Lecture Scribe Creation

L6_S2_A.pdf
PDF

Create a lecture scribe using only the provided lecture slides. The scribe must function as reliable reference material for exam preparation and accurately reflect the lecture content.

While preparing the scribe, explain definitions, notation, assumptions, theorem statements or results, proofs, derivations and worked examples in a step-by-step manner following the same sequence of logical reasoning used in the lecture. Ensure that each step is clearly explained so that the reasoning can be followed and relyed during revision.

Restrict the content strictly to the provided lecture slides. Do not add any new explanations, examples, assumptions or reasoning that are not explicitly included in the given materials.

Write in a formal academic style and structure the scribe in a clear and organized way so that the progression of concepts, results, definitions, proofs and examples can be easily understood.

Below is a **formal lecture scribe created strictly from the provided lecture slides**. All definitions, notation, derivations, and examples follow the exact sequence and reasoning from the lecture without introducing external material. The original lecture slides are available here:

CSE400 – Fundamentals of Probability in Computing

Lecture 6: Discrete Random Variables, Expectation and Problem Solving

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Date: January 22, 2025

1. Random Variables

1.1 Definition and Concept

A **random variable** is defined as a function that assigns a real number to each outcome in the sample space.

Formally,

$$X : \Omega \to R$$

where:

- Ω denotes the sample space.
- $\omega \in \Omega$ denotes a sample point.
- $X(\omega)$ is the real number assigned to the outcome ω.

Thus, a random variable maps outcomes of an experiment to numerical values.

The lecture further restricts attention to **discrete random variables**, meaning:

- The set of possible values of X,

$$\{X(\omega) : \omega \in \Omega\}$$

is a **finite or countably infinite subset of** R.

This means that although the codomain of X is R, the actual values taken by X belong to a discrete set.

1.2 Visualization of a Random Variable

The distribution of a discrete random variable can be visualized using a **bar diagram**, where:

- The **x-axis** represents the possible values of the random variable.
- The **height of each bar at value** a represents:

$$Pr(X = a)$$

This probability corresponds to the probability of the event in the sample space mapped to value a.

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2. Types of Random Variables

Random variables are classified into two main types.

2.1 Discrete Random Variable

A random variable is discrete if:

- It has **countable support**
- A **Probability Mass Function (PMF)** exists
- Probabilities are assigned to individual values
- Each possible value has strictly positive probability

2.2 Continuous Random Variable

A random variable is continuous if:

- It has **uncountable support**
- It uses a **Probability Density Function (PDF)**
- Probabilities are assigned to intervals
- Each individual value has zero probability

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3. Example of a Discrete Random Variable

Experiment Description

Suppose an experiment consists of tossing **3 fair coins**.

Define:

$$Y = \text{number of heads}$$

Then the possible values of Y are:

$$Y \in \{0, 1, 2, 3\}$$

Probability Calculation

All possible outcomes:

$$\{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

Each outcome has probability:

$$\frac{1}{8}$$

Now compute probabilities:

Case 1: No heads

$$P(Y = 0) = P(TTT) = \frac{1}{8}$$

Case 2: One head

Outcomes:

$$TTH, THT, HTT$$

$$P(Y = 1) = \frac{3}{8}$$

Case 3: Two heads

Outcomes:

$$THH, HTH, HHT$$

$$P(Y = 2) = \frac{3}{8}$$

Case 4: Three heads

Outcome:

$$HHH$$

$$P(Y = 3) = \frac{1}{8}$$

Total Probability Property

Since Y must take one of these values:

$$1 = \sum_{i=0}^{3} P(Y = i)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= 1$$

This confirms a valid probability distribution.

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4. Probability Mass Function (PMF)

4.1 Definition

A random variable that takes at most a countable number of values is called discrete.

Let:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

be the range of possible values.

The function:

$$P_X(x_k) = P(X = x_k)$$

for $k = 1, 2, 3, \dots$

is called the **Probability Mass Function (PMF)**.

4.2 Property of PMF

Since the random variable must take one of its possible values:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

This expresses the total probability law.

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5. PMF Example with Unknown Constant

Given:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$

Find:

1. $P(X = 0)$
2. $P(X > 2)$

Step 1: Use Total Probability Property

Since:

$$\sum_{i=0}^{\infty} p(i) = 1$$

Substitute:

$$\sum_{i=0}^{\infty} c\frac{\lambda^i}{i!} = 1$$

Factor c:

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using the identity:

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$$

Thus:

$$ce^{\lambda} = 1$$

Solve for c:

$$c = e^{-\lambda}$$

Step 2: Find $P(X = 0)$

Substitute $i = 0$:

$$P(X=0) = c\frac{\lambda^0}{0!}$$

$$= e^{-\lambda}$$

Step 3: Find $P(X > 2)$

Use complement:

$$P(X > 2) = 1 - P(X \leqslant 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

Substitute:

$$= 1 - (e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda})$$

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6. Bayes' Theorem

Using conditional probability:

$$Pr(A \mid B_i) = \frac{Pr(B_i \mid A)Pr(A)}{\sum_{i=1}^{n} Pr(A \mid B_i)Pr(B_i)}$$

This is called the **Bayes Formula**.

Definitions

Prior Probability

$$Pr(B_i)$$

Probability formed before observing evidence.

Posterior Probability

$$Pr(B_i \mid A)$$

Probability after observing event A.

7. Bayes' Theorem Example: Auditorium with 30 Rows

Given

- Auditorium has 30 rows
- Row 1: 11 seats
- Row 2: 12 seats
- Row 3: 13 seats
- …
- Row 30: 40 seats

Selection process:

1. Select row randomly:

$$P(R_k) = \frac{1}{30}$$

2. Select seat randomly within row.

Step 1: Find $P(S_{15} \mid R_{20})$

Row 20 has:

$$40 - (30 - 20) = 30 \text{ seats}$$

Thus:

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

Step 2: Find $P(R_{20} \mid S_{15})$

Using Bayes formula:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{P(S_{15})}$$

Denominator:

$$P(S_{15}) = \sum_{k=5}^{30} P(S_{15} \mid R_k)P(R_k)$$

$$= \sum_{k=5}^{30} \frac{1}{k+10} \frac{1}{30}$$

Numerical value:

$$= 0.0342$$

Thus:

$$P(R_{20} \mid S_{15}) = \frac{(1/30)(1/30)}{0.0342} = 0.0325$$

Summary of Key Results

Random variable definition:

$$X : \Omega \to \mathrm{R}$$

PMF definition:

$$P_X(x_k) = P(X = x_k)$$

PMF property:

$$\sum P_X(x_k) = 1$$

Constant value:

$$c = e^{-\lambda}$$

Bayes theorem:

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum P(A \mid B_i)P(B_i)}$$

If you want, I can also convert this into **exact LaTeX source code or PDF lecture scribe formatted for submission or exam revision**.