Example 2: portfolio modeling

Suppose that you're trying to construct a portfolio: that is, to decide how to allocate your wealth among D financial assets. Things you want might to track include:

- the expected value of your portfolio at some point in the future (e.g. when you retire).
- the variance of your portfolio's value at some point in the future.
- the probability of losing some specific amount of money (10K, 20% of total value, etc)
- some measure of "tail risk," i.e. what a bad week/month/year might look like.

Key idea: **use the bootstrap to simulate portfolio performance.**

Example 2: portfolio modeling

Notation:

- Let T be our investing horizon (e.g. T = 20 days, T = 40 years, etc), and let t index discrete time steps along the way.

- Let $X_{t,j}$ be the value of asset $j = 1, \ldots, D$ at time period t.

- Let $R_{t,j}$ be the *return* of asset j in period t, so that we have the following recursive update:

$$X_{t,j} = X_{t-1,j} \cdot (1 + R_{t,j})$$

Example 2: portfolio modeling

Notation:

- A portfolio is a set of investment weights over assets: $(w_{t1}, w_{t2}, \ldots, w_{tD})$. Note: these weights might be fixed, or they might change over time.

- The value of your portfolio is the weighted sum of the value of your assets:

$$W_t = \sum_{j=1}^{D} w_{t,j} X_{t,j}$$

Example 2: portfolio modeling

We care about W_T: the random variable describing your terminal wealth after T investment periods.

Problem: this random variable is a super-complicated, nonlinear function of $T \times D$ individual asset returns:

$$W_T = f(R) \quad \text{where} \quad R = \{R_{t,j} : t = 1, \ldots, T; j = 1, \ldots, D\}$$

Example 2: portfolio modeling

If we knew the asset returns, we could evaluate this function recursively, starting with initial wealth W_0 at time $t = 0$ and sweeping through time $t = T$:

Starting with initial wealth $X^{(i)}_{1,j}$ in each asset, we sweep through from $t = 1$ to $t = T$:

$$
\begin{aligned}
X^{(f)}_{t,j} &= X^{(i)}_{t,j} \cdot (1 + R_{t,j}) && \text{(Update each asset)} \\
W_t &= \sum_{j=1}^{D} w_{t,j} X^{(f)}_{t,j} && \text{(Sum over assets)} \\
X^{(i)}_{t+1,j} &= w_{t+1,j} \cdot W_t && \text{(Rebalance)}
\end{aligned}
$$

Example 2: portfolio modeling

But of course, we don't know the asset returns! This suggests that we should use a Monte Carlo simulation, where we repeat the following `for` loop many times.

For $t = 1, \ldots, T$:

1. Simulate $R_t = (R_{t1}, R_{t2}, \ldots, R_{tD})$ from the joint probability distribution of asset returns at time t.
2. Use these returns to update $X_{j,t}$, the value of your holdings in each asset at step t.
3. Rebalance your portfolio to the target allocation.

The precise math of the update and rebalance steps are on the previous slide.

Example 2: portfolio modeling

The difficult step here is (1): simulate a high-dimensional vector of asset returns from its joint probability distribution.

- very complicated correlation structure
- probably not something simple like a Gaussian!

In general, using simple parametric probability models (e.g. multivariate Gaussian) to describe high-dimensional joint distributions is a very dicey proposition.

A simple approach: bootstrap resampling

Suppose we have M past samples of the asset returns, stacked in a matrix:

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1D} \\ R_{21} & R_{22} & \cdots & R_{2D} \\ \vdots & & & \\ R_{M1} & R_{M2} & \cdots & R_{MD} \end{pmatrix}$$

where R_{tj} is the return of asset j in period t.

A simple approach: bootstrap resampling

The key idea of bootstrap resampling is the following:

- We may not be able to describe what the joint distribution $P(R_1, \ldots, R_D)$ is.
- But *we do know that every row of this R matrix is a sample from this joint distribution.*
- So instead of sampling from some theoretical joint distribution, we will sample from the sample—i.e. we will bootstrap the past data.
- Thus every time we need a new draw from the joint distribution $P(R_1, \ldots, R_D)$, we randomly sample (with replacement) a single row of R.

A simple approach: bootstrap resampling

Thus our Monte Carlo simulation looks like the following at each draw.

For $t = 1, \dots, T$:

1. Simulate $R_t = (R_{t1}, R_{t2}, \dots, R_{tD})$ by drawing a whole row, with replacement, from our matrix of past returns.
2. Use these returns to update $X_{j,t}$, the value of your holdings in each asset at step t.
3. Rebalance your portfolio to the target allocation.

Example

Let's go to the R code! See `portfolio.R` on the website.

Key discussion question

Why do we draw an entire row of $\mathbf{R}$ at a time?