Lecture Scribe Notes: Random Variables

L6_S2_A

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Generate a lecture scribe notes which can be used as exam reference material .Use only the provided context (lecture slides) dont add anything of you own . While writing the scribe reason step by step through all definitions , notation, assumptions , theorem statements , proofs ,derivations and worked examples exactly as given in the lecture slides. Make the logical flow explicit so that a student can follow the reasoning during exam revision . Include proofs, derivations and examples only if they appear in the provided context. Do not introduce new explanations examples or material beyond the lecture scope . Maintain a formal and academic tone and strict faithfulness to the lecture structure

Course: CSE400 - Fundamentals of Probability in Computing **Lecture:** 6 & 7 - Discrete Random Variables, Expectation, and Problem Solving **Instructor:** Dhaval Patel, PhD **Date:** January 22, 2025

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1. Random Variables (Recap and Concepts)

1.1 Definition

A random variable (RV) X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each sample point $\omega \in \Omega$.

- **Discrete Random Variables:** We restrict attention to RVs that are discrete. This means they take values in a range that is finite or countably infinite. Even though X maps to $\mathbb{R}$, the set of values $\{X(\omega) : \omega \in \Omega\}$ is a discrete subset of $\mathbb{R}$.

1.2 Probability Mass Function (PMF)

For a discrete random variable X with a range (possible values) $R_X = x_1, x_2, x_3, ...$, the Probability Mass Function (PMF) is defined as:

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, ...$$

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- **Legitimacy Property:** Since X must take on one of the values x_k, the sum of probabilities must equal 1:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

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- **Visualization:** The PMF can be visualized as a bar diagram where the x-axis represents the values the variable can take (a), and the height of the bar is the probability $Pr[X = a]$.

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2. Worked Example: Solving for a Constant in a PMF

Problem: The PMF of a random variable X is given by $p(i) = c\frac{\lambda^i}{i!}$ for $i = 0, 1, 2, ...$ where λ is a positive value. Find the constant c, then find $P\{X = 0\}$ and $P\{X > 2\}$.

Solution Step-by-Step:

1. **Find c:** We know that for a legitimate PMF, $\sum p(i) = 1$.

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

. Using the Taylor series expansion $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, we substitute e^λ for the summation.

$$c \cdot e^{\lambda} = 1 \implies c = e^{-\lambda}$$

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2. **Find $P\{X = 0\}$:** Substitute $i = 0$ and $c = e^{-\lambda}$ into the PMF:

$$P\{X = 0\} = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}$$

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3. **Find $P\{X > 2\}$:** Using the complement rule:

$$P\{X > 2\} = 1 - P\{X \leq 2\}$$

$$P\{X > 2\} = 1 - P\{X = 0\} - P\{X = 1\} - P\{X = 2\}$$

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$$P\{X > 2\} = 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2}$$

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3. Bayes' Theorem and Independent Events

3.1 Bayes' Theorem Recap

The Bayes Formula is given by:

$$Pr(B_i|A) = \frac{Pr(A|B_i)Pr(B_i)}{\sum_{i=1}^{n} Pr(A|B_i)Pr(B_i)}$$

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- $Pr(B_i)$: A priori probability (presupposed models).
- $Pr(B_i|A)$: Posteriori probability (calculated after observing events).

3.2 Worked Example: Communication System

Scenario: A system sends binary data (0 or 1).

- $Pr(0 \text{ received} \mid 0 \text{ transmitted}) = 0.95$
- $Pr(1 \text{ received} \mid 0 \text{ transmitted}) = 0.05$
- $Pr(0 \text{ received} \mid 1 \text{ transmitted}) = 0.10$
- $Pr(1 \text{ received} \mid 1 \text{ transmitted}) = 0.90$. Assume $Pr(\text{"}0\text{"}) = 1/2$ (a priori).

Problem: Find $Pr(0 \text{ received})$.

Solution: Using the law of total probability:

$$Pr(0_{Rx}) = Pr(0_{Rx}|0_{Tx})Pr(0_{Tx}) + Pr(0_{Rx}|1_{Tx})Pr(1_{Tx})$$

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$$Pr(0_{Rx}) = (0.95)(0.5) + (0.10)(0.5) = 0.525$$

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Problem: If a 0 is detected, what is the probability a 1 was transmitted? ($Pr(1_{Tx}|0_{Rx})[cite_start]$).

$$Pr(1_{Tx}|0_{Rx}) = \frac{Pr(0_{Rx}|1_{Tx})Pr(1_{Tx})}{Pr(0_{Rx})} = \frac{(0.10)(0.5)}{0.525}$$

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3.3 Independent Events

Two events A and B are independent if $Pr(A|B) = Pr(A)$. This implies the joint probability is:

$$Pr(A, B) = Pr(A)Pr(B)$$

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Example (Communication Network): Consider a network with nodes A, B, C, D and links a_1, a_2, a_3, a_4 where p is the probability of a link being available. Path from A to D requires available links. Probability of path $A \to D$:

$$Pr(A \to D) = Pr(\{a_3 \cap a_4\} \cup \{a_1 \cap a_2 \cap a_4\})$$

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Using Inclusion-Exclusion Principle:

$$= Pr(a_3 \cap a_4) + Pr(a_1 \cap a_2 \cap a_4) - Pr(a_1 \cap a_2 \cap a_3 \cap a_4)$$

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$$= p^2 + p^3 - p^4$$

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4. Types of Discrete Random Variables

4.1 Bernoulli Random Variable

- **Experiment:** An experiment with two outcomes: Success or Failure (e.g., coin toss).
- **Definition:** Let $X = 1$ if Success, $X = 0$ if Failure.
- **PMF:**

$$P_X(1) = p$$

$$P_X(0) = 1 - p$$

  where $p \in (0, 1)$ is the success probability.
- **Applications:**
  - Single coin toss.
  - Randomly selected email is spam or not.

4.2 Binomial Random Variable

- **Experiment:** n independent trials, each results in success (p) or failure ($1 - p$).
- **Definition:** X denotes the number of successes in n trials. Notation: $B(n, p)$.
- **PMF:**

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \text{ for } i = 0, 1, ..., n$$

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- **Applications:**
  - Number of correct answers on a multiple-choice test (random guessing).
  - Number of defective items in a sample size n.

4.3 Geometric Random Variable

- **Experiment:** Independent trials ($0 < p < 1$) performed *until* a success occurs.
- **Definition:** X denotes the number of trials required for success.
- **PMF:**

$$P_X(X = n) = (1-p)^{n-1} \times p$$

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- **Validation (Legitimate PMF):** Summation of successive trials forms a geometric series:

$$\sum_{n=1}^{\infty} P\{X = n\} = p \sum_{n=1}^{\infty} (1-p)^{n-1} = \frac{p}{1-(1-p)} = 1$$

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- **Worked Example (Urn Problem):** Urn with N white and M black balls. Select with replacement until black is found. X is number of draws. Success probability $p = M/(M+N)$.

$$P\{X = n\} = \left(\frac{N}{M+N}\right)^{n-1} \frac{M}{M+N} = \frac{MN^{n-1}}{(M+N)^n}$$

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4.4 Poisson Random Variable

- **Definition:** A random variable X taking values $0, 1, 2, \ldots$ with parameter $\lambda > 0$.
- **PMF:**

$$p(i) = P\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!} \text{ for } i = 0, 1, 2, \ldots$$

  .
- **Validation:**

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

  .
- **Approximation:** Poisson RV can approximate a Binomial RV $B(n, p)$ when n is large and p is small such that $n \times p$ is of moderate size.
- **Applications:**
  - Number of misprints on a page.
  - Number of wrong telephone numbers dialed in a day.