CSE400 – Lecture 6 Scribe

Discrete Random Variables, PMF, Bayes' Theorem & Standard Discrete Distributions

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1 Random Variables – Motivation and Definition

Step 1: Start from the sample space

Let:

- Ω = sample space (set of all possible outcomes of an experiment)
- A **random variable** is a function:

$$X : \Omega \to \mathbb{R}$$

This means:

$$\Rightarrow \text{Each outcome } \omega \in \Omega \text{ is mapped to a real number } X(\omega)$$

Step 2: Restriction to discrete random variables

In this lecture:

- We focus on **discrete random variables**

This means:

- The set of values taken by X

$$\{X(\omega) : \omega \in \Omega\}$$

is:

- finite OR
- countably infinite

So even though X maps into real numbers, only **discrete points** occur.

Step 3: Distribution visualization (PMF bars)

For a discrete RV:

- x-axis $\to$ possible values of the RV
- bar height at value $a \to P(X = a)$

Each probability comes from the probability of the corresponding event in the sample space.

2 Discrete vs Continuous (conceptual distinction from slides)

Discrete RV:

- Countable support
- Uses **Probability Mass Function (PMF)**
- Probabilities assigned to single values
- Each value has positive probability

Continuous RV:

- Uncountable support
- Uses **Probability Density Function (PDF)**
- Probabilities assigned to intervals
- Each exact value has probability zero

(Only discrete RVs are used in this lecture's worked material.)

3 Example: Tossing 3 Fair Coins

Step 1: Define experiment

Toss 3 fair coins.

Each outcome equally likely.

Let:

$$Y = \text{number of heads}$$

Possible values:

$$Y \in \{0, 1, 2, 3\}$$

Step 2: List outcomes by value of Y

$Y = 0$

Only outcome:

$$(t, t, t)$$

So:

$$P(Y = 0) = \frac{1}{8}$$

$Y = 1$

Outcomes:

$$(t, t, h), (t, h, t), (h, t, t)$$

Total = 3 outcomes

$$P(Y = 1) = \frac{3}{8}$$

$Y = 2$

Outcomes:

$$(t, h, h), (h, t, h), (h, h, t)$$

Total = 3 outcomes

$$P(Y = 2) = \frac{3}{8}$$

$Y = 3$

Outcome:

$$(h, h, h)$$

$$P(Y = 3) = \frac{1}{8}$$

Step 3: Probability must sum to 1

$$\sum_{i=0}^{3} P(Y = i) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Valid PMF

4 Probability Mass Function (PMF) — Formal Definition

Step 1: Discrete RV

A random variable is **discrete** if it takes at most countably many values.

Let possible values:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

Step 2: Define PMF

$$P_X(x_k) = P(X = x_k)$$

for $k = 1, 2, 3, \dots$

This function is the **Probability Mass Function (PMF)**.

Step 3: Required property

Since X must take one of these values:

$$\sum_k P_X(x_k) = 1$$

5 PMF Example Using Exponential Series

Given:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

Step 1: Use total probability = 1

$$\sum_{i=0}^{\infty} p(i) = c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Step 2: Recognize series

From slides:

$$e^\lambda = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

So:

$$ce^\lambda = 1 \Rightarrow c = e^{-\lambda}$$

Step 3: Find probabilities

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

6 Bayes' Theorem (from recap)

Step 1: Start from joint probability

$$P(A, B_i) = P(B_i|A)P(A)$$

Step 2: Solve for conditional probability

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$$

Step 3: Interpretation

- $P(B_i)$ = prior probability
- $P(B_i|A)$ = posterior probability (after observing A)

7 Bayes Example — Auditorium

Given:

- 30 rows
- Row 1 has 11 seats, Row 2 has 12 seats, ... Row 30 has 40 seats

Procedure:

1. Randomly select a row (each with probability 1/30)
2. Randomly select a seat within that row

(a) Probability Seat 15 given Row 20

Row 20 has 30 seats.

Each seat equally likely:

$$P(\text{Seat 15}|\text{Row 20}) = \frac{1}{30}$$

(b) Probability Row 20 given Seat 15

Use Bayes:

$$P(R_{20}|S_{15}) = \frac{P(S_{15}|R_{20})P(R_{20})}{P(S_{15})}$$

Where:

- $P(R_{20}) = 1/30$
- $P(S_{15}|R_{20}) = 1/30$
- $P(S_{15}) = \sum_{k=15}^{30} \frac{1}{30} \cdot \frac{1}{(k+10)}$

(from rows that actually contain seat 15)

8 Independent Events

Definition (from slides)

Two events A and B are independent if:

$$P(A|B) = P(A), \quad P(B|A) = P(B)$$

which implies:

$$P(A, B) = P(A)P(B)$$

For three events:

A, B, C are mutually independent if:

$$P(A, B) = P(A)P(B)$$

$$P(A, C) = P(A)P(C)$$

$$P(B, C) = P(B)P(C)$$

$$P(A, B, C) = P(A)P(B)P(C)$$

9 Bernoulli Random Variable

Step 1: Define experiment

Outcome = Success or Failure

Define RV:

$$X = 1 \text{ if success}, \quad X = 0 \text{ if failure}$$

Step 2: PMF

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

where $0 < p < 1$

Applications (from slides):

- single coin toss
- spam/not spam
- classification outcomes

10 Binomial Random Variable

Step 1: Experiment

- n independent trials
- each success probability = p

Step 2: Define RV

$$X = \text{number of successes in n trials}$$

Notation:

$$X \sim B(n, p)$$

Step 3: PMF

$$P(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, 1, \ldots, n$$

Reason:

- $\binom{n}{i}$ ways to choose i successes
- p^i probability of successes

- $(1-p)^{n-i}$ probability of failures

11 Geometric Random Variable

Step 1: Experiment

Repeated independent trials until first success Each trial success probability = p

Step 2: Define RV

$$X = \text{number of trials until first success}$$

Step 3: PMF

To get success on trial n:

- first n1 must be failures $\to (1-p)^{n-1}$
- nth must be success $\to p$

So:

$$P(X = n) = (1-p)^{n-1}p$$

Example (urn)

With replacement:

$$p = \frac{M}{M+N}$$

$$P(X = n) = \left(\frac{N}{M+N}\right)^{n-1} \frac{M}{M+N}$$

12 Poisson Random Variable

Step 1: Values

$$X = 0, 1, 2, \ldots$$

Parameter:

$$\lambda > 0$$

Step 2: PMF

$$P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$$

Step 3: Validity

$$\sum_{i=0}^{\infty} P(X = i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

Proper PMF

Note from slides:

Poisson approximates Binomial when:

- n large
- p small
- np moderate

Final Logical Flow for Revision

1. Random variable = mapping outcomes → numbers
2. Discrete RV → finite/countable values
3. PMF assigns probability to each value
4. Probabilities must sum to 1
5. Examples reinforce PMF construction
6. Bayes' theorem updates probabilities using evidence
7. Independence formalized via conditional probabilities
8. Bernoulli → single trial
9. Binomial → fixed number of trials
10. Geometric → trials until first success
11. Poisson → counts of rare events