Mean-reversion trading strategy using Co-Integration

Mean reversion is a statistical arbitrage trading strategy where we expect the returns of an asset follows a stationary process and reverts to its long-term mean μ when returns have diverged significantly away from the mean.

Stationarity

Before exploring further, let us first understand what is a stationary process. Suppose we observe a time series (could be stock prices, daily weather, number of people visiting a shopping mall at each hour, etc), the fluctuations of these values appear random, but often there exist the same type of stochastic behavior from one time period to the next (e.g. every Sunday, 12pm, we observed that it is highly likely that the number of visitors are higher than other times). We can use this property, although seemingly random but predictable over the same time window, to predict or estimate future values of the same time window.

Mathematically, a process (which is a series of random variables) is stationary if the its stochastic property is not affected by a change of time origin, that is, if for any time s and t, the probability distribution of a sequence $Y_1, \ldots, Y_t$ and $Y_{1+s}, \ldots, Y_{t+s}$ are the same, then the process is said to be stationary.

We consider a weakly stationary time series of order 2, that is its mean, variance and covariance structure does not change with time shifts:

- Constant mean $E[Y_t] = \mu$
- Constant variance $var[Y_t] = E[(Y_t - \mu)^2] = \sigma^2$
- Constant autocovariance structure $Corr[Y_s, Y_t] = \rho_{|s-t|}, \ \forall s, t$

The mean and variance do not change with time and the covariance structure (and correlation) between two observations of the time series depend only on the time-lag, of the time distance between the two observations.

Example of non-stationary process:

Example of a stationary process:
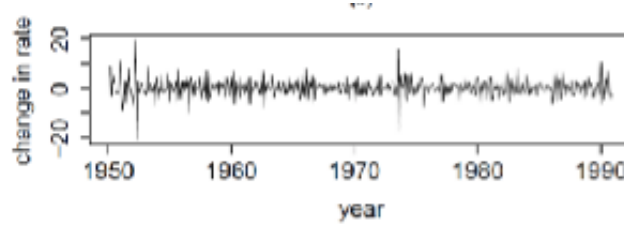
Note: These are by no means a formal and rigorous definition of stationarity. Refer to textbooks for more detailed mathematical definition.

Determine non-stationarity (or unit root non-stationarity)

<u>1-D Time Series:</u>

For an AR(1) model, $y_t = \mu + \phi y_{t-1} + \epsilon_t$, where ϵ_t is a white noise process, the time series is stationary iff $-1 < \phi < 1$.

If $\phi = 1$ then the time series has unit root non-stationarity.

Generally for an AR(p) model, $y_t = \mu + a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \epsilon_t$, the time series is stationary if **all** the roots of its characteristics polynomial lie outside the unit circle. Using B as the backshift operator, i.e. $Ly_t = y_{t-1}$, the AR(p) model can be written in terms of its AR characteristics polynomial $a(B)$:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \epsilon_t \Rightarrow a(B)y_t = \mu + \epsilon_t \tag{2}$$

where $a(B) = 1 - a_1 B - a_2 B^2 - \cdots - a_p B^p$.

The roots of the AR characteristic polynomial is obtained by solving the reverse characteristic equation $a(\lambda) = 0$.

<u>n-D Time Series:</u>

For VAR(1) model,

$$r_t = \Phi_0 + \Phi_1 r_{t-1} + \epsilon_t \tag{3}$$

it is stationary iff $\rho(\Phi_1) < 1$, i.e. the eigenvalues of Φ_1 are less than 1.

For VAR(p) model,

$$r_t = \Phi_0 + \Phi_1 r_{t-1} + \Phi_2 r_{t-2} + \cdots + \Phi_p r_{t-p} \epsilon_t \tag{4}$$

there are 2 ways to determine stationarity:

- Roots of $det(\Phi(x)) = 0$ lie outside the unit circle, where $\Phi(x) = I - \Phi_1 x - \Phi_2 x^2 - \cdots - \Phi_p x^p$
- Convert VAR(p) to VAR(1) model:
  $$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} \epsilon_t \Rightarrow x_t = \Phi^* x_{t-1} + b_t$$
  where Φ^* is the companion matrix of $\Phi(B)$. VAR(p) model is stationary iff eigenvalues of Φ^* is less than 1, i.e. $\rho(\Phi^*) < 1$.

Mean reversion strategy

Now that we understand what a stationary process is, we can then search for any stock returns that are stationary and set up a mean-reversion trading strategy as follows:

1. Identify an asset whose returns are stationary

2. Set up a threshold δ (e.g. $\delta = \pm(2 \times \sigma)$) where σ is the standard deviation of the asset returns from its mean

3. When the return of the asset crosses above (below) the threshold $+\delta$ ($-\delta$), then we can long (short) the asset at the prevalent market price.

4. Unwind the position when the asset return crosses below (above) the opposite threshold $-\delta$ ($+\delta$)

We often use asset returns rather than the asset price itself for this mean-reversion strategy as the asset price process are more likely than not non-stationary.

Co-integration

Co-integration is the concept of linearly combining 2 different assets to derive a stationary process for our mean-reversion trading strategy, often known as pair-trading strategy. The idea is similar to the case of a 1 asset strategy.

Mathematically, let x_t, y_t be the price returns of two assets. We want to find a γ s.t. the portfolio $w_t = x_t + \gamma y_t$ is stationary.

A brief explanation on autoregression. Autoregression (AR) is a regression model where we conduct regression analysis on its own time-lagged variable. For example, let x_t denote the return of an asset at time t. Then an AR(1) regression model is $x_t = x_{t-1} + a_t$ where a_t is a white noise with mean 0 and constant variance. Let's extend this further to 2 time series, then we have Vector Autoregression, VAR.

The construction of a portfolio through co-integration is best described with an example. Consider a VAR(1) process:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} a_t \\ b_t \end{bmatrix} \tag{5}$$

Consider the transformation

$$w_t = x_t - 2y_t, v_t = \frac{1}{2}x_t + y_t, c_t = a_t - 2b_t, d_t = \frac{1}{2}a_t + b_t \tag{6}$$

Then we get that

$$\begin{bmatrix} w_t \\ v_t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} x_{t-1} - 2y_{t-1} \\ 0 \end{bmatrix} + \begin{bmatrix} c_t \\ d_t \end{bmatrix} = \begin{bmatrix} w_{t-1} \\ 0 \cdot v_{t-1} \end{bmatrix} + \begin{bmatrix} c_t \\ d_t \end{bmatrix} \tag{7}$$

Notice that $w_t = w_{t-1} + c_t$ is a unit-root nonstationary (coefficient of $w_{t-1} = 1$) but $v_t = d_t$ is a stationary process. In summary, we applied a transformation and obtained a linear combination of the original time series that is stationary: $v_t = \frac{1}{2}x_t + y_t$. The vector $[\frac{1}{2}, 1]$ is called a cointegration vector.

Finding the co-integration vector

Let's consider again a generic VAR(1) model

$$\boldsymbol{x_t} = \boldsymbol{\phi_0} + \boldsymbol{\Phi x_{t-1}} + \boldsymbol{a_t} \tag{8}$$

We want to obtain a linear transformation $\mathbf{y_t} = L\mathbf{x_t}$, applying to our VAR(1) model, we get that

$$\boldsymbol{y_t} = L\boldsymbol{\phi_0} + L\boldsymbol{\Phi}\boldsymbol{x_{t-1}} + L\mathbf{a_t} \tag{9}$$
$$= L\boldsymbol{\phi_0} + L\boldsymbol{\Phi}L^{-1}\boldsymbol{y_{t-1}} + L\mathbf{a_t} \tag{10}$$

Suppose $L\boldsymbol{\Phi}L^{-1} = \begin{bmatrix} \boldsymbol{\Phi_1} & 0 \\ 0 & \boldsymbol{\Phi_2} \end{bmatrix}$ is a block diagonal matrix, where $\rho(\boldsymbol{\Phi_1}) < 1$ and $\rho(\boldsymbol{\Phi_2}) = 1$ and $\rho(\cdot)$ denotes the eigenvalues of any matrix.

We use Jordan diagonalization to obtain the block diagonal matrix $L\boldsymbol{\Phi}L^{-1} = \begin{bmatrix} \boldsymbol{\Phi_1} & 0 \\ 0 & \boldsymbol{\Phi_2} \end{bmatrix} = J$.

In Jordan normal form, $\boldsymbol{\Phi} = L^{-1}JL$ where the rows of L correspond to left eigenvectors of $\boldsymbol{\Phi}$. The steps to find the matrix L is thus:

1. Find eigenvalues λ_i of $\boldsymbol{\Phi}$ s.t. $det(\lambda_i I - \boldsymbol{\Phi})) = 0$

2. Find eigenvectors v_i^T s.t. $v_i^T \boldsymbol{\Phi} = \lambda_i v_i^T$

3. Arrange the k distinct eigenvalues from smallest to largest into a diagonal matrix, $J = diag\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ where $\lambda_1 < \lambda_2 < \ldots \lambda_k$

4. Arrange the left eigenvector matrix $L = [v_1, \ldots, v_k]^T$ where the rows of L correspond to left eigenvectors of $\boldsymbol{\Phi}$, in other words, the first row of L correspond to eigenvector v_1^T, second row of L correspond to eigenvector v_2^T and so on until the last eigenvector v_k^T corresponding to the largest eigenvalue λ_k.

Now that we have the matrix L s.t. $L\boldsymbol{\Phi}L^{-1} = \begin{bmatrix} \boldsymbol{\Phi_1} & 0 \\ 0 & \boldsymbol{\Phi_2} \end{bmatrix} = J$, substituting into our transformed VAR(1) model gives us

$$\boldsymbol{y_t} = L\boldsymbol{\phi_0} + L\boldsymbol{\Phi}L^{-1}\boldsymbol{y_{t-1}} + L\mathbf{a_t} \tag{14}$$
$$= L\boldsymbol{\phi_0} + J\boldsymbol{y_{t-1}} + L\mathbf{a_t} \tag{15}$$

Expanding each rows individually, we get that (for example, using first row of L):

$$y_{1,t} = v_1^T \boldsymbol{\phi_0} + \lambda_1 y_{1,t-1} + v_1^T \boldsymbol{a}_{,t} \tag{16}$$

which is the same as multiplying the original VAR(1) model with the eigenvector v_1^T:

$$y_{1,t} = v_1^T \boldsymbol{x_t} = v_1^T \boldsymbol{\phi_0} + v_1^T \boldsymbol{\Phi}\boldsymbol{x_{t-1}} + v_1^T \boldsymbol{a_t} \tag{17}$$
$$= v_1^T \boldsymbol{\phi_0} + \lambda_1 v_1^T \boldsymbol{x_{t-1}} + v_1^T \boldsymbol{a_t} \tag{18}$$
$$= v_1^T \boldsymbol{\phi_0} + \lambda_1 y_{1,t-1} + v_1^T \boldsymbol{a_t} \tag{19}$$

$y_{1,t}$ is an AR(1) process with $\lambda_1 < 1$, so it is a stationary process.

The corresponding eigenvector v_1^T is our co-integration vector

Construction of trading portfolio using co-integration

We build a trading portfolio using eigenvectors corresponding to eigenvalues less than 1:

$$\Pi_t = v_1^T \boldsymbol{x_t} = v_1^{(1)} x_{1,t} + v_1^{(2)} x_{2,t} + \cdots + v_1^{(n)} x_{n,t} \tag{20}$$

for $\boldsymbol{x_t} \in \mathbb{R}^{n\times 1}$.

We find the mean and standard deviation of the portfolio Π_t and set up a trading strategy as mentioend above.

Caveat

Notice that in our discussion above, it is assumed that the AR coefficients, $\phi_1, \phi_2, \ldots$ and constants ϕ_0, are given.

The estimation of these coefficients and constants require using techniques such as Ordinary Least Squares method or Maximum Likelihood Estimation method. These are left as a separate topic of discussion.