Statistics for Data Science
Unit 4 Homework Solutions: Random Variables

September 27, 2016

1. **The Warranty is Worth It**

   Suppose the life span of a particular (shoddy) server is a continuous random variable, with a uniform probability distribution between 0 and 1 year. The server comes with a contract that guarantees you money if the server lasts less than 1 year. In particular, if the server lasts t years, the manufacturer will pay you $x = \$100(1-t)^{1/2}$. Let X be the random variable representing the payout from the contract.

   (a) Given that the server lasts 6 months without failing, what is the conditional probability that it will last another 3 months?

   Answer: Let L6 be the event that the server lasts 6 months without failing, L9 be the event that it lasts 9 months without failing. We can express the probability we want as

   $$P(L9|L6) = \frac{P(L9 \cap L6)}{P(L6)}$$

   Note that L6 is contained inside L9 because there is no way the server can last 9 months but not 6 (i.e. event L6 is a subset of L9). This means that the numerator is equal to $P(L9)$.

   Since t is distributed uniformly between 0 and 1 year, we know the probability density must be 1 inside this interval (and 0 outside it). This means that the probability that t falls in some interval $[a, b]$ where $0 \leq a < b \leq 1$ is just $\int_a^b 1dt = b - a$. This tells us that $P(L9) = 1/4$ and $P(L6) = 1/2$. Therefore, we can write the answer as,

   $$P(L9|L6) = \frac{P(L9)}{P(L6)} = 1/2$$

   .

   (b) Write down an expression for the cumulative probability function of the payout from the contract. That is, what is $F(x)$, the probability that X is less than x? (Hint: make sure that $F(0) = 0$, $F(\$100) = 1$).

   Answer: Let X be the random variable representing the payout from the contract, and let T be the random variable representing how long the contract lasts. When X takes the value x, $0 \leq x \leq \$100$, we can find the lifespan of the server that corresponds to that payoff using the relationship $x = \$100(1-t)^{1/2}$. Solving for t,

   $$t = 1 - (\frac{x}{\$100})^2 = 1 - \frac{x^2}{\$^2 10000}$$

Notice the strange units in the denominator - we have do divide by square dollars since we?re squaring a dollar amount in the numerator. This is a decreasing function of x, so X is less than the value x whenever T is greater than the corresponding value t. The probability that T is greater than this value (and therefore in the interval $[t, 1]$) is $F(x) = 1-(1-(\frac{x}{\$100})^2) = \frac{x^2}{\$^2 10000}$. We can therefore write down the cumulative distribution function as follows.

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{\$^2 10000}, & 0 \leq x < \$100 \\ 1, & \$100 \leq x \end{cases}$$

(c) Compute the expected payout from the contract, $E(X)$.

Answer: One solution would be to differentiate F to find the probability density function for X, then plug this into the definition of expectation. However, a simpler solution is to use the formula for the expectation of a random variable. Since $X = h(T)$ where $h(t) = \$100(1-t)^{1/2}$, we know

$$E(X) = \int_0^1 f(x)h(x)dx \tag{0.0.1}$$
$$= \int_0^1 1 \cdot \$100(1-t)^{1/2}dx \tag{0.0.2}$$
$$= -\frac{2}{3}\$100(1-t)^{3/2}|_0^1 \tag{0.0.3}$$
$$= \frac{2}{3}\$100 = \$66.6 \tag{0.0.4}$$