

Partial Outer Convexification for Compressor Cost Optimization in a Gas-to-Power Network

Final presentation of Master Thesis

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Goal of the Thesis

Goal: Optimize compressor costs in a Gas-to-Power network.

Methods: We will compare the **three-step approach** based on **Partial Outer Convexification (POC) Reformulation** and the **direct solver Bonmin** working with an NLP-based Branch-and-Bound Algorithm.

Three-Step Approach



Figure 1: Gas Network

In this gas-network we have **two compressors** and **four compressor combinations**

- compressor 0 is **off** and compressor 1 is **off**: (0,0)
- compressor 0 is **off** and compressor 1 is **on**: (0,1)
- compressor 0 is **on** and compressor 1 is **off**: (1,0)
- compressor 0 is **on** and compressor 1 is **on** : (1,1)

If n_{com} denotes the number of compressors, we have $n_{oc} = 2^{n_{com}}$ number of combinations.

We introduce **convex multiplier controls** $w^t \in \{0, 1\}^{n_{oc}}$ that tell which combination holds in t : e.g. if combination (0, 1) in t holds, then $w_1^t = 1$ and $w_k^t = 0$ for $k \neq 1$.

Three-Step Approach

1. Formulate Problem into the form (POC formulation):

$$\min_{u^t} f(x^t, u^t) \quad (1)$$

$$\text{subject to: } x^{t+1} = x^t + \Delta t \Phi(x^t, u^t) w^t \quad (\text{PDE}) \quad (2)$$

$$\sum_{s=1}^{n_{oc}} w_s^t = 1 \text{ with } w^t \in \{0, 1\}^{n_{oc}} \quad (\text{SOS1}) \quad (3)$$

$$(+ \text{ some additional constraints for } x, u \text{ and } w) \quad (4)$$

2. Apply the following Algorithm:

Algorithm: Three-Step Approach

Discretize problem 1 - 4 with appropriate step sizes Δt and Δx fulfilling the CFL-condition.

Step 1: Relax the integrality conditions

$$w_s^t \in \{0, 1\} \rightarrow \tilde{w}_s^t \in [0, 1], \forall t \in \{0, \dots, n_T\}, \forall s \in \{0, \dots, n_{oc}\} \quad (5)$$

and solve the arising continuous NLP. This yields the optimal vector \tilde{w} .

Step 2: Compute a feasible binary solution w out of \tilde{w} by solving CIAP (Combinatorial Integral Approximation Problem).

Step 3: Simulate the dynamics again with w to obtain a feasible trajectory and a corresponding objective value.

Approximation Theorem for POC

Theorem

Let $\mathcal{I} = \{0, \dots, n_T - 1\}$, $\mathcal{D} \subseteq \mathbb{R}^N$ and $\Phi : \mathcal{I} \times \mathcal{D} \rightarrow \mathbb{R}^{n \times n_{oc}}$ be a matrix-valued function that is continuous with respect to the second argument and satisfies

$$\|\Phi(t, \mu)\nu\|_X \leq M_{oc}\|\nu\|_\Omega, \quad \forall t \in \mathcal{I}, \mu \in \mathcal{D}, \nu \in \mathbb{R}^{n_{oc}} \quad (6)$$

$$\|(\Phi(t, \mu) - \Phi(t, \eta))\alpha\|_X \leq L_{oc}\|\mu - \eta\|_X, \quad \forall t \in \mathcal{I}, \mu, \eta \in \mathcal{D}, \alpha \in H \quad (7)$$

for constants $M_{oc}, L_{oc} < \infty$. Furthermore, for each $t \in \mathcal{I}$ let $h_t > 0$, $\alpha^t, \beta^t \in H$ such that $T = \sum_{t=0}^{n_T-1} h_t$ and for each $t \in \mathcal{I} \cup \{n_T\}$ let $\mu^t, \eta^t \in \mathcal{D}$ be given such that for all $t \in \mathcal{I}$

$$\mu^{t+1} = \mu^t + h_t \Phi(t, \mu^t) \alpha^t \quad \text{and} \quad \eta^{t+1} = \eta^t + h_t \Phi(t, \eta^t) \beta^t. \quad (8)$$

If for some set $\mathcal{I}' \subseteq \mathcal{I}$, constants $C_{oc}, \epsilon < \infty$ and some vector $\delta^0 \in \mathbb{R}^{n_{oc}}$ it holds that

$$\|(\Phi(t+1, \mu^{t+1}) - \Phi(t, \mu^t))\nu\|_X \leq h_t C_{oc} \|\nu\|_\Omega, \quad \forall t \in \mathcal{I}', \nu \in \mathbb{R}^{n_{oc}} \quad (9)$$

$$\|\delta^0 + \sum_{t=0}^{k-1} h_t (\alpha^t - \beta^t)\|_\Omega \leq \epsilon, \quad \forall k \in \mathcal{I} \cup \{n_T\} \quad (10)$$

then it follows with $T'_k = k \max\{h_t | t = 0, \dots, k-1\}$ and $n_{jump} = |\mathcal{I} \setminus (\mathcal{I}' \cup \{n_T - 1\})|$ that for all $k \in \mathcal{I} \cup \{n_T\}$

$$\sum_{t=0}^k h_t \|\mu^t - \eta^t\|_X \leq \frac{\exp(T'_k L_{oc}) - 1}{L_{oc}} (\|\mu^0 - \eta^0\|_X + (2M_{oc}(1 + n_{jump}) + TC_{oc})\epsilon). \quad (11)$$

Step 2: CIAP (General)

CIAP is denoted as follows:

$$\begin{aligned}
 & \min_{\substack{\beta^t \in H \cap \{0,1\}^{n_{oc}}, \\ t=0,\dots,n_T-1, \\ \delta^0 \in \mathbb{R}^{n_{oc}}, \epsilon \in \mathbb{R}_{\geq 0}}} \epsilon \\
 & \text{subject to: } \left\| \delta^0 + \sum_{t=0}^{k-1} \Delta t (\alpha^t - \beta^t) \right\|_{\Omega} \leq \epsilon \quad \text{for all } k = 0, \dots, n_T
 \end{aligned} \tag{12}$$

Without additional constraints: Solve with **Sum Up Rounding (SUR)**

SUR can be solved in polynomial time and its objective value ϵ is proportional to Δt

With additional constraints that couple over time:

Solve MILP: (12) + additional constraints

ϵ cannot be driven to zero by decreasing Δt

Power Network (1)

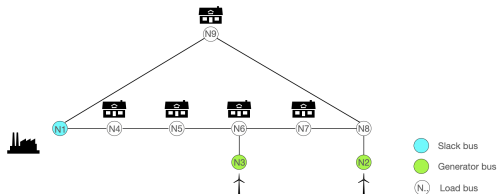


Figure 2: Power Network with one slack bus, two generator buses and six load buses

Power Flow Equations:

$$P_k = \sum_{j=1}^N |V_k| |V_j| (G_{kj} \cos(\phi_{k,j}) + B_{kj} \sin(\phi_{k,j}))$$

$$Q_k = \sum_{j=1}^N |V_k| |V_j| (G_{kj} \sin(\phi_{k,j}) - B_{kj} \cos(\phi_{k,j}))$$
(13)

with P : real power, Q : reactive power, V : voltage amplitude, ϕ : phase, G_{kj}/B_{kj} : conductance / susceptance between bus k and j .

At each bus: two variables of P , Q , V , ϕ are known \rightarrow 18 known and unknown variables.

\rightarrow Solve Power-Flow Equations with **Levenberg-Marquard Algorithm**.

Power Network (2)

Convert gas to power: Take P at slack bus $N1$ and convert gas to power with the following rule

$$\epsilon(P) = a_0 + a_1 P + a_2 P^2 \quad (14)$$

with $a_0 = 2$, $a_1 = 5$ and $a_2 = 5$.

1D Euler Equation

The dynamics in a single can be described by the **Isothermal Euler Equation**:

$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} q = 0 \quad (15)$$

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} p + \frac{\partial}{\partial x} \frac{q^2}{\rho} = -g\rho s - \frac{\lambda(q)|q|q}{2D\rho} \quad (16)$$

ρ : density (kg/m^3), q : mass flow (kg/s), p : pressure (bar) with relation $p = c^2 \rho$,
 $c = 340m/s$, g : gravitational constant, s : inclination angle of pipe, λ : friction factor,
 D : diameter of pipe

The **Weymouth Equation** is a simplification of **Isothermal Euler Equation** for high-pressure (approx. 70 bar and $q/\rho \approx 10m/s$) in gas pipes:

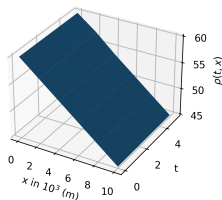
$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} q = 0 \quad (17)$$

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} p = -g\rho s - \frac{\lambda(q)|q|q}{2D\rho}. \quad (18)$$

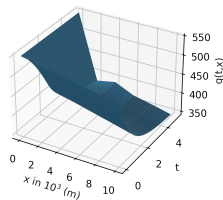
Note: $\frac{\partial}{\partial x} \frac{q^2}{\rho}$ is omitted.

Euler Equation vs. Weymouth Equation

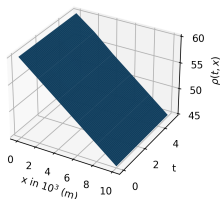
Discretized with Simple Upwind Scheme



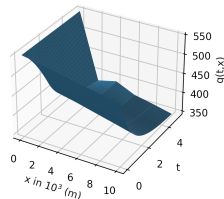
Euler Equation: $\rho(t, x)$



Euler Equation: $q(t, x)$



Weymouth Equation: $\rho(t, x)$



Weymouth Equation: $q(t, x)$

Note: No difference can be spotted!

Notation for Gas-Node Conditions

V : set of all *gas nodes*

V_q : set of all *source nodes*

V_p : set of all *sink nodes*

V_c : set of all *compressor nodes*

$\delta^+(v)$: set of *outgoing edges* at node $v \in V$

$\delta^-(v)$: set of *ingoing edges* at node $v \in V$

${}^e q(x^n, t)$: flow in pipe e in point (x^n, t)

Note: Similiar for ${}^e p(x^n, t)$.

Conditions at Gas Nodes (1)

Gas Coupling Conditions

Sum of ingoing flows is equal to sum of outgoing flows:

$$\sum_{e \in \delta^-(v)} {}^e q(x^n, t) = \sum_{h \in \delta^+(v)} {}^h q(x^0, t), \quad \forall v \in V \setminus (V_q \cup V_p), \quad t = 0, \dots, T. \quad (19)$$

No pressure loss near the node:

For a fix but arbitrary node in the innodes $v \in V \setminus (V_q \cup V_c \cup V_p)$ and a pressure $p(v, t)$ at node v in time t we state

$$\begin{aligned} {}^e p(x^n, t) &= p(v, t), \quad \forall e \in \delta^-(v), \quad t = 0, \dots, T \\ {}^h p(x^0, t) &= p(v, t), \quad \forall h \in \delta^+(v), \quad t = 0, \dots, T. \end{aligned} \quad (20)$$

Gas-Power Coupling at Slack Node s

s is the gas node connected to slack bus via edge h_{slack} , ϵ^t is flow taken out of the gas network. At slack node s it holds:

$$\sum_{e \in \delta^-(s)} {}^e q(x^n, t) = \sum_{h \in \delta^+(s) \setminus h_{slack}} {}^h q(x^0, t) + \epsilon^t, \quad t = 0, \dots, T. \quad (21)$$

Conditions at Gas-Nodes (2)

Compressor Conditions

Conditions at Compressor Node:

$${}^e p(x^n, t) = {}^h p(x^0, t) + u_v^t, \quad \forall v \in V_c, e \in \delta^-(v), h \in \delta^+(v), t = 0, \dots, T \quad (22)$$

where u_v^t denotes pressure increase (*bar*) at compressor node v .

Non-negativity constraints

Pressure and pressure increase cannot be negative:

$$u_v^t \geq 0, \quad \forall v \in V_c, t = 0, \dots, m \quad (23)$$

$$p_j^t \geq 0, \quad j = 0, \dots, n, t = 0, \dots, m, \text{ for all pipes} \quad (24)$$

Compressor Cost Optimization Problem (1)

Optimization Problem with **POC reformulation**:

$$\min_u \quad \frac{1}{2} \sum_{t=0}^m \sum_{v \in V_c} (u_v^t)^2$$

subject to:

1. **Weymouth Equation discretized with Simple Upwind Scheme**
2. **gas coupling conditions**
3. **compressor conditions**

$${}^e p_n^t = {}^h p_0^t + u_v^t, \quad \forall v \in V_c, e \in \delta^-(v), h \in \delta^+(v), t = 0, \dots, m \quad (25)$$

$$u_v^t = \sum_{s=1}^{n_{oc}} w_s^t c_v^s u_v^t, \quad \forall v \in V_c, t = 0, \dots, m$$

where $c_v^s \in \{0, 1\}$ denotes state of compressor v for configuration s

$$\sum_{s=1}^{n_{oc}} w_s^t = 1 \quad (SOS-Type1)$$

4. **non – negativity constraints**
5. **gas – power coupling condition**

Initial conditions: $p_j^0, q_j^0, j = 0, \dots, n$ for each pipe and $\epsilon^t, t = 0, \dots, m$.

Solution: $p_j^t, q_j^t, j = 0, \dots, n, t = 0, \dots, m$ for each pipe, $u_v^t, \forall v \in V_c$ and configuration $w^t, t = 0, \dots, m$.

Compressor Cost Optimization Problem (2)

Optimization Problem **without POC reformulation:**

$$\min_u \quad \frac{1}{2} \sum_{t=0}^m \sum_{v \in V_c} (u_v^t)^2$$

subject to:

1. **Weymouth Equation discretized with Simple Upwind Scheme**
2. **gas coupling conditions**
3. **compressor conditions**

(26)

$$^e p_n^t = ^h p_0^t + u_v^t, \quad \forall v \in V_c, e \in \delta^-(v), h \in \delta^+(v), t = 0, \dots, m$$

$$u_v^t = \beta_v^t \cdot u_v^t, \quad \forall v \in V_c, t = 0, \dots, m \text{ and } \beta_v^t \in \{0, 1\}$$

4. **non – negativity constraints**
5. **gas – power coupling condition**

Initial conditions: p_j^0, q_j^0 $j = 0, \dots, n$, for each pipe and ϵ^t , $t = 0, \dots, m$.

Solution: p_j^t, q_j^t , $j = 0, \dots, n$, $t = 0, \dots, m$, for each pipe, u_v^t and configuration β_v^t $v \in V_c$ and $t = 0, \dots, m$.

Additional Constraints (1)

Additional Constraints: Type 1 - Compressor has to keep its state for at least a time M_1 , M_2 for every compressor node $v \in V_c$.

$$\sum_{e=k+1}^{k+\left\lfloor \frac{M_1}{\Delta t} \right\rfloor} \sum_{s=1}^{n_{oc}} c_v^s w_s^e \geq \sum_{s=1}^{n_{oc}} c_v^s \left\lfloor \frac{M_1}{\Delta t} \right\rfloor (-w_s^k + w_s^{k+1}) \quad \forall k \leq m - \left\lfloor \frac{M_1}{\Delta t} \right\rfloor \quad (27)$$

$$\sum_{e=k+1}^{k+\left\lfloor \frac{M_2}{\Delta t} \right\rfloor} \left(1 - \sum_{s=1}^{n_{oc}} c_v^s w_s^e \right) \geq \sum_{s=1}^{n_{oc}} c_v^s \left\lfloor \frac{M_2}{\Delta t} \right\rfloor (w_s^k - w_s^{k+1}) \quad \forall k \leq m - \left\lfloor \frac{M_2}{\Delta t} \right\rfloor \quad (28)$$

Additional Constraints: Type 2 - Compressor can only switch maximum $r \in \mathbb{N}$ times.

$$\sum_{t=1}^m \sum_{s=1}^{n_{oc}} |w_s^t c_v^s - w_s^{t-1} c_v^s| \leq r, \quad \forall v \in V_c. \quad (29)$$

Additional Constraints (2)

Solve CIAP by solving MILP

$$\begin{aligned}
 & \min_{\substack{\beta^t \in H \cap \{0,1\}^{n_{oc}}, \\ t=0, \dots, m-1, \\ \delta^0 \in \mathbb{R}^{n_{oc}}, \epsilon \in \mathbb{R}}} \epsilon \\
 & \text{subject to: } \left\| \delta^0 + \sum_{t=0}^{k-1} \Delta t (\alpha^t - w^t) \right\|_{\Omega} \leq \epsilon \quad \text{for all } k = 0, \dots, m \\
 & \quad + \text{additional constraints type 1 OR } + \text{additional constraints type 2} \\
 & \hspace{15em} (30)
 \end{aligned}$$

where α^t are the relaxed binary variables in time t .

Algorithms used

	Method	Solver	Programming Language
Power Model	Levenberg-Marquardt	<i>fsolve</i>	Matlab
Three-Step Approach	Interior Point Method (+ Sum Up Rounding or MILP)	IPOPT 3.12.3 with ma27 integrated in CasADi Gurobi 9.1.2	Python 3.7.6
Direct Solver	NLP-Based Branch-and-Bound Algorithm	BONMIN integrated in CasADi	Python 3.7.6

PC: MacBook Pro, 16 GB RAM, Intel-Quad Core i5, 2,4 GHz CPU

Numerical Analysis: Simple Gas Network



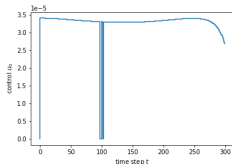
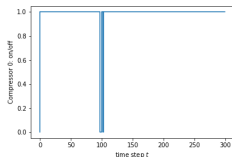
Figure 3: Simple Model

Constants: $\lambda = 0.11$, $D = 1 \text{ m}$ and $c = 340 \text{ m/s}$, pipe of length 12 km , total execution time of 5 sec , spacial step size $\Delta x = 2000$, time step size $\Delta t = 1/60$

Initial data: $\epsilon^t = 0$, $t = 0, \dots, m$, $p(0, x) = 60 \text{ bar}$ and $q(0, x) = 500 \text{ kg/s}$ for each pipe at every pipe section

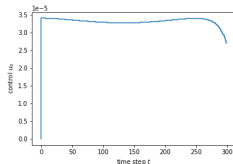
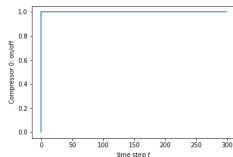
Numerical Analysis: Simple Model

Bonmin (no POC)



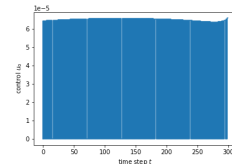
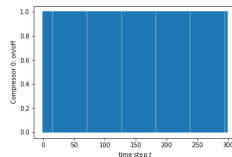
OFV: $1.6350264 \cdot 10^{-7}$
 Runtime: 107.35 sec
 (≈ 1.8 min)

Bonmin (POC)



OFV: $1.64764 \cdot 10^{-7}$
 Runtime: 176.13 sec
 (≈ 2.9 min)

Three-Step Approach



OFV: $3.1675 \cdot 10^{-7}$
 Runtime: 7.59 sec

Additional Constraints: Simple Model

Three-Step Approach with and without Additional Constraints

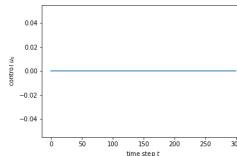
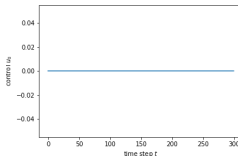
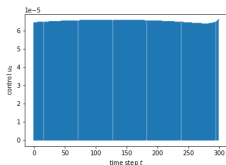
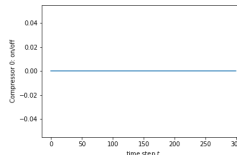
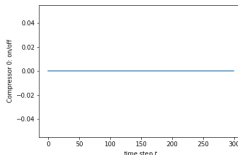
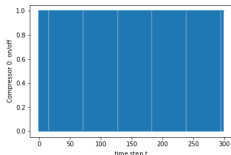
No Add.Constraints

Add. Constraint Type 1

Add. Constraint Type 2

Compressor can only switch 5 times in 5 sec

Compressor has to keep its state for at least 3 time steps



OFV: $3.1675 \cdot 10^{-7}$
Runtime: 7.59 sec

OFV: $1.5 \cdot 10^{-14}$
Runtime: 12.41 sec

OFV: $1.5 \cdot 10^{-14}$
Runtime: 18.21 sec

Numerical Analysis: Advanced Model

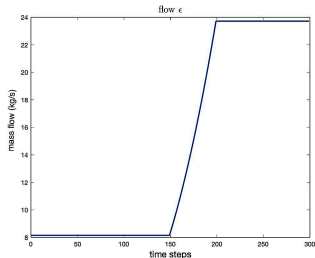
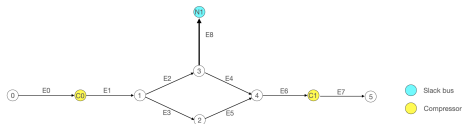


Figure 8



Advanced Gas Network

Constants: $\lambda = 0.11$, $D = 1 \text{ m}$ and $c = 340 \text{ m/s}$, pipe of length 12 km , total execution time of 5 sec , spacial step size $\Delta x = 2000$, time step size $\Delta t = 1/60$

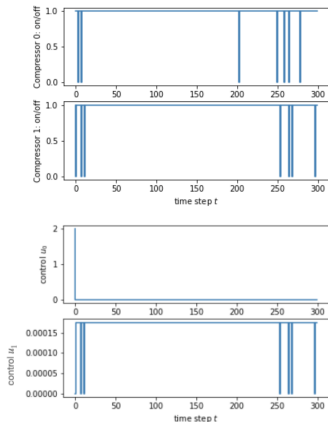
Initial data:

	$E0$	$E1$	$E2, E3$	$E4$	$E5$	$E6, E7$
$p(x, 0) \text{ (bar)}$	60	62	62	62	62	62
$q(x, 0) \text{ (kg/s)}$	500	500	250	241.847	250	491.847

ϵ^t (mass taken out of gas network is shown in Figure 8)

Numerical Analysis: Advanced Model

Three-Step Approach



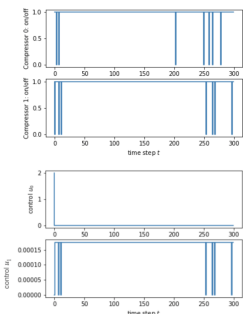
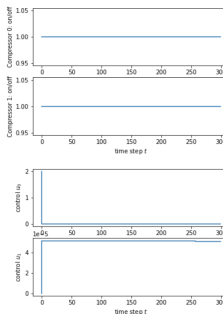
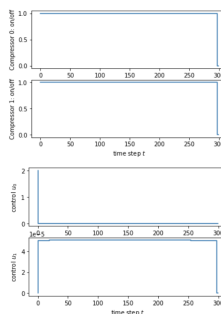
OFV: 2.00000

Runtime: 1259.03 sec (≈ 21 min)

- solved to 'acceptable level' (tolerance level 10^{-6})
- direct solver bonmin could not find any result in 2 hours - freezing behaviour.

Additional Constraints: Advanced Model

Three-Step Approach with and without Additional Constraints

No Add.Constraints	Add. Constraint Type 1 Compressor can only switch 10 times in 5 sec	Add. Constraint Type 2 Compressor has to keep its state for at least 3 time steps
 <p>OFV: 2.0000 Runtime: 1259 sec (≈ 21 min)</p>	 <p>OFV: 2.0000 Runtime: 1941.63 sec (≈ 32.3 min)</p>	 <p>OFV: 2.0000 Runtime: 1920.15 sec (≈ 32 min)</p>

Conclusion

- Three Step Approach delivers good optimization results and is up to 14 times faster than direct solver bonmin, which works with the NLP-based Brand-and-Bound algorithm
- With additional constraints we can prevent the compressor from switching too often and can achieve even better OFV
- Due to the CFL Condition we had to choose a very small time step ($\Delta t = \frac{1}{60}$) and therefore considered a simulation of 5 sec. For realistic simulations (e.g. 1 day) computation will likely take several hours.
Interesting question: How does run time and solutions vary for implicit scheme and explicit scheme (with Three Step approach)?

Thank you for your attention!