Partial Outer Convexification for Compressor Cost Optimization in a Gas-to-Power Network

Final presentation of Master Thesis

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Goal of the Thesis

Goal: Optimize compressor costs in a Gas-to-Power network.

Methods: We will compare the three-step approach based on Partial Outer Convexification (POC) Reformulation and the direct solver Bonmin working with an NLP-based Branch-and-Bound Algorithm.

Three-Step Approach

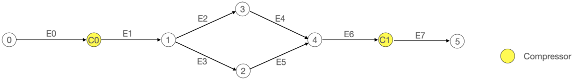


Figure 1: Gas Network

In this gas-network we have two compressors and four compressor combinations

- compressor 0 is **off** and compressor 1 is **off**: (0,0)
- compressor 0 is **off** and compressor 1 is **on**: (0,1)
- compressor 0 is **on** and compressor 1 is **off**: (1,0)
- compressor 0 is **on** and compressor 1 is **on** : (1,1)

If n_{com} denotes the number of compressors, we have $n_{oc} = 2^{n_{com}}$ number of combinations.

We introduce convex multiplier controls $w^t \in \{0,1\}^{n_{oc}}$ that tell which combination holds in t: e.g. if combination $(0,1)$ in t holds, then $w_1^t = 1$ and $w_k^t = 0$ for $k \neq 1$.

Three-Step Approach

1. Formulate Problem into the form (POC formulation):

$$\min_{u^t} \quad f(x^t, u^t) \tag{1}$$

$$\text{subject to:} \quad x^{t+1} = x^t + \Delta t \Phi(x^t, u^t) w^t \qquad \text{(PDE)} \tag{2}$$

$$\sum_{s=1}^{n_{oc}} w_s^t = 1 \text{ with } w^t \in \{0,1\}^{n_{oc}} \qquad \text{(SOS1)} \tag{3}$$

$$(\,+\text{ some additional constraints for } x,\ u \text{ and } w) \tag{4}$$

2. Apply the following Algorithm:

Algorithm: Three-Step Approach

Discretize problem 1 - 4 with appropriate step sizes Δt and Δx fulfilling the CFL-condition.

Step 1: Relax the integrality conditions

$$w_s^t \in \{0,1\} \to \tilde{w}_s^t \in [0,1], \forall t \in \{0,\dots,n_T\},\ \forall s \in \{0,\dots,n_{oc}\} \tag{5}$$

and solve the arising continuous NLP. This yields the optimal vector $\tilde{w}$.

Step 2: Compute a feasible binary solution w out of $\tilde{w}$ by solving CIAP (Combinatorial Integral Approximation Problem).

Step 3: Simulate the dynamics again with w to obtain a feasible trajectory and a corresponding objective value.

Approximation Theorem for POC

Theorem

Let $\mathcal{I} = \{0, \ldots, n_T - 1\}$, $\mathcal{D} \subseteq \mathbb{R}^N$ and $\Phi : \mathcal{I} \times \mathcal{D} \to \mathbb{R}^{n \times n_{oc}}$ be a matrix-valued function that is continuous with respect to the second argument and satisfies

$$\|\Phi(t, \mu)\nu\|_X \leq M_{oc}\|\nu\|_\Omega, \qquad \forall t \in \mathcal{I}, \mu \in \mathcal{D}, \nu \in \mathbb{R}^{n_{oc}} \tag{6}$$

$$\|(\Phi(t, \mu) - \Phi(t, \eta))\alpha\|_X \leq L_{oc}\|\mu - \eta\|_X, \quad \forall t \in \mathcal{I}, \mu, \eta \in \mathcal{D}, \alpha \in H \tag{7}$$

for constants $M_{oc}, L_{oc} < \infty$. Furthermore, for each $t \in \mathcal{I}$ let $h_t > 0$, $\alpha^t, \beta^t \in H$ such that $T = \sum_{t=0}^{n_T - 1} h_t$ and for each $t \in \mathcal{I} \cup \{n_T\}$ let $\mu^t, \eta^t \in \mathcal{D}$ be given such that for all $t \in \mathcal{I}$

$$\mu^{t+1} = \mu^t + h_t\Phi(t, \mu^t)\alpha^t \quad and \quad \eta^{t+1} = \eta^t + h_t\Phi(t, \eta^t)\beta^t. \tag{8}$$

If for some set $\mathcal{I}' \subseteq \mathcal{I}$, constants $C_{oc}, \epsilon < \infty$ and some vector $\delta^0 \in \mathbb{R}^{n_{oc}}$ it holds that

$$\|(\Phi(t+1, \mu^{t+1}) - \Phi(t, \mu^t))\nu\|_X \leq h_t C_{oc}\|\nu\|_\Omega, \quad \forall t \in \mathcal{I}', \nu \in \mathbb{R}^{n_{oc}} \tag{9}$$

$$\|\delta^0 + \sum_{t=0}^{k-1} h_t(\alpha^t - \beta^t)\|_\Omega \leq \epsilon, \qquad \forall k \in \mathcal{I} \cup \{n_T\} \tag{10}$$

then it follows with $T'_k = k\, max\{h_t | t = 0, \ldots, k-1\}$ and $n_{jump} = |\mathcal{I} \setminus (\mathcal{I}' \cup \{n_T - 1\})|$ that for all $k \in \mathcal{I} \cup \{n_T\}$

$$\sum_{t=0}^{k} h_t\|\mu^t - \eta^t\|_X \leq \frac{exp(T'_k L_{oc}) - 1}{L_{oc}}(\|\mu^0 - \eta^0\|_X + (2M_{oc}(1 + n_{jump}) + TC_{oc})\epsilon). \tag{11}$$

Step 2: CIAP (General)

CIAP is denoted as follows:

$$\begin{aligned} \min_{\substack{\beta^t \in H \cap \{0,1\}^{n_{oc}},\\ t=0,\dots,n_T-1,\\ \delta^0 \in \mathbb{R}^{n_{oc}}, \epsilon \in \mathbb{R}_{\geq 0}}} \quad & \epsilon \\ \text{subject to:} \quad & \left\| \delta^0 + \sum_{t=0}^{k-1} \Delta t(\alpha^t - \beta^t) \right\|_{\Omega} \leq \epsilon \quad \text{for all } k = 0, \dots, n_T \end{aligned} \tag{12}$$

Without additional constraints: Solve with **Sum Up Rounding (SUR)**
SUR can be solved in polynomial time and its objective value ϵ is proportional to Δt

With additional constraints that couple over time:
Solve MILP: (12) + additional constraints
ϵ cannot be driven to zero by decreasing Δt

Power Network (1)

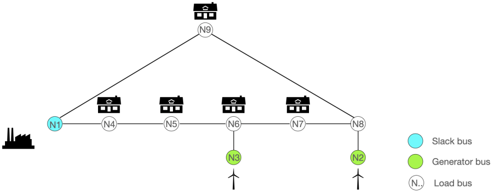


Figure 2: Power Network with one slack bus, two generator buses and six load buses

Power Flow Equations:

$$\begin{aligned} P_k &= \sum_{j=1}^{N} |V_k||V_j|(G_{kj}cos(\phi_{k,j}) + B_{kj}sin(\phi_{k,j})) \\ Q_k &= \sum_{j=1}^{N} |V_k||V_j|(G_{kj}sin(\phi_{k,j}) - B_{kj}cos(\phi_{k,j})) \end{aligned} \tag{13}$$

with P: real power, Q: reactive power, V: voltage amplitude, ϕ: phase, G_{kj}/B_{kj}: conductance / susceptance between bus k and j.
At each bus: two variables of P, Q, V, ϕ are known $\rightarrow$ 18 known and unknown variables.
$\rightarrow$ Solve Power-Flow Equations with Levenberg-Marquard Algorithm.

Power Network (2)

Convert gas to power: Take P at slack bus $N1$ and convert gas to power with the following rule

$$\epsilon(P) = a_0 + a_1 P + a_2 P^2 \tag{14}$$

with $a_0 = 2$, $a_1 = 5$ and $a_2 = 5$.

1D Euler Equation

The dynamics in a single can be described by the Isothermal Euler Equation:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}q = 0 \tag{15}$$

$$\frac{\partial}{\partial t}q + \frac{\partial}{\partial x}p + \frac{\partial}{\partial x}\frac{q^2}{\rho} = -g\rho s - \frac{\lambda(q)|q|q}{2D\rho} \tag{16}$$

ρ: density (kg/m^3), q: mass flow (kg/s), p: pressure (bar) with relation $p = c^2\rho$, $c = 340m/s$, g: gravitational constant, s: inclination angle of pipe, λ: friction factor, D: diameter of pipe

The Weymouth Equation is a simplification of Isothermal Euler Equation for high-pressure (approx. 70 bar and $q/\rho \approx 10m/s$) in gas pipes:

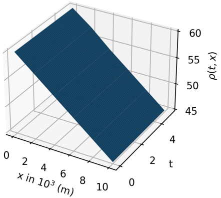
$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}q = 0 \tag{17}$$

$$\frac{\partial}{\partial t}q + \frac{\partial}{\partial x}p = -g\rho s - \frac{\lambda(q)|q|q}{2D\rho}. \tag{18}$$

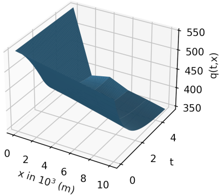
Note: $\frac{\partial}{\partial x}\frac{q^2}{\rho}$ is omitted.

Euler Equation vs. Weymouth Equation

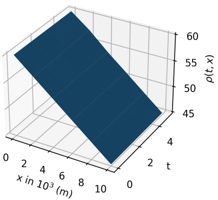
Discretized with Simple Upwind Scheme



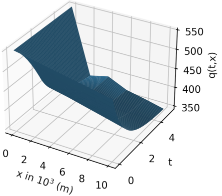
Euler Equation: $\rho(t,x)$



Euler Equation: $q(t,x)$



Weymouth Equation: $\rho(t,x)$



Weymouth Equation: $q(t,x)$

Note: No difference can be spotted!

Notation for Gas-Node Conditions

V: set of all *gas nodes*
V_q: set of all *source nodes*
V_p: set of all *sink nodes*
V_c: set of all *compressor nodes*

$\delta^+(v)$: set of *outgoing edges* at node $v \in V$
$\delta^-(v)$: set of *ingoing edges* at node $v \in V$

${}^e q(x^n, t)$: flow in pipe e in point (x^n, t)
Note: Similiar for ${}^e p(x^n, t)$.

Conditions at Gas Nodes (1)

Gas Coupling Conditions

Sum of ingoing flows is equal to sum of outgoing flows:

$$\sum_{e\in\delta^-(v)} {}^{\mathrm{e}}q(x^n, t) = \sum_{h\in\delta^+(v)} {}^{\mathrm{h}}q(x^0, t), \quad \forall v \in V \setminus (V_q \cup V_p),\ t = 0, \dots, T. \tag{19}$$

No pressure loss near the node:
For a fix but arbitrary node in the innodes $v \in V \setminus (V_q \cup V_c \cup V_p)$ and a pressure $p(v, t)$ at node v in time t we state

$$\begin{aligned} {}^{\mathrm{e}}p(x^n, t) &= p(v, t), \quad \forall e \in \delta^-(v),\ t = 0, \dots, T \\ {}^{\mathrm{h}}p(x^0, t) &= p(v, t), \quad \forall h \in \delta^+(v),\ t = 0, \dots, T. \end{aligned} \tag{20}$$

Gas-Power Coupling at Slack Node s

s is the gas node connected to slack bus via edge h_{slack}, ϵ^t is flow taken out of the gas network. At slack node s it holds:

$$\sum_{e\in\delta^-(s)} {}^{\mathrm{e}}q(x^n, t) = \sum_{h\in\delta^+(s)\setminus h_{slack}} {}^{\mathrm{h}}q(x^0, t) + \epsilon^t, \quad t = 0, \dots, T. \tag{21}$$

Conditions at Gas-Nodes (2)

Compressor Conditions

Conditions at Compressor Node:

$$ {}^{\mathrm{e}}p(x^n, t) = {}^{\mathrm{h}}p(x^0, t) + u_v^t, \quad \forall v \in V_c,\ e \in \delta^-(v),\ h \in \delta^+(v),\ t = 0, ..., T \tag{22} $$

where u_v^t denotes pressure increase (bar) at compressor node v.

Non-negativity constraints

Pressure and pressure increase cannot be negative:

$$ u_v^t \geq 0, \quad \forall v \in V_c,\ t = 0, ..., m \tag{23} $$

$$ p_j^t \geq 0, \quad j = 0, ..., n,\ t = 0, ..., m, \text{ for all pipes} \tag{24} $$

Compessor Cost Optimization Problem (1)

Optimization Problem with POC reformulation:

$$\min_{u} \quad \frac{1}{2}\sum_{t=0}^{m}\sum_{v\in V_c}(u_v^t)^2$$

subject to:

1. **Weymouth Equation discretized with Simple Upwind Scheme**
2. **gas coupling conditions**
3. **compressor conditions**

$$\begin{aligned} {}^{e}p_n^t &= {}^{h}p_0^t + u_v^t, \quad \forall v \in V_c,\ e \in \delta^-(v),\ h \in \delta^+(v),\ t = 0,\dots,m \\ u_v^t &= \sum_{s=1}^{n_{OC}} w_s^t c_v^s u_v^t, \quad \forall v \in V_c,\ t = 0,\dots,m \end{aligned} \tag{25}$$

where $c_v^s \in \{0,1\}$ denotes state of compressor v for configuration s

$$\sum_{s=1}^{n_{OC}} w_s^t = 1 \quad \textit{(SOS-Type1)}$$

4. **non − negativity constraints**
5. **gas − power coupling condition**

Initial conditions: $p_j^0,\ q_j^0,\ j=0,\dots,n$ for each pipe and $\epsilon^t,\ t=0,\dots,m$.
Solution: $p_j^t,\ q_j^t\ \ j=0,\dots,n,\ t=0,\dots,m$ for each pipe, u_v^t, $\forall v \in V_c$ and configuration w^t $t=0,\dots,m$.

Compessor Cost Optimization Problem (2)

Optimization Problem without POC reformulation:

$$\min_{u} \quad \frac{1}{2}\sum_{t=0}^{m}\sum_{v\in V_c}(u_v^t)^2$$

subject to:

1. **Weymouth Equation discretized with Simple Upwind Scheme**
2. **gas coupling conditions**
3. **compressor conditions**

   $${}^{e}p_n^t = {}^{h}p_0^t + u_v^t, \quad \forall v \in V_c,\ e \in \delta^-(v),\ h \in \delta^+(v),\ t = 0, ..., m$$

   $$u_v^t = \beta_v^t \cdot u_v^t, \quad \forall v \in V_c,\ t = 0, ..., m \text{ and } \beta_v^t \in \{0, 1\}$$
4. **non − negativity constraints**
5. **gas − power coupling condition**

(26)

Initial conditions: p_j^0, q_j^0 $j = 0, ..., n$, for each pipe and ϵ^t, $t = 0, ..., m$.
Solution: p_j^t, q_j^t, $j = 0, ..., n$, $t = 0, ..., m$, for each pipe, u_v^t and configuration β_v^t $v \in V_c$ and $t = 0, ..., m$.

Additional Constraints (1)

Additional Constraints: Type 1 - Compressor has to keep its state for at least a time M_1, M_2 for every compressor node $v \in V_c$.

$$\sum_{e=k+1}^{k+\left\lfloor \frac{M_1}{\Delta t} \right\rfloor} \sum_{s=1}^{n_{oc}} c_v^s w_s^e \geq \sum_{s=1}^{n_{oc}} c_v^s \left\lfloor \frac{M_1}{\Delta t} \right\rfloor \left(-w_s^k + w_s^{k+1} \right) \quad \forall k \leq m - \left\lfloor \frac{M_1}{\Delta t} \right\rfloor \tag{27}$$

$$\sum_{e=k+1}^{k+\left\lfloor \frac{M_2}{\Delta t} \right\rfloor} \left(1 - \sum_{s=1}^{n_{oc}} c_v^s w_s^e \right) \geq \sum_{s=1}^{n_{oc}} c_v^s \left\lfloor \frac{M_2}{\Delta t} \right\rfloor \left(w_s^k - w_s^{k+1} \right) \quad \forall k \leq m - \left\lfloor \frac{M_2}{\Delta t} \right\rfloor \tag{28}$$

Additional Constraints: Type 2 - Compressor can only switch maximum $r \in \mathbb{N}$ times.

$$\sum_{t=1}^{m} \sum_{s=1}^{n_{oc}} |w_s^t c_v^s - w_s^{t-1} c_v^s| \leq r, \quad \forall v \in V_c. \tag{29}$$

Additional Constraints (2)

Solve CIAP by solving MILP

$$
\begin{aligned}
&\min_{\substack{\beta^t \in H \cap \{0,1\}^{n_{oc}},\\ t=0,\dots,m-1,\\ \delta^0 \in \mathbb{R}^{n_{oc}}, \epsilon \in \mathbb{R}}} \epsilon \\
&\text{subject to:} \quad \left\| \delta^0 + \sum_{t=0}^{k-1} \Delta t(\alpha^t - w^t) \right\|_{\Omega} \leq \epsilon \quad \text{for all } k = 0, \dots, m \\
&\qquad\qquad + \textit{additional constraints type } 1 \text{ OR } + \textit{additional constraints type } 2
\end{aligned}
\tag{30}
$$

where α^t are the relaxed binary variables in time t.

Algorithms used

	Method	Solver	Programming Language
Power Model	Levenberg-Marquardt	*fsolve*	Matlab
Three-Step Approach	Interior Point Method	IPOPT 3.12.3 with ma27 integrated in CasADi	Python 3.7.6
	(+ Sum Up Rounding or MILP)	Gurobi 9.1.2	
Direct Solver	NLP-Based	BONMIN integrated in CasADi	Python 3.7.6
	Branch-and-Bound Algorithm		

PC: MacBook Pro, 16 GB RAM, Intel-Quad Core i5, 2,4 GHz CPU

Numerical Analysis: Simple Gas Network

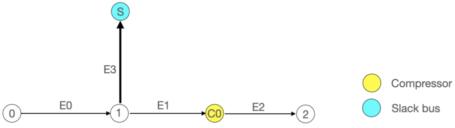
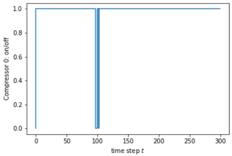
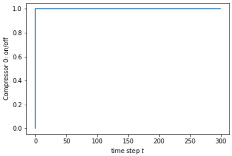
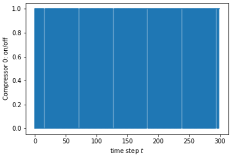
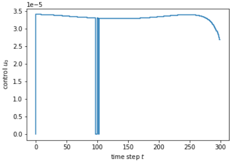
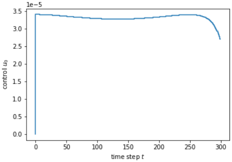
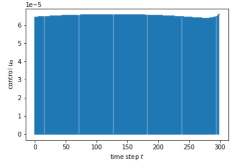


Figure 3: Simple Model

Constants: $\lambda = 0.11$, $D = 1\,m$ and $c = 340\,m/s$, pipe of length 12 km, total execution time of 5 sec, spacial step size $\Delta x = 2000$, time step size $\Delta t = 1/60$

Initial data: $\epsilon^t = 0,\ t = 0, ..., m,\ p(0, x) = 60\,bar$ and $q(0, x) = 500\,kg/s$ for each pipe at every pipe section

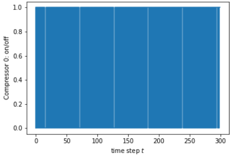
Numerical Analysis: Simple Model

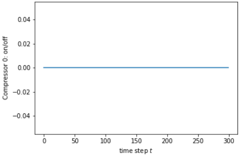
Bonmin (no POC)	Bonmin (POC)	Three-Step Approach
		
		
OFV: $1.6350264 \cdot 10^{-7}$ Runtime: 107.35 *sec* ($\approx$ 1.8 *min*)	OFV: $1.64764 \cdot 10^{-7}$ Runtime: 176.13 *sec* ($\approx$ 2.9 *min*)	OFV: $3.1675 \cdot 10^{-7}$ Runtime: 7.59 *sec*

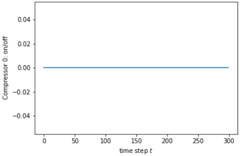
Additional Constraints: Simple Model

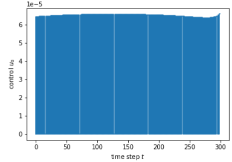
Three-Step Approach with and without Additional Constraints

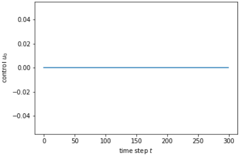
No Add.Constraints	Add. Constraint Type 1	Add. Constraint Type 2
	Compressor can only switch 5 times in 5 *sec*	Compressor has to keep its state for at least 3 time steps

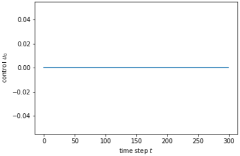












No Add.Constraints	Add. Constraint Type 1	Add. Constraint Type 2
OFV: $3.1675 \cdot 10^{-7}$	OFV: $1.5 \cdot 10^{-14}$	OFV: $1.5 \cdot 10^{-14}$
Runtime: 7.59 *sec*	Runtime: 12.41 *sec*	Runtime: 18.21 *sec*

Numerical Analysis: Advanced Model

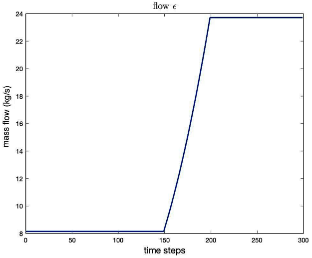
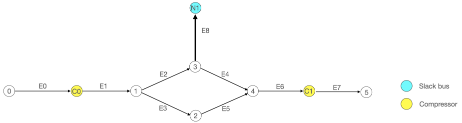


Figure 8



Advanced Gas Network

Constants: $\lambda = 0.11$, $D = 1\,m$ and $c = 340\,m/s$, pipe of length $12\,km$, total execution time of $5\,sec$, spacial step size $\Delta x = 2000$, time step size $\Delta t = 1/60$

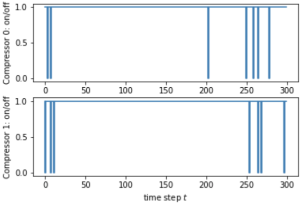
Initial data:

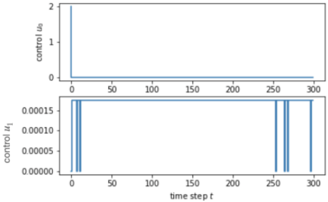
	$E0$	$E1$	$E2,\ E3$	$E4$	$E5$	$E6,\ E7$
$p(x,0)\,(bar)$	60	62	62	62	62	62
$q(x,0)\,(kg/s)$	500	500	250	241.847	250	491.847

ϵ^t (mass taken out of gas network is shown in Figure 8)

Numerical Analysis: Advanced Model

Three-Step Approach



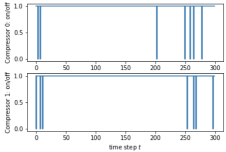
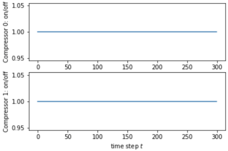
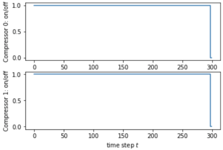
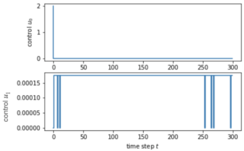
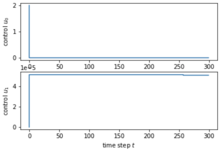
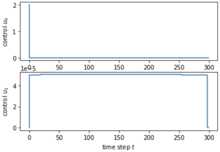


OFV: 2.00000
Runtime: 1259.03 *sec* (≈ 21 *min*)

- solved to 'acceptable level' (tolerance level 10^{-6})
- direct solver bonmin could not find any result in 2 *hours* - freezing behaviour.

Additional Constraints: Advanced Model

Three-Step Approach with and without Additional Constraints

No Add.Constraints	Add. Constraint Type 1	Add. Constraint Type 2
	Compressor can only switch 10 times in 5 *sec*	Compressor has to keep its state for at least 3 time steps
		
		
OFV: 2.0000	OFV: 2.0000	OFV: 2.0000
Runtime: 1259 *sec* (≈ 21 *min*)	Runtime: 1941.63 *sec* (≈ 32.3 *min*)	Runtime: 1920.15 *sec* (≈ 32 *min*)

Conclusion

- Three Step Approach delivers good optimization results and is up to 14 times faster than direct solver bonmin, which works with the NLP-based Brand-and-Bound algorithm
- With additional constraints we can prevent the compressor from switching too often and can achieve even better OFV
- Due to the CFL Condition we had to choose a very small time step ($\Delta t = \frac{1}{60}$) and therefore considered a simulation of $5\,sec$. For realistic simulations (e.g. 1 day) computation will likely take several hours.
  Interesting question: How does run time and solutions vary for implicit scheme and explicit scheme (with Three Step approach)?

Thank you for your attention!