Introduction to the Theory of Computation: Hints

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1 Chapter 1: Regular Languages

*1.45 Let $A/B = \{w \mid wx \in A \text{ for some } x \in B\}$. Show that if A is regular and B is any language, then A/B is regular.

Hint. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = A$. Construct

$$F' = \{q \in Q \mid \exists x \in B,\ \delta(q, x) \in F\}.$$

*1.56 If A is a set of natural numbers and k is a natural number greater than 1, let

$$B_k(A) = \{w \mid w \text{ is the representation in base } k \text{ of some number in } A\}.$$

Here, we do not allow leading 0s in the representation of a number. For example, $B_2(\{3,5\}) = \{\mathtt{11}, \mathtt{101}\}$ and $B_3(\{3,5\}) = \{\mathtt{10}, \mathtt{12}\}$. Give an example of a set A for which $B_2(A)$ is regular but $B_3(A)$ is not regular. Prove that your example works.

Hint. Let $A = \{2^n \mid n \in \mathbb{N}\}$. Then use pumping lemma on $B_3(A)$, after which do a little bit of algebra or number theory.

*1.57 If A is any language, let $A_{\frac{1}{2}-}$ be the set of all first halves of strings in A so that

$$A_{\frac{1}{2}-} = \{x \mid \text{for some } y,\ |x| = |y| \text{ and } xy \in A\}.$$

Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

Hint. Let M be a DFA such that $L(M) = A$. Utilizing M, design an NFA which accepts $A_{\frac{1}{2}-}$ by guessing the k-th character from the end of y while reading the k-th character of x.

*1.58 If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ be the set of all strings in A with their middle thirds removed so that

$$A_{\frac{1}{3}-\frac{1}{3}} = \{xz \mid \text{for some } y,\ |x| = |y| = |z| \text{ and } xyz \in A\}.$$

Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular

Hint. Let $A = \mathtt{a}^*\mathtt{\#}\mathtt{b}^*$. Prove $A_{\frac{1}{3}-\frac{1}{3}} \cap \mathtt{a}^*\mathtt{b}^*$ is not regular by pumping lemma.

*1.59 Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let h be a state of M called its "home". A ***synchronizing sequence*** for M and h is a string $s \in \Sigma^*$ where $\delta(q, s) = h$ for every $q \in Q$. (Here we have extended δ to strings, so that $\delta(q, s)$ equals the state where M ends up when M starts at state q and reads input s.) Say that M is ***synchronizable*** if it has a synchronizing sequence for some state h. Prove that if M is a k-state synchronizable DFA, then it has a synchronizing sequence of length at most k^3. Can you improve upon this bound?

Hint. Let Q' stands for some subset of Q. Try to design a method that finds a relatively short string w such that $|\delta(Q', w)| < |Q'|$, where $\delta(Q', w) = \{\delta(q, w) \mid q \in Q'\}$.

*1.63
a. Let A be an infinite regular language. Prove that A can be split into two infinite disjoint regular subsets.
b. Let B and D be two languages. Write $B \Subset D$ if $B \subseteq D$ and D contains infinitely many strings that are not in B. Show that if B and D are two regular languages where $B \Subset D$, then we can find a regular language C where $B \Subset C \Subset D$.

Hint.

a. A's being infinite implies that there is an infinite regular language $xy^*z \subseteq A$ given by pumping lemma. Then how about $x(yy)^*z$?
b. Plug $A = D - B$ into a. to get $A = A_1 \cup A_2$, and then let $C = B \cup A_1$.

*1.65 Prove that for each $n > 0$, a language B_n exists where

a. B_n is recognizable by an NFA that has n states, and

b. if $B_n = A_1 \cup \cdots \cup A_k$ for regular languages A_i, then at least one of the A_i requires a DFA with exponentially many states.

Hint. Consider $B_{n+2} = \Sigma^* \mathtt{1} \mathtt{0}^n$. Then use Myhill–Nerode theorem. However the theorem is not necessary for a proof, though applying it can simplify the work.

*1.67 Let the ***rotational closure*** of language A be $RC(A) = \{yx \mid xy \in A\}$.

a. Show that for any language A, we have $RC(A) = RC(RC(A))$.

b. Show that the class of regular languages is closed under rotational closure.

Hint. For b., let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = A$. Utilizing M, design an NFA which accepts $RC(A)$ by guessing $\delta(q_0, x)$ and where y ends in yx.

*1.68 In the traditional method for cutting a deck of playing cards, the deck is arbitrarily split two parts, which are exchanged before reassembling the deck. In a more complex cut, called Scarnes cut, the deck is broken into three parts and the middle part in placed first in the reassembly. Well take Scarnes cut as the inspiration for an operation on languages. For a language A, let $CUT(A) = \{yxz \mid xyz \in A\}$.

a. Exhibit a language B for which $CUT(B) \neq CUT(CUT(B))$.

b. Show that the class of regular languages is closed under CUT.

Hint.

a. $B = \mathtt{123}$.

b. Just do like what we do in problem 1.67.