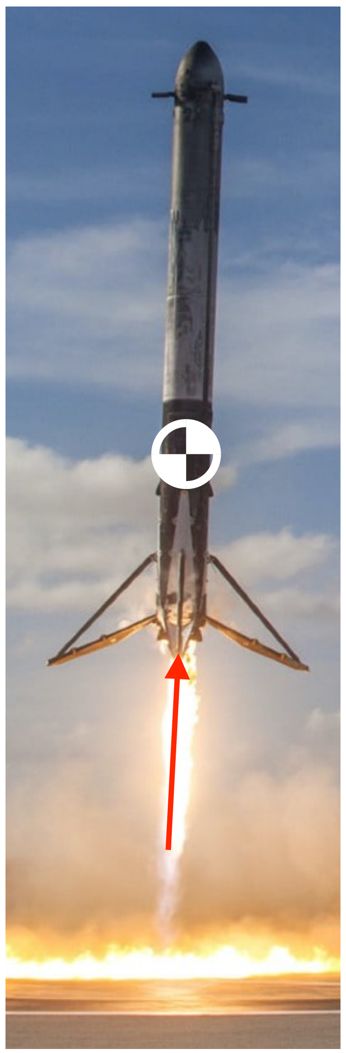
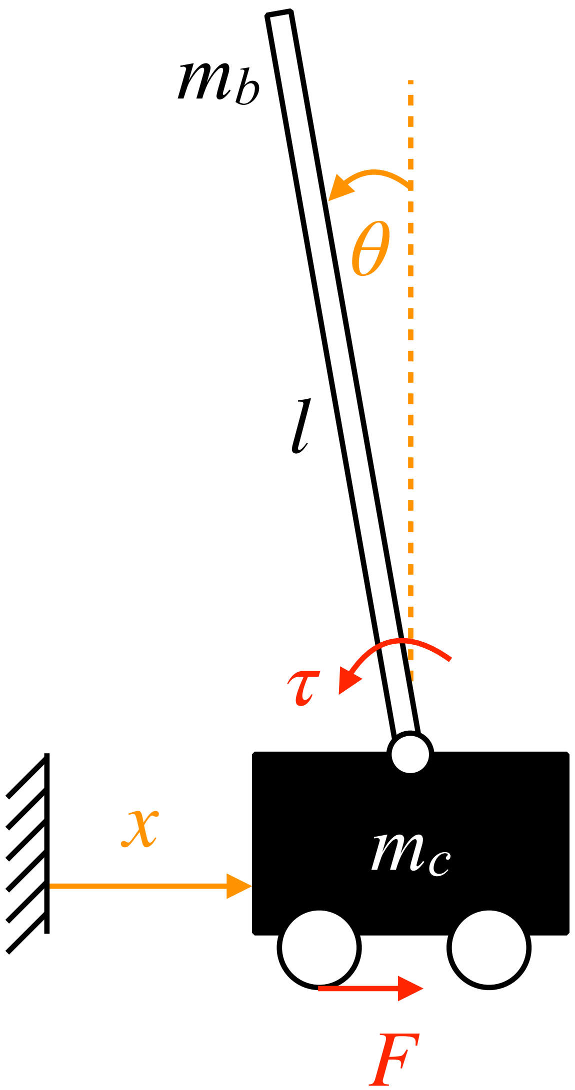
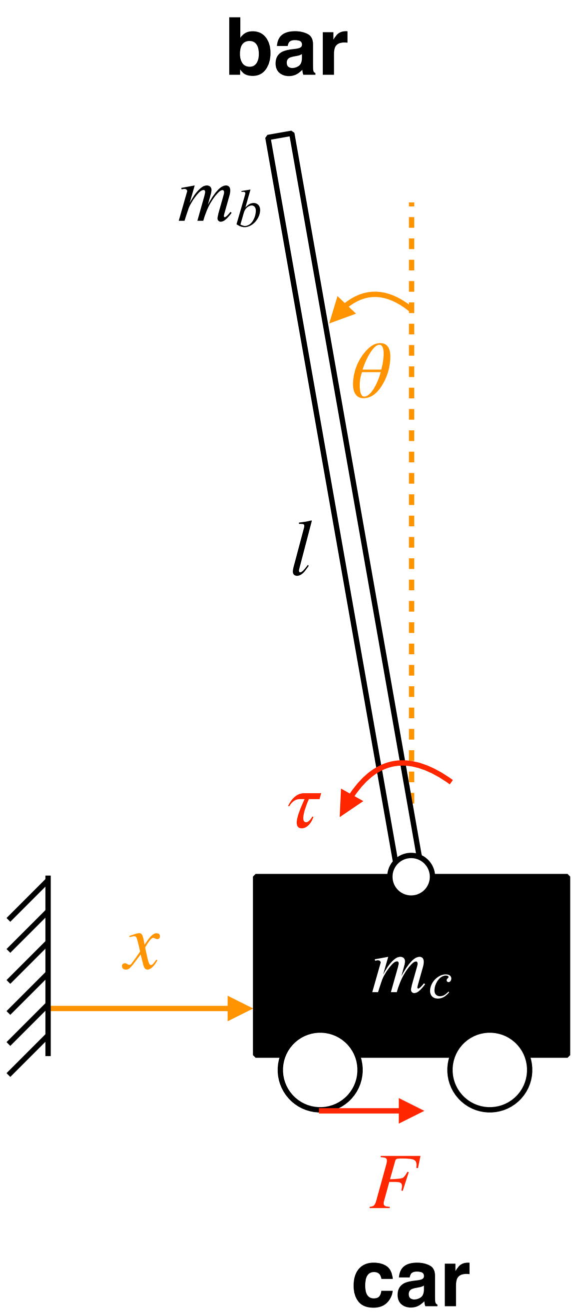




倒立摆的控制

1. 倒立摆的运动微分方程
2. 固有频率和模态
3. 倒立状态的控制率设计
    3.1. 双电机
    3.2. 单电机
    3.3. 平移操作

问题描述：倒立摆的控制



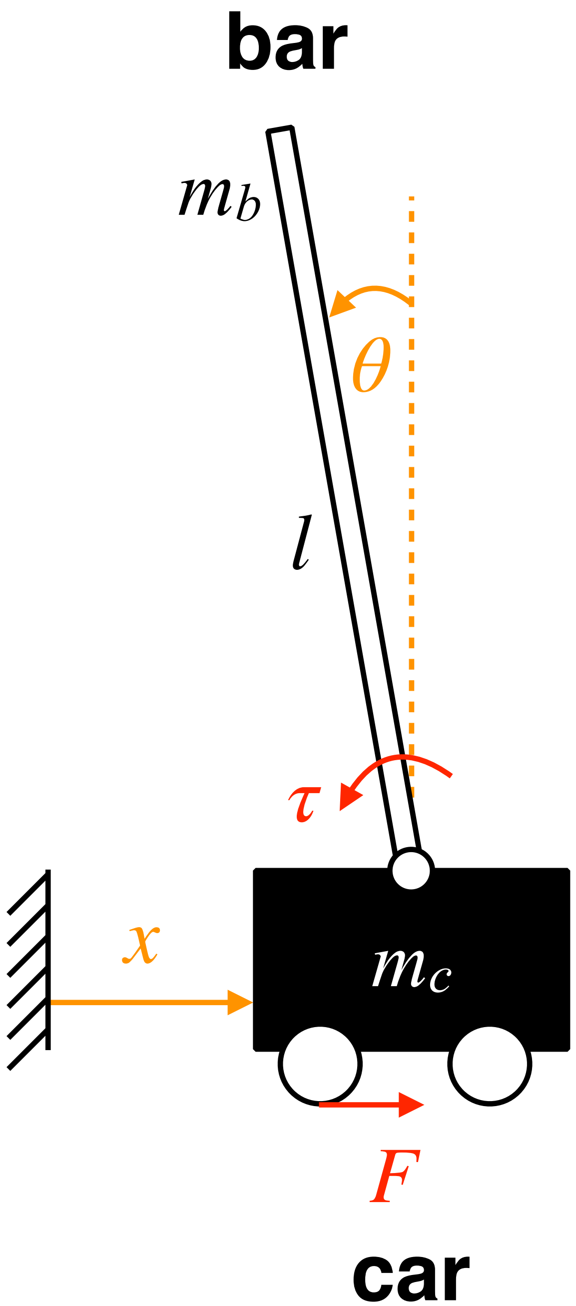
如图所示，质量 $m_c = 0.5$ kg 的活动小车上铰接一根长 $l = 0.3$ m ，质量$m_b = 0.1$ kg的竖杆。一车轮受电机驱动，地面粗糙时，无论轮胎是否打滑，电机力矩被车轮转化为摩擦驱动力 F。倒立摆与小车的连接处也可能有电机，输出力矩为 τ。杆和车轮的质量和转动惯量均忽略不计。

测量杆的摆角 θ，车轮转角即小车位置 x，以及 $\dot{x}$ 和 $\dot{\theta}$，制定电机的控制策略

1、保持竖杆不倒

2、小车按设定的轨迹运动

两种情况：轮胎不打滑和打滑



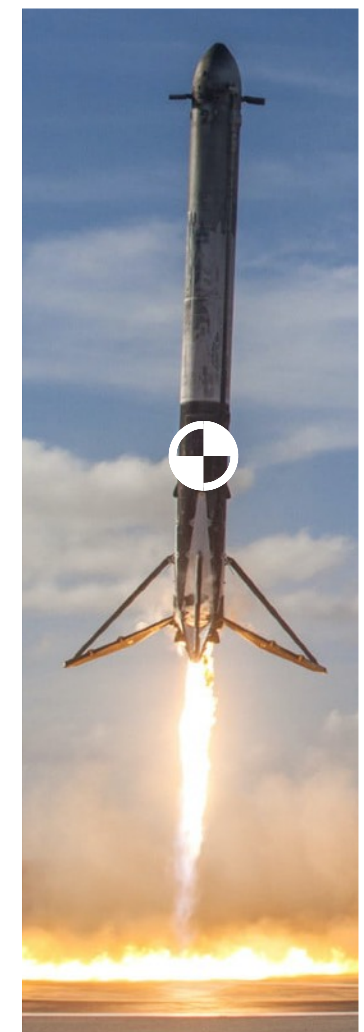
一、轮胎不打滑

控制电机的转速，直接驱动$\dot{x}$

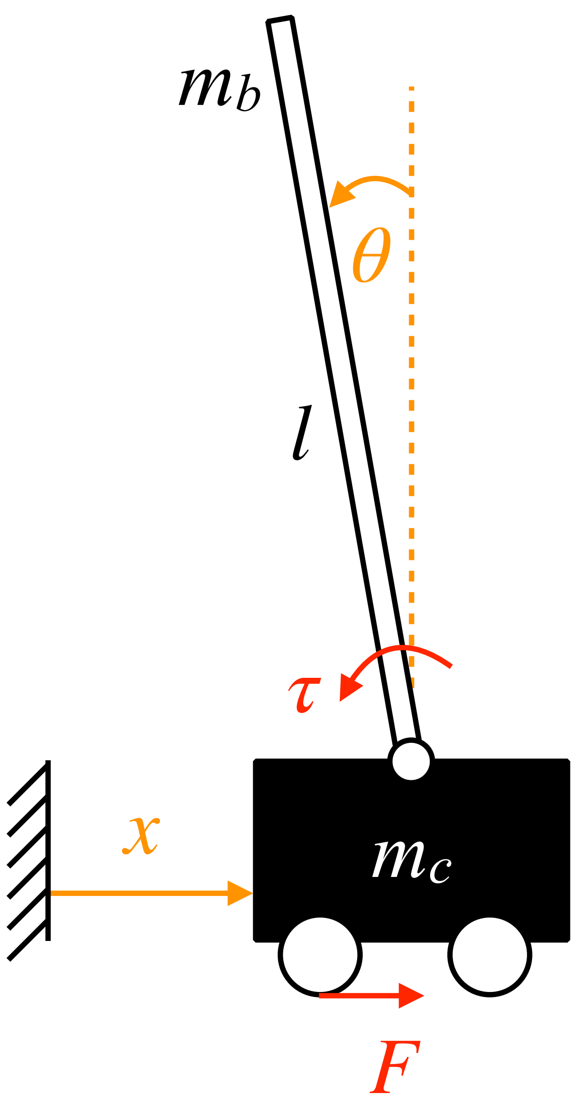
二、轮胎打滑

控制电机的输出力矩，即控制F。

对应火箭控制模型



两种情况：轮胎不打滑和打滑



一、轮胎不打滑

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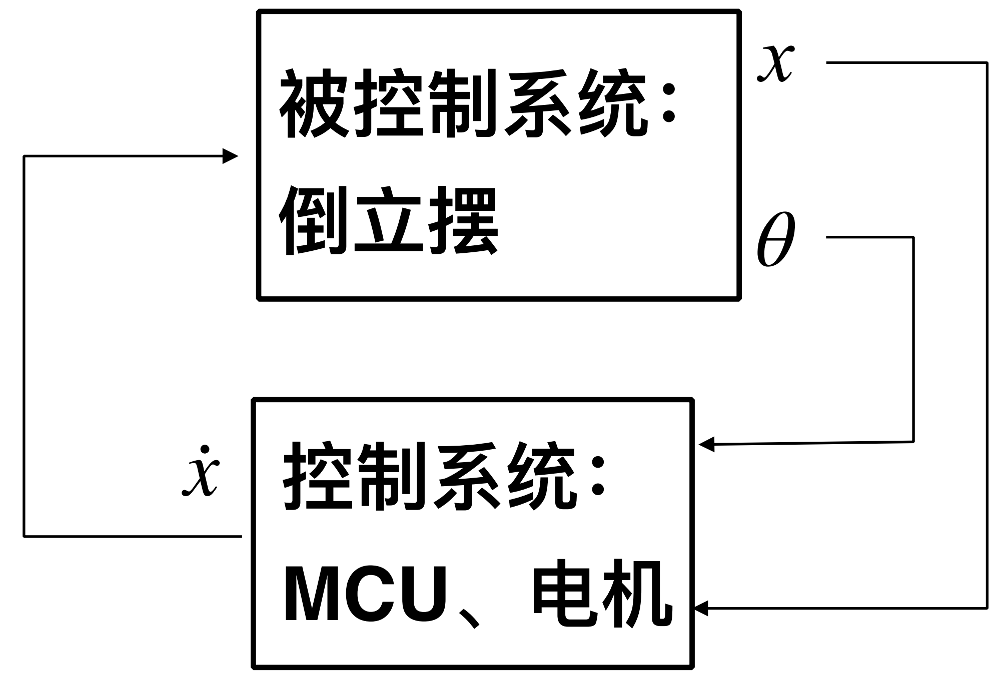
二、轮胎打滑

控制电机的输出力矩，即控制F。

对应火箭控制模型

相关的软硬件系统

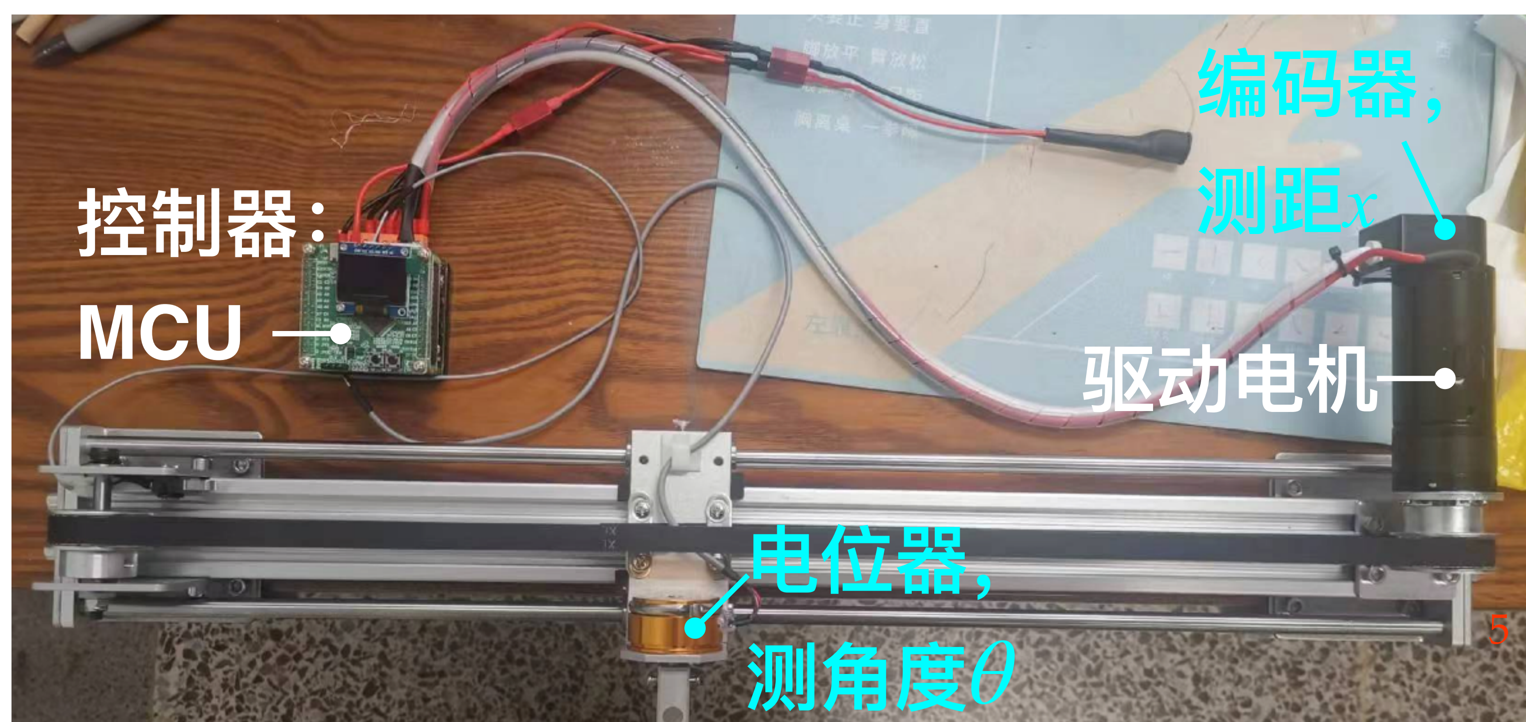
控制系统改变了系统的运动微分方程，及解的响应特性



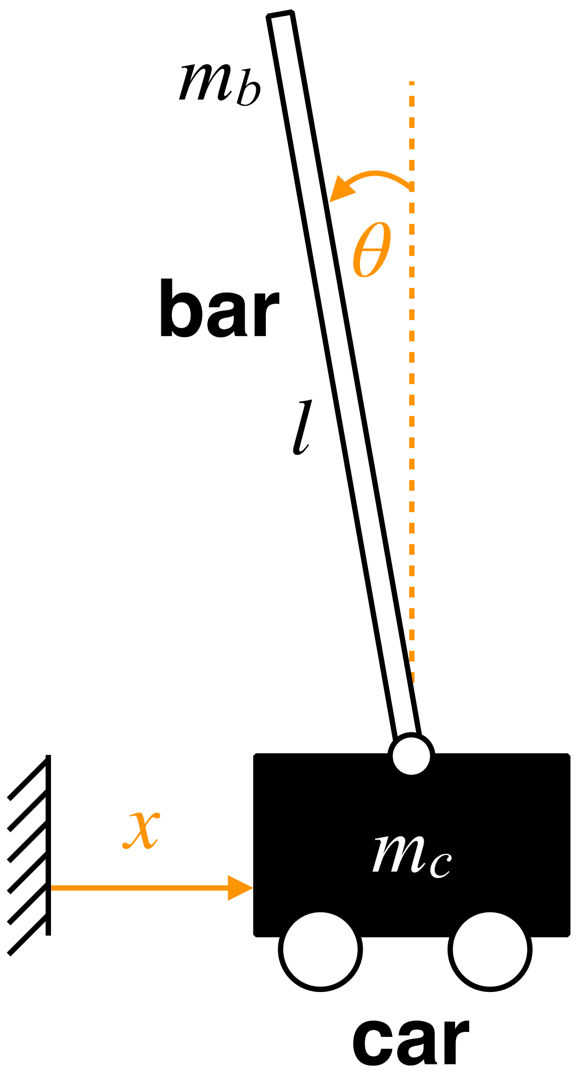
理论： 动力学方程（有无控制系统）

软件： C/C++，高级语言但能兼容低级功能，既能抽象类又能直接操作硬件（寄存器）

硬件： MCU如STM32；传感器



1. 轮胎不打滑 — 倒立摆的运动微分方程



控制电机的转速，直接驱动$\dot{x}$

小车非惯性系下看杆

$$\frac{1}{3}m_b l^2\ddot{\theta} = m_b g\frac{l}{2}\sin\theta + m_b\ddot{x}\frac{l}{2}\cos\theta$$

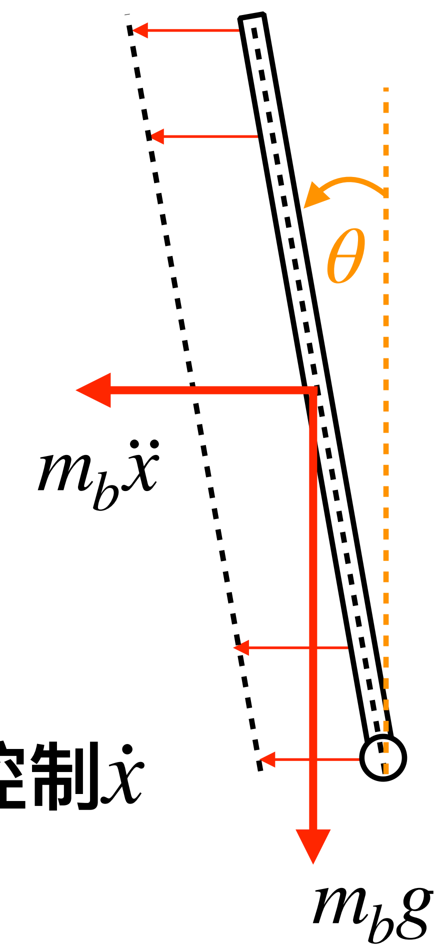
在倒立状态$\theta = 0$的线化方程

$$\ddot{\theta} - \frac{3g}{2l}\theta = \boxed{\frac{3}{2l}\ddot{x}} = -k_d\dot{\theta} - k_p\theta$$

如果不打滑，通过控制电机的转速，可以控制$\dot{x}$

也就是 $\frac{3}{2l}\dot{x} = -k_d\theta - k_p\int_0^t \theta\,\mathrm{d}t$

$$\ddot{\theta} + k_d\dot{\theta} + (k_p - \frac{3g}{2l})\theta = 0$$



渐进稳定两个条件：刚度为正，阻尼为正

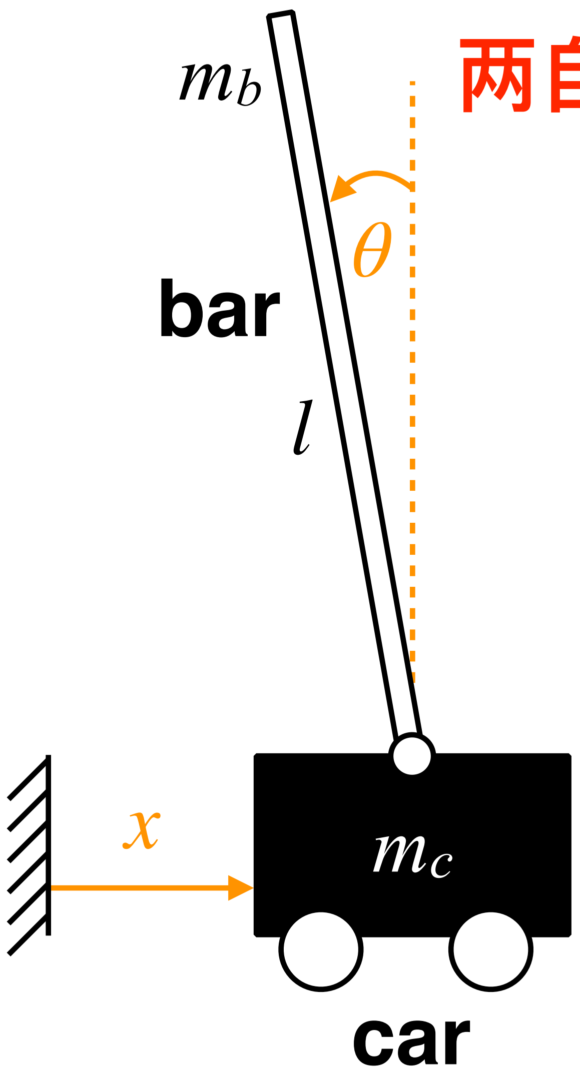
为衰减的快，可以取临界阻尼

1. 轮胎不打滑 — 控制率只依赖于角度

保持匀速运动

当$\theta, \dot{\theta}$趋于零，$\dot{x}$趋于常数

在倒立状态$\theta = 0$的线化方程



两自由度

$$\boxed{\ddot{\theta} - \frac{3g}{2l}\theta = \frac{3}{2l}\ddot{x} \qquad \frac{3}{2l}\dot{x} = -k_d\theta - k_p\int_0^t \theta\,\mathrm{d}t}$$

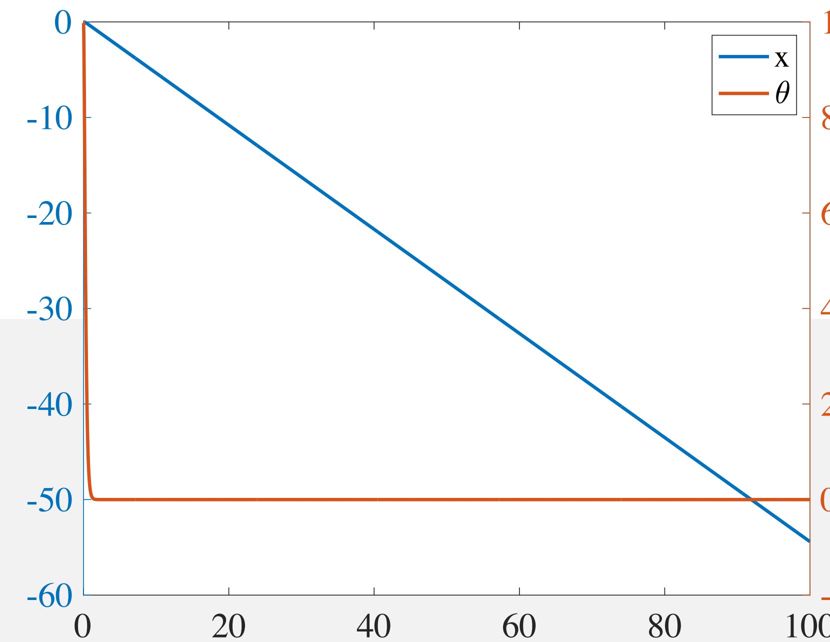
控制率

$$\ddot{\theta} + k_d\dot{\theta} + (k_p - \frac{3g}{2l})\theta = 0$$

为衰减的快，可以取临界阻尼

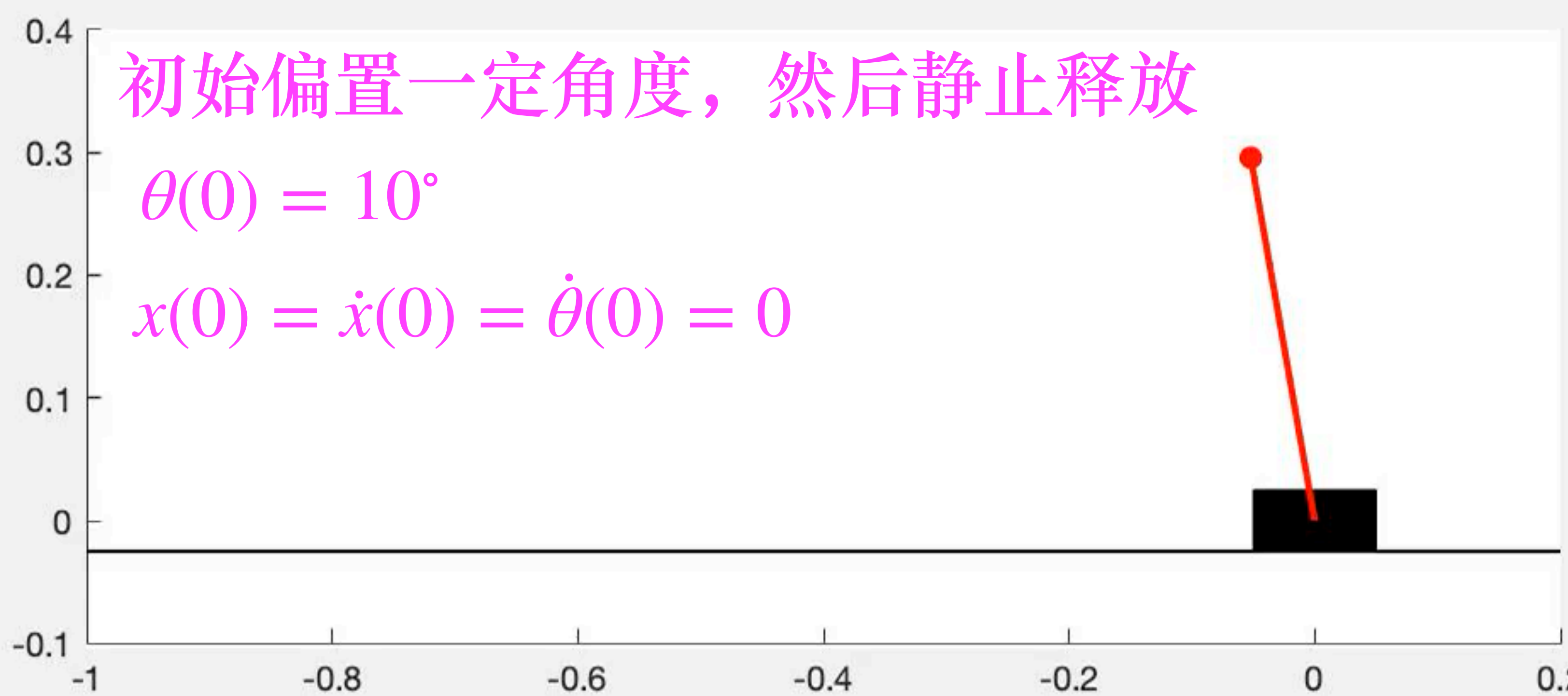
不妨取：$k_p - \dfrac{3g}{2l} = (2\pi \times 1)^2$

$$k_d = 4\pi \times 1$$



控制电机的转速，直接驱动$\dot{x}$

被控制后系统的自由度是多少?

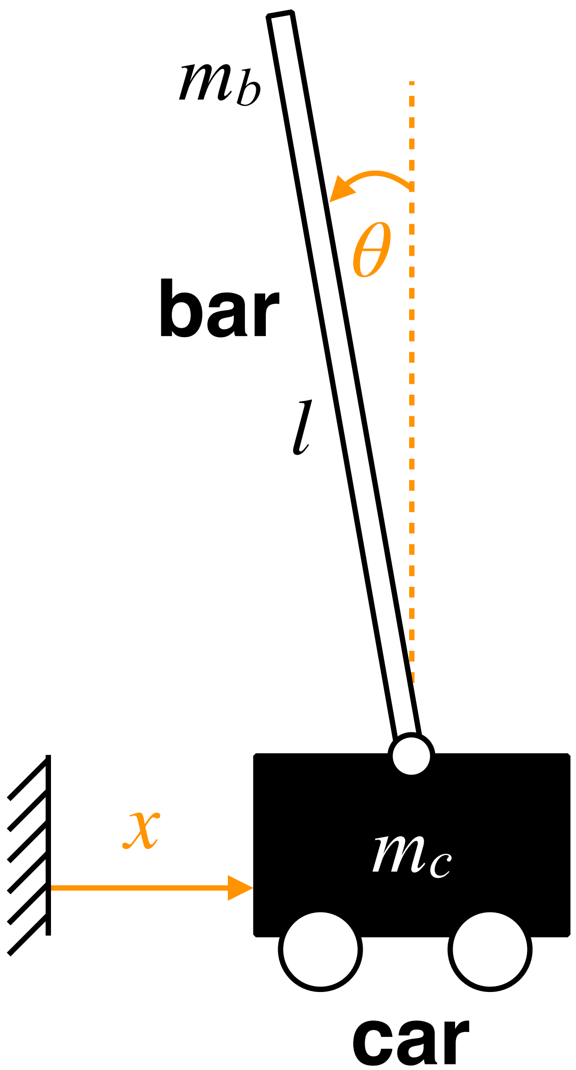


初始偏置一定角度，然后静止释放

$\theta(0) = 10^\circ$

$x(0) = \dot{x}(0) = \dot{\theta}(0) = 0$

1. 轮胎不打滑 — 控制率依赖于角度和位置



$$\ddot{\theta}-\frac{3g}{2l}\theta=\frac{3}{2l}\ddot{x}$$

$$\frac{3}{2l}\ddot{x}=-k_d\dot{\theta}-k_p\theta-\frac{3}{2l}k_d^x\dot{x}-\frac{3}{2l}k_p^x x$$

两自由度，两方程，要在$x=0,\theta=0$处平衡且渐进稳定

写成矩阵向量形式

$$\begin{pmatrix}-\frac{3}{2l} & 1\\ \frac{3}{2l} & 0\end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{\theta}\end{pmatrix}+\begin{pmatrix}0 & 0\\ \frac{3}{2l}k_d^x & k_d\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{\theta}\end{pmatrix}+\begin{pmatrix}0 & -\frac{3g}{2l}\\ \frac{3}{2l}k_p^x & k_p\end{pmatrix}\underbrace{\begin{pmatrix}x\\ \theta\end{pmatrix}}_{\triangleq \boldsymbol{s}}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\boldsymbol{M}_s\ddot{\boldsymbol{s}}+\boldsymbol{C}_s\dot{\boldsymbol{s}}+\boldsymbol{K}_s\boldsymbol{s}=\boldsymbol{0}$$

假设有振动模态 $\boldsymbol{s}(t)=\boldsymbol{\phi}e^{\beta t}$

特征方程: $(\beta^2\boldsymbol{M}_s+\beta\boldsymbol{C}_s+\boldsymbol{K}_s)\boldsymbol{\phi}=\boldsymbol{0}\quad \beta\in\mathbb{C}$

控制电机的转速，直接驱动$\dot{x}$

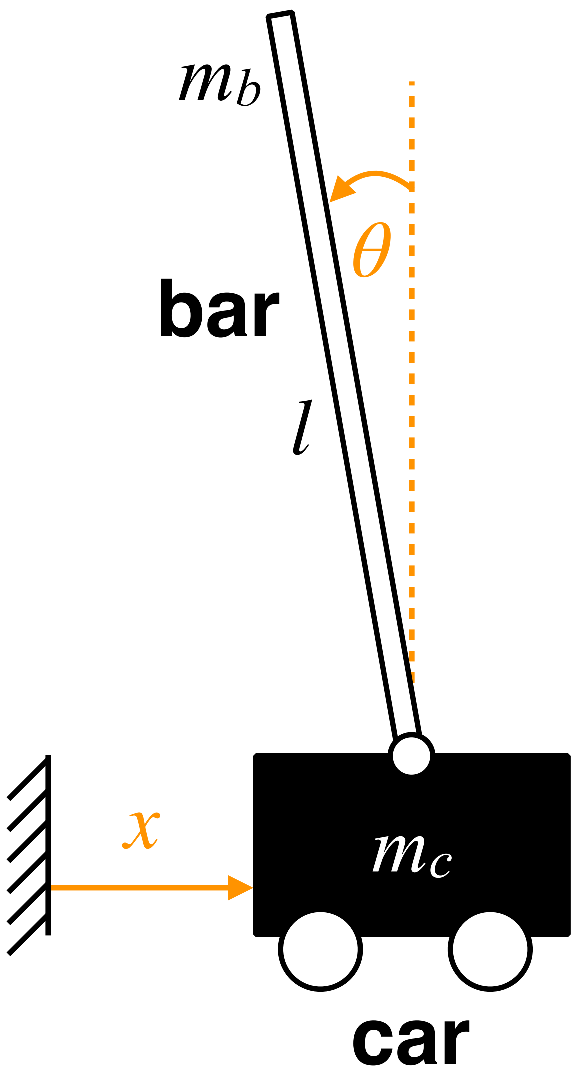
如果某个特征值的实部为正，则平衡位置不稳定

所有特征根的实部都为负，则平衡位置渐进稳定

$$\frac{3}{2l}\dot{x}=-k_d\theta-k_p\int_0^t\theta\,\mathrm{d}t-\left(k_d^x x+k_p^x\int_0^t x\,\mathrm{d}t\right)\frac{3}{2l}$$

增加位置依赖项

1. 轮胎不打滑 — 控制率依赖于角度和位置



$$\begin{pmatrix} -\frac{3}{2l} & 1 \\ \frac{3}{2l} & 0 \end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{\theta}\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{3}{2l}k_d^x & k_d\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{\theta}\end{pmatrix} + \begin{pmatrix} 0 & -\frac{3g}{2l} \\ \frac{3}{2l}k_p^x & k_p\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\boldsymbol{M}_s\ddot{\boldsymbol{s}} + \boldsymbol{C}_s\dot{\boldsymbol{s}} + \boldsymbol{K}_s\boldsymbol{s} = \boldsymbol{0}$ 假设有振动模态 $\boldsymbol{s}(t) = \boldsymbol{\phi}e^{\beta t}$

$$\beta^4 + (k_d + k_d^x)\beta^3 + (k_p + k_p^x - \frac{3g}{2l})\beta^2 - \frac{3g}{2l}k_d^x\beta - \frac{3g}{2l}k_p^x = 0$$

如果 $\beta = a + bj$，$\boldsymbol{\phi}$ 是一个特征解对

复数解成对出现

那么 $\beta^* = a - bj$，$\boldsymbol{\phi}^*$ 也是一个特征解对

$$(\beta-\beta_1)(\beta-\beta_2)(\beta-\beta_3)(\beta-\beta_4) = 0$$

参数不能随意选择

稳定要求所有的 β_i 的实部都小于零

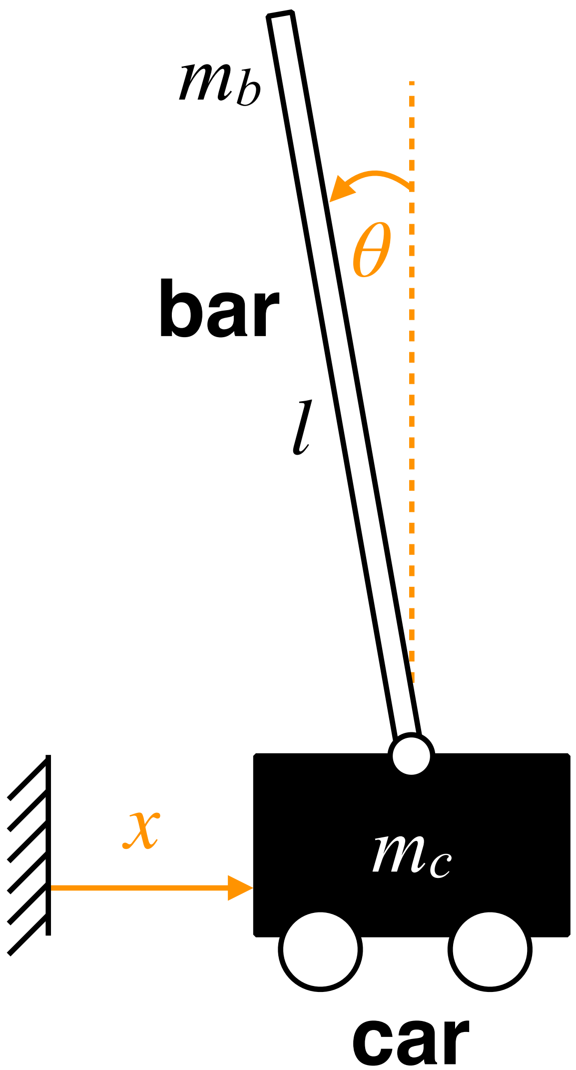
控制电机的转速，直接驱动$\dot{x}$

参照单自由度时临界阻尼是衰减最快的解，假设系统有两个重实根，为负数。

$$(\beta-\beta_1)^2(\beta-\beta_2)^2 = 0 \qquad \beta_1 < 0, \quad \beta_2 < 0$$

$$\beta^4 - 2(\beta_1+\beta_2)\beta^3 + (\beta_1^2+\beta_2^2+4\beta_1\beta_2)\beta^2 - 2\beta_1\beta_2(\beta_1+\beta_2)\beta + \beta_1^2\beta_2^2 = 0$$

1. 轮胎不打滑 — 控制率依赖于角度和位置



控制电机的转速，直接驱动$\dot{x}$

$$\begin{pmatrix} -\frac{3}{2l} & 1 \\ \frac{3}{2l} & 0 \end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{\theta}\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{3}{2l}k_d^x & k_d\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{\theta}\end{pmatrix} + \begin{pmatrix}0 & -\frac{3g}{2l}\\ \frac{3}{2l}k_p^x & k_p\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\boldsymbol{M}_s\ddot{\boldsymbol{s}} + \boldsymbol{C}_s\dot{\boldsymbol{s}} + \boldsymbol{K}_s\boldsymbol{s} = \boldsymbol{0}$$

假设有振动模态 $\boldsymbol{s}(t) = \boldsymbol{\phi}e^{\beta t}$

$$\beta^4 + (k_d + k_d^x)\beta^3 + (k_p + k_p^x - \frac{3g}{2l})\beta^2 - \frac{3g}{2l}k_d^x\beta - \frac{3g}{2l}k_p^x = 0$$

$$\beta^4 - 2(\beta_1 + \beta_2)\beta^3 + (\beta_1^2 + \beta_2^2 + 4\beta_1\beta_2)\beta^2 - 2\beta_1\beta_2(\beta_1 + \beta_2)\beta + \beta_1^2\beta_2^2 = 0$$

$$-2(\beta_1 + \beta_2) = \overset{>0}{\underline{k_d}} + k_d^x$$

$$\beta_1^2 + \beta_2^2 + 4\beta_1\beta_2 = \overset{>0}{\underline{k_p}} + k_p^x - \frac{3g}{2l}$$

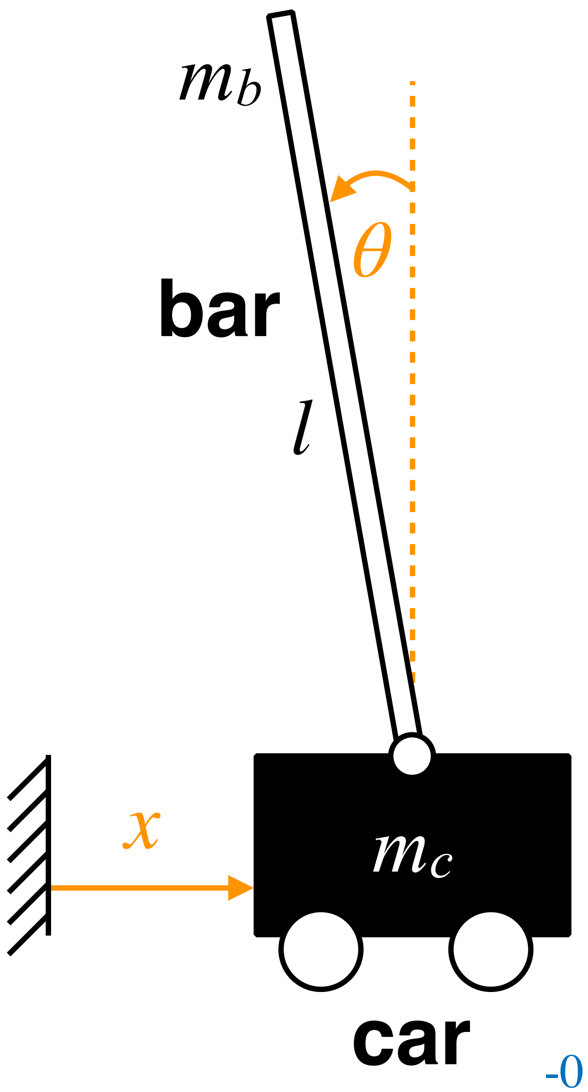
$$-2\beta_1\beta_2(\beta_1 + \beta_2) = -\frac{3g}{2l}\underline{k_d^x} < 0$$

$$\beta_1^2\beta_2^2 = -\frac{3g}{2l}\underline{k_p^x} < 0$$

给定β_1和β_2就能求出一组 k_p^x, k_d^x, k_p, k_d

$$\ddot{\theta} - \frac{3g}{2l}\theta = \frac{3}{2l}\ddot{x} \qquad \frac{3}{2l}\ddot{x} = -\overset{>0}{k_d}\dot{\theta} - \overset{>0}{k_p}\theta - \frac{3}{2l}\overset{<0}{k_d^x}\dot{x} - \frac{3}{2l}\overset{<0}{k_p^x}x$$

1. 轮胎不打滑 — 控制率依赖于角度和位置



控制电机的转速，直接驱动$\dot{x}$

$$-2(\beta_1+\beta_2)=k_d+k_d^x$$

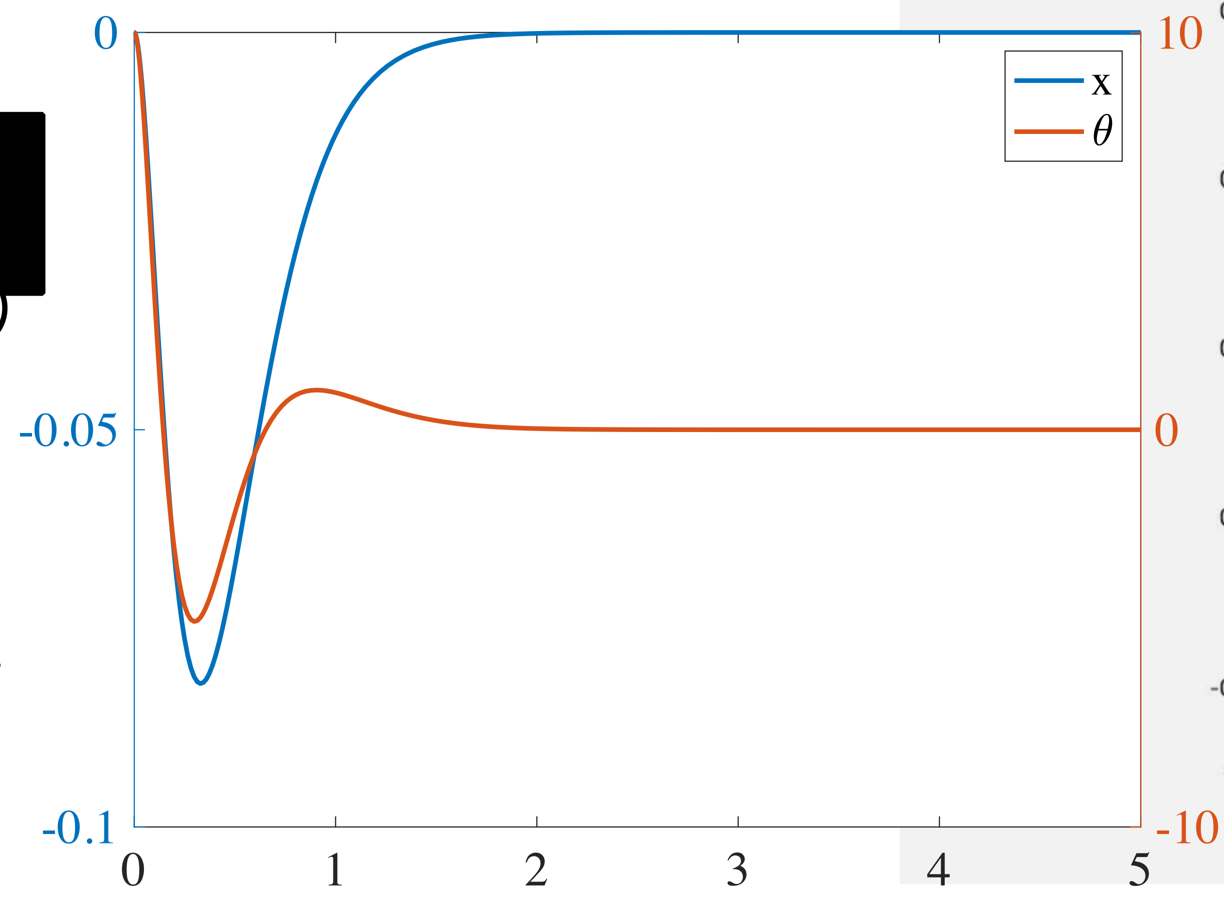
$$\beta_1^2+\beta_2^2+4\beta_1\beta_2=k_p+k_p^x-\frac{3g}{2l}$$

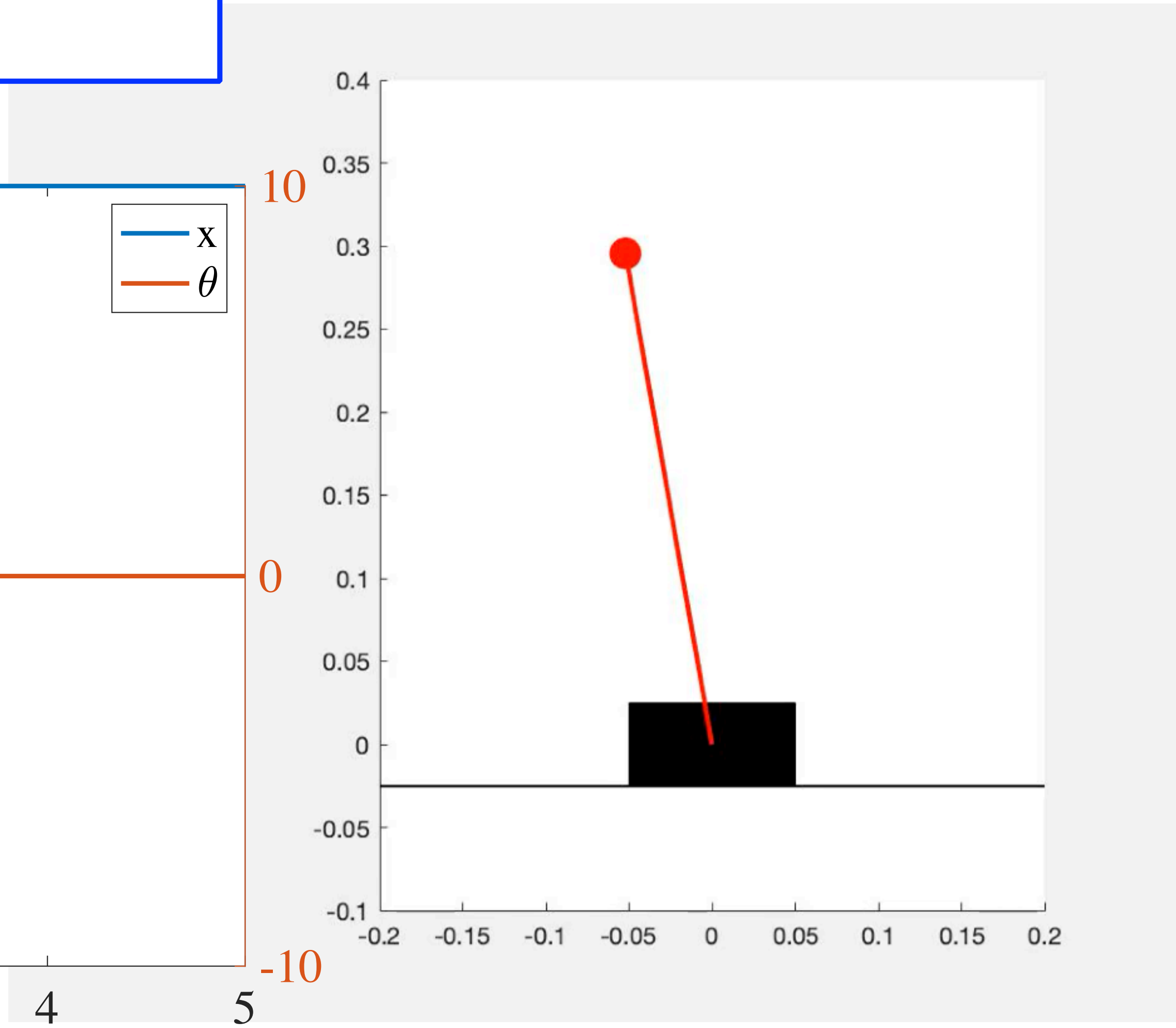
$$-2\beta_1\beta_2(\beta_1+\beta_2)=-\frac{3g}{2l}k_d^x$$

$$\beta_1^2\beta_2^2=-\frac{3g}{2l}k_p^x$$

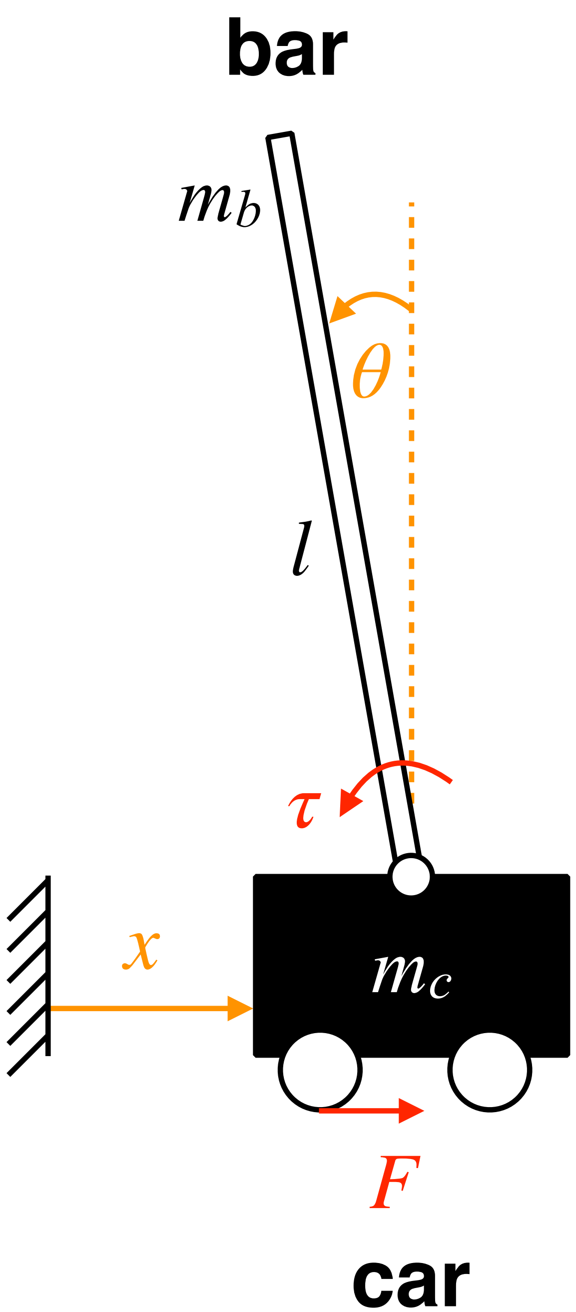
$$\beta_1=-2\pi\times 1$$

$$\beta_2=-2\pi\times 1$$





两种情况：轮胎不打滑和打滑



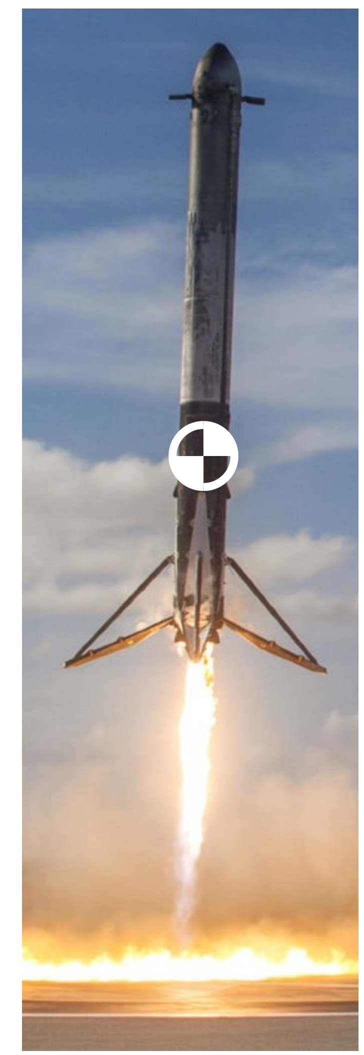
一、轮胎不打滑

控制电机的转速，
直接驱动$\dot{x}$

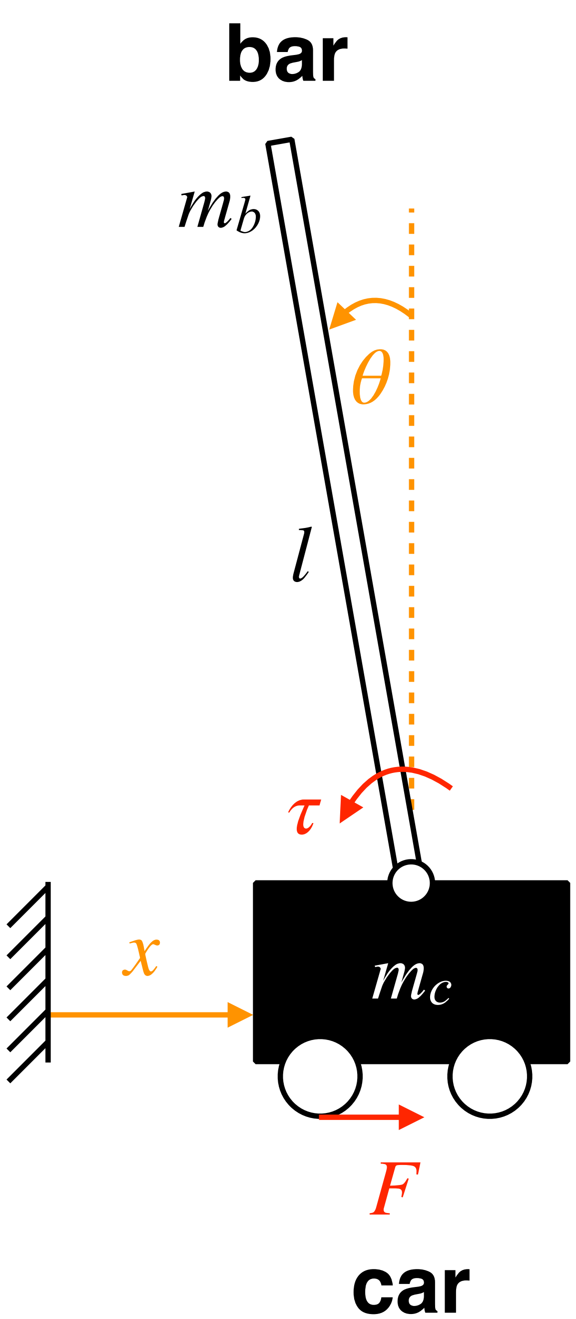
二、轮胎打滑

控制电机的输出力矩，即控制F。

对应火箭控制模型



倒立摆的控制理论 — 动力学方程



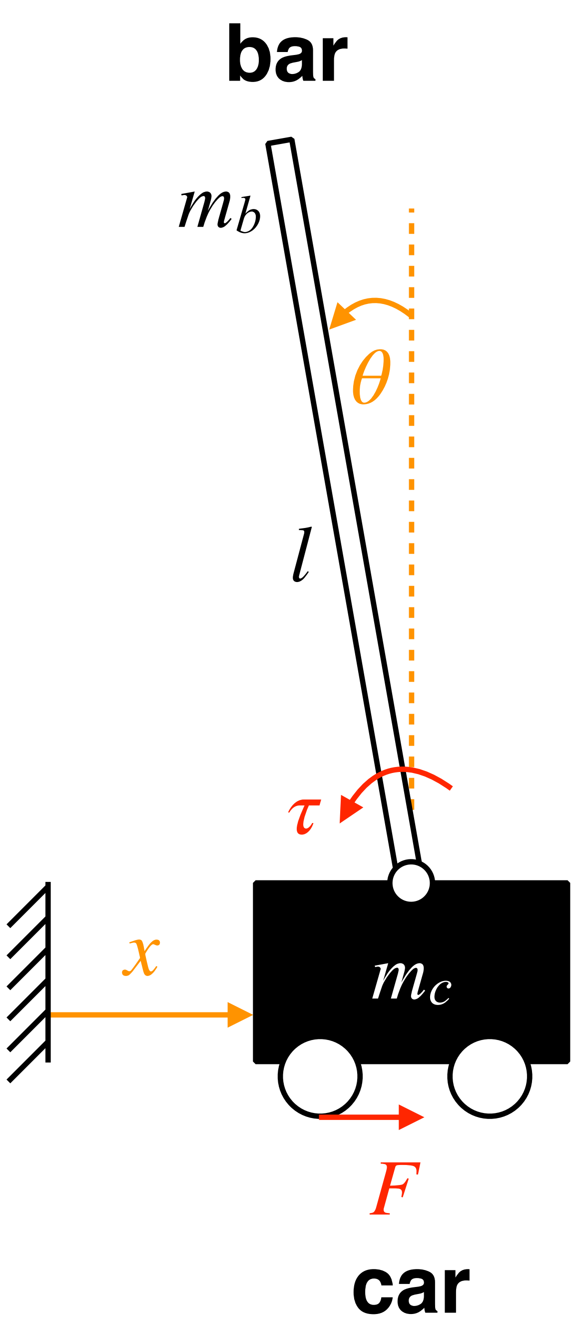
1、$\boldsymbol{q}=(x,\theta)^{\intercal}$，用拉格朗日方程列出系统的运动微分方程，求解得动力学响应

2、分析所有平衡位置的稳定性，特别是倒立状态。

3、在倒立状态下，选择模态坐标为广义坐标，得到无外力时解耦的方程，然后看如何设计外力能让系统稳定在倒立状态。

4、让小车移动一个预先设计好的路径，同时维持单摆倒立不倒。

1. 倒立摆的运动微分方程



动能 $T=\frac{1}{2}m_c\dot{x}^2+\frac{1}{2}m_b(\dot{x}^2-l\dot{x}\dot{\theta}\cos\theta+\frac{1}{4}l^2\dot{\theta}^2)+\frac{1}{2}\frac{1}{12}m_bl^2\dot{\theta}^2$

势能 $V=m_bg\frac{l}{2}\cos\theta$

利用第二类拉格朗日方程得到系统运动微分方程

$$F=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=(m_c+m_b)\ddot{x}-\frac{1}{2}m_bl\ddot{\theta}\cos\theta+\frac{1}{2}m_bl\dot{\theta}^2\sin\theta$$

$$\tau=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\frac{1}{3}m_bl^2\ddot{\theta}-\frac{1}{2}m_bl\ddot{x}\cos\theta-m_bg\frac{l}{2}\sin\theta$$

写成矩阵向量形式

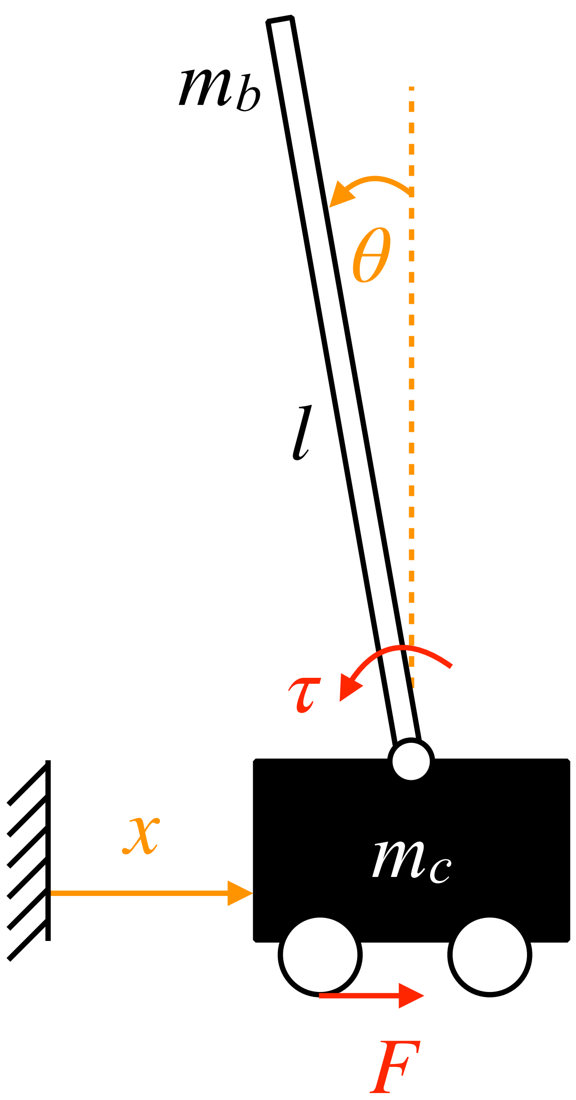
$$\boldsymbol{M}\ddot{\boldsymbol{q}}=\boldsymbol{F}+\boldsymbol{F}_e \qquad \boldsymbol{q}(t)=\begin{pmatrix}x\\ \theta\end{pmatrix} \qquad \boldsymbol{F}=\begin{pmatrix}F\\ \tau\end{pmatrix}$$

$$\boldsymbol{M}=\begin{pmatrix}m_c+m_b & -m_b\frac{l}{2}\cos\theta\\ -m_b\frac{l}{2}\cos\theta & \frac{1}{3}m_bl^2\end{pmatrix} \qquad \boldsymbol{F}_e=m_b\frac{l}{2}\begin{pmatrix}-\dot{\theta}^2\sin\theta\\ g\sin\theta\end{pmatrix}$$

1. 倒立摆的运动微分方程 – 无速度释放

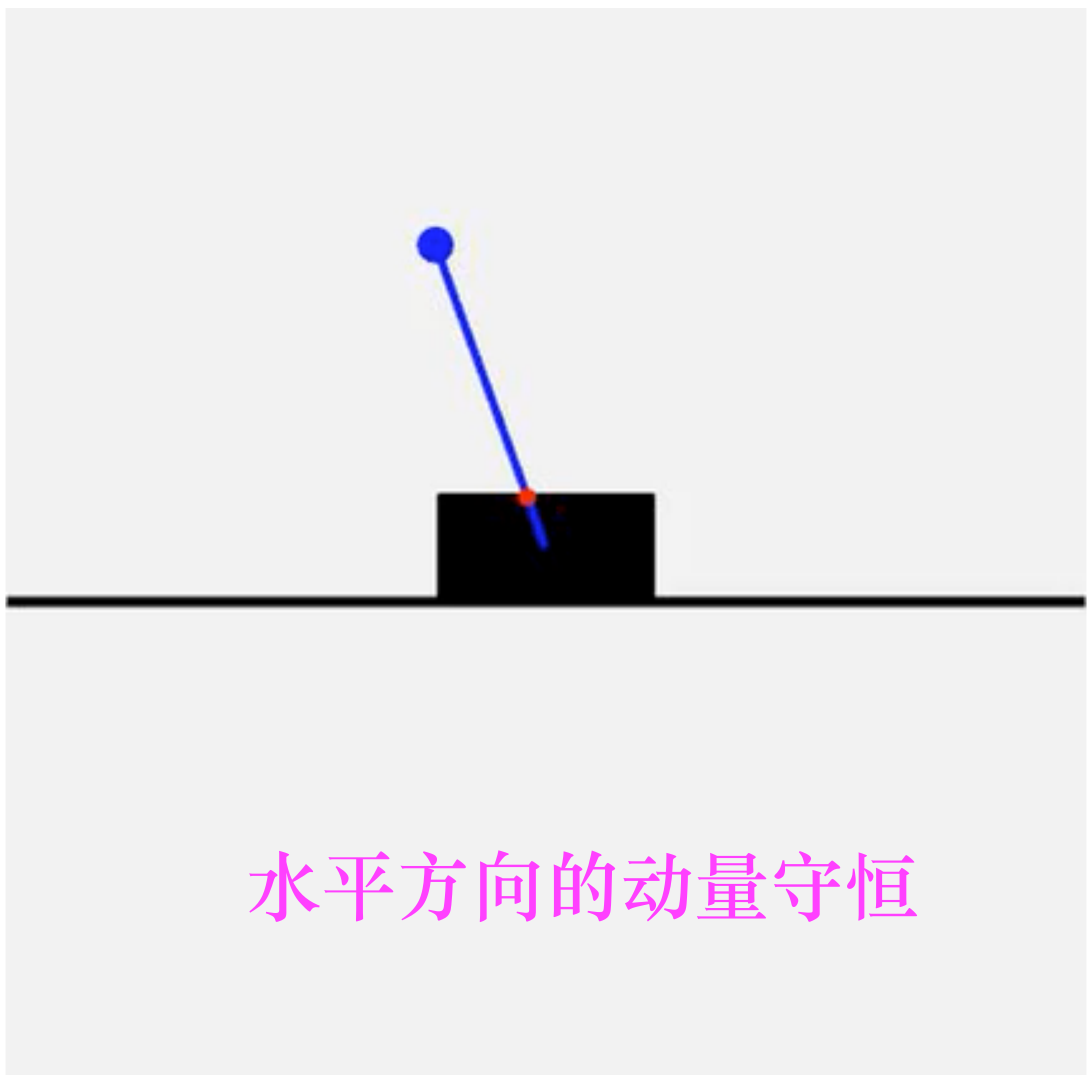
$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \cancel{\boldsymbol{F}} + \boldsymbol{F}_e \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b\frac{l}{2}\cos\theta \\ -m_b\frac{l}{2}\cos\theta & \frac{1}{3}m_b l^2 \end{pmatrix} \qquad \boldsymbol{F}_e = m_b\frac{l}{2}\begin{pmatrix} -\dot{\theta}^2\sin\theta \\ g\sin\theta \end{pmatrix}$$



初始偏置一定角度，然后静止释放 $\theta(0) = 10°$

$x(0) = \dot{x}(0) = \dot{\theta}(0) = 0$



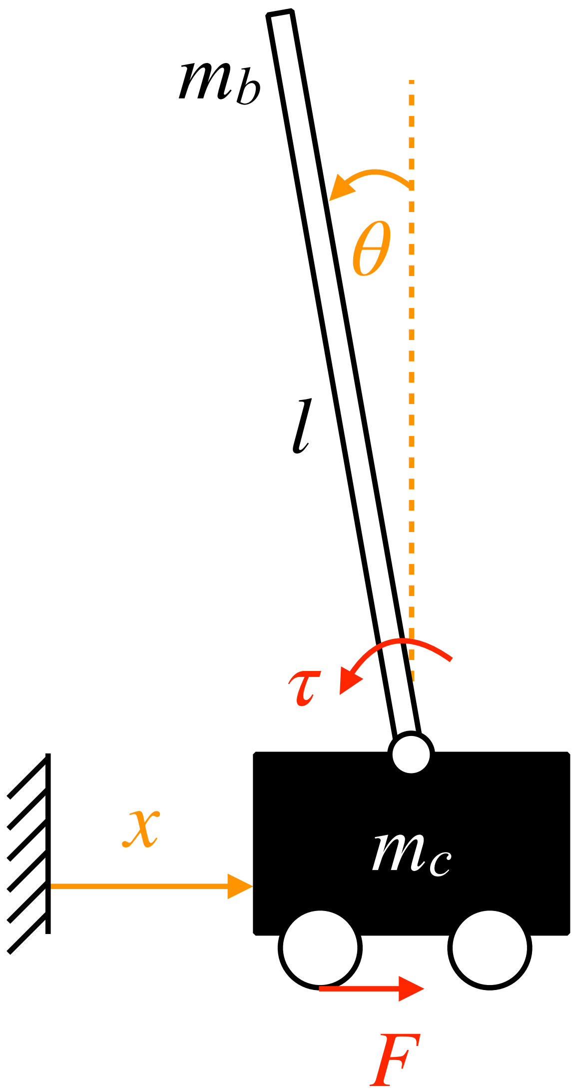
水平方向的动量守恒

降阶

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix} \quad \dot{\boldsymbol{Y}} = \begin{bmatrix} y(3) \\ y(4) \\ \boldsymbol{M}^{-1}\boldsymbol{F}_e \end{bmatrix}$$

调用matlab的ode45可求解

1. 倒立摆的运动微分方程 — 平衡位置

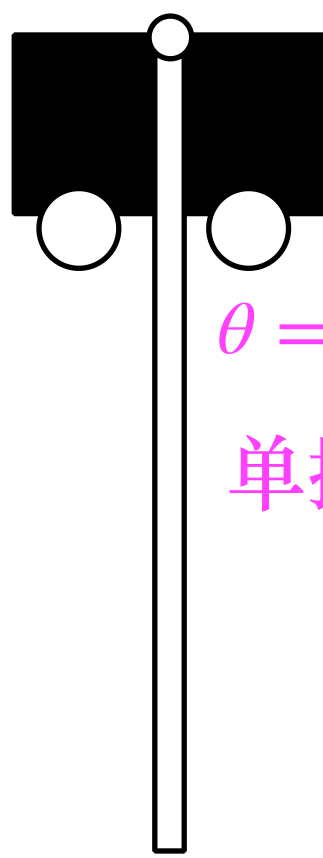


$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \cancel{\boldsymbol{F}} + \boldsymbol{F}_e \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b \frac{l}{2}\cos\theta \\ -m_b \frac{l}{2}\cos\theta & \frac{1}{3} m_b l^2 \end{pmatrix} \quad \boldsymbol{F}_e = m_b \frac{l}{2} \begin{pmatrix} -\dot{\theta}^2 \sin\theta \\ g \sin\theta \end{pmatrix}$$

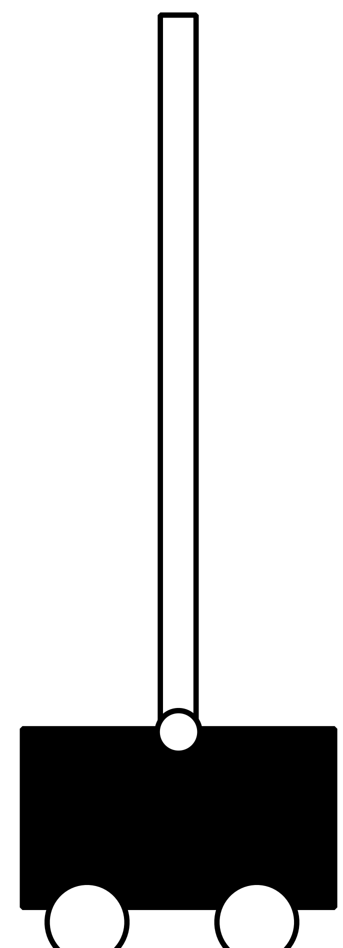
不受外力时的平衡位置满足 $m_b g l/2 \sin\theta = 0$

有两个平衡位置



$\theta = \pi$

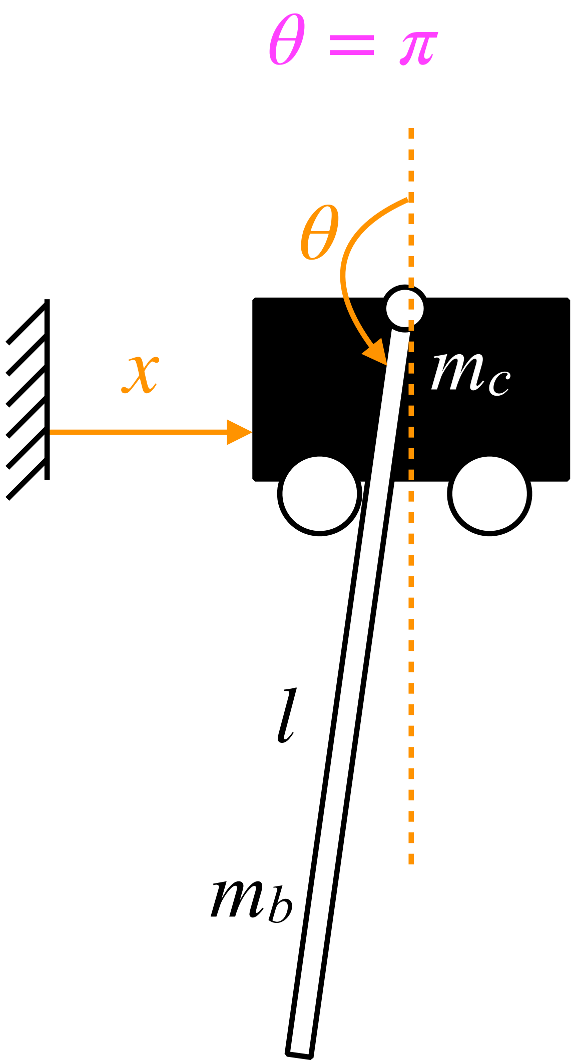
单摆



$\theta = 0$

倒立摆

2. 固有频率和模态 — 平衡位置1



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \cancel{\boldsymbol{F}} + \boldsymbol{F}_e \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b\frac{l}{2}\cos\theta \\ -m_b\frac{l}{2}\cos\theta & \frac{1}{3}m_b l^2 \end{pmatrix} \qquad \boldsymbol{F}_e = m_b\frac{l}{2}\begin{pmatrix} -\dot{\theta}^2\sin\theta \\ g\sin\theta \end{pmatrix}$$

平衡位置附近做小幅运动时的控制方程

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{F} \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \Delta\theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & m_b\frac{l}{2} \\ m_b\frac{l}{2} & \frac{1}{3}m_b l^2 \end{pmatrix} \qquad \boldsymbol{K} = \frac{m_b g l}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

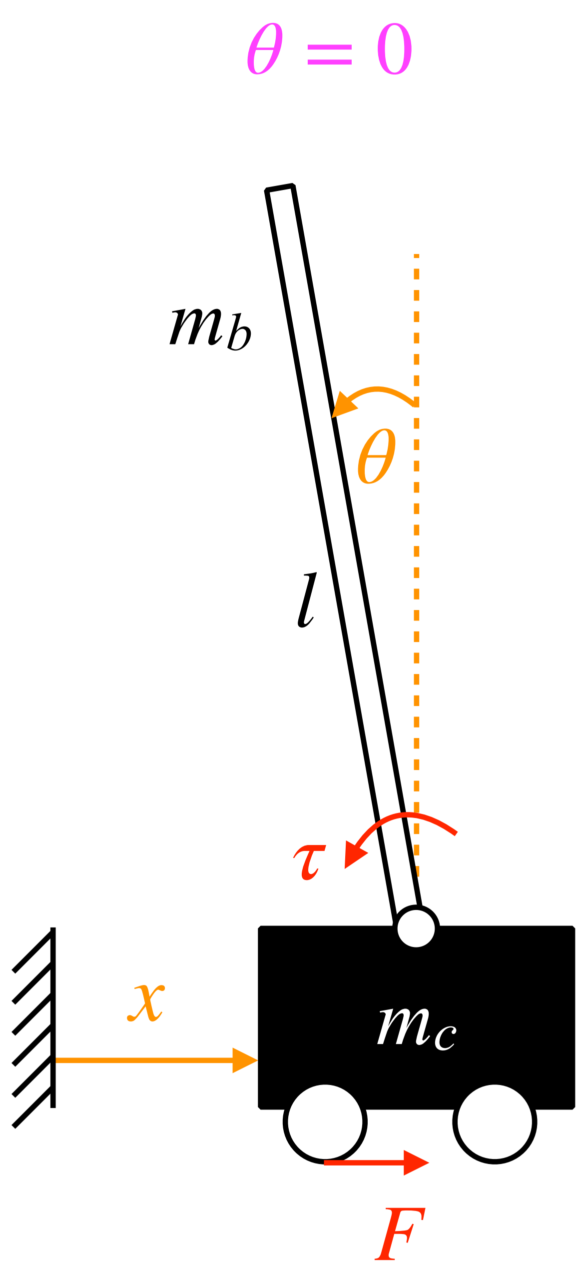
$$\boldsymbol{q}(t) = \boldsymbol{\varphi}e^{\lambda t} \qquad (\lambda^2\boldsymbol{M} + \boldsymbol{K})\boldsymbol{\varphi} = \boldsymbol{0}$$

$$\lambda_1 = 0 \qquad \lambda_2 = \pm 7.487j$$

刚体模态 $\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix}$ $\qquad \boldsymbol{\varphi}_2 = \begin{pmatrix} -0.488 \\ 19.518 \end{pmatrix}$

0频没有重根 平衡位置1稳定

2. 固有频率和模态 — 平衡位置2



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \cancel{\boldsymbol{F}} + \boldsymbol{F}_e \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b \frac{l}{2}\cos\theta \\ -m_b \frac{l}{2}\cos\theta & \frac{1}{3} m_b l^2 \end{pmatrix} \qquad \boldsymbol{F}_e = m_b \frac{l}{2} \begin{pmatrix} -\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{pmatrix}$$

平衡位置附近做小幅运动时的控制方程

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{F} \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \Delta\theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b \frac{l}{2} \\ -m_b \frac{l}{2} & \frac{1}{3} m_b l^2 \end{pmatrix} \qquad \boldsymbol{K} = \frac{m_b g l}{2} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{q}(t) = \boldsymbol{\varphi} e^{\lambda t} \qquad (\lambda^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{\varphi} = \boldsymbol{0}$$

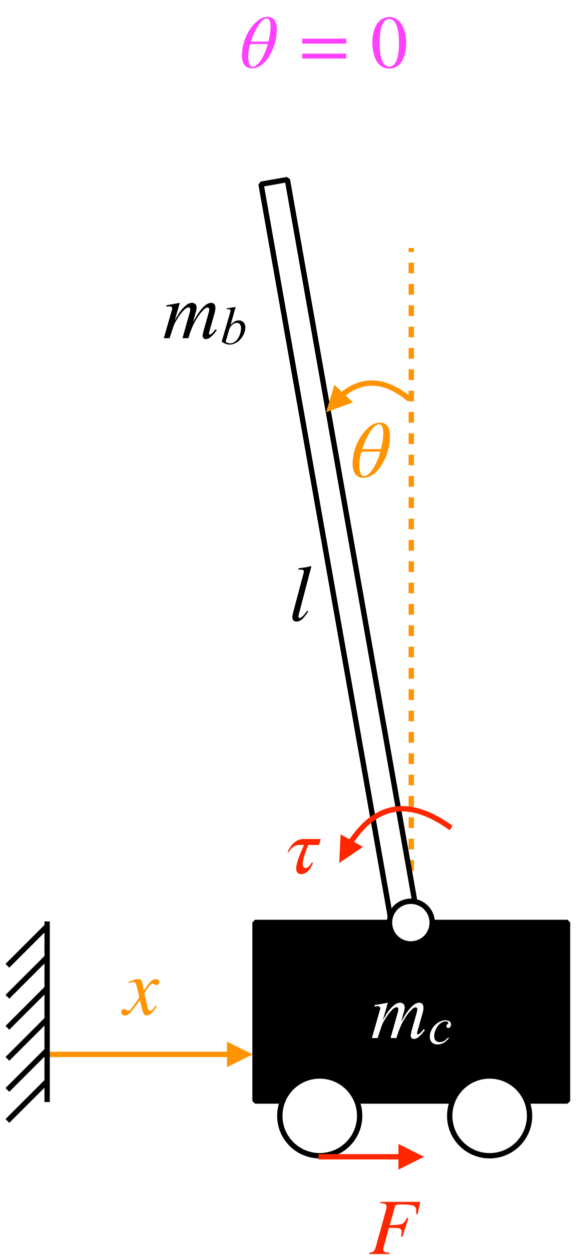
$$\lambda_1 = 0 \qquad \lambda_2 = \pm 7.487 \propto \sqrt{\frac{g}{l}}$$

l 越长发散的越慢

刚体模态 $\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix}$ $\qquad \boldsymbol{\varphi}_2 = \begin{pmatrix} 0.488 \\ 19.518 \end{pmatrix}$

平衡位置2不稳定

3. 倒立状态的控制率设计 — 模态叠加法



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{F} \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \Delta\theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ \tau \end{pmatrix}$$

$$\boldsymbol{q}(t) = \boldsymbol{\varphi} e^{\lambda t} \qquad (\lambda^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{\varphi} = \boldsymbol{0}$$

平衡位置2不稳定

$$\lambda_1 = 0 \qquad \lambda_2 = \pm 7.487$$

刚体模态

$$\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \varphi_{21} \end{pmatrix} \qquad \boldsymbol{\varphi}_2 = \begin{pmatrix} 0.488 \\ 19.518 \end{pmatrix} = \begin{pmatrix} \varphi_{12} \\ \varphi_{22} \end{pmatrix}$$

定义 $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]$，用模态叠加法求解 $\boldsymbol{q}(t) = \boldsymbol{\Phi}\boldsymbol{s}(t)$

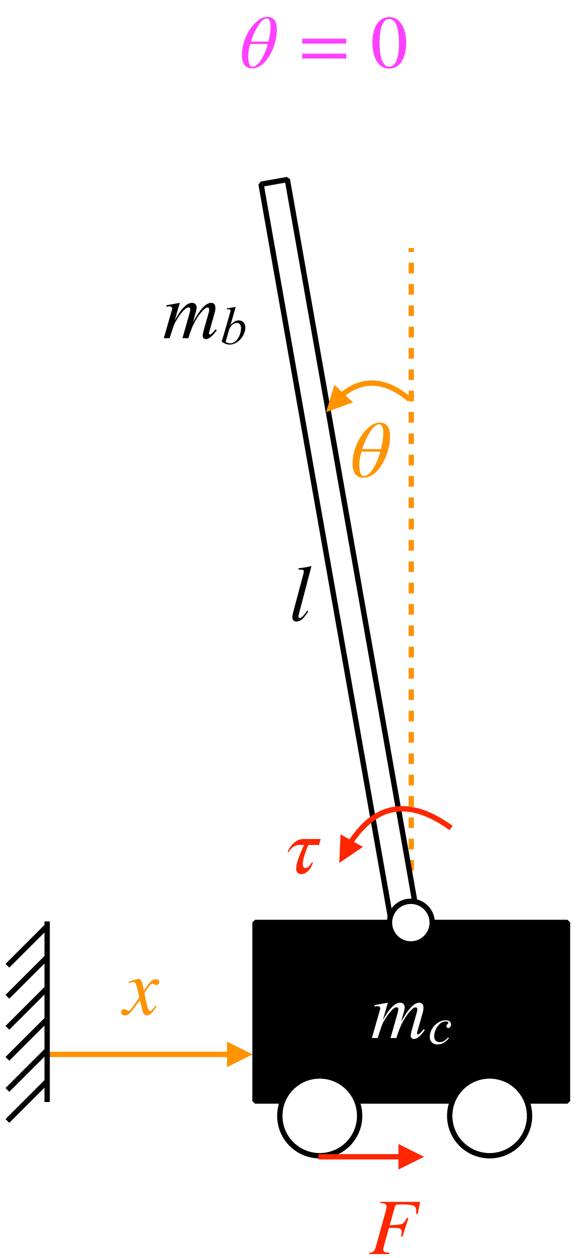
$$\ddot{s}_1 - \lambda_1^2 s_1 = \boldsymbol{\varphi}_1^{\mathrm{T}} \boldsymbol{F} = \varphi_{11} F + \varphi_{21}\tau$$ 刚体模态对应的方程

$$\ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^{\mathrm{T}} \boldsymbol{F} = \varphi_{12} F + \varphi_{22}\tau$$ 不稳定的模态

本来可以把具体值代入，但这里留着符号，让后面的推导适用范围更广。

3.1 倒立状态的控制率设计 — 双电机

$\boldsymbol{q}(t)=\boldsymbol{\Phi}\boldsymbol{s}(t)$



$m_c=0.5$ kg

$m_b=0.1$ kg

$l=0.3$ m

$\lambda_1=0$ $\lambda_2=\pm 7.487$

$$\boldsymbol{\varphi}_1=\begin{pmatrix}1.291\\0\end{pmatrix}=\begin{pmatrix}\varphi_{11}\\\varphi_{21}\end{pmatrix}\qquad \boldsymbol{\varphi}_2=\begin{pmatrix}0.488\\19.518\end{pmatrix}=\begin{pmatrix}\varphi_{12}\\\varphi_{22}\end{pmatrix}$$

假设有两个电机， 可设计 F 和 τ 让两个模态坐标都稳定

$$\ddot{s}_1-\lambda_1^2 s_1=\boldsymbol{\varphi}_1^{\mathrm{T}}\boldsymbol{F}=\boxed{\varphi_{11}F+\varphi_{21}\tau=-k_1 s_1-c_1\dot{s}_1-\lambda_1^2 s_1}$$

$$\ddot{s}_1+c_1\dot{s}_1+k_1 s_1=0 \qquad c_1=2\sqrt{k_1}$$

为了能够衰减的快，可以取临界阻尼

$$\ddot{s}_2-\lambda_2^2 s_2=\boldsymbol{\varphi}_2^{\mathrm{T}}\boldsymbol{F}=\boxed{\varphi_{12}F+\varphi_{22}\tau=-k_2 s_2-c_2\dot{s}_2-\lambda_2^2 s_2}$$

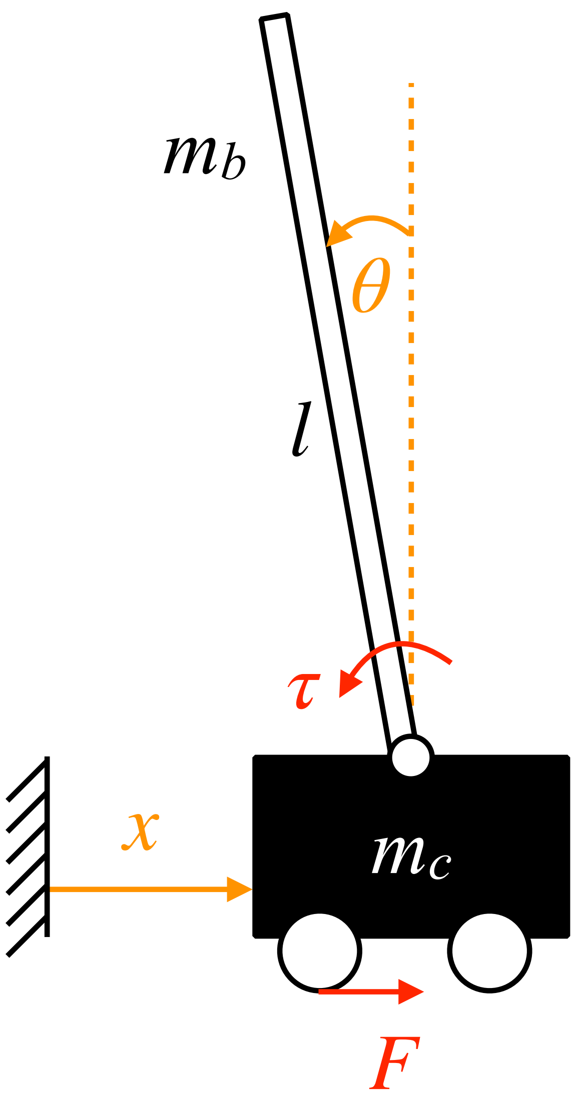
$$\ddot{s}_2+c_2\dot{s}_2+k_2 s_2=0 \qquad c_2=2\sqrt{k_2}$$

为了能够衰减的快，可以取临界阻尼

3.1 倒立状态的控制率设计 — 双电机

$q(t) = \boldsymbol{\Phi} s(t)$

$\theta = 0$



$m_c = 0.5$ kg
$m_b = 0.1$ kg
$l = 0.3$ m

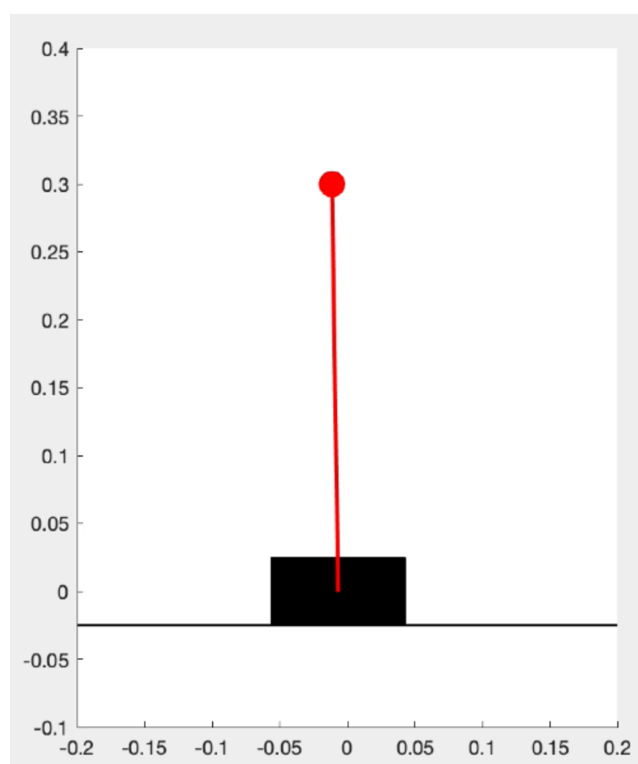
$$\boxed{\varphi_{11}F + \varphi_{21}\tau = -k_1 s_1 - c_1\dot{s}_1 - \lambda_1^2 s_1}$$

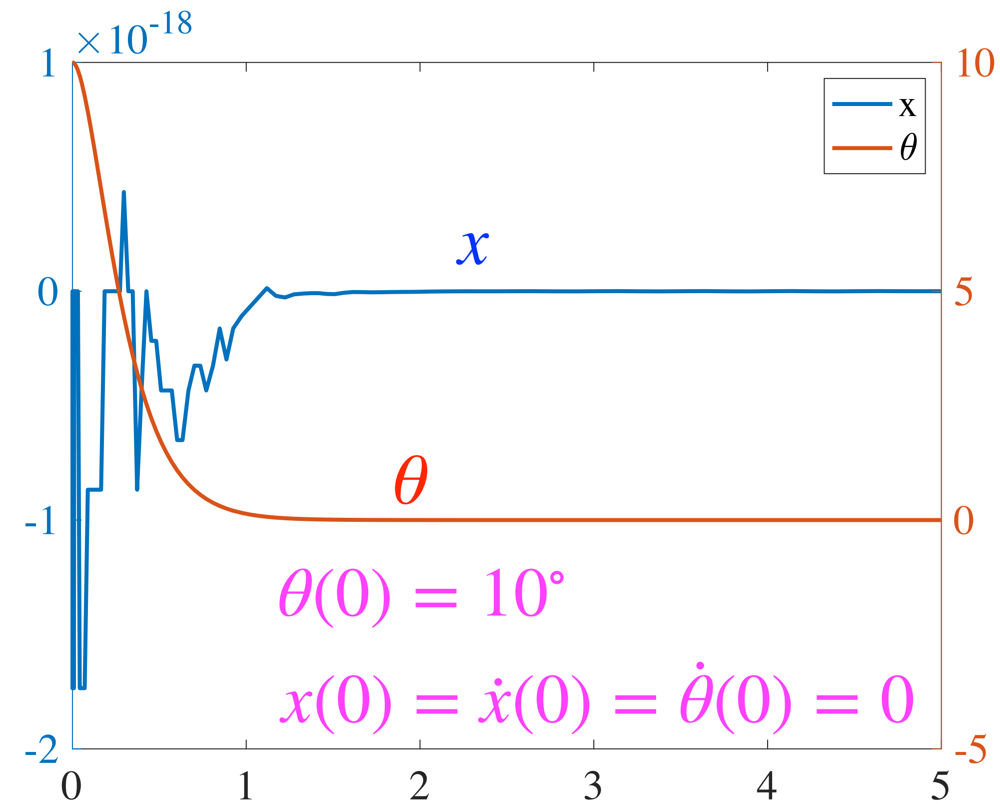
$$\boxed{\varphi_{12}F + \varphi_{22}\tau = -k_2 s_2 - c_2\dot{s}_2 - \lambda_2^2 s_2}$$

$$\boldsymbol{\Phi}^{\intercal}\boldsymbol{F} = \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix}\begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} -k_1 s_1 - c_1\dot{s}_1 - \lambda_1^2 s_1 \\ -k_2 s_2 - c_2\dot{s}_2 - \lambda_2^2 s_2 \end{pmatrix}$$

$$(\boldsymbol{\Phi}^{\intercal})^{-1} = \boldsymbol{M}\boldsymbol{\Phi}$$

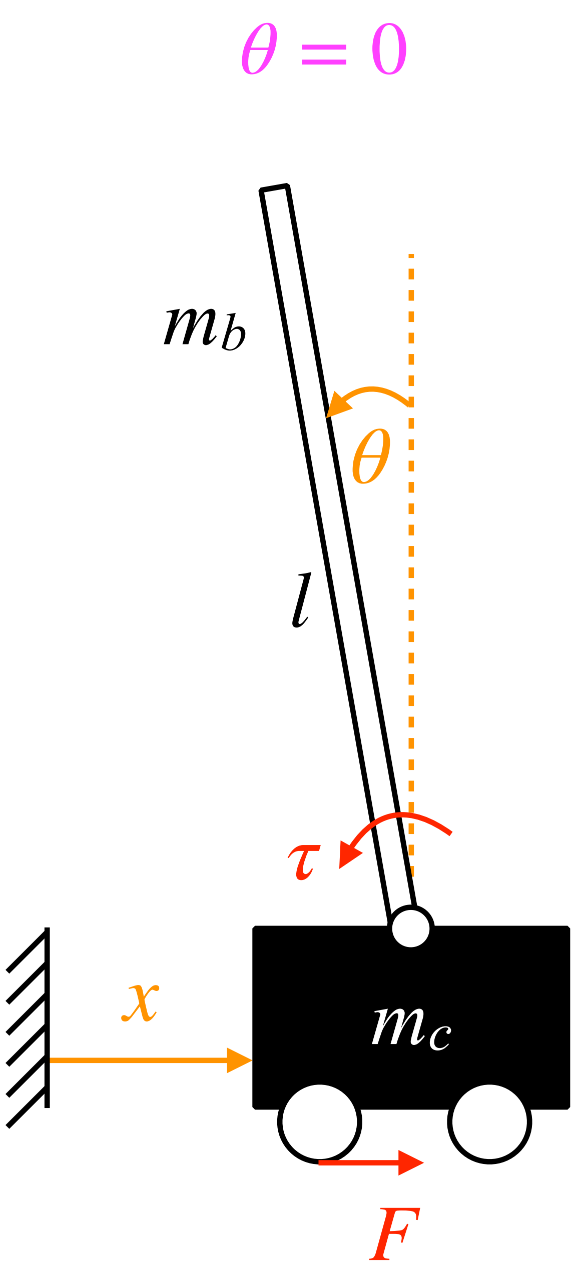
取临界阻尼 $k_1 = k_2 = (2\pi \times 1)^2 \quad c_1 = c_2 = 4\pi \times 1$





3.2 倒立状态的控制率设计 — 单电机

$$\boldsymbol{q}(t)=\boldsymbol{\Phi}\boldsymbol{s}(t)$$



$m_c=0.5$ kg

$m_b=0.1$ kg

$l=0.3$ m

$$\lambda_1=0 \qquad \lambda_2=\pm 7.487$$

$$\boldsymbol{\varphi}_1=\begin{pmatrix}1.291\\0\end{pmatrix}=\begin{pmatrix}\varphi_{11}\\\varphi_{21}\end{pmatrix} \qquad \boldsymbol{\varphi}_2=\begin{pmatrix}0.488\\19.518\end{pmatrix}=\begin{pmatrix}\varphi_{12}\\\varphi_{22}\end{pmatrix}$$

假设只有轮子上有电机， 欠驱动，两个目标量，一个输入量

刚体模态对应的方程

$$\ddot{s}_1-\cancel{\lambda_1^2 s_1}=\boldsymbol{\varphi}_1^{\mathrm{T}}\boldsymbol{F}=\varphi_{11}F+\cancel{\varphi_{21}\tau}=\varphi_{11}F$$

当 s_2 趋于零，F 趋于零，$\ddot{s}_1=0$ s_1 可以保持匀速运动

倒立不稳定的模态对应的方程

$$\ddot{s}_2-\lambda_2^2 s_2=\boldsymbol{\varphi}_2^{\mathrm{T}}\boldsymbol{F}=\varphi_{12}F+\cancel{\varphi_{22}\tau}=\varphi_{12}F=-k_2s_2-c_2\dot{s}_2-\lambda_2^2 s_2$$

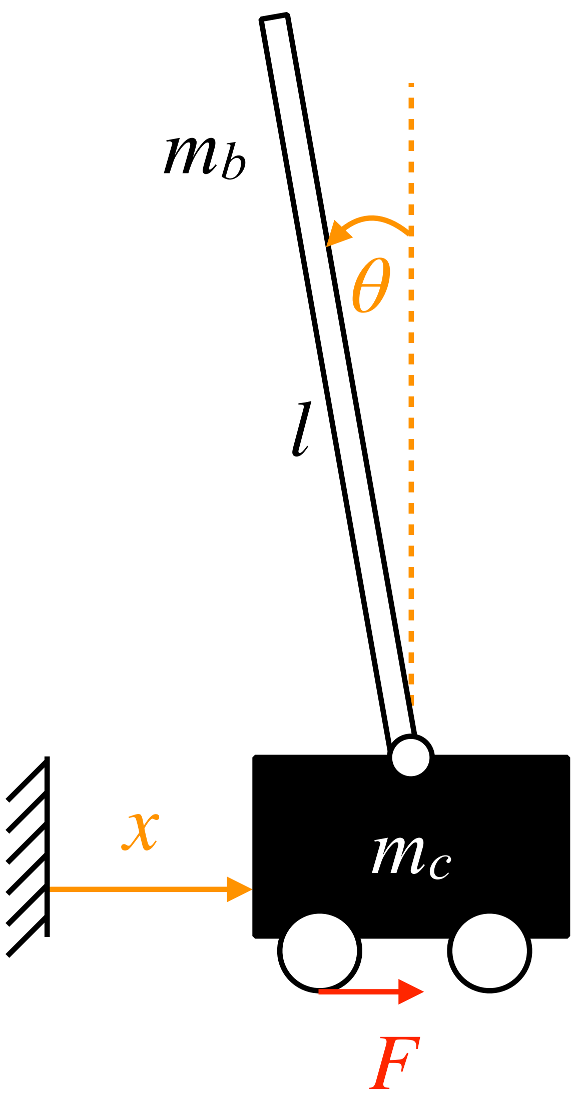
$$\ddot{s}_2+c_2\dot{s}_2+k_2s_2=0 \qquad c_2=2\sqrt{k_2}$$

为衰减的快，可以取临界阻尼

3.2 倒立状态的控制率设计 — 单电机

$$\boldsymbol{q}(t) = \boldsymbol{\Phi}\boldsymbol{s}(t)$$

$\theta = 0$



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\lambda_1 = 0 \qquad \lambda_2 = \pm 7.487$$

$$\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \varphi_{21} \end{pmatrix} \qquad \boldsymbol{\varphi}_2 = \begin{pmatrix} 0.488 \\ 19.518 \end{pmatrix} = \begin{pmatrix} \varphi_{12} \\ \varphi_{22} \end{pmatrix}$$

假设只有轮子上有电机， 欠驱动，两个目标量，一个输入量

$$\ddot{s}_1 - \cancel{\lambda_1^2 s_1} = \boldsymbol{\varphi}_1^{\mathsf{T}}\boldsymbol{F} = \varphi_{11}F + \cancel{\varphi_{21}\tau} = \varphi_{11}F$$

s_1 可以保持匀速运动

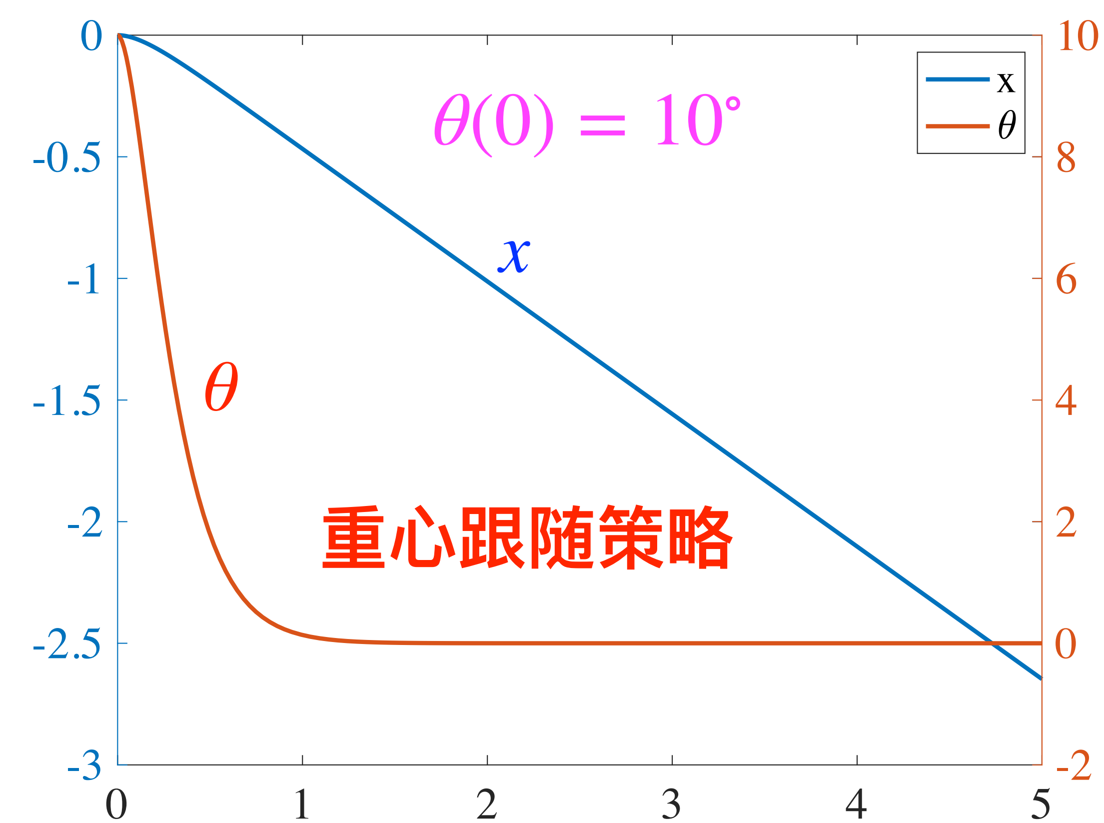
$$\ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^{\mathsf{T}}\boldsymbol{F} = \varphi_{12}F + \cancel{\varphi_{22}\tau} = \varphi_{12}F = -k_2 s_2 - c_2\dot{s}_2 - \lambda_2^2 s_2$$

取临界阻尼

$$k_2 = (2\pi \times 1)^2$$

$$c_2 = 2\sqrt{k_2}$$

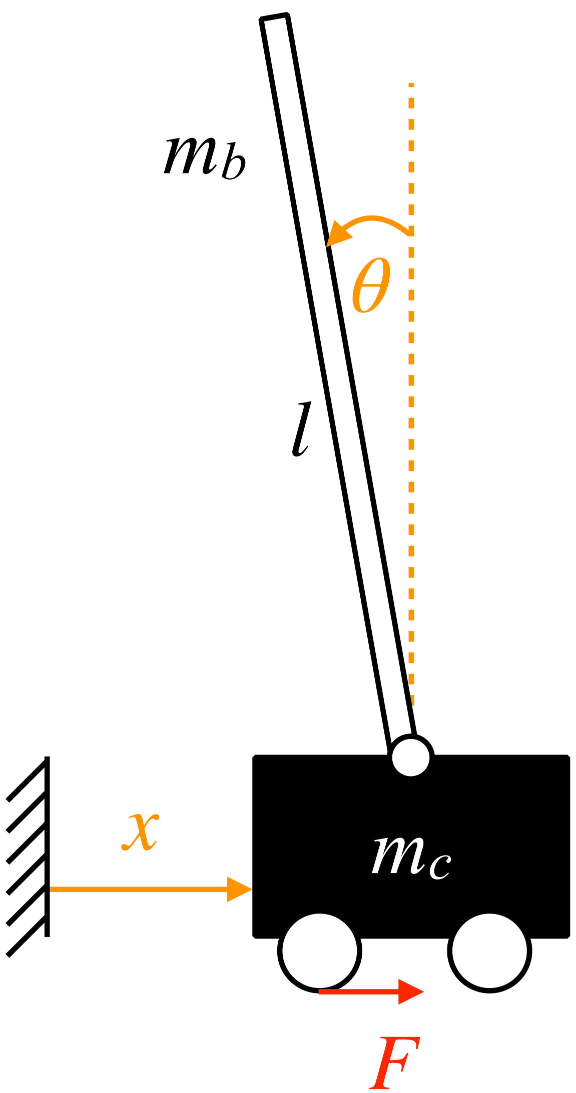
$$\ddot{s}_2 + c_2\dot{s}_2 + k_2 s_2 = 0$$



3.2 倒立状态的控制率设计 — 单电机

$q(t) = \Phi s(t)$

$\theta = 0$



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$\lambda_1 = 0$ $\quad\quad$ $\lambda_2 = \pm 7.487$

$$\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \varphi_{21} \end{pmatrix} \quad\quad \boldsymbol{\varphi}_2 = \begin{pmatrix} 0.488 \\ 19.518 \end{pmatrix} = \begin{pmatrix} \varphi_{12} \\ \varphi_{22} \end{pmatrix}$$

假设只有轮子上有电机， 欠驱动，两个目标量，一个输入量

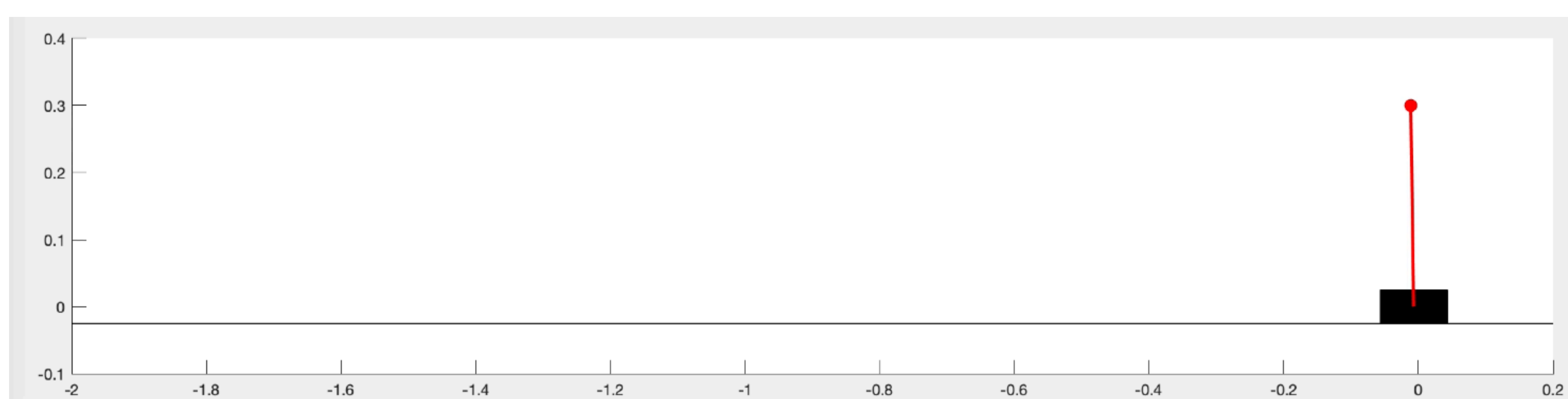
$$\ddot{s}_1 - \cancel{\lambda_1^2 s_1} = \boldsymbol{\varphi}_1^{\mathsf{T}} \boldsymbol{F} = \varphi_{11} F + \cancel{\varphi_{21}} \tau = \varphi_{11} F$$

s_1 可以保持匀速运动

$$\ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^{\mathsf{T}} \boldsymbol{F} = \varphi_{12} F + \cancel{\varphi_{22}} \tau = \varphi_{12} F = -k_2 s_2 - c_2 \dot{s}_2 - \lambda_2^2 s_2$$

$k_2 = (2\pi \times 1)^2 \quad c_2 = 2\sqrt{k_2} \quad \theta(0) = 10^\circ$

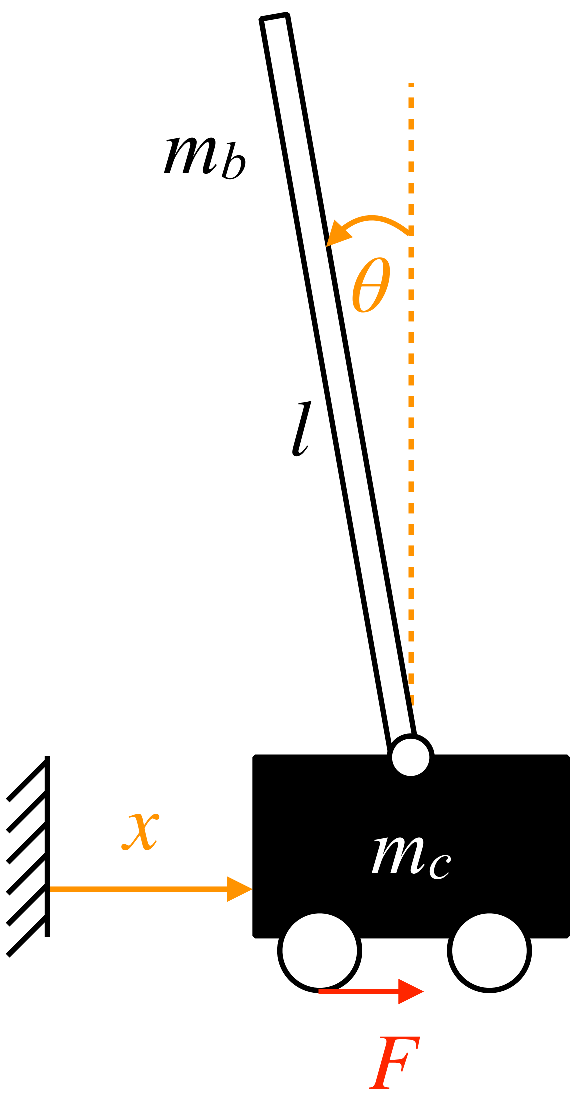
下一步让 $s_1 = 0$ 稳定



3.2 倒立状态的控制率设计 — 单电机

$\boldsymbol{q}(t) = \boldsymbol{\Phi s}(t)$

$\theta = 0$



$m_c = 0.5$ kg
$m_b = 0.1$ kg
$l = 0.3$ m

$\lambda_1 = 0$ $\qquad\qquad$ $\lambda_2 = \pm 7.487$

$$\boldsymbol{\varphi}_1 = \begin{pmatrix} 1.291 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \varphi_{21} \end{pmatrix} \qquad \boldsymbol{\varphi}_2 = \begin{pmatrix} 0.488 \\ 19.518 \end{pmatrix} = \begin{pmatrix} \varphi_{12} \\ \varphi_{22} \end{pmatrix}$$

假设只有轮子上有电机， 欠驱动，两个目标量，一个输入量

$$\ddot{s}_1 - \cancel{\lambda_1^2 s_1} = \boldsymbol{\varphi}_1^{\mathrm{T}}\boldsymbol{F} = \varphi_{11}F + \cancel{\varphi_{21}}\tau = \varphi_{11}F = \alpha f \qquad \alpha = \varphi_{11}/\varphi_{12}$$

$$\ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^{\mathrm{T}}\boldsymbol{F} = \varphi_{12}F + \cancel{\varphi_{22}}\tau = \varphi_{12}F = -k_2 s_2 - c_2\dot{s}_2 - \lambda_2^2 s_2$$

$$= f \qquad\qquad -k_1 s_1 - c_1\dot{s}_1$$

补充和 s_1 相关的量

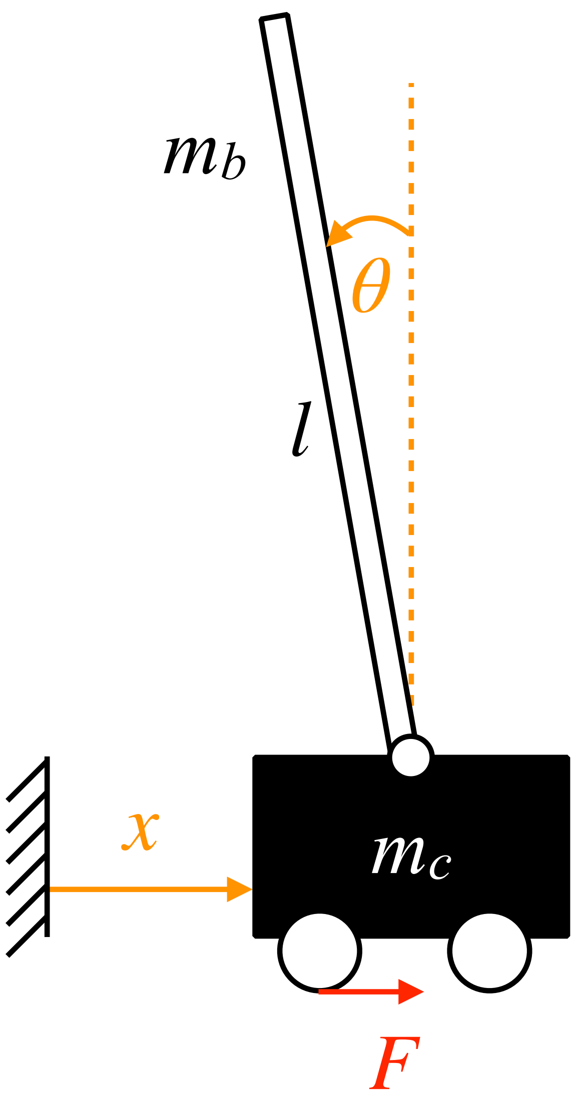
s_1 和 s_2 彼此不独立

下一步让 $s_1 = 0$ 稳定

$$\begin{cases} \ddot{s}_1 - \lambda_1^2 s_1 = \alpha f = \alpha(\ddot{s}_2 - \lambda_2^2 s_2) \\ \ddot{s}_2 + k_2 s_2 + c_2\dot{s}_2 + k_1 s_1 + c_1\dot{s}_1 = 0 \end{cases}$$

复模态

$\theta = 0$



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\begin{cases} \ddot{s}_1 - \cancel{\lambda_1^2} s_1 = \alpha f = \alpha(\ddot{s}_2 - \lambda_2^2 s_2) \\ \ddot{s}_2 + k_2 s_2 + c_2 \dot{s}_2 + k_1 s_1 + c_1 \dot{s}_1 = 0 \end{cases}$$

写成矩阵向量形式

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha\lambda_2^2 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boldsymbol{M}_s \ddot{\boldsymbol{s}} + \boldsymbol{C}_s \dot{\boldsymbol{s}} + \boldsymbol{K}_s \boldsymbol{s} = \boldsymbol{0}$ 假设有振动模态 $\boldsymbol{s}(t) = \boldsymbol{\phi} e^{\beta t}$

特征方程: $(\beta^2 \boldsymbol{M}_s + \beta \boldsymbol{C}_s + \boldsymbol{K}_s)\boldsymbol{\phi} = \boldsymbol{0}$

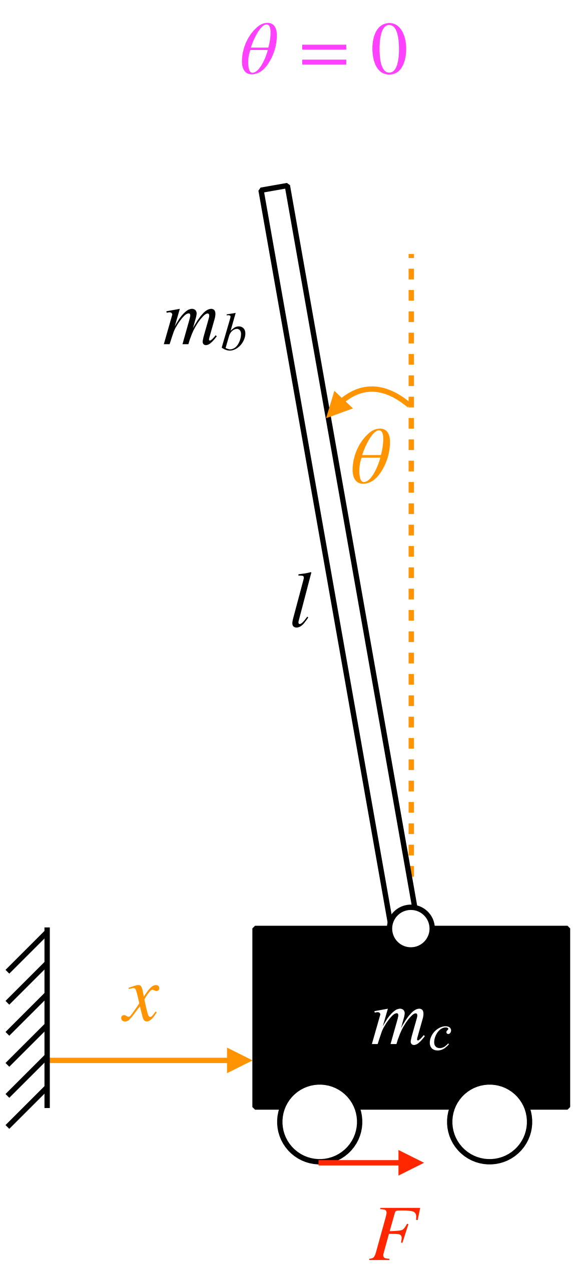
存在非平凡解: $\left| \beta^2 \boldsymbol{M}_s + \beta \boldsymbol{C}_s + \boldsymbol{K}_s \right| = 0$

$\beta \in \mathbb{C}$ $\beta = a + bj$ $\boldsymbol{\phi} \in \mathbb{C}^2$

如果某个特征值的实部为正，则平衡位置不稳定

所有特征根的实部为负，则平衡位置渐进稳定

3.2 倒立状态的控制率设计 — 单电机控制率设计



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha\lambda_2^2 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{M}_s\ddot{\boldsymbol{s}} + \boldsymbol{C}_s\dot{\boldsymbol{s}} + \boldsymbol{K}_s\boldsymbol{s} = \boldsymbol{0} \qquad \boldsymbol{s}(t) = \boldsymbol{\phi} e^{\beta t} \qquad \left|\beta^2\boldsymbol{M}_s + \beta\boldsymbol{C}_s + \boldsymbol{K}_s\right| = 0$$

$$\begin{vmatrix} \beta^2 & -\alpha\beta^2 + \alpha\lambda_2^2 \\ k_1 + c_1\beta & \beta^2 + \beta c_2 + k_2 \end{vmatrix} = 0$$

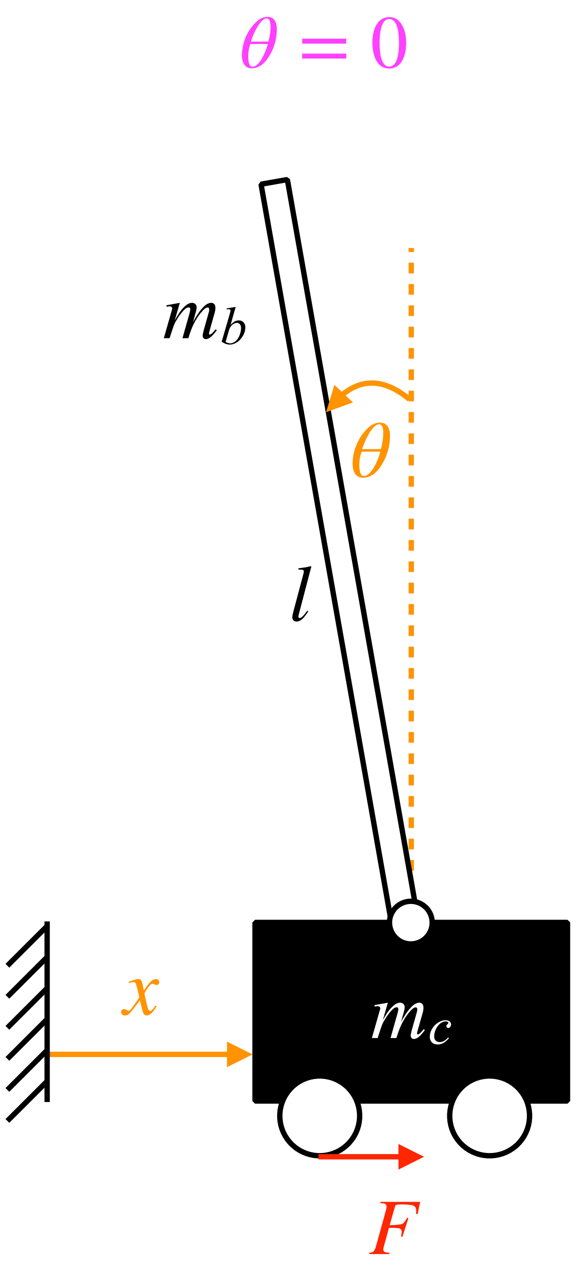
$$\beta^4 + (c_2 + \alpha c_1)\beta^3 + (k_2 + \alpha k_1)\beta^2 - \alpha\lambda_2^2 c_1\beta - k_1\lambda_2^2\alpha = 0$$

参照单自由度时临界阻尼是衰减最快的解，假设系统有两个重实根，为负数。

$$(\beta - \beta_1)^2(\beta - \beta_2)^2 = 0 \qquad \beta_1 < 0, \quad \beta_2 < 0$$

$$\beta^4 - 2(\beta_1 + \beta_2)\beta^3 + (\beta_1^2 + \beta_2^2 + 4\beta_1\beta_2)\beta^2 - 2\beta_1\beta_2(\beta_1 + \beta_2)\beta + \beta_1^2\beta_2^2 = 0$$

3.2 倒立状态的控制率设计 — 单电机控制率设计



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix}\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix}+\begin{pmatrix} 0 & \alpha\lambda_2^2 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{M}_s\ddot{\boldsymbol{s}}+\boldsymbol{C}_s\dot{\boldsymbol{s}}+\boldsymbol{K}_s\boldsymbol{s}=\boldsymbol{0} \quad \boldsymbol{s}(t)=\boldsymbol{\phi}e^{\beta t} \quad \left|\beta^2\boldsymbol{M}_s+\beta\boldsymbol{C}_s+\boldsymbol{K}_s\right|=0$$

$$\beta^4+(c_2+\alpha c_1)\beta^3+(k_2+\alpha k_1)\beta^2-\alpha\lambda_2^2c_1\beta-k_1\lambda_2^2\alpha=0$$

$$\beta^4-2(\beta_1+\beta_2)\beta^3+(\beta_1^2+\beta_2^2+4\beta_1\beta_2)\beta^2-2\beta_1\beta_2(\beta_1+\beta_2)\beta+\beta_1^2\beta_2^2=0$$

$$\beta_1<0, \quad \beta_2<0$$

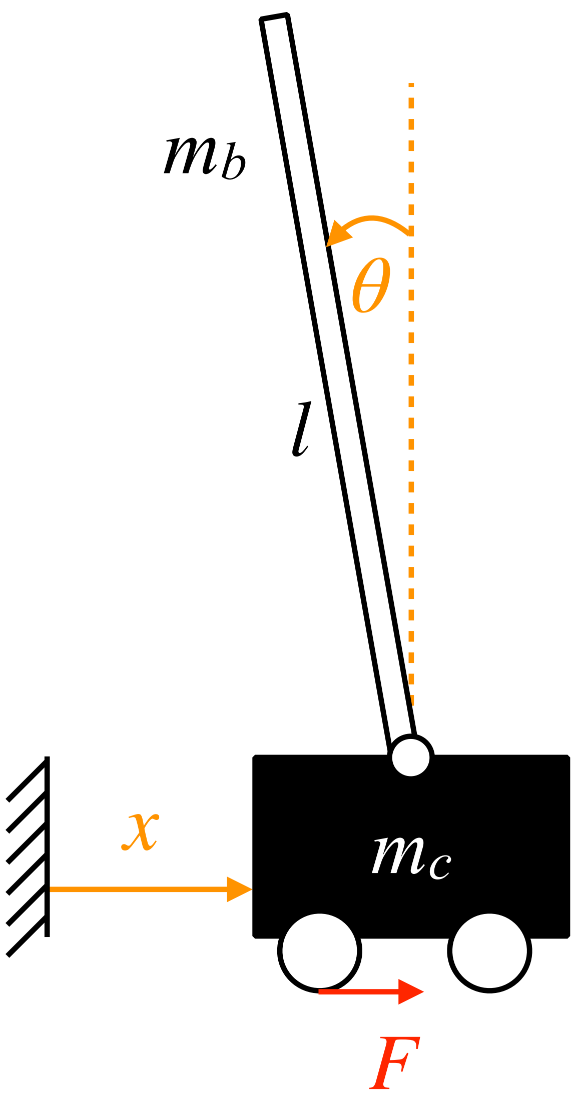
$$\boxed{\begin{aligned} &-2(\beta_1+\beta_2)=c_2+\alpha c_1 \\ &\beta_1^2+\beta_2^2+4\beta_1\beta_2=k_2+\alpha k_1 \\ &-2\beta_1\beta_2(\beta_1+\beta_2)=-\alpha\lambda_2^2c_1 \\ &\beta_1^2\beta_2^2=-k_1\lambda_2^2\alpha \end{aligned}}$$

给定β_1和β_2就能求出一组k_1, c_1, k_2, c_2

$k_1<0, c_1<0$

3.2 倒立状态的控制率设计 — 单电机控制率设计

$\theta = 0$



$m_c = 0.5$ kg

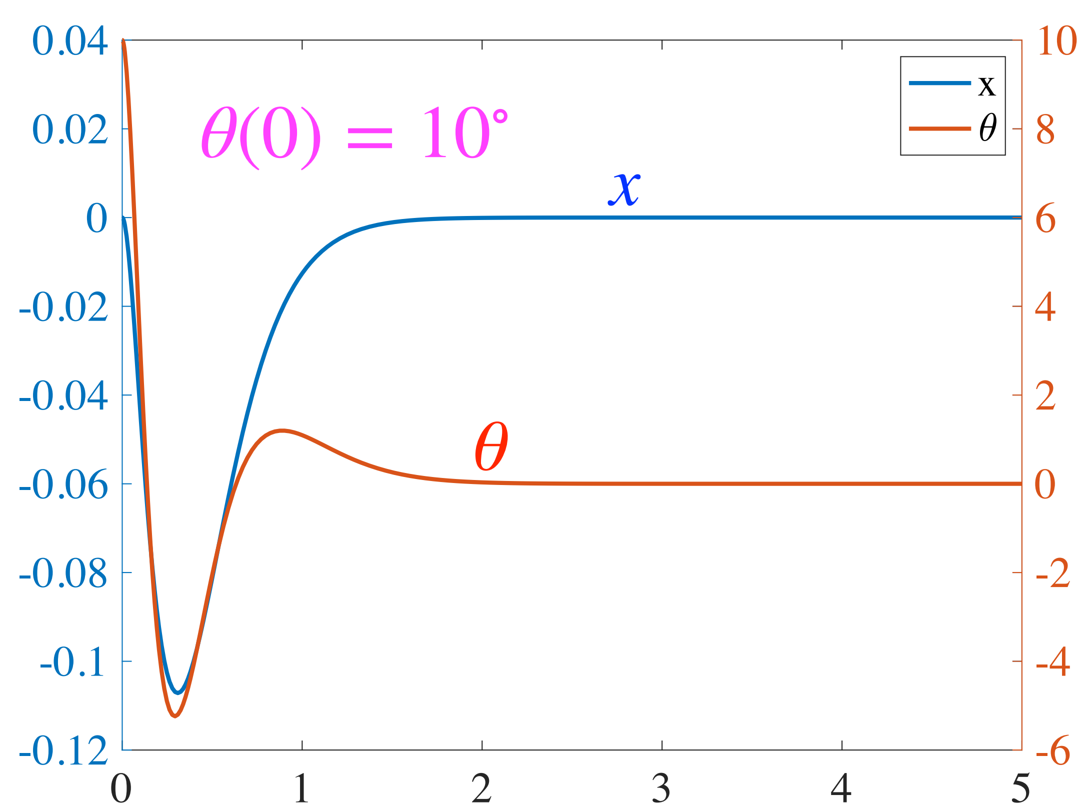
$m_b = 0.1$ kg

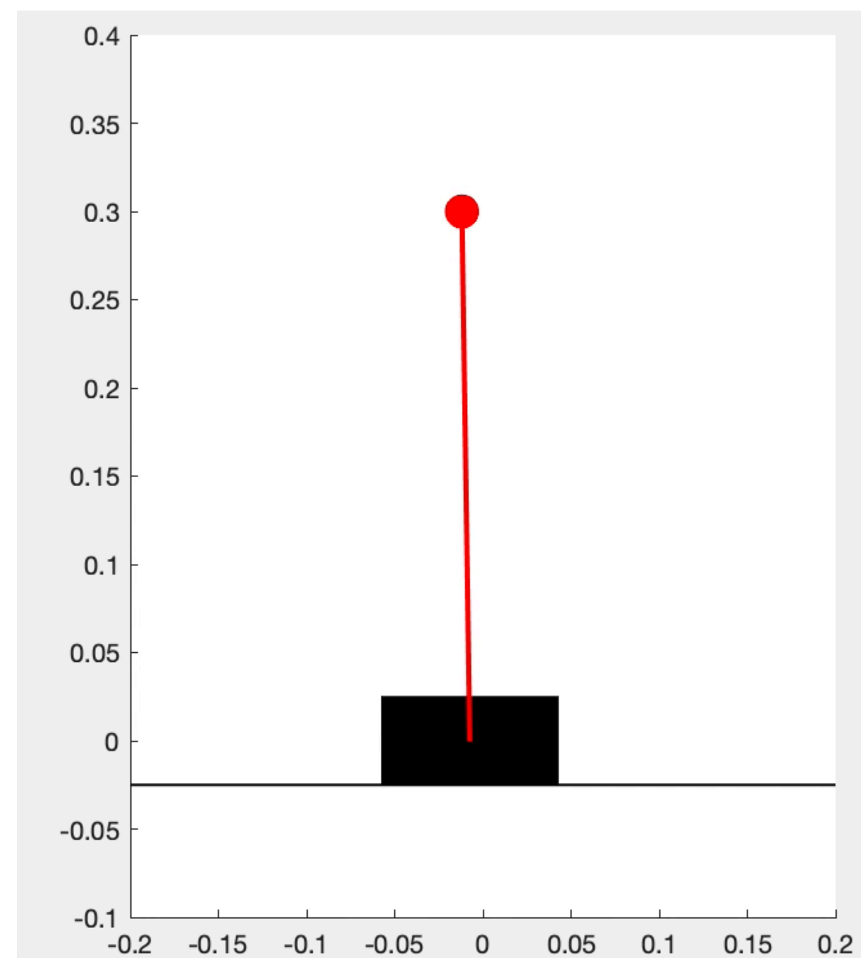
$l = 0.3$ m

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha\lambda_2^2 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

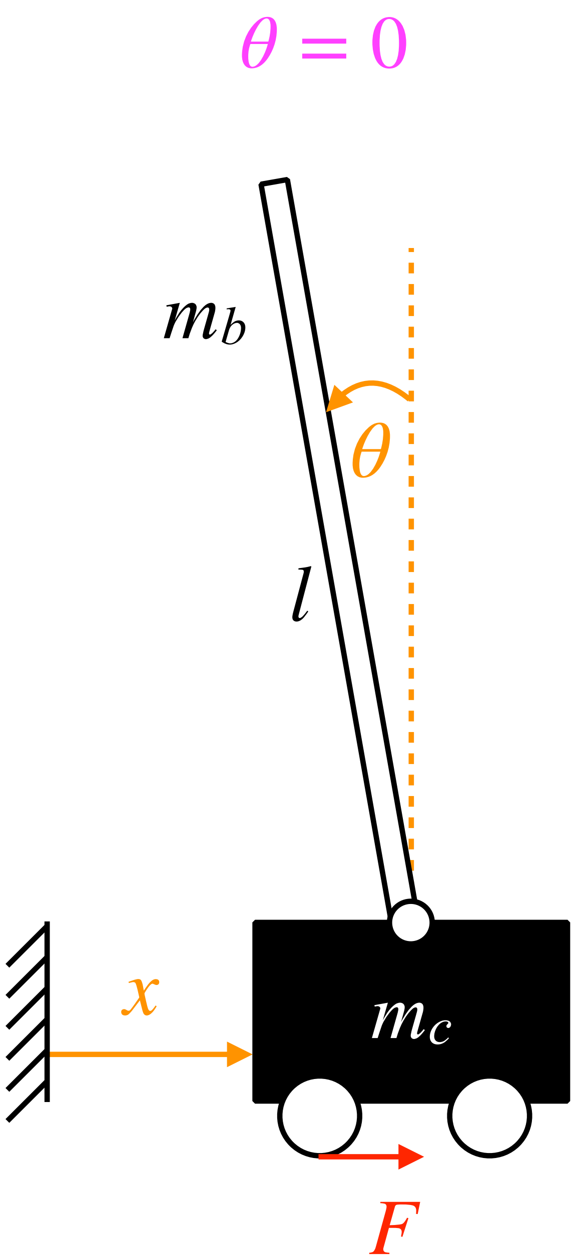
$$\boxed{\begin{aligned} -2(\beta_1 + \beta_2) &= c_2 + \alpha c_1 \\ \beta_1^2 + \beta_2^2 + 4\beta_1\beta_2 &= k_2 + \alpha k_1 \\ -2\beta_1\beta_2(\beta_1 + \beta_2) &= -\alpha\lambda_2^2 c_1 \\ \beta_1^2\beta_2^2 &= -k_1\lambda_2^2\alpha \end{aligned}}$$

$\beta_1 = -2\pi \times 1 \quad \beta_2 = -2\pi \times 1$





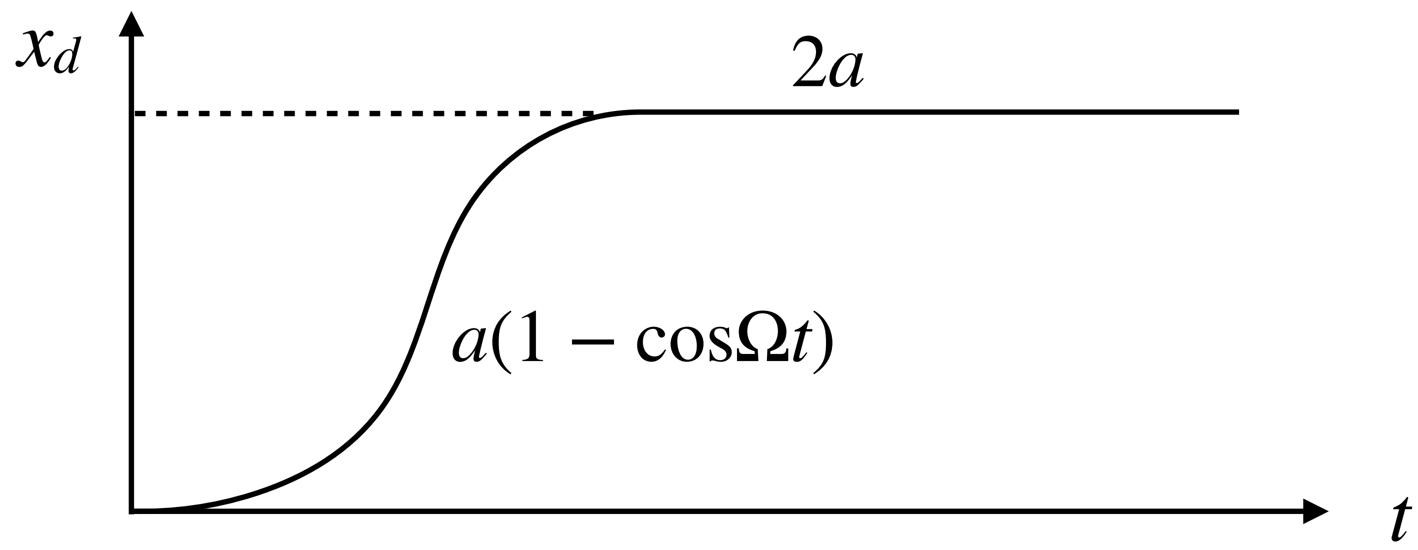
3.3 倒立状态的控制率设计 — 平移



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m



$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{F} \qquad \boldsymbol{q}(t) = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

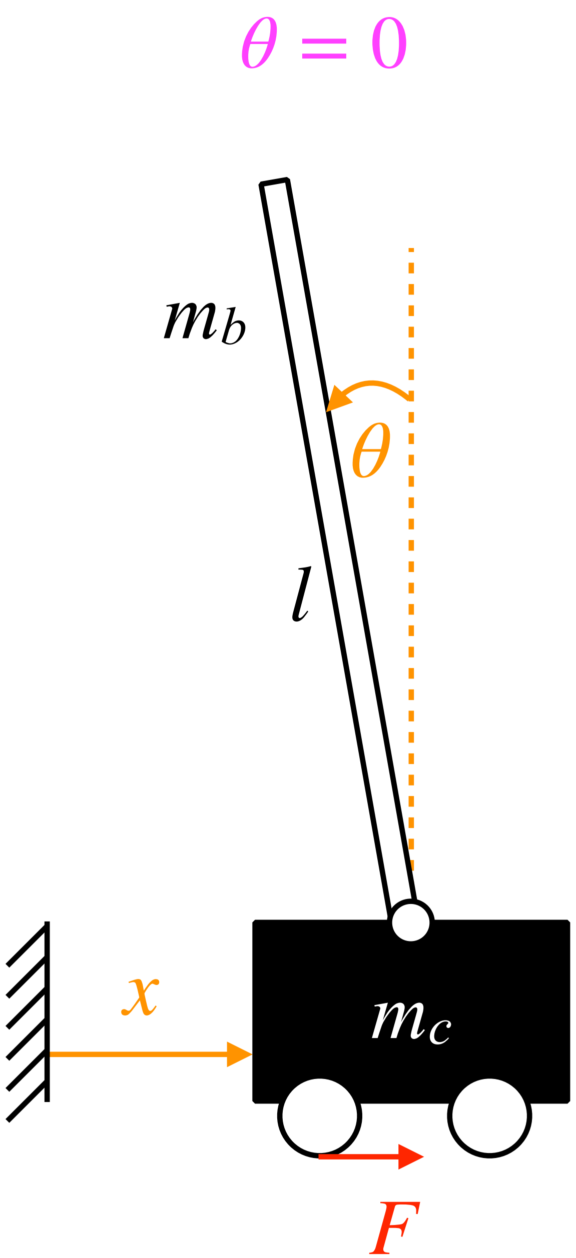
$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b l/2 \\ -m_b l/2 & m_b l^2/3 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} 0 & 0 \\ 0 & -m_b g l/2 \end{pmatrix}$$

$$\boldsymbol{M}\begin{pmatrix} \ddot{x} - \ddot{x}_d \\ \ddot{\theta} \end{pmatrix} + \boldsymbol{M}\begin{pmatrix} \ddot{x}_d \\ 0 \end{pmatrix} + \boldsymbol{K}\begin{pmatrix} x - x_d \\ \theta \end{pmatrix} + \cancel{\boldsymbol{K}\begin{pmatrix} x_d \\ 0 \end{pmatrix}} = \boldsymbol{F}$$

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{q}} + \boldsymbol{K}\Delta\boldsymbol{q} = \boldsymbol{F} - \boldsymbol{M}\begin{pmatrix} \ddot{x}_d \\ 0 \end{pmatrix} \qquad \Delta\boldsymbol{q} = \begin{pmatrix} x - x_d \\ \theta \end{pmatrix}$$

目标是让偏差量 $\Delta \boldsymbol{q}$ 在零位置稳定

3.3 倒立状态的控制率设计 — 平移



$m_c = 0.5$ kg

$m_b = 0.1$ kg

$l = 0.3$ m

$$\boldsymbol{M} = \begin{pmatrix} m_c + m_b & -m_b l/2 \\ -m_b l/2 & m_b l^2/3 \end{pmatrix} \quad \boldsymbol{K} = \begin{pmatrix} 0 & 0 \\ 0 & -m_b g l/2 \end{pmatrix} \quad \boldsymbol{F} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{q}} + \boldsymbol{K}\Delta\boldsymbol{q} = \boldsymbol{F} - \boldsymbol{M}\begin{pmatrix} \ddot{x}_d \\ 0 \end{pmatrix} \triangleq \bar{\boldsymbol{F}} \qquad \Delta\boldsymbol{q} = \begin{pmatrix} x - x_d \\ \theta \end{pmatrix}$$

定义 $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]$，用模态叠加法求解 $\Delta\boldsymbol{q}(t) = \boldsymbol{\Phi}\boldsymbol{s}(t)$

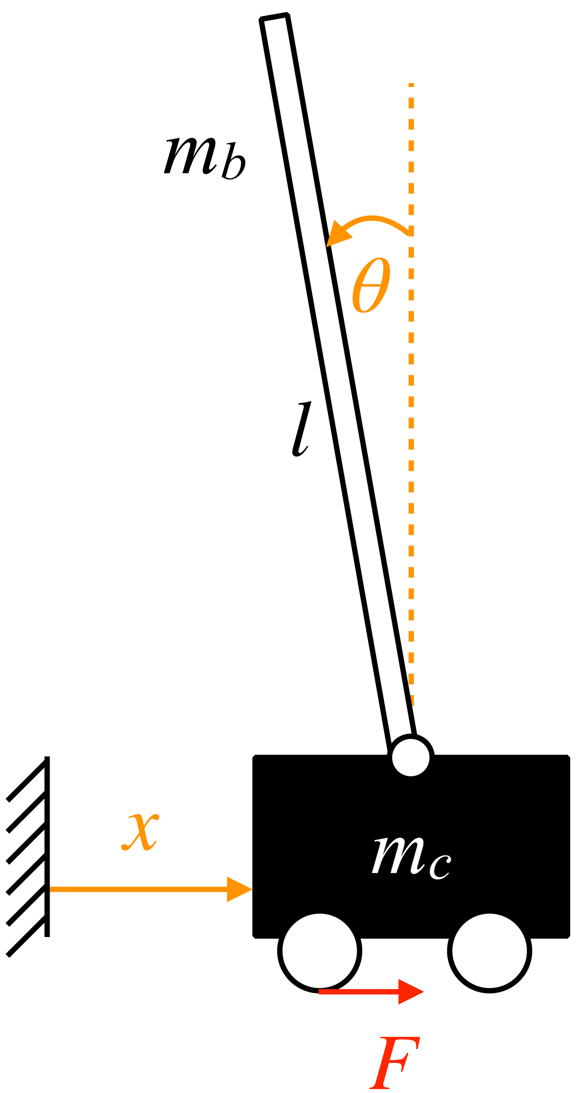
$$\begin{cases} \ddot{s}_1 - \lambda_1^2 s_1 = \boldsymbol{\varphi}_1^\intercal \bar{\boldsymbol{F}} \\ \\ \ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^\intercal \bar{\boldsymbol{F}} \end{cases}$$

同前配置 $\bar{\boldsymbol{F}}$，使得 s_1，s_2 稳定

目标是让偏差量 $\Delta\boldsymbol{q}$ 在零位置稳定

3.3 倒立状态的控制率设计 — 平移

$\theta = 0$



$m_c = 0.5$ kg

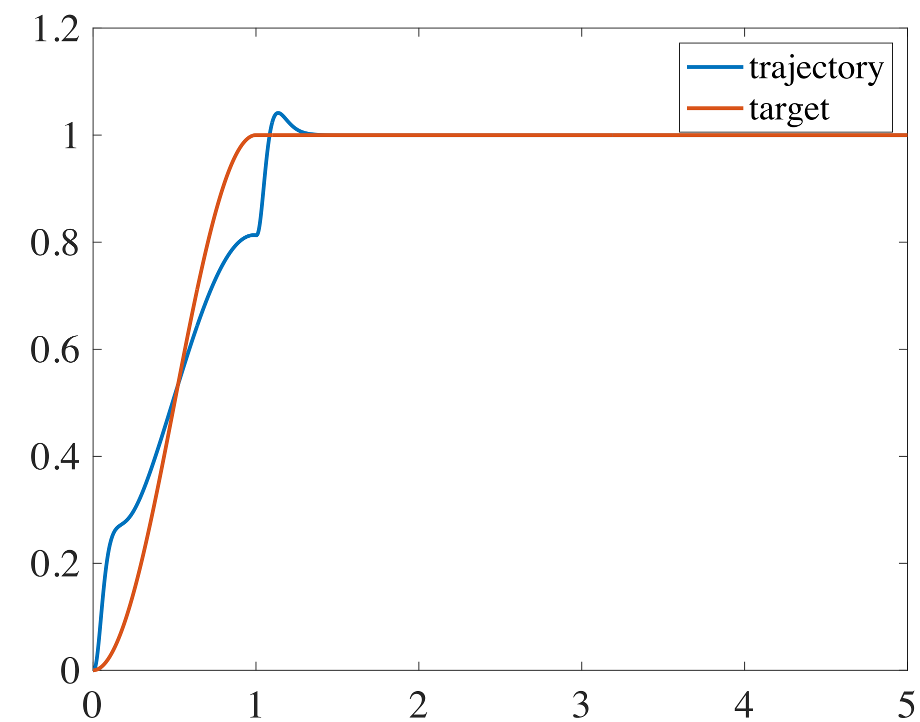
$m_b = 0.1$ kg

$l = 0.3$ m

同前配置 $\bar{F}$，使得 s_1，s_2 稳定

$$
\begin{cases}
\ddot{s}_1 - \lambda_1^2 s_1 = \boldsymbol{\varphi}_1^{\mathsf{T}} \bar{\boldsymbol{F}} \\
\ddot{s}_2 - \lambda_2^2 s_2 = \boldsymbol{\varphi}_2^{\mathsf{T}} \bar{\boldsymbol{F}}
\end{cases}
$$

$\beta_1 = \beta_2 = -2\pi \times 5$



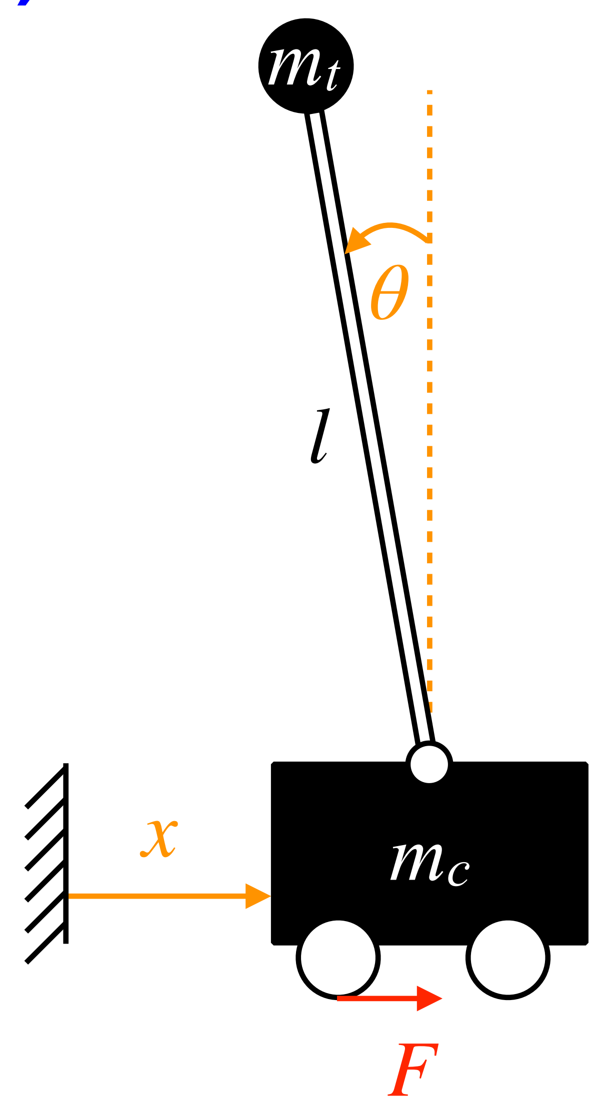
4. 小结 & 作业

自由选做，请在网络学堂上提交。截止日期1月23日23:59

倒立摆控制的总体思路：先求平衡位置，利用特征值信息判断稳定性，然后设计控制率增稳(还是利用特征值判稳)。

作业：针对右图所示的倒立摆系统，杆质量不计

1. 推导系统的运动微分方程。
2. 用**matlab**编写程序，计算电机不工作时，系统无初速度自由释放的动响应 $\theta(0)=10°$，$x(0)=\dot{x}(0)=\dot{\theta}(0)=0$。画出 $x(t)$ 和 $\theta(t)$ 随时间的变化曲线图，并作动画展示运动情况。
3. 设计控制率驱动电机，让系统可以稳定在 $x=\theta=0$ 处。并用**matlab**编写程序求解系统在该控制率下的动响应（$\theta(0)=10°$，$x(0)=\dot{x}(0)=\dot{\theta}(0)=0$）画出 $x(t)$ 和 $\theta(t)$ 随时间的变化曲线图，并作动画展示运动情况。



$m_c=0.5$ kg

$m_t=0.1$ kg

$l=0.3$ m