CSE400 – Fundamentals of Probability in Computing
Lecture 6: Discrete Random Variables, Expectation and Problem Solving

Instructor: Dhaval Patel, PhD

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1 Random Variables

1.1 Definition and Concept

A random variable is defined as a function that assigns a real number to each outcome in the sample space.

Formally,

$$X : \Omega \to \mathbb{R}$$

where:

- Ω denotes the sample space.
- $\omega \in \Omega$ denotes a sample point.
- $X(\omega)$ is the real number assigned to the outcome ω.

Thus, a random variable maps outcomes of an experiment to numerical values.

The lecture further restricts attention to discrete random variables, meaning:

$$\{X(\omega) : \omega \in \Omega\}$$

is a finite or countably infinite subset of $\mathbb{R}$.

This means that although the codomain of X is $\mathbb{R}$, the actual values taken by X belong to a discrete set.

1.2 Visualization of a Random Variable

The distribution of a discrete random variable can be visualized using a bar diagram, where:

- The x-axis represents the possible values of the random variable.

- The height of each bar at value a represents:

$$Pr(X = a)$$

This probability corresponds to the probability of the event in the sample space mapped to value a.

2 Types of Random Variables

Random variables are classified into two main types.

2.1 Discrete Random Variable

A random variable is discrete if:

- It has countable support
- A Probability Mass Function (PMF) exists
- Probabilities are assigned to individual values
- Each possible value has strictly positive probability

2.2 Continuous Random Variable

A random variable is continuous if:

- It has uncountable support
- It uses a Probability Density Function (PDF)
- Probabilities are assigned to intervals
- Each individual value has zero probability

3 Example of a Discrete Random Variable

3.1 Experiment Description

Suppose an experiment consists of tossing 3 fair coins.

Define:

$$Y = \text{number of heads}$$

Then the possible values of Y are:

$$Y \in \{0, 1, 2, 3\}$$

3.2 Probability Calculation

All possible outcomes:

$$\{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

Each outcome has probability:

$$\frac{1}{8}$$

Case 1: No heads

$$P(Y = 0) = P(TTT) = \frac{1}{8}$$

Case 2: One head
Outcomes:

$$TTH, THT, HTT$$

$$P(Y = 1) = \frac{3}{8}$$

Case 3: Two heads
Outcomes:

$$THH, HTH, HHT$$

$$P(Y = 2) = \frac{3}{8}$$

Case 4: Three heads
Outcome:

$$HHH$$

$$P(Y = 3) = \frac{1}{8}$$

3.3 Total Probability Property

Since Y must take one of these values:

$$1 = \sum_{i=0}^{3} P(Y = i)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= 1$$

This confirms a valid probability distribution.

4 Probability Mass Function (PMF)

4.1 Definition

A random variable that takes at most a countable number of values is called discrete.

Let:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

be the range of possible values.

The function:

$$P_X(x_k) = P(X = x_k)$$

for

$$k = 1, 2, 3, \dots$$

is called the Probability Mass Function (PMF).

4.2 Property of PMF

Since the random variable must take one of its possible values:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

This expresses the total probability law.

5 PMF Example with Unknown Constant

Given:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$

Find:

$$P(X = 0)$$

$$P(X > 2)$$

5.1 Step 1: Use Total Probability Property

Since:

$$\sum_{i=0}^{\infty} p(i) = 1$$

Substitute:

$$\sum_{i=0}^{\infty} c\frac{\lambda^i}{i!} = 1$$

Factor c:

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using the identity:

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$$

Thus:

$$ce^{\lambda} = 1$$

Solve for c:

$$c = e^{-\lambda}$$

5.2 Step 2: Find $P(X = 0)$

Substitute $i = 0$:

$$P(X = 0) = c\frac{\lambda^0}{0!}$$

$$= e^{-\lambda}$$

5.3 Step 3: Find $P(X > 2)$

Use complement:

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

Substitute:

$$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda} \right)$$

6 Bayes' Theorem

Using conditional probability:

$$Pr(A|B_i) = \frac{Pr(B_i|A)Pr(A)}{\sum_{i=1}^{n} Pr(A|B_i)Pr(B_i)}$$

This is called the Bayes Formula.

6.1 Definitions

Prior Probability $Pr(B_i)$
Probability formed before observing evidence.
Posterior Probability $Pr(B_i|A)$
Probability after observing event A.

7 Bayes' Theorem Example: Auditorium with 30 Rows

Given
Auditorium has 30 rows
Row 1: 11 seats
Row 2: 12 seats
Row 3: 13 seats
...
Row 30: 40 seats
Selection process:
Select row randomly:

$$P(R_k) = \frac{1}{30}$$

Select seat randomly within row.

7.1 Step 1: Find $P(S_{15}|R_{20})$

Row 20 has:

$$20 + 10 = 30 \text{ seats}$$

Thus:

$$P(S_{15}|R_{20}) = \frac{1}{30}$$

7.2 Step 2: Find $P(R_{20}|S_{15})$

Using Bayes formula:

$$P(R_{20}|S_{15}) = \frac{P(S_{15}|R_{20})P(R_{20})}{P(S_{15})}$$

Denominator:

$$P(S_{15}) = \sum_{k=5}^{30} P(S_{15}|R_k)P(R_k)$$

$$= \sum_{k=5}^{30} \frac{1}{k+10}\frac{1}{30}$$

Numerical value:

$$= 0.0342$$

Thus:

$$P(R_{20}|S_{15}) = \frac{(1/30)(1/30)}{0.0342}$$

$$= 0.0325$$

8 Summary of Key Results

Random variable definition:

$$X : \Omega \to \mathbb{R}$$

PMF definition:

$$P_X(x_k) = P(X = x_k)$$

PMF property:

$$\sum P_X(x_k) = 1$$

Constant value:

$$c = e^{-\lambda}$$

Bayes theorem:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum P(A|B_i)P(B_i)}$$