Calibration of the Heston Stochastic Volatility Model

Introduction

In financial markets, accurate option pricing and risk management are crucial for investors and traders. The Black-Scholes model falls short in capturing real-world market phenomena such as volatility smiles and skews. The Heston model, introduced by Heston (1993), extends the Black-Scholes framework by allowing for stochastic volatility, thereby providing a more realistic representation of market dynamics.

This mini-project focuses on the implementation and calibration of the Heston model using option data from Apple Inc (AAPL). The primary objective is calibrating the Heston model parameters to match observed market option prices. This process minimises the difference between model-generated and market prices across a range of strike prices and maturities. By doing so, the obtained set of parameters best describes the underlying volatility process and captures the volatility surface observed in the market.

Pricing Dynamics

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. On this space, we define two correlated Brownian motions $W^{\mathbb{P}}$ and $Z^{\mathbb{P}}$ with correlation coefficient $\rho \in [-1, 1]$.

The Heston model describes the dynamics of an asset price S_t and its instantaneous variance v_t under the real-world measure $\mathbb{P}$ as follows:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{\mathbb{P}} \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_t^{\mathbb{P}} \tag{2}$$

where:

- S_t is the asset price at time t
- v_t is the instantaneous variance at time t
- μ is the drift of the asset price
- $\kappa > 0$ is the rate of mean reversion of the variance
- $\theta > 0$ is the long-term mean of the variance
- $\sigma > 0$ is the volatility of volatility
- $dW_t^{\mathbb{P}}$ and $dZ_t^{\mathbb{P}}$ are increments of the correlated Brownian motions under $\mathbb{P}$

The correlation between the two Brownian motions is given by:

$$d\langle W^{\mathbb{P}}, Z^{\mathbb{P}}\rangle_t = \rho dt \tag{3}$$

This formulation captures both the stochastic nature of the asset price and its volatility, allowing for a more realistic modelling of financial markets.

Change of Measure

To price derivatives, we need to move from the real-world measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. We apply Girsanov's theorem to effect this change of measure (Kevin (2021)).

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be our new probability space, where $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Define the market price of asset risk φ_S and the market price of volatility risk φ_v as:

$$\varphi_s = \frac{\mu - r}{\sqrt{v_t}}$$
$$\varphi_v = \frac{\lambda}{\sigma}\sqrt{v_t}$$

where r is the risk-free rate and λ represents the premium for volatility risk.

By Girsanov's theorem, we can define new Brownian motions under $\mathbb{Q}$:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \varphi_S dt$$
$$dZ_t^{\mathbb{Q}} = dZ_t^{\mathbb{P}} + \varphi_v dt$$

The dynamics under the risk-neutral measure $\mathbb{Q}$ then become:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^{\mathbb{Q}}$$
$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t} dZ_t^{\mathbb{Q}}$$

where:

- $\kappa^* = \kappa + \lambda$ is the risk-neutral rate of mean reversion
- $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$ is the risk-neutral long-term mean of variance
- $dW_t^{\mathbb{Q}}$ and $dZ_t^{\mathbb{Q}}$ are increments of the Brownian motions under $\mathbb{Q}$

The correlation between the two Brownian motions remains unchanged:

$$d\langle W^{\mathbb{Q}}, Z^{\mathbb{Q}}\rangle_t = \rho dt$$

This risk-neutral formulation of the Heston model provides the basis for option pricing and other derivative valuation under the model.

Heston's partial differential equation

Let V(S_t, v_t, t), a function of stock price, variance, and time, be the value of an option. By Ito's lemma, the value of dV is given by

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS \cdot dS + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}dv \cdot dv + \frac{\partial V}{\partial S \partial v}dS \cdot dv \tag{4}$$

Substituting the equations 1 and 2 in the equation 4, we get

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}\,dt + S_t\left(r\,dt + \sqrt{v_t}\,dW_t^{\mathbb{Q}}\right)\frac{\partial V}{\partial S} + \left(\kappa^*(\theta^* - v_t)\,dt + \sigma\sqrt{v_t}\,dZ_t^{\mathbb{Q}}\right)\frac{\partial V}{\partial v} \\
&\quad + \frac{1}{2}S_t^2 v_t\,dt\,\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v_t\,dt\,\frac{\partial^2 V}{\partial v^2} + \rho\sigma S_t v_t\,dt\,\frac{\partial^2 V}{\partial S\partial v} \\
&= \left(\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \kappa^*(\theta^* - v_t)\frac{\partial V}{\partial v} + \frac{1}{2}v_t S_t^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma v_t S_t\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 V}{\partial v^2}\right)dt \\
&\quad + \frac{\partial V}{\partial S}\sqrt{v_t}S_t\,dW_t^{\mathbb{Q}} + \frac{\partial V}{\partial v}\sigma\sqrt{v_t}\,dZ_t^{\mathbb{Q}}.
\end{aligned}
$$

To satisfy the no-arbitrage condition, the evolution of the option price at time t must equate to the risk-free rate.

$$
\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa^*(\theta^* - v)\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} = rV.
$$

Thus, the Heston model yields the following partial differential equation for the value of the option:

$$
\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa^*(\theta^* - v)\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} - rV = 0 \tag{5}
$$

Heston's Solution

The analytical formula for calculating the price $C(S_t, v_t, t)$ of a European call option involves a complex integral (Pironneau (2023)):

$$
C = \frac{1}{2}(S_0 - Ke^{-rT}) + \frac{1}{\pi}\Re\left\{\int_0^\infty \left(\frac{e^{rT}\phi(u-i)}{iuK^{iu}} - \frac{\phi(u)}{iuK^{iu}}\right)du\right\} \tag{6}
$$

where,

$$
\phi(u) = e^{rT}S_0^{iu}\left(\frac{1-ge^{-dT}}{1-g}\right)^{-2\theta^*\kappa^*/\sigma^2}\exp\left(\frac{\theta^*\kappa^*T}{\sigma^2}(\kappa^* - \rho\sigma iu - d) + \frac{v_0}{\sigma^2}(\kappa^* - \rho\sigma iu + d)\frac{1-e^{dT}}{1-ge^{dT}}\right) \tag{7}
$$

$$
d = \sqrt{(\rho\sigma iu - \kappa^*)^2 + \sigma^2(iu + u^2)} \tag{8}
$$

$$
g = \frac{\kappa^* - \rho\sigma iu - d}{\kappa^* - \rho\sigma iu + d} \tag{9}
$$

Calibration recipe

The objective of the calibration process is to find the set of parameters $(\kappa^*, \theta^*, \sigma, \rho, v_0, \lambda)$ that minimises the difference between the market prices of options and the model prices. Mathematically, it is expressed as:

$$
\min_{\kappa^*,\theta^*,\sigma,\rho,v_0}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}\left(C_{\text{market}}(T_i, K_j) - C_{\text{model}}(T_i, K_j; \kappa^*, \theta^*, \sigma, \rho, v_0, \lambda)\right)^2, \tag{10}
$$

where, $C_{\text{market}}(T_i, K_j)$ are the observed market prices of options with strike price K_j and maturity T_i, $w_{i,j}$ is the weight, and $C_{\text{model}}(T_i, K_j; \kappa^*, \theta^*, \sigma, \rho, v_0, \lambda)$ are the model prices given the parameters.

For this study, all option prices have been assigned equal weights for the sake of simplicity.

Feller's condition is a constraint that ensures the positivity of the variance process in the Heston model. The condition is given by:

$$2\kappa^*\theta^* \geq \sigma^2. \tag{11}$$

This constraint must be satisfied during calibration to ensure the variance remains strictly positive, preventing unrealistic scenarios where the variance could become negative.

Incorporating Feller's condition into the calibration process introduces a non-convex constraint. According to Mrázek & Pospíšil (2017), the non-convexity arises because Feller's condition creates a non-linear relationship between the parameters κ^*, θ^*, and σ.

Methodology

The following section describes the methodology used in this project, with Python extensively employed for all tasks.

1. **Data Collection**: Data used in this study is fetched from Yahoo Finance, providing historical market data for various financial instruments.
2. **Interest Rate Data**: Interest rates are sourced from the US department of Treasury.
3. **Interest Rate Calibration**: The Nelson-Siegel-Svensson (NSS) model is employed to capture interest rate dynamics across maturities. The interest rate under NSS framework is calibrated as follows:

$$R(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} + \beta_2 \left(\frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} - e^{-\tau/\lambda_1} \right) + \beta_3 \left(\frac{1 - e^{-\tau/\lambda_2}}{\tau/\lambda_2} - e^{-\tau/\lambda_2} \right)$$

   where:

   - $R(\tau)$ is the interest rate for maturity τ.
   - β_0 represents the long-term interest rate level.
   - β_1 controls the short-term component.
   - β_2 influences the medium-term component.
   - β_3 adjusts the curve to better fit long-term data.
   - λ_1 is a decay factor for the medium-term component.
   - λ_2 is a decay factor for the long-term component.

Model Performance and Results[1]

The Heston model is successfully calibrated using SciPy's 'L-BFGS-B' optimiser. L-BFGS-B is an iterative algorithm deployed for solving large-scale optimisation problems with bound constraints. It is particularly well-suited for problems where the objective function is smooth and differentiable and efficiently handles high-dimensional parameter spaces. The L-BFGS-B algorithm can effectively navigate through non-convex objective functions, which may exhibit multiple local minima. The calibration process demonstrated positive outcomes, as evidenced by the convergence plot 1 of the variables, which show the progression of the parameter estimates towards their optimal values. Table 1 compares the market and Heston model prices. The value on the top of each table indicates the time

[1] The following calculations were performed on July 13, 2024, with the spot price of AAPL at \$230.53

to maturity in years, while the index is the strike prices across different maturities. The percentage difference is calculated as:

$$\text{pct\_diff} = \frac{\mathrm{C_{market}} - \mathrm{C_{model}}}{\mathrm{C_{market}}} \times 100\%$$

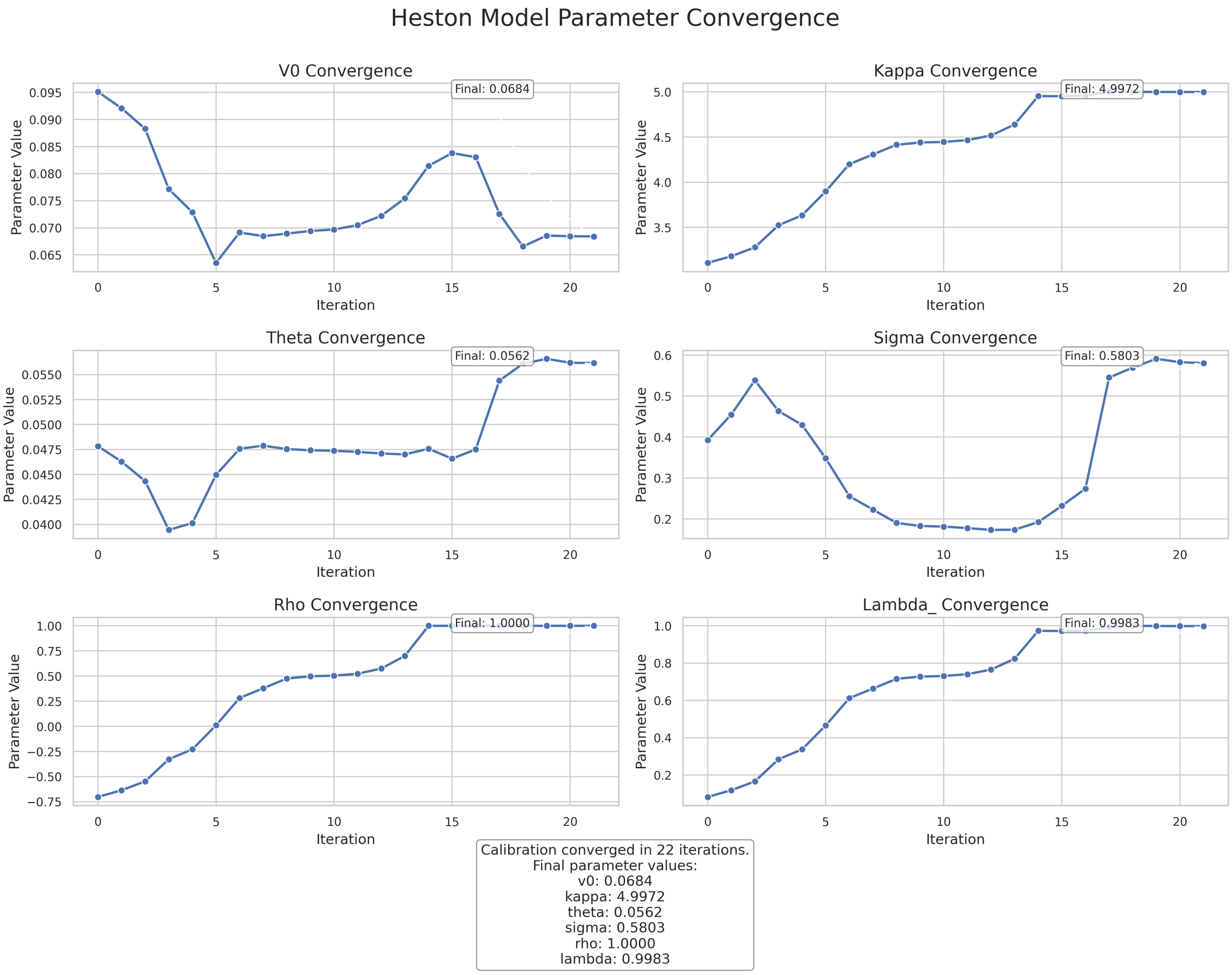

Figure 1: Convergence plot

	0.11			0.18			0.26			0.34		
	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff
220.0	14.75	14.35	2.70	16.64	16.58	0.35	18.62	18.53	0.51	21.40	20.32	5.06
225.0	11.42	11.05	3.28	13.55	13.45	0.75	15.60	15.45	0.98	17.90	17.26	3.59
230.0	8.20	8.33	-1.63	10.65	10.79	-1.35	12.50	12.80	-2.43	15.15	14.59	3.69
235.0	5.97	6.16	-3.21	8.17	8.59	-5.15	10.05	10.55	-4.98	12.64	12.28	2.83
240.0	4.25	4.48	-5.35	6.00	6.77	-12.85	7.95	8.64	-8.67	10.40	10.30	0.99
245.0	2.95	3.18	-7.80	4.45	5.28	-18.57	6.03	7.03	-16.65	8.40	8.59	-2.30
250.0	1.85	2.19	-18.50	3.25	4.07	-25.28	4.70	5.69	-20.97	6.92	7.14	-3.11
260.0	0.84	0.93	-10.18	1.71	2.30	-34.79	3.00	3.62	-20.55	4.42	4.84	-9.40
270.0	0.47	0.24	49.71	0.91	1.18	-29.89	1.59	2.18	-37.38	2.87	3.17	-10.50
	0.43			0.51			0.68			0.93		
	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff
220.0	23.45	22.44	4.30	24.43	24.07	1.45	28.45	27.60	3.00	32.70	32.41	0.88
225.0	19.76	19.37	1.96	21.30	20.99	1.45	25.88	24.48	5.43	30.65	29.24	4.61
230.0	17.14	16.66	2.78	18.55	18.25	1.64	21.91	21.65	1.18	26.66	26.32	1.27
235.0	14.43	14.29	1.00	15.95	15.81	0.87	19.98	19.11	4.38	24.25	23.65	2.48
240.0	12.00	12.21	-1.72	13.45	13.66	-1.59	17.30	16.82	2.76	21.90	21.21	3.16
245.0	10.15	10.40	-2.42	11.45	11.78	-2.84	15.50	14.78	4.66	19.65	18.99	3.38
250.0	8.40	8.82	-5.03	9.60	10.12	-5.39	12.98	12.95	0.20	17.30	16.97	1.92
260.0	5.70	6.28	-10.10	6.92	7.40	-6.87	9.70	9.88	-1.87	14.35	13.48	6.06
270.0	4.25	4.37	-2.84	4.80	5.32	-10.76	7.20	7.45	-3.54	10.85	10.63	2.04
	1.18			1.43			1.51			1.93		
	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff	Market	Model	%Diff
220.0	36.50	37.00	-1.36	41.42	41.41	0.02	41.25	42.75	-3.63	46.20	49.86	-7.93
225.0	34.53	33.78	2.18	37.21	38.16	-2.55	39.44	39.48	-0.10	44.39	46.55	-4.86
230.0	31.09	30.79	0.98	34.40	35.11	-2.05	35.30	36.41	-3.15	40.95	43.40	-5.99
235.0	28.20	28.02	0.66	32.50	32.26	0.75	33.80	33.54	0.77	39.66	40.43	-1.95
240.0	26.42	25.45	3.65	28.10	29.60	-5.33	31.70	30.86	2.66	36.88	37.63	-2.04
245.0	22.35	23.09	-3.33	27.47	27.13	1.25	29.40	28.36	3.55	34.25	34.99	-2.17
250.0	21.50	20.92	2.68	25.50	24.83	2.61	26.05	26.03	0.08	31.95	32.51	-1.76
260.0	18.52	17.10	7.65	21.44	20.74	3.26	22.05	21.86	0.85	28.16	28.00	0.57
270.0	14.85	13.90	6.38	18.80	17.25	8.26	19.48	18.29	6.12	24.58	24.04	2.19

Table 1: Comparison of market and model values with percentage differences for different strikes and maturities.

Analysis of Option Pricing Model Performance

Based on the data presented in Table 1, we can draw several insights about the performance of the option pricing model:

1. **Increasing percentage differences at higher strikes:** For lower maturities (e.g., 0.11, 0.18), the percentage difference between market and model values tends to increase dramatically as the strike price increases. For example, at maturity 0.11, the difference goes from 2.70% at strike 220 to 49.71% at strike 270.

2. **Maturity effect on accuracy:** As maturity increases, the model's accuracy improves for higher strike prices. The extreme percentage differences at short maturities become less pronounced at longer maturities. There's an interesting oscillating pattern in the accuracy as maturity increases. For some strike prices, the model alternates between overestimation and underestimation across different maturities.

3. **Extreme mispricings:** There are extreme mispricings, particularly for deep out-of-the-money options at short maturities. For example, at maturity 0.11 and strike 270, the model overprices by 49.71%.

4. **Volatility smile effect:** The increasing percentage differences at higher and lower strikes, with better accuracy in the middle, suggest that the model might not fully capture the volatility smile effect observed in market data.

These insights suggest that while the model performs reasonably well for at-the-money options and mid-range maturities, it struggles with accurately pricing deep out-of-the-money options, especially at very short and very long maturities. This could indicate that the model might benefit from refinements to better capture market dynamics across a range of strike prices and maturities.

Volatility Surface [2]

The volatility surface in finance refers to a three-dimensional plot of implied volatilities as a function of both strike prices and time to maturity. It provides a comprehensive view of market expectations regarding future volatility levels, which is crucial for pricing and risk management.

In the context of the Heston model, the stochastic volatility model is calculated by matching model prices (C_{Heston}) with observed market prices (C_{market}) of options. This involves solving the equation derived from the Black-Scholes model under the assumption of stochastic volatility:

$$\begin{aligned} C_{\text{model}}(T_i, K_j; \Theta) &= C_{\text{BS}}(K, T, S_0, r, \sigma_{\text{imp}}) \\ &= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2), \end{aligned}$$

Here, S_0 is the spot price of the underlying asset, K is the strike price of the option, T is the time to maturity, r is the risk-free interest rate, and Θ represents the parameters of the Heston model governing volatility dynamics.

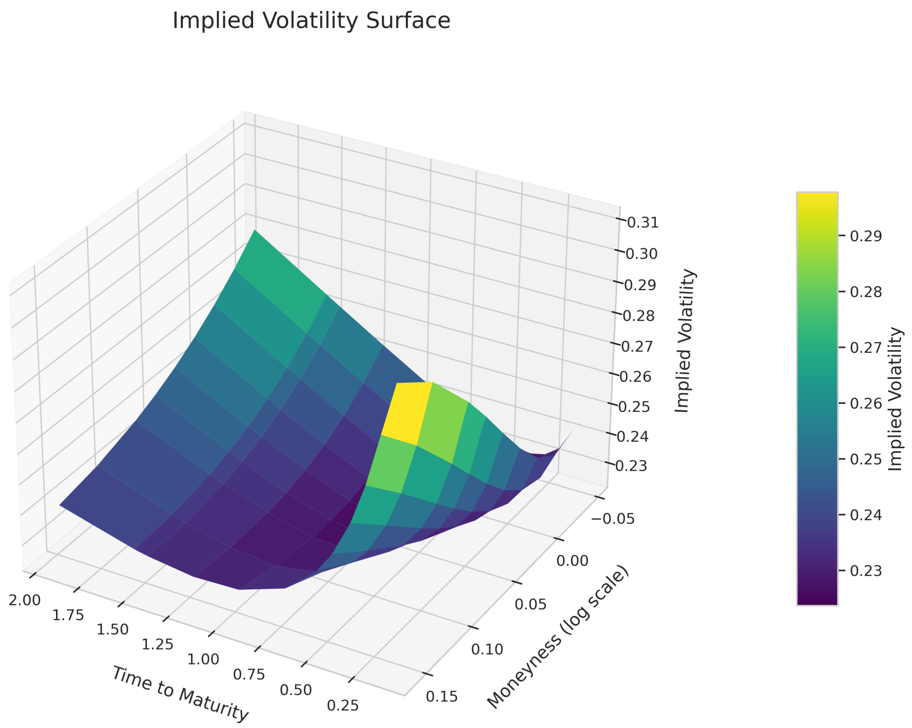

Figure 2: Implied Volatility Surface

Conclusion

In conclusion, the calibration of the Heston model on AAPL data has provided valuable insights into the dynamics of option pricing and volatility estimation. By leveraging Python's computational capabilities, the study achieved decent parameter estimates that explain the evolution of stock price and variance under the Heston model.

[2]This section is referenced from Stack Exchange.

Bibliography

Heston, S. L. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *The Review of Financial Studies* **6**(2), 327–343.

Kevin (2021), 'Heston stochastic volatility, girsanov's theorem', https://quant.stackexchange.com/questions/61927/heston-stochastic-volatility-girsanov-theorem/61931#61931. Accessed: 2024-07-13.

Mrázek, M. & Pospíšil, J. (2017), 'Calibration and simulation of heston model', *Open Mathematics* **15**(1), 679–704.
URL: *https://doi.org/10.1515/math-2017-0058*

Pironneau, O. (2023), 'Heston model calibration using keras', *Example Journal* . Accessed: 2024-07-14.
URL: *http://www.example.com/heston-calibration*