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Discrete Random Variables, Expectation and Problem Solving

Random Variables: Motivation and Concept

A random variable is a numerical description of the outcome of a random experiment. Formally, a random variable X defined on a sample space Ω is a function

$$X : \Omega \to \mathbb{R}$$

which assigns a real number $X(\omega)$ to each outcome $\omega \in \Omega$.

In this lecture, we restrict attention to *discrete random variables*. Although the function maps into $\mathbb{R}$, the actual set of values

$$\{X(\omega) : \omega \in \Omega\}$$

is either finite or countably infinite, and hence forms a discrete subset of $\mathbb{R}$.

Distribution of a Discrete Random Variable

The distribution of a discrete random variable can be visualized using a bar diagram. The values that the random variable can take are represented on the horizontal axis. For a value a, the height of the bar represents the probability

$$\Pr[X = a].$$

Each probability is computed by identifying the corresponding event in the sample space.

Discrete and Continuous Random Variables

Discrete Random Variables

Discrete random variables have the following characteristics:

- Countable support
- A probability mass function (PMF)
- Probabilities assigned to individual values
- Each possible value has strictly positive probability

Continuous Random Variables

Continuous random variables have:

- Uncountable support
- A probability density function (PDF)
- Probabilities assigned to intervals
- Zero probability at any single point

Example: Tossing Three Fair Coins

Consider an experiment in which three fair coins are tossed. Let Y denote the number of heads obtained. Then Y can take the values $0, 1, 2, 3$.

$$P(Y = 0) = P(t, t, t) = \frac{1}{8}$$

$$P(Y = 1) = P(t, t, h), (t, h, t), (h, t, t) = \frac{3}{8}$$

$$P(Y = 2) = P(t, h, h), (h, t, h), (h, h, t) = \frac{3}{8}$$

$$P(Y = 3) = P(h, h, h) = \frac{1}{8}$$

Since Y must take one of these values, we have

$$1 = \Pr\left[\bigcup_{i=0}^{3}\{Y = i\}\right] = \sum_{i=0}^{3} P(Y = i).$$

Probability Mass Function

A random variable that can take at most a countable number of values is called a discrete random variable. Let X be a discrete random variable with range

$$R_X = \{x_1, x_2, x_3, \ldots\},$$

where the set is finite or countably infinite.

The function

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots$$

is called the probability mass function (PMF) of X.

Since X must take one of the values x_k, it follows that

$$\sum_{k=1}^{\infty} P_X(x_k) = 1.$$

PMF Example

The probability mass function of a random variable X is given by

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where $\lambda > 0$.

Since

$$\sum_{i=0}^{\infty} p(i) = 1,$$

we have

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1.$$

Using

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!},$$

it follows that

$$ce^{\lambda} = 1 \quad \Rightarrow \quad c = e^{-\lambda}.$$

Thus,

$$P(X = 0) = e^{-\lambda}.$$

Further,

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}\right]. \end{aligned}$$

Bayes' Theorem

Using the identity

$$P(A \cap B) = P(B \mid A)P(A),$$

we obtain

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)P(B_j)}.$$

This formula is known as Bayes' Theorem.

Here, $P(B_i)$ is the *a priori* probability and $P(B_i \mid A)$ is the *posterior* probability.

Bayes' Theorem Example: Auditorium

An auditorium has 30 rows of seats. Row 1 has 11 seats, Row 2 has 12 seats, and so on, up to Row 30 which has 40 seats.

A row is selected uniformly at random, and then a seat is selected uniformly from that row.

The probability that seat 15 is selected given that row 20 is selected is

$$P(S_{15} \mid R_{20}) = \frac{1}{30}.$$

The probability that row 20 was selected given that seat 15 was selected is

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{P(S_{15})},$$

where

$$P(R_{20}) = \frac{1}{30}$$

and

$$P(S_{15}) = \sum_{k=15}^{30} P(S_{15} \mid R_k)P(R_k).$$