Report for Numerical Methods in Geophysics. The Heat Equation.

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Contents

1 Introduction

In this report the solution for the 1D Heat Equation is presented. The initial setting is given in the Table 1. The main task is to a find solution for the original problem for $f = 0$ with regard to daily and yearly changes of temperature. The analytical solution will be compared with numerical methods "Forward Euler Method", "Backward Euler Method" and additionally "Crank-Nicholson Method ". The next aim is to find the depth at which positive temperatures first occur, when the surface has a temperature minimum of -8°C. Finally, the practicability of the results with regards to the placement of piping inside the earth is discussed.

Parameter	Value
f	0
k	$2 \cdot 10^{-7} \frac{m^2}{s}$
ω	2π
T_0	$12°C$
dT	$20°K$
Z_{min}	0 m
Z_{max}	10 m
$T_{Z_{min}}$	$T = T_0 + dT \cdot sin(\omega\pi)$
$T_{Z_{max}}$	$T = T_0$

Table 1: Initial data.

2 The Heat Equation

2.1 Theoretical basics

The Heat Equation is a parabolic partial differential equation(1).

$$\Delta T - \kappa \frac{\partial T}{\partial t} = f, \tag{1}$$

where κ is thermal diffusivity, T is temperature and t is time. In our case, since $f = 0$ we obtain:

$$\Delta T - \kappa \frac{\partial T}{\partial t} = 0 \tag{2}$$

We would like to study how heat will change itself over time(during the day and the year) in the earth from the surface to the depth of 10 meters.The problem is considered to be 1D, that is, it accounts change with regard to time and depth. Hence, we look for the function for the temperature T(z,t).

Firstly, we look on the known values of the temperature. On the surface the temperature will change periodically(Table 1) and at the maximum depth it equals 12°C and remains constant in time.These values are called boundary conditions(BC). We denote them as $T(Z_{min} = 0, t)$ and $T(Z_{max} = 10, t)$, respectively. Secondly, we know the analytic solution for this problem(3):

$$T(z,t) = T_0 + dT \cdot exp(-z\sqrt{\frac{\omega}{2\kappa}}) \cdot sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}}), \tag{3}$$

where ω is frequency.

Deriving the analytic solution for $T(z, t = 0)$ we can set initial conditions (IC) which are necessary for solving the problem numerically. We proceed with aforementioned numerical methods to solve the original problem.

3 Numerical methods

In this section we inspect numerical methods: "Forward Euler Method", "Backward Euler Method" and additionally "Crank-Nicholson Method ".It is important to discuss the Finite Difference Method which is often used for the discretization of PDEs. The first step in the Finite Difference Method is to construct a grid with points on which we are interested in solving the equation. We introduce a discrete model $z \to z_i$ and $T(z_i;t) \to T_i$, with spatial step size Δ z and $i = 1 \dots M$. Since $f = 0$, we get :

$$\kappa \cdot \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} \tag{4}$$

Here, in contrast to the case when k=1, we have to incorporate k and account its impact.We approximate derivatives using their definition.

$$\frac{\partial^2 T}{\partial z^2} \to \mathbf{k} \cdot DT, \tag{5}$$

with $D \in R^{M \times M}$ and $T \in T^M$ Then we replace this with nested central differences:

$$\frac{\partial^2 T}{\partial z^2} \approx \kappa \cdot \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta z^2} \tag{6}$$

Now it's possible to introduce the operator for Dirichlet Boundary conditions $\mathbf{D} \in R^{M \times M}$:

$$D = \frac{1}{\Delta z^2} \begin{bmatrix} -2 & 1 & \dots & \dots \\ 1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & 1 & -2 \end{bmatrix} \tag{7}$$

Having that, we can construct the system of linear equations for our solution. It is important to truncate the matrices correctly and account κ. The κ is a factor for $\mathbf{D}$ and T_{BC}. We reduce the size of the linear system by removing columns and lines corresponding to boundary values and shift Dirichlet values to the right-hand side. The operator $\mathbf{D} \in R^{M-2 \times M-2}$, vector of temperatures $\mathbf{T} \in R^{M-2}$. We obtain:

$$\frac{k}{\Delta z^2} \begin{bmatrix} -2 & 1 & \dots & \dots \\ 1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & 1 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{M-1} \end{bmatrix} + \begin{bmatrix} \frac{\kappa T_0}{\Delta z^2} \\ 0 \\ \vdots \\ \frac{\kappa T_M}{\Delta z^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 T_1}{\partial z^2} \\ \frac{\partial^2 T_2}{\partial z^2} \\ \vdots \\ \frac{\partial^2 T_{M-1}}{\partial z^2} \end{bmatrix} \tag{8}$$

It is also possible to implement Neumann Boundary Conditions. In case of Neumann BC the size of system does not change. The next step before we start with particular methods is the temporal discretization. That is, we introduce the discrete model $t \to t^n$ and $T(t^n) \to T_i^n$ with the time step Δt.

Finally, we can approximate the 1^{st} order time derivative for each numeric method.The temporal discretization scheme is different for each method. However, the grid remains the same(1)

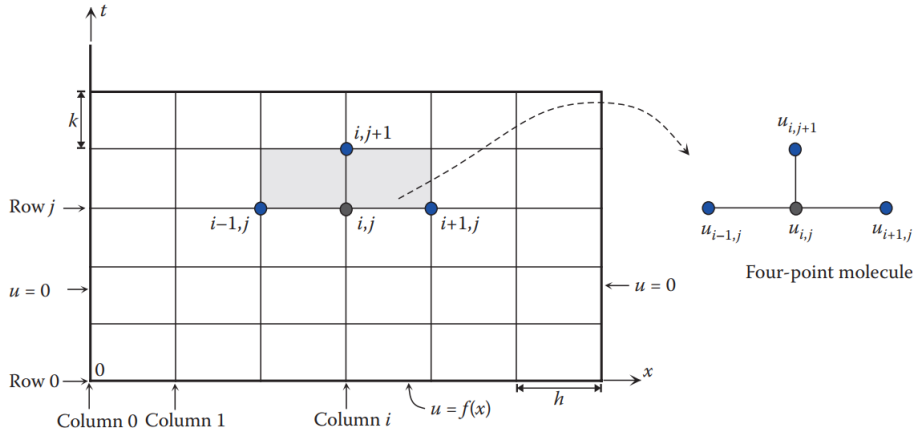

Figure 1: An example of the discretization grid.

3.1 Forward Euler method

We begin temporal discretization with replacing the time derivatives with their approximations(9:

$$\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^n}{\Delta t} = f(t^n, T^n) \tag{9}$$

Next stage is to rearrange the equation for T^{n+1} -temperature for the next future time step.

$$T_i^{n+1} = T_i^n + \kappa \cdot \Delta t \cdot f(t^n, T^n) \tag{10}$$

Finally, we obtain the equation, where $f(t^n, T^n)$ is replaced by a central finite difference approximation(12).

$$T_i^{n+1} = T_i^n + \kappa \cdot \Delta t \cdot \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta z^2} \tag{11}$$

This could be represented in a matrix form as well.

$$T_i^{n+1} = (I + \kappa \Delta t D)T^n + \kappa \Delta t T_{BC}, \tag{12}$$

where I is an Identity matrix.

The explicit time advancement scheme has the notable advantage that no matrix inversion (or solution of simultaneous equations) is necessary to determine either a transient or a steady-state solution to the governing equation. Not only does it make the computational algorithm straightforward to implement, but it also results in a tremendous reduction in runtime memory because the coefficient matrix no longer needs to be stored. If the number of grid points is large, storage of the coefficient matrix, even considering its sparseness, may often be a showstopper.

A critical issue with any time advancement scheme is its underlying stability. The inequality shows the stability criteria for the explicit Euler method, so-called **grid Fourier number** (12).

$$r = \frac{\kappa \cdot \Delta t}{\Delta z^2} \leq \frac{1}{2} \tag{13}$$

This condition severely limits the size of the time step: whenever one refines the space grid by increasing the number of grid points by a factor m, the number of grid points in the time grid has to increase by a factor m^2 for the method to remain stable (at least asymptotically for large m). This is particularly bad, if one solves a parabolic, time-dependent problem only to get to the stationary solution of a related elliptic problem, because then one would like to take large time steps (for fast convergence), but have a fine space discretization (for accuracy of the stationary solution).

3.2 Backward Euler method

Although the explicit (forward Euler) method is easy to implement and requires little memory, time step restriction can often result in a huge penalty on computational time. In particular, if a nonuniform mesh is used, the maximum allowable time step is dictated by the smallest grid

spacing, which can often be an order of magnitude less than that computed for the average grid spacing. In the backward Euler method, the difference approximation to the time derivative is derived by employing a backward Taylor series expansion in time and it leads to the following scheme(14):

$$T^{n+1} = T^n + \kappa \cdot \Delta t \cdot f(t^{n+1}, T^{n+1}) \tag{14}$$

Analogously to the explicit method we rearrange terms:

$$T_i^{n+1} = T_i^n + \kappa \cdot \Delta t \cdot \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta z^2} \tag{15}$$

Matrix form representation :

$$(I - \kappa \Delta t D) T_i^{n+1} = T^n + \kappa \Delta t T_{BC}, \tag{16}$$

where I is an Identity matrix.

Even though the implicit (backward Euler) method allows use of arbitrarily large time steps from a stability perspective, one must keep in mind that a large time step implies large temporal truncation error and inaccurate solution. Thus, although large time step size is acceptable from a stability perspective, it is not desirable from an accuracy perspective. Nonetheless, one of the notable advantages of the implicit time advancement method is its unconditional stability. On account of this favorable attribute, it is almost always the default method of time advancement in general purpose unsteady PDE solvers, such as those used in commercially available finite element or computational fluid dynamics codes.

3.3 Crank-Nicolson Method

Both the explicit (forward Euler) and implicit (backward Euler) methods have temporal truncation errors that are first order. This means that small time steps must be used to obtain an accurate solution. Further, the explicit method has additional restrictions due to its stability. For many practical applications, using excessively small time steps is not affordable from a computational efficiency standpoint. The Crank–Nicolson method was proposed in 1947 to address this critical shortcoming of the forward and backward Euler methods. It is a higher order (in time) implicit method.The idea behind the Crank–Nicolson method is to employ a six-point system as opposed to the four-point system (Figure 2) used with the Finite Difference method. One can consider this method as an "average" of the explicit and implicit Euler method,i.e. time derivative at midpoint between . We get the following formula(17):

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \kappa \cdot \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{2\Delta z^2} + \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{2\Delta z^2}\right) \tag{17}$$

where I is an Identity matrix.

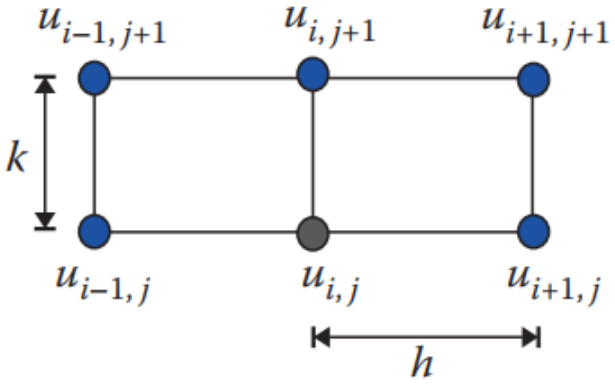

Figure 2: A six-point system.

The Crank–Nicolson method is unconditionally stable for the unsteady diffusion equation. This makes it an attractive choice for computing unsteady problems since accuracy can be enhanced without loss of stability at almost the same computational cost per time step. Another way of interpreting the increased order of accuracy of the method is that larger time steps may be used to get comparable accuracy with the forward or backward Euler methods. This is a significant advantage for problems where a large number of time steps have to be executed to reach the desired instant of time.

4 Results

The results of computations of analytical as well of numerical methods for daily and annual temperature changes are presented below(Figures (3,5). One can observe the obvious temperature oscillations at the shallow depth approximately in the interval of $[0, 0.5]$ m. Also the deviation of numerical methods are plotted as well. One can admit that the implicit Euler method has the least deviations from analytical solution and the explicit Euler method has the largest deviation, while Crank-Nicolson method is in the middle(Figures (4,6). This result correspond to the properties of these methods.

4.1 Daily temperature oscillations

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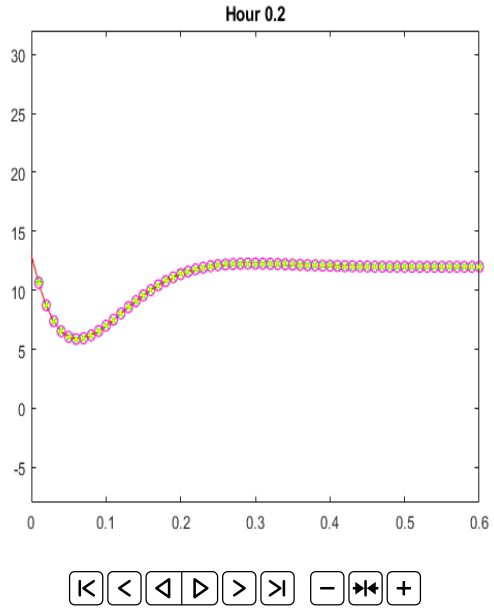

Figure 3: Daily temperature oscillations.

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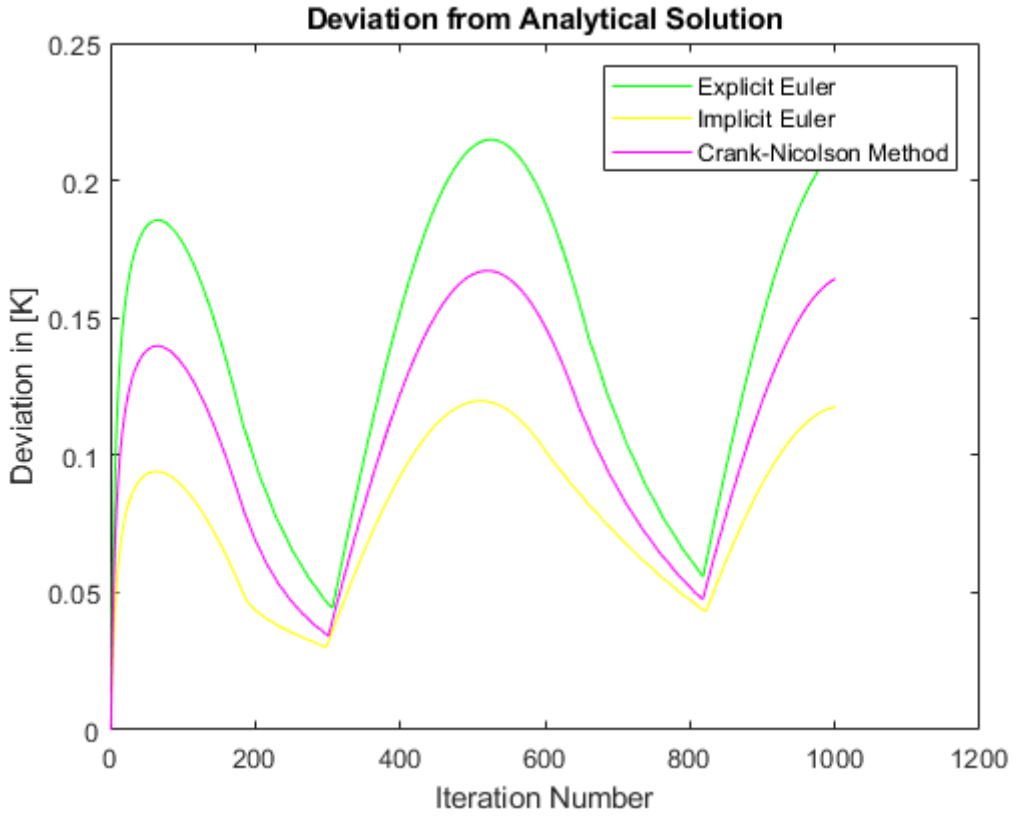

Figure 4: Deviation from the analytical solution (Daily).

4.2 Yearly temperature oscillations

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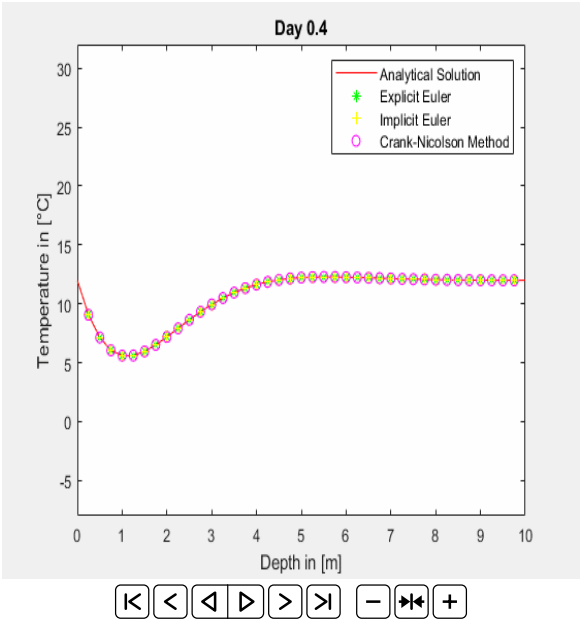

Figure 5: Yearly temperature oscillations.

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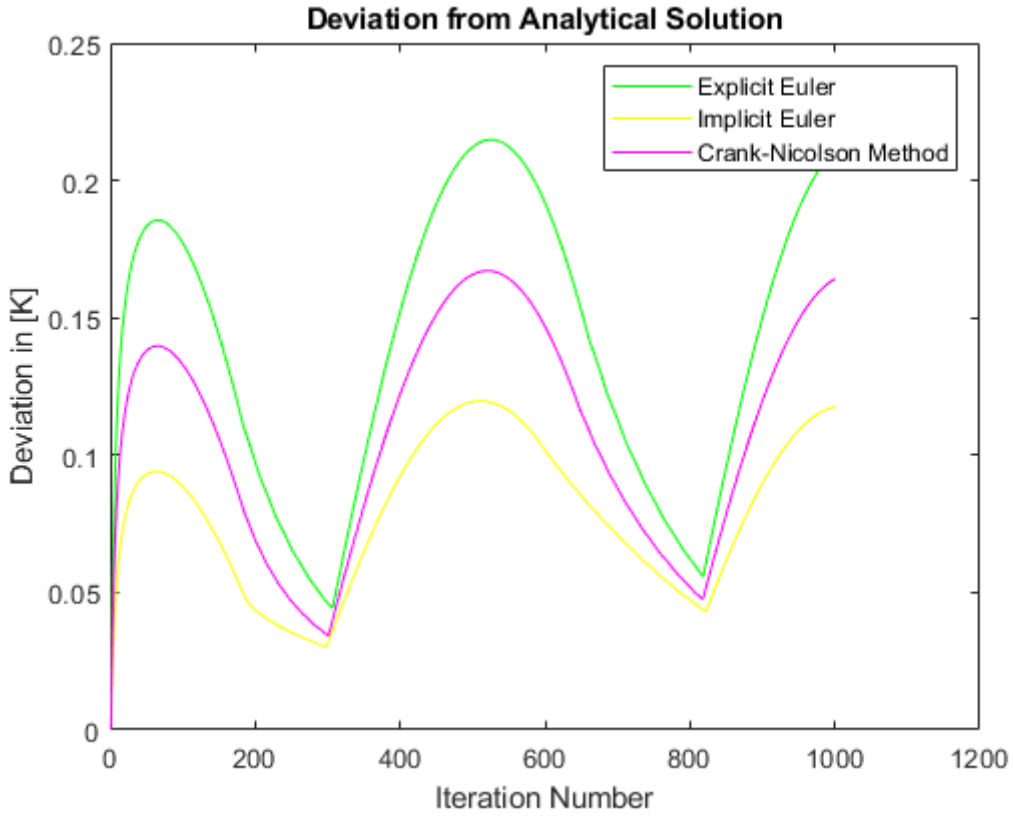

Figure 6: Deviation from the analytical solution (yearly).

4.3 First occurrence of positive temperatures

The depth for the first occurrence of positive temperatures when the surface has a temperature minimum of -8°C was also computed for daily and annual temperature oscillation. While the daily value of depth has too little significance, it's better to consider annual one (although both were computed in the attached Matlab script). In this case the first occurrence of positive temperatures when the surface has a temperature minimum of -8°C happens at the depth 1 m.

4.4 Practicability of the results with regards to the placement of piping inside the earth

These results have a quite significant meaning with regards to the the placement of piping inside the earth. In order to prevent or avoid the rupture of the pipeline due to significant temperature changes in the surrounding environment or contrast between the temperature of fluid which flows along the pipeline, one has to consider depth, where the temperature remains positive over the year and changes very slightly. The second problem one can face comes from Geomechanics. The heating of the usually frozen soil could lead to the landslides and instability of soils. The reason for that is frozen water(which is contained in the soil)starts to melt and change the properties of surrounding ground. Thus, it is a good practice to pay attention on the temperature of intervals. In general, most of the pipelines are placed below 1-2 meters to avoid such issues.

5 Bibliography

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