



- N clearly accepts everything N_A or N_B accept, and nothing else.
- Therefore by recognition, N accepts everything in A or in B, and nothing else.
- Therefore by union, N accepts everything in $A \cup B$, and nothing else.
- Therefore by recognition, N recognizes $A \cup B$.
- Therefore, there is an *NFA* recognizing $A \cup B$.
- Therefore, $A \cup B$ is regular. □

Proof: Closure of Regular Languages – Concatenation





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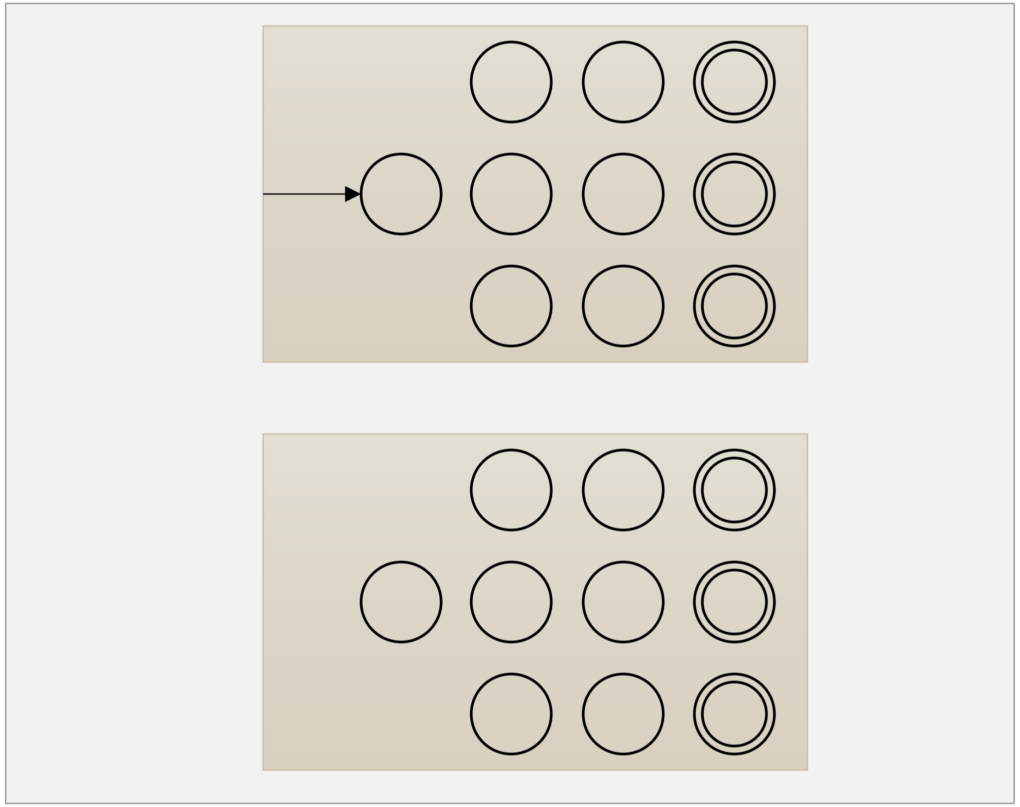

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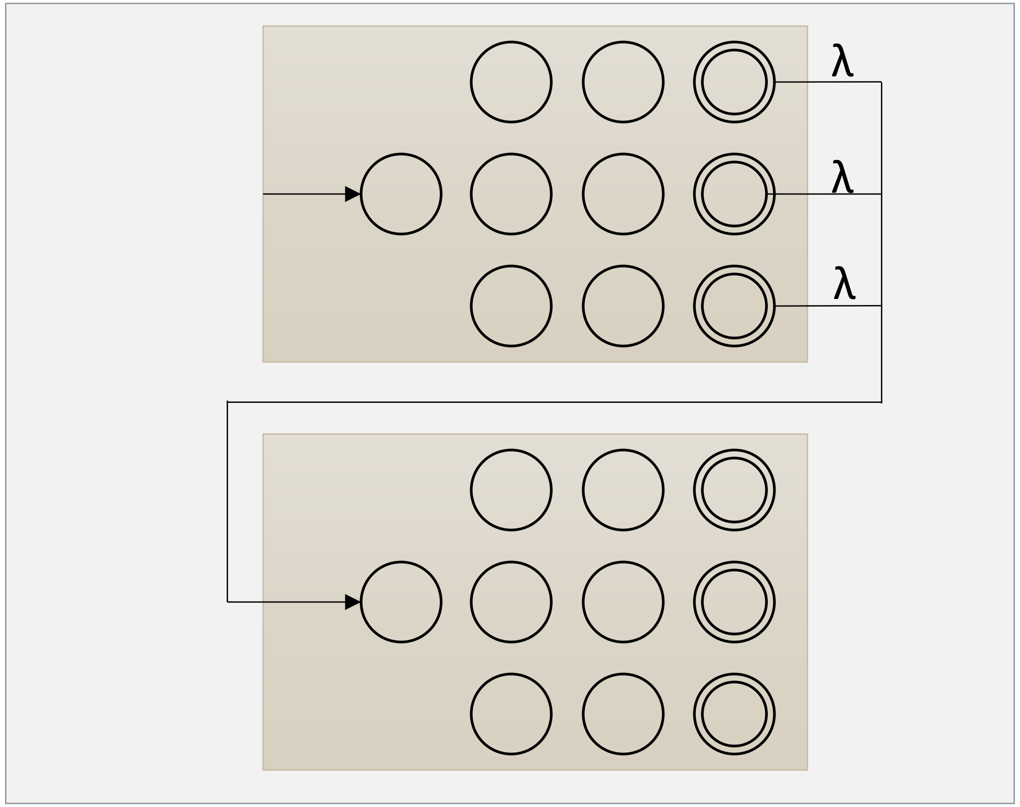

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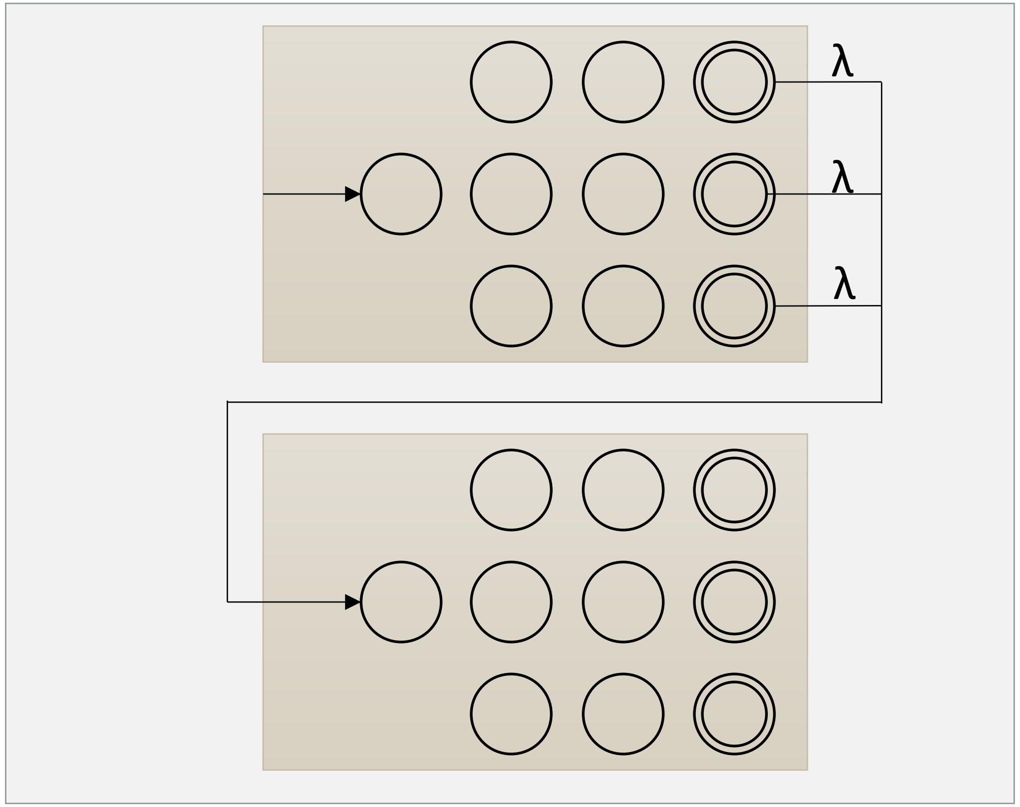
- Start state q_{0A}
- $F = F_B$
- $\delta(q,a) = \begin{cases} \delta_B(q,a) & q \in Q_B \\ \delta_A(q,a) & q \in Q_A \text{ and } q \notin F_A \\ \delta_A(q,a) & q \in F_A \text{ and } a \neq \lambda \\ \delta_A(q,\varepsilon) \cup \{q_{0B}\} & q \in F_A \text{ and } a = \lambda \end{cases}$

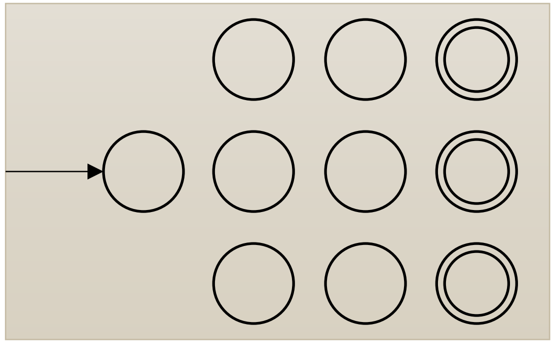
Proof: Closure of Regular Languages – Concatenation



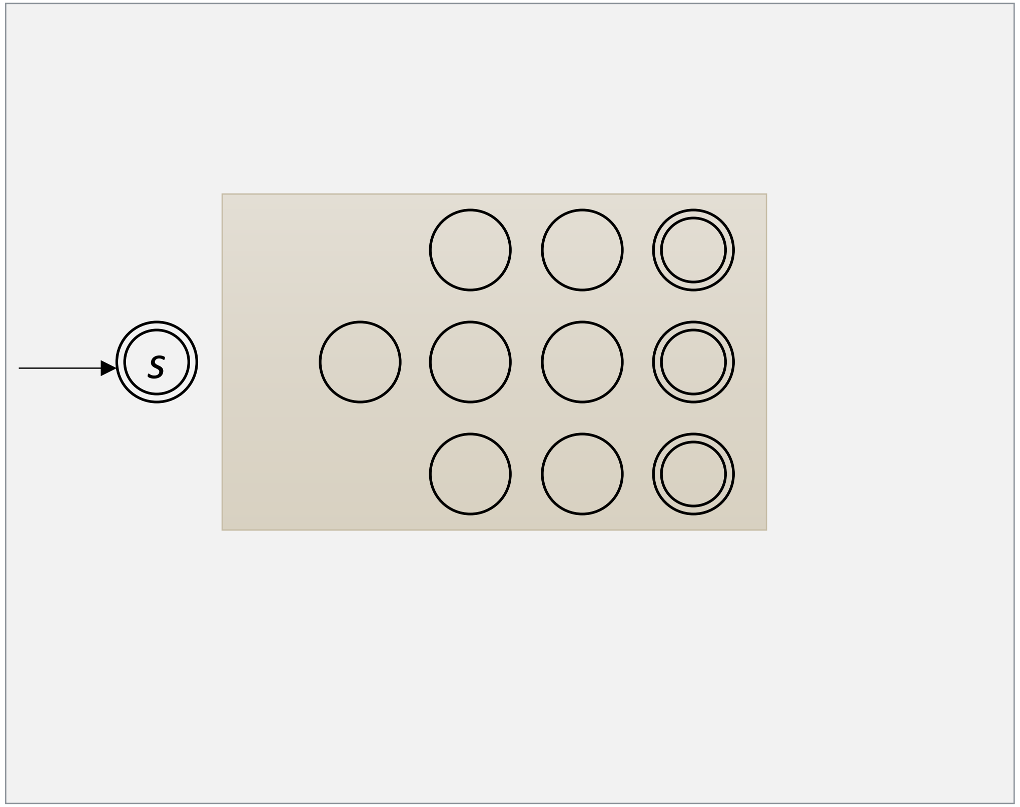
- N clearly accepts every string consisting of a string accepted by N_A followed by a string accepted by N_B, and only those strings.
- Therefore by recognition, N accepts every string that is a string in A followed by a string in B, and nothing else.
- Therefore by concatenation, N accepts every string in AB, and nothing else.
- Therefore by recognition, N recognizes AB.
- Therefore, there is an *NFA* recognizing AB.
- Therefore, AB is regular. □

Proof: Closure of Regular Languages – Star

Let $N_A = \{ Q_A, \Sigma, \delta_A, q_{0A}, F_A \}$ be an NFA recognizing regular language A.



Proof: Closure of Regular Languages – Star

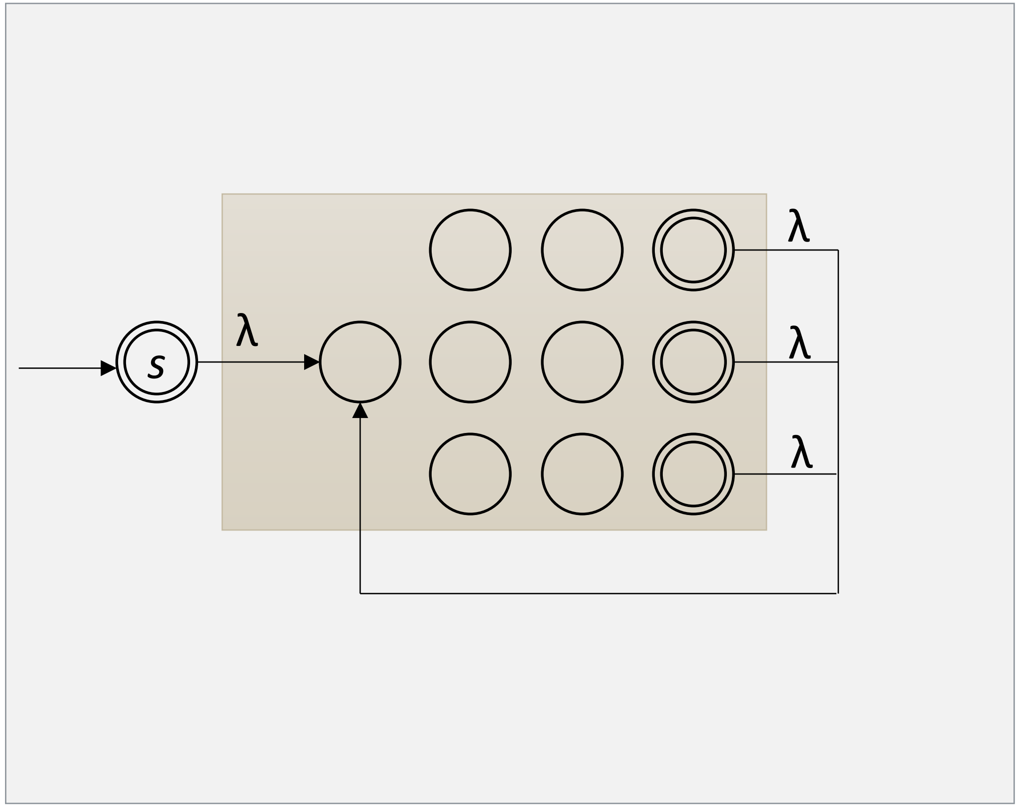


Let $N_A = \{\, Q_A, \Sigma, \delta_A, q_{0A}, F_A \,\}$ be an NFA recognizing regular language A.

Construct a new NFA $N = \{\, Q, \Sigma, \delta, s, F \,\}$ with:

- $Q = Q_A \cup \{\, s \,\}$
- Start state s
- $F = F_A \cup \{\, s \,\}$

Proof: Closure of Regular Languages – Star



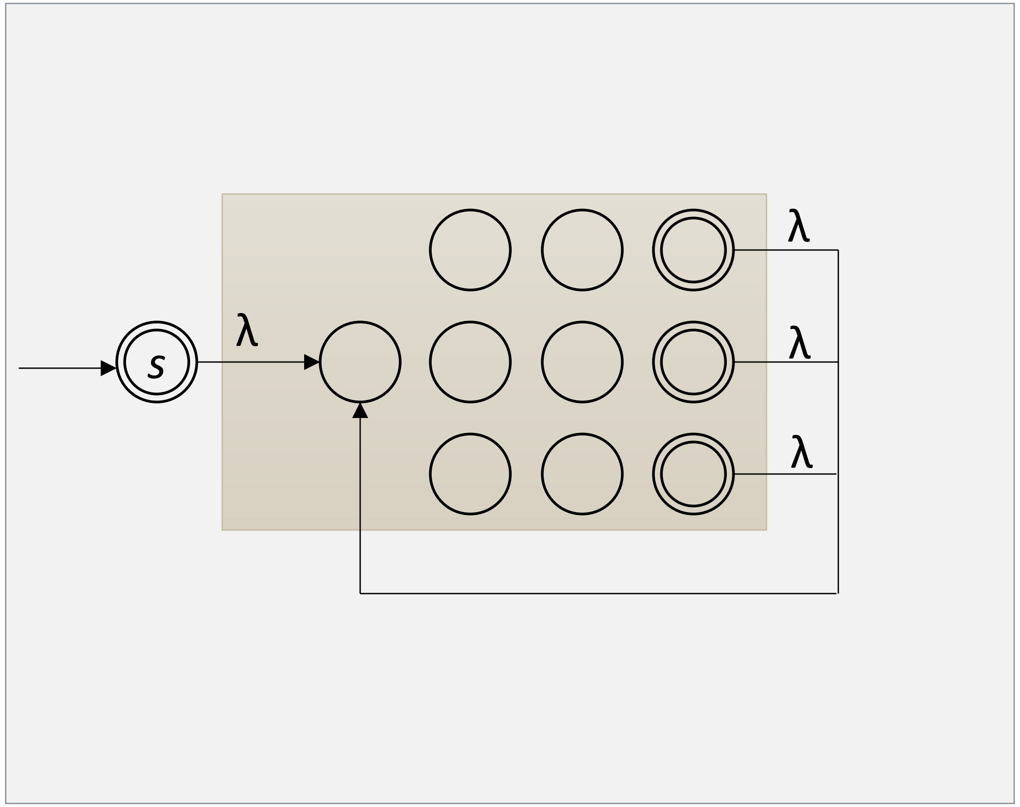
Let $N_A = \{ Q_A, \Sigma, \delta_A, q_{0A}, F_A \}$ be an NFA recognizing regular language A.

Construct a new NFA $N = \{ Q, \Sigma, \delta, s, F \}$ with:

- $Q = Q_A \cup \{ s \}$
- Start state s
- $F = F_A \cup \{ s \}$

- $\delta(q,a) = \begin{cases} \delta_A(q,a) & q \in Q_A \text{ and } q \notin F_A \\ \delta_A(q,a) & q \in F_A \text{ and } a \neq \lambda \\ \delta_A(q,\lambda) \cup \{q_{0A}\} & q \in F_A \text{ and } a = \lambda \\ \{q_{0A}\} & q = s \text{ and } a = \lambda \\ \emptyset & \text{otherwise} \end{cases}$

Proof: Closure of Regular Languages – Star



- *N* clearly accepts every string consisting of zero or more strings accepted by N_A.
- Therefore by recognition, *N* accepts every string consisting of zero or more strings in *A*.
- Therefore by star, *N* accepts every string in *A**.
- Therefore by recognition, *N* recognizes *A**.
- Therefore, there is an *NFA* recognizing *A**.
- Therefore, *A** is regular. □

DFA vs NFA Differences

Deterministic Finite Automata	Nondeterministic Finite Automata
Each state must have a transition for each symbol in a given alphabet Σ.	Each state is not required to have a transition for each symbol in a given alphabet Σ.
Each state can only have 1 transition for each symbol in a given alphabet Σ.	Each state can have multiples transitions for each symbol in a given alphabet Σ.
λ/ϵ transitions are NOT allowed.	λ/ϵ transitions are allowed.

Acknowledgement

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