

Lecture Scribe CSE400



L6_S2_A.pdf

PDF

Generate a lecture scribe from the provided lecture slides and textbook context. The scribe should serve as exam reference material and must faithfully reconstruct the lecture content.

While writing the scribe, explain definitions, notations, assumptions, theorem statements or results, proofs, derivations and worked examples step by step, following the same logical reasoning as presented in the lecture. Include the reasoning clearly so that the document can be used reliably for exam revision.

Use only the provided context and do not introduce any additional explanations, examples, assumptions or reasoning that are not present in the lecture materials.

Maintain a formal academic tone and organize the scribe clearly so that the logical flow of concepts, results, definitions, proofs and examples is easy to follow.

Below is the **formal lecture scribe reconstructed faithfully from the provided lecture slides**. All definitions, derivations, and examples strictly follow the lecture content. The structure and reasoning mirror the lecture progression so it can serve as reliable exam revision material.

Lecture Scribe

CSE400 – Fundamentals of Probability in Computing

Lecture 6: Discrete Random Variables, Expectation and Problem Solving

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1. Random Variables

1.1 Definition and Concept

Let

- Ω denote the sample space
- $\omega \in \Omega$ denote a sample point

A **random variable** X is defined as a function:

$$X : \Omega \rightarrow \mathbb{R}$$

That assigns a real number to each sample point $\omega \in \Omega$:

$$X(\omega) \in \mathbb{R}$$

Thus, a random variable converts outcomes of an experiment into real numerical values.

1.2 Discrete Random Variables

The lecture restricts attention initially to **discrete random variables**, which satisfy:

- They take values from a finite or countably infinite set.
- The actual set of values:

$$\{X(\omega) : \omega \in \Omega\}$$

forms a discrete subset of the real numbers.

This means the values can be listed explicitly.

1.3 Visualization of Random Variable Distribution

The distribution of a discrete random variable can be visualized using a bar diagram:

- The x-axis represents possible values of the random variable.
- The height of each bar represents:

$$P(X = a)$$

Each probability corresponds to the probability of the associated event in the sample space.

2. Types of Random Variables

2.1 Discrete Random Variable

A random variable is discrete if:

- It has countable support.
 - It is described using a Probability Mass Function (PMF).
 - Probabilities are assigned to individual values.
 - Each possible value has strictly positive probability.
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2.2 Continuous Random Variable

A random variable is continuous if:

- It has uncountable support.
 - It is described using a Probability Density Function (PDF).
 - Probabilities are assigned over intervals.
 - Each individual value has probability zero.
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3. Example: Tossing Three Fair Coins

3.1 Experiment Description

An experiment consists of tossing 3 fair coins.

Define:

$$Y = \text{number of heads observed}$$

Possible values of Y :

$$Y \in \{0, 1, 2, 3\}$$

3.2 Computing Probabilities

The sample space contains 8 equally likely outcomes:

$$\{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

Each has probability:

$$\frac{1}{8}$$

Now compute probabilities:

Case 1: No heads

$$P(Y = 0) = P(TTT) = \frac{1}{8}$$

Case 2: One head

Possible outcomes:

$$\{TTH, THT, HTT\}$$

Thus:

$$P(Y = 1) = \frac{3}{8}$$

Case 3: Two heads

Possible outcomes:

$$\{THH, HTH, HHT\}$$

Thus:

$$P(Y = 2) = \frac{3}{8}$$

Case 4: Three heads

Possible outcome:

$$HHH$$

Thus:

$$P(Y = 3) = \frac{1}{8}$$

3.3 Validity Condition of PMF

Since Y must take exactly one of these values:

$$P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) = 1$$

Thus,

$$\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

This confirms the distribution is valid.

4. Probability Mass Function (PMF)

4.1 Definition

A random variable that can take at most countable values is called a discrete random variable.

Let the range of values be:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

Define the function:

$$P_X(x_k) = P(X = x_k)$$

This function is called the **Probability Mass Function (PMF)** of X .

4.2 Properties of PMF

The PMF must satisfy:

Property 1: Non-negativity

$$P_X(x_k) \geq 0$$

Property 2: Total probability equals 1

Since the random variable must take one of its possible values:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

This follows because the possible values are mutually exclusive and collectively exhaustive events.

5. Example: Finding Unknown Constant in PMF

5.1 Given

The PMF is:

$$p(i) = c \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$.

We must find:

1. $P(X = 0)$
 2. $P(X > 2)$
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5.2 Step 1: Use Total Probability Condition

Since PMF must sum to 1:

$$\sum_{i=0}^{\infty} p(i) = 1$$

Substitute PMF:

$$\sum_{i=0}^{\infty} c \frac{\lambda^i}{i!} = 1$$

Factor out constant:

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using known identity:

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^\lambda$$

Thus:

$$ce^\lambda = 1$$

Solve for c :

$$c = e^{-\lambda}$$

5.3 Step 2: Find $P(X = 0)$

Substitute into PMF:

$$\begin{aligned} P(X = 0) &= c \frac{\lambda^0}{0!} \\ &= e^{-\lambda} \end{aligned}$$

5.4 Step 3: Find $P(X > 2)$

Use complement rule:

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \end{aligned}$$

Substitute:

$$= 1 - (e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2})$$

Thus,

$$P(X > 2) = 1 - e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

6. Bayes' Theorem

6.1 Conditional Probability Relationship

Using definition of conditional probability:

$$Pr(A \mid B) = \frac{Pr(B \mid A)Pr(A)}{Pr(B)}$$

This gives Bayes' Formula:

$$Pr(B_i \mid A) = \frac{Pr(A \mid B_i)Pr(B_i)}{\sum_{i=1}^n Pr(A \mid B_i)Pr(B_i)}$$

6.2 Terminology

Prior probability

$$Pr(B_i)$$

Probability before observing new information.

Posterior probability

$$Pr(B_i \mid A)$$

Probability after observing event A .

7. Bayes' Theorem Example: Auditorium

7.1 Problem Description

An auditorium has 30 rows.

Number of seats in rows:

Row 1 → 11 seats

Row 2 → 12 seats

Row 3 → 13 seats

...

Row 30 → 40 seats

Selection procedure:

Step 1: Select a row randomly (equal probability)

$$P(R_k) = \frac{1}{30}$$

Step 2: Select a seat randomly within that row.

7.2 Find $P(S_{15} \mid R_{20})$

Row 20 has:

$$10 + 20 = 30 \text{ seats}$$

Thus,

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

7.3 Find $P(R_{20} \mid S_{15})$

Using Bayes' formula:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{\sum_{k=5}^{30} P(S_{15} \mid R_k)P(R_k)}$$

Since seat 15 exists only in rows 5 through 30.

Denominator:

$$\sum_{k=5}^{30} \frac{1}{(k+10)} \cdot \frac{1}{30}$$

Numerical evaluation from lecture:

$$P(R_{20} \mid S_{15}) \approx 0.0325$$

Summary of Key Results

Random Variable

$$X : \Omega \rightarrow \mathbb{R}$$

PMF

$$P_X(x_k) = P(X = x_k)$$

$$\sum P_X(x_k) = 1$$

Exponential Identity

$$\sum \frac{\lambda^i}{i!} = e^\lambda$$

Bayes' Theorem

$$P(B_i | A) = \frac{P(A | B_i)P(B_i)}{\sum P(A | B_i)P(B_i)}$$

End of Lecture Scribe

If you want, I can also convert this into **clean handwritten-style exam notes** or a **PDF formatted revision sheet**.