ENGG 2430 / ESTR 2004: Probability and Statistics

Spring 2019

3. Independence and Random Variables

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Independence of two events

Let E_1 be "first coin comes up $\mathtt{H}$"

E_2 be "second coin comes up $\mathtt{H}$"

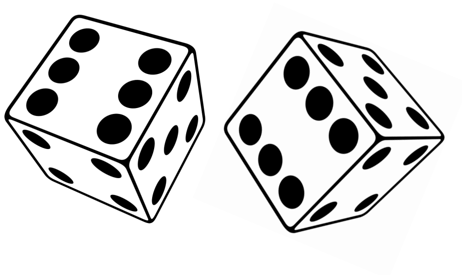
Then $\mathbf{P}(E_2 \mid E_1) = \mathbf{P}(E_2)$

$\mathbf{P}(E_2 \cap E_1) = \mathbf{P}(E_2)\mathbf{P}(E_1)$

Events A and B are independent if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$$

Examples of (in)dependence



Let E_1 be "first die is a 4"

S_6 be "sum of dice is a 6"

S_7 be "sum of dice is a 7"

E_1 and S_6?

$$P(E_1 \cap S_6) \stackrel{?}{=} P(E_1) \cdot P(S_6)$$

$$1/36 \neq 1/6 \quad 5/36$$

NOT IND.

E_1 and S_7?

$$P(E_1 \cap S_7) = P(E_1) \cdot P(S_7)$$

$$1/36 \quad 1/6 \quad 1/6$$

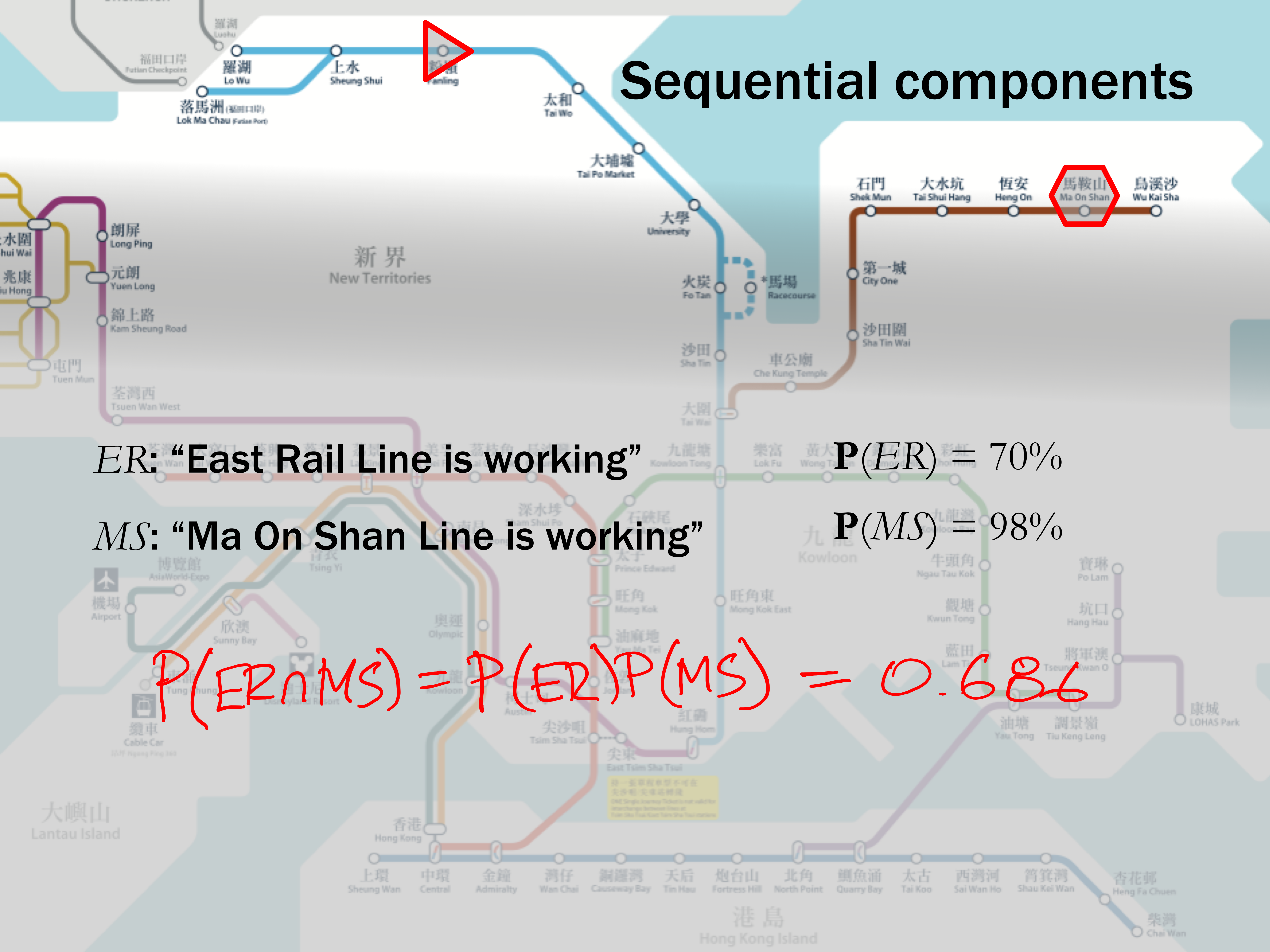
IND.

S_6 and S_7?

$$P(S_6 \cap S_7) \stackrel{?}{=} P(S_6) \cdot P(S_7)$$

$$0 \neq >0 \quad >0$$

Sequential components



ER: "East Rail Line is working" $\quad \mathbf{P}(ER) = 70\%$

MS: "Ma On Shan Line is working" $\quad \mathbf{P}(MS) = 98\%$

$P(ER \cap MS) = P(ER)P(MS) = 0.686$

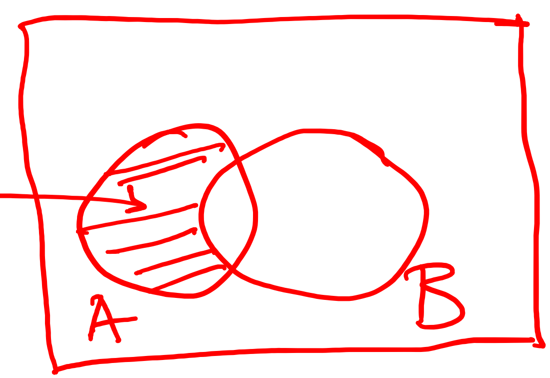
Algebra of independent events

If A and B are independent, then A and B^c are also independent.

Proof:

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$
$$= P(A)(1 - P(B))$$
$$= P(A)P(B^c)$$

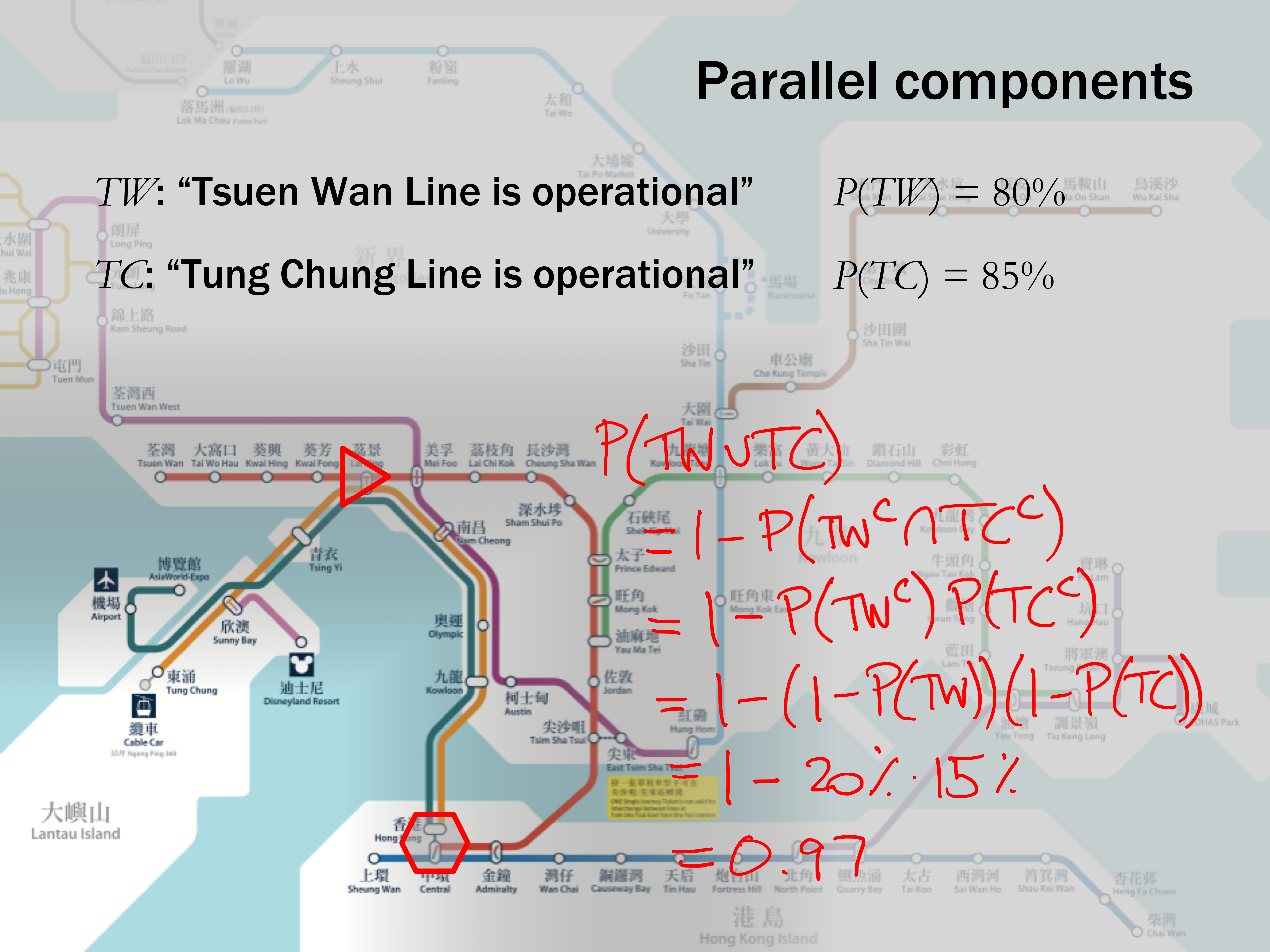
$A \cap B^c$



Parallel components

TW: **"Tsuen Wan Line is operational"** $\qquad P(TW) = 80\%$

TC: **"Tung Chung Line is operational"** $\qquad P(TC) = 85\%$



$$P(TW \cup TC)$$
$$= 1 - P(TW^c \cap TC^c)$$
$$= 1 - P(TW^c)\,P(TC^c)$$
$$= 1 - (1 - P(TW))(1 - P(TC))$$
$$= 1 - 20\% \cdot 15\%$$
$$= 0.97$$

Independence of three events

Events A, B, and C are **independent** if

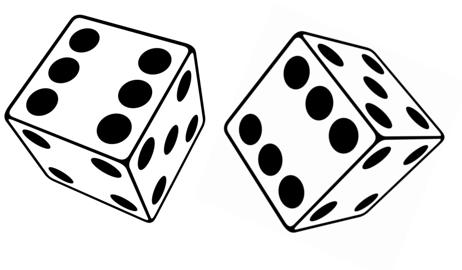
$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$$

$$\mathbf{P}(B \cap C) = \mathbf{P}(B)\,\mathbf{P}(C)$$

$$\mathbf{P}(A \cap C) = \mathbf{P}(B)\,\mathbf{P}(C)$$

and $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$.

(In)dependence of three events



Let E_1 **be "first die is a** 4**"**

E_2 **be "second die is a** 3**"**

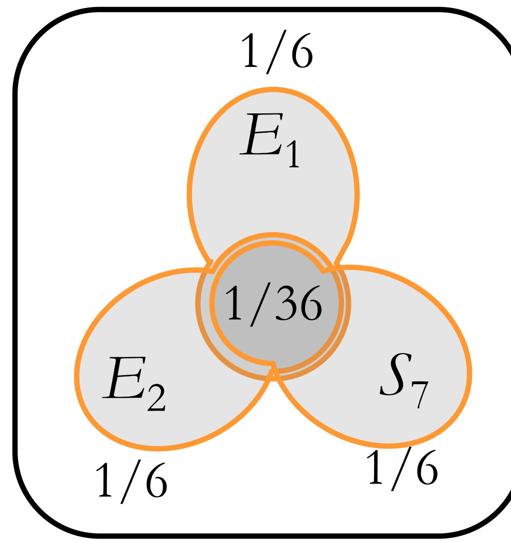
S_7 **be "sum of dice is a** 7**"**

E_1, E_2**?** $\quad \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ ✓

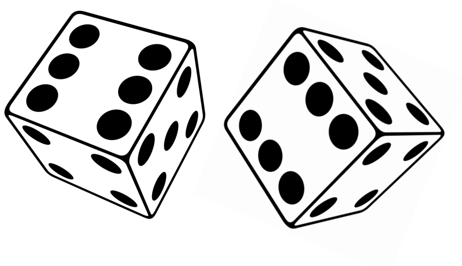
E_1, S_7**?** $\quad \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ ✓

E_2, S_7**?** $\quad \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ ✓

E_1, E_2, S_7**?** $\quad P(E_1 \cap E_2 \cap E_7) = \frac{1}{36}$ X



(In)dependence of three events



Let A **be "first roll is** 1**,** 2**, or** 3 **"** $1/2$

B **be "first roll is** 3**,** 4**, or** 5**"** $1/2$

C **be "sum of rolls is** 9**"** $1/9$

A, B**?** $\frac{1}{2} \times \frac{1}{2} \neq \frac{1}{6}$

A, C**?** $\frac{1}{2} \times \frac{1}{9} \neq \frac{1}{12}$

B, C**?** $\frac{1}{2} \times \frac{1}{9} \neq \frac{1}{12}$

A, B, C**?** $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{9} = \frac{1}{36}$

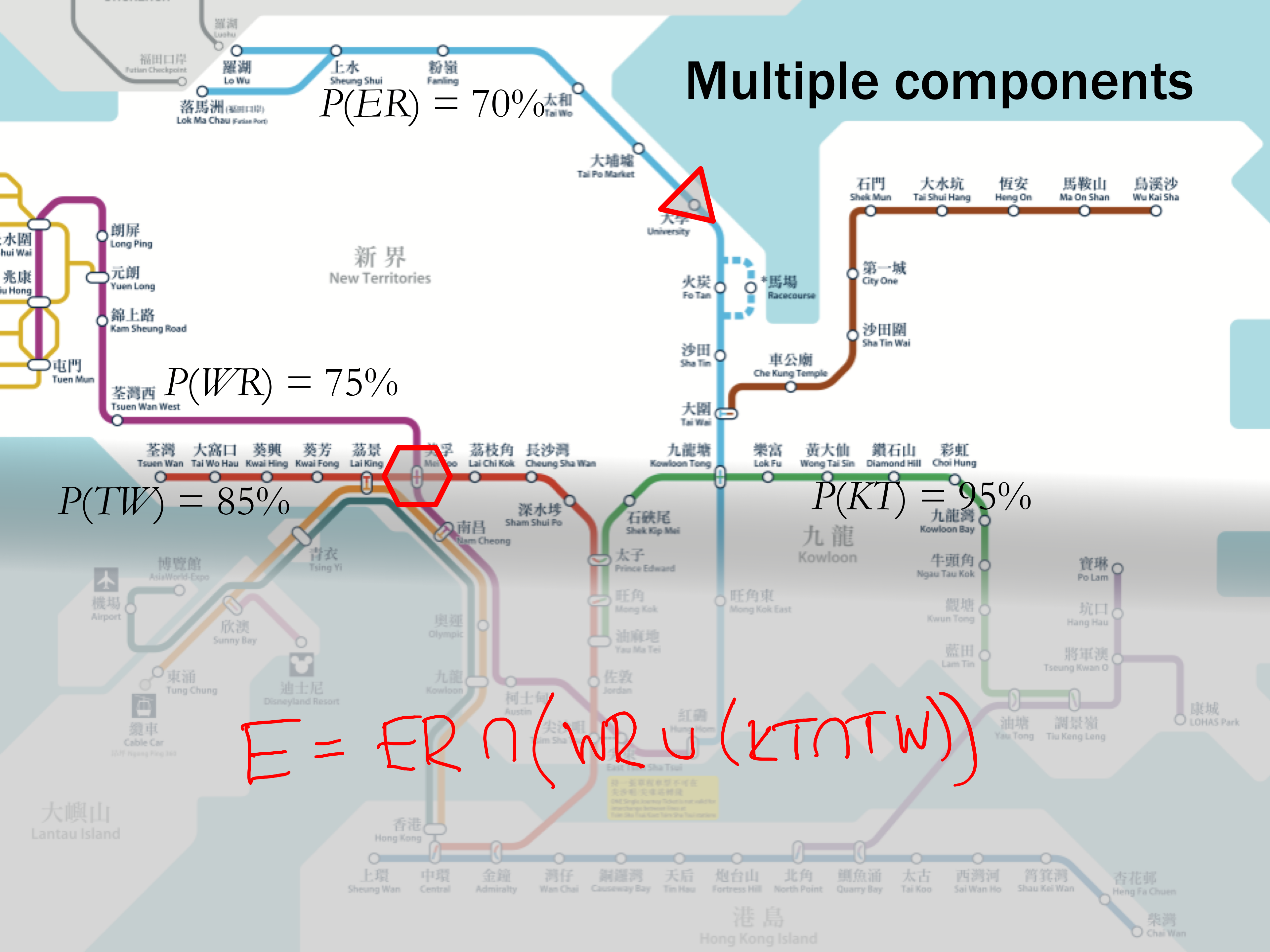
Independence of many events

Events $A_1, A_2, \ldots$ are independent if for every subset of the events, the probability of the intersection is the product of their probabilities.

Algebra of independent events

Independence is preserved if we replace some event(s) by their complements, intersections, unions

Multiple components



$P(ER) = 70\%$

$P(WR) = 75\%$

$P(TW) = 85\%$

$P(KT) = 95\%$

$E = ER \cap (WR \cup (KT \cap TW))$

Multiple components

$P(ER) = 70\%$

$P(WR) = 75\%$

$P(KT) = 95\%$

$P(TW) = 85\%$

$$P(E) = P(ER \cap (WR \cup (KT \cap TW)))$$

$$= \underbrace{P(ER)}_{70\%} \cdot P(WR \cup (KT \cap TW))$$

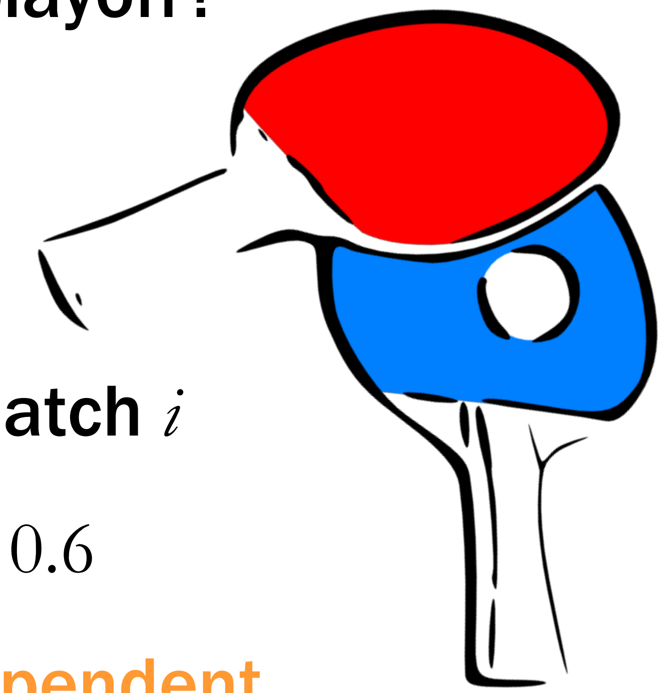
$$P(WR \cup (KT \cap TW)) = 1 - \overbrace{(1 - P(WR))}^{25\%} \cdot \underbrace{(1 - \underbrace{P(KT)P(TW)}_{\approx 80\%})}_{\approx 20\%}$$

$$\approx 95\%$$

$$P(E) \approx 70\% \cdot 95\% \approx 67\%$$

Playoffs

Alice wins 60% of her ping pong matches against Bob. They meet for a 3 match playoff. What are the chances that Alice will win the playoff?



Probability model

Let A_i be the event Alice wins match i

Assume $P(A_1) = P(A_2) = P(A_3) = 0.6$

Also assume A_1, A_2, A_3 are independent

Playoffs

outcome	probability
AAA	0.6^3
AAB	$0.6^2 \cdot 0.4$
ABA	$0.6^2 \cdot 0.4$
BAA	$0.6^2 \cdot 0.4$

$$\mathbf{P}(\mathcal{A}) = 0.6^3 + 3 \cdot 0.6^2 \cdot 0.4 = 0.648$$

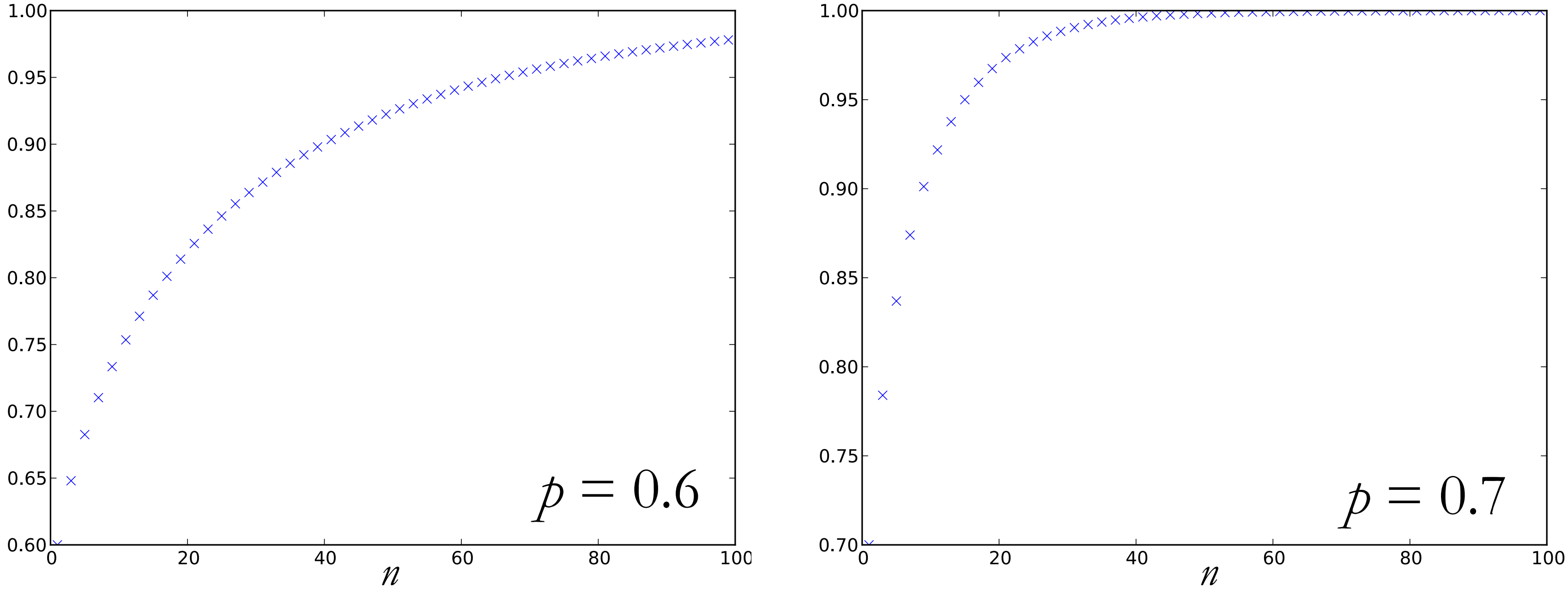
Bernoulli trials

n trials, each succeeds independently with probability p

The probability at least k out of n succeed is

$$\binom{n}{k} p^{n-k}(1-p)^{k} + \binom{n}{k+1} p^{n-k+1}(1-p)^{k-1} + \dots + \binom{n}{n} p^{n}$$

Playoffs



The probability that Alice wins an n game tournament

The Lakers and the Celtics meet for a 7-game playoff. They play until one team wins four games.

Suppose the Lakers win 60% of the time. What is the probability that all 7 games are played?

ALL 7 PLAYED

FIRST 6 HAVE 3 LAKERS WINS +

3 LAKERS LOSSES

$$P(E) = \binom{6}{3} 0.6^3 \cdot 0.4^3$$

Conditional independence

A **and** B **are independent conditioned on** F **if**

$$\mathbf{P}(A \cap B \mid F) = \mathbf{P}(A \mid F)\, \mathbf{P}(B \mid F)$$

Alternative definition:

$$\mathbf{P}(A \mid B \cap F) = \mathbf{P}(A \mid F)$$

today	tomorrow
☀️	80% ☀️, 20% 🌧️
🌧️	40% ☀️, 60% 🌧️

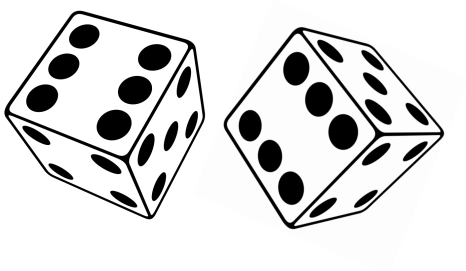
It is ☀️ on Monday. Will it 🌧️ on Wednesday?

$$P(W \mid M \cap T) = P(W \mid T)$$

$$P(T) = \underbrace{P(T \mid M)}_{80\%}\underbrace{P(M)}_{1} + P(T \mid M^c)\underbrace{P(M^c)}_{0} = 0.8$$

$$P(W) = \underbrace{P(W \mid T)}_{80\%}\underbrace{P(T)}_{0.8} + \underbrace{P(W \mid T^c)}_{40\%}\underbrace{P(T^c)}_{0.2} = 0.72$$

Conditioning does not preserve independence



Let E_1 be "first die is a 4"

E_2 be "second die is a 3"

S_7 be "sum of dice is a 7"

E_1, E_2 INDEPENDENT BUT

$$P(E_1 \cap E_2 | S_7) \neq P(E_1 | S_7) P(E_2 | S_7)$$

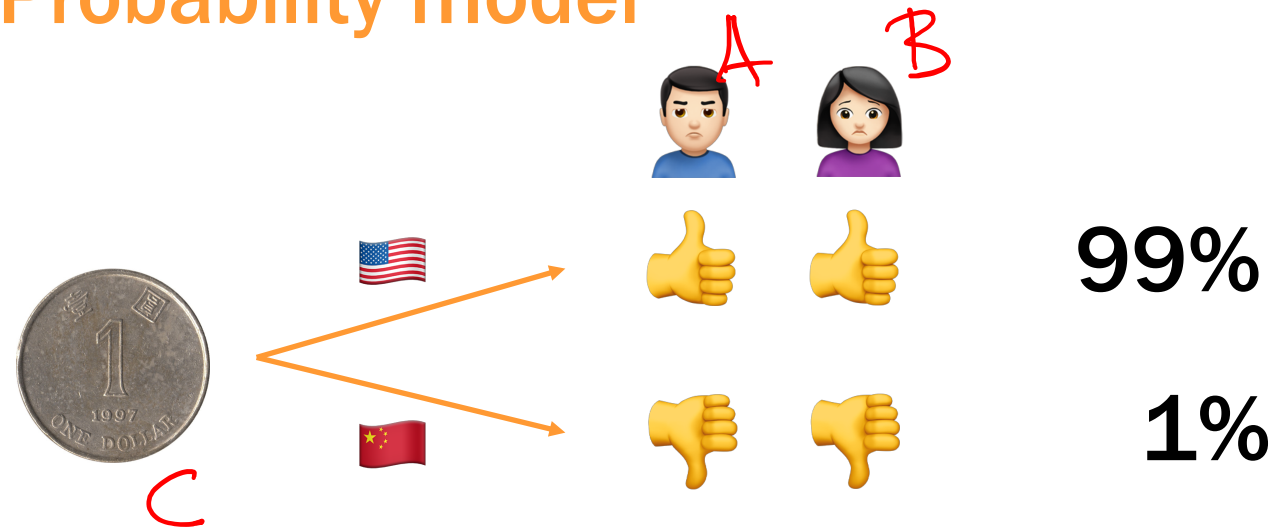
$1/6$ $\quad$ $1/6$ $\quad$ $1/6$

The New York Times

'Crazy Rich Asians' Has Soared, but It May Not Fly in China

Conditioning may destroy dependence

Probability model



A, B INDEPENDENT <u>GIVEN C</u> BUT

$$P(A) = \frac{1}{2}\cdot 99\% + \frac{1}{2}\cdot 1\% = \frac{1}{2}$$

$$P(B) = \frac{1}{2}\cdot 99\% + \frac{1}{2}\cdot 1\% = \frac{1}{2}$$

$$P(A\cap B) = \frac{1}{2}\cdot 0.99^2 + \frac{1}{2}\cdot 0.01^2 \approx 49\%$$

Random variable

A discrete random variable assigns a discrete value to every outcome in the sample space.

Example 

$\{ \mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT} \}$

$N =$ number of $\mathtt{H}$s

Probability mass function

The probability mass function (p.m.f.) of discrete random variable X is the function

$$p(x) = P(X = x)$$

Example 

$$\{\ \mathtt{HH},\ \mathtt{HT},\ \mathtt{TH},\ \mathtt{TT}\ \}$$

$$\tfrac{1}{4}\quad \tfrac{1}{4}\quad \tfrac{1}{4}\quad \tfrac{1}{4}$$

$N =$ **number of** $\mathtt{H}$**s**

$$p(0) = P(N = 0) = P(\{\mathtt{TT}\}) = 1/4$$

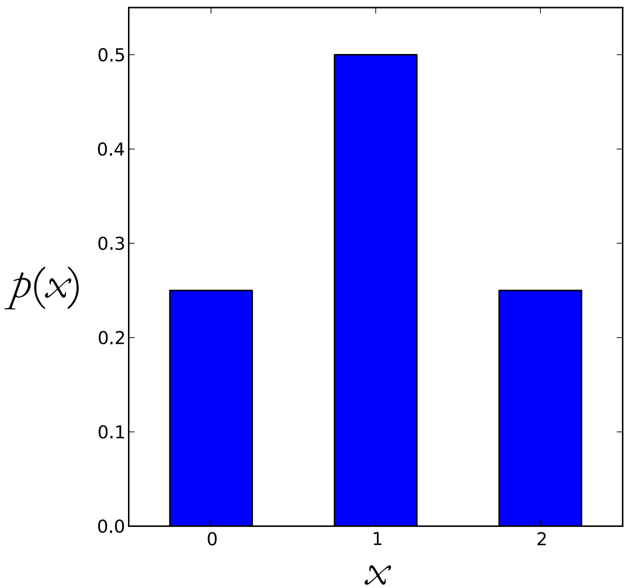
$$p(1) = P(N = 1) = P(\{\mathtt{HT}, \mathtt{TH}\}) = 1/2$$

$$p(2) = P(N = 2) = P(\{\mathtt{HH}\}) = 1/4$$

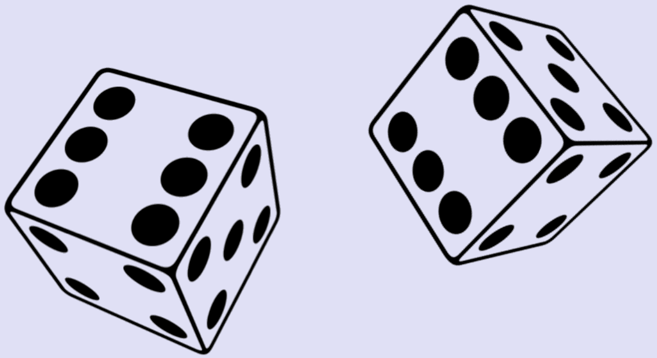
Probability mass function

We can describe the p.m.f. by a table or by a chart.

x	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$



Two six-sided dice are tossed. Calculate the p.m.f. of the difference D of the rolls.



What is the probability that $D > 1$? D is odd?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

d	$P(D=d)$
-5	$1/36$
-4	$2/36$
-3	$3/36$
-2	$4/36$
-1	$5/36$
0	$6/36$
1	$5/36$
2	$4/36$
3	$3/36$
4	$2/36$
5	$1/36$

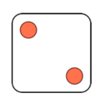
The binomial random variable

$\mathrm{Binomial}(n, p)$: **Perform n independent trials, each of which succeeds with probability p.**

$X =$ **number of successes**

Examples

Toss n coins. “number of heads” is $\mathrm{Binomial}(n, ½)$.

Toss n dice. “Number of  s” is $\mathrm{Binomial}(n, 1/6)$.

A less obvious example

Toss n coins. Let C be the number of consecutive changes (HT or TH).

Examples:

ω	$C(\omega)$
HHHHHHH	0
THHHHHT	2
HTHHHHT	3

Then C is $\text{Binomial}(n-1, \frac{1}{2})$.

A non-example

Draw a 10-card hand from a 52-card deck.

Let $N =$ number of aces among the drawn cards

Is N a $\mathrm{Binomial}(10, 1/13)$ random variable?

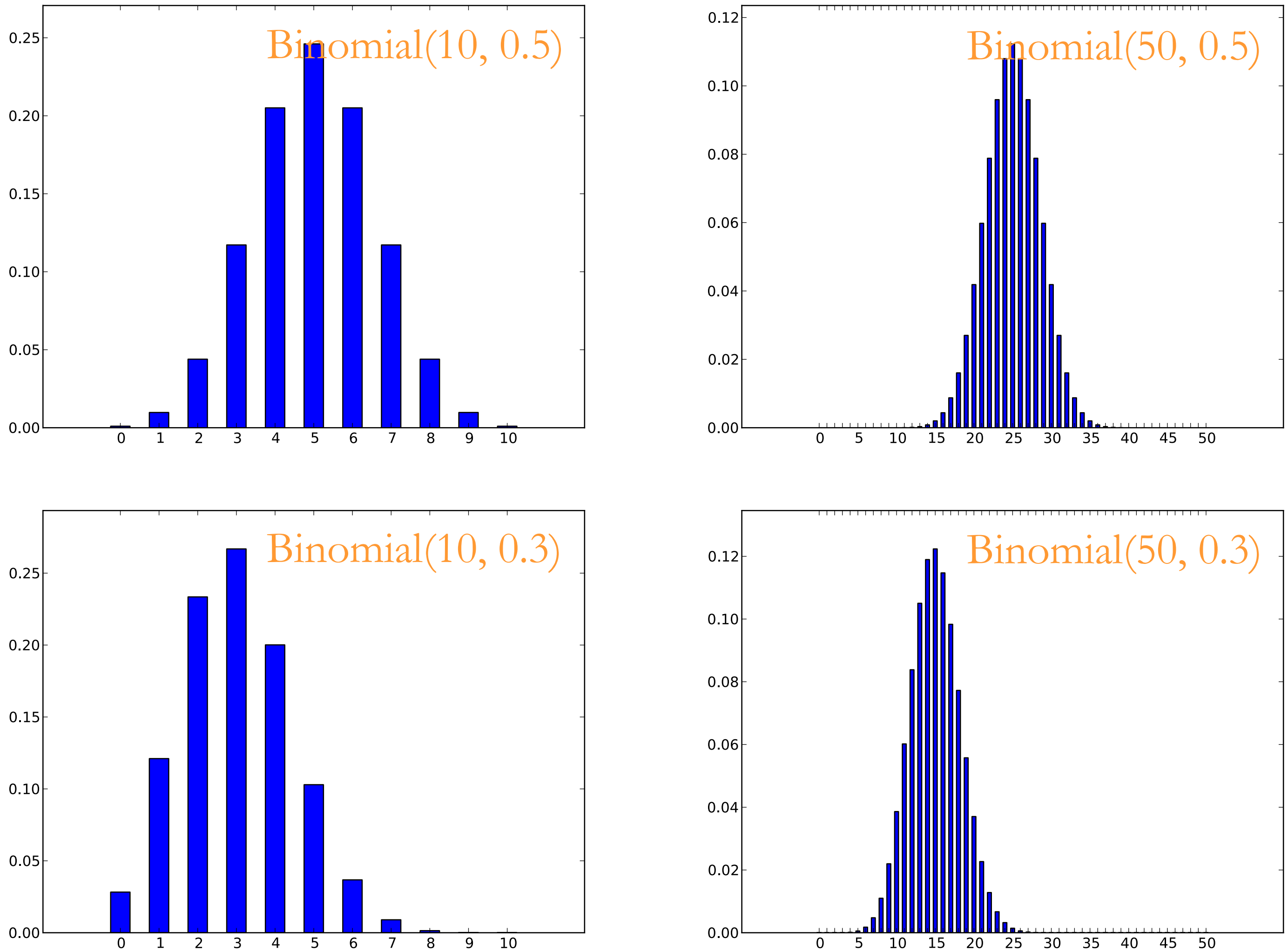
No!

Trial outcomes are not independent.

Probability mass function

If X **is** $\text{Binomial}(n, p)$**, its p.m.f. is**

$$\boxed{p(k) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}}$$


Binomial(10, 0.5)
Binomial(50, 0.5)
Binomial(10, 0.3)
Binomial(50, 0.3)
0.00
0.05
0.10
0.15
0.20
0.25
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0 1 2 3 4 5 6 7 8 9 10
0 5 10 15 20 25 30 35 40 45 50

Geometric random variable

Let $X_1, X_2, \ldots$ be independent trials with success p.

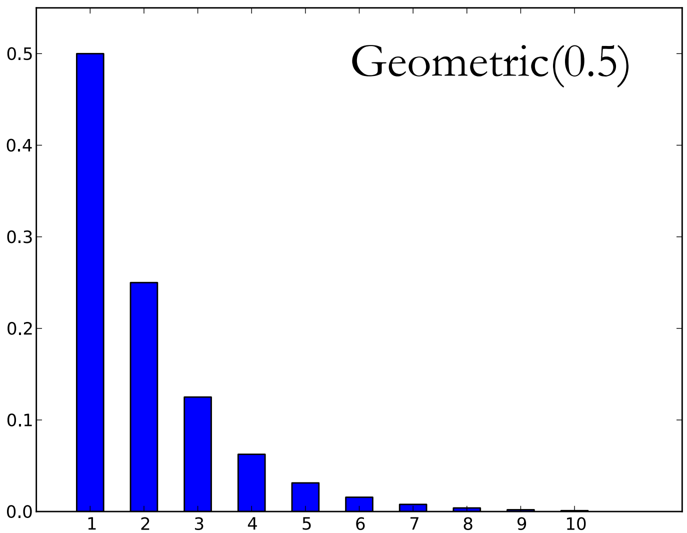
A $\text{Geometric}(p)$ random variable N is the time of the first success among $X_1, X_2, \ldots$:

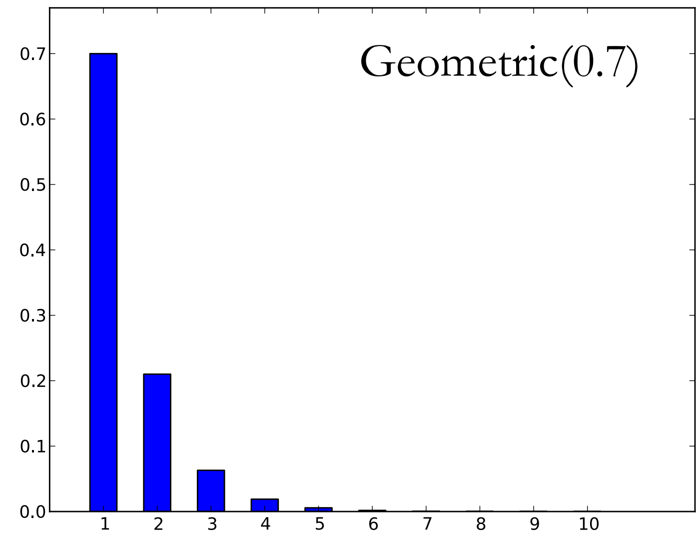
$$N = \text{first (smallest) } n \text{ such that } X_n = 1.$$

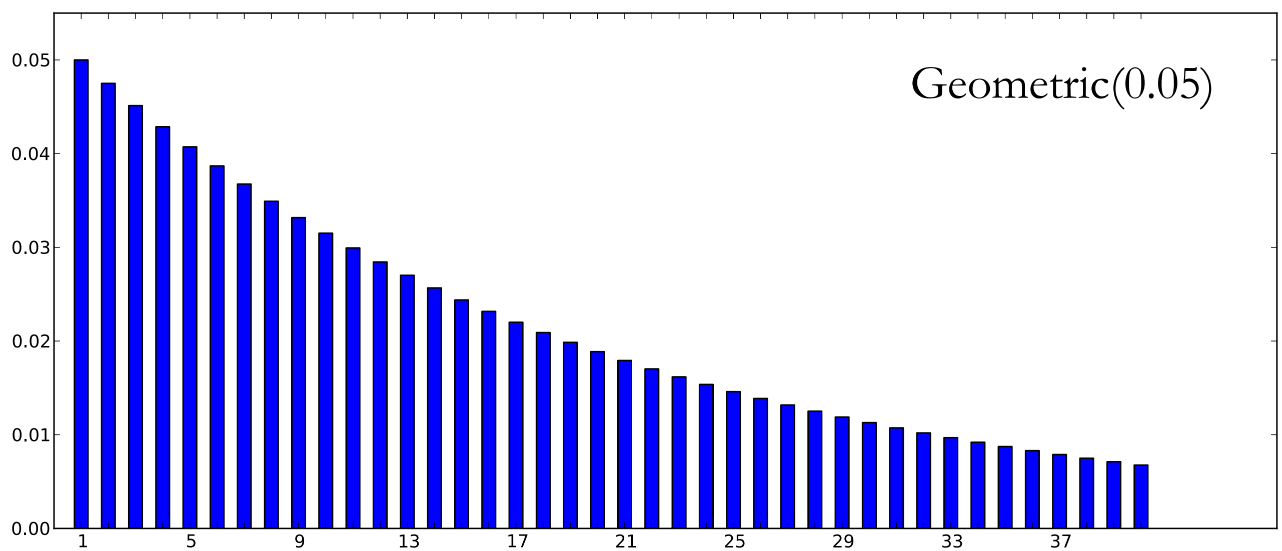
So $P(N = n) = P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1)$

$$= (1 - p)^{n-1}p.$$

This is the p.m.f. of N.


Geometric(0.5)
0.0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5
6
7
8
9
10


Geometric(0.7)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1
2
3
4
5
6
7
8
9
10


Geometric(0.05)
0.00
0.01
0.02
0.03
0.04
0.05
1
5
9
13
17
21
25
29
33
37

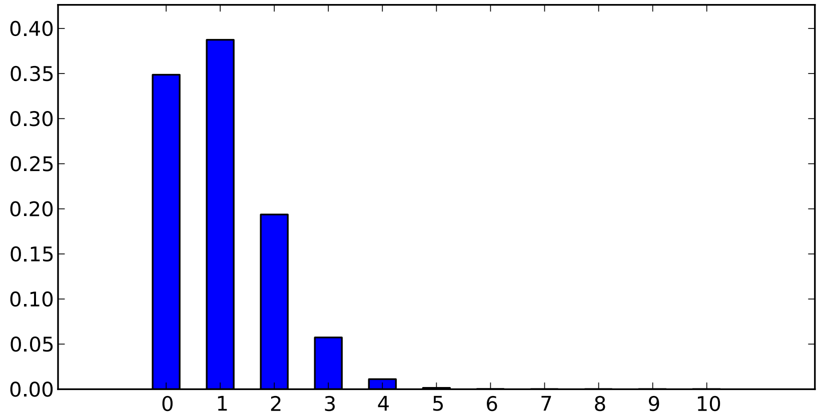
Apples

About 10% of the apples on your farm are rotten.

You sell 10 apples. How many are rotten?

Probability model

N**umber of rotten apples you sold is** $\text{Binomial}(n = 10, p = 1/10)$.

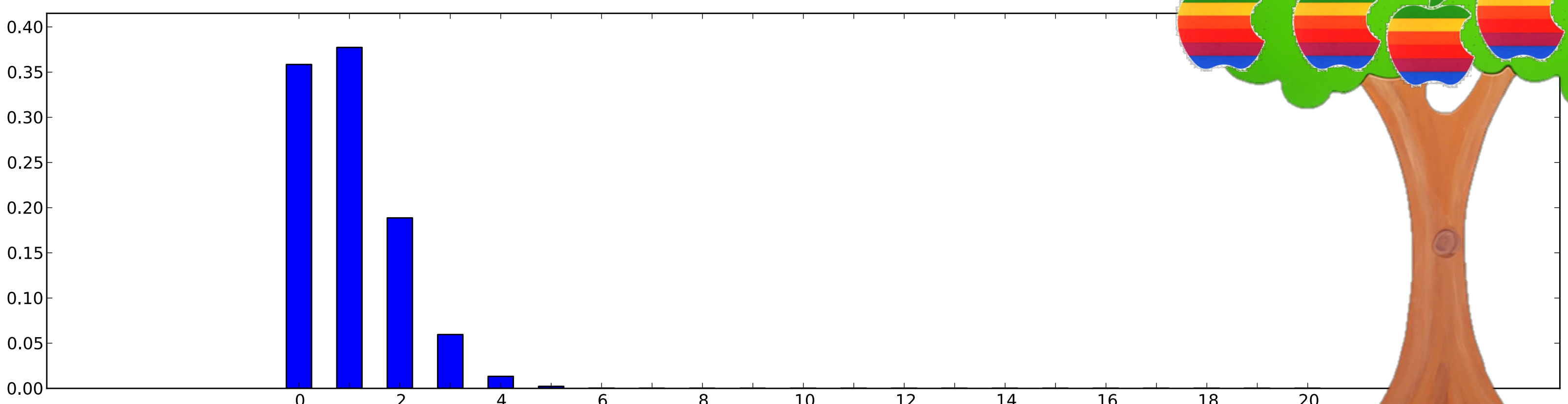


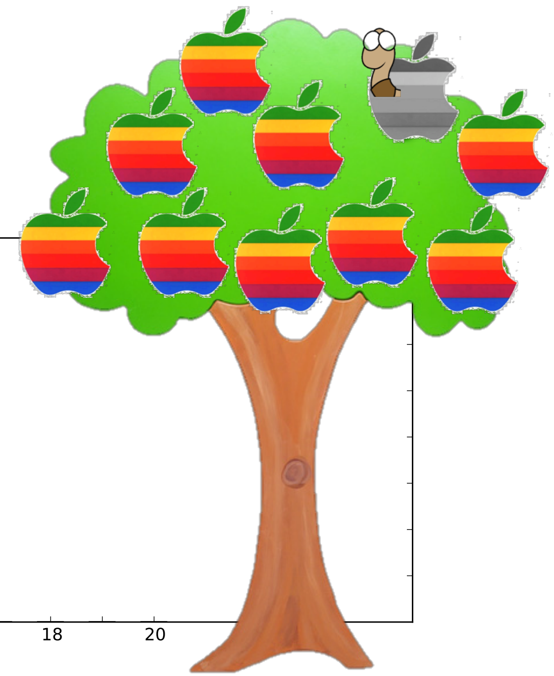
Apples

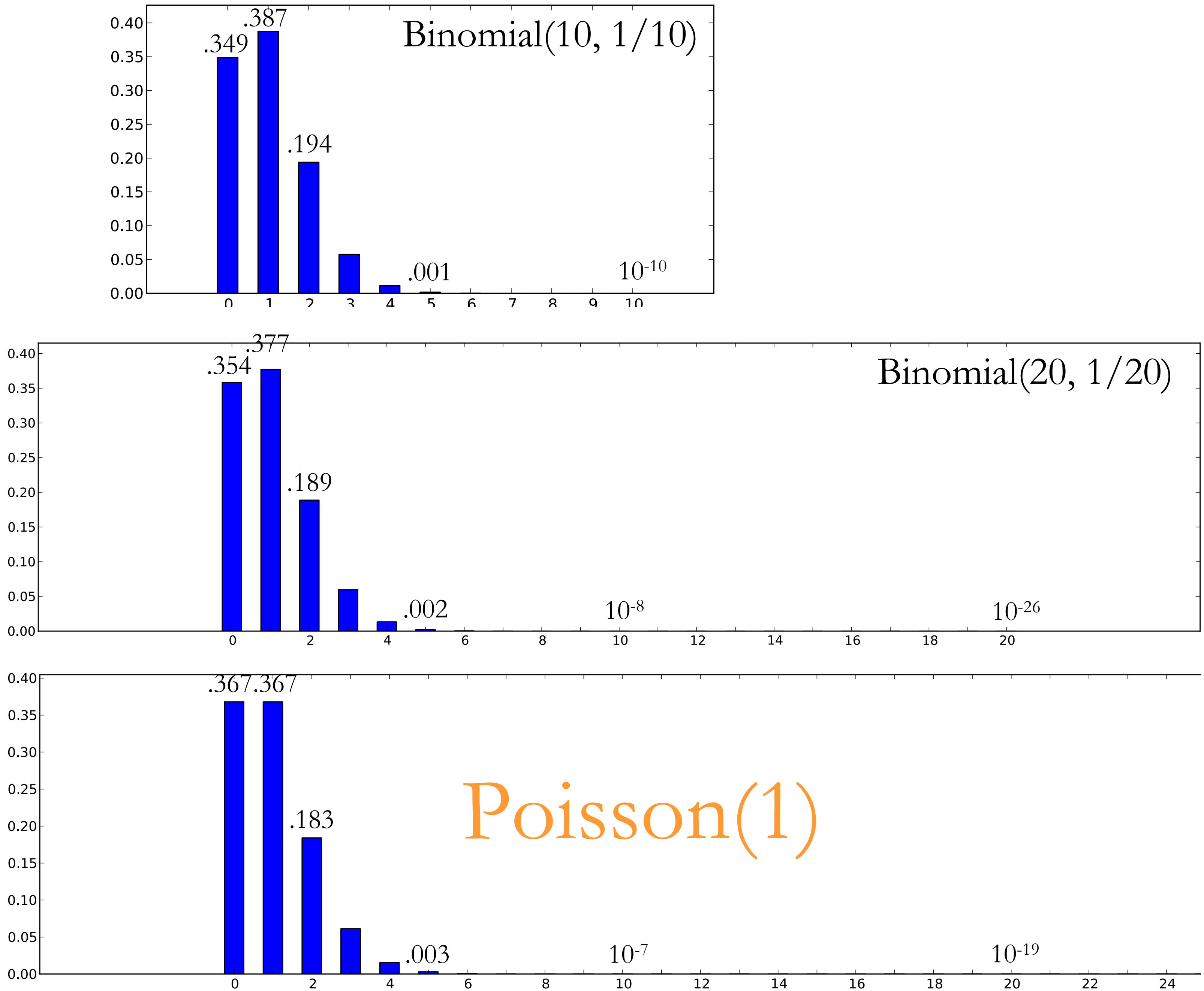
You improve productivity; now only 5% apples rot.

You can now sell 20 apples.

N **is now** $\text{Binomial}(20, 1/20)$.






Binomial(10, 1/10)
.349
.387
.194
.001
10^{-10}
Binomial(20, 1/20)
.354
.377
.189
.002
10^{-8}
10^{-26}
Poisson(1)
.367
.367
.183
.003
10^{-7}
10^{-19}

The Poisson random variable

A $\text{Poisson}(\lambda)$ random variable has this p.m.f.:

$$\boxed{p(k) = e^{-\lambda}\, \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

Poisson random variables do not occur "naturally" in the sample spaces we have seen.

They approximate $\text{Binomial}(n, p)$ random variables when $\lambda = np$ is fixed and n is large (so p is small)

$$p_{\text{Poisson}(\lambda)}(k) = \lim_{n \to \infty} p_{\text{Binomial}(n,\, \lambda/n)}(k)$$

Functions of random variables

p.m.f. of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

p.m.f. of $X - 1$?

x	-1	0	1
q(x)	1/3	1/3	1/3

p.m.f. of $(X - 1)^2$?

x	0	1
r(x)	1/3	2/3

If X is a random variable with p.m.f. p_X, then $Y = f(X)$ is a random variable with p.m.f.

$$p_Y(y) = \sum_{x:\, f(x) = y} p_X(x).$$

Two six-sided dice are tossed. D is the difference of rolls. Calculate the p.m.f. of $|D|$.

d	$P(D=d)$
-5	$1/36$
-4	$2/36$
$\vdots$	$\vdots$
-1	$5/36$
0	$6/36$
1	$5/36$
$\vdots$	$\vdots$
5	$1/36$

d	$P(\lvert D\rvert=d)$
0	$6/36$
1	$10/36$
2	$8/36$
3	$6/36$
4	$4/36$
5	$2/36$