Discrete Response Model
Lecture 4

datascience@berkeley

Ordinal Logistical Regression Model

Introduction

Suppose that the response categories are **ordered** in the following way:

category 1 < category 2 < ⋯ < category J

- For example, a response variable may be measured using a scale with categories strongly disagree, disagree, neutral, agree, or strongly agree.
- There is a natural ordering here!
- Logit transformations of the probabilities can incorporate these orderings in a variety of ways.
- In this section, we will focus on one way where probabilities are cumulated based on these orderings.

Introduction

The cumulative probability for Y is

$$P(Y \mid j) = \pi_1 + \ldots + \pi_j$$

for j = 1, …, J. Note that P(Y | J) = 1. The logit of the cumulative probabilities can be written as

$$\operatorname{logit}[P(Y \leq j)] = \log\left[\frac{P(Y \leq j)}{1 - P(Y \leq j)}\right] = \log\left[\frac{\pi_1 + \qquad + \pi_j}{\pi_{j+1} + \qquad + \pi_J}\right]$$

When there is only one explanatory variable x, we can allow the log odds to vary by using a <u>**proportional odds model**</u>:

$$\operatorname{logit}[P(Y \leq j)] = \beta_{j0} + \beta_1 x$$

for j = 1, …, J – 1. Equivalently, the model is written as

$$P(Y \leq j) = \frac{\exp(\beta_{j0} + \beta_1 x)}{1 + \exp(\beta_{j0} + \beta_1 x)}$$

Proportional Odds

- The model assumes that **the effects of the explanatory variables are the same regardless of which cumulative probabilities are used to form the log odds**
- The proportional odds name comes from there being no j subscripts on the β parameter.
- This means these parameters are the same for each possible log-odds.
- This leads to each odds being a multiple of $\exp(\beta_{j0})$.

- $\beta_{10} < \cdots < \beta_{J0}$ due to the cumulative probabilities. Thus, the odds increasingly become larger for $j = 1, \ldots, J - 1$.
- A proportional odds model actually is a special case of a cumulative probability model, which allows the parameter coefficient on each explanatory variable to vary as a function of j.

More Than One Explanatory Variable

For more than one explanatory variable, the model becomes:

$$\text{logit}[P(Y \leq j)] = \beta_{j0} + \beta_1 x_1 + \quad + \beta_p x_p$$

for j = 1, …, J – 1

What is π_j only? Consider the case of one explanatory variable x again:

$$\begin{aligned}\pi_j &= P(Y = j) \\ &= P(Y \leq j) - P(Y \leq j-1) \\ &= \frac{e^{\beta_{j0} + \beta_1 x}}{1 + e^{\beta_{j0} + \beta_1 x}} - \frac{e^{\beta_{j-1,0} + \beta_1 x}}{1 + e^{\beta_{j-1,0} + \beta_1 x}}\end{aligned}$$

for j = 2, …, J – 1

Proportional Odds

For j = 1, π_1 = P(Y = 1) = P(Y | 1) = $e^{\beta_{10}+\beta x}/(1+e^{\beta_{10}+\beta x})$

For j = J, π_J = P(Y = J) = P(Y | J) − P(Y | J − 1)
= 1 − P(Y | J − 1)
= $1-e^{\beta_{J-1,0}+\beta_1 x}/(1+e^{\beta_{J-1,0}+\beta_1 x})$.

Example

Examine the shape of the proportional odds model.

$$\text{logit}[P(Y \le j)] = \beta_{j0} + 2x_1$$

where $\beta_{10} = 0$, $\beta_{20} = 2$, $\beta_{30} = 4$, and $J = 4$