3Y03

let $\mathbb{N}^0 \equiv \{0\} \cup \mathbb{N}$

Week 1 | Events and Manipulating Events

- Stats is the science of *collecting, analyzing, and inferring* from **data**
- **Probability** = mathematics of random events, intimately related to statistics
- **Experiment** = anything that produces **data**, while a ***random experiment*** is an experiement that can produce *different* outcomes from the same process
- **Sample Space** $:= S$ = the set of all outcomes of a **random experiment**
  - **Discrete** iff *finite* or *countably infinite*
  - **Continuous** iff *infinite*
- **Event** $:= E \subseteq S$
- An *event* E, is a subset of the sample space *S* , where E is a set of outcomes
  - *E = {HHH, HHT, HTH, THH} ⊆⊆ {HHH, HHT, ...} = S*
- Given some events, new events can be defined:
- Given E, E_1, $E_2 \subseteq S$
  - **Union**: $E_1 \cup E_2 := \{x \in S : x \in E_1 \vee x \in E_2\}$
- **Intersection**: $E_1 \cap E_2 := \{x \in S : x \in E_1 \wedge x \in E_2\}$
  - **Complement:** $E' = \{x \in S : x \notin E\}$

If $E \subseteq S$ is any event, then $E \cup E' = S$ and $E \cap E' = \emptyset$

S and $\emptyset$ are events, and $S' = \emptyset$, given sample space S

- **Mutualy Exclusivity =** Given $E_1, E_2, \subseteq S$, two events are **mutually exclusive** if:
  - $E_1 \cap E_2 = \emptyset$
  - Inutition is that events cannot happen simultaneously, which using the coin exmaple with space $S = \{H, T\}$ would be the events $E_1 = \{H\}$ and $E_2 = \{T\}$
- Useful rules for **manipulating events** algebraically using events $A, B, C, \subseteq S$
- (A')' = A
- **Distributivity**:
  - (A ∪∪ B) ∩∩ C = (A ∩∩ C) ∪∪ (B ∩∩ C)
  - (A ∩∩ B) ∪∪ C = (A ∪∪ C) ∩∩ (B ∪∪ C)
- **DeMorgan's Laws:**
  - (A ∪∪ B)' = A' ∩∩ B'
  - (A ∩∩ B)' = A' ∪∪ B'

Counting Techniques

- **Basic Counting Principle**:
  - Suppose we have *r-many* experiments, and suppose the *ith* experiment has n_i possible outcomes
  - The ***total number of outcomes*** from running all the experiments ***consecutively*** is

$$\prod_{i=1}^{r} n_i = n_1 * n_2 \ldots n_{r-1} n_r$$

- **ex. coin flip:** if we toss a coin **3 times**, and each coin toss has **2 outcomes**, the the total number of outcomes if $\prod_{i=1}^{3} 2 = 2 * 2 * 2 = 2^3 = 8$

Permuatations

- **Permutations**: $n! = n(n-1)(n-2) \ldots 3 * 2 * 1$
  - Given *n* **distinct** objects, the number of ways to permute them is *n* **factorial**
  - In general, the formula for a set of n distinct objects, the number of ways to permute $r \leq n$ of them is:

    $$P_r^n = nPr = n(n-1) \ldots . (n-r+1) = \frac{n!}{(n-r)!}$$

    - r = length of sequence, n = size of set to draw from.
- Ex. number of 3 letter words with no repeats: $P_3^{26} = \frac{26!}{(26-3)!}$ = $26 * 25 * 24$
- Not all objects may be unique, ex. "**BANANA**"
- If all the letters were unique, there would be $6!$ perumtations, but since there are 3 As, and 2 Ns, some of the permutations leave the word unchanged
- There are 3! ways to permute the A's and 2! ways to permute the Ns. If we "cancel" the permuations that do nothing, we have $\frac{6!}{3!2!}$ = 60 many unique permutations
- Given $n = n_1 + n_2 + \ldots + n_r$ many objects with n_i identical many objects of type i, there are $\frac{n!}{n_1!n_2! \ldots n_r!}$

Combinations

Given a group of n distinct objects, the number of ways to choose $r \leq n$ of them is

$$C_r^n = nCr = (\frac{n}{r}) = \frac{n(n-1)..(n-r+1)}{(n-r)!} = \frac{n!}{(n-r)!r!}$$

n "choose" r

- **Binomial Coefficients** = $(\frac{n}{r})$
  - Appear in many places, such as the binomial theorem

    $$\forall x, y: \ (x+y)^n = \sum_{r=0}^{n} (\frac{n}{r})$$

  - read as *n choose r*, or how many ways are there to choose r items from n elements

Week 2 | Interpretations and Axioms of Probability

- Probabliity is used to quantify the likelihood, or chance that an outcome of a **random experiment** will occur
  - interpreted as the *limiting value* of the proportion of times the outcome occurs in *n* repetitions of the random experiment as n increases beyond all bounds
- **Equally Likely Outcomes**

    - Whenever a **sample space** consists of N possible outcomes that *are equally likely*, the probability of each outcome is $\frac{1}{N}$
- **Probability of an Event**
    - For a **discrete sample space**, the probability of an event **E**, denoted at P(E), equals the *sum* of the probabilities of the outcomes in **E**

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment, that satisfies the following:

- $P(S) = 1$, where S is the **sample space**
- $0 \leq P(E) \leq 1$ *for any event*
- For two events, E_1 and E_2, with $E_1 \cap E_2 = \emptyset$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

The property that $0 \leq P(E) \leq 1$ is equivalent to the requirement that a relative frequency must be between 0 and 1

$P(\emptyset) = 0$

$P(E') = 1 - P(E)$

$P(E_1) \leq P(E_2) \iff E_1 \subseteq E_2$

Unions of Events and Addition Rules

Joint Events are generated by applying basic set operations at individual levels:

Probability of a Union

if A and B are **mutually exclusive**, then their intersection is ***zero***

- For 2 sets:
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- For 3 or more sets:
    - $P(A \cup B \cup C) = P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$
    - $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Mutual Exclusion

- **Mutual Exclusive** - for all pairs $E_i \cap E_j = \emptyset$, for $E_i, E_j \subseteq S$
    - $P(E_i \cap E_j) = P(E_i) + P(E_j)$

Intersection of Events and Mulitplication Total Probability Rules

Total Probability Rule (2 Events)

For any events, A and B

$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$

Total Probability Rule (Multiple Events)

Assume $E_1, E_2, \ldots, E_k$ are *k* mutually exclusive

$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k)$

Random Variables

- **Random Variable =** function that assigns a real number to each outcome in the sample space of a random experiement
    - denoted in capital, like X
    - After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as x

      Discrete random variable = random variable with finite (or countably infinite) range

      Continuous Random Variable = random variable with an interval (either finite or infinite) of real numbers for its range

Probability Distributions and Probability Mass Functions

- **Probability Distribution** = description of the probabilities associated with the possible values of X.
    - For a **discrete** random variable, the distribution is often specified by just a list of the possible values along with the probabilities of each
- **Probability Mass Function** = function that gives the probability that a discrete random variable is exactly equal to some value. Given:
    - $f(x_i) \geq 0$
    - $\displaystyle\sum_{i=1}^{n} f(x_i) = 1$
    - $f(x_i) = P(X = x_i)$

Week 3

Cumulative Distribution Functions

- **Cumulative Distribution Functions** = is the probability that X will take a value less than or equal to x
    - $F : \mathbb{R} \rightarrow [0, 1], \forall x \in \mathbb{R}$
    - $F(x) = P(X \leq x)$
    - Suppose that X is a DRV with range $\{x_1, x_2, \ldots\}$. pmf $f(x)$ and cdf $F(x)$. Then $F(x)$ satisfies
        - $\forall x \in \mathbb{R}, F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
        - $\forall x \in \mathbb{R}, 0 \leq F(x) \leq 1$
        - $\forall x, y \in \mathbb{R}, \text{if } x \leq y, \text{then } F(x) \leq F(y)$
    - NOTE: if $f(x)$ and $F(X)$ are the ***pmf*** and ***cdf*** of X, they are "inter-defineable":

      $$F(x_n) = \sum_{i=0}^{n} f(x_i)$$

      $$f(x_n) = F(x_n) - F(x_{n-1})$$

Mean and Variance of a Discrete Random Variable

- **Mean (Expected Value)** = is the average outcome. Weighted sum of all outcomes with their probabilities

  $$\mu = E[X] := \sum_{i=1}^{n} x_i f(x_i)$$

- **Variance** = Average distance from the mean

  $$\sigma^2 = V[X] := E[(X-u)^2] = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$$

  - Other formula which may be useful:

    $$\begin{aligned} V[X] &= \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2 \\ &= E(X^2) - E(X)^2 \end{aligned}$$

Binomial Distribution

- **Bernouli Trial** = experiment with *two possible outcomes*; failure or success
  - p - probability trial is successful
  - $(1-p)$ - probability trial fails
- **Binomial Distribution** = written as $X \sim \text{Bin}(n, p)$, it is experiment where a **Bernouli Trial** is run ***n***-times with fixed probability p of success
  - n - number of trials run
  - p - fixed probability of success
  - trials occur ***independently***
  - X - the number of successful outcomes in the ***n***-many **independent** Bernouli trials
- **Probability Mass Function**:
  - When $X \sim \text{Bin}(n, p)$

    $$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

    - X - $x \in (1, n)$
- **Mean** = $E[X] = \mu = np$
- **Variance** = $\text{Var}(X) = \sigma^2 = np(1-p)$

Geometric Distribution

- **Geometric distribution** = The probability distribution of the number *X* of [Bernoulli trials] needed to get one success, supported on the set { 1, 2, 3, ... }
  - X $\sim \text{G}(p)$
  - X counts the number of succesful Bernoulli trials in a set of *n* independent trials
  - p - probability of Bernoulli trial success
  - ***memoryless property*** = $P\{X > m + n | X \geq m\} = P\{X > n\}$
- **probability mass function:** $f(x) = (1-p)^{(x-1)} p$
  - $f(n)$ is the *probability* that there are $n-1$ many failures(probability $(1-p)$) followed by a success (probability p)
- **Mean** $= E[X] = \mu = \dfrac{1}{p}$
- **Variance** $= \text{Var}(X) = \dfrac{1-p}{p^2}$

- The geometric distribution is an appropriate model if the following assumptions are true:
  - The phenomenon being modeled is a sequence of independent trials.
  - There are only two possible outcomes for each trial, often designated success or failure.
  - The probability of success, p, is the same for every trial.

The Inverse (Negative) Binomial Distribution

- **Inverse Binomial Distribution** = Generalization of the geometric distribution
  - Suppose we have bernoulli trial with probability p of success, and let $r \geq 1$ be an integer
  - Let X be the number of trails we need to run until we reach $r - \text{many}$ successful trials
  - $X \sim \text{Bin}^{-1}(p, r)$
    - Observe that a ***geometric random variable*** with parameter p is precisely an inverse ***binomial random variable*** with parameters p and $r = 1$
    - In otherwords, $X \sim \text{Geo}(p) \equiv X \sim \text{Bin}^{-1}(p, 1)$
    - If X_i is the number of trials required to get the $i\text{th}$ success, then X_i is **geometric** and $X = X_1 + X_2 + X_3 + ... + X_r$
- **Probability Mass Function(p.m.f)** $= f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$
- **Mean** $= E[X] = \mu = \frac{r}{p}$
- **Variance** $= \text{Var}(X) = \sigma^2 = \frac{r(1-p)}{p^2}$

Week 4

Hypergeometric Distribution

- **Hypergeometric Distribution =** the distribution of a random variable X that represents the number of successes selecting a correct object from a sample of size n, given there are K correct objects of the N total objects.
  - $X \sim \text{Hypergeomeric}(N, K, n)$
  - N - length of the set of objects
  - K - number of objects classified as success, conversely there are $(N - K)$ classified as failures
  - n - size of sample drawn from the set of N objects
  - $p = \frac{K}{N}$
  - $K \leq N$ and $n \leq N$
- **Probability Mass Function (p.m.f)** $= f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$
  - $x = \max\{0, n + K - N\} \text{ to } \min\{K, n\}$
- **Mean** $= E[X] = \mu = np$
- **Variance** $= \text{Var}(X) = \sigma^2 = np(1-p)(\frac{N-n}{N-1})$

Poisson Distribution

- **Poisson Distribution** = describes the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known ***constant mean rate*** (λ) and ***independently*** of the time since the last event.
  - $X \sim \text{Pois}(\lambda)$
  - X - random variable that equals the number of events in a Poisson process
  - λ - average rate by which the independent events of the Poisson process occur
    - $0 < \lambda$
  - T - length of the interval X spans
  -
- **Probability Mass Function (p.m.f)** = $f(x) = \frac{e^{-\lambda T}(\lambda T)^x}{x!}, x \in \mathbb{N}^0$
- **Mean** = $E[X] = \mu = \lambda T$
- **Variance =** $= \text{Var}(X) = \sigma^2 = \lambda T \equiv E[X]$

Continuous Random Variables

- **Continuous Random Variable** = a random variable with an interval (either *finite* or *infinite*) of real numbers for its range

Probability Density Functions

- For a *continuous random variable* X, a ***probability density function*** **(p.m.f)** is a function such that:
  - $f(x) \geq 0 \; \forall \; x \in X$ - a continuous random variable has a probability at every step, there are no holes in the graph
  - $\int_{-\infty}^{\infty} f(x)dx = 1$ - The sum of all the values of the probability density function is one. All the probabilities sum to unity
  - $P\{a \leq X \leq b\} = \int_a^b f(x)dx =$ area under $f(x)$ from a to $b \; \forall \; a, b$
- Property of *continuous random variable*: boundary edges are all equivalent.
  - $P\{a \leq X \leq b\} = P\{a < X \leq b\} = P\{a \leq X < b\} = P\{a < X < b\}, \; \forall \; a, b \in X$
- **Histogram** = an *approximation* of a **probability density function**

Cumulative Distribution Function

- **Cumulative Distribution Function** = function of a continuous random variable X, when evaluated at x, is the *probability* that X will take a value ***less than or equal*** to x.

  $$F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(u)\mathrm{d}u \;, \forall \; x \in \mathbb{R}$$

  - Sums the probability of X taking each infinitesmal value $f(x) \; \forall \; x \in (-\infty, x)$, i.e. the probability X takes a value less than or equal to x, or the sum of the area under $F(x)$ up until x
  - $f(x) = \frac{dF(x)}{dx}$ - consequently, the **p.d.f** can be determined from the **c.d.f** by differentiating it, so long as the derivative exists

Mean and Variance

- **Mean** = $E[X] = \mu = \int_{-\infty}^{\infty} xf(x)dx$ - the probability of an event occuring, times its value summed over the interval of the variable yeilds its average. Weighted sum of events
- **Variance** = $\text{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Expected Value of *Function* of a Continuous Variable

- If X is a continuous random variable with p.d.f $f(x)$:

  let $g(X)$ be a function of a continuouse random variable X

  $$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

  - Recall: the expected value or *mean* is the weighted sum of the value of $x \in X$, times the probability of X taking the value x $f(x)$. Therefore, if $g(X)$ is a function of X, then its mean can be expressed as the weighted sum of the functions value for every value x in X, $g(x)$, times the probability of X taking that value x, which is $f(x)$. As its continuous, this sum is taken as the integral over the range of X

Distributions of Continous Random Variables

Continuous Uniform Distribution

- **Continuous Uniform Distribution** = distribution of a *continuous random variable* X with *p.m.f*

  $$f(x) = \frac{1}{b-a},\ a \leq x \leq b, 0 \text{ otherwise}$$

  - i.e. every value in X has equal probability of occuring
  - $b - a$ = range of the interval of X
    - in essence, $f(x) = |X|^{-1}$
- **Mean** = $E[X] = \mu = \int_a^b \frac{x}{b-a}dx = \frac{a+b}{2}$
  - in otherwords, $E[X] = \frac{\max(X) + \min(X)}{2}$
- **Variance** = $\text{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} \frac{(x - \frac{a+b}{2})^2}{b-a}dx = \frac{(b-a)^2}{12}$
- **Cumulative Distribution Function** = $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$
  - Outside of the interval (a, b) the **p.d.f** is 0, therefore 0 until a, and is 1 only once b is reached, and remains 1 thereafter

Normal Distribution

- **Normal (Gaussian) Distribution =** a distribution that represents the distribution of many random variables as symmetric about its mean, with the distribution more frequent around the mean
  - $X \sim \mathcal{N}(\mu, \sigma^2)$

- Normal distribution is a function of ***variance*** and ***mean***

- **Probability density function (p.d.f)** = $f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $x \in (-\infty, \infty)$, $\sigma > 0$
- **Mean** $= E[X] = \mu$
- **Variance** $= \sigma^2$

For any normal random variable,

$$
\begin{aligned}
&P(\mu - \sigma < X < \mu + \sigma) = 0.6827 \\
&P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545 \\
&P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973 \\
&P(X < \mu) = P(X > \mu) = 0.5.
\end{aligned}
$$

Standard Normal Distribution

- **Standard normal variable** = a *normal random variable* with $\mu = 0$, $\sigma^2 = 1$, and is denoted Z
  - **Cumulative Distribution Function (c.d.f)** = $\phi(z) = P(Z \leq z)$

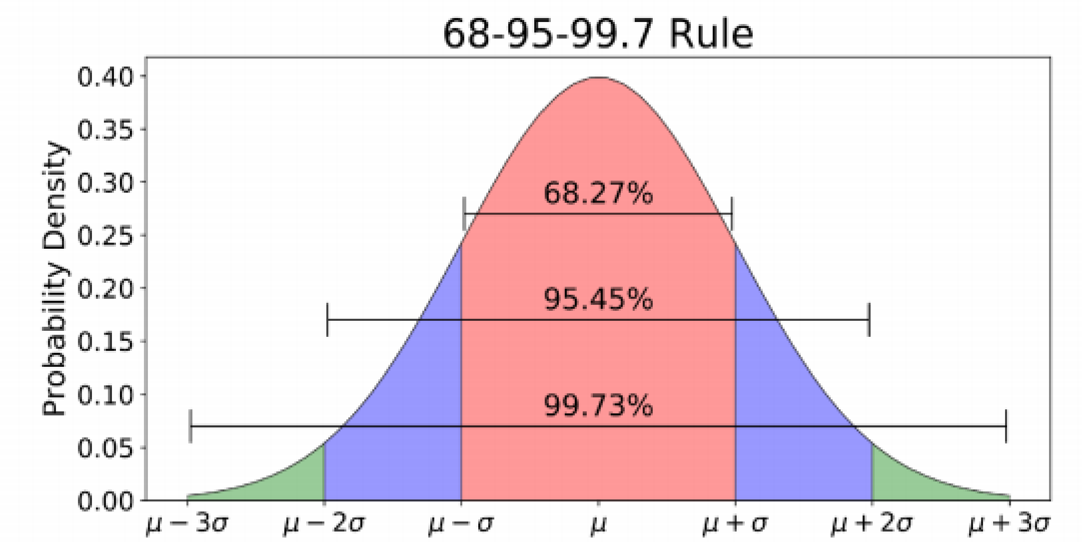
Standardizing a Normal Random Variable

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma}$ is a ***normal random variable*** with $E[Z] = 0$ and $\mathrm{Var}(Z) = 1$
  - **Mean:** 0
  - **Variance:** 1
  - Subtracts mean from every possible value of X, then divides by the standard deviation. Centres the mean around 0, and normalizes by the standard deviation to set variance to 1.

Standardizing to Calculate Probability

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then the probability of an event in X can be expressed in terms of Z by:
  $P\{X \leq x\} = P\{\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\} = P(Z \leq z)$
  - Z - standard normal random variable
  - z - $z = \frac{x - \mu}{\sigma}$ is the value obtained by standardizing X

Here is is the density function of a normal distribution with mean μ and variance σ^2. We can see that an outcome occurs within three standard deviations of the mean with probability over 99%:



Week 5

Normal Approximation to the Binomial Distribution

- Binomial distribution is roughly *bell-shaped*, and can be approximated with the normal distribution
  - binomial random variables are ***discrete***, and normal random variables are ***continuous*** but special circumstances allow for approximation
  - Binomial distribution can have expensive computations and modelling as a normal distribution can reduce this
- Suppose $X \sim \text{Bin}(n, p)$ is the binomial random
  - $\sigma^2 = np(1-p)$
  - $\mu = np$
  - if n is "*large enough*", then $x \approx \mathcal{N}(\mu, \sigma^2) \approx \mathcal{N}(np, np(1-p))$
  - Approximation usually good when $np > 5$ ***and*** $n(1-p) > 5$
    - *this is what we mean by "n sufficiently large"*
- From above, if n *sufficiently large* :

$$\frac{X - np}{\sqrt{np(1-p)}} \approx Z = (0, 1)$$

Continuity Correction

Fact (Continuity Correction)

Let X be binomial with parameters n and p. If n is sufficiently large then, for $x \in \mathbb{R}$:

$$P(X \leq x) = P(X \leq x + 0.5) \approx P\left(Z \leq \frac{x - np - 0.5}{\sqrt{np(1-p)}}\right),$$

where the first equality holds because X is discrete.

Exponential Distribution

- **Exponential Distribution** = continuous probability distribution of the time between events in a Poission point process
  - ***memory-less*** = $P\{X > t+s | X > t\} = P\{X > s\}$
    - If t time has elapsed, you can just examine s
    - Continuous analogue of ***geometric distribution***
    - Rather than counting trials until the first "success", for example, we may be marking time until the arrival of the first job to a queue.
  - Events occur *continuously* and *independently* at a constant average rate λ
- Consider an interval $[0, x], x \in \mathbb{R}$
  - Suppose that on $[0, x]$ there is some event that occurs according to $\text{Pois}(\lambda)$
  - X is then defined to be the distance in $[0, x]$ from zero to the first event
    - X is an ***exponential random variable***
  - Consider $P\{X > x\}$, the probability that *no events* occur in interval $[0, x]$
    - Since we assumed the events are $\sim \text{Pois}(\lambda)$

$$P\{X > x\} = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

$$P\{X \leq x\} = 1 - P\{X > x\} = 1 - e^{-\lambda x}$$

$$\therefore X \sim \exp(\lambda) \implies F(x) = 1 - e^{-\lambda x}$$

$$f(x) = F'(x) \implies f(x) = \lambda e^{-\lambda x}$$

- **Cumulative Distribution Function (c.d.f)** = $F(x) = 1 - e^{-\lambda x}$
- **Probability Density Function (p.m.f)** = $f(x) = \lambda e^{-\lambda x}$
- **Mean** = $\mu = E[X] = \lambda^{-1} = \dfrac{1}{\lambda}$
- **Variance** = $\sigma^2 = \lambda^{-2} = \dfrac{1}{\lambda} = \mu^2$
- **Standard Deviation** = $\sigma = \dfrac{1}{\lambda} = \mu$
  - It is interesting to note that the standard deviation is equal to the mean

Joint Probability Distributions

Let X and Y be ***continuous random variables***

- **Joint Probability Distribution** = is a probability distribution, that gives the probability of each $X, Y \ldots$, falls in any particular range specified for that variable
  - ***bi-variate or multi-variate distributions***
  - **(Bi-variate) Joint Probability Density function**: $f_{X,Y}(x, y)$ satisfies:
    - $f_{X,Y}(x, y) \geq 0 \; \forall \; x, y \in \mathbb{R}$
    - $\displaystyle\int\int_{\mathbb{R}^2} f_{X,Y} dx dy = 1$
    - Any region A is a subset of $\mathbb{R}^2$, $A \subseteq \mathbb{R}^2$

$$P\{(X, Y) \in A\} = \int\int_A f_{X,Y}(x, y) dx dy$$

Marginal Probability Density Functions

Let X and Y be *continuous random variables* defined in a region $R \subseteq \mathbb{R}^2$

Assumed $f_{X,Y}(x, y)$ is 0 outside of R

- Suppose that we are given a ***joint pdf*** $f_{x,y}(x, y)$ such that
  - $\int\int_{\mathbb{R}^2} f_{X,Y} dxdy = 1$
- **Marginal Probability Density Function** = seperation of a ***joint pdf*** to describe the distributions of its component random variables

  for $a, b \in \mathbb{R}$

  $$R_X(a) := \{y \in \mathbb{R} : (a, y) \in R\}$$
  $$R_Y(b) := x \in \mathbb{R} : (x, b) \in R$$

  then the ***marginal probability densities are***:

  $$f_X(x) = \int_{R_X(x)} f_{X,Y}(x, y)dy$$
  $$f_Y(y) = \int_{R_Y(y)} f_{X,Y}(x, y)dy$$

- **Mean** = $\mu_X = E(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \iint_{\mathbb{R}^2} xf_{X,Y}(x, y)dxdy$
- **Variance** = The variance of X and Y can be computed from the usual formula: $V(X) = E((X - \mu_X)^2) = E(X^2) - E(X)^2$.

Week 7

Independent Random Variables

- **Independence:** A and B are independent events if $P(A \cap B) = P(A)P(B)$
- **Independence for Continuos Random Variables**: For CRVs X and Y with ***joint pdf*** $f_{X,Y}(x, y)$

  $$X \perp\!\!\!\perp Y \iff f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

  where $f_X(x)$ and $f_Y(y)$ are the ***marginal density functions*** of X and Y respectively
  - If X and Y are *independent* and *jointly distributed* on a region $R \subseteq \mathbb{R}^2$ *then R must be **rectangular***
    - i.e. $R = l_1 * l_2$ for intervals l_1 and l_2 (the converse is false!)
    - $1 = \int\int_{l_1 * l_2} f_{X,Y}(x, y)dxdy = (\int_{l_1} f_X(x)dx)(\int_{l2} f_Y(y)dy)$
      - Since the sum of each respective marginal density function over its over interval should be 1 if they're independent, $1 * 1 = 1$

Covariance

- **Covariance** = a measure of the joint variability of two random variables
  - Suppose X and Y are jointly distributed with joint pdf $f_{X,Y}(x, y)$

    for any *nice* function $h(x, y)$, $h(X, Y)$ is a random variable, and we have the expected value of $h(X, Y)$:

$$E[h(X,Y)] = \int\int_{\mathbb{R}^2} h(x,y) f_{X,Y}(x,y)dxdy$$

- Measure of the ***linear relationship*** between X and $Y

- **Covariance =** $\sigma_{XY} := E[XY] - E[X]E[Y]$
  - When $X \perp\!\!\!\perp Y$:

    $$\begin{aligned} E[XY] &= \int\int_{\mathbb{R}^2} xyf_{X,Y}(x,y)dxdy \\ &= (\int_{\mathbb{R}} xf_X(x)dx)(\int_{\mathbb{R}} yf_Y(y)dy) \\ &= E[X]E[Y] \end{aligned}$$

    $$X \perp\!\!\!\perp Y \implies \sigma_{XY} = 0$$

- Note: $\sigma_{XY} = 0 \not\Longrightarrow X \perp\!\!\!\perp Y$

Correlation

- **Correlation =** degree to which a pair of variables are linearly related
  - $\rho_{XY} = \dfrac{\text{cov(C,Y)}}{\sqrt{V(X)V(Y)}} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$
  - $-1 \leq \rho_{XY} \leq 1$
    - If the points in the joint probability distribution of X and Y that recieve positive probability tend to fall along a line of positive (or negative) slope, ρ_{XY} is near $+1$ or (1)
  - $\rho_{XY} \geq 0 \implies X$ and Y are ***correlated***
  - $X \perp\!\!\!\perp Y \implies \rho_{XY} = \sigma_{XY} = 0$

Linear Functions of Random Variables

- **Linear Function**: given random Variables $X_1, .. X_p$ and constants $c_0, c_1, .. c_p$, a **linear function** of $X_1 ... X_p$ is defined:

  $$Y = c + 0 + c_1 X_1 + ... + c_p X_p$$

  - **Mean:** $E[Y] = c_0 + c_1 E[X_1] + ... + c_p E[X_p]$
  - **Variance:** $\text{Var[Y]} = c_1^2 \text{Var}[X_1] + ... + c_p^2 \text{Var}(X_p) + 2 \sum \sum_{i<j} c_i c_j \text{cov}(X_i, X_j)$

Mean and Variance of an Average

- If $\bar{X} = \dfrac{(X_1 + ... + X_p)}{p}$ with $E[X_i] = \mu$
  - **Mean:** $E[\bar{X}] = \mu$
  - **Variance:** $V[\bar{X}] = \dfrac{\sigma^2}{p}$
    - if all $X \in \bar{X}$ are independent

Reproductive Property of the Normal Distribution

- If $X_1, X_2, ... X_p$ are independent, normal random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ for $i = 1, 2, .., p$ then:

  $$Y = c + 0 + c_1 X_1 + c_2 X_2 + ... + c_p X_p$$

  is a *normal random variable* with

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \ldots + c_p\mu_p$$

and

$$V(Y) = c_1^2\sigma_1^2 + \ldots + c_1^2\sigma_1^2$$

- This is to say, if Y is a ***linear combination*** of other random variables, its **mean** and **variance** are related in the same way

Numerical Summaries of Data

Sample Statistics

- **Sample Mean** = $\bar{X} = \dfrac{\sum_{i=1}^{n} x_i}{n}$
  - The average of all the values in a sample of a random variable
  - ***average value of all oversations***
- **Sample Variances** = $s^2 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n}$
  - **Sample Standard Deviations** = s , the positive square root of sample vairance
- **Sample Range** = $r = \max(\mathbf{X}) - \min(\mathbf{X})$

Stem and Leaf Diagrams

- **Steam and Leaf Diagram** = visual display of a data set where each x_i consists of at least 2 digits
  - Steps:
    1. Divide each number x_i into two parts: a **stem** consisting of one or more leading digits, and a **leaf** consisting of the remaining digit
    2. List the **stem values** in a vertical column
    3. Record the **leaf** for each observation beside its stem
    4. Write the units for stems and leaves on the display

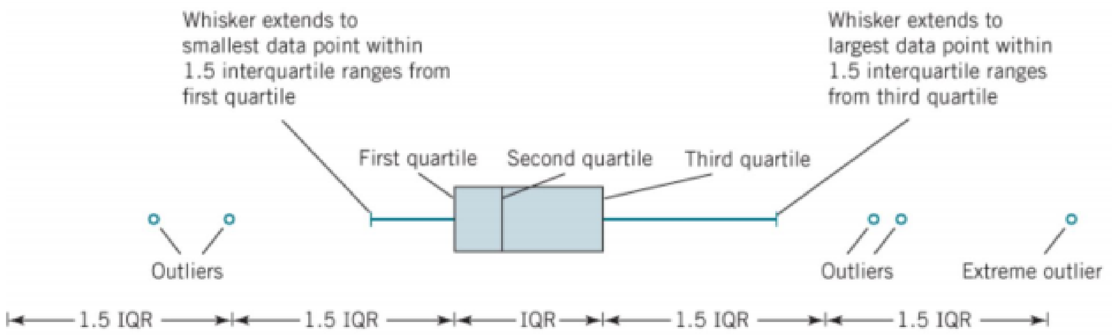
Frequency Distributions and Histograms

- **Frequency Distribution** = compact summary of data , with the range divided into intervals called class intervals, cells, or bins
  - Choose the number of bins approximately equal to the square root of the number of observations
  - # bins = $\sqrt{(N)}$
  - Relative frequencies are found by dividing the observed frequency in each bin by the total number of observation
    - Provides the % of the population the class represents
- **Histogram** = visual display of the frequency distribution
  - Steps
    1. Label the bin boundaries on a horizontal scale
    2. Mark and label the vertical scale with the frequencies or the relative frequencies
    3. Above each bin, draw a rectangle where height is equal to the frequency corresponding to that bin

$$\text{Rectangle height} = \frac{\text{Bin frequency}}{\text{Din Width}}$$

Box Plots

- **Box Plot** = Graphical display that simultaneously describes several important features of a data set, such as **centre, spread, departure from symmetry,** and **outliers**



Week 8

Probability Plots

- **Probability Plots** = empirical way to determine if data fits a particular distribution
  - Suppose $S = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ is some sample data
  - Imagine we have a random variable X, and we want to know if the distribution of X fits the observed data
  - Let $F(X)$ be the **cdf** of the function
  1. Arrange the data points in increasing order and rename them if necessary
  2. For each $1 \leq i \leq n$, choose a value $y_i \in \mathbb{R}$ such that
     1. $F(y_i) = P(X \leq y_i) = \dfrac{i - 0.5}{n}$
  3. Plot the pairs (x_i, y_i)
  4. Draw a line of best fit

  Conclusion: If all the points lay on or near the line, we can conclude that the distribution of X fits the data well, otherwise the fit is not so good

  Most of the time we will be concerned of discovering if our data fits the **standard normal distribution** $F(z) = \phi(z)$

Point Estimation of Parameters

- Make prediction based on limited data
- **Parameter Estimation** = a parameter, θ is any **numerical feature** of a **population**
  - ex. we may want to estimate the **mean** or the **variance**
  - Given a particular paramter, an estimator for θ is a **sample statistic** $\hat{\Theta} = h(X_1, \ldots, X_n)$ which we want to use to estimate θ
  - If $\hat{\Theta} = h(X_1, \ldots, X_n)$ is an estimator for θ and $(x_1, x_2, \ldots, x_n)$ is some data, then the number $\hat{\theta} = h(x_1, \ldots, x_n)$ is called a **point estimate for** θ

Sample Distributions & the Central Limit Theorem

- If $Y = h(X_1, ..., X_n)$ is a ***statistic*** then Y is also a random variable
- If the distribution associated to such a statistic is called a ***sampling distribution***:
  - ex. if X is a random variable and $X_1, ..., X_n$ are all independent, with the same distribution as X, then the distribution of $\bar{X} = \frac{X_1 + ... + X_n}{n}$
  - A **sampling distribution** depends on many factors, including the **sample size**, the **sample method**, the distribution of X, etc.

Central Limit Theorem

Suppose $X_1, ... X_{n+1}$ is a random sample (so *i.i.d*) taken from a distribtuion with **mean** μ and **variance** σ^2. IF:

$$\bar{X_n} = \frac{X_1 + ... + X_n}{n}$$

then

$$\frac{\bar{X_n} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

converges in probability to $N(0, 1)$, meaning

$$\lim_{n \to \infty} P(\frac{\bar{X_n} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq z) = \phi(z)$$

- For any sequence of *i.i.d* random variables $X_1, ... X_n$ (*discrete* or *continuous*) if *n* is large enough we have

  $\bar{X_n} = \frac{X_1 + ... + X_n}{n}$ ~= $N(\mu, \frac{\sigma^2}{n})$

  and so we may use the normal distributions in our calculations
- In practice, taking $n \geq 30$ is often large enough to get a good approximation
- If the distribution of X_i is nice enough (eg. symmetric, unimodal) then $\bar{X_n}$ is often approximately normal for $n \geq 5$

Unbiased Estimators

- Let θ be a parameter for some population
- Let $\hat{\Theta}$ be an estimator for θ
  - ***bias of*** $\hat{\Theta}$ = $E(\hat{\Theta}) - \theta$
  - ***bias*** = measure of how far away an estimator is from being correct ***on average***
  - For any random sample, the average $\bar{X}$ is an ***unbiased estimator*** of the mean μ of $X = X_i$
    - sample viaraince is also an ***unbiased estimator*** of variance
  - However, sample standard deviation *is not* ***unbiased***\
    - $E(S) - \sigma$ gets small as the sample gets large, and so S is still a pretty good estimator

Variance of Estimators

- For a given parameter, θ , there is not *necessarily* a unique unbiased estimator of θ
    - ex. both **sample mean** and the **median** are unbiased estimators of the **mean**
    - How should we pick the "best" estimator of a given parameter?
- Given $\hat{\Theta}$ is *itself* a **random variable**, we can take the ***variance*** as a measure of the probability *mass* of the estimator away from $E(\hat{\Theta}) = \theta$
    - if $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are two unbiased estimators of θ, and $V(\hat{\Theta}_1) < V(\hat{\Theta}_2)$ then (*on average*) a **point estimate** for θ via $\hat{\Theta}_1$ *will be closer* to θ than from $\hat{\Theta}_2$
    - **to choose estimator: *minimize the variance***

Definition

Let θ be a parameter. Then an estimator of θ, $\widehat{\Theta}$, with minimal variance is called the minimum variance unibiased estimator (MVUE) of θ.

- For example: if $\{X_1, \ldots, X_n\}$ is a random sample of a distribution with mean μ and variance σ^2, then X_1 and $\overline{X}$ are both unbiased estimators of μ, but $E(X_1) = \sigma^2$ and $E(\overline{X}) = \sigma^2/n$.
- There are general techniques for finding MVUE's, but they are beyond the scope of this course.
- Important Fact: If $X_1, \ldots, X_n$ is a random sample of a RV X, then $\overline{X}$ is the MVUE of $E(X)$.

Estimate Error

- Another way to choose a good estimator is to minimize the **error**
- Given $\hat{\Theta}$, the ***standard error*** is:

$$\sigma_{\hat{\Theta}} = \sqrt{V(\hat{\Theta})}$$

    - If the **standard error** contains unknown parameters, they may be estimated and substituted into the standard error to obtain the ***estimated standard error***, denoted $S_{\hat{\Theta}}$

Biased Estimators

- Sometimes, we have no choice but to use a **biased estimator**
    - S is often used for σ
- When using a **biased estimator**, it is often useful to measure error
- **Mean Square Error** = $\mathrm{MSE}(\hat{\Theta}) = E[(\hat{\Theta} - \theta)^2]$
    - $\mathrm{MSE}(\hat{\Theta}) = V(\hat{\Theta}) + (E(\hat{\Theta}) - \theta)^2$

Week 9

Confidence Interval on the Mean of a Normal Distribution

- An **interval estimate** for a **population parameter** is called a ***confidence interval***
    - Suppose that $X_1, X_2, \ldots, X_n$ is a **random sample** from a **normal distribution** with unknown mean, μ and known variance σ^2, then the sample mean $\bar{X}$ is **normally distributed** with μ, σ^2 ,

- We may **standardize** $\bar{X}$ by $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt(n)}$

- **confidence interval estimate for** μ = interval of the form $l \leq \mu \leq u$
  - endpoints l, u are computed from the sample data
  - Because different samples will produce different values, these are random variables L and U
    - $P\{L \leq \mu \leq U\} = 1 - \alpha$
    - **Confidence Coefficient**

Fact (Confidence Interval on the Mean, Variance Known)

If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval on μ is given by

$$\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the value such that $\Phi(z_{\alpha/2}) = \alpha/2$.

Fact (Sample Size for Specified Error on the Mean, Variance Known)

If $\bar{x}$ is used as an estimate of , we can be $100(1-\alpha)\%$ confident that the error $|\bar{x} - \mu$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2$$

Fact (One-Sided Confidence Bounds on the Mean, Variance Known)

A $100(1-\alpha)\%$ upper-confidence bound for μ is

$$\mu \leq \bar{x} + z_\alpha \sigma/\sqrt{n}$$

and a $100(1-\alpha)\%$ lower-confidence bound for μ is

$$\bar{x} - z_\alpha \sigma/\sqrt{n} \leq \mu$$

Confidence Interval: Mean & Variance Unknown

Fact (t Distribution)

Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a t distribution with $n-1$ degrees of freedom.

Fact (Confidence Interval on the Mean, Variance Unknown)

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval on μ is given by

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of the t distribution with $n-1$ degrees of freedom.

95 /

Normal Approximation for Binomial Proportion

Fact (Normal Approximation for a Binomial Proportion)

If n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\widehat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately standard normal, where $\widehat{P} = X/n$.

Week 10

Hypothesis Testing for a Single Variable

- **Statistical Hypothesis** = a statement about parameters of one or more populations
  - general structure consists of a ***guess*** and an ***alternate guess***

Structure of a Hypothesis Test

- H_0 = ***Null Hypothesis*** (statement initially assumed true)
  - Usually of the form $\theta = r$, for some r
  - We may guess $\mu = 50$
- H_1 = ***Alternate Hypothesis*** (statement that contradicts the *null hypothesis*)
  - If $H_0 = \theta = r$, H_1 may be
    - $\theta \neq r$ (two sided alternate hypothesis)
    - $\theta \geq r$ (upper one-sided alternate hypothesis)
    - $\theta \leq r$ (lower one-sided alternate hypothesis)

Procedure

1. Assume H_0
2. Consider the data: is there something that is very unlikely, implausible given H_0
3. If YES: reject *null hypothesis* in favour of H_1
4. If NO, accept H_0

Objectives

- Testing if a parameter has changed
- Testing a theory
- Conformance testing

Method

1. Assume H_0
2. Pick an estimator for your parameter $\hat{\Theta}(X_1, \ldots, X_n)$
    1. For example, if the paramater of interest is μ, we could choose the sample mean $\bar{X}$
3. Choose a ***critical region***, determined by some critical values
    1. For example, for $\theta \neq r$, choose critical region to be the complement
        1. $C = \mathbb{R}/(r-l, r+u)$ for critical values $r-l$ and $r+u$
4. Take a sample $x_1, \ldots, x_n$ and compute a point estimate $\hat{\theta} = \hat{\Theta}(x_1, \ldots, x_n)$
5. Is the point estimate in the critical region? In other words, $\hat{\theta} \in C$

Outcomes

There are four possible outcomes of a hypothesis test:

1. If H_0 is true and we accept H_0, then we are happy!
2. If H_1 is true and we reject H_0, then we are happy!
3. If H_0 is true and we reject H_0 in favour of H_1, then we say that we have made a "*Type I Error*".
4. IF H_0 is false (so H_1 is true) and we accept H_0, then we say that we have made a "*Type II Error*".

- We define the "*significance level*" of the test to be the probability of making a Type I Error, which we call $\alpha = P(\text{Type I Error})$.
- The "*power*" of the test is $1 - \beta$, where $\beta = P(\text{Type II Error})$.

Fixed Significance Level Testing

- One way to determine a **critical region** is to require that our test has a ***fixed significance level*** α
    - Recall: α is the probability that we incorrectly reject H_0, given that H_0 is true
- Suppose that $\hat{\Theta}$ is an estimator for a parameter θ and that we have a null hypothesis H_0 : $\theta = r$ and a $H_1 : \theta \neq r$
    - We fix a ***significance level*** α
    - Choose critical values based on this value
    - We choose aim to find a symmetric region defined by critical values $r-a$ and $r+a$

- $$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is True}) \\ &= P(\widehat{\Theta} \in (-\infty, r-a] \cup [r+a, \infty) \mid H_0 \text{ is True}).\end{aligned}$$

P- Values

- **P-Values** give a more dynamic approach to hypothesis testing
  - Suppose we have a test with null hypothesis H_0
  - **P-Value of the test** = smallest *significance level* that would lead to a rejection of H_0 with the sample
    - P-value as the probability of being in a sort of variable critical region
    - Sometimes called **observed significance**
  - Consider $H_0 : \theta = r$ and $H_1 : \theta \neq r$, and $\hat{\Theta}(X_1, \dots, X_n)$
    - suppose $\hat{\theta} = \hat{\Theta}(x_1, \dots, x_n)$
    - assuming H_0, our "observed" critical values are $r \pm |r - \hat{\theta}|$
    - Then the **P-Value Test** is:

$$P\text{-value} = P(\widehat{\Theta} \in (-\infty, r - |r-\hat{\theta}|] \cup [r + |r-\hat{\theta}|, \infty) \mid \theta = r)$$

  - P-value of the observation is the probability of making an observation ***at least as far from*** $\theta = r$ ***as*** $\hat{\Theta}$
    - Measure of the *risk* that we make an incorrect decision if we reject H_0 based on sample data
    - If we make an observation / point estimate that has a very high P-value, then we are at ***high risk*** of making a Type 1 error if we reject H_0
- We can use P-values to refine the *fixed significance level testing procedure*
  - Fix a significance level α and construct a **critical region** based on α
  - The observed value $\hat{\theta}$ in the **critical region** iff ($\iff$)the P-value is ***at most*** α, and so we reject H_0 iff $P - value \leq \alpha$
  - Accept $H_0 \iff P - value > \alpha$

Relationship between Hypothesis Testing and Confidence Intervals

- Let θ be an unknown paramter
  - There is a close relationship between **hypothesis test** and **confidence intervals** for θ
  - suppose (L, U) is a $100(1-\alpha)$ confidence interval for θ constructed around a point estimate $\bar{\theta}$
    - Consider a $H_0 : \theta = r, H_1 : \theta \neq r$
    - Then the observation $\hat{\theta}$ leads to a rejection of H_0 , iff ($\iff$)$r \notin (L, U)$
    - This gives us an equivalent way of performing **fixed significance testing**
      1. Given a test $H_0 : \theta = r, H_1 : \theta \neq r$
         1. Choose a test statistic $\hat{\Theta}$ and a **significance level** α
         2. For a point estimate, $\hat{\theta} = \hat{\Theta}(x_1, \dots, x_n)$ construct the $100(1-\alpha)\%$ **confidence interval** around $\hat{\theta}$
         3. If $r \in (L, U)$, accept H_0
         4. If $r \notin (L, U)$, reject H_0 in favour of H_1

Week 11

Tests on the Mean of a Normal Distribution Variance Known

Summary of Tests on the Mean, Variance Known

Testing Hypotheses on the Mean, Variance Known (Z-Tests)		
Null hypothesis: $H_0 : \mu = \mu_0$ Test statistic: $Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$		
Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
H_1: $\mu \neq \mu_0$	Probability above $\lvert z_0 \rvert$ and probability below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(\lvert z_0 \rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
H_1: $\mu > \mu_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
H_1: $\mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Type II Error and Choice of Sample Size

$H_0 = \mu = \mu_0, H_1 : \mu \neq \mu_0$

- Suppose that the *null hypothesis* is **false**
- Assume true value of the mean is $\mu = \mu_0 + \delta$
  - Therefore, δ = $\mu - \mu_0$, or **predicted mean** minus **sample mean**
- Then, when H_1 is true, the **distribution of the test statistic** Z_0 is
  - $Z_0 \sim N(\frac{\delta * \sqrt{n}}{\sigma}, 1)$

Fact (Probability of a Type II Error for a Two-Sided Test on the Mean, Variance Known)

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

One may easily obtain formulas that determine the appropriate sample size to obtain a particular value of β for a given α and δ

Fact (Sample Size for a Two-Sided Test on the Mean, Variance Known)

$$n \approx \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} \qquad \text{where } \delta = \mu - \mu_0$$

Fact (Sample Size for a One-Sided Test on the Mean, Variance Known)

$$n \approx \frac{(z_{\alpha} + z_\beta)^2 \sigma^2}{\delta^2} \qquad \text{where } \delta = \mu - \mu_0$$

Tests on the Mean of a Normal Distribution, Variance Unknown

- We n is *large enough*, the **sample standard deviation**, S, can be substituted for σ in the test procedures *with little effect*
  - Although we have given a test for the **mean** of a normal distribution with known σ^2, it can be easily converted into a ***large-sample test procedure*** for unknown σ^2
    - Valid regardless of the form of the distribution of the population
    - Relies on the **Central Limit Theorem**, just as the ***large-sample confidence interval on*** μ did
- **Test Statistic** = $T_0 = \dfrac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}}$

Summary

Summary for the One-Sample *t*-test

Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown		
Null hypothesis: $H_0 : \mu = \mu_0$ Test statistic: $T_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$		
Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1: \mu \neq \mu_0$	Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1: \mu > \mu_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1: \mu < \mu_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

Week 12

Tests on a Population Proportion

- Modelling the occurrence of defectives with the **binomial distribution** is usually reasonable when the **binomial parameter *p* represents the proportion of defective items produces**
    - Consequently, many engineering decision problems involve **hypothesis testing** about *p*
    - An approximate test based on the **normal approximation** to the **binomial** is given
        - Procedure will be valid as long as p **is not extremely close to 0 or 1**
        - **sample size** is relatively large
    - Let X be the *number of observations in a random sample of size* n that belongs to the **class** associated with p
        - If $H_0 : p = p_0$ is true:
            - $X \sim N[np_0, np_0(1p_0)]$
            - $Z_0 = \dfrac{X - np_0}{\sqrt{np_0(1-p_0)}}$

Summary of Approximate Tests on a Binomial Proportion

<u>Testing Hypotheses on a Binomial Proportion</u>

Null hypotheses: $H_0 : p = p_0$

Test statistic: $Z_0 = \frac{X-np_0}{\sqrt{np_0(1-p_0)}}$

Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1: p \neq p_0$	Probability above $\lvert z_0 \rvert$ and probability below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(z_0)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: p > p_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: p < p_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Type II Error and Choice of Sample Size

Fact (Approximate Sample Size for a Two-Sided Test on a Binomial Proportion)

$$n = \left[\frac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta \sqrt{p(1-p)}}{p - p_0} \right]^2$$

Fact (Approximate Sample Size for a One-Sided Test on a Binomial Proportion)

$$n = \left[\frac{z_{\alpha}\sqrt{p_0(1-p_0)} + z_\beta \sqrt{p(1-p)}}{p - p_0} \right]^2$$

Hypotheses Tests on the Difference in Means, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

- This means, the standard deviation of the two random variables is the same

$H_0 : \mu_1 - \mu_2 = \Delta_0, H_1 : \mu_1 - \mu_2 \neq \Delta_0$

- Let $X_{11}, \ldots, X_{1n_1}$ be a random sample of n_1 observations from *first* population
- Let $X_{21}, \ldots, X_{2n_2}$ be a random sample of n_1 observations from the *second* population
- Let $\bar{X}_1, \bar{X}_2, S_1^2, S_2^2$ be the *sample means* and *sample variances* respectively
- **Expected Value** of the ***difference in means*** = $\bar{X}_1 - \bar{X}_2 = E(\bar{X}_1 - \bar{X}_2)$
  - Therefore $\bar{X}_1 - \bar{X}_2$ is an ***unbiased estimator*** for the **difference in means**
- **Variance** of $\bar{X}_1 - \bar{X}_2 = V(\bar{X}_1 - \bar{X}_2) = \sigma^2(\frac{1}{n_2} + \frac{1}{n_2})$

Fact (Pooled Estimator of Variance)

The pooled estimator of σ^2, denoted by S_p^2, is defined by

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Because test statistic

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1),$$

then the t statistic

$$T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

has a t distribution with $n_1 + n_2 - 2$ degree of freedom.

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal*

$$\text{Null hypothesis:} \quad H_0: \mu_1 - \mu_2 = \Delta_0 \tag{10.14}$$

$$\text{Test statistic:} \quad T_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$	$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha, n_1+n_2-2}$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha, n_1+n_2-2}$

Case 2: $\sigma_1^2 \neq \sigma_2^2$

Test Statistic for the Difference in Means: Variance Unknown and Not Assumed Equal

Fact (Case 2: Test Statistic for the Difference in Means, Variances Unknown and Not Assumed Equal)

If $H_0: \mu_1 - \mu_2 = \Delta_0$ is true, the statistic

$$T_0^* = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

is distributed approximately as t with degrees of freedom given by

$$v = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

If v is not an integer, round down to the nearest integer.

Empirical Models

- **Regression Analysis** = The collection of statistical tools are used to **model** and *explore* relationship between variables in a non-deterministic manner
    - Only one independent or predictor variable, x
        - Study the relationship with the response, y, which is assumed to be linear
        - Reasonable to assume the mean of Y is related to x by

$Y = \beta_0 + \beta_1 x + \epsilon$

- Slope and intercept of the line are called **regression coefficients**
- ϵ is the **random error term**

- We call this model the **simple linear regression** model
  - It only has one independent variable, or ***regressor***
- Suppose that the **mean** and **variance** of ϵ are 0 and σ^2

  Then:

  $E(Y|x) = \beta_0 + \beta_1 x)$ and $V(Y|x) = \sigma^2$

Suppose that the true relationship between Y and x is a straight line and that the observation Y at each level of x is a random variable.

$$Y = \beta_0 + \beta_1 x + \epsilon,$$

where the intercept β_0 and the slope β_1 are unknown regression coefficients.
Suppose that we have n pairs of observations $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$.The estimates of β_0 and β_1 should result in a line that is (in some sense) a "best fit" to the data. German scientist Karl Gauss (1777–1855) proposed estimating the parameters to minimize the sum of the squares of the vertical deviations. We call this criterion for estimating the regression coefficients the method of least squares.

We may express the n observations in the sample as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \cdots, n$$

and the sum of the squares of the deviations of the observations from the true regression line is

$$L = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 + \beta_1 x_i)^2$$

Fact (Least Squares Estimates)

The least squares estimates of the intercept and slope in the simple linear regression model are $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - \frac{(\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}},$$

where $\bar{y} = (\sum_{i=1}^{n} y_i)/n$ *and* $\bar{x} = (\sum_{i=1}^{n} x_i)/n$.

The fitted or estimated regression line is therefore

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Note that each pair of observations satisfies the relationship

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i, i = 1, 2, \cdots, n$$

where $e_i = y_i - \hat{y}_i$ is called the residual. The residual describes the error in the fit of the model to the ith observation y_i.
Given data $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, let

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}$$

and

$$S_{xy} = \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n} x_i y_i - \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n}$$

$\hat{\beta}_1$ is an unbiased estimator in simple linear regression of the true slope β_1 and $V(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$.
$\hat{\beta}_0$ is an unbiased estimator of the intercept β_0 and $V(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]$

Fact (Estimated Standard Errors)

In simple linear regression, the estimated standard error of the slope and the estimated standard error of the intercept are

$$se(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \text{ and } se(\hat{\beta}_0) = \sqrt{\hat{\sigma}^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_x x}\right]}$$