

The Meridional Circulation of the Sun: Observations, Theory and Connections with the Solar Dynamo

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Abstract

The meridional circulation of the Sun, which is observed to be poleward at the surface, should have a return flow at some depth. Since large-scale flows like the differential rotation and the meridional circulation are driven by turbulent stresses in the convection zone, these flows are expected to remain confined within this zone. Current observational (based on helioseismology) and theoretical (based on dynamo theory) evidences point towards an equatorward return flow of the meridional circulation at the bottom of the convection zone. Assuming the mean values of various quantities averaged over turbulence to be axisymmetric, we study the large-scale flows in solar-like stars on the basis of a 2D mean field theory. Turbulent stresses in a rotating star can transport angular momentum, setting up a differential rotation. The meridional circulation arises from a slight imbalance between two terms which try to drive it in opposite directions: a thermal wind term (arising out of the higher efficiency of convective heat transport in the polar regions) and a centrifugal term (arising out of the differential rotation). To make these terms comparable, the poles of the Sun should be slightly hotter than the equator. We discuss the important role played by the meridional circulation in the flux transport dynamo model. The poloidal field generated by the Babcock–Leighton process at the surface is advected poleward, whereas the toroidal field produced at the bottom of the convection zone is advected equatorward. The fluctuations in the meridional circulation (with coherence time of about 30–40 yr) help in explaining many aspects of the irregularities in the solar cycle. Finally, we discuss how the Lorentz force of the dynamo-generated magnetic field can cause periodic variations in the large-scale flows with the solar cycle.

Astrophysical plasma, Solar physics, Sun: solar magnetism, Sun: helioseismology, Fluid flow: rotational

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1 Introduction

There is an intriguing fluid flow pattern inside the Sun (and probably inside other solar-like stars): the *meridional circulation*. It is known for nearly half a century that matter at the solar surface moves continuously from the equator to the poles in both the hemispheres—the maximum speed of this motion at mid-latitudes being of order 20 m s^{-1} . Since we do not expect matter to be piled up near the poles, there has to be a return flow at some depth underneath the Sun’s surface bringing back the matter from the polar regions to the equatorial region. Apart from the intrinsic interest we may have in a such a flow from a purely fluid dynamical point of view, it is realized in the last few years that this flow plays a crucial role in the dynamo process producing the 11-year sunspot cycle. Let us begin with a discussion of the mathematical definition of the meridional circulation.

Any plane passing through the rotation axis of a rotating, self-gravitating body (such as a star or a planet) is referred to as a *meridional plane*. If we introduce spherical coordinates with the origin at the centre of the body and with the rotation axis as the polar axis, it is easy to see that a meridional plane would be an (r, θ) plane over which ϕ is constant. Let us begin by considering a simple kind of fluid flow which is axisymmetric (i.e. independent of ϕ). The fluid velocity can be written as

$$\mathbf{v} = v_r(r, \theta, t) \mathbf{e}_r + v_\theta(r, \theta, t) \mathbf{e}_\theta + v_\phi(r, \theta, t) \mathbf{e}_\phi. \quad (1)$$

The part $v_\phi(r, \theta, t) \mathbf{e}_\phi$ is called the *azimuthal* or *zonal circulation*, whereas the part lying in the meridional plane, i.e.

$$\mathbf{v}_m = v_r(r, \theta, t) \mathbf{e}_r + v_\theta(r, \theta, t) \mathbf{e}_\theta \quad (2)$$

is called the meridional circulation. Writing $v_\phi = r \sin \theta \Omega$, where Ω is the angular velocity, we can put (1) in the form

$$\mathbf{v} = \mathbf{v}_m + r \sin \theta \Omega(r, \theta, t) \mathbf{e}_\phi. \quad (3)$$

Very often we consider flows which are time-independent in addition to being axisymmetric. It then easily follows from the equation of continuity that $\nabla \cdot (\rho \mathbf{v}_m) = 0$, implying that flows in the meridional plane should be of the nature of circulation with closed streamlines.

We are aware of fluid flows existing in the interiors and atmospheres of many stars and planets. A state of strict hydrostatic equilibrium without any motions is often unstable or is continuously disturbed by forces driving the flows. The flows inside stars and planets are sometimes of the nature of turbulent flows, which means that they are neither axisymmetric nor time-independent. However, by suitable spatial and temporal averaging, we can often get a mean flow pattern which may be approximated as axisymmetric and time-independent—at least over a certain regime of space and time. Considering such flows is the natural first step in understanding the complex physics of this subject. In the theoretical portions of this basic review, we shall always restrict our discussion to mean meridional circulations which are axisymmetric, but we shall discuss certain aspects of time variations.

Before getting into a discussion of the meridional circulation of the Sun, let us consider a simple meridional circulation in the Earth’s atmosphere. Suppose

the equatorial region of the atmosphere is heated by the Sun's rays. The air there expands and becomes lighter, causing it to be buoyant and to rise up. The colder air from higher latitudes would rush to the equatorial region. The hot air, which rises in the equatorial region, will cool as it rises and then will flow to the higher latitudes through the upper layers of the atmosphere, thereby setting up a meridional circulation pattern. At first sight, it may seem that the physics of this problem is straightforward. After all, it involves only thermodynamics and fluid mechanics. We invite those readers who are familiar with thermodynamics and fluid mechanics, but have not studied the theory of meridional circulation earlier, to set up the mathematical equations of this problem. As soon as we try to make a mathematical formulation of this problem, we realize that it is much more complicated than what we may initially think.

The best way of handling such a fluid flow problem is to consider the *vorticity*

$$\omega = \nabla \times \mathbf{v}. \quad (4)$$

It is easy to see that the meridional circulation given by (2) would produce a ϕ component of vorticity. One can try to obtain an equation for ω_ϕ from the basic equations of fluid mechanics. Usually the equation for ω_ϕ turns out to have the form

$$\frac{\partial \omega_\phi}{\partial t} = (\text{source terms}) + (\text{dissipation term}). \quad (5)$$

As we shall see later, the meridional circulation of the Sun really satisfies an equation like this. The source terms, which may involve thermodynamic considerations, drive the meridional circulation, whereas the dissipative term tries to damp it. If these terms somehow manage to balance each other, then we may get a time-independent meridional circulation.

At the outset, let us point out an important result of stellar structure modelling that the heat generated by nuclear reactions at the centre of the Sun is transported outward by radiative transfer till about $r = 0.7R_\odot$ (where R_\odot is the solar radius), whereas heat is transported by convection from $r = 0.7R_\odot$ to $r = R_\odot$ [1, 2]. In other words, we have a turbulent convection zone just below the Sun's surface. The convection cells at the solar surface known as *granules* can be observed through telescopes. As we shall discuss later, the turbulent stresses in the convection zone play a crucial role in driving the meridional circulation. So, it is assumed that the streamlines of the meridional circulation would remain confined within the convection zone. While developing the theory of the meridional circulation of the Sun, we shall see that this theory is intimately connected with the theory of differential rotation (a non-constant Ω varying with r and θ is called *differential rotation*). It has been known from the mid-nineteenth century that the angular velocity at the solar surface near the equator is more than that at higher latitudes [3]. As we shall point out later, the new science of helioseismology has provided the crucial information of how the angular velocity $\Omega(r, \theta)$ varies under the solar surface. Helioseismology also provides information about the meridional circulation underneath the solar surface. However, as we shall discuss, this information becomes less and less reliable as we go deeper down from the solar surface, and although there is now strong observational evidence that the return flow of the meridional circulation (bringing back matter from the polar regions to the equatorial region) takes place at the bottom of the convection zone, there is still not a complete consensus on this. A poleward flow at the solar surface and an equatorward flow

deeper down give rise to negative ω_ϕ within the core region of the meridional circulation in the northern hemisphere. Hence, when we develop the theory of the solar meridional circulation, the sum of the source terms in (5) is expected to be negative in much of the northern hemisphere.

Sunspots are regions of concentrated magnetic field (typically of order 3000 G) and the 11-year sunspot cycle (also called the *solar cycle*) is the magnetic cycle of the Sun. This cycle is believed to be caused by a magnetohydrodynamic or MHD process known as the *dynamo process*. The early models of the solar dynamo were developed at a time when the existence of the meridional circulation was not known and these early models naturally did not include the meridional circulation. Over the years, it became clear that these earlier models of the solar dynamo without the meridional circulation had many difficulties. From the 1990s, a new kind model known as the *flux transport dynamo model*—in which the meridional circulation plays a crucial role—has been developed. This model has been successful in explaining various aspects of the solar cycle, leading to an increased interest in the science of the meridional circulation.

It may be mentioned that, in the last few years, there have been some impressive numerical simulations of convection inside rotating stars, showing that the turbulent stresses can drive the large-scale flow patterns. The discussion of simulations will be rather limited in this review, the focus being on the mean field theory obtained by averaging over turbulence—for the following two reasons. Firstly, the primary aim of this review is to elucidate the basic physics, which can be understood better from the mean field theory rather than from a description of the results of simulations. Secondly, the author personally is not particularly qualified to discuss the intricacies and subtleties of numerical simulations. We shall highlight some results of simulations which throw light on our discussions based on the mean field model, but we shall not attempt to present any systematic account of the simulations done by different groups.

We shall summarize the salient features of the observational data about the meridional circulation of the Sun in the next Section. Then §3 will be devoted to discussing the basic theory of the meridional circulation—along with the basic theory of differential rotation—presenting some modelling efforts. The role of the meridional circulation and its irregularities in the flux transport dynamo model of the Sun will be discussed in §4. Then §5 will be devoted to the back reaction of the dynamo on the large-scale flow patterns of the Sun. Finally, we shall present some concluding remarks in §6.

2 Relevant observational data

One of the challenges of observing the meridional circulation at the solar surface is that it involves fluid flows which are much weaker than other kinds of fluid flows present there. We have mentioned that the maximum speed of the meridional circulation at the mid-latitudes is about 20 m s^{-1} , which means that the time to traverse a quadrant of the Sun's circumference (from the equator to the pole) would be of the order of about 1.7 yr. The solar surface has other kinds of fluid flows which are much faster with shorter time scales. The convective velocities associated with granules at the solar surface are of order $1\text{--}2 \text{ km s}^{-1}$, the typical lifetimes of granules being of the order of a few minutes. The rotation period of about 25 days near the solar equator gives rise to an azimuthal

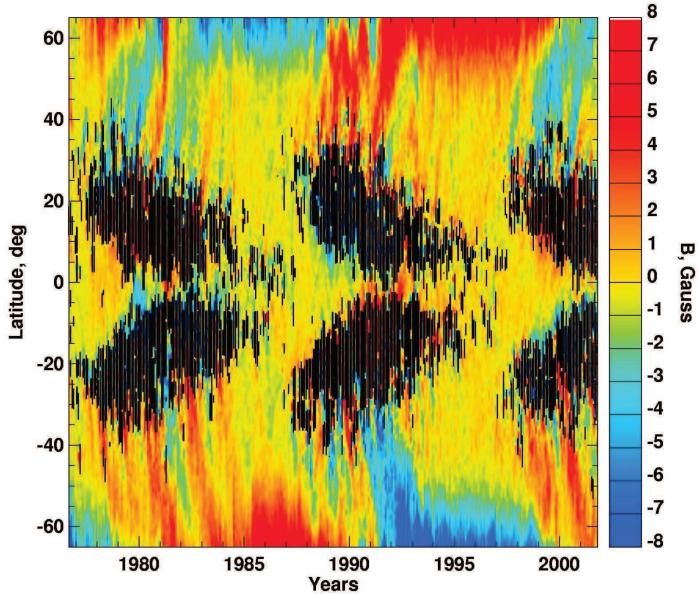


Figure 1: A time-latitude plot in which the shaded regions indicate the solar latitudes at which sunspots were seen as different times. The colours indicate values of the longitude-averaged magnetic field outside the sunspots.

velocity of about 2 km s^{-1} . Thus, to measure the velocity of the meridional circulation directly, we have to pick up a signal much weaker than the other signals present.

2.1 Meridional circulation at the solar surface

Although the meridional circulation is much weaker than other fluid flows at the solar surface, it can be identified by something special that it does. It carries various surface features with it poleward. We may mention that the turbulent velocities of convection near the solar surface make things spread out, giving rise to an effective diffusion, which is very important in the flux transport dynamo model. This diffusion also can cause a poleward spread of various things. However, the role of the meridional circulation in the poleward transport of surface features is much more direct and effective (the spread by diffusion goes as \sqrt{t} , whereas the transport by a flow goes as t). Historically, the meridional circulation was discovered from observations of such poleward transport.

Sunspots, regions of strong magnetic field on the solar surface, appear around latitudes 30° – 40° at the beginning of a solar cycle. As the cycle progresses, sunspots appear at lower and lower latitudes. The shaded portions in Figure 1 indicate the regions in a time-latitude plot where sunspots appeared during the span of a little more than two solar cycles. The colours in Figure 1 indicate the longitude-averaged values of the magnetic field outside sunspots in this time-latitude plot. The magnetic field outside sunspots consists of latitude belts in which this field is predominantly of a particular sign. In contrast to the sunspot

belts which shift equatorward with the solar cycle, the field outside sunspots seems to be advected poleward, suggesting a poleward flow of matter which carries this magnetic field with it. The existence of the meridional circulation was first inferred from the observation that there were unipolar patches of magnetic field in certain latitude belts [4] and that these patches shifted poleward with time [5, 6]. The polar field of the Sun, which gets built up as the magnetic fields from the lower latitudes are brought to the polar region, reverses its direction around the time of the sunspot maximum and is clearly tied to the solar cycle. The early measurements with low-resolution magnetograms suggested that the polar field is of order 10 G (see the colour code in Figure 1). We know for several decades now that this magnetic field outside sunspots is actually concentrated inside highly intermittent flux tubes with magnetic field of order 1000 G [7] and the values measured by the early low-resolution magnetograms are merely values averaged over patches of the solar surface when the flux tubes are not resolved. The meridional circulation could be estimated also from the poleward displacements of small magnetic features [8].

Apart from the poleward shift of unipolar magnetic patches, there is another important proxy which gave a lot of information about the meridional circulation in the early years of research in this field. There must be a neutral boundary line between the regions of opposite magnetic polarity on the solar surface. When we observe the Sun using an H_{α} filter, we often see dark filaments above the neutral line—presumably made out of cool gas resting on the magnetic canopy that must exist above the neutral line. Positions of the dark filaments in an H_{α} plate would indicate the neutral line and one can draw inferences about the meridional circulation from a study of how the neutral line shifts poleward with time. From an analysis of the H_{α} plates of the Sun taken at the Kodaikanal Observatory over several decades, the existence of the meridional circulation in the early decades of the twentieth century, when there were no measurements of the magnetic field outside sunspots, could be established [9, 10].

Some attempts to measure the meridional circulation at the surface directly through the Doppler shifts of spectral lines have also been made [11, 12].

2.2 Sub-surface results from helioseismology

After the existence of the meridional circulation at the solar surface was established, the important question was whether we can determine its nature underneath the surface—especially whether we can find where the return flow from the poles to the equator occurs. During the last few years, we have some information about it from helioseismology, which is the study of solar oscillations first discovered at the solar surface in the 1960s [13]. These surface oscillations are caused by acoustic waves propagating underneath the solar surface and buffering the surface. If there are large-scale fluid flows underneath the surface, they affect the propagation of the acoustic waves and it is possible to draw inferences about these flows from the analysis of the oscillations data. This is a highly technical subject and the details of how this is done are beyond the scope of this review. We refer the interested readers to standard reviews of this subject [14, 15, 16] and present only the results here.

The effect of the meridional circulation on the solar oscillations at the surface is a very small effect and it becomes increasingly difficult to make inferences about the meridional circulation in deeper layers of the Sun underneath the

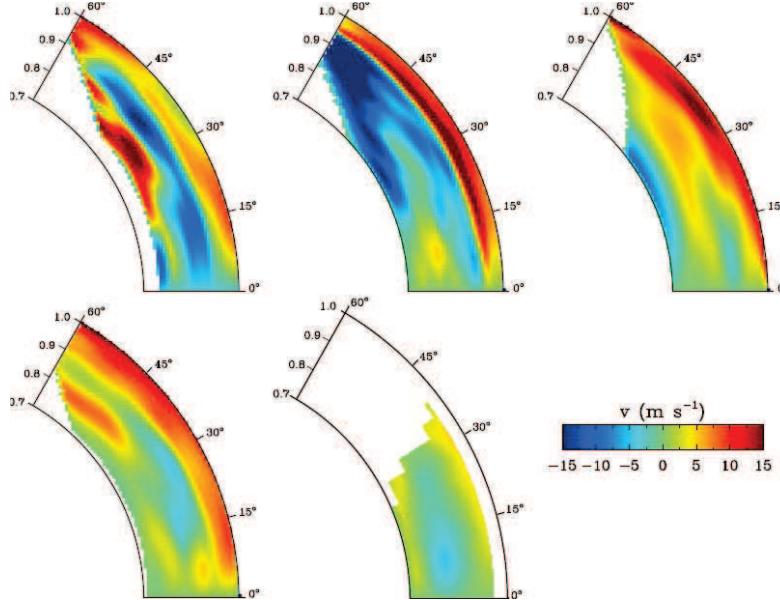


Figure 2: Hemispherically symmetrized profiles of the meridional circulation component v_θ obtained by (a) Zhao et al. [19], (b) Jackiewicz et al. [21], (c) Rajaguru and Antia [20], (d) Chen and Zhao [22], and (e) Lin and Chou [23]. Poleward flows are positive and equatorward flows are negative. Figure courtesy of Junwei Zhao.

surface from this small effect. The first results of helioseismology were about the nature of the meridional circulation in the layers immediately underneath the solar surface [17, 18]. Only within the last few years, there have been serious efforts to look for the return flow from the poles to the equator. As we shall point out in our discussion of the flux transport dynamo model, we get the best results if we assume that there is only one cell of the meridional circulation spanning the entire convection zone, with the return flow at the bottom of the convection zone. Whether helioseismic studies can either confirm or contradict this view has become a very important question. Some authors claim that they find evidence for a return flow at the middle of the convection zone rather than at the bottom [19], whereas other authors, analyzing the same data, conclude that a single-cell meridional circulation spanning the whole of the convection zone is consistent with the data [20]. Figure 2 shows results presented by different groups about the nature of the meridional circulation underneath the solar surface. As we can easily see, there are large divergences among the results of different groups for the meridional circulation in the deeper layers of the convection zone. A very recent analysis of data from different sources led Gizon et al. [24] to conclude that the meridional circulation consists of a single cell in each hemisphere, with the return flow at the bottom of the convection zone.

2.3 The differential rotation of the Sun

As we have pointed out in the Introduction and shall again see in §3, the theory of the meridional circulation is intimately connected with the theory of the other

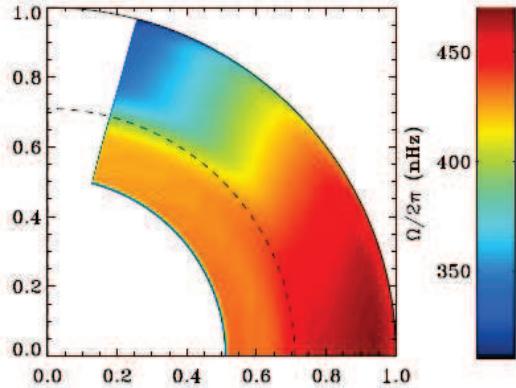


Figure 3: A profile of the differential rotation inside the Sun obtained by helioseismology. From Howe et al. [27], as presented in Basu [16].

large-scale fluid pattern inside the Sun: the differential rotation given by a non-constant $\Omega(r, \theta)$. Hence, a basic knowledge about the nature of the differential rotation is essential for our discussion. One of the remarkable achievements of helioseismology is that it has provided a map of $\Omega(r, \theta)$ underneath the solar surface. The early maps obtained in the 1980s eventually converged to a robust map by the mid-1990s [25, 26]. Figure 3 is a map of meridional circulation inside the Sun.

We find that the differential rotation is confined within the convection zone, with indications that the radiative core of the Sun may be rotating like a solid body. This result is along theoretically expected lines because we think that the differential rotation also, like the meridional circulation, is driven by turbulent stresses in the convection zone, as we shall discuss in §3.2. Within the convection zone, $\Omega(r, \theta)$ appears to be approximately constant on conical surfaces, with the regions near the equator having higher values of $\Omega(r, \theta)$ compared to the regions near the pole. Such a distribution of $\Omega(r, \theta)$ within the main body of the convection zone results in a strong radial gradient of $\Omega(r, \theta)$ at the bottom of the convection zone, which can be seen in Figure 3. This region of strong gradient of $\Omega(r, \theta)$ at the bottom of the convection zone is called the *tachocline*.

We may point out that asteroseismology (i.e. the study of stellar oscillations) has now started giving some results of differential rotation in solar-like stars [28].

2.4 Variations of the meridional circulation with time

We now come to the important question whether there are variations of the meridional circulation with time. On simple theoretical grounds, we may expect a systematic variation with the solar cycle. Presumably, the magnetic field in the solar interior is strongest at the time of the sunspot maximum and the Lorentz force due to this magnetic field also must be strongest. This Lorentz force may act on the large-scale flows and may cause a variation with the solar cycle. The variation of the meridional circulation with the solar cycle has indeed been found—both from helioseismology [29, 30, 31, 32] and from the tracking of surface traces [33]. Figure 4 shows how the meridional circulation at a mid-

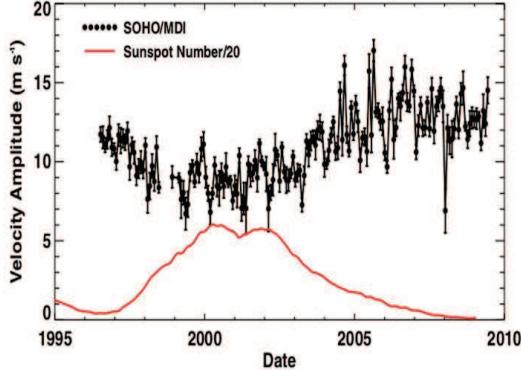


Figure 4: The variation with time of the meridional circulation at the surface at mid-latitudes, plotted along with the sunspot number. From Hathaway and Rightmire [33].

latitude point on the solar surface varied with time during a solar cycle, with the sunspot number plotted along with it. It is clear that the meridional circulation becomes weaker at the time of the sunspot maximum. The equatorward meridional circulation at the bottom of the convection zone is also now found to be weaker at the time of the solar maximum [24]. We expect the Lorentz force of the solar magnetic field to act on the differential rotation also. The variations of the differential rotation with the solar cycle, known as *torsional oscillations*, have been studied extensively. Although we shall make a few comments on torsional oscillations in §5, a detailed discussion of torsional oscillations is outside the scope of this review (see [34] and references therein). We point out another intriguing aspect of the meridional circulation at the sunspot maximum. There seems to be an inward flow towards the sunspot belt superposed on the overall flow pattern [35, 32].

Apart from the systematic variation with the solar cycle, are there non-systematic random fluctuations in the meridional circulation? Since we have reliable observational data of the meridional circulation for a period not longer than a quarter century, we cannot directly conclude from these data whether there had been fluctuations in the meridional circulation with longer coherence times. However, we can try to draw some inferences about this from indirect considerations. As we shall discuss in §4.2, the period of the flux transport dynamo decreases with the amplitude of the meridional circulation. This means that the solar cycle durations will be shorter when the meridional circulation is stronger and vice versa. Figure 5 is a plot of the durations of last 23 solar cycles spanning over more than a couple of centuries. There have been epochs when successive solar cycles had durations shorter than the average, suggesting that the meridional circulation was stronger during such epochs. From such indirect considerations, we can conclude that there have been fluctuations in the meridional circulation in the past with coherence times of order 30–40 yr [36]. Passos and Lopes [37] reconstructed the history of the meridional circulation in the past 250 years on the basis of a low order dynamo model and arrived at very similar conclusions.

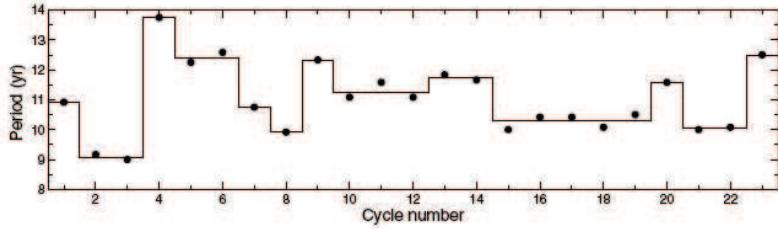


Figure 5: The points show the durations of the last 23 solar cycles against the cycle number. The solid line is indicative of the trend in variations of the cycle durations. From Karak and Choudhuri [36].

3 Theory of meridional circulation

The theory of the meridional circulation happens to be a somewhat complicated subject. While the majority of the research papers on this subject would appear fairly forbidding and inaccessible to the uninitiated, we are also not aware of any convenient textbooks or pedagogical reviews from which a beginner can learn this subject. The two classic monographs by Tassoul [38] and Rüdiger [39] which discussed large-scale flows inside stars in some detail are now very much outdated. Both these monographs give comprehensive historical summaries of early research in this field before reliable observational data for the meridional circulation and the internal differential rotation of the Sun became available and before the currently held theoretical viewpoint emerged. The present review mainly focuses on the current theoretical viewpoint based on observational data, without much discussion of the earlier efforts. We refer the readers to a couple of excellent short reviews by Kitchatinov [40, 41], on which we draw heavily in our presentation. Since the meridional circulation now appears to be so important in many solar phenomena, it is desirable that a solar physicist should have a rough, qualitative idea about the theory of how the meridional circulation arises. Our aim is to provide that in this Section. We assume that readers are familiar with the basic principles of fluid mechanics and MHD (will be needed in the next two sections), which are discussed in many well-known books.

3.1 Governing equations for large-scale fluid motions

Any discussion of the dynamics of fluid flows should begin with the Navier–Stokes equation, which we write in the following form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} + \rho F_i + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial v_i}{\partial x_j} \right). \quad (6)$$

(See for example [42], §7, §15; [43], §4.3, §5.1). Here all the symbols have their usual meanings, \mathbf{F} being the body force per unit mass (like gravity). We are also using the summation convention that an index repeated twice implies summation over the spatial directions. We shall restrict our discussion to unmagnetized fluids in this Section, with discussions about the magnetic field postponed to the next two Sections.

When we deal with turbulent fluid motions (as within the convection zone of the Sun), we can write the velocity in the following manner

$$v_i = \bar{v}_i + v'_i, \quad (7)$$

where \bar{v}_i is the mean value of v_i averaged over turbulence and v'_i is the fluctuation around the mean. Let us now write v_i and v_j in (6) in this manner and average over turbulence. Keeping in mind that $\bar{v}'_i = 0$, we are led to

$$\frac{\partial}{\partial t}(\rho \bar{v}_i) + \frac{\partial}{\partial x_j}(\rho \bar{v}_i \bar{v}_j + \rho \bar{v}'_i \bar{v}'_j) = -\frac{\partial p}{\partial x_i} + \rho F_i + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \bar{v}_i}{\partial x_j} \right). \quad (8)$$

Subtracting the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{v}_j) = 0$$

multiplied by \bar{v}_i from the left side of (8), we get

$$\rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} = -\nabla p + \rho \mathbf{F} + \mathbf{K}, \quad (9)$$

where \mathbf{K} is a term of which the i -th component given by

$$K_i = \frac{\partial}{\partial x_j} \left(-\rho \bar{v}'_i \bar{v}'_j + \mu \frac{\partial \bar{v}_i}{\partial x_j} \right) \quad (10)$$

involves the turbulent stress tensor $\rho \bar{v}'_i \bar{v}'_j$, which plays a crucial role in the theory of large-scale flows inside the Sun. Lebedinski [44] appears to be the first person to realize in 1941 that turbulent stresses may drive large-scale flows. “To remind us of his contributions”, the driving of mean large-scale flows by turbulent stresses has been christened as the Λ -effect by Rüdiger ([39], p. 37). This idea was further developed by Wasiutynski [45] and Biermann [46].

Since we shall be primarily dealing with mean fluid flows, let us drop the overline sign henceforth and write $\bar{\mathbf{v}}$ as \mathbf{v} , keeping in mind that from now onwards \mathbf{v} would refer to the mean flow. We write (9) as follows

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{\nabla p}{\rho} + \mathbf{F} + \frac{\mathbf{K}}{\rho}. \quad (11)$$

This is going to be the central equation in our theoretical discussions. Since the turbulent stress term $\rho \bar{v}'_i \bar{v}'_j$ in (10) is usually several orders of magnitude larger than the viscous stress term $\mu (\partial v_i / \partial x_j)$ inside a stellar convection zone, often the viscous stress term is neglected in (10). However, turbulence itself gives rise to an effective viscosity and sometimes the turbulent stress term is taken to be as follows

$$\rho \bar{v}'_i \bar{v}'_j = -\mu_T \frac{\partial v_i}{\partial x_j}, \quad (12)$$

where μ_T is the turbulent viscosity. (Strictly speaking, we should symmetrize any viscous tensor term to exclude rotation, e.g. [43], p. 79—we are being somewhat hand-waving here). It follows from (10) and (12) that

$$\mathbf{K} = \mu_T \nabla^2 \mathbf{v}, \quad (13)$$

where we have neglected the term due to the viscous stress tensor (arising out of ‘molecular’ viscosity). However, we should stress that (12) and (13) are approximations which sometimes miss out some of the essential physics connected with the large-scale flows inside the Sun. When we want to make a realistic model of the large-scale flows inside the Sun, we have to determine the viscous stress tensor $\rho \overline{v'_i v'_j}$ more carefully to look for non-dissipative parts. Still, in our discussions, we shall sometimes point out as to what happens if \mathbf{K} is given by (13), since this simplification often gives us quite a bit of insight into the nature of the problem.

In a 1963 pioneering paper, Kippenhahn [47] took the turbulent stress term to be of the form (13). However, he assumed the coefficient $\mu_{T,r}$ for the radial transport of momentum to be different from the coefficient $\mu_{T,h}$ for the horizontal transport of momentum. This already gave very interesting results which we shall discuss in §3.2. Durney and Spruit [48] were among the first authors to attempt a detailed calculation of the turbulent stress tensor for convection inside a rotating star. Later, Kitchatinov and Rüdiger [49, 50] calculated this tensor from their model of turbulence and constructed details models of large-scale flows inside rotating stars. A look at these papers [48, 49, 50] shows the complexity of the expressions of the turbulent stress tensor which these authors arrived at. We shall try to discuss some of the basic physics of the problem without getting into the details of how to calculate the turbulent stress tensor. In §3.2 we shall indicate how to compute only one crucial component $\overline{v'_r v'_\theta}$ of the turbulent stress tensor.

Let us now consider the ϕ component of (11) in spherical coordinates. When we assume axisymmetry (i.e. $\partial/\partial\phi = 0$ everywhere), the gradient terms do not have any component in the ϕ direction and a body force like gravity would also have no ϕ component. Then we get

$$\frac{\partial v_\phi}{\partial t} - [\mathbf{v} \times (\nabla \times \mathbf{v})]_\phi = \frac{K_\phi}{\rho}. \quad (14)$$

This is the basic equation governing the dynamics of the differential rotation.

As pointed out in the Introduction, we need to find an equation for ω_ϕ of the form (5) in order to develop a theory of the meridional circulation. We need to take the curl of (11) and consider its ϕ component. This gives

$$\frac{\partial \omega_\phi}{\partial t} = [\nabla \times \{\mathbf{v} \times (\nabla \times \mathbf{v})\}]_\phi - \frac{1}{\rho^2} [\nabla p \times \nabla \rho]_\phi + \left[\nabla \times \left(\frac{\mathbf{K}}{\rho} \right) \right]_\phi. \quad (15)$$

This is the basic equation governing the dynamics of the meridional circulation.

As we shall point out in §3.2 and §3.3, the terms $[\mathbf{v} \times (\nabla \times \mathbf{v})]_\phi$ and $[\nabla \times \{\mathbf{v} \times (\nabla \times \mathbf{v})\}]_\phi$ appearing in (14) and (15) involve both the meridional circulation and the differential rotation. Because of these terms, (14) and (15) get coupled to each other and we cannot solve one of them in isolation. Both of them have to be solved together, showing that the theories of the differential rotation and the meridional circulation are intimately connected to each other. To solve these equations, we need to know the turbulent stress $\rho \overline{v'_i v'_j}$ so that we can calculate \mathbf{K} by using (10). So, in order to develop theories of the differential rotation and the meridional circulation, we need to proceed as follows. We first have to evaluate the turbulent stresses from some suitable theory of turbulence. Then we can solve (14) and (15) together. Often we may be interested in the steady

large-scale flows in the interior of a star. Then the time evolution terms in (14) and (15) can be set to zero. Still, it is an immensely difficult problem to solve (14) and (15). We discuss some basic physics issues connected with this problem in the next two subsections.

3.2 Driving the differential rotation

Although this is a review primarily devoted to the meridional circulation, we shall see in §3.3 that we need to know the profile of the differential rotation to calculate the main driving term for the meridional circulation. So we begin with a discussion of the theory of differential rotation. As we already pointed out, (14) gives the dynamics of the differential rotation. For an axisymmetric velocity field given by (1), we can easily work out the expression for $[\mathbf{v} \times (\nabla \times \mathbf{v})]_\phi$ so that (14) leads to

$$\frac{\partial v_\phi}{\partial t} + \frac{v_r}{r} \frac{\partial}{\partial r}(r v_\phi) + \frac{v_\theta}{r \sin \theta} \frac{\partial}{\partial \theta}(\sin \theta v_\phi) = \frac{K_\phi}{\rho}. \quad (16)$$

The second and third terms in this equation correspond to the meridional circulation carrying the angular momentum with it and thereby altering the profile of v_ϕ . We can get an equation for specific angular momentum (i.e. angular momentum per unit mass) $\mathcal{L} = r \sin \theta v_\phi$ by multiplying (16) by $r \sin \theta$, which gives

$$\frac{\partial \mathcal{L}}{\partial t} + \mathbf{v}_m \cdot \nabla \mathcal{L} = r \sin \theta \frac{K_\phi}{\rho}. \quad (17)$$

Using the relation $v_\phi = r \sin \theta \Omega$, it is also easy to cast (16) into the form of an equation of Ω . Readers will find that sometimes in the literature the equation of azimuthal dynamics is written in terms of Ω rather than v_ϕ . To solve (16), we need to evaluate the turbulent stress $\rho \bar{v'_i v'_j}$ required for obtaining \mathbf{K} through (10). Since the mathematical theory of the turbulent stress is extremely complicated, we now discuss the basic physics of the problem qualitatively without getting into the details of the mathematical theory.

Within a stellar convection zone, hot blobs of gas move upward and cold blobs of gas move downward. We may naively expect these blobs to carry their angular momentum with them when they move upward or downward. This suggests that angular momentum may get well mixed within the convection zone, such that the specific angular momentum is constant throughout the convection zone. Taking $s = r \sin \theta$ as the outward distance from the rotation axis, specific angular momentum in a region of the convection zone would be $\mathcal{L} = s^2 \Omega$. If this were to be constant throughout the convection zone, then Ω would fall off as we go further from the rotation axis. However, we find the opposite of this in the Sun, as seen in Figure 3. In order for the equatorial regions of the Sun to have higher Ω , we need some mechanism to continuously pump angular momentum away from the rotation axis so that it can get piled up in the equatorial regions, making those regions to rotate faster. Let us now consider what kind of turbulent stresses can do this.

If the ϕ -component of momentum ρv_ϕ has to be advected in the radial direction, it is easy to argue that $\rho \bar{v_r v_\phi}$ would do the job and would contribute to a radial flux of angular momentum. The crucial question is whether this will be positive or negative. To address this question, it is convenient to look at the

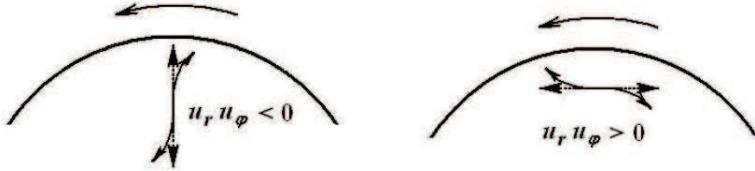


Figure 6: A sketch illustrating angular momentum transport in the equatorial plane by turbulent mixing. The direction of rotation is indicated at the tops. The left and right panels indicate how radially moving and horizontally moving fluid blobs are deflected by the Coriolis force. From Kitchatinov [40].

system (i.e. the star) from the frame of its average angular velocity. We had written down our dynamical equation (11) with respect to an inertial frame. When the variation of Ω over a star like the Sun is small compared to the average value of Ω , it is indeed often useful to introduce a frame rotating with the average Ω . It is well known that, in such a frame, we shall have an additional Coriolis force term $-2\boldsymbol{\Omega} \times \mathbf{v}$ appearing on the right hand side of (11). Now, look at the left panel of Figure 6, indicating the direction of motion of a convective blob moving radially (upward or downward) near the equatorial plane. Assuming that we looking down from the rotation axis, it is easy to show that the Coriolis force would make the bob move as indicated in the figure. Clearly $\rho v_r v_\phi$ is negative for such a blob, indicating that radially moving convective blobs would transport angular momentum downward. Presumably, such transport would tend to make $s^2\Omega$ constant within the convection zone. Now, consider a horizontally moving turbulent blob shown in the right panel of Figure 6. It is easy to check that that the Coriolis force would make the blob move as shown in Figure 6, leading to positive $\rho v_r v_\phi$. We conclude that such turbulent blobs would transport the angular momentum outward. Hence, in the solar convection zone, we shall have the required outward pumping of angular momentum if horizontal turbulent motions are more dominant within the solar convection zone compared to radial turbulent motions. Can this be the case under some circumstances?

While we do want to get into a full discussion of the complicated problem of calculating turbulent stress tensors, we point out how the deflections caused by the Coriolis force, as indicated in Figure 6, enter into the calculation of $\overline{v_r v_\phi}$ for a weakly rotating star. Let us now consider a convective blob which would have the velocity

$$\mathbf{v}_0 = v_{0,r} \mathbf{e}_r + v_{0,\phi} \mathbf{e}_\phi \quad (18)$$

associated with it in the absence of the Coriolis force. If the Coriolis force acts on the blob during its coherence time τ , then the velocity induced by the Coriolis force will be

$$\mathbf{v}_1 = -2\boldsymbol{\Omega} \times \mathbf{v}_0 \tau \quad (19)$$

so that the velocity with the Coriolis deflection becomes

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1.$$

Using (18) and keeping in mind that $\boldsymbol{\Omega} = \Omega \cos \theta \mathbf{e}_r - \Omega \sin \theta \mathbf{e}_\theta$ at the colatitude θ , from (19) we get

$$\mathbf{v}_1 = 2\Omega\tau v_{0,\phi} \sin \theta \mathbf{e}_r + 2\Omega\tau v_{0,\phi} \cos \theta \mathbf{e}_\theta - 2\Omega\tau v_{0,r} \sin \theta \mathbf{e}_\phi. \quad (20)$$

The turbulent stress term we are interested in is given by

$$\overline{v_r v_\phi} = \overline{(v_{0,r} + v_{1,r})(v_{0,\phi} + v_{1,\phi})} = \overline{v_{0,r} v_{0,\phi}} + \overline{v_{1,r} v_{0,\phi}} + \overline{v_{0,r} v_{1,\phi}} + \overline{v_{1,r} v_{1,\phi}}. \quad (21)$$

If we make the simplifying assumption in this discussion that $v_{0,r}$ and $v_{0,\phi}$ would be uncorrelated in the absence of the Coriolis force, then $\overline{v_{0,r} v_{0,\phi}} = \overline{v_{0,r}} \overline{v_{0,\phi}} = 0$. Also, we expect $\Omega\tau$ to be small for weak rotation, so that we can neglect $\overline{v_{1,r} v_{1,\phi}}$, which will be quadratic in $\Omega\tau$. Substituting for $v_{1,r}$ and $v_{1,\phi}$ from (20) into (21), we get

$$\overline{v_r v_\phi} = 2\Omega\tau (\overline{v_{0,\phi}^2} - \overline{v_{0,r}^2}) \sin \theta. \quad (22)$$

It is clear that $\overline{v_r v_\phi}$ is positive when $\overline{v_{0,\phi}^2} > \overline{v_{0,r}^2}$, leading to outward transport of angular momentum, and is negative when $\overline{v_{0,r}^2} > \overline{v_{0,\phi}^2}$, leading to inward transport of angular momentum, in conformity with the discussion accompanying Figure 6.¹

The crucial question now is whether $\overline{v_{0,\phi}^2} - \overline{v_{0,r}^2}$ appearing in (22) is positive or negative. One standard result in fluid mechanics is the Taylor–Proudman theorem (see [43], p. 183), according to which fluid phenomena tend to be aligned parallel to the rotation axis. In accordance with this theorem, stellar convection tends to take place in banana-shaped convection rolls parallel to the rotation axis if the star rotates sufficiently fast, as found in numerical simulations (see Figure 3 of [51] or Figure 5 of [52]). If the star rotating slowly, we expect radial turbulent motions to dominate. This will lead to negative $\overline{v_r v_\phi}$ according to (22) and presumably a downward pumping of angular momentum such that the equatorial regions may rotate slower. However, when a star rotates rapidly and banana-shaped convection cells form, horizontal convective motions may become more and more important making $\overline{v_r v_\phi}$ given by (22) positive. This is likely to cause outward pumping of angular momentum, which may make the equatorial regions rotate faster. Presumably, something like this is happening in the Sun, although we should caution the reader that the statement we just made is an over-simplification of a complex situation. It is clear from (17) that the meridional circulation also carries angular momentum with it. The final distribution of angular velocity inside the convection zone follows from a complicated interplay of various angular momentum transfer terms. Still, it is interesting to note that numerical simulations show that solar-like stars have anti-solar differential rotation when rotating slowly and solar-like differential rotation when rotating fast [51, 53], in qualitative agreement with the idea that, when stellar convection is affected more by rotation, there is a higher tendency of angular momentum getting transferred outward. Whether rotation affects stellar convection significantly depends on the dimensionless number $\Omega\tau$ appearing in (22). This dimensionless number (or rather $2\Omega\tau$) is often called the *Coriolis number* and is essentially the inverse of what is known as the *Rossby number*. A full theory of turbulent stresses would involve calculating $\overline{v_{0,\phi}^2}/\overline{v_{0,r}^2}$.

¹This expression of $\overline{v_r v_\phi}$ was derived by Lebedinski [44] in his paper written in Russian. I am grateful to Leonid Kitchatinov for bringing this derivation to my attention.

as a function of $\Omega\tau$, which will make $\overline{v_r v_\phi}$ given by (22) a more complicated nonlinear function of $\Omega\tau$. We do not go into the details of this complicated subject.

Certainly (22) is not of the form (12). In general, one should write

$$\overline{v'_i v'_j} = Q_{ij}^\Lambda - \mathcal{N}_{ijkl} \frac{\partial \overline{v_k}}{\partial x_l} \quad (23)$$

[49, 50]. The term Q_{ij}^Λ , which would incorporate expressions like (22), is the essence of the Λ -effect, as indicated by the superscript Λ . However, we would like to emphasize that, even by taking the turbulent stress to be of the simple form (12), Kippenhahn [47] succeeded in driving the differential rotation by assuming the coefficient $\mu_{T,r}$ for the radial viscous transport to be different from the coefficient $\mu_{T,h}$ for the horizontal viscous transport. We naively expect that the viscous drag may oppose relative motions between fluid layers inside a star, leading to solid body rotation. This is indeed found to be the case when $\mu_{T,r} = \mu_{T,h}$. However, when these coefficients were taken to be unequal, Kippenhahn [47] found the following intriguing results: a larger $\mu_{T,r}$ led to lower angular velocity near the equatorial region, whereas a larger $\mu_{T,h}$ led to higher angular velocity there. Presumably, the case of radially moving fluid blobs discussed above corresponds to higher $\mu_{T,r}$, whereas the case of horizontally moving fluid blobs corresponds to higher $\mu_{T,h}$. Thus, Kippenhahn's conclusions are in agreement with the physics encapsulated in Figure 6 as we discussed above. Afterwards, Kitchatinov and Rüdiger [49, 50] calculated the turbulent stress tensor from their model of turbulence and constructed more detailed models of stellar rotation. They also found that angular velocity is higher in the equatorial regions (as we see for the Sun) when the turbulent stress due to the horizontal motions dominates. We shall discuss these solutions in §3.4.

3.3 Driving the meridional circulation

We are now ready to discuss on the basis of (15) how the meridional circulation inside a star is driven. If \mathbf{K} is taken to be as given in (13), then $\nabla \times \mathbf{K}$ would equal $\mu_T \nabla^2 \omega$ and it is clear that the last term in (15) would be a term giving dissipation of vorticity. We easily see that (15) is an equation of the nature of (5), with the first two terms on the right hand side of (15) as the source terms which drive the meridional circulation. We now discuss the significance of these crucial source terms.

The term $-\nabla p \times \nabla \rho / \rho^2$ is called the *thermal wind* term. Within the convection zone of a star like the Sun, the ascending and descending convective blobs are deflected by the Coriolis force. This effect is less on the convective blobs moving near the polar regions. Due to this, convective heat transport is expected to be more efficient in the polar regions. This is likely to make the polar temperature slightly higher than the temperature at the equator. It was realized in the 1970s that the effect of rotation on convection may make the heat transport latitude-dependent and that this may give rise to large-scale flows [54, 55]. A higher temperature (and a higher pressure) at the poles would drive a meridional circulation which is equatorward near the surface. However, this is exactly the opposite of what we observe! This means that the other source term has to overcome this effect to drive the meridional circulation in the correct direction. Let us point out that the thermal wind term indeed mathematically

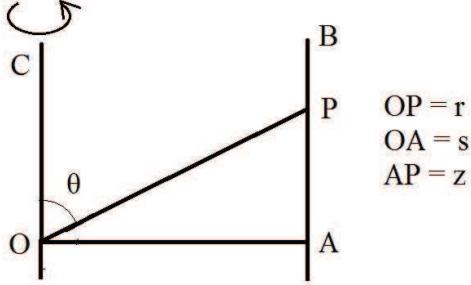


Figure 7: A sketch indicating the relation of the (s, z) coordinates with the (r, θ) coordinates.

leads to a meridional circulation opposite to what is seen in the Sun. The thermal wind term arises when the contours of constant ρ and constant p do not coincide. The solar surface can be taken as a surface of constant ρ . If the polar region is hotter, then a surface of constant p which intersects the solar surface at mid-latitudes would be above the solar surface near the poles and would be below the solar surface near the equator. It is easy to check that $-\nabla p \times \nabla \rho$ will be positive in the northern hemisphere. It then follows from (15) that this term will tend to drive a meridional circulation with positive vorticity in the northern hemisphere, opposite to what we find in the Sun, as pointed out in §1.

Let us now turn our attention to the other source term $[\nabla \times \{\mathbf{v} \times (\nabla \times \mathbf{v})\}]_\phi$ in (15). This term is clearly quadratic in \mathbf{v} . When we substitute the velocity field given by (2) and (3) in this, we find that there are some terms which are quadratic in meridional components v_r, v_θ and some terms which are quadratic in Ω . We have already pointed out the velocities associated with the meridional circulation are much smaller than the velocities connected with the solar rotation over much of the Sun. We now make the approximation of keeping only the terms quadratic in Ω . Then, a few lines of easy algebra give us

$$[\nabla \times \{\mathbf{v} \times (\nabla \times \mathbf{v})\}]_\phi = r \sin \theta \cos \theta \frac{\partial}{\partial r} \Omega^2 - \sin^2 \theta \frac{\partial}{\partial \theta} \Omega^2 \quad (24)$$

To understand the significance of this expression, let us consider a straight line APB parallel to the rotation axis OC at a distance $s = r \sin \theta$ from it, as shown in Figure 7. If z is measured upward from the equatorial plane OA, then $z = r \cos \theta$. We can use s and z as our two independent spatial coordinates in the place of r and θ . This means that

$$\left(\frac{\partial}{\partial z} \right)_s = \left(\frac{\partial r}{\partial z} \right)_s \frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial z} \right)_s \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}. \quad (25)$$

It then follows from (24) that

$$[\nabla \times \{\mathbf{v} \times (\nabla \times \mathbf{v})\}]_\phi = r \sin \theta \frac{\partial}{\partial z} \Omega^2. \quad (26)$$

Substituting this in (15), we get

$$\frac{\partial \omega_\phi}{\partial t} = r \sin \theta \frac{\partial}{\partial z} \Omega^2 - \frac{1}{\rho^2} [\nabla p \times \nabla \rho]_\phi + \left[\nabla \times \left(\frac{\mathbf{K}}{\rho} \right) \right]_\phi, \quad (27)$$

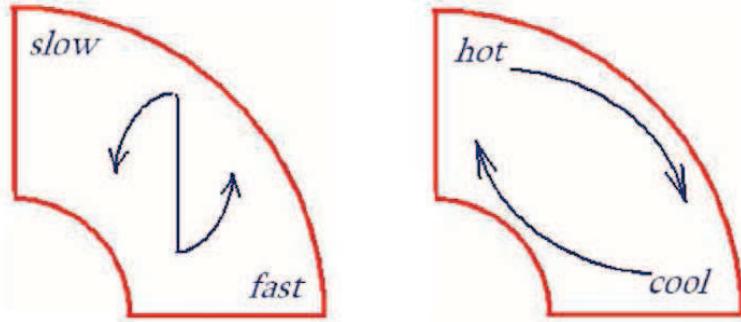


Figure 8: A figure indicating the directions in which the source terms would tend to drive the meridional circulation. The left panel indicates the centrifugal term and the right panel the thermal wind term. From Kitchatinov [41].

which is our crucial equation. We may point out that, if we had not neglected the terms quadratic in the meridional circulation velocities, then there would have been the following additional term in the left hand side of (27)

$$+s\nabla \cdot \left(\mathbf{v}_m \frac{\omega_\phi}{s} \right) \quad (28)$$

which corresponds to the meridional circulation carrying the vorticity with it.

Let us now discuss the physical significance of the first source term in the right hand side of (27). The appearance of Ω^2 suggests that this term may be connected with the centrifugal force, which turns out to be the case. This term is naturally called the *centrifugal term*. Suppose we consider a straight line parallel to the rotation axis inside the solar convection zone, like the line APB shown in Figure 7. It is obvious that the centrifugal force near the equatorial region is larger than the centrifugal force at higher latitudes, if the rotation profile inside the solar convection zone is as shown in Figure 3—with higher Ω near the equatorial region. If we subtract some mean centrifugal force averaged along the line APB, then the net force near the equator would be in the outward direction and the net force at higher latitudes in the inward direction. This would tend to drive a meridional circulation which is in the same sense as the meridional circulation of the Sun, as indicated in the left panel of Figure 8. The right panel of Figure 8 shows the kind of meridional circulation that the thermal wind term would tend to drive. It may be noted that, if Ω were constant along lines parallel to the rotation axis like APB, then it is easy to write the centrifugal force as a gradient, showing that it is a conservative force in this situation. Only when Ω varies with z , the centrifugal force becomes non-conservative and can drive a circulation. When we turn to mathematics after understanding the physical concepts, it is easy to check that $\partial \Omega^2 / \partial z$ in the solar convection zone in the northern hemisphere is negative, showing that the centrifugal term in (27) would tend to produce a meridional circulation with negative vorticity, which is the case for the Sun in the northern hemisphere.

We thus conclude that the meridional circulation in the Sun or similar stars arises out of the interplay between the two source terms. The thermal wind term would try to drive a meridional circulation in the sense opposite to what is

seen in the Sun. Presumably, the centrifugal term overcomes this and drives the meridional circulation in the correct direction. Is it possible that the entire solar surface is at the same temperature so that the thermal wind term is zero and the centrifugal term alone drives the meridional circulation of the Sun in the right direction? As shown in the Appendix, if we make an order of magnitude estimate for the solar convection zone, the dissipation term (i.e. the last term) in (27) turns out to be several orders of magnitude smaller compared to the centrifugal term, when we use typical values of the large-scale flow velocities in the Sun. If the centrifugal term alone was driving the meridional circulation, then the centrifugal term arising out of the solar rotation profile would drive a much stronger meridional circulation. The only possibility is that the thermal wind term must be nearly comparable to the centrifugal term and should balance it. The small leftover part of the centrifugal term must be driving the solar meridional circulation. Kitchatinov and Rüdiger [50] estimated that the solar pole has to be hotter by about 4 K compared to the equator to give rise to a thermal wind term comparable to the centrifugal term. There have been some attempts to measure if there is any temperature variation on the solar surface from the equator to the pole [56, 57]. This is a difficult measurement and, although the results may not be completely conclusive, there are indications that the poles of the Sun are indeed slightly hotter. We discuss in the Appendix how an order of magnitude estimate of the pole-equator temperature difference can be made.

It is an intriguing question why the two source terms in (27) are comparable in magnitude. Presumably, this is not an accident. Let us consider what would happen if the centrifugal term becomes much larger. Then it would drive a much stronger meridional circulation. We see in (17) that the meridional circulation can carry angular momentum with it, changing the profile of Ω . A stronger meridional circulation would change the profile of Ω in such a manner that the centrifugal term given by (26) is reduced, thereby decreasing the meridional circulation. We believe that there is such a feedback mechanism in the Sun which keeps the two source terms in (27) comparable in amplitude.

We now show how to cast the thermal wind term in a different form involving the specific entropy S per unit mass, since readers may often encounter the thermal wind term written in this form in the literature. We have

$$\nabla p \times \nabla \rho = \left(\frac{\partial p}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \mathbf{e}_\theta \right) \times \left(\frac{\partial \rho}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \rho}{\partial \theta} \mathbf{e}_\theta \right) = \frac{1}{r} \left(\frac{\partial p}{\partial r} \frac{\partial \rho}{\partial \theta} - \frac{\partial \rho}{\partial r} \frac{\partial p}{\partial \theta} \right) \mathbf{e}_\phi. \quad (29)$$

For a parcel of gas, we have the basic thermodynamic relation

$$T dS = C_V dT + p d \left(\frac{1}{\rho} \right), \quad (30)$$

where C_V is the specific heat per unit mass. Eliminating T by using the ideal gas law $p = (\gamma - 1) C_V \rho T$, (30) can easily be put in the form

$$d\rho = \frac{\rho}{\gamma p} dp - \frac{\rho}{\gamma C_V} dS. \quad (31)$$

Substituting this for the differential of ρ in (29), we arrive at

$$\nabla p \times \nabla \rho = \frac{1}{r} \frac{\rho}{\gamma C_V} \left(\frac{\partial S}{\partial r} \frac{\partial p}{\partial \theta} - \frac{\partial p}{\partial r} \frac{\partial S}{\partial \theta} \right) \mathbf{e}_\phi. \quad (32)$$

Now, convection tends to equalize entropy in the radial direction so that we have

$$\frac{\partial S}{\partial r} \approx 0 \quad (33)$$

within the convection zone. Also, the hydrostatic equilibrium condition is

$$\frac{\partial p}{\partial r} = -\rho g, \quad (34)$$

where g is the acceleration due to gravity. Substituting (33) and (34) in (32), the dominant term is

$$\frac{\nabla p \times \nabla \rho}{\rho^2} = \frac{1}{r} \frac{g}{\gamma C_V} \frac{\partial S}{\partial \theta} \mathbf{e}_\phi. \quad (35)$$

Substituting this in (27), we get

$$\frac{\partial \omega_\phi}{\partial t} = r \sin \theta \frac{\partial}{\partial z} \Omega^2 - \frac{1}{r} \frac{g}{\gamma C_V} \frac{\partial S}{\partial \theta} + \left[\nabla \times \left(\frac{\mathbf{K}}{\rho} \right) \right]_\phi. \quad (36)$$

If the poles are hotter, then clearly $\partial S / \partial \theta$ is negative and the thermal wind term tends to create positive vorticity in agreement with our earlier discussion.

When the meridional circulation is maintained in a steady by a balance between the two large terms in (36), we have the thermal wind balance equation

$$r \sin \theta \frac{\partial}{\partial z} \Omega^2 = \frac{1}{r} \frac{g}{\gamma C_V} \frac{\partial S}{\partial \theta}. \quad (37)$$

Balbus et al. [58] pointed out that one can get a profile of the differential rotation matching observations remarkably well by integrating (37) with the assumption that the S is constant over contours of constant Ω so that we can write $S = f(\Omega^2)$. This bypasses the need for evaluating the turbulent stress terms. However, the justifications for the assumption $S = f(\Omega^2)$ do not appear particularly compelling to us.

3.4 Large-scale fluid flows inside solar-like stars

In the previous subsections §3.1–3, we have presented the basic physical ideas of how we can theoretically calculate large-scale fluid flows like the differential rotation and the meridional circulation inside stars. We basically need to solve (16) and (27) simultaneously, with the time derivative terms set to zero when we deal with a steady state, and accompanied by an equation for convective heat transport to provide latitudinal variation of temperature that gives rise to the thermal wind term. We often make the statement that the meridional circulation is driven by the turbulent stresses in the convection zone. We should explain what precisely we mean by this. We have pointed out that turbulent stress terms like $\overline{v_r v_\phi}$ estimated in (22) drive the differential rotation. While the turbulent stresses may not explicitly appear in (27), they are the ultimate causes of both the Coriolis term and the thermal wind term, the two drivers of the meridional circulation. That is why we expect the meridional circulation to be confined to the convection zone.

The anisotropic viscosity model of Kippenhahn [47] gave rise to a meridional circulation along with differential rotation due to the centrifugal term, although

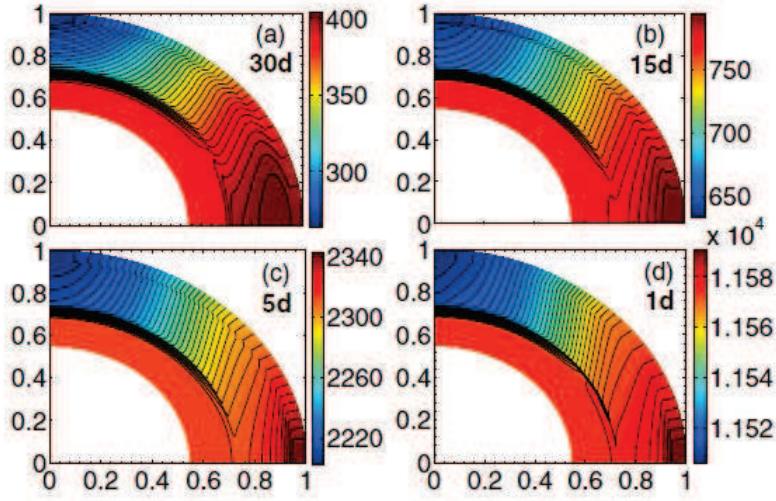


Figure 9: Theoretically computed profiles of angular velocity $\Omega(r, \theta)$ in the poloidal planes of solar-mass stars with rotation periods of 30, 15, 5, and 1 days. The rotational frequencies in nHz are indicated by the different colours. From Karak, Kitchatinov, and Choudhuri [62], based on the model of Kitchatinov and Olemskoy [61].

the thermal wind term was not included in this model. Köhler [59] presented detailed computations of the meridional circulation based on this model. As we already pointed out, Kitchatinov and Rüdiger [50] calculated both the differential rotation and the meridional circulation based on their mean field model. Due to many uncertainties in the parameters of the mean field theory, it is difficult to say conclusively whether the meridional circulation should consist of a single cell in a hemisphere or should have a more complicated structure [60]. For example, Kitchatinov and Rüdiger [50] found two radially stacked cells of the meridional circulation (see their Figure 1), whereas slight modifications in the model led Kitchatinov and Olemskoy [61] to obtain a single cell. Now we briefly describe some results presented by Karak et al. [62] based on the model of Kitchatinov and Olemskoy [61].

As we pointed out in §3.2, the nature of the differential rotation induced depends on the nature of the turbulent stress. If the star is weakly rotating, then presumably radial turbulent motions dominate and Ω near the equator ends up with a lower value. On the other hand, if the star is rotating fast, horizontal turbulent motions may become more dominant and Ω near the equator ends up with a higher value. Figure 9 shows theoretically computed angular velocity patterns inside the convection zone of stars having mass equal to the solar mass, but rotating with different rotation periods [62]. All the cases shown in Figure 9 correspond to situations in which the rotation of the star is sufficiently fast and the equatorial region has the higher Ω (like the Sun). However, within this regime, we see a clear trend. If the rotation is made faster (i.e. rotation period shorter), then the contours of constant Ω tend to become cylinders parallel to the rotation axis. On the other hand, slower rotation tends to give contours constant over cones, as in the Sun. Presumably, the Sun is rotating fast enough

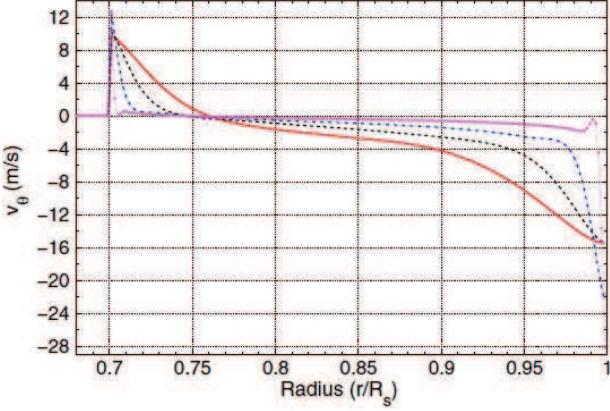


Figure 10: Theoretically computed component v_θ (in m s^{-1}) of meridional circulation at 45° latitude of solar-mass stars with different rotation periods. Solid (red), dashed (black), dash-dotted (blue), and dot-pointed (magenta) lines correspond to stars with rotation periods of 30, 15, 5, and 1 days, respectively. From Karak, Kitchatinov, and Choudhuri [62], based on the model of Kitchatinov and Olemskoy [61].

to make the horizontal turbulent motions important within the convection zone so that $\overline{v_r v_\phi}$ given by (22) is positive, but not fast enough to make Ω constant over cylinders.

As Ω tends to become constant over cylinders for stars rotating fast, it is obvious that $\partial \Omega^2 / \partial z$ will tend to become smaller. Since the main driver of the meridional circulation becomes weaker for faster rotating stars, detailed computations show that the meridional circulation is weaker in faster rotating stars and tends to be confined to the edges of the convection zone where the condition of constancy over cylinders is expected to be violated in thin boundary layers. Some results [62] are shown in Figure 10. As we shall discuss later, this result that the meridional circulation becomes weaker for faster rotating stars poses some problems in modelling stellar dynamos.

Although we do not intend to present a full discussion of numerical simulations in this paper, we describe a few main results. As already pointed out in §3.2, simulations of stellar convection showed that slowly rotating stars have anti-solar differential rotation with the equatorial region having lower Ω and rapidly rotating stars have solar-like differential rotation with the equatorial region having higher Ω [51, 53]. However, rapidly rotating stars with accelerated equatorial regions tend to have angular velocity constant over cylinders in most of the simulations, indicating that the Taylor–Proudman constraint is quite strong. Getting the angular velocity constant over cones (as found by helioseismology) rather than over cylinders has proved particularly difficult in numerical simulations [63, 64]. If Ω is constant over cylinders, then the centrifugal term given by (26) would be much smaller than what it is inside the Sun and the meridional circulation which one gets from such simulations should be interpreted with caution. Careful simulations of the meridional circulation showed that it is possible to get single cell meridional circulations for slowly rotating stars with decelerated equatorial regions, but rapidly rotating stars with

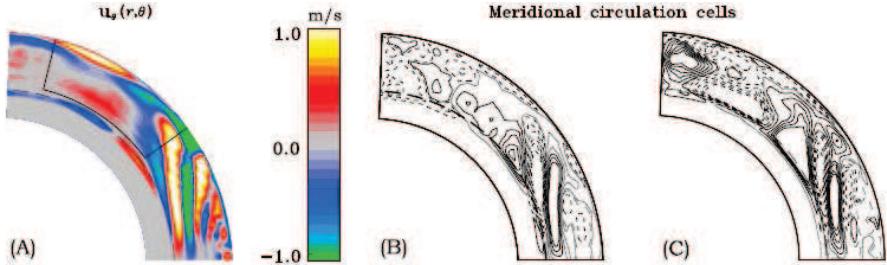


Figure 11: The mean meridional circulation (averaged over turbulence) as found by Passos, Charbonneau, and Miesch [65] from their numerical simulation of the solar convection. (A) shows how the θ component of the meridional circulation varies over the meridional plane. This can be compared with the observational Figure 2. (B) and (C) give streamlines of the meridional circulation at the times of the solar minimum and the solar maximum. The dashed and solid lines respectively indicate regions of negative and positive vorticity.

accelerated equatorial regions tend to produce multiple cells of the meridional circulation [52, 53, 65]. A typical result is shown in Figure 11. It may be noted that this result is based on an MHD code including the dynamo action, leading to a variation of the meridional circulation with the solar cycle. We shall discuss this problem in detail in §5. If we really have single-cell meridional circulation in a hemisphere as indicated by the most recent observational analysis [24], we have to confess that we do not have simple, compelling arguments at the present time to explain why it is so.

While discussing the basic mathematical theory of the differential rotation, we refrained from a discussion of the boundary conditions at the top and the bottom of the convection zone which we need to impose while solving (16). If the observed conical isorotation contours have to match with the solid body rotation in the radiation zone, then there has to be a boundary layer at the interface. The tachocline is such a boundary layer. Why the tachocline is so thin remains poorly understood. Whether the meridional circulation or even magnetic fields play a role in keeping the tachocline confined in a thin layer is an intriguing question [66, 67]. Rempel [68] developed a model of large-scale flows by assuming simple forms of the turbulent stress tensors and argued that the tachocline may play an important role in breaking the Taylor–Proudman constraint even within the convection zone. A careful look at Figure 3 shows a boundary layer of strong shear even at the solar surface. There have been attempts to model this shear layer though simulations of the upper convection zone [64, 69].

4 The role of meridional circulation in solar dynamo models

After discussing the relevant observations and basic theoretical ideas connected with the meridional circulation, we now turn our attention to the solar dynamo problem and the role of the meridional circulation in it. Because of the paucity

of good pedagogical introductions to the theory of the meridional circulation, we have discussed the basic theoretical ideas about the meridional circulation in a pedagogical manner in §3. There are, however, convenient pedagogical introductions to dynamo theory ([43], Chapter 16; [70]; [71], Chapter 8; [72]) and also comprehensive reviews [73, 74, 75]. In view of this, our discussion of the basics of dynamo theory will be very brief, assuming that the readers are familiar with the fundamentals of MHD.

4.1 Basics of solar dynamo theory

Just as an axisymmetric velocity field can be written in the form (1–3), an axisymmetric magnetic field can be written as

$$\mathbf{B} = B_\phi(r, \theta, t) \mathbf{e}_\phi + \nabla \times [A(r, \theta, t) \mathbf{e}_\phi], \quad (38)$$

where $B_\phi(r, \theta)$ is called the *azimuthal magnetic field* and

$$\mathbf{B}_p = \nabla \times [A(r, \theta, t) \mathbf{e}_\phi] \quad (39)$$

is called the *poloidal magnetic field*. The basic idea of dynamo theory is that the toroidal and the poloidal fields sustain each other through a feedback loop. As we shall discuss below, it is easy to see that the differential rotation can stretch the poloidal field lines to create the toroidal field. How the poloidal field can be generated back from the toroidal field is more complicated. A crucial idea was due to Parker [76], who suggested that turbulent helical motions can twist the toroidal field to produce the poloidal field. Since the Coriolis force due to the Sun's rotation would cause the convective blobs in the Sun's convection zone to rotate, we clearly have helical turbulence there which could conceivably twist the toroidal field to produce the poloidal field.

The basic evolution equation of the magnetic field in MHD is the well-known induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (40)$$

where

$$\eta = \frac{1}{\mu_0 \sigma} \quad (41)$$

is often referred to as the *magnetic diffusivity*, σ being the electrical conductivity. To study the behaviour of the magnetic field inside a turbulent fluid, we have to split both \mathbf{B} and \mathbf{v} into a mean part and a fluctuating part as we did for the velocity field in (7). We write

$$\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}', \quad \mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'. \quad (42)$$

Substituting into (40) and averaging, we get

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{v}} \times \overline{\mathbf{B}}) + \nabla \times \mathcal{E} + \eta \nabla^2 \overline{\mathbf{B}}, \quad (43)$$

where

$$\mathcal{E} = \overline{\mathbf{v}' \times \mathbf{B}'} \quad (44)$$

is known as the *mean EMF*. Just as the turbulent stress $\rho \overline{v'_i v'_j}$ appearing in (10) is crucial in the theory of large-scale flows, this mean EMF is crucial in dynamo

theory. Steenbeck et al. [77] developed the systematic mean field theory of MHD in a turbulent situation, from which \mathcal{E} can be calculated. If the turbulence is isotropic, then \mathcal{E} can be written in the form

$$\mathcal{E} = \alpha \overline{\mathbf{B}} - \eta_T \nabla \times \overline{\mathbf{B}}, \quad (45)$$

where

$$\alpha = -\frac{1}{3} \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \tau, \quad \eta_T = \frac{1}{3} \overline{\mathbf{v}' \cdot \mathbf{v}'} \tau, \quad (46)$$

where τ is the correlation time (see [43], §16.5 for a derivation). It is obvious from the expression of α in (46) that α is a measure of the helical turbulence in the fluid. Substituting (45) in (43), we arrive at

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{v}} \times \overline{\mathbf{B}}) + \nabla \times (\alpha \overline{\mathbf{B}}) + (\eta + \eta_T) \nabla^2 \overline{\mathbf{B}}, \quad (47)$$

Clearly η_T is of the nature of a diffusion coefficient, and (46) makes it clear that it arises out of turbulence. This turbulent diffusion coefficient η_T is usually much larger than η , which can be neglected compared to η_T . Also, as we shall be dealing only with mean fields now onwards, we simplify the notation by dropping the overline to indicate the mean, as we did from (11) onwards in §3. Then we write (47) as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\alpha \mathbf{B}) + \eta_T \nabla^2 \mathbf{B}. \quad (48)$$

This is the fundamental equation of the turbulent dynamo.

We shall assume that both the mean velocity field and the mean magnetic field are axisymmetric. Then we can substitute (3) and (38) for \mathbf{v} and \mathbf{B} in (48). A few steps of easy algebra give us the following evolution equations of the toroidal and the poloidal fields

$$\frac{\partial B_\phi}{\partial t} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_r B_\phi) + \frac{\partial}{\partial \theta} (v_\theta B_\phi) \right] = \eta_T \left(\nabla^2 - \frac{1}{s^2} \right) B_\phi + s (\mathbf{B}_p \cdot \nabla) \Omega, \quad (49)$$

$$\frac{\partial A}{\partial t} + \frac{1}{s} (\mathbf{v}_m \cdot \nabla) (s A) = \eta_T \left(\nabla^2 - \frac{1}{s^2} \right) A + \alpha B_\phi, \quad (50)$$

where $s = r \sin \theta$. We see in (49) that the source term for the toroidal field is $s(\mathbf{B}_p \cdot \nabla) \Omega$ which corresponds to the stretching of the poloidal field by the differential rotation to produce the toroidal field. The source term for the poloidal field in (50) is αB_ϕ which, in conjunction with the expression of α given in (46), encapsulates Parker's idea of helical turbulence twisting the toroidal field to produce the poloidal field [76].

When the first solar dynamo models were constructed in the 1960s and 1970s, the existence of the meridional circulation was not yet established and the dynamo modellers of that era did not realize that such a flow may have important consequences for the solar dynamo. Also, the helioseismology results of Ω were not available at time. Dynamo modellers of that era would assume a 'reasonable' profile of Ω and then solve (49) and (50) after setting \mathbf{v}_m (and its components v_r, v_θ) to zero. It was found that one can obtain a dynamo wave propagating towards the equator if the condition

$$\alpha \frac{\partial \Omega}{\partial r} < 0 \quad (51)$$

known as the *Parker–Yoshimura sign rule* is satisfied in the northern hemisphere [76, 78]. It has been mentioned in §2.1 that sunspots appear at increasingly lower latitudes with the progress of the solar cycle, leading to the butterfly diagram of sunspots shown by the shaded areas in Figure 1. The equatorward propagation of the dynamo wave was believed to provide the theoretical explanation for this drift of sunspots with the solar cycle, and the solar dynamo models of that era could give nice butterfly diagrams. We are aware of only one paper of that era [79] which studied some effects of the meridional circulation on the solar dynamo and pointed out that the meridional circulation could change the appearance of the butterfly diagram.

While these older models of the solar dynamo seemed reasonably successful at that time, certain new developments in solar physics made their inadequacies clear, paving the way to the formulation of the flux transport dynamo model, in which the meridional circulation plays a crucial role. We turn to these developments now.

4.2 The flux transport dynamo model

Large sunspots often appear on the solar surface in pairs, with the two members of the pair having opposite magnetic polarities [80]. The appearance of such bipolar sunspot pairs is explained by Parker’s famous idea of magnetic buoyancy [81]. The line joining two sunspots in a pair is usually approximately parallel to the solar equator, but with a tilt which tends to increase with latitude in spite of a large statistical scatter [80, 82]. This dependence of the tilt on latitude is called *Joy’s law*. This tilt is produced by the action of the Coriolis force on the rising magnetic flux tubes [83].

As seen in Figure 3, the differential rotation is concentrated in the tachocline at the bottom of the solar convection zone. We believe that this is the region where strong toroidal field is produced due to the action of the differential rotation on the poloidal field. Presumably, this toroidal field can be stored in a stable layer there and parts of it break out in the form of magnetic flux tubes which rise through the convection zone to produce sunspots at the solar surface. Using the thin flux tube equation [84, 85], simulations have been carried out to study the buoyant rise of flux tubes through the convection zone to produce sunspots [86, 87, 83, 88, 89]. These simulations fit observational data well only if the magnetic field inside the flux tubes at the bottom of the convection zone is taken to be of order 10^5 G. However, helical turbulence will be unable to twist such strong magnetic fields and the traditional α -effect arising out of the α -coefficient given by (46) cannot be operational. The poloidal field has to be generated in some other manner.

Recent solar dynamo models usually invoke an idea due to Babcock [90] and Leighton [91] for the generation of the poloidal field. They pointed out that, when tilted bipolar sunspots decay and the magnetic field in them spreads around, the magnetic field from the sunspot at the higher latitude contributes more in building up the overall magnetic field at higher latitudes. Like the traditional α -effect, the Babcock–Leighton process also can be described by an α -coefficient which is concentrated near the solar surface and which appears in (50) in exactly the same manner. Surface observations of sunspot pair tilts suggest that α due to the Babcock–Leighton process is positive in the northern hemisphere. When combined with profile of Ω determined by helioseismology,

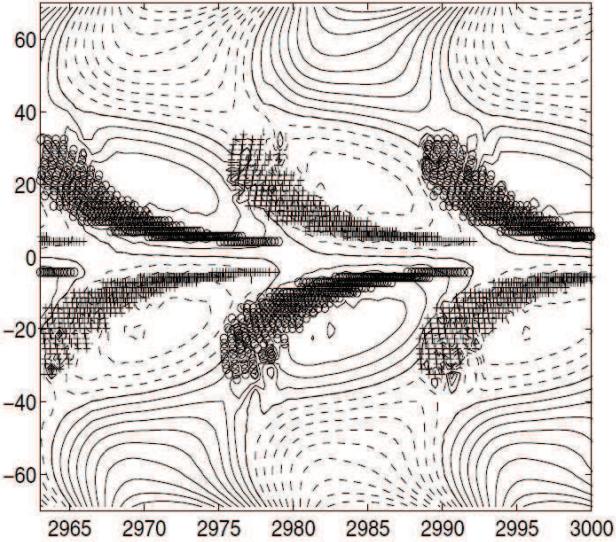


Figure 12: Theoretical butterfly diagram from a flux transport dynamo simulation by Chatterjee, Nandy, and Choudhuri [103]. The latitudes where sunspots are seen at a certain time are shaded, shown along with contours of constant B_r in the time-latitude plot.

it was found that the Parker–Yoshimura sign rule (51) is not satisfied at the low latitudes where sunspots are seen. This suggests that the dynamo wave should propagate poleward, in contradiction to the observations. We need something to turn around the dynamo wave. Choudhuri et al. [92] showed that the meridional circulation can do the job.

From the late 1980s, there were studies of how the poleward meridional circulation near the solar surface advects solar magnetic fields [6, 93, 94, 95], and Wang et al. [96] constructed a 1D dynamo model incorporating the meridional circulation. However, proper 2D models of the dynamo with the meridional circulation were first constructed in 1995 by Choudhuri et al. [92] and Durney [97], and developed further in many subsequent papers [98, 99, 100, 101, 102, 103]. If the toroidal field is generated by the differential rotation in the tachocline and there is equatorward meridional circulation there, then the toroidal field can be advected equatorward, in spite of the Parker–Yoshimura sign rule being violated, as shown by Choudhuri et al. [92]. This would cause sunspots to form at increasingly lower latitudes with the progress of the solar cycle, whereas the poloidal field near the solar surface is advected poleward by the poleward meridional circulation there—in agreement with the observational data discussed in §2.1. Figure 12 taken from Chatterjee et al. [103] presents a theoretical butterfly diagram obtained from a dynamo model, along with contours of constant B_r at the solar surface in a time-latitude plot. This theoretical figure can be compared favourably with the observational Figure 1. Such a remarkable agreement with observational data would be impossible without incorporating the meridional circulation in this kind of dynamo model, known as the *flux transport dynamo model*. While different authors sometimes use this term to mean slightly dif-

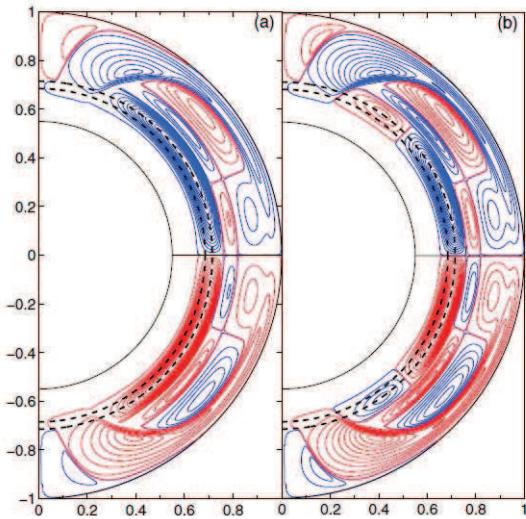


Figure 13: Two arbitrarily complicated meridional circulations with which Hazra, Karak, and Choudhuri [104] solved the equations of the flux transport dynamo. Blue contours imply counter-clockwise circulation, whereas red contours imply clockwise circulation. One obtains reasonable butterfly diagrams with these meridional circulations.

ferent things, we would refer to a dynamo model as a flux transport dynamo model if the poloidal field generation takes place by the Babcock–Leighton process and the meridional circulation plays a crucial role in advecting the toroidal field at the bottom of the convection zone and the poloidal field at the surface. The meridional circulation even decides the period of the dynamo cycle. The dynamo period turns out to be essentially equal to the time taken by a fluid element to travel from higher latitudes to lower latitudes at the bottom of the convection zone. If the meridional circulation is made stronger in the model, the period becomes shorter [99].

It may be pointed out that most of the dynamo models were worked out by assuming a single-cell meridional circulation encompassing one full hemisphere, with the equatorward flow at the bottom of the convection zone. We mentioned in §2.2 that the nature of the meridional circulation deeper down inside the convection zone remains uncertain. Hazra et al. [104] addressed the important question of whether the flux transport dynamo model can match observational data if the meridional circulation is more complicated. They solved the equations of the flux transport dynamo for some arbitrarily complicated meridional circulations, two of which are shown in Figure 13. They concluded that the flux transport dynamo can work as long as there is a layer of equatorward flow at low latitudes at the bottom of the convection zone. Jouve and Brun [105] also presented solutions of the flux transport dynamo with multi-cell meridional circulation. If there is sufficiently strong downward pumping as suggested in some convection simulations, that also can give rise to an appropriate dynamo wave at the bottom of the convection zone even if the return flow of the meridional circulation occurs well above the bottom [106].

The flux transport dynamo models work best if the meridional circulation is assumed to penetrate a little bit below the bottom of the convection zone where the toroidal flux can be stored in a stable layer [102, 107]. How much penetration is possible remains controversial—some authors arguing that the meridional circulation cannot penetrate much into the stable layer [108], whereas others have argued in favour of a considerable penetration [109]. It has also been suggested that the meridional circulation may play an important role in storing the strong toroidal field in the stable layer underneath the bottom of the convection zone [110].

With indications that many other stars have cycles like the Sun, one important question is whether flux transport dynamos work in other solar-like stars [111]. We have discussed in §3 how the large-scale flow patterns like the differential rotation and the meridional circulation can be theoretically calculated for stars rotating with different rotation periods. Using such theoretically computed flow patterns, Karak et al. [62] constructed flux transport dynamo models of solar-mass stars rotating with different rotational velocities. They could explain such observational features as the enhanced activity for faster rotating stars. However, these models have some difficulty in explaining the observational data that faster rotating stars have shorter activity cycles. As we pointed out in §3.4, theoretical considerations suggest that faster rotating stars have weaker meridional circulation, which would lead to longer cycle periods [112, 62]. Hazra et al. [113] have suggested that the inclusion of the downward turbulent pumping may help in closing the gap between observations and theory.

Although the 2D models of the flux transport dynamo have been reasonably successful in modelling many aspects of the solar cycle, one limitation of such models is that magnetic buoyancy and the Babcock–Leighton process are inherently 3D processes. They are treated in 2D models with rather drastic approximations [100, 114, 115]. Of late, there have been attempts of developing 3D kinematic models of the flux transport dynamo, in which the large-scale flows are assumed to be given and the magnetic field is treated in a 3D manner [116, 117, 118, 119]. Another approach of treating the non-axisymmetric nature of the Babcock–Leighton process is to study the evolution of B_r on the solar surface, under the action of diffusion and the meridional circulation. See Jiang et al. [120] for a survey of such surface flux transport models. While these models can handle the Babcock–Leighton process at the solar surface quite satisfactorily, they cannot treat the evolution of the magnetic fields in the polar regions realistically by not including the submergence of the meridional circulation underneath the surface near the polar regions [118]. There have also been efforts of combining 2D flux transport dynamo model (in r and θ) with the 2D surface flux transport model (in θ and ϕ) [121].

4.3 Modelling irregularities in the solar cycle

We now turn to the important question of how the various irregularities in the solar cycle arise (reviewed in [122]) and shall see that the meridional circulation plays quite an important role in this problem also. We first point out how several time scales in the flux transport dynamo are related, since an understanding of this will be necessary for our discussions.

If l is the thickness of the tachocline within which the magnetic diffusivity is η_{tach} , then the diffusion time within the tachocline is l^2/η_{tach} . In order for

magnetic fields to be advected within the tachocline by the meridional flow velocity of order v , the time scale R_\odot/v of the meridional circulation has to be shorter than this. Since the magnetic diffusivity η_T inside the convection zone is expected to be several orders of magnitude larger than that in the tachocline, we expect the diffusion time scale L^2/η_T within the convection zone (L is the thickness of the convection zone) to be much shorter than that within the tachocline. We basically have two possible ordering of these various times scales

$$L^2/\eta_T < R_\odot/v < l^2/\eta_{\text{tach}}, \quad (52)$$

or

$$R_\odot/v < L^2/\eta_T < l^2/\eta_{\text{tach}}, \quad (53)$$

Dynamo models have been constructed both satisfying (52) [103, 123, 124] and satisfying (53) [99, 125]. As long as we are interested only in modelling periodic features of the solar cycle, both types of models had reasonable success. However, when we introduce fluctuations in the dynamo model for modelling the irregularities of the cycle, models satisfying (52) and (53) behave very differently [124, 126], as we shall discuss below.

Since magnetic stresses can quench the flows driving the dynamo (treated in kinematic models by introducing some heuristic quenching terms), the dynamo problem is essentially nonlinear. It was initially thought that the nonlinearities cause the irregularities in the dynamo cycles [127]. After it was realized that the most obvious types of nonlinearities cannot produce sustained irregularities, the attention in the last few years has been turned to stochastic fluctuations in the dynamo [128]. However, the nonlinearities are probably responsible for certain kinds of irregularities. For example, the Gnevyshev–Ohl rule that the odd cycle tends to be stronger than the preceding even cycle is likely be a manifestation of period doubling in a nonlinear system [129, 130].

Let us now turn to the possible sources of stochastic fluctuations in the solar dynamo. One finds a scatter in the tilt angles of sunspot pairs around the mean tilt satisfying Joy’s law [82]. While the mean tilt results from the action of the Coriolis force on rising flux tubes [83], the scatter is presumably caused by the buffeting due to turbulence when the flux tubes rise through the convection zone [131]. This implies that the Babcock–Leighton process for generating the poloidal field from tilted bipolar sunspots should involve fluctuations [123]. If such fluctuations are introduced in theoretical dynamo models satisfying (52), then their effects spread through the convection zone in a few years. On the other hand, fluctuations introduced in models satisfying (53) do not diffuse much, but get carried with the meridional circulation. The strength of the cycle 24 predicted by Dikpati and Gilman [125] based on a dynamo model satisfying (53) completely failed to match observations, whereas the strength predicted by Choudhuri et al. [123] based on a dynamo model satisfying (52) turned out to be the first successful dynamo-based prediction of a solar cycle before its onset. The higher turbulent diffusivity of the convection zone corresponding to the condition (52) also helps in explaining the preferred dipolar parity of the Sun [103, 132] and the lack of hemispheric asymmetry [133, 134]. It appears that the solar situation corresponds to the condition (52) rather than (53). Dynamo models with stochastic fluctuations in the Babcock–Leighton process can also produce grand minima like the Maunder minimum [135].

As we pointed out in §2.4, there is evidence of random fluctuations in the meridional circulation of the Sun having correlation time of order 30–40 yr. This is a second important source of fluctuations which is expected to affect the dynamo. Let us try to figure out as to what will happen if the meridional circulation becomes weaker during an epoch due to these fluctuations. As discussed in §4.2, a slower meridional circulation will make the period of the dynamo longer. This will give rise to two competing effects. The diffusion will have a longer time to act, thereby trying to make the magnetic fields weaker. The differential rotation also will have a longer time to produce a stronger toroidal magnetic field. Which of these two effects dominates will depend on whether the condition (52) or the condition (53) is satisfied. If the condition (52) holds, then diffusion will be dominant and longer cycles will be weaker. On the other hand, if the condition (53) holds, then the differential rotation generating more toroidal field will be the dominant process, making longer cycles stronger. Observational data indicate that it the first possibility—longer cycles are weaker—which holds for the Sun, again suggesting the condition (52) is the appropriate condition for the Sun. One observational fact known as the *Waldmeier effect*—that shorter cycles rise faster—is a consequence of this. If shorter cycles are stronger, they are certainly expected to rise faster. Only by considering fluctuations in the meridional circulation causing the durations of different cycles unequal, it has been possible to provide a theoretical explanation of the Waldmeier effect [36].

Taking the fluctuations in the meridional circulation to be the only fluctuations in the solar dynamo process, Karak [136] succeeded in modelling the irregularities of the solar cycle to some extent. Since fluctuations in the Babcock–Leighton process are also present, a full theory should be based on the combined effect of both of these types of fluctuations. Choudhuri et al. [123] made their prediction for cycle 24 at a time when the importance of fluctuations in the meridional circulation was not realized and these fluctuations were not taken into account. Presumably, this prediction turned out to be so successful because there was no big random change in the meridional circulation between the time of the prediction and the peak of the next cycle. A prediction of a future cycle should take into account of both fluctuations in the Babcock–Leighton process and fluctuations in the meridional circulation. Observational data suggest that changes in the meridional circulation may take a few years to change the strength of the solar cycle [137]. This delay in the effect of the meridional circulation enables us to use the value of the meridional circulation at the end of a cycle (from the rate of decline of the cycle at that time) which is appropriate for determining the strength of the next cycle. Also, the poloidal field at the end of the cycle provides information about the fluctuations in the Babcock–Leighton process that is needed for predicting the next cycle. Hazra and Choudhuri [138] have developed a formula for predicting the next cycle by using the decline rate and the poloidal field at the end of the previous cycle.

At last, we come to the question of explaining the most extreme events in the irregularities of the solar cycle—the grand minima when sunspots may disappear for several decades. From the analysis ^{10}Be concentration in polar ice cores, it has been possible to infer that there had been about 27 grand minima in the last 11,000 years [139]. If the fluctuations in the Babcock–Leighton process make the poloidal field at the end of a cycle too weak, or if the fluctuations in the meridional circulation make it too slow (keep in mind that a slower meridional circulation leads to a longer cycle of weaker strength), then that may drive the

dynamo into a grand minimum. Dynamo simulations suggest that it is possible to produce grand minima by considering fluctuations in the Babcock–Leighton process alone [135] or in the meridional circulation alone [136], if the fluctuations are assumed to be sufficiently large. Considering both kinds of fluctuations simultaneously and choosing the statistical parameters of these fluctuations on the basis of past observations, Choudhuri and Karak [140] succeeded in explaining the statistical properties of grand minima reasonably well. Presumably, the grand minima are produced by the combined effect of fluctuations in both the Babcock–Leighton process and the meridional circulation [140, 141].

5 Back reaction of the dynamo on the meridional circulation

We presented our discussion of large-scale fluid motions in §3 by assuming that there is no magnetic field present. If a magnetic field is present in the fluid, then it gives rise to the Lorentz force, which has to be included in the basic dynamical equation (11) such that it becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{\nabla p}{\rho} + \mathbf{F} + \frac{\mathbf{K}}{\rho} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho}. \quad (54)$$

The Lorentz force term can be written as

$$\mathbf{F}_L = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho} = -\frac{1}{\rho} \nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{\mu_0 \rho}. \quad (55).$$

The first term on the right hand side indicates that the magnetic field has a pressure $B^2/2\mu_0$ associated with it. The other term is of the nature of *magnetic tension*. If magnetic field lines are straight and parallel in a region, it is easy to see that $(\mathbf{B} \cdot \nabla) \mathbf{B}$ will be zero in that region. This term arises when the magnetic field lines are bent and tries to straighten the field lines. Another effect of the magnetic tension is that it tries to shorten the lengths of magnetic field lines.

The magnetic field generated in the Sun by the dynamo action will certainly have a Lorentz force associated with it. This Lorentz force, appearing in (54) as shown above, would affect both the turbulent motions and the mean motions. The analytical theory of how turbulent motions are affected by the Lorentz force is extremely complicated [142, 143]. Here we focus our attention on the action of the Lorentz force on the large-scale mean motions. In §2.4 we mentioned torsional oscillations and variations of the meridional circulation with the solar cycle. Before getting into the mathematical theory of these, let us present the main physical ideas qualitatively. One important ingredient of the dynamo process is that the poloidal field lines get stretched by the differential rotation to create the toroidal field. We expect a field line to look as shown in Figure 14(a). Such a field line would have a tension force in the direction of the thick arrow, inducing motions in the azimuthal direction. We believe that this is how torsional oscillations are driven. To figure out the effect of the magnetic field on the meridional circulation, we keep in mind that the toroidal magnetic field in the tachocline at the bottom of the convection zone, as shown in Figure 14(b), would be the dominant component of the magnetic field at the time of the solar maximum. Due to magnetic tension, the toroidal field lines

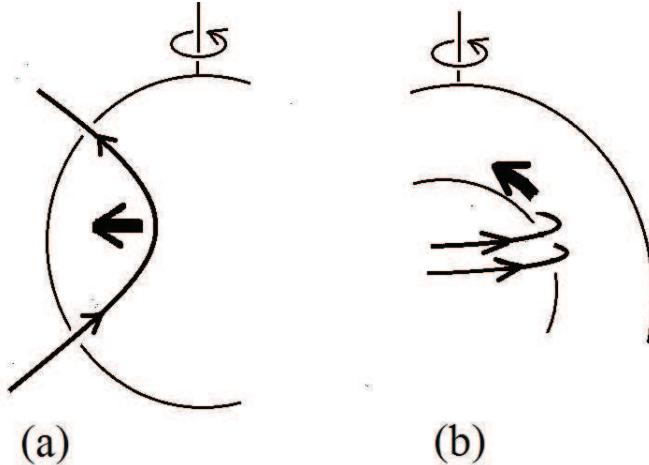


Figure 14: Sketches of (a) a typical magnetic field line inside the Sun and (b) a band of strong toroidal field at the bottom of the solar convection zone. The thick arrows indicate the parts of the Lorentz force which drive (a) torsional oscillations and (b) variations in the meridional circulation with solar cycle.

will try to shorten their lengths. Motions in the radial direction would be inhibited if the layer of tachocline where the toroidal field is stored is stable. The easiest way for the toroidal field lines to shorten their lengths is to slip in the poleward direction, which means that we would have a force as indicated by the thick arrow in Figure 14(b). This force would clearly oppose the equatorward meridional circulation at the bottom of the convection zone, causing a decrease in the meridional circulation speed at the time of the solar maximum, as seen in Figure 4. We now turn to a discussion of the mathematical formulation of these qualitative ideas.

To calculate the back reaction of the dynamo-generated magnetic fields on the large-scale flows crucial for driving the dynamo, we need to solve (54) along with the dynamo equations (49) and (50), which give the magnetic fields required to evaluate the Lorentz force term in (54). Rempel [144] followed such a procedure and showed that the back reaction gives rise to torsional oscillations and variations in the meridional circulation. Here we shall discuss the action of the Lorentz force on azimuthal motions and meridional motions separately, since that makes the basic physics of the problem clearer. One simple way of incorporating the fact that the meridional circulation becomes weaker when there are strong magnetic fields is to include a quenching due to magnetic fields in the expression of the meridional circulation [145]. Such a quenching has a tendency of making dynamos satisfying condition (53) unstable, adding additional support to our contention that the condition (52) is the appropriate condition for the solar dynamo.

Let us first consider the azimuthal motions which may be driven by the Lorentz force. We have to consider the ϕ component of (54). This is nothing but (16) with an additional term corresponding to the ϕ component of \mathbf{F}_L given by (55). If we calculate $(\nabla \times \mathbf{B}) \times \mathbf{B}$ by using the expression (38) of the magnetic

field, this additional term is found to have an elegant form

$$(\mathbf{F}_L)_\phi = \frac{1}{\mu_0 \rho s^3} J \left(\frac{sB_\phi, sA}{r, \theta} \right) \quad (56)$$

involving a Jacobian which essentially is a product of terms having toroidal and poloidal components. This is in agreement with Figure 14(a) which suggests that the Lorentz force driving azimuthal motions should involve both the toroidal and the poloidal components. Chakraborty et al. [34] solved the dynamo equations (49) and (50) along with (16) with the additional term given by (56). They were able to develop a model of torsional oscillations which agreed reasonably well with observational data. Earlier theoretical efforts are cited in this paper.

We now turn to the problem which is of central interest to us: how variations in the meridional circulation with the solar cycle are produced by the Lorentz force. For this purpose, we have to take the curl of (54) and consider its ϕ component. This leads to (27) with the additional term $(\nabla \times \mathbf{F}_L)_\phi$. Taking the magnetic field as given by (38) and using the expression (55) for \mathbf{F}_L , we find that the dominant terms in the expression of this additional term are

$$(\nabla \times \mathbf{F}_L)_\phi = \frac{1}{\mu_0 \rho} \left[\frac{1}{r^2} \frac{\partial}{\partial \theta} (B_\phi^2) - \frac{\cot \theta}{r} \frac{\partial}{\partial r} (B_\phi^2) \right]. \quad (57)$$

We note that the dominant terms in the part of the Lorentz force that causes variations in the meridional circulation arise from the toroidal component, in accordance with Figure 14(b). To find out how the meridional circulation varies with the solar cycle, we need to solve the dynamo equations (49) and (50) along with (27) with the additional term given by (57). Hazra and Choudhuri [146] solved this problem by following a perturbative approach, in which \mathbf{v}_m and its vorticity ω_ϕ were divided into a time-independent average part denoted by subscript 0 and a part varying with the solar cycle denoted by subscript 1, i.e.

$$\mathbf{v}_m = \mathbf{v}_0 + \mathbf{v}_1, \quad \omega_\phi = \omega_0 + \omega_1. \quad (58)$$

Substituting this in (27) with the additional term given (57) and subtracting from it the equation for ω_0 , we end up with the equation for the perturbed part

$$\frac{\partial \omega_1}{\partial t} + s \nabla \cdot (\mathbf{v}_0 \frac{\omega_1}{s}) + s \nabla \cdot (\mathbf{v}_1 \frac{\omega_0}{s}) = \frac{1}{\mu_0 \rho} \left[\frac{1}{r^2} \frac{\partial}{\partial \theta} (B_\phi^2) - \frac{\cot \theta}{r} \frac{\partial}{\partial r} (B_\phi^2) \right] + \left[\nabla \times \left(\frac{\mathbf{K}_1}{\rho} \right) \right]_\phi. \quad (59)$$

Note that we have included the small term given by (28), since we are dealing with the equation of a small perturbed quantity. We have also assumed that the turbulent stress term \mathbf{K} can be written in a form linear in the mean velocity as in (13) and write the part of \mathbf{K} associated with \mathbf{v}_1 as \mathbf{K}_1 . It is clear from (59) that a part of the Lorentz force associated with the toroidal magnetic field causes the variations in the meridional circulation with the solar cycle. Solving the full equation (27) with the additional term (57) would be a particularly challenging problem, since it would involve evaluating the thermal wind term which requires thermodynamics in addition to fluid mechanics. When we subtract the equation for ω_0 from the full equation, the thermal wind term drops out and it becomes a much more tractable problem. Hazra and Choudhuri [146] solved (59) with the dynamo equations (49) and (50) to develop a theory of the variations of the meridional circulation with the solar cycle.

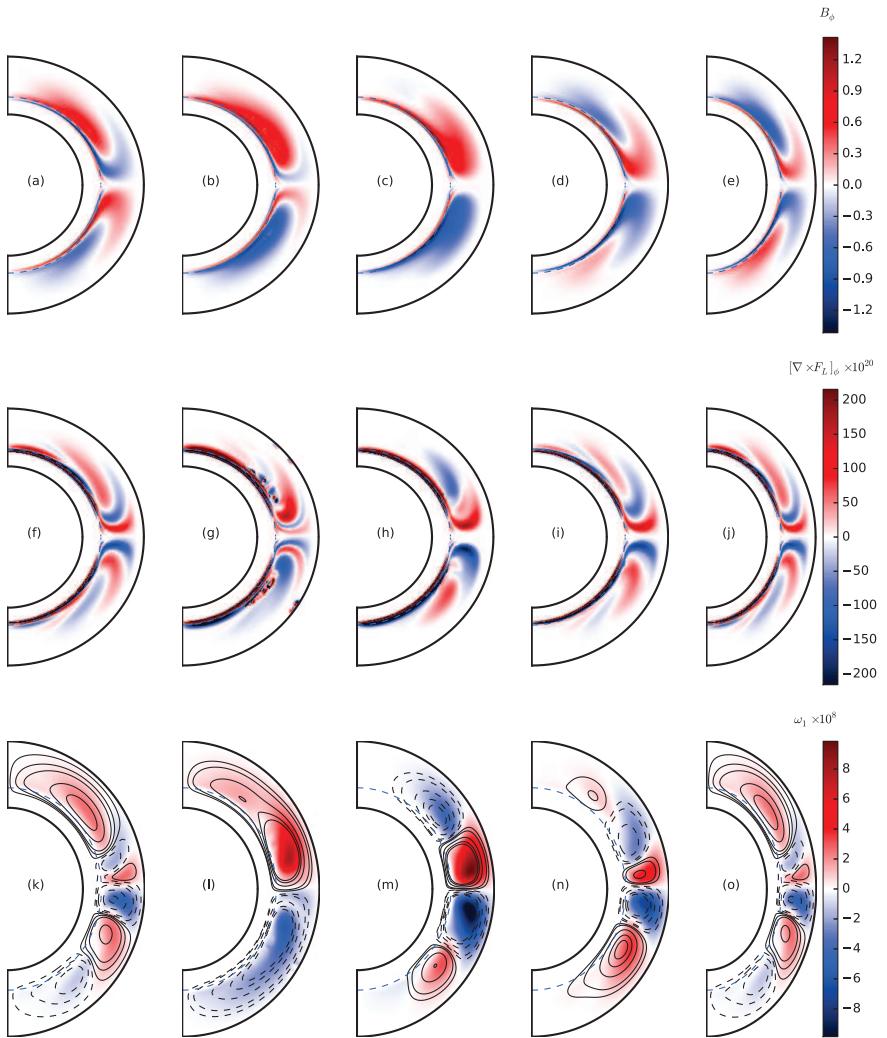


Figure 15: The evolution within the solar convection zone of the toroidal magnetic component B_ϕ (top row), the part $(\nabla \times \mathbf{F}_L)_\phi$ of the Lorentz force driving the variations in the meridional circulation (middle row) and the perturbed vorticity ω_1 associated with the meridional circulation variations (bottom row). The successive columns correspond to the profiles at intervals of $T/8$ (T is the dynamo period), the second column corresponding to the solar maximum and the fourth column to the solar minimum. From Hazra and Choudhuri [146].

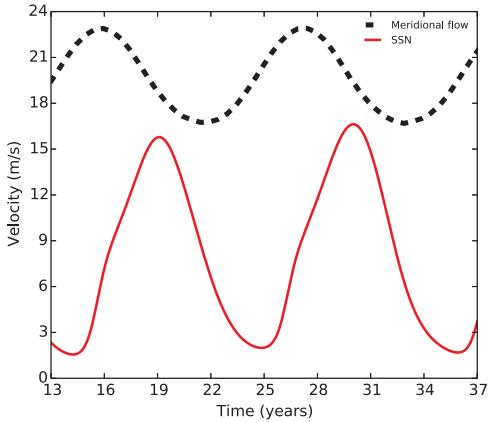


Figure 16: The variation of the meridional circulation just below the solar surface at latitude 25° (black dashed line), as computed from a theoretical model. The theoretical sunspot number (red solid line) is also shown. From Hazra and Choudhuri [146].

Let us now point out one puzzle, which still remains unresolved. It is clear from (56) that the part of the Lorentz force driving torsional oscillations is quadratic in toroidal and poloidal components, whereas (57) indicates that the part driving variations in the meridional circulation involves a simple square of the toroidal component. Since the toroidal component is much stronger than the poloidal component in the Sun, we conclude that the driver of the variations in the meridional circulation is much stronger than the driver of the torsional oscillations. We then expect the variations in the meridional circulation to have a much larger amplitude than that of the torsional oscillations. Observationally, however, both these amplitudes are found to be comparable—of order 5 m s^{-1} in the top layers of the convection zone. Even simple order of magnitude estimates suggest that the variations in the meridional circulation with the solar cycle should be much larger than what they are [146]. We still do not have a resolution of this puzzle. If the Lorentz force appearing in (59) is divided by a suitable factor, then our theoretical model gives variations in the meridional circulation agreeing with observational data reasonably well.

Since ω_0 for the meridional circulation in the northern hemisphere is negative, we want ω_1 to be positive at least in some regions of the northern hemisphere at the time of the solar maximum so that the meridional circulation becomes weaker at that time. Figure 15 taken from Hazra and Choudhuri [146] shows how B_ϕ , $(\nabla \times \mathbf{F}_L)_\phi$ and ω_1 vary during the solar cycle, the second column corresponding the solar maximum. It can be seen that the relevant part $(\nabla \times \mathbf{F}_L)_\phi$ of the Lorentz force and ω_1 driven by it are both predominantly positive in the northern hemisphere at that time. We expect this to weaken the meridional circulation at the time of the solar maximum. Figure 16 shows how the meridional circulation at the mid-latitude at the surface varies with time, along with the sunspot number calculated from the theoretical dynamo model. This figure compares favourably with Figure 4.

MHD simulations of stellar convection zone dynamics give rise to dynamo cycles and the meridional circulation varying with these cycles. This is clearly seen in Figure 11. The variation of the meridional circulation with the dynamo cycle was studied very carefully and thoroughly by Passos et al. [147]. Apart from directly exerting a Lorentz force on the large-scale flows like the differential rotation and the meridional circulation, the dynamo-generated magnetic fields can also modify the turbulent stresses which drive the large-scale flows. This second effect, which is usually not included in the mean field models, is automatically taken into account in the MHD simulations. However, these simulations so far have certain other limitations. These MHD simulations have demonstrated that dynamo action indeed takes place in stellar convection zones and it is possible to get periodic solutions. However, so far these simulations have not yielded solutions which can be regarded as realistic representations of the flux transport dynamo. These simulations are still far from matching actual solar cycle data, which a mean field model can do with a suitable specification of parameters, as seen in Figure 12. The meridional circulation obtained in these simulations also has a multi-cell structure as shown in Figure 11. Simulations can study the variations in such a meridional circulation with the dynamo cycle, even though either the dynamo cycle or the mean meridional circulation may not look very solar-like. The advantage of the mean field approach outlined in this Section is that one can choose the various parameters appropriately to have solar-like cycles and solar-like \mathbf{v}_0 , from which the time-varying part \mathbf{v}_1 can be calculated. Thus, both the mean field approach and the simulations approach have their relative advantages and limitations while modelling the variations of the large-scale flows in the Sun. Both these complementary approaches should be pursued to gain a deeper insight into this complex problem.

There have been efforts of explaining the variations of the meridional circulation with the solar cycle on the basis of inward flows towards active region belts—presumably driven by the fall in gas pressure in such belts [35]. Now that such cycle variations of the meridional circulation have been confirmed observationally even at the bottom of the convection zone [24], such an explanation based on a local surface phenomenon does not appear convincing to us.

6 Conclusion

This review focuses on the meridional circulation of the Sun—driven presumably by turbulent stresses in the solar convection zone. Well before the current era of research in this field, Eddington [148] and Sweet [149] pointed out the possibility of meridional circulations inside rotating stars. Due to the polar flattening in a rotating star, the temperature gradient tends to be steeper in the polar region. It may not be possible to reconcile this with the nuclear energy generation process without incorporating a meridional circulation in the star, even in the radiatively stable regions. Since the rotational flattening of the Sun is very small, the Eddington–Sweet circulation inside the Sun would be very slow with a time scale of order 10^{12} yr—much larger than the age of the Universe ([150], §42.5). The meridional circulation that we observe in the Sun is certainly a very different thing.

The meridional circulation, which was first observed at the solar surface, is expected to be confined within the convection zone. This circulation is now real-

ized to be an important component of the solar dynamo process which generates the solar magnetic field and its cycle. We believe that such a circulation exists in other solar-like stars as well, in which the dynamo cycles are probably generated in the same manner [111]. Even for compact stars like neutron stars accreting matter from a companion, flows in the meridional plane play a crucial role in the evolution of their magnetic fields [151, 152]. While helioseismology has thrown considerable light on the nature of the meridional circulation underneath the solar surface, its nature in lower regions of the convection zone still remains uncertain, although recent results support the view that the equatorward flow exists at the bottom of the convection zone [24]. Theoretical dynamo models work best if the meridional circulation is assumed to have a single cell spanning the whole of the convection zone in a hemisphere, although more complicated circulations satisfying certain criteria can also be accommodated [104].

The theoretical discussions in this review are primarily based on 2D mean field models, since such models make the physics of the problem clear. The theory of the meridional circulation is intimately connected with the theory of the other large-scale fluid flow pattern inside the Sun: the differential rotation. The Coriolis force due to the Sun's rotation induces horizontal motions within the convection cells, which may give rise to a transport of angular momentum away from the rotation axis—leading presumably to a faster rotating equatorial region. Such a pattern of differential rotation gives rise to a centrifugal term driving the meridional circulation in the direction consistent with observations. However, this term is opposed by a thermal wind term arising out of the fact that the Sun's poles are probably slightly hotter (about 4 K according to [50]) due to the more efficient convection in the polar regions. It seems that these two terms are comparable in magnitude and a slight imbalance between them drives the meridional circulation.

Solar dynamo models initially started being developed at a time when even the existence of the meridional circulation was not known. The early models without the meridional circulation had various difficulties which led to the formulation in the 1990s of the flux transport dynamo model, in which the meridional circulation plays a central role and even determines the period of the dynamo cycle. Irregular fluctuations in the meridional circulation (which seem to have a coherence time of about 30–40 yr [36]) are important in explaining many aspects of the irregularities in the solar cycle, in making comprehensive models of grand minima [140] and in predicting future cycles [138]. The Lorentz force of the magnetic fields generated by the dynamo can react back on the large-scale flows like the differential rotation and the meridional circulation causing their periodic variations with the solar cycle.

Since the meridional circulation, the differential rotation and the dynamo action are all related to each other, a full 2D mean field model should require simultaneous solution of (16), (27), (49) and (50) with the additional terms (56) and (57) added to (16) and (27) respectively. Since a calculation of the thermal wind term in (27) needs a realistic model of convective heat transport in which the effect of the Coriolis force on convection cells is included, an equation of heat transport also has to be solved along with the equations listed above. This is a formidable problem. Much of our theoretical understanding of this field has come from solutions of parts of this full problem. This review focuses on such studies of parts of the full problem, which elucidate many aspects of basic physics. In spite of the major advances in the last few years, many issues remain

poorly understood. We hope that in the near future observations, mean field models and numerical simulations will go hand in hand to solve many of the remaining puzzles.

This review is dedicated to the memory of late Bernard Durney, who kindled my first interest in the theory of the meridional circulation many years ago and whose seminal contributions in this field are often not sufficiently recognized. Peng-Fei Chen urged me to write this review. I thank Gopal Hazra, Bidya Karak and Leonid Kitchatinov for valuable inputs and suggestions on a preliminary version of the manuscript.

Appendix. Order of magnitude estimates of various terms in the equation driving the meridional circulation

We pointed out in §3.3 that there should be pole-equator temperature difference to give rise to a thermal wind term comparable to the centrifugal term and that the dissipation term should be negligible compared to these source terms. Now we present some order of magnitude estimates of these terms.

Let us first proceed with the assumption that the dissipation term is negligible and the thermal wind balance condition (37) holds within the convection zone. It is easy to argue that the left hand side of (37) is approximately equal to

$$r \sin \theta \frac{\partial}{\partial z} \Omega^2 \approx -[\Omega_{\text{eq}}^2 - \Omega_{\text{mid}}^2], \quad (\text{A1})$$

where Ω_{eq} and Ω_{mid} are the surface values of Ω at the equator and at the mid-latitude respectively. The respective values of frequency at these points are 440 and 400 nHz (see Figure 3), from which the values of Ω can be obtained by a multiplication with 2π . We thus have

$$r \sin \theta \frac{\partial}{\partial z} \Omega^2 \approx -[(440)^2 - (400)^2] \times (2\pi 10^{-9})^2 \text{ s}^{-2} \approx 1.4 \times 10^{-12} \text{ s}^{-2}. \quad (\text{A2})$$

To make an estimate of the right hand side of (37), we note that the specific entropy of an ideal gas is given by

$$S = C_V \ln T - (\gamma - 1) C_V \ln \rho + K,$$

where K is a constant. The entropy difference between the equator and the pole at the solar surface (which is surface of constant ρ) is

$$\Delta S = C_V \ln \left(\frac{T_{\text{eq}}}{T_{\text{pole}}} \right).$$

Taking ΔT to be the temperature excess of the pole with respect to equator, we have

$$\Delta S \approx -C_V \frac{\Delta T}{T_S}, \quad (\text{A3})$$

where T_S is the temperature of the solar surface and we have made use of the approximation $\ln(1 + x) \approx x$ for $|x| \ll 1$. Since this entropy difference takes

place over an angular separation $\pi/2$, we have

$$\frac{\partial S}{\partial \theta} \approx -2C_V \frac{\Delta T}{\pi T_S}, \quad (A4)$$

Substituting this in the right hand side of (37), we get

$$\frac{1}{r} \frac{g}{\gamma C_V} \frac{\partial S}{\partial \theta} \approx -\frac{2}{\pi \gamma} \frac{GM_\odot}{(0.85R_\odot)^3} \frac{\Delta T}{T_S}, \quad (A5)$$

where we have taken r to be given by $0.85R_\odot$ corresponding to the middle of the convection zone and have also used this to calculate g . If we now use the standard values of solar mass and radius, then we get (taking $\gamma = 1.4$)

$$\frac{1}{r} \frac{g}{\gamma C_V} \frac{\partial S}{\partial \theta} \approx 2.8 \times 10^{-7} \frac{\Delta T}{T_S} \text{ s}^{-2}. \quad (A6)$$

Finally, if we equate (A2) and (A6) as required by the thermal wind balance condition (37), we arrive at

$$\frac{\Delta T}{T_S} \approx 5.0 \times 10^{-6}. \quad (A7)$$

If we take T_S equal to the temperature 5800 K at the photospheric surface, then we get a rather low value $\Delta T \approx 2.9 \times 10^{-2}$ K. But, should we use the photospheric temperature for T_S in (A7)? Choudhuri [153] has argued that we should use a temperature deeper in the convection zone for T_S and pointed out that these order of magnitude estimates provide a clue for understanding the origin of the near-surface shear layer seen in Figure 3.

We now make an order of magnitude estimate of the last term in the equation (27) of the meridional circulation, the dissipation term, to show that it would be negligible compared to the centrifugal term. If \mathbf{K} is given by (13), then the last term in (27) is of order

$$\frac{\mu_T |v_m|}{\rho L^3},$$

where we can take the length scale L to be equal to the thickness 2×10^{10} cm of the convection zone. The quantity μ_T/ρ , called the kinematic viscosity, is estimated to be about 10^{12} within the convection zone [124]. Taking $|v_m| \approx 10^3$ cm s $^{-1}$, the value of the last term in (27) comes out to be of order 1.3×10^{-16} s $^{-6}$. Comparing with (A2), we point out that this term clearly cannot balance the centrifugal term, which has to be balanced by the thermal wind term, as can be seen in (27).

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