1. a). $P(T) = \cancel{}\ 0.01$ T: event that you select the trick coin.

$\cancel{P(H_k)}$ H_k: event that comes up heads all k times.

$P(H_k|T) = 1$ since the trick coin always comes up heads.

We want to know $\cancel{P}$ $P(T|H_k) = \frac{P(H_k|T)\,P(T)}{\underbrace{P(H_k)}_{\text{unknown}}}$

$P(H_k) = P(H_k|T) \cdot P(T) + P(H_k|T^c)\,P(T^c)$ According to LOTP.

$P(H_k) = 1 \times 0.01 + \left(\frac{1}{2}\right)^k \times 0.99.$

$= \frac{1}{100}\left(1 + \frac{99}{2^k}\right)$

$P(T|H_k) = \frac{P(H_k|T)\,P(T)}{P(H_k)}$

$= \frac{0.01 \times 1}{\frac{1}{100}\left(1 + \frac{99}{2^k}\right)} = \frac{2^k}{2^k + 99}$ //

b) $P(T|H_k) > 99\%$. what is k?

write a simple R code.

```
for (i in 1:20) {
    print ( 2^i / (2^i + 99))
}
```

~~When~~ $P(T|H_{13}) = 0.988$, $P(T|H_{14}) = 0.99399$.

If 14 heads in a row, the probability of having the trick coin is higher than 99%.
//

2. a) X : R.V. total # of unicorns.

$$X_i = \begin{cases} 1 & \text{w.p. } 3/4 \\ 0 & \text{w.p. } 1/4 \end{cases} \qquad E[X_i] = \frac{3}{4}, \quad E[X_i^2] = \frac{3}{4}$$

$n = 2$, $X = X_1 + X_2$

$$\text{PMF } X = \begin{cases} 0 & \text{w.p. } \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \\ 1 & \text{w.p. } \frac{6}{16} \\ 2. & \text{w.p. } \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \end{cases}$$

b) $$\text{CDF } F_X(x) = \begin{cases} x > 0. & \text{w.p. } 0 \\ 0 \le x < 1 & \text{w.p. } \frac{1}{16} \\ 1 \le x < 2 & \text{w.p. } \frac{7}{16} \\ 2 < x & \text{w.p. } 1 \end{cases}$$

c). $E[X] = E[X_1] + E[X_2] = 2 \cdot E[X_i] = 2 \cdot \frac{3}{4} = \frac{3}{2}$

d) $Var(X) = Var(X_1 + X_2)$ ($\because X_1, X_2$ are independent)

$= Var(X_1) + Var(X_2)$

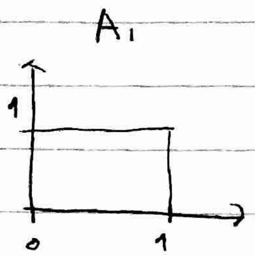
$= 2\, Var(X_i)$

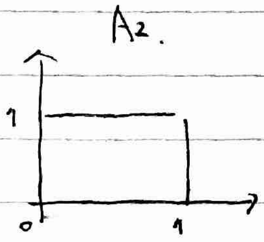
$= 2\left(E(X_i^2) - (E[X_i])^2 \right)$

$= 2\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2 \right)$

$= \frac{3}{8}$ //

3.



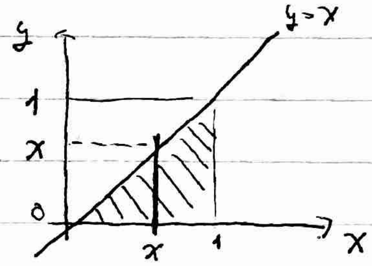


$X = \max(A_1, A_2)$

$Y = \min(A_1, A_2)$

$$f(x, y) = \begin{cases} 2, & 0<y<x<1 \\ 0, & o/w \end{cases}$$

a)



b) $F_X(x) = \mathbb{P}(X \le x)$

$= \mathbb{P}(\max(A_1, A_2) \le x)$

$= \mathbb{P}(A_1 \le x, A_2 \le x)$

$= \mathbb{P}(A_1 \le x)\, \mathbb{P}(A_2 \le x)$

$= F_{A_1}(x)\, F_{A_2}(x)$

$= x^2$

$$f_X(x) = \begin{cases} 0 & o/w. \\ 2x & 0<x<1 \end{cases}$$

c). $E[X] = \int_0^1 x f_X(x)\,dx$

$= \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$

3 d) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2}{2x} = \dfrac{1}{x}$

e) $E[Y|X] = \displaystyle\int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_0^x y \frac{1}{x}\, dy = \frac{1}{x}\left(\frac{y^2}{2}\Big|_0^x \right)$

$= \dfrac{x}{2}$

f) $E[XY] = E\left[E[XY|X]\right]$

$= E[X E[Y|X]]$

$= E\left[\dfrac{X^2}{2}\right]$

$= \dfrac{1}{2}\displaystyle\int_0^1 x^2 f_X(x)\, dx$

$= \dfrac{1}{4}$

g) $\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$

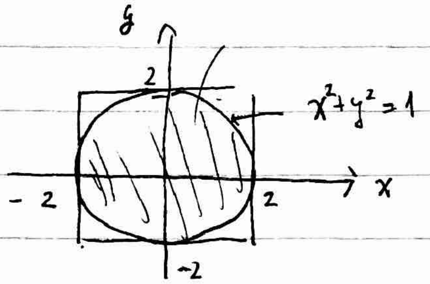
$E[Y] = E[E[Y|X]] = E\left[\dfrac{X}{2}\right] = \dfrac{1}{2}\cdot\dfrac{2}{3} = \dfrac{1}{3}$

$\operatorname{cov}(X,Y) = \dfrac{1}{4} - \dfrac{2}{3}\cdot\dfrac{1}{3}$

$= \dfrac{1}{36}$

4 a) X_i, Y_i, $1 \le i \le n$ both uniform distribution on $[-1, 1]$

$$D_i = \begin{cases} 1 & p = P(X_i^2 + Y_i^2 < 1) = \dfrac{\pi \cdot 1^2}{2 \times 2} = \dfrac{\pi}{4} \\ 0 & 1 - p = 1 - \dfrac{\pi}{4} \end{cases}$$



$$E(D_i) = p = \frac{\pi}{4}$$

$$E(D_i^2) = \frac{\pi}{4}$$

b) $$\sigma = \sqrt{Var(D_i)}$$

$$= \sqrt{E(D_i^2) - [E(D_i)]^2}$$

$$= \sqrt{\frac{\pi}{4}\left(1 - \frac{\pi}{4}\right)}$$

c) $$\bar{D} = \frac{D_1 + D_2 + \cdots + D_n}{n}$$

$$E(\bar{D}) = E\left(\frac{D_1 + D_2 + \cdots + D_n}{n}\right)$$

$$= \frac{1}{n} \cdot n E(D_i) = \frac{\pi}{4}$$

$$Var(\bar{D}) = Var\left(\frac{D_1 + D_2 + \cdots + D_n}{n}\right)$$

$$= \frac{1}{n^2} \cdot n Var(D_i) = \frac{\pi}{4n}\left(1 - \frac{\pi}{4}\right)$$

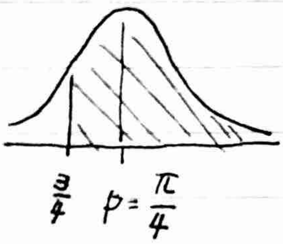
Standard Error = Standard Deviation of $\bar{D}$ $= \sqrt{Var(\bar{D})} = \sqrt{\frac{\pi}{4n}\left(1 - \frac{\pi}{4}\right)}$

4 d) Sufficiently large n. $\bar{D}$ has normal distribution of $\bar{D} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$ regardless of D's distribution, according to CLT.

This enables us to compute the answer for this problem because normal distribution is easy to compute.

We look for $P(\bar{D} > \frac{3}{4})$

$$= 1 - P(\bar{D} \leq \frac{3}{4}) = \cancel{0.5702422} \quad 0.8057173$$



pnorm (3/4 , mean = E($\bar{D}$), sd= standard error)

calculated in 4 c)