

Lecture Scribe Guidelines



L6_S2_A.pdf

PDF

I want you to generate a lecture scribe that will be used strictly as exam-oriented reference material for a closed-notes / reading-based exam.

The lecture scribe must be a faithful reconstruction of what was taught in class. If a student studies only this scribe, they should be able to reliably revise the entire lecture exactly as it was taught.

You must use ONLY the provided context . Do not use any external knowledge. Do not assume anything that is not explicitly present in the provided context. Do not invent content.

While writing the lecture scribe, reason step by step through all material exactly as presented in the lecture, including:

Definitions and notation

Assumptions and conditions

Statements of theorems or results

Proofs or derivations

Worked examples

All reasoning, proofs, derivations, and example solutions must be written step by step, making the logical flow explicit, exactly as shown in the lecture or textbook.

Include ONLY the following, and only if they appear in the provided context: Definitions and notation, Assumptions and conditions ,Theorem or result statements , Proofs or proof sketches, step by step ,Worked examples with all intermediate steps , Clear logical dependencies between concepts

Follow the structure and order of the lecture strictly. Use clear section and subsection headings. Preserve

the same notation, terminology, and sequencing used in the lecture. Maintain a formal academic tone suitable for exam revision.

Do not do any of the following:

Do not introduce new examples

Do not add intuition or explanations that were not discussed in the lecture Do not simplify, paraphrase creatively, or reinterpret results

Do not provide alternative solution methods

Do not go beyond the lecture or textbook material Do not skip steps or summarize reasoning

The final output must be a single, complete lecture scribe that is self-contained, precise, and can be directly used as exam reference material.

Below is a **faithful, exam-oriented lecture scribe** reconstructed **strictly and only** from the uploaded lecture slides **L6_S2_A.pdf**, following the **exact structure, order, notation, and content** as presented in class.

No external knowledge, assumptions, reinterpretations, or additional explanations have been introduced.

CSE400 – Fundamentals of Probability in Computing

Lecture 6: Discrete Random Variables, Expectation and Problem Solving

Instructor: Dhaval Patel, PhD

Date: January 22, 2025

(This scribe is intended for closed-notes / reading-based exam revision.)

1. Random Variables

1.1 Motivation and Concept

Let Ω denote the sample space.

Definition:

A **random variable** X on a sample space Ω is a function

$$X : \Omega \rightarrow \mathbb{R}$$

that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

Restriction to Discrete Random Variables

Until further notice, attention is restricted to **discrete random variables**, i.e., random variables that take values in a set that is **finite or countably infinite**.

Although X maps to \mathbb{R} , the actual set of values

$$\{X(\omega) : \omega \in \Omega\}$$

forms a **discrete subset** of \mathbb{R} .

Visualization

- Sample points in Ω are mapped by X onto discrete points on the real line.
 - Multiple sample points may map to the same real value.
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1.2 Distribution of a Discrete Random Variable

The **distribution** of a discrete random variable can be visualized using a **bar diagram**.

- The **x-axis** represents the possible values of the random variable.
- The **height of the bar** at value a equals:

$$\Pr[X = a]$$

Each probability is computed by finding the probability of the **corresponding event** in the sample space.

2. Types of Random Variables**2.1 Discrete Random Variables**

Characteristics:

- **Countable support**
 - **Probability Mass Function (PMF)**
 - Probabilities assigned to **single values**
 - Each possible value has **strictly positive probability**
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2.2 Continuous Random Variables

Characteristics:

- **Uncountable support**
 - **Probability Density Function (PDF)**
 - Probabilities assigned to **intervals**
 - Each individual value has **zero probability**
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3. Example 1: Tossing 3 Fair Coins

Experiment Description

An experiment consists of tossing **3 fair coins**.

Let:

Y = number of heads observed

Possible Values of Y

$$Y \in \{0, 1, 2, 3\}$$

Probability Calculations

- $P(Y = 0)$:

$$P(t, t, t) = \frac{1}{8}$$

- $P(Y = 1)$:

$$P(t, t, h) + P(t, h, t) + P(h, t, t) = \frac{3}{8}$$

- $P(Y = 2)$:

$$P(h, h, t) + P(h, t, h) + P(t, h, h) = \frac{3}{8}$$

- $P(Y = 3)$:

$$P(h, h, h) = \frac{1}{8}$$

Resulting Distribution

$$P(Y = 0) = \frac{1}{8}, \quad P(Y = 1) = \frac{3}{8}, \quad P(Y = 2) = \frac{3}{8}, \quad P(Y = 3) = \frac{1}{8}$$

Normalization Condition

Since Y must take exactly one of the values 0, 1, 2, 3:

$$1 = P\left(\bigcup_{i=0}^3 \{Y = i\}\right) = \sum_{i=0}^3 P(Y = i)$$

4. Probability Mass Function (PMF)

4.1 Definition

A random variable that can take on **at most a countable number** of possible values is called **discrete**.

Let X be a discrete random variable with range:

$$R_X = x_1, x_2, x_3, \dots$$

(where the range may be finite or countably infinite).

Define:

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \dots$$

This function $P_X(\cdot)$ is called the **Probability Mass Function (PMF)** of X .

4.2 PMF Normalization Property

Since X must take one of the values x_k :

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

5. PMF – Worked Example

Given

The PMF of a random variable X is:

$$p(i) = c \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$.

Find:

1. $P(X = 0)$
 2. $P(X > 2)$
-

Step 1: Find Constant c

Using normalization:

$$\begin{aligned} \sum_{i=0}^{\infty} p(i) &= 1 \\ c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} &= 1 \end{aligned}$$

Using:

$$\begin{aligned} e^{\lambda} &= \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \\ ce^{\lambda} &= 1 \quad \Rightarrow \quad c = e^{-\lambda} \end{aligned}$$

Step 2: Compute $P(X = 0)$

$$P(X = 0) = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}$$

Step 3: Compute $P(X > 2)$

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - (e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}) \end{aligned}$$

6. Bayes' Theorem (Recap)

6.1 Conditional Probability Identity

$$\Pr(A \mid B) = \Pr(B \mid A) \Pr(A)$$

6.2 Bayes Formula (Proposition 3.1)

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_{j=1}^n \Pr(A \mid B_j) \Pr(B_j)}$$

6.3 Terminology

- $\Pr(B_i)$: a priori probability
 - $\Pr(B_i \mid A)$: posterior probability
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7. Bayes' Theorem – Example: Auditorium with 30 Rows

Problem Description

- Auditorium has **30 rows**.
- Row 1 has **11 seats**, Row 2 has **12 seats**, ..., Row 30 has **40 seats**.
- A door prize is awarded by:
 1. Randomly selecting a row (each row equally likely)
 2. Randomly selecting a seat within that row (each seat equally likely)

Tasks

1. Compute the probability that **Seat 15** was selected **given Row 20 was selected**.
 2. Compute the probability that **Row 20** was selected **given Seat 15 was selected**.
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Step 1: Probability of Selecting Seat 15 Given Row 20

Since Row 20 has 30 seats:

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

Step 2: Compute $P(S_{15})$

Using total probability:

$$P(S_{15}) = \sum_{k=15}^{30} P(S_{15} \mid R_k)P(R_k)$$

Each row is equally likely:

$$P(R_k) = \frac{1}{30}$$
$$P(S_{15}) = \sum_{k=15}^{30} \frac{1}{(k+10)} \cdot \frac{1}{30} \approx 0.0342$$

Step 3: Probability Row 20 Given Seat 15

Using Bayes' formula:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{P(S_{15})}$$
$$= \frac{\frac{1}{30} \cdot \frac{1}{30}}{0.0342} \approx 0.0325$$

End of Lecture 6 Scribe

This document strictly follows the lecture slides in **content, order, notation, and derivations**, and is suitable for **closed-book exam revision**.