School of Engineering and Applied Science (SEAS), Ahmedabad University
CSE 400: Fundamentals of Probability in Computing
Lecture Scribe Notes: Random Variables
Lecture 6 & 7 – Discrete Random Variables, Expectation, and Problem Solving
Instructor: Dhaval Patel, PhD
Date: January 22, 2025

1. Random Variables (Recap and Concepts)

1.1 Definition

A random variable (RV) on a sample space is a function that assigns a real number to each sample point.

$$X : \Omega \to \mathbb{R}, \quad X(\omega),\ \omega \in \Omega$$

Discrete Random Variables: We restrict attention to RVs that are discrete. This means they take values in a range that is finite or countably infinite. Even though X maps to $\mathbb{R}$, the set of values $\{X(\omega) : \omega \in \Omega\}$ is a discrete subset of $\mathbb{R}$.

1.2 Probability Mass Function (PMF)

For a discrete random variable X with a range (possible values) $R_X = \{x_1, x_2, x_3, \ldots\}$, the Probability Mass Function (PMF) is defined as:

$$P_X(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3, \ldots$$

Legitimacy Property: Since X must take on one of the values x_k, the sum of probabilities must equal 1:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

Visualization: The PMF can be visualized as a bar diagram where the x-axis represents the values the variable can take, and the height of the bar is the probability.

2. Worked Example: Solving for a Constant in a PMF

The PMF of a random variable is given by

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where λ is a positive value.

Solution

For a legitimate PMF,

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using the Taylor series expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

we obtain

$$ce^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$$

$$P\{X = 0\} = e^{-\lambda}$$

$$P\{X > 2\} = 1 - P\{X \leq 2\}$$
$$P\{X > 2\} = 1 - P\{X = 0\} - P\{X = 1\} - P\{X = 2\}$$

3. Bayes' Theorem and Independent Events

3.1 Bayes' Theorem

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

A priori probability: presupposed models. Posteriori probability: calculated after observing events.

3.2 Communication System Example

$$P(0Rx) = P(0Rx|0Tx)P(0Tx) + P(0Rx|1Tx)P(1Tx)$$
$$P(0Rx) = (0.95)(0.5) + (0.10)(0.5) = 0.525$$

3.3 Independent Events

Two events A and B are independent if

$$P(A|B) = P(A)$$

$$P(A, B) = P(A)P(B)$$

4. Types of Discrete Random Variables

4.1 Bernoulli Random Variable

$$X = \begin{cases} 1 & \text{Success} \\ 0 & \text{Failure} \end{cases}$$

$$P_X(1) = p, \quad P_X(0) = 1 - p$$

4.2 Binomial Random Variable

$$X \sim B(n, p)$$

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots, n$$

4.3 Geometric Random Variable

$$P(X = n) = (1-p)^{n-1} p$$

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = 1$$

4.4 Poisson Random Variable

$$p(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

$$\sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = 1$$

Poisson random variables can approximate Binomial random variables when n is large and p is small.