Biomedical Engineering
生醫工程

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Partial Differential Equations Part I – Heat Equation

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(1) What is a partial differential equation (PDE)?

(2) Derive the heat equation

(3) Stationary temperature distributions

(4) Numerical methods to obtain the stationary state

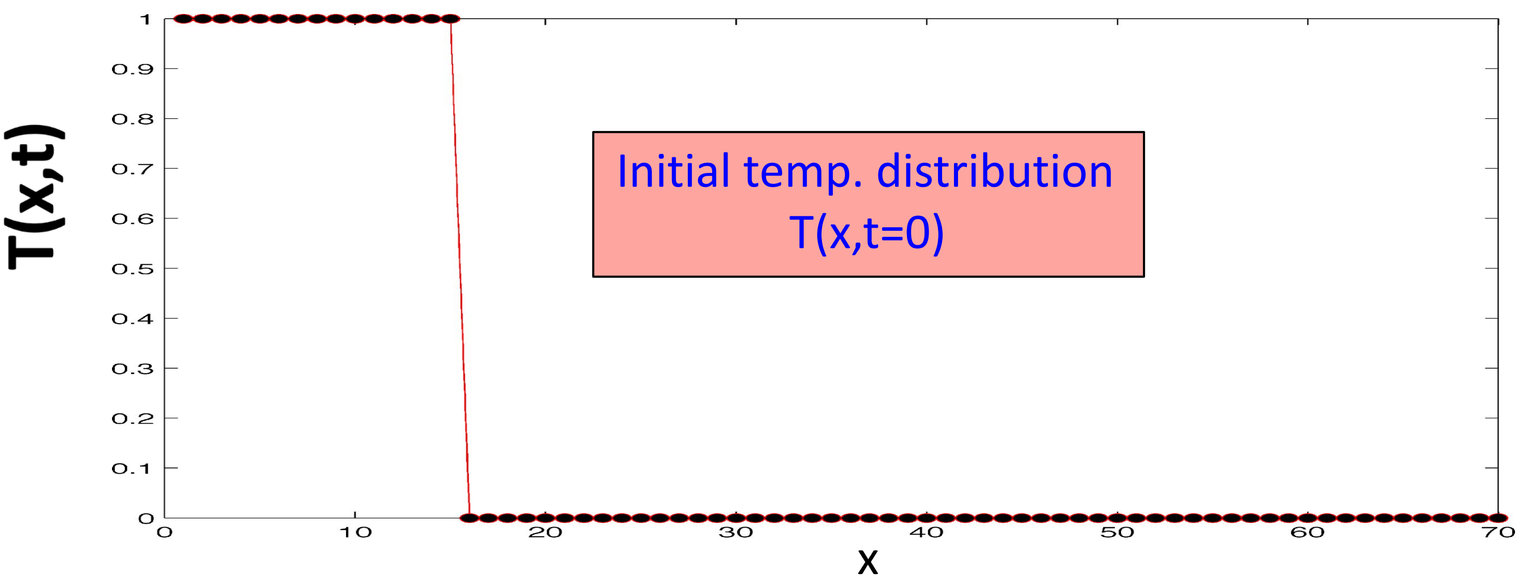
3 mechanisms that transport heat

- Diffusion
- Convection
- Radiation

Initial temperature distribution inside the wall of a house T(x,t=0)

Inside temp. (const)

Outside temperature (const)

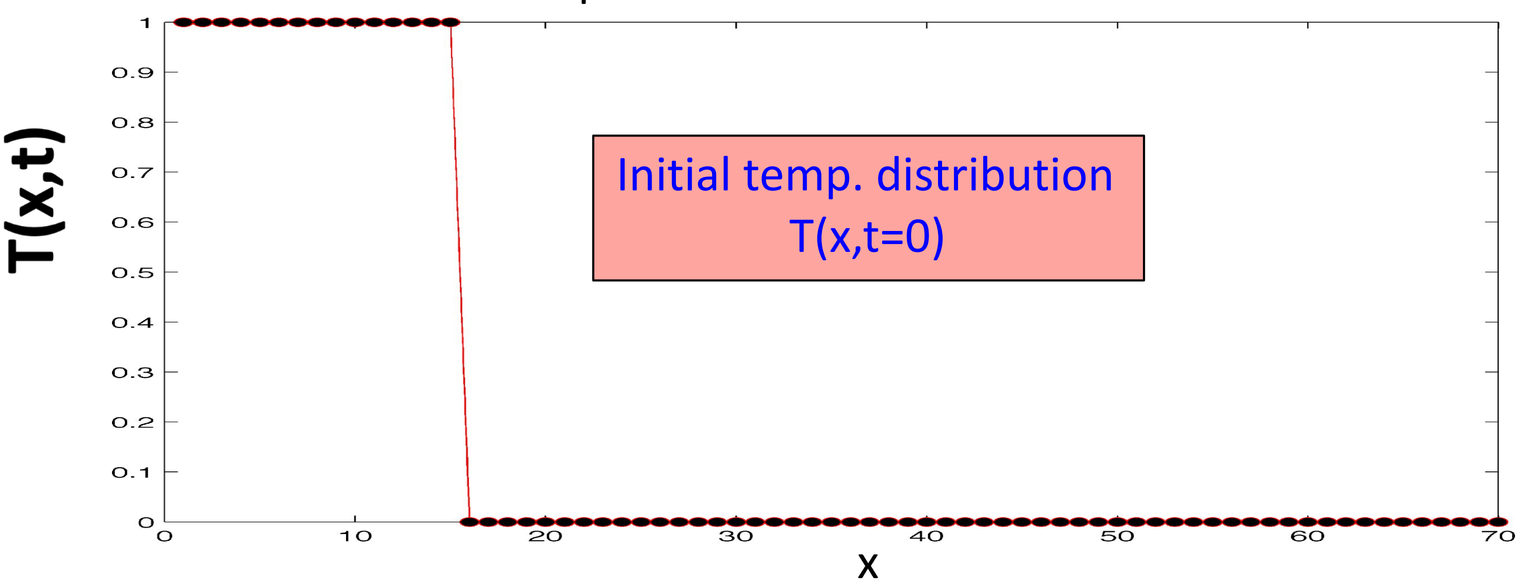


Initial temperature distribution inside the wall of a house T(x,t=0)

Inside temp. (const)

temperature distribution inside wall

Outside temperature (const)



Evolution of the Temperature distribution inside the wall of a house T(x,t)

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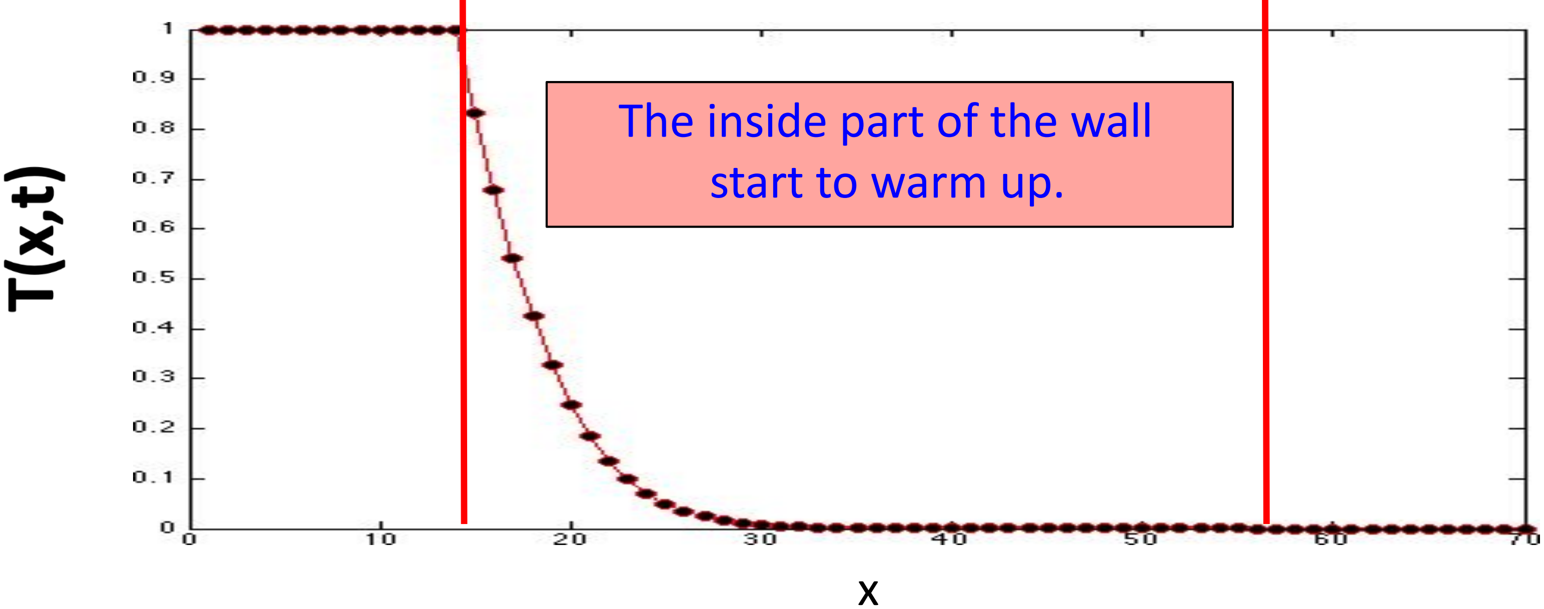
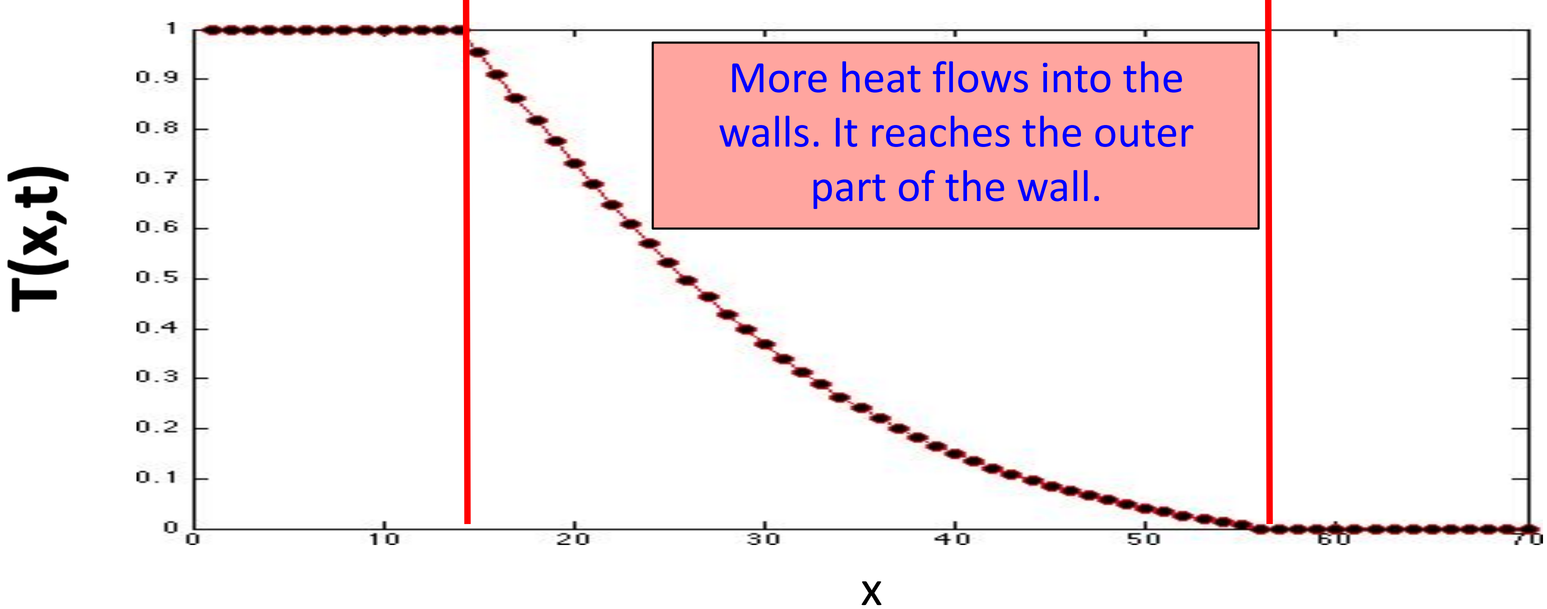
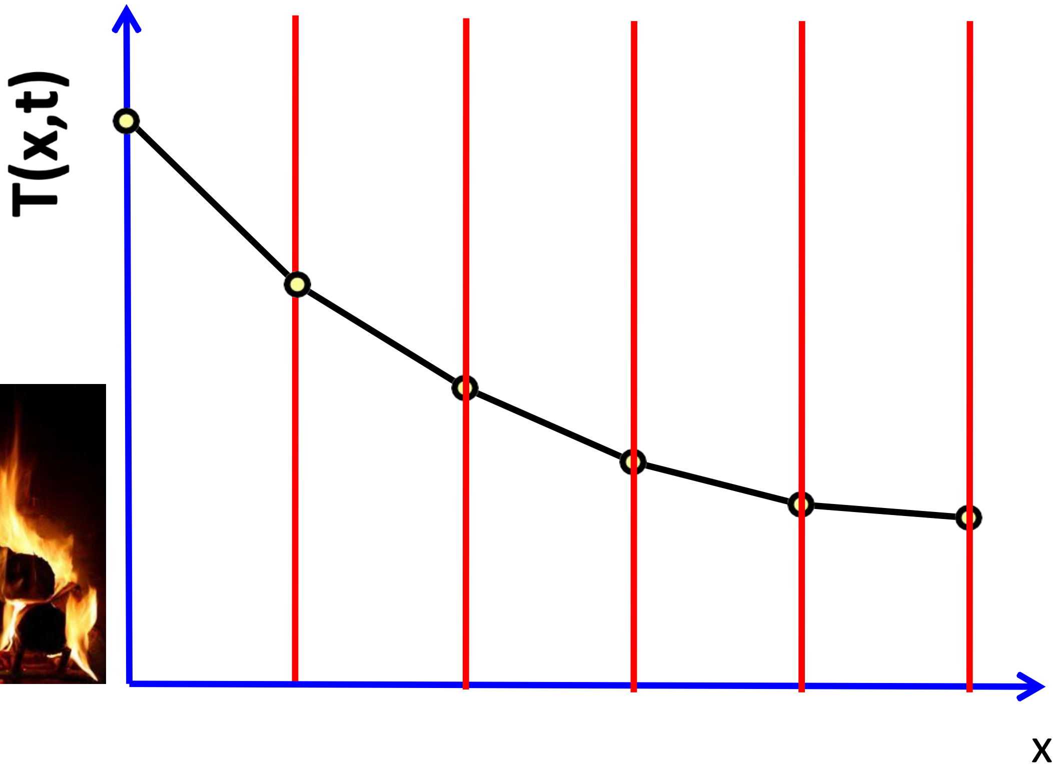
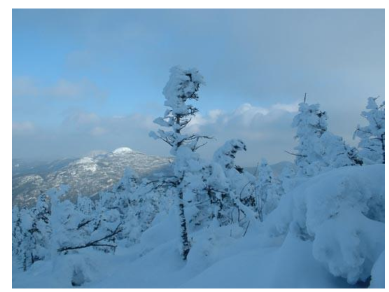


Evolution of the Temperature distribution inside the wall of a house T(x,t)

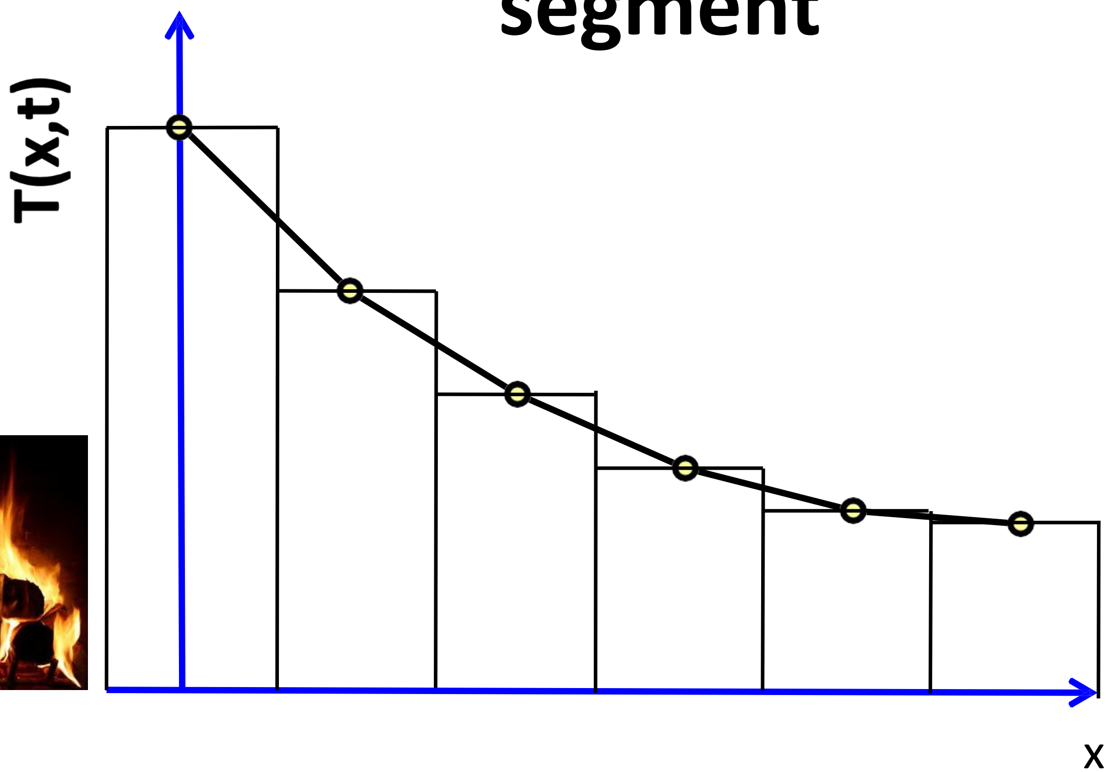


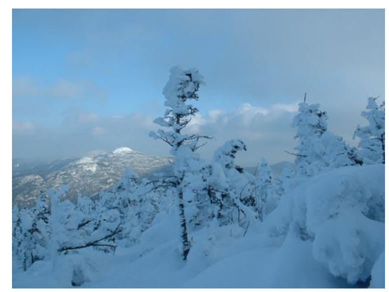
We discretize the temperature distribution in the wall by introducing a grid of x points



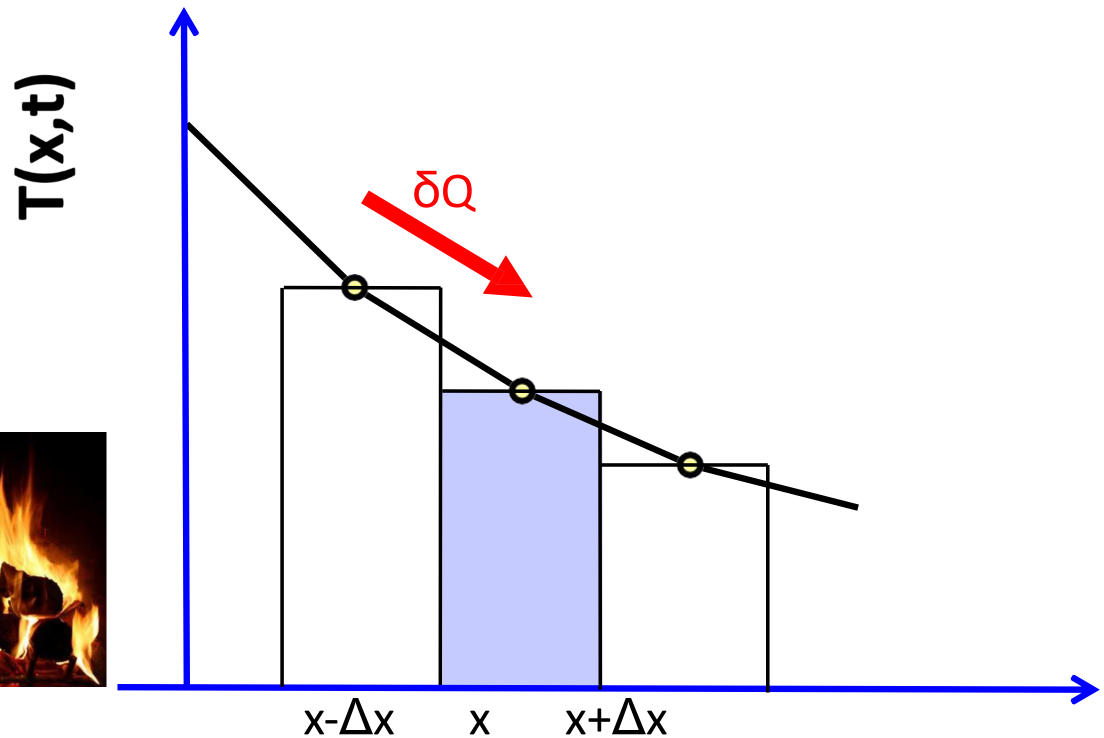


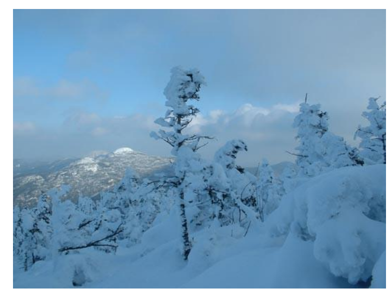
Assume the temperature is constant inside every wall segment





How is the temperature T(x,t) at one point x going to evolve with time?





$$\delta Q = -\kappa\,[T(x) - T(x-\Delta x)]\,\Delta t\,/\Delta x$$

κ … thermal conductivity

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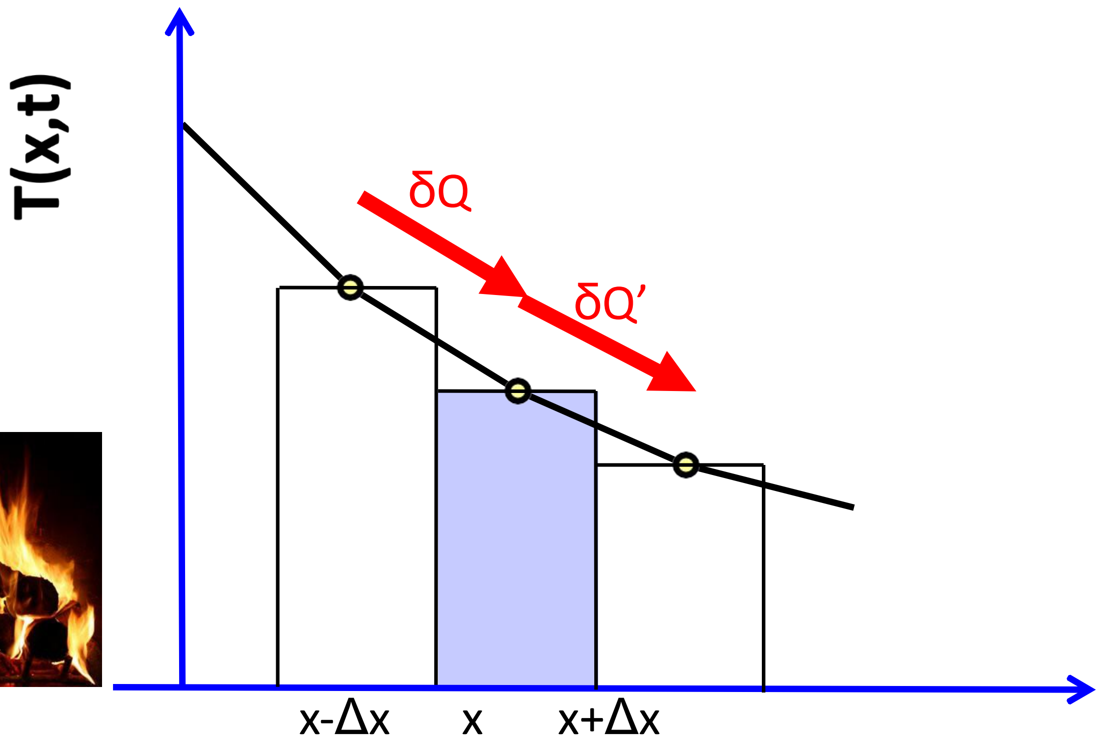

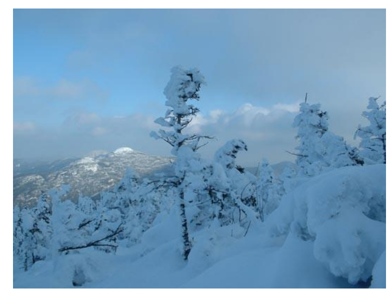

$$\delta Q = -\kappa\,[T(x) - T(x-\Delta x)]\,\Delta t / \Delta x$$

$$\delta Q' = -\kappa\,[T(x+\Delta x) - T(x)]\,\Delta t / \Delta x$$

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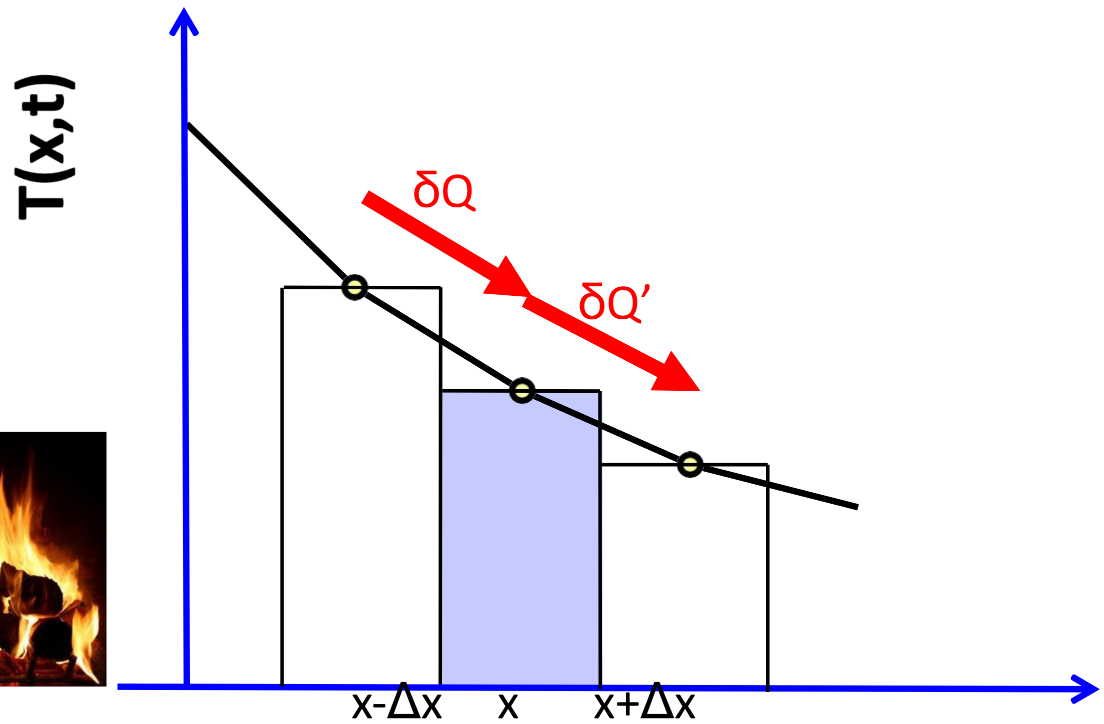
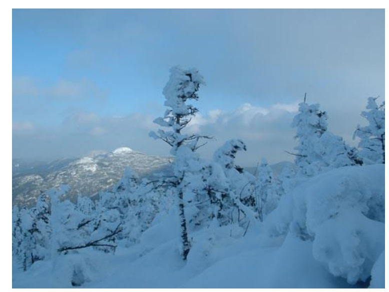




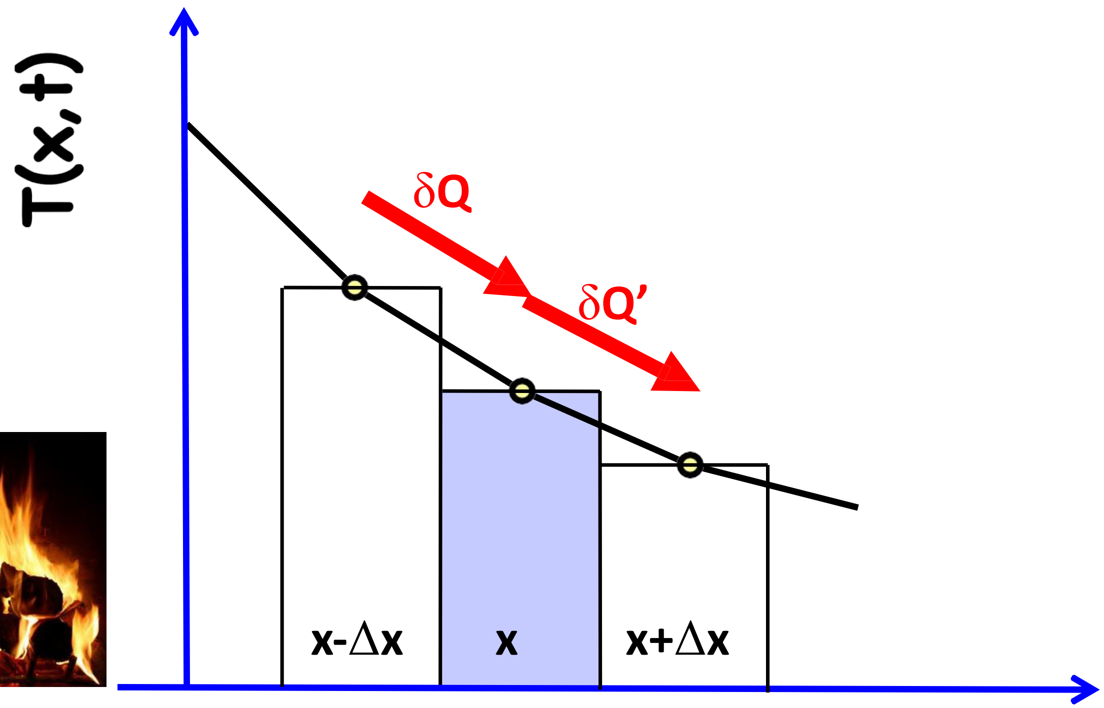
κ … thermal conductivity

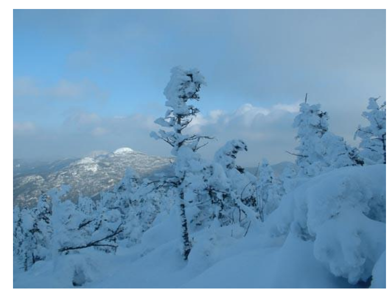
$$\delta Q = -\kappa\,[T(x) - T(x-\Delta x)]\ \Delta t \,/\, \Delta x$$

$$\delta Q' = -\kappa\,[T(x+\Delta x) - T(x)]\ \Delta t \,/\, \Delta x$$

$$\Delta E = \delta Q - \delta Q' = c_p m\ \Delta T$$

Temperature distribution in the wall of a house T(x,t)





$$\delta Q = -\kappa\,[T(x) - T(x-\Delta x)]\,\Delta t\,/\,\Delta x$$

κ … thermal conductivity

$$\delta Q' = -\kappa\,[T(x+\Delta x) - T(x)]\,\Delta t\,/\,\Delta x$$

$$\Delta E = \delta Q - \delta Q' = c_p m\,\Delta T$$

Derivation of the heat equation

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$$\delta Q = -\kappa \left[T(x) - T(x - \Delta x) \right] \Delta t / \Delta x$$

$$\delta Q' = -\kappa \left[T(x + \Delta x) - T(x) \right] \Delta t / \Delta x$$

$$\Delta E = \delta Q - \delta Q' = c_P m \, \Delta T$$

$$c_P m \, \Delta T \quad = -\kappa \left[T(x) - T(x - \Delta x) + T(x) - T(x + \Delta x) \right] \Delta t / \Delta x$$

Derivation of the heat equation

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$$\delta Q = -\kappa \left[T(x) - T(x - \Delta x) \right] \Delta t / \Delta x$$

$$\delta Q' = -\kappa \left[T(x + \Delta x) - T(x) \right] \Delta t / \Delta x$$

$$\Delta E = \delta Q - \delta Q' = c_P m \, \Delta T$$

$$c_P m \, \Delta T \quad = -\kappa \left[T(x) - T(x - \Delta x) + T(x) - T(x + \Delta x) \right] \Delta t / \Delta x$$

$$c_P m \, \Delta T \quad = +\kappa \left[T(x + \Delta x) - 2T(x) + T(x - \Delta x) \right] \Delta t / \Delta x$$

Derivation of the heat equation

$$\delta Q = -\kappa \left[T(x) - T(x-\Delta x)\right] \Delta t / \Delta x$$

$$\delta Q' = -\kappa \left[T(x+\Delta x) - T(x)\right] \Delta t / \Delta x$$

$$\Delta E = \delta Q - \delta Q' = c_p m \, \Delta T$$

$$c_p m \, \Delta T \quad = -\kappa \left[T(x) - T(x-\Delta x) + T(x) - T(x+\Delta x)\right] \Delta t / \Delta x$$

$$c_p m \, \Delta T \quad = +\kappa \left[T(x+\Delta x) - 2T(x) + T(x-\Delta x)\right] \Delta t / \Delta x$$

$$\frac{\Delta T}{\Delta t} = +\frac{\kappa}{c_p m}\left[\frac{\partial^2 T}{\partial x^2}\right](\Delta x)^3 = \frac{\kappa}{c_p \rho}\left[\frac{\partial^2 T}{\partial x^2}\right]$$

$$\boxed{\frac{\partial T}{\partial t} = +\frac{\kappa}{c_p \rho}\frac{\partial^2 T}{\partial x^2} = +k\frac{\partial^2 T}{\partial x^2}}$$

k ... thermal diffusivity
κ ... thermal conductivity
ρ ... *density (mass per unit volume)*
c_p ... heat capacity (energy needed to change temperature by 1K per unit mass)