Biomedical Engineering
生醫工程

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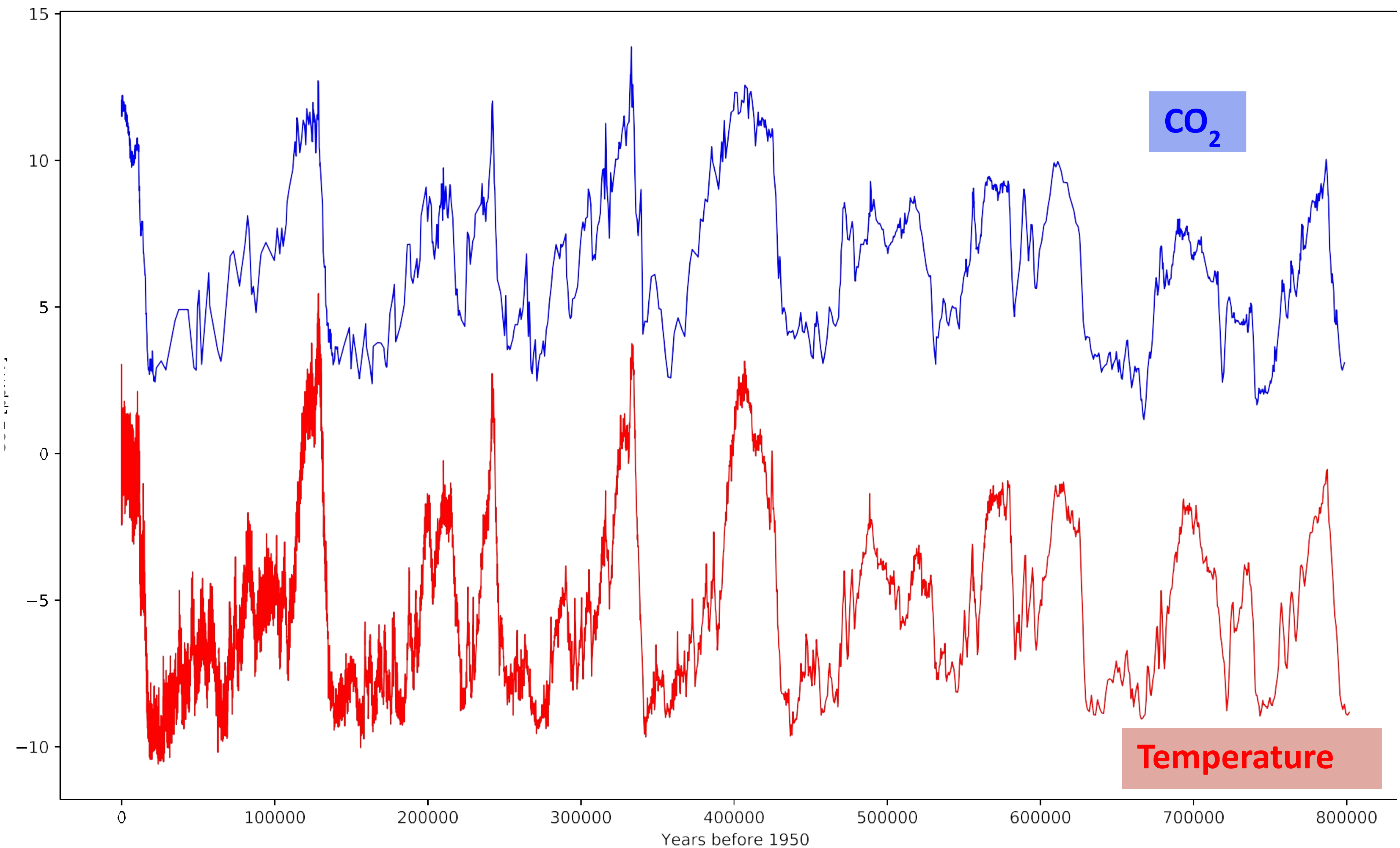
Fourier Transforms

Correlation Of Temperature and CO_2 Record



Pearson Correlation Coefficient r

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

r = +1 for (positive) correlation

r ≈ 0 no correlation

r = -1 for anti correlation

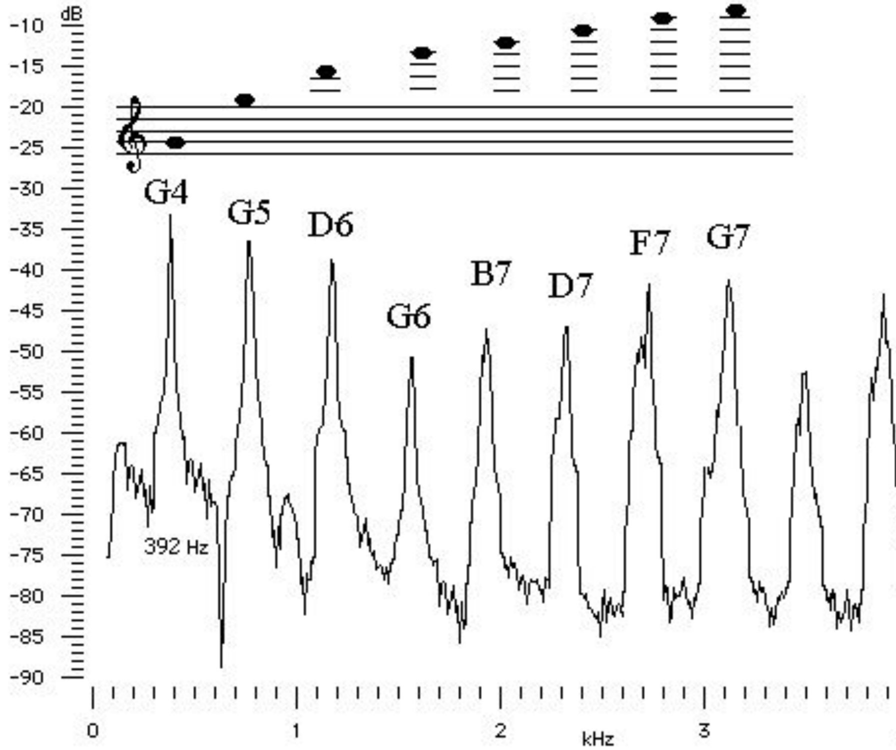
Pearson Correlation Coefficient r

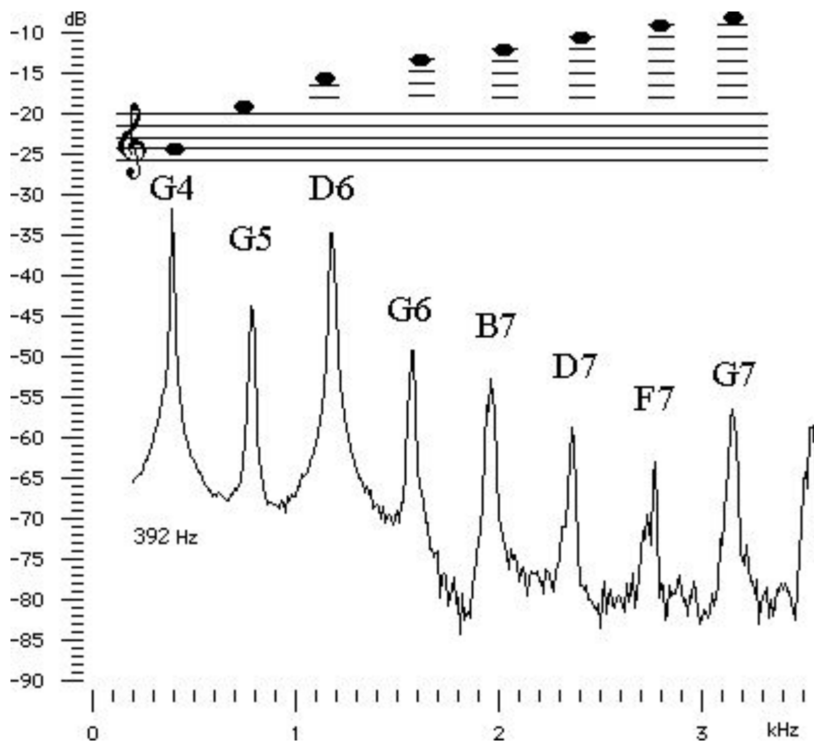
$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Special case: If $\bar{x} = 0$ and $\bar{y} = 0$ then

$r \sim \sum x_i y_i$

(proportional)


dB
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
-65
-70
-75
-80
-85
-90
G4
G5
D6
G6
B7
D7
F7
G7
392 Hz
0
1
2
3
4
5
kHz


dB
-10
-15
-20
-25
-30
-35
-40
-45
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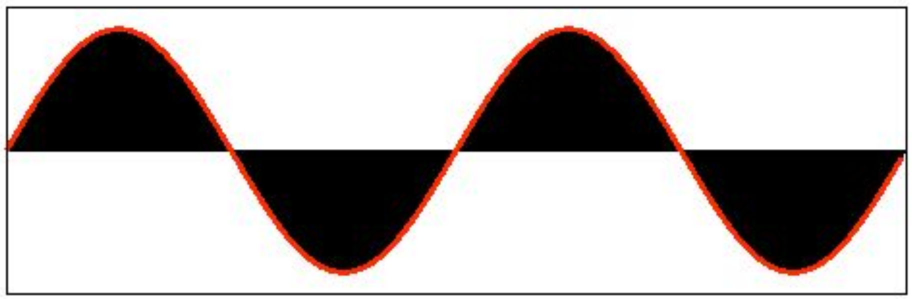
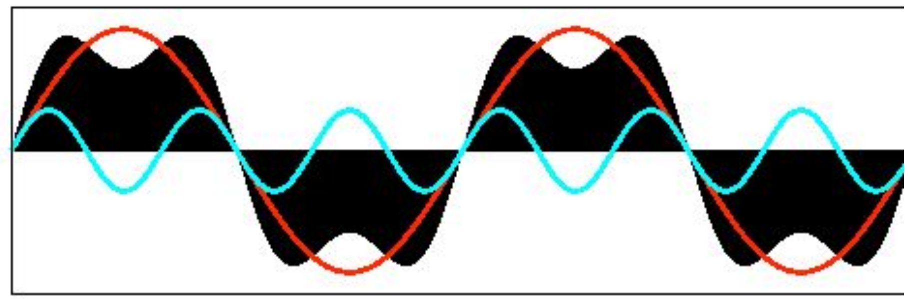
-55
-60
-65
-70
-75
-80
-85
-90
G4
D6
G5
G6
B7
D7
F7
G7
392 Hz
0
1
2
3
4
5
kHz

Square Wave

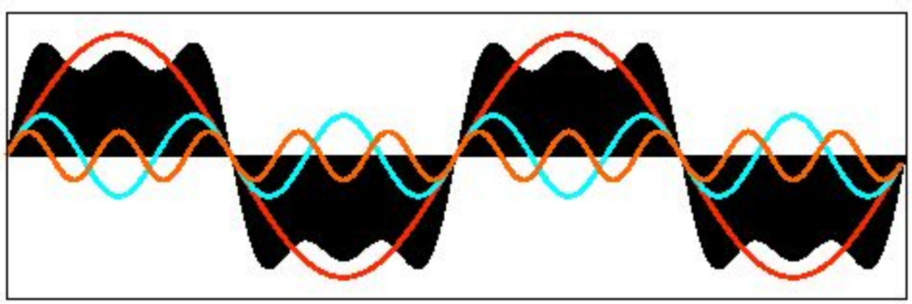
Frequencies: *f*



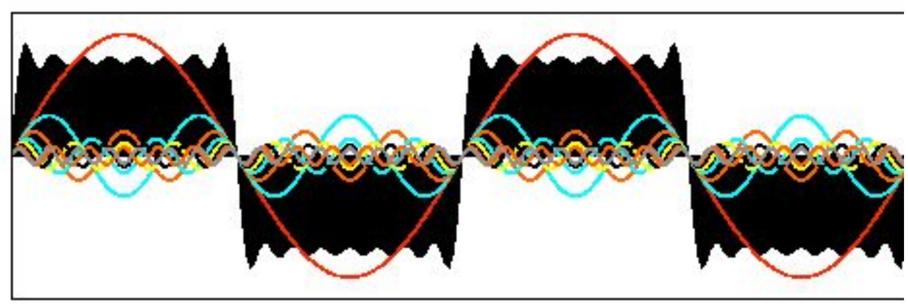
Frequencies: *f* + 3*f*



Frequencies: *f* + 3*f* + 5*f*

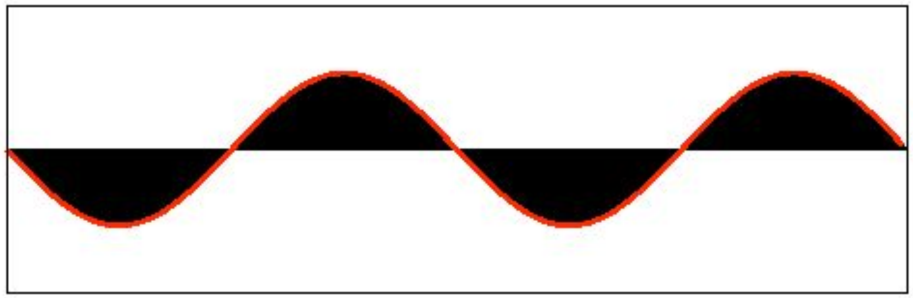


Frequencies: *f* + 3*f* + 5*f* + … + 15*f*

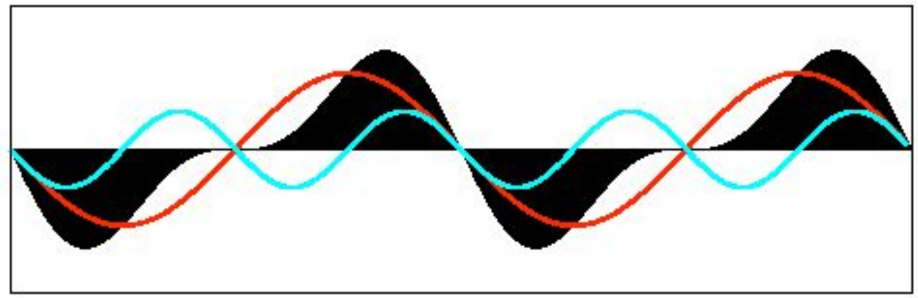


Sawtooth Wave

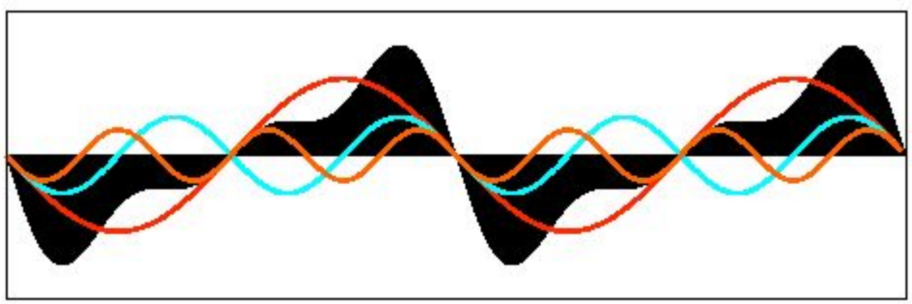
Frequencies: *f*



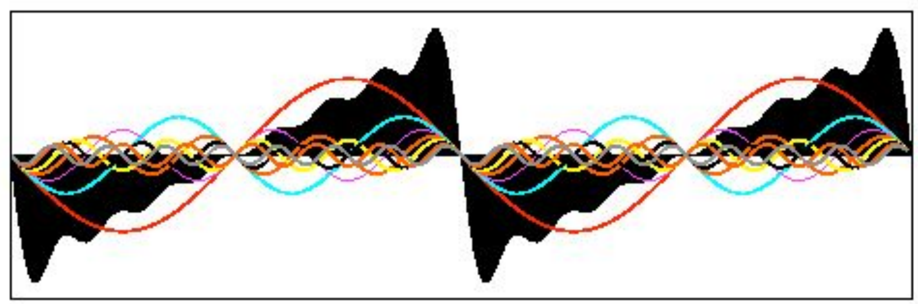
Frequencies: *f* + 2*f*



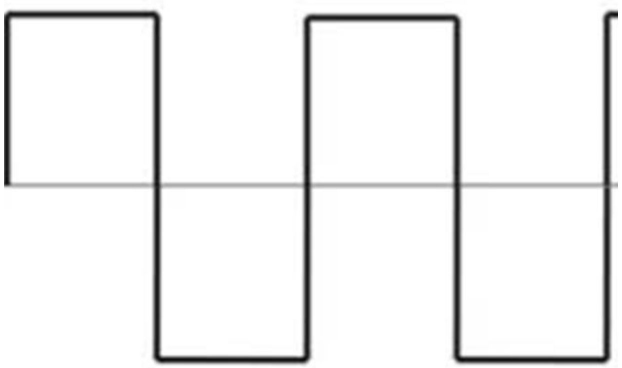
Frequencies: *f* + 2*f* + 3*f*



Frequencies: *f* + 2*f* + 3*f* + … + 8*f*



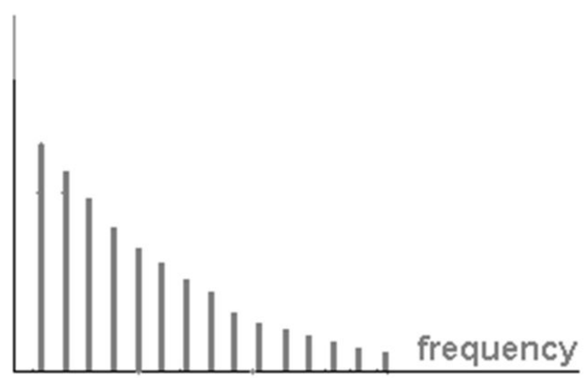
Fourier series for a square wave



=

$$f(x) = \sum 1/n * \sin(nx)$$

$$n = 1, 3, 5\ldots$$



Fourier Series

A function $f(x)$ can be expressed as a series of sines and cosines:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

where:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

$$n = 1, 2, 3, \ldots$$

Credit: Mark Hedl

Fourier Transform

- Fourier Series can be generalized to complex numbers, and further generalized to derive the *Fourier Transform*.

Forward Fourier Transform:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dk$$

Inverse Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x} dk$$

Note: $e^{xi} = \cos(x) + i \sin(x)$

Fourier Transform

- Fourier Transform maps a time series (eg audio samples) into the series of frequencies (their amplitudes and phases) that composed the time series.

- Inverse Fourier Transform maps the series of frequencies (their amplitudes and phases) back into the corresponding time series.

- The two functions are inverses of each other.

Discrete Fourier Transform

- If we wish to find the frequency spectrum of a function that we have *sampled*, the continuous Fourier Transform is not so useful.
- We need a discrete version:
  - *Discrete Fourier Transform*

Discrete Fourier Transform

Forward DFT:

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N}$$

The complex numbers $f_0 \ldots f_N$ are transformed into complex numbers $F_0 \ldots F_n$

Inverse DFT:

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-2\pi i k n/N}$$

The complex numbers $F_0 \ldots F_n$ are transformed into complex numbers $f_0 \ldots f_N$

Fourier Transform of a Cosine

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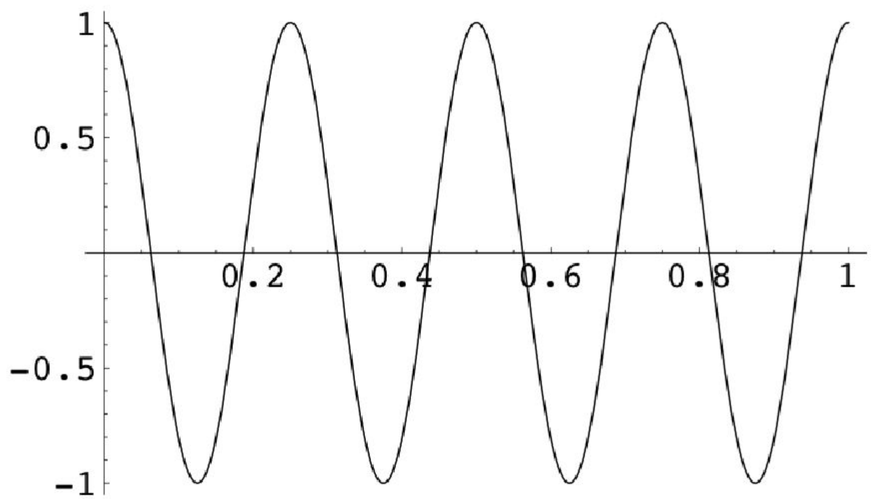
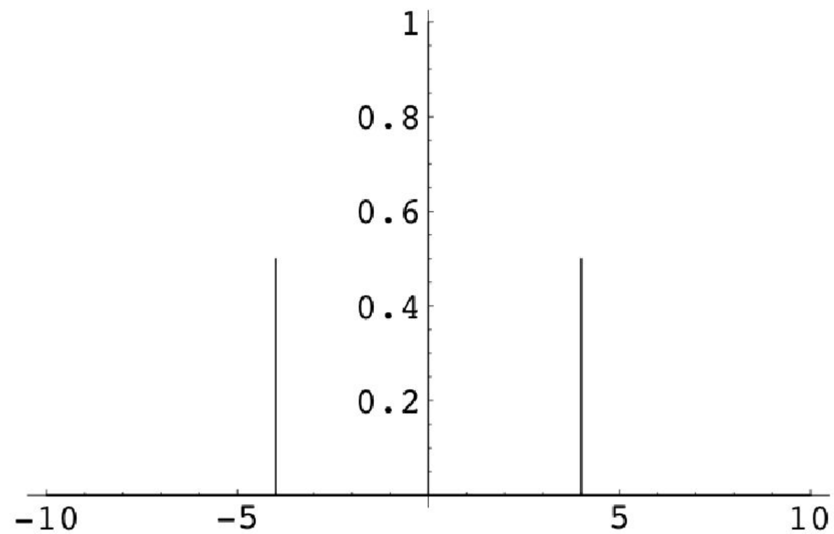
Spatial Domain

$$\cos(2\pi st)$$



Frequency Domain

$$\frac{1}{2}\delta(u-s)+\frac{1}{2}\delta(u+s)$$

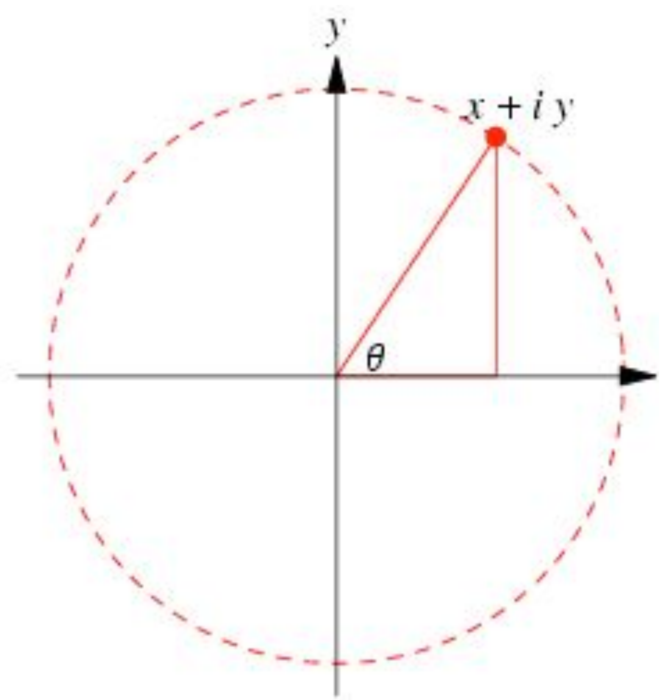


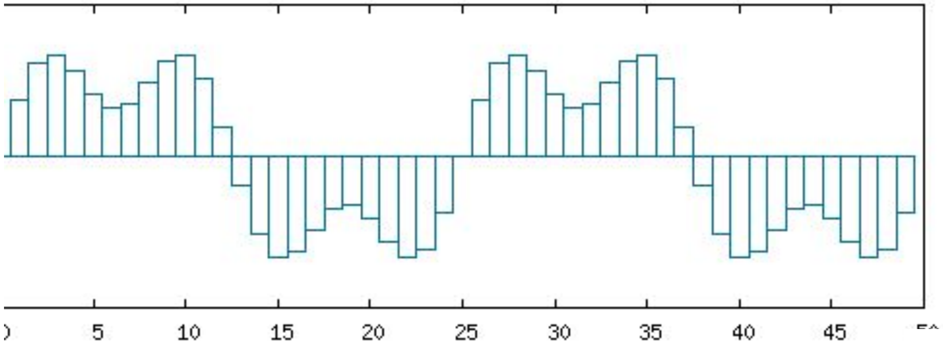
$$\cos(k\omega_0 t)=\frac{1}{2}\left(e^{jk\omega_0 t}+e^{-jk\omega_0 t}\right)$$

$$\sin(k\omega_0 t)=\frac{1}{2j}\left(e^{jk\omega_0 t}-e^{-jk\omega_0 t}\right)$$

DFT Example

- Interpreting a DFT can be slightly difficult, because the DFT of real data includes complex numbers.
- Basically:
  - The magnitude of the complex number for a DFT component is the power at that frequency.
  - The phase θ of the waveform can be determined from the relative values of the real and imaginary coefficients.
- Also both positive and “negative” frequencies show up.

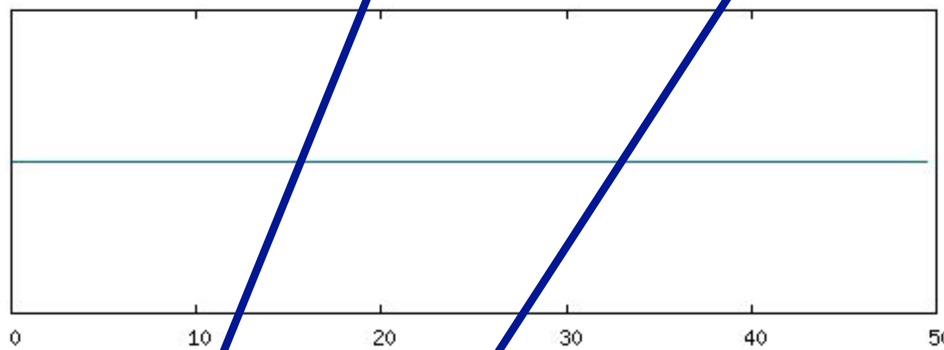




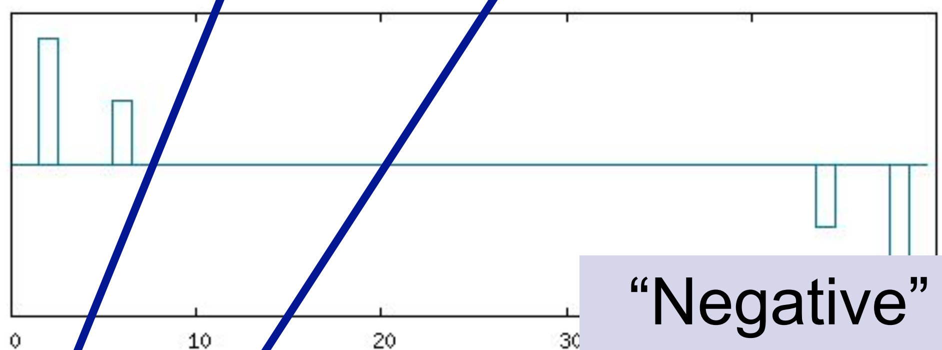
Sampled data:

$f(x) = 2\sin(x) + \sin(3x)$

DFT: Real Components

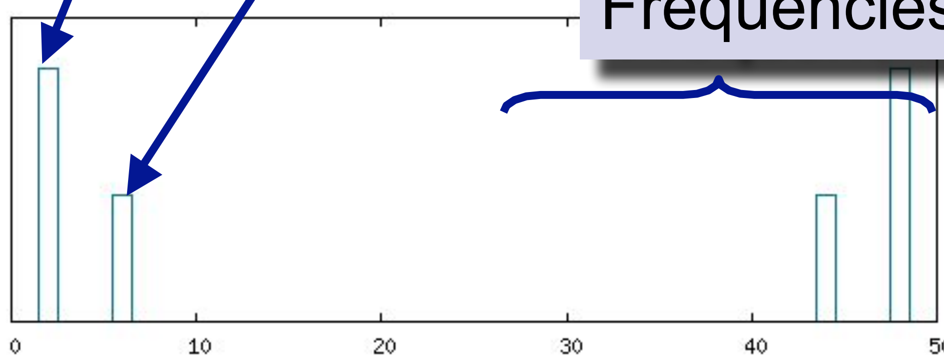


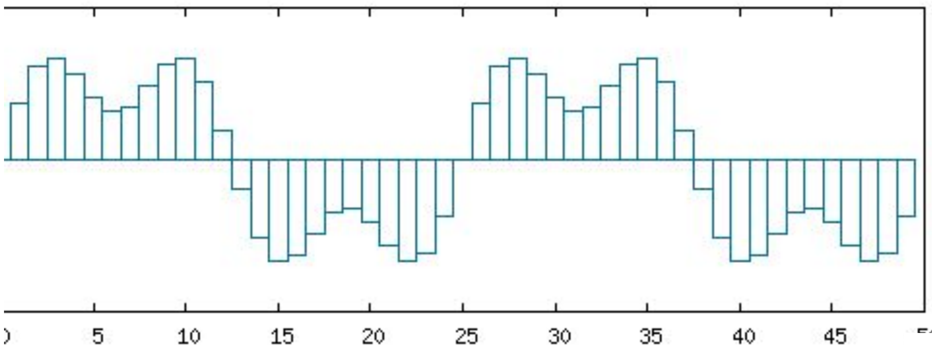
DFT: Imaginary Components



"Negative" Frequencies

DFT: Magnitude





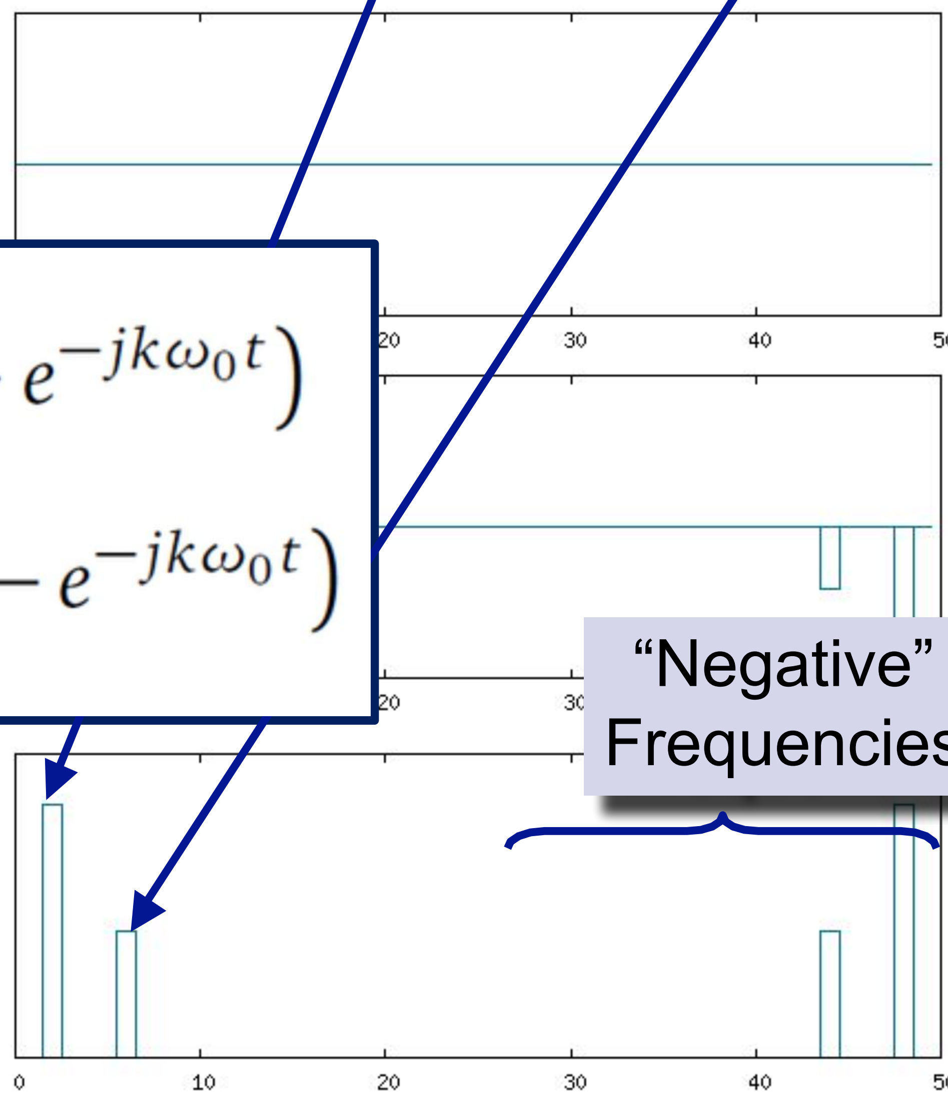
Sampled data:

$$f(x) = \underbrace{2\sin(x)} + \underbrace{\sin(3x)}$$

DFT: Real Components

$$\cos(k\omega_0 t) = \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right)$$

$$\sin(k\omega_0 t) = \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$



“Negative” Frequencies

DFT: Magnitude

Lab Assignment

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re are clearly a number of frequencies hidden in this file. You can hear several of them. Now we want t discrete Fourier transforms to identify those frequencies ν.

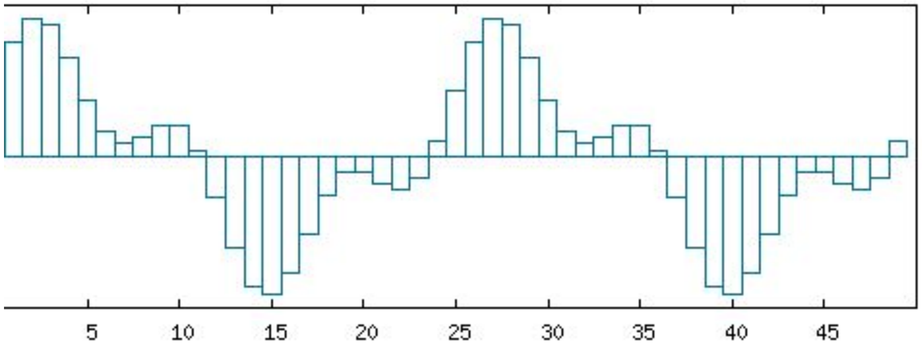
$$f^{(sin)}(\nu) = \int_{-\infty}^{+\infty} dt\, x(t)\ \sin(2\pi\nu t) \qquad \text{and} \qquad f^{(cos)}(\nu) = \int_{-\infty}^{+\infty} dt\, x(t)\ \cos(2\pi\nu t)\,.$$

ead of continuous function, $f(t)$, our audio signal is provides as a series discrete points $x_{j=0\ldots n-1}$. So and cosine transform assume the following discrete forms: (n is the number of points.)

$$f_k^{(sin)} = \sum_{j=0}^{n-1} x_j\ \sin\left\{2\pi\frac{jk}{n}\right\} \qquad \text{and} \qquad f_k^{(cos)} = \sum_{j=0}^{n-1} x_j\ \cos\left\{2\pi\frac{jk}{n}\right\}$$

a sheet of paper, work out how you convert between time t and index j. More importantly work ot the version from frequency index k to frequency ν.

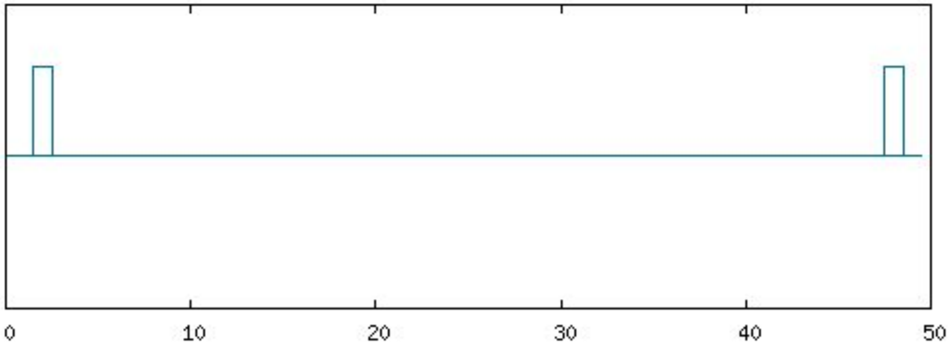
e a Python function that computes $\left[f_k^{(sin)}\right]^2 + \left[f_k^{(cos)}\right]^2$ for an arbitrary data set and frequency index



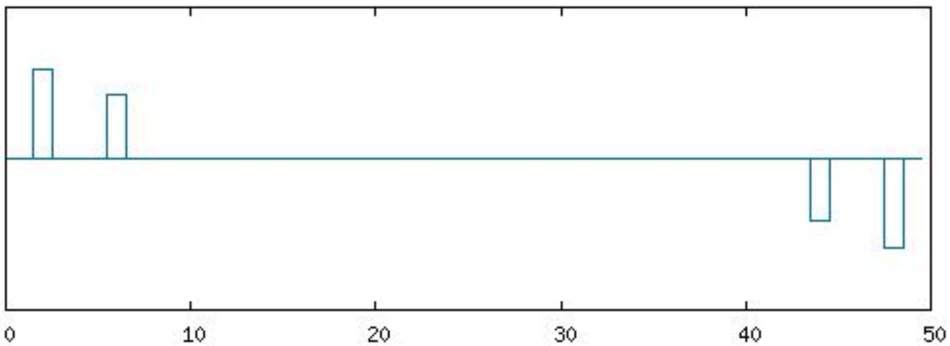
Sampled data:

$f(x) = 2\sin(x+45) + \sin(3x)$

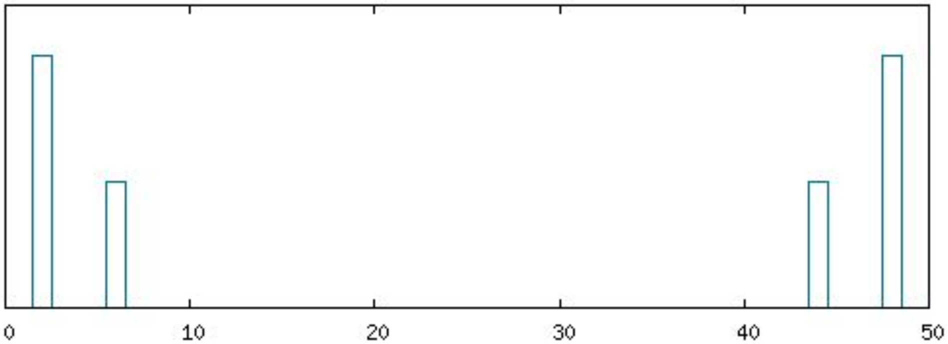
DFT: Real Components



DFT: Imaginary Components



DFT: Magnitude



Fast Fourier Transforms (FFT)

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The DFT requires N^2 complex multiplications. At each stage of the FFT (i.e. each halving) $\frac{N}{2}$ complex multiplications are required to combine the results of the previous stage. Since there are $(log_2 N)$ stages, the number of complex multiplications required to evaluate an N -point DFT with the FFT is approximately $N/2log_2N$ (approximately because multiplications by factors such as , W_N^0, $W_N^{\frac{N}{2}}$ are $W_N^{\frac{N}{4}}$ and $W_N^{\frac{3N}{4}}$ really just complex additions and subtractions).

N	N^2 (DFT)	$\frac{N}{2}log_2N$ (FFT)	saving
32	1,024	80	92%
256	65,536	1,024	98%
1,024	1,048,576	5,120	99.5%

7.3.3 Practical considerations

If N is not a power of 2 there are 2 strategies available to complete an N-point FFT.