Report for Numerical Methods in Geophysics. DC Problem.

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Contents

1 Introduction

In this report the solution for the 2D DC problem is presented. The simulation of a dipole-dipole measurement over conductive anomalies within a homogeneous half-space is carried out according to the following instructions and initial data:

1. Define a model domain $-500m \leq x \leq 500m$; $-500m \leq y \leq 0m$ with $\sigma_h = 0.1 \ \frac{m}{S}$ including a vertical dike at $3m \leq x \leq 4m$; $-500m \leq y \leq 0m$ with $\sigma_h = 10 \ \frac{m}{S}$.

2. Consider 21 electrodes at positions x = [-10; 10]m with separation $d_x^e = 1m$, each at height y = 0 and injection currents of $I_A = 1A$; $I_B = -1A$.

3. The offset $d_x^e = 1m$ between injection (A; B) as well as potential electrodes (M; N) are kept fixed for the entire measurement, hence, the separation between the dipoles can be $d_x^p = [1; 18]m$.

4. Simulate the apparent resistivity given by the Neumann formula(1):

$$\rho_a^{ABMN} = \frac{U_{MN}}{I_a} \frac{\pi}{(\ln r_{AM} - \ln r_{BM} - \ln r_{AN} + \ln r_{BN})} \tag{1}$$

   Why does this formula differ from the typical definition(2) given in all the textbooks?

$$\rho_a^{ABMN} = \frac{U_{MN}}{I_a} \frac{1}{(r_{AM} - \frac{1}{r_{BM}} - \frac{1}{r_{AN}} + \frac{1}{r_{BN}})} \tag{2}$$

5. Write a function to visualize the entire measurement as pseudosection.

6. Now repeat this process with a rectangular anomaly at $3m \leq x \leq 4m$; $-4m \leq y \leq -3m$. What difference do you find? How would you explain the differences?

2 Theoretical basics

2.1 Direct Current Resistivity

In a DC resistivity survey, a generator is used to inject current into the subsurface. Currents flow, and the path they follow depends on the conductivity structure of the subsurface. Charges build up on conductivity interfaces and these change the electric potentials that are measured on the surface or in boreholes[3].

The generator inputs electrical current into the ground and a voltmeter measures the potential difference between two locations. The implemented dipole-dipole 4-electrode array is shown in Fig.1. However different configurations can be used due to specific task and conditions.

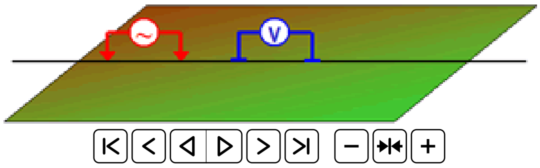

Figure 1: Dipole-dipole configuration.

2.2 General principles and rules for designing DC survey

- **Excite the target of interest:** This requires positioning source electrodes so that significant current flows through the target. This sets up the electrical charge distribution.
- **Measure a signal:** The potential electrodes must be close enough to the electrical charges, and configured so that they can measure a potential difference that is above the noise level.

For co-linear arrays, the depth of penetration depends upon the size of the array. The depth to which significant current flows depends upon the distance between the source electrodes. A target at depth can be excited only when the current electrodes are significantly farther apart than the depth of the target. Since a datum is a potential difference, and since deeper targets are associated with smaller electrical charges (there's only small currents going through), detecting meaningful signal requires that the potential electrodes have significant separation[3].

Assembling the above information leads to a general statement that depth of penetration progressively decreases as one proceeds from pole-pole, pole- dipole, to dipole-dipole. This is a reasonable rule of thumb and is applicable for surface arrays or for co-linear arrays in borehole measurements.

2.3 Description of the 2D DC problem

We start with the continuity equation which describes electrical potential distribution ϕ due to current injection(3), where I is the current, σ is the conductivity:

$$\nabla \cdot (\sigma \nabla U) = -I\delta(x)\delta(y) \text{ in } \Omega \tag{3}$$

We also have BC(4):

$$\begin{aligned} &\phi(x, y) = 0 \text{ at } \Gamma_0 \text{ in earth} \\ &\frac{\partial \phi(x, y)}{\partial n} = 0 \text{ at } \Gamma_1 \text{ at surface} \end{aligned} \tag{4}$$

Since the domain has a rectangular shape we get:

$$\begin{aligned} \phi(x_{min}, y) = \phi(x_{max}, y) = \phi(x, y_{min}) = 0 \text{ at } \Gamma_0 \\ \frac{\partial \phi(x, y_{max})}{\partial y} = 0 \text{ at } \Gamma_1 \end{aligned} \tag{5}$$

Analytical solution for point source $I\delta(x; y)$ at the top of homogeneous half-space $\sigma = const$, where r_0 is a scaling factor, which defines the zero of logarithm function[4].

$$\phi(x, y) = -\frac{I}{\sigma \pi} ln(\frac{|[x - x_0; y - y_0]^T|}{r_0}) \tag{6}$$

2.4 The derivation of the general Neumann formula

We measure the voltage, ΔU_{MN}, at points M and N on the surface of the earth. This voltage, according to Ohm's law, will be proportional to the strength of the current, I, and also, be dependent on the electrical properties of the earth. For example, in the simple case of a homogeneous half-space with resistivity ρ, the potential at the point M will be the sum of contributions from the currents flowing through point contacts A and B as follows:

$$\begin{aligned} U_M &= \frac{I\rho}{2\sigma\pi}(\frac{1}{r_{AM}} - \frac{1}{r_{BM}}) \\ U_N &= \frac{I\rho}{2\sigma\pi}(\frac{1}{r_{AN}} - \frac{1}{r_{BN}}) \\ U_{MN} &= \frac{I\rho}{2\sigma\pi}(\frac{1}{r_{AM}} - \frac{1}{r_{BM}} - \frac{1}{r_{AN}} + \frac{1}{r_{BN}}) \end{aligned} \tag{7}$$

In this model, ΔU_{MN}, is proportional to the resistivity, ρ. The equation can be inverted to yield the resistivity of the medium in terms of measured quantities:

$$\rho_a^{ABMN} = \frac{U_{MN}}{I_A} \frac{2\pi}{(\frac{1}{r_{AM}} - \frac{1}{r_{BM}} - \frac{1}{r_{AN}} + \frac{1}{r_{BN}})} = K_g \frac{U_{MN}}{I_a} \tag{8}$$

and K_g is the geometric factor for the array of electrodes being used to measure resistivity. The ratio U_{MN}/I is called the mutual resistance, R_m, and is measured in ohms. The mutual resistance is not a resistance in the ordinary sense inasmuch as the voltage and the current are measured in separate circuits. For the mutual resistance to be meaningful, the two circuits must be linearly coupled so that the ratio is a constant. But this formula is valid for the **3D half-space**, so we have to refine the formula for the 2D problem.

2.5 The derivation of the Neumann formula for the 2D case

We implement the Neumann formula to compute the apparent resistivity ρ_a^{ABMN}

$$\rho_a^{ABMN} = \frac{U_{MN}}{I_a} \cdot \frac{\pi}{(\ln r_{AM} - \ln r_{BM} - \ln r_{AN} + \ln r_{BN})} \tag{9}$$

The peculiarity of the implemented Neumann formula lies within the shape of studied half-space. Here we consider a 2D half-space what means the sphere's surface as a function of the current flow is reduced to the circumference. Thus, we have a modified Neumann formula with another scaling factor. Since the analytical solution for potential of the point source is known, this formula could be derived as follows:

$$E = -\Delta U \tag{10}$$

We also know that:

$$\phi(r) = \int_{-\infty}^{r} E(r)dr \tag{11}$$

One can express E in terms of Ohm's Law, $E = j\rho$, where j stands the for current density. For 3D problem we have:

$$j = \frac{I}{4\pi r^2} \tag{12}$$

Meanwhile we consider the 2D case, so the sphere turns into a circumference, which leads to the following expression:

$$j = \frac{I}{2\pi r} \tag{13}$$

We insert 13 in 11 and obtain:

$$\phi(r) = \int_{-\infty}^{r} \frac{\rho I}{2\pi r} dr \tag{14}$$

Since the domain is defined as a half-space($r > 0$) and the current does not flow above the surface due to $\rho_{air} = \infty$, we set the boundary of the integral to 0. Thus, it leads to the simplified expression:

$$\phi(r) = \int_{0}^{r} \frac{\rho I}{2\pi r} dr = \frac{\rho I}{2\pi ln(r)} \tag{15}$$

Applying the similar principle just like in the 3D case we get:

$$U_{MN} = \phi_M(r) - \phi_N(r) = \frac{\rho I}{\pi r}(\ln r_{AM} - \ln r_{BM} - \ln r_{AN} + \ln r_{BN}) \tag{16}$$

Finally, we rearrange this equation for the apparent resistivity ρ and obtain the reqired expression:

$$\rho_a^{ABMN} = \frac{U_{MN}}{I_a} \cdot \frac{\pi}{(\ln r_{AM} - \ln r_{BM} - \ln r_{AN} + \ln r_{BN})} \tag{17}$$

2.6 Spatial Dicretization in 2D

The grid for the spatial discretization is presented below:

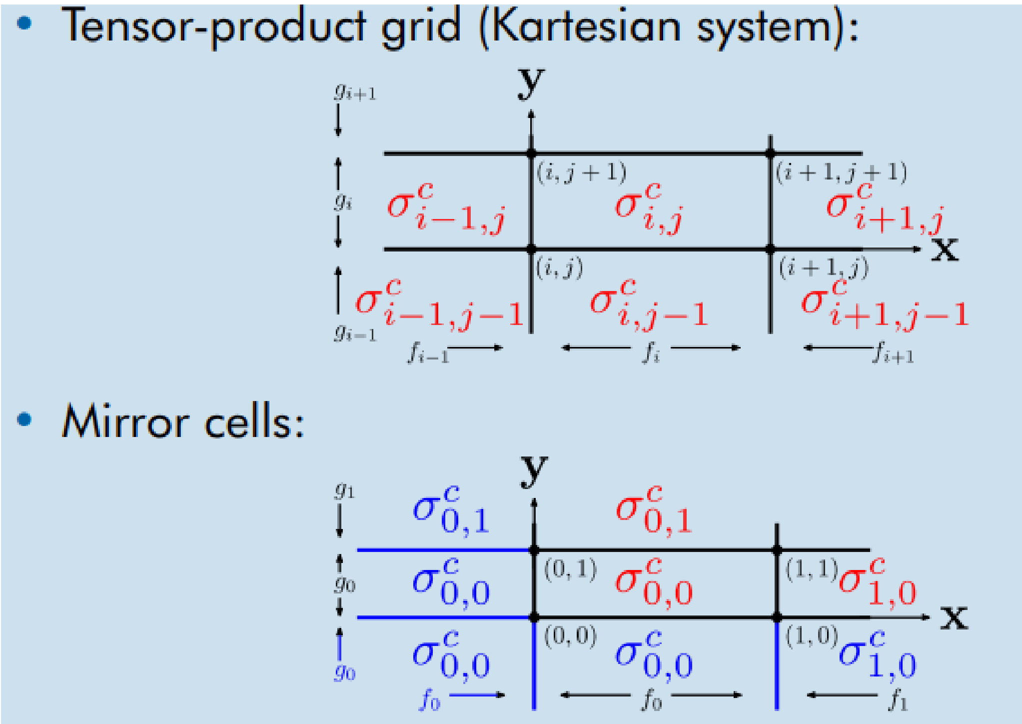

Figure 2: Tensor grid and mirror cells.

We store all conductivity values in the matrix $K \in R^{(n_x-1)\times(n_y-1)}$ which is flattened then into a vector $\sigma \in R^{(n_x-1)(n_y-1)\times 1}$:

$$K = \begin{bmatrix} \sigma_{0,0} & \sigma_{0,1} & \dots & \sigma_{0,n_y} \\ \sigma_{1,0} & \sigma_{1,1} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n_x,0} & \sigma_{n_x,1} & \dots & \sigma_{n_x,n_y} \end{bmatrix} \implies \sigma = \begin{bmatrix} \sigma_{0,0} \\ \sigma_{0,1} \\ \vdots \\ \sigma_{1,0} \\ \sigma_{1,1} \\ \vdots \\ \sigma_{n_x,n_y} \end{bmatrix} \tag{18}$$

Next step is to derive the conductivity gradient in X and Y directions, which are computed with attached supporting function for gradients:

$$\begin{aligned} \partial_x \sigma_{i,j} &= \frac{\frac{\Sigma_{m=j-1}^{j} f_i g_m \sigma_{i,j}^c}{\Sigma_{m=j-1}^{j} f_i g_m} - \frac{\Sigma_{m=j-1}^{j} f_{i-1} g_m \sigma_{i-1,j}^c}{\Sigma_{m=j-1}^{j} f_{i-1} g_m}}{\frac{f_i + f_{i+1}}{2}} \\ \partial_y \sigma_{i,j} &= \frac{\frac{\Sigma_{m=j-1}^{j} f_i g_m \sigma_{i,j}^c}{\Sigma_{m=j-1}^{j} f_i g_m} - \frac{\Sigma_{m=j-1}^{j} f_i g_{m-1} \sigma_{i,j-1}^c}{\Sigma_{m=j-1}^{j} f_i g_{m-1}}}{\frac{g_m + g_{m+1}}{2}} \end{aligned} \tag{19}$$

Then we assemble vector $b = [0, ..b_{i,j}..0]^T$ for ρ with entries:

$$b_{i,j} = \frac{I}{\frac{f_i + f_{i-1}}{2} + \frac{g_j + g_{j-1}}{2}} \tag{20}$$

The mirror cells are excluded. We obtain values from the function "$assemble\_coefficients.m$", which invokes "$get\_sig\_grad.m$".

After that we can consider the Neumann and Dirichlet BC.
It's important to apply Neumann BC first and only then Dirichlet BC. The supporting function "$apply\_NBC.m$" and "$apply\_DBC.m$" are attached to the main script.

2.7 Finite-difference discretization according Dey and Morrison

This method is one of several methods for finite - difference discretization, e.g Brewitt-Taylor. We start with the equation of continuity(21):

$$\nabla \cdot (\sigma \nabla U) = -I\delta(x)\delta(y) \tag{21}$$

in Ω, where I is the current, σ is the conductivity. Then we obtain using vector identities:

$$\nabla\sigma\nabla U + \sigma\nabla^2 U = -I\delta(x)\delta(y) \tag{22}$$

Using (23),

$$\nabla^2(\sigma U) = U\nabla^2\sigma^2 + 2\nabla\sigma\nabla U + \sigma\nabla^2 U \tag{23}$$

we obtain the following expression (24):

$$\nabla\sigma\nabla U = \frac{1}{2}(\nabla^2(\sigma U) - U\nabla^2\sigma - \sigma\nabla^2 U) \tag{24}$$

and insert it into (21) Then we rearrange the equation and obtain:

$$\nabla^2(\sigma U) - U\nabla^2\sigma + \sigma\nabla^2 U = -2I\delta(x)\delta(y) \tag{25}$$

Now this contains only Laplacians and we can proceed to defining the conductivity in grid points. We define $P_i, j = \sigma_{i,j}U_{i,j}$:

$$\nabla^2 P_{i,j} = \frac{2}{f_i + f_{i-1}} \cdot [\frac{P_{i-1,j} - P_{i,j}}{f_{i-1}} + \frac{P_{i+1,j} - P_{i,j}}{f_i}] + \frac{2}{g_j + g_{j-1}} \cdot [\frac{P_{i,j-1} - P_{i,j}}{g_{j-1}} + \frac{P_{i,j+1} - P_{i,j}}{g_j}] \tag{26}$$

For the simplicity we drop index j and consider the X-direction:

$$\frac{\partial^2 P_i}{\partial x^2} = 2\frac{f_{i-1}P_{i+1} - (f_i + f_{i-1})P_i + f_i P_{i-1}}{f_{i-1}f_i(f_{i-1} + f_i)} \tag{27}$$

Having that, we insert this result into equation(25) and sort it for U_{i-1}, U_i and U_{i+1}:

$$\begin{aligned} U_i\frac{[-2\sigma_i(f_i + f_{i-1}) - 2f_i\sigma_{i-1} + 2(f_i + f_{i-1})\sigma_i - 2f_{i-1}\sigma i + 1 - 2(f_i + f_{i-1})\sigma_i]}{f_{i-1}f_i(f_{i-1} + f_i)} + \\ + U_{i-1}\frac{(2f_i\sigma_{i-1} + 2f_i\sigma_i)}{f_{i-1}f_i(f_{i-1} + f_i)} + \\ + \quad U_{i+1}\frac{(2f_{i-1}\sigma_{i+1} + 2f_{i-1}\sigma_i)}{f_{i-1}f_i(f_{i-1} + f_i)} \quad = \quad -2I\delta(x)\delta(y) \end{aligned} \tag{28}$$

From this equation we can now derive coupling coefficients in the X-direction:

$$\begin{aligned} C_{1i,j} &= \frac{\sigma_{i-1,j} + \sigma_{i,j}}{f_{i-1}(f_{i-1} + f_i)} \\ C_{2i,j} &= \frac{\sigma_{i+1,j} + \sigma_{i,j}}{f_i(f_{i-1} + f_i)} \end{aligned} \tag{29}$$

The same story for the Y-direction, we drop the i-index and consider Y-direction, replacing f with g:

$$\begin{aligned} \frac{\partial^2 P_j}{\partial y^2} &= 2\frac{g_{j-1}P_{j+1} - (g_j + g_{j-1})P_j + g_j P_{j-1}}{g_{j-1}g_j(g_{j-1}+g_j)} \\ C_{3i,j} &= \frac{\sigma_{i,j-1} + \sigma_{i,j}}{g_{j-1}(g_{j-1}+g_j)} \\ C_{4i,j} &= \frac{\sigma_{i,j+1} + \sigma_{i,j}}{g_j(g_{j-1}+g_j)} \end{aligned} \tag{30}$$

Now we have derived all required coefficients for solving the 2D problem, except self-coupling coefficient C_0, which represent negative sum of all coefficients:

$$\begin{aligned} C_{1i,j} &= \frac{\sigma_{i-1,j} + \sigma_{i,j}}{f_{i-1}(f_{i-1}+f_i)} \\ C_{2i,j} &= \frac{\sigma_{i+1,j} + \sigma_{i,j}}{f_i(f_{i-1}+f_i)} \\ C_{3i,j} &= \frac{\sigma_{i,j-1} + \sigma_{i,j}}{g_{j-1}(g_{j-1}+g_j)} \\ C_{4i,j} &= \frac{\sigma_{i,j+1} + \sigma_{i,j}}{g_j(g_{j-1}+g_j)} \\ C_{0i,j} &= -\Sigma_{l=1}^{4} C_l \end{aligned} \tag{31}$$

If we'd consider the 3D problem,the same procedure has to be carried out for Z-axis and self-coupling coefficient C_0 has 6 terms[4].

This method is realized via the supporting functions "*assemble_system.m*" and are invoked in the main script"*DC_report.m*".

3 Results

3.1 Pseudosection

Pseudosections are often used to visualize data from 2D profiles. To account for the fact that measurements with larger electrode separations sample deeper portions of the earth, lines at 45° degree angles, are drawn from the mid-points of the current and potential electrode pairs and the datum is plotted at the intersection of these lines. In cases where a pole transmitter or receiver is used, the 45° lines are drawn directly from the electrode location. The function "*plot_pseudosection.m*" plots such image for the dipole-dipole configuration. The vertical axis represents separation distance.

3.2 Initial anomaly

The first pseudosection is obtained for the initial anomaly with coordinates $3m \leq x \leq 4m$; $-500m \leq y \leq 0m$.It represents a slab elongated along axis y, that is, x $\ll$ y. Such result shows us that the separation between electrodes is not sufficient enough comparing with the size of the dike[2].

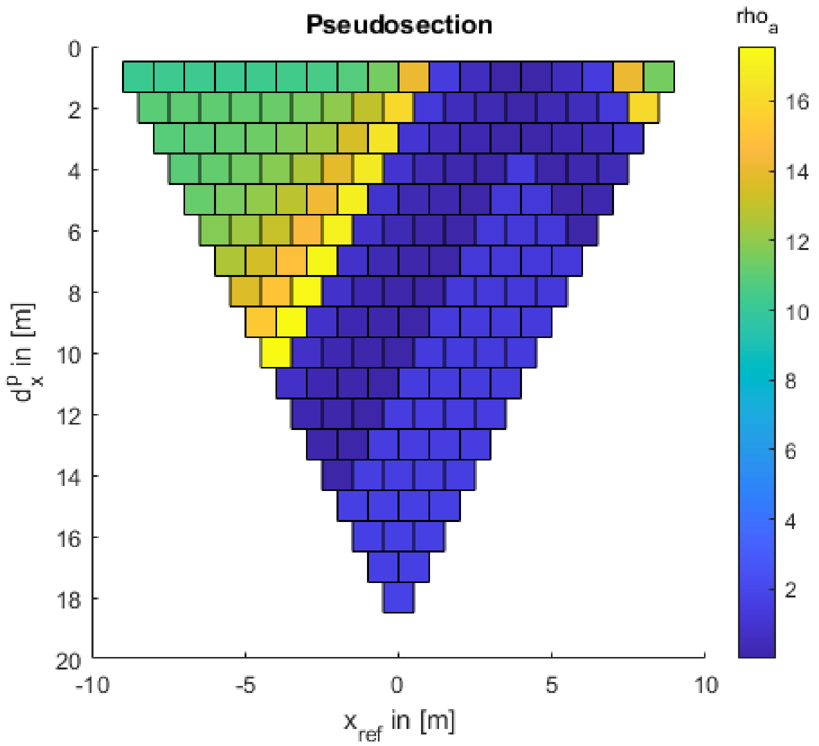

Figure 3: Pseudosection of the apparent resistivity,$[\Omega \cdot m]$ for the first anomaly.

3.3 Rectangular anomaly

This pseudosection is obtained for the rectangular anomaly with coordinates $3m \leq x \leq 4m$; $-4 \leq y \leq -3m$. Due to limited sizes of the anomaly and appropriate separation between the electrodes we obtain this plot.

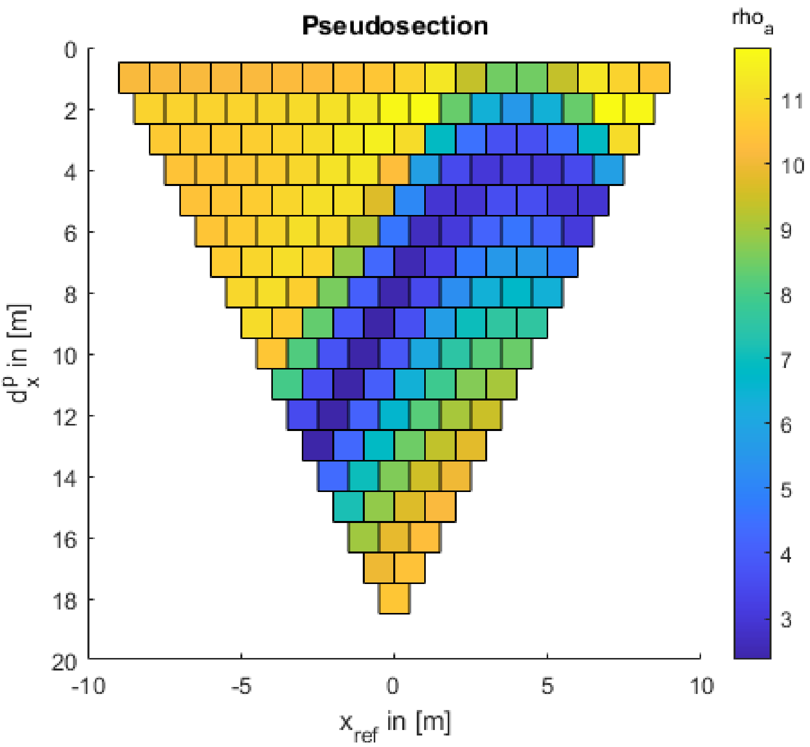

Figure 4: Pseudosection of the apparent resistivity,$[\Omega \cdot m]$ for the rectangular anomaly.

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