CSE400 – Fundamentals of Probability in Computing
Lecture 6: Discrete Random Variables, Expectation and Problem Solving
Instructor: Dhaval Patel, PhD
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(This scribe is intended for closed-notes / reading-based exam revision.)

L6_S2_A

1 Random Variables

1.1 Motivation and Concept

Let Ω denote the sample space.

Definition: A random variable X on a sample space Ω is a function

$$X : \Omega \to \mathbb{R}$$

that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

Restriction to Discrete Random Variables

Until further notice, attention is restricted to discrete random variables, i.e., random variables that take values in a set that is finite or countably infinite.

Although X maps to $\mathbb{R}$, the actual set of values

$$\{X(\omega) : \omega \in \Omega\}$$

forms a discrete subset of $\mathbb{R}$.

Visualization

Sample points in Ω are mapped by X onto discrete points on the real line.

Multiple sample points may map to the same real value.

1.2 Distribution of a Discrete Random Variable

The distribution of a discrete random variable can be visualized using a bar diagram.

The x-axis represents the possible values of the random variable.

The height of the bar at value a equals:

$$\Pr[X = a]$$

Each probability is computed by finding the probability of the corresponding event in the sample space.

2 Types of Random Variables

2.1 Discrete Random Variables

Characteristics:

- Countable support
- Probability Mass Function (PMF)
- Probabilities assigned to single values
- Each possible value has strictly positive probability

2.2 Continuous Random Variables

Characteristics:

- Uncountable support
- Probability Density Function (PDF)
- Probabilities assigned to intervals
- Each individual value has zero probability

3 Example 1: Tossing 3 Fair Coins

Experiment Description

An experiment consists of tossing 3 fair coins.

Let:

$$Y = \text{number of heads observed}$$

Possible Values of Y

$$Y \in \{0, 1, 2, 3\}$$

Probability Calculations

$$P(Y = 0) = P(t, t, t) = \frac{1}{8}$$

$$P(Y = 1) = P(t, t, h) + P(t, h, t) + P(h, t, t) = \frac{3}{8}$$

$$P(Y = 2) = P(h, h, t) + P(h, t, h) + P(t, h, h) = \frac{3}{8}$$

$$P(Y = 3) = P(h, h, h) = \frac{1}{8}$$

Resulting Distribution

$$P(Y = 0) = \frac{1}{8}, \quad P(Y = 1) = \frac{3}{8}, \quad P(Y = 2) = \frac{3}{8}, \quad P(Y = 3) = \frac{1}{8}$$

Normalization Condition

Since Y must take exactly one of the values $0, 1, 2, 3$:

$$1 = P\left(\bigcup_{i=0}^{3} \{Y = i\}\right) = \sum_{i=0}^{3} P(Y = i)$$

4 Probability Mass Function (PMF)

4.1 Definition

A random variable that can take on at most a countable number of possible values is called discrete.

Let X be a discrete random variable with range:

$$R_X = x_1, x_2, x_3, \ldots$$

(where the range may be finite or countably infinite).

Define:

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots$$

This function $P_X(\cdot)$ is called the Probability Mass Function (PMF) of X.

4.2 PMF Normalization Property

Since X must take one of the values x_k:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

5 PMF – Worked Example

Given

The PMF of a random variable X is:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$.

Find:

1. $P(X = 0)$
2. $P(X > 2)$

Step 1: Find Constant c

Using normalization:

$$\sum_{i=0}^{\infty} p(i) = 1$$

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using:

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$ce^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$$

Step 2: Compute $P(X = 0)$

$$P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$$

Step 3: Compute $P(X > 2)$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda} \right)$$

6 Bayes' Theorem (Recap)

6.1 Conditional Probability Identity

$$\Pr(A \mid B) = \Pr(B \mid A) \Pr(A)$$

6.2 Bayes Formula (Proposition 3.1)

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_{j=1}^{n} \Pr(A \mid B_j) \Pr(B_j)}$$

6.3 Terminology

$$\Pr(B_i) : \text{a priori probability}$$

$$\Pr(B_i \mid A) : \text{posterior probability}$$

7 Bayes' Theorem – Example: Auditorium with 30 Rows

Problem Description

Auditorium has 30 rows.

Row 1 has 11 seats, Row 2 has 12 seats, ..., Row 30 has 40 seats.

A door prize is awarded by:

1. Randomly selecting a row (each row equally likely)
2. Randomly selecting a seat within that row (each seat equally likely)

Tasks

1. Compute the probability that Seat 15 was selected given Row 20 was selected.
2. Compute the probability that Row 20 was selected given Seat 15 was selected.

Step 1: Probability of Selecting Seat 15 Given Row 20

Since Row 20 has 30 seats:

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

Step 2: Compute $P(S_{15})$

Using total probability:

$$P(S_{15}) = \sum_{k=15}^{30} P(S_{15} \mid R_k) P(R_k)$$

Each row is equally likely:

$$P(R_k) = \frac{1}{30}$$

$$P(S_{15}) = \sum_{k=15}^{30} \frac{1}{k+10} \cdot \frac{1}{30} \approx 0.0342$$

Step 3: Probability Row 20 Given Seat 15

Using Bayes' formula:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20}) P(R_{20})}{P(S_{15})}$$

$$= \frac{\frac{1}{30} \cdot \frac{1}{30}}{0.0342} \approx 0.0325$$