# Optimal Funded Pension for Consumers with Heterogeneous Self-Control

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August 3, 2022

**Abstract**

This study designs an optimally funded pension scheme for consumers with self-control problems. The model assumes consumers' self-control costs vary, remain private information, and they can borrow against their future pension benefits. Pension plans are offered to consumers to maximize social welfare, including self-control costs. Results demonstrate that, under certain assumptions, an optimal pension scheme is a simplified approach that invests the premiums paid by consumers, delivering returns directly. Under this scheme, consumers with higher self-control costs choose smaller premiums. The higher the premium, the larger the amount that can be borrowed, and the greater the temptation. Smaller premiums help

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*JEL Classification*: D03; D82; H55

*Key Words*: Self-Control; Overconsumption; Funded Pension

# 1 Introduction

This study seeks to present a funded pension scheme that maximizes social welfare, including self-control costs of various magnitudes. When average consumers have money at hand, they are often tempted to spend more than planned. Though they may try to resist temptation, this has a psychological cost. Bucciol (2012) demonstrates this self-control problem using the US Survey of Consumer Finances. The individual magnitude of this cost can vary widely. One way to achieve a desirable consumption plan while reducing self-control cost is to establish commitment. A publicly-funded pension scheme is considered to offer such a commitment device.

One of the objectives of social security is forced savings, and funded pensions are particularly strong in this regard. However, it is established that in standard economic models, funded pensions do not improve social welfare (e.g., Samuelson, 1975). For example, suppose that a consumer decides that the optimal amount of savings is \$1000 per month. When there is no pension policy, consumers privately save. When a pension is introduced and the premium paid is \$500 per month, the consumer pays it, privately saving \$500 because the optimal savings remains \$1000. Since the total amount of savings does not change before and after introducing the pension, the consumer's lifetime utility is unchanged. If the premium is larger than \$1000, the consumer may over-save, which decreases the lifetime utility. Consequently, the pension does not improve social welfare.

One factor to justify funded pensions is temptation, which leads consumers to overconsume (Diamond, 1977). However, most contemporary pension schemes are unsuitable in terms of temptation. Naturally, degrees of temptation differ among consumers, as research suggests (e.g., Ameriks et al., 2007). While some consumers spend much of their income as soon as it is received, regretting their myopic behavior, other consumers save their incomes farsightedly. Since pension schemes affect consumers' behavior, they should be designed from this perspective. Existing pension schemes determine the premium that each consumer pays based on income levels, not the degrees of temptation. This study aims to design and assess a pension scheme to maximize social welfare while recognizing differences in the degree of temptation.

In this model, The government creates a list of pension plans that combine pension premiums and benefits. This list is called the pension schedule. Each consumer selects one plan from the presented pension schedule and pays a premium according to the plan before consuming (facing temptation). In this way, the pension works as forced saving. [1] Consumers are also allowed to borrow against future pension benefits.

The main contribution of this study is the concrete example of an optimal pension scheme under a log-type utility function of consumption. This scheme has some interesting features. First, the optimal schedule offers plans that invest premiums paid by each consumer and return them as is, implying no income redistribution. Second, consumers with higher self-control costs choose smaller premiums. The higher the premium, the larger the amount that can be borrowed, and the greater the temptation. Smaller premiums are chosen to avoid this. If borrowing is not possible, then the optimal schedule includes plans that only collect the savings chosen when there is no temptation or pension plan.

[1] This model focuses on sophisticated consumers; that is, consumers that know that they will face temptation at the point of consumption decision.

Third, the optimal schedule does not depend on the distribution of types or even the types present. This means that the government does not need to know this information, which is a good feature from a practical perspective.

As we need a model that describes decisions with temptation and self-control, we follow Gul and Pesendorfer (2001), henceforth GP. GP considers a two-stage choice problem in which, after choosing a menu, one chooses an option from the menu. The utility function proposed by GP allows us to deal explicitly with restraining costs.

Some literature investigates policies as a means to resist temptation. Gul and Pesendorfer (2004) shows that funded pensions improve social welfare, enabling consumers to avoid temptation with less self-control cost. Also, pay-as-you-go pensions improve social welfare if the temptation is sufficiently large (Kumru and Thanopoulos, 2008). Another way to mitigate self-control cost is to make consumption relatively unattractive. This works because the cost arises from the gap between the attractiveness of normatively desirable alternatives and that of a tempting alternative. Krusell et al. (2010) show that subsidies for savings improve social welfare for this reason. Though these studies are important, the research assumes economies in which the same degrees of temptation apply to all consumers. In contrast, the model of this study analyzes situations in which degrees of temptation differ for various consumers.

Galperti (2015) investigates optimal contracts as a commitment device. The agent is either consistent or inconsistent, which is randomly determined and is the agent's private information. If inconsistent, the agent values consumption less in the second period than in the first period. A characteristic of the model is that there is a preference for flexibility. This is notable, especially when the problem is long-term. Conversely, we assumed that no offer interferes with a commitment device. An advantage of the model in this study is that agents (consumers) are able to assume debt after choosing devices. As discussed in section 5.1, the government can offer a pension scheme to which consumers commit completely when consumers can assume debt. In this way, governments can achieve an optimal outcome with no self-control cost, as some ways to borrow using pension income as collateral remain. For example, Japan's Welfare and Medical Service Agency offers a loan based on pension as security.

As an alternative approach to analyzing myopic behaviors, Laibson (1997) uses a (quasi) hyperbolic discounting model. Some literature on economic policies applies this approach (e.g., Roeder, 2014). The model describes a situation in which preferences are time-inconsistent, although the model cannot explicitly capture self-control cost. The GP model is appropriate to the objective of this study, as it endeavors to address the effects of temptation and self-control separately.

Next, section 2 introduces the notations and assumptions used in the study, investigating the consumption-saving decision of a consumer with a preference for self-control. Section 3 presents identical and two-type economies as benchmarks, revealing the monotonicity of an optimal pension. Section 4 generalizes the results of section 3 to a model of many and continuous types, revealing robust results for generalization. Section 5 presents a discussion regarding the effect of a borrowing constraint, including analyses and an extended model analyzing income differences. Section 6 concludes.

# 2 Model

The difference between standard pension models and this study's model is the inclusion of consumers' preferences between temptation and self-control. At first, the government offers pairs of pension benefits and pension premiums to consumers before the consumption decision at working age. Consumers choose one pair from the set. After payments of premium, consumers decide how much to consume in working age.

### 2.1 Budget Constraint

The population of consumers is standardized to 1. Each consumer is endowed with an identical income of $I \in \mathbb{R}_{++}$. Let $R \in \mathbb{R}_{+}$ and $P \in [0, I]$ indicate a pension payout and a pension premium, respectively. $P$'s upper bound is assumed to rule out situations in which consumers need to borrow to pay premiums. A pair of $R$ and $P$ represents a pension plan. The set of possible pension plans is denoted as $T \equiv \mathbb{R}_{+} \times [0, 1]$. A pension schedule is a set of pension plans $S \subseteq T$.

Consumers choose a pension plan $\tau \in S$ before consumption. According to the plan, a pension premium must be paid to the government. In period 1, consumers decide the amount of consumption $c_1 \in \mathbb{R}_{+}$ and saving $I - P - c_1$ at working age. Consumers can also borrow in period 1, using pension income as collateral. Thus, if $I - P - c_1$ is strictly positive (negative) then it represents the amount of saving (borrowing). Let $\hat{r}$, $\rho \in \mathbb{R}_{++}$ be the interest rate for saving and borrowing, respectively. Thus, a consumer choosing $\tau = (R, P)$ can borrow up to $\frac{R}{1+\rho}$. $\rho > \hat{r} > 0$ is assumed. In period 2, the consumer receives the pension income $R$ and determines consumption in old age $c_2 \in \mathbb{R}_{+}$.

The budget set for the consumer that choose $\tau = (R, P)$ is

$$B(\tau) \equiv \left\{ (c_1, c_2) \in \mathbb{R}_{+}^{2} : \begin{array}{c} c_1 \leqslant I - P + \frac{R}{1+\rho}, \\ I - P - c_1 \geqslant 0 \Rightarrow c_2 \leqslant (1 + r)(I - P - c_1) + R, \\ I - P - c_1 < 0 \Rightarrow c_2 \leqslant (1 + \rho)(I - P - c_1) + R \end{array} \right\}.$$

Let $\mathcal{B}$ denote the set of all possible budget sets.

## 2.2 Preference

Consumers have self-control preferences introduced by GP. Two kinds of utility functions representing preferences of consumption plans are assumed. One is a normative utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ and the other is a temptation utility function $v\colon \mathbb{R}_+ \to \mathbb{R}$. To express temptation, $v$ is specialized as $v \equiv \lambda u$, wherein $\lambda$ is a non-negative real number denoting the strength of temptation. Let $\Lambda \in \mathbb{R}_+$ be a set of all $\lambda$. This form of temptation utility is also used in Gul and Pesendorfer (2004). The value $\lambda$ is assumed to differ across consumers and is private information. For any $\lambda \in \Lambda$, let $n_\lambda$ be the proportion of consumers with $\lambda$, where $0 \leqslant n_\lambda \leqslant 1$ and $\sum_{\lambda\in\Lambda} n_\lambda = 1$.

A few assumptions are imposed regarding the normative utility function $u$.

**Assumption 2.1.**

*(i)* *$u$ is twice continuously differentiable.*

*(ii)* *$u'(c) > 0$ and $u''(c) < 0$.*

*(iii)* $\lim_{c\to 0} u'(c) = \infty$.

For example, this assumption is satisfied if normative utility functions are $u(c) = \log(c)$ or $u(c) = c^\alpha$ with $\alpha \in (0, 1)$. In the latter half of this chapter, $u(c) \equiv \log(c)$ is assumed to derive sharper results. Applying the GP utility, preferences of budget sets are delineated by the function $W\colon \mathscr{B} \times \mathbb{R}_+ \to \mathbb{R}$

$$\begin{aligned} W(\tau;\lambda) &= \max_{(c_1,c_2)\in B(\tau)} \big[u(c_1) + \delta u(c_2) + \lambda u(c_1)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) \\ &= \max_{(c_1,c_2)\in B(\tau)} \big[(1+\lambda)u(c_1) + \delta u(c_2)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1), \end{aligned}$$

where $\delta \in (0, 1)$ is a discount factor.

Assumption 2.1 (ii) implies

$$\max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) = \lambda u\left(I - P + \frac{R}{1+\rho}\right).$$

This is obvious, as $u'(c) > 0$ and the constraint that $c_1 \leqslant I - P + \frac{R}{1+\rho}$. $W(\tau;\lambda)$ is the private welfare of the consumer with $\lambda$.

## 2.3 Consumption

This section considers the consumption problem contained in the first term of $W(\tau;\lambda)$. Let $c_t^\lambda(\tau)$ be the optimal consumption in period $t$ when a consumer with type $\lambda$ has chosen pension plan $\tau \in S$.

The pension plan that the consumer chose affects the optimum amount of savings. A usual method of utility maximization implies

$$\begin{cases} (1+\lambda)u'(c_1(\tau)) = (1+r)\delta u'(c_2(\tau)) & \text{if } c_1(\tau) < I - P \\ (1+r)\delta u'(c_2(\tau)) < (1+\lambda)u'(c_1(\tau)) < (1+\rho)\delta u'(c_2(\tau)) & \text{if } c_1(\tau) = I - P \\ (1+\lambda)u'(c_1(\tau)) = (1+\rho)\delta u'(c_2(\tau)) & \text{if } c_1(\tau) > I - P. \end{cases}$$

## 2.4 Government

Let $\tau_\lambda$ indicate a pension plan consumers with $\lambda \in \Lambda$ intended to choose. The government's challenge is to choose a pension schedule $S \equiv \{\tau_\lambda \in T : \lambda \in \Lambda\} \subseteq T$. The government's objective is to maximize consumers' (expected) aggregated welfare. The constraints are

$$W(\tau_\lambda;\lambda) \geqslant W(\tau';\lambda),\ \forall\lambda \in \Lambda,\ \forall\tau' \in S \tag{IC}$$

$$(1+r)\sum_{\lambda\in\Lambda} n_\lambda P_\lambda \geqslant \sum_{\lambda\in\Lambda} n_\lambda R_\lambda., \tag{FB}$$

where $r$ is an interest rate for pension and $\hat{r} \leqslant r < \rho$ is assumed. The first condition is unnecessary when types are public information. Consumers of type $\lambda \in \Lambda$ will choose a plan $\tau_\lambda$ on their own initiative; in other words, this is the condition incentivizing consumers to report their types truthfully. The second condition requires that $S$ is feasible, i.e., that the pension be paid out within the amount of premiums collected and invested. Individual rationality is not included in the constraints. This is because the pension scheme here is run by the government, which has the power to enforce it.

# 3 Benchmark

## 3.1 Identical type consumers

This section considers a simple case in which all consumers have identical type; that is, $\Lambda = \{\lambda\}$ and it is common knowledge, and condition (IC) can be ignored. The condition (FB) is rewritten as $(1 + r)P_\lambda \geqslant R_\lambda$.

Regarding the individual welfare function $W(\tau; \lambda)$, let $\sigma_\lambda \colon T \to \mathbb{R}$ indicate the marginal rate of substitution of $R$ for $P$ at $\tau \in T$. $\sigma_\lambda$ has different properties in the amount of savings and affects the characterization of consumption as in section 2.3. The following lemma is useful.

**Lemma 3.1.** *An indifference curve corresponding to the level on $R$-$P$ plane satisfies the following properties for any individual welfare level.*

*(i)* $\sigma_\lambda(\tau) \geqslant \frac{1}{1+r}$ *if* $c_1(\tau) < I - P$, *where equality holds if and only if* $\lambda = 0$ *and* $r = \hat{r}$.

*(ii)* $\sigma_\lambda(\tau) = \frac{1}{1+\rho} < \frac{1}{1+r}$ *if* $c_1(\tau) > I - P$.

*(iii) Every indifference curve is continuous and differentiable.*

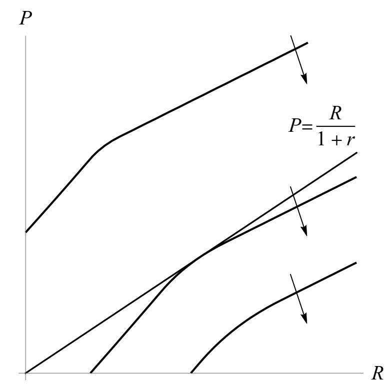


Figure 1: Indifference curves on $R-P$ plane. Arrows indicate a more desirable direction. There exists an indifference curve tangent to the straight line $P = R/(1+r)$.

This lemma states that the indifference curves of $W(\tau;\lambda)$ are smooth. The indifference curve is shown in Figure 1.

Optimal plans are determined wherein the indifference curve is tangent to the line of $P = \frac{R}{1+r}$. The consumer has no saving and borrowing at the point by lemma 3.1 except in the case of $\lambda = 0$ and $r = \hat{r}$.

Considering optimal $\tau_\lambda = (R_\lambda, P_\lambda)$ when $\lambda > 0$, it satisfies

$$\sigma_\lambda(\tau) = \frac{-\delta u'(R_\lambda) + \frac{\lambda}{1+\rho} u'\left(I - P_\lambda + \frac{R_\lambda}{1+\rho}\right)}{-(1+\lambda)u'(I-P_\lambda) + \lambda u'\left(I - P_\lambda + \frac{R_\lambda}{1+\rho}\right)} = \frac{1}{1+r}.$$

or using $R_\lambda = (1+r)P_\lambda$,

$$\begin{aligned}
&\frac{-\delta u'((1+r)P_\lambda) + \frac{\lambda}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P_\lambda\right)}{-(1+\lambda)u'(I-P_\lambda) + \lambda u'\left(I - \frac{\rho - r}{1+\rho} P_\lambda\right)} = \frac{1}{1+r} \\
\Rightarrow \lambda = &\frac{(1+r)\delta u'((1+r)P_\lambda) - u'(I-P_\lambda)}{u'(I-P_\lambda) - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho}\right)}.
\end{aligned}$$

Let $\mathscr{P}$ be a set of $P$ that are optimal for some $\lambda$ and define $Q\colon \mathscr{P} \to \Lambda$ as

$$Q(P) \equiv \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P\right)}. \tag{1}$$

We will next explore some properties of $Q$.

**Theorem 3.1.** *$Q$ satisfies the following properties.*

*(i) $Q$ is strictly decreasing.*

*(ii) $Q(P) \to \infty$ as $P \to 0$.*

*(iii) $Q(P) = 0$ if $P$ satisfies $(1+r)\delta u'((1+r)P) - u'(I-P) = 0$.*

*(iv) $Q$ is onto function.*

*Proof.* Properties (ii) and (iii) follow the assumptions of $u$. Properties (i)-(iii) and continuity of $u$ imply property (iv). Property (i) is shown. Let $P$ and $P'$ be arbitrary premiums that satisfy $P' > P$. We have

$$u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P'\right) > u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P\right)$$

and

$$(1+r)\delta u'((1+r)P') - u'(I-P') < (1+r)\delta u'((1+r)P) - u'(I-P).$$

It follows that

$$\begin{aligned} Q(P) = \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P\right)} &> \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P'\right)} \\ > \frac{(1+r)\delta u'((1+r)P') - u'(I-P')}{u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I - \frac{\rho-r}{1+\rho}P'\right)} &= Q(P'). \end{aligned}$$

Therefore, $Q$ is strictly decreasing. $\square$

Since $Q$ is one-to-one and onto function, the inverse function of $Q$ characterizes the optimal plan for $\lambda \in \Lambda$. Property (i) in theorem 3.1 states that small premium is optimal for a consumer with high self-control cost. This can be understood by focusing on a case wherein $R = (1+r)P$. Then a consumer chooses $c_1 = I - P$ and $c_2 = R$. The consumer's welfare is

$$\begin{aligned} W(\tau;\lambda) &= (1+\lambda)u(I-P) + \delta u((1+r)P) - \lambda u\left(I - \frac{\rho - r}{1+\rho}P\right) \\ &= u(I-P) + \delta u((1+r)P) - \lambda\left[u\left(I - \frac{\rho - r}{1+\rho}P\right) - u(I-P)\right]. \end{aligned}$$

The first and second terms are the utility of consumption and the third term represents self-control cost. The marginal effect of $P$ for $W$ is

$$\begin{aligned} \frac{\partial W(\tau;\lambda)}{\partial P} &= -u'(I-P) + (1+r)\delta u'((1+r)P) \\ &\quad - \lambda\left[-u'\left(I - \frac{\rho - r}{1+\rho}P\right)\left(\frac{\rho - r}{1+\rho}\right) + u'(I-P)\right]. \end{aligned}$$

Optimal $P$ makes this 0; that is, marginal utility equals marginal cost. The second-order derivative is

$$\begin{aligned} \frac{\partial^2 W(\tau;\lambda)}{\partial P^2} =& u''(I-P) + (1+r)^2\delta u''((1+r)P) \\ & - \lambda\left[u''\left(I - \frac{\rho - r}{1+\rho}P\right)\left(\frac{\rho - r}{1+\rho}\right)^2 - u''(I-P)\right]. \end{aligned}$$

The marginal utility decreases as $P$ increases. Conversely, marginal cost increases with $P$ (see the proof of the theorem 3.1). Therefore, for $P$ be optimal for the consumer, the $\lambda$ must be small.

The increase in $P$ has two effects; one reduces disposable income, and the other increases borrowing facility. This is because the optimal plan satisfies $R = (1+r)P$. The second effect raises consumers' temptation to borrow, and

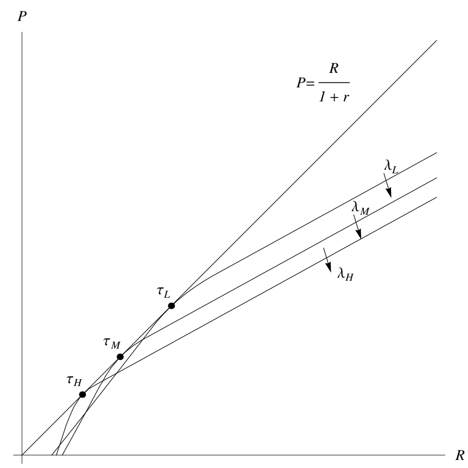


Figure 2: Optimal plans. Consumers with larger $\lambda$ choose the lower-left plan (the plan with smaller premiums and benefits).

the pension payment causes higher self-control costs. This effect is stronger for consumers with large $\lambda$. To avoid a high self-control cost, it is inappropriate to apply a large pension premium for a consumer with a large $\lambda$.

Figure 2 shows indifference curves of three consumers with different $\lambda$ ($\lambda_L < \lambda_M < \lambda_H$). Three points $\tau_L$, $\tau_M$ and $\tau_H$, represent the optimal plans. As $\lambda$ increases, the optimal plan is determined in the lower-left area of the plane.

At the end of this subsection, the condition that $(1+r)\delta u'((1+r)P) - u'(I - P) = 0$ in property (iii) is considered. This is an Euler equation for the simple case in which there is no temptation ($\lambda = 0$) and no pension policy, wherein the interest rate is $r$. This implies that the pension does not affect the saving behavior of consumers with no self-control problem.

This is a well-known result. If the pension earns the same interest as private savings, these two ways of saving are indifferent and the pension does not improve welfare (for example, see Samuelson, 1975). However, when a consumer has $\lambda > 0$, this result does not hold. Even if interest rates are equal, private savings and the pensions are not equivalent.

## 3.2 Two types of consumers

In this section, two types of heterogeneous consumers are considered. First, a benchmark situation in which the government knows consumers' types allows transfers between types. The types are denoted as $\lambda_L$ and $\lambda_H$, where $0 < \lambda_L < \lambda_H$. The population of $\lambda_L$ and $\lambda_H$ are $n_L$ and $n_H$, respectively, where we assume that $n_L > 0$, $n_H > 0$ and $n_L + n_H = 1$. $R_i$ and $P_i$ $(i = L, H)$ indicate income and payment that the government wants type $i$ consumers to choose, respectively.

Note that $n_L P_L + n_H P_H > \frac{n_L R_L + n_H R_H}{1+r}$ is not optimal because the marginal welfare of $R$ is positive. Also, the following lemma holds.

**Lemma 3.2.** *If a pension schedule* $\{\hat{\tau}_L, \hat{\tau}_H\}$ *is feasible, then the schedules* $\{\tau_L, \tau_H\}$ *such that*

$$\left[P_L \geqslant \frac{R_L - \hat{R}_L}{1+r} + \hat{P}_L, \tau_H = \hat{\tau}_H\right] \tag{2}$$

*or*

$$\left[P_H \geqslant \frac{R_H - \hat{R}_H}{1+r} + \hat{P}_H, \tau_L = \hat{\tau}_L\right] \tag{3}$$

*are also feasible.*

For a given plan $\hat{\tau}_\lambda$, we call plans $\tau_\lambda$ that satisfy $P_\lambda \geqslant \frac{R_\lambda - \hat{R}_\lambda}{1+r} + \hat{P}_\lambda$ individually feasible plans. Lemma 3.2 is useful for finding solutions. For example, in Figure 3, $\sigma_\lambda(\tau)$ is lower than $\frac{1}{1+r}$. We ignore incentive compatibility conditions because the government knows the types of consumers. Let us fix a pension plan for another consumer arbitrarily. $\tau_\lambda$ is not optimal because we can find individually feasible and better plans in the gray area. This change is a Pareto improvement because it works without transfer between types.

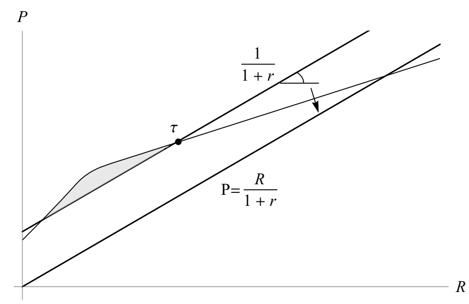


Figure 3: Graphical explanation of improvement. $\tau$ is not optimal for a consumer that is choosing $\tau$ because better plans exist in the gray area. The change to a better plan does not harm other consumers, so it improves social welfare.

The next lemma is important to prove lemma 3.4.

**Lemma 3.3.** *For any $\lambda \in \Lambda \subseteq \mathbb{R}_+$, $\tau$ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ satisfies $R > 0$.*

The following lemma, derived from the previous one, presents the condition for optimality.

**Lemma 3.4.** *$\sigma_L(\tau_L)$ and $\sigma_H(\tau_H)$ are equal to $\frac{1}{1+r}$ for the optimal schedule under complete information.*

By lemma 3.4, if a pension schedule is optimal, it holds that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ for all $\lambda \in \Lambda$. Together with lemma 3.3, if a pension schedule is optimal, we have $R_\lambda > 0$ for all $\lambda \in \Lambda$; that is, if a consumer faces temptation, the funded pension improves social welfare.

Here, we have the following result. Thus, it is not optimal to assign a common pension plan to consumers with different $\lambda$.

**Theorem 3.2.** *If $\lambda \neq \lambda'$, it is not optimal that $\tau_\lambda = \tau_{\lambda'}$.*

*Proof.* Consider an arbitrary $\lambda$, $\lambda'$. By lemma 3.4, if the singleton schedule $\{\tau\}$

is optimal, it is satisfied that

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r}$$

and

$$\frac{-\delta u'(R) + \frac{\lambda'}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P) + \lambda' u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r},$$

then it must be follow that

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{-\delta u'(R) + \frac{\lambda'}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P) + \lambda' u'\left(I - P + \frac{R}{1+\rho}\right)}. \tag{4}$$

This is a necessary condition for $\tau$ to be optimal. Some parts of (4) are abbreviated as follows:

$$\begin{aligned}
A &\equiv -\delta u'(R) \\
B &\equiv \frac{1}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right) \\
C &\equiv u'(I-P) \\
D &\equiv u'(I-P) + u'\left(I - P + \frac{R}{1+\rho}\right).
\end{aligned}$$

Then (4) is rewritten to,

$$\begin{aligned}
&\frac{A+\lambda B}{C+\lambda D} = \frac{A+\lambda' B}{C+\lambda' D} \\
\Leftrightarrow &(A+\lambda B)(C+\lambda' D) = (A+\lambda' B)(C+\lambda D) \\
\Leftrightarrow &(\lambda - \lambda')(AD - BC) = 0.
\end{aligned}$$

By assuming 2.1 (ii), $AD = -\delta u'(R)\left[u'(I-P) + u'\left(I - P + \frac{R}{1+\rho}\right)\right]$ is strictly negative and $BC = \frac{1}{1+\rho}u'\left(\ I - P + \frac{R}{1+\rho}\right)u'(I-P)$ is strictly positive. Therefore, $\lambda = \lambda'$. This implies (4), which is necessary condition for singleton schedule to be optimal, is satisfied only if $\lambda = \lambda'$. $\square$

Next, the optimal schedule in this situation is constructed. To obtain a precise result, henceforth the normative utility function is specialized as $u(c) = \log c$ for all $c \in \mathbb{R}_+$. The following consumption levels are derived for this specialized utility function:

$$c_1(\tau) = \begin{cases} \frac{(1+\lambda)\left((1+\hat{r})(I-P)+R\right)}{(1+\hat{r})(1+\delta+\lambda)} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{(1+\hat{r})\delta} \\ I - P & \text{if } I - \frac{(1+\lambda)R}{(1+\hat{r})\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta} \\ \frac{(1+\lambda)\left((1+\rho)(I-P)+R\right)}{(1+\rho)(1+\delta+\lambda)} & \text{otherwise,} \end{cases}$$

$$c_2(\tau) = \begin{cases} \frac{\delta\left((1+\hat{r})(I-P)+R\right)}{1+\delta+\lambda} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{(1+\hat{r})\delta} \\ R & \text{if } I - \frac{(1+\lambda)R}{(1+\hat{r})\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta} \\ \frac{\delta\left((1+\rho)(I-P)+R\right)}{1+\delta+\lambda} & \text{otherwise.} \end{cases}$$

Next, the locus of points at which $\sigma_\lambda(\tau)$ equals $\frac{1}{1+r}$ are considered. According to lemma 3.1, it is enough to obtain the locus to consider $\tau$ such that $I - \frac{(1+\lambda)R}{(1+\hat{r})\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta}$. Consider an arbitrary $\lambda \in \Lambda$. Then the individual welfare for such $\tau$ is

$$W(\tau;\lambda) = (1+\lambda)\log(I-P) + \delta\log(R) - \lambda\log\left(I - P + \frac{R}{1+\rho}\right).$$

We have

$$\sigma_\lambda(\tau) = \frac{-\frac{\delta}{R} + \frac{\lambda}{(1+\rho)(I-P)+R}}{-\frac{1+\lambda}{I-P} + \frac{(1+\rho)\lambda}{(1+\rho)(I-P)+R}}.$$

The relation between $P$ and $R$ satisfying $\sigma_\lambda(\tau) = \frac{1}{1+r}$ can be written as

$$P = I - K_\lambda R,$$

using the coefficients determined by $\lambda$.

**Proposition 3.1.** *For any $\lambda > 0$ and $\lambda' > 0$ such that $\lambda \neq \lambda'$, if $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$, then $\tau \neq \tau'$.*

*Proof.* Fix $\lambda > 0$ and $\lambda' > 0$ arbitrarily such that $\lambda \neq \lambda'$. Suppose that for $\tau$ and $\tau'$, $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$, then $P = I - K_\lambda R$ and $P' = I - K_{\lambda'} R'$. The relations between $P$ and $R$ and between $P'$ and $R'$ are linear, with the gradients of $-K_\lambda$ and $-K_{\lambda'}$, respectively. If $\lambda > \lambda'$, then $K_\lambda > K_{\lambda'}$. We can check whether $K_\lambda > K_{\lambda'}$, though $\tau = \tau'$ holds only if $R = R' = 0$, $R > 0$ and $R' > 0$ by lemma 3.3. Therefore $\tau \neq \tau'$. $\square$

It is proposed that the point at which the indifference curve of each consumer is constant with the feasibility frontier differs according to type.

By lemma 3.4, the optimal schedule satisfies that $\sigma_L(\tau_L) = \sigma_H(\tau_H) = \frac{1}{1+r}$, so we have

$$P_L = I - K_L R_L$$
$$P_H = I - K_H R_H.$$

The feasibility condition is satisfied with equality for the optimal schedule, as previously noted. Hence, according to the feasibility condition, we have

$$
\begin{aligned}
&(1+r)[n_L P_L + n_H P_H] = n_L R_L + n_H R_H \\
\Leftrightarrow &R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}
$$

Thus the summarized problem is that

$$
\begin{aligned}
\max_{\{\tau_\lambda\}_{\lambda=L,H}} \ & n_L \left[(1+\lambda_L)\log(I-P_L) + \delta \log R_L - \lambda_L \log\left(I - P_L + \frac{R_L}{1+\rho}\right)\right] \\
& + n_H \left[(1+\lambda_H)\log(I-P_H) + \delta \log R_H - \lambda_H \log\left(I - P_H + \frac{R_H}{1+\rho}\right)\right] \\
\text{s.t. } & P_L = I - K_L R_L, \ P_H = I - K_H R_H \\
& R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}
$$

The solution to this problem is as follows:

$$
\begin{aligned}
R_L &= \frac{(1+r)I}{1+(1+r)K_L}, P_L = \frac{I}{1+(1+r)K_L} \\
R_H &= \frac{(1+r)I}{1+(1+r)K_H}, P_H = \frac{I}{1+(1+r)K_H}.
\end{aligned}
$$

This shows that the optimal schedule satisfies $R_\lambda = (1+r)P_\lambda$ for $\lambda \in \Lambda$. In this situation, the incentive compatibility condition is strictly satisfied. Therefore, the following theorem is concluded.

**Theorem 3.3.** *Assume $u(c) = \log c$ and $|\Lambda| = 2$. The optimal pension schedule has the following form: for any $\lambda \in \Lambda$,*

$$
\begin{aligned}
P_\lambda &= \frac{I}{1+(1+r)K_\lambda}, \\
R_\lambda &= (1+r)P_\lambda.
\end{aligned}
$$

The result reveals three points. First, no monetary transfer occurs among types because the optimal plans are balanced for each type. Second, mirroring the identical type case, consumers neither save nor borrow privately, implying that the monetary market is balanced. Third, the optimal schedule does not depend on the distribution of types. Indeed, it does not contain $n_L$ and $n_H$. Consider the mechanism in which the government presents a pension schedule such that

$$\left\{(R, P) \in T\colon P = \frac{R}{1+r}\right\}$$

letting each consumer choose from this schedule. Though formal proof is omitted, this mechanism implements the optimal schedule with a weakly dominant strategy. This follows the convexity of individual welfare and not the existing externality between types. This mechanism is costless, as no information regarding types is required.

# 4 Generalization

In this section, we generalize the number of types. The first case is in which there are finitely many types. The second case is in which types distribute continuously over an interval. As in the previous section, the normative utility function is specialized as $\log(c)$.

## 4.1 Finitely many types

Consider the set of types $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, where $m < \infty$ denotes the population of type $\lambda_i$ by $n_i$, wherein it is assumed that $n_i > 0$ for all $\lambda_i \in \Lambda$

and $\sum_{i=1}^{m} n_i = 1$. The government solves

$$\max_{S} \sum_{i=1}^{m} n_i \left\{ (1+\lambda_i)\log(I-P_i) + \delta \log R_i - \lambda_i \log\left(I - P_i + \frac{R_i}{1+\rho}\right)\right\}$$
$$\text{s.t. } (1+r)\sum_{i=1}^{m} n_i P_i \geqslant \sum_{i=1}^{m} n_i R_i.$$

**Proposition 4.1.** *The optimal schedule for the case in which there are finitely many types consists of plans that satisfy*

$$R_i = \frac{(1+r)I}{(1+r)K_i+1}, P_i = \frac{I}{(1+r)K_i+1}$$

Thus, $R_j = (1+r)P_j$ is satisfied for any type $\lambda_j \in \Lambda$. This echoes the result of the two types case.

## 4.2 Continuous types

Let $\Lambda = [\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda} \geqslant 0$. $\lambda \in \Lambda$ distribute according to the distribution function $F$. $(R(\lambda), P(\lambda))$ denotes the pension plan for a type $\lambda \in \Lambda$ consumer. We assume that $R(\lambda)$ and $P(\lambda)$ are continuous and differentiable.

It follows that $P(\lambda) = I - K(\lambda)R(\lambda)$, $c_1^\lambda = I - P(\lambda)$ and $c_2^\lambda = R(\lambda)$ for each $\lambda$, where $K(\lambda)$ is a constant with the same form as $K_\lambda$ in the previous section. The problem is that

$$\max_{S} \int_{\underline{\lambda}}^{\overline{\lambda}} \left[(1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\right] dF(\lambda)$$
$$\text{s.t. } \int_{\underline{\lambda}}^{\overline{\lambda}} [(1+r)P(\lambda) - R(\lambda)]\, dF(\lambda) = 0$$
$$P(\lambda) = I - K(\lambda)R(\lambda),$$

where we assume that the solution satisfies the feasibility condition with equality, mirroring the previous section. The feasibility constraint is reconsidered, as it includes an integration. A new variable $h$ is defined as

$$h(\lambda) \equiv \int_{\underline{\lambda}}^{\lambda} [(1+r)P(x) - R(x)]\, dF(x).$$

Then, $h$ satisfies the following conditions, conversely implying the feasibility condition.

$$h'(\lambda) = (1+r)P(\lambda) - R(\lambda),$$
$$h(\underline{\lambda}) = 0 \text{ and } h(\overline{\lambda}) = 0.$$

Thus, a problem of another form arises:

$$\max_{S} \int_{\underline{\lambda}}^{\overline{\lambda}} \left[ (1+\lambda)\log(I - P(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left( I - P(\lambda) + \frac{R(\lambda)}{1+\rho} \right) \right] dF(\lambda)$$

s.t. $h'(\lambda) = (1+r)P(\lambda) - R(\lambda)$,

$h(\underline{\lambda}) = 0$ and $h(\overline{\lambda}) = 0$

$P(\lambda) = I - K(\lambda)R(\lambda)$.

The following theorem summarizes the solution to this problem.

**Theorem 4.1.** *Assume* $u(c) = \log(c)$ *and* $\Lambda = [\underline{\lambda}, \overline{\lambda}]$. *The optimal pension schedule has the following form: for any* $\lambda \in \Lambda$,

$$P(\lambda) = \frac{I}{(1+r)K(\lambda) + 1}$$
$$R(\lambda) = (1+r)P(\lambda).$$

The relation that $R(\lambda) = (1+r)P(\lambda)$ is obtained again.

*Proof.* The associated Hamiltonian is

$$
\begin{aligned}
\mathscr{H} \equiv& (1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I-P(\lambda)+\frac{R(\lambda)}{1+\rho}\right) + \mu(\lambda)h'(\lambda) \\
=& (1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I-P(\lambda)+\frac{R(\lambda)}{1+\rho}\right) \\
&+ \mu(\lambda)\left[(1+r)P(\lambda)-R(\lambda)\right],
\end{aligned}
$$

where $\mu$ is the co-state variable. Substituting $P(\lambda) = I - K(\lambda)R(\lambda)$, we have,

$$
\begin{aligned}
\mathscr{H} =& (1+\lambda)\log(K(\lambda)R(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(\left(K(\lambda)+\frac{1}{1+\rho}\right)R(\lambda)\right) \\
&+ \mu(\lambda)\left[(1+r)I - ((1+r)K(\lambda)+1)R(\lambda)\right].
\end{aligned}
$$

By Pontryagin's principle,

$$
\begin{aligned}
\frac{\partial\mathscr{H}}{\partial R(\lambda)} &= \frac{1+\lambda}{R(\lambda)} + \frac{\delta}{R(\lambda)} - \frac{\lambda}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1)R(\lambda) \\
&= \frac{1+\delta}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1 = 0 \qquad (5) \\
\mu'(\lambda) &= -\frac{\partial\mathscr{H}}{\partial h(\lambda)}.
\end{aligned}
$$

Note that $u'(\lambda) = 0$ because $-\partial\mathscr{H}/\partial h(\lambda) = 0$. This implies that $u(\lambda)$ does not depend on $\lambda$. Hence we can rewrite $\mu(\lambda) = \mu$. (5) can be rearranged as

$$
R(\lambda) = \frac{1+\delta}{((1+r)K(\lambda)+1)\mu}.
$$

By the feasibility condition,

$$
\begin{aligned}
&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)(I-K(\lambda)R(\lambda))-R(\lambda)dF(\lambda)\\
=&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)I-((1+r)K(\lambda)+1)\,\frac{1+\delta}{((1+r)K(\lambda)+1)\,\mu}dF(\lambda)\\
=&\left((1+r)I-\frac{1+\delta}{\mu}\right)\int_{\underline{\lambda}}^{\overline{\lambda}} dF(\lambda)=(1+r)I-\frac{1+\delta}{\mu}=0\\
\Leftrightarrow \mu =&\frac{1+\delta}{(1+r)I}
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
R(\lambda)&=\frac{1+\delta}{((1+r)K(\lambda)+1)\mu}=\frac{(1+r)I}{(1+r)K(\lambda)+1},\\
P(\lambda)&=I-K(\lambda)R(\lambda)=\frac{I}{(1+r)K(\lambda)+1}.
\end{aligned}
$$

□

# 5 Discussion

## 5.1 The effect of a borrowing constraint

Thus far, we have assumed that consumers can borrow money in the first period. In this section, we consider a case wherein consumers cannot borrow, that is, $s \geqslant 0$[2]. The presence or absence of borrowing constraints has a great deal to do with the function of the pension as a commitment device. Let $r$ equal $\hat{r}$ here to focus on the benefit of commitment.

[2] There may be various strengths of the constraint, but here only strongest borrowing constraint is considered.

In this case, the budget constraint is

$$B(\tau) = \{(c_1, c_2) \in \mathbb{R}^2_+ \colon c_1 + s \leqslant I - P, c_2 \leqslant (1+\hat{r})(I - P - c_1) + R\}.$$

We specialize a normative utility function as $u(c) = \log(c)$. Then the consumption in period 1 is

$$c_1(\tau) = \begin{cases} \frac{(1+\lambda)[(1+\hat{r})(I-P)+R]}{(1+\hat{r})(1+\delta+\lambda)} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{\delta(1+\hat{r})} \\ I - P & \text{if } I - \frac{(1+\lambda)R}{\delta(1+\hat{r})} \leqslant P \leqslant I. \end{cases}$$

Similar to section 3.1, identical type is assumed, then $R = (1+r)P$ follows.

An optimal plan is

$$P_\lambda = \begin{cases} \text{any number } P \in [0, \frac{\delta I}{1+\delta}] & \text{if } \lambda = 0 \\ \frac{\delta I}{1+\delta} & \text{if } \lambda > 0 \end{cases}$$

$$R_\lambda = (1+r)P_\lambda, \ \forall \lambda \geqslant 0.$$

Note that $P$ does not depend on the type if $\lambda > 0$. This optimal $P$ equals the optimal amount of savings when neither temptation nor a pension scheme exists. The government reclaims it to constrain consumers who cannot save money independently due to overspending. The pension premium decreases the self-control cost, as the pension is not collateral.

## 5.2 CRRA utility function

We refer to the result wherein the utility function is CRRA form. The utility function is

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

where $\gamma > 0$. Since this function satisfies assumption 2.1, the analyses in section 3.1 and complete information results in section 3.2 continue to hold, however, the solution for the problem of complete information does not satisfy incentive compatibility conditions. We will present a numerical example because it is difficult to derive a closed-form solution for this setting.

Let $n_L = n_H = 0.5$, $\lambda_L = 1$, $\lambda_H = 3$, $\delta = 0.99$, $I = 10$, $r = \hat{r} = 0.1$ and $\rho = 0.5$. Note that interest rates differ from the previous section to obtain a clear result. Additionally, $\gamma = 2$ , following Kumru and Thanopoulos (2008). The optimal schedule is $R_L = 4.38817$, $P_L = 4.05319$, $R_H = 3.56167$ and $P_H = 3.17394$. As shown in Figure 4, incentive compatibility for $\lambda_L$ binds. Therefore, theorem 3.3 is not robust for the utility function other than $u(c) = \log(c)$; however, welfare for all consumers improves without a pension scheme. In this example, welfare for low type consumers changes from 1.53167 to 1.54391 and from 1.42053 to 1.45638 for high type consumers following the introduction of the optimal pension scheme.

## 5.3 Income differences

This section considers a situation wherein consumers' incomes vary. Income is one of the elements in $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ and the degree of temptation derives from $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, where $0 < I_1 < I_2 < \cdots < I_m < \infty$ and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \infty$. Consumers are characterized by the pair

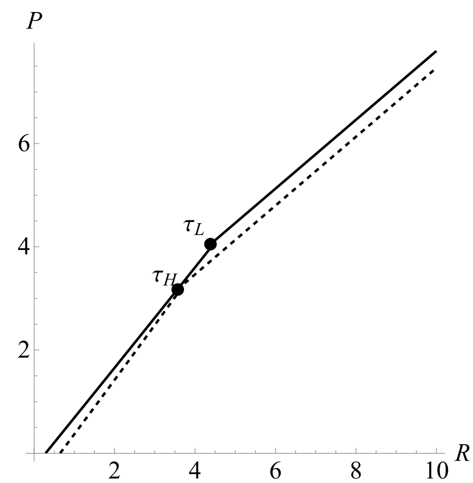


Figure 4: Optimal schedule for CRRA utility function. The dashed line is an indifference curve of $\lambda_H$ consumer. Unlike the case of $u(c) = \log(c)$, the incentive compatibility condition is an equality sign. First-best cannot be achieved under asymmetric information.

$(I_s, \lambda_t) \in \Theta \subseteq \mathcal{I} \times \Lambda$. $n_{st}$ represents the proportion of consumers with $(I_s, \lambda_t) \in \Theta$. A pension plan for consumers with $(I_s, \lambda_t)$ is the pair $(R_{st}, P_{st})$. It is assumed that $R_{st}$ and $P_{st}$ are not negative and consumers can pay the premium for their pension plan, that is, $P_{st} \leqslant I_s$.

Unlike the degree of temptation, the government can observe the income of consumers; however, the problem of complete information is first considered, as in the previous section. Then, lemma 3.4 is used again, as a result does not depend on income, so the focus is pension plans that satisfy $P_{st} = I_s - K_t R_{st}$. The constraint that $0 \leqslant P_{st} \leqslant I_s$ is rewritten as $0 \leqslant R_{st} \leqslant I_s/K_t$. The feasibility condition can also be rearranged as

$$
\begin{aligned}
&(1+r)\sum_{(s,t)\in\Theta} n_{st}P_{st} \geqslant \sum_{(s,t)\in\Theta} n_{st}R_{st}\\
\iff &(1+r)\sum_{(s,t)\in\Theta} n_{st}(I_s - K_tR_{st}) \geqslant \sum_{(s,t)\in\Theta} n_{st}R_{st}\\
\iff &\sum_{(s,t)\in\Theta} n_{st}\left\{(1+r)I_s - (1+(1+r)K_t)R_{st}\right\} \geqslant 0.
\end{aligned}
$$

The government's problem is

$$
\begin{aligned}
\max_{S} \quad & \sum_{(s,t)\in\Theta} n_{st}\left\{(1+\lambda_t)\log K_tR_{st} + \delta\log R_{st} - \lambda_t\log\left(K_t + \frac{1}{1+\rho}\right)R_{st}\right\}\\
\text{s.t} \quad & \sum_{(s,t)\in\Theta} n_{st}\left\{(1+r)I_s - (1+(1+r)K_t)R_{st}\right\} \geqslant 0\\
& 0 \leqslant R_{st} \leqslant I_s/K_t,\ \forall (s,t)\in\Theta.
\end{aligned}
$$

It can be concluded that the solution derived not only applies to the problem with complete information but also to the problem with incomplete information.

**Theorem 5.1.** *In the unique optimal schedule, the plan of consumer with* $(I_s, \lambda_t)$ *is*

$$
\begin{aligned}
R_{st} &= \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}}I_{\tilde{s}}}{1+(1+r)K_t}\\
P_{st} &= \frac{I_s + (1+r)K_t\sum_{(s',t')\neq(s,t)} n_{s't'}(I_s - I_{s'})}{1+(1+r)K_t}.
\end{aligned}
$$

Let us check consumers' incentive to report degrees of temptation untruthfully. As their optimal plans derived are determined on the locus of $\sigma_{st}(\tau_{st}) = 1/(1+r)$, one sufficient condition for no deviation is that plans for two arbitrary consumers with the same income are on the same line with the gradient of $1/(1+r)$.

**Proposition 5.1.** *For any* $(s,t)$ *and* $(s,t')$*, it holds that*

$$\frac{P_{st}-P_{st'}}{R_{st}-R_{st'}}=\frac{1}{1+r}.$$

*Proof.* Calculating $R_{st}-R_{st'}$ and $P_{st}-P_{st'}$ in practice,

$$\begin{aligned}
R_{st}-R_{st'} &= \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t} - \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_{t'}} \\
&= \frac{(1+r)(K_{t'}-K_t)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{[1+(1+r)K_t][1+(1+r)K_{t'}]}, \\
P_{st}-P_{st'} &= (I_s-K_tR_{st})-(I_s-K_t'R_{st'}) \\
&= \frac{K_{t'}((1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}})}{1+(1+r)K_{t'}} - \frac{K_t((1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}})}{1+(1+r)K_t} \\
&= \frac{K_{t'}[1+(1+r)K_t]-K_t[1+(1+r)K_{t'}]}{[1+(1+r)K_t][1+(1+r)K_{t'}]} \sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}} \\
&= \frac{(K_{t'}-K_t)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{[1+(1+r)K_t][1+(1+r)K_{t'}]}.
\end{aligned}$$

Hence, the intended result is acquired. □

Figure 5 presents an example of the optimal schedule when $\mathcal{I}=\{I_1,I_2\}$ and $\Lambda=\{\lambda_1,\lambda_2\}$. Four points are the elements of an optimal schedule. Consumers can only lie about their degree of temptation; thus, the deviation is between $(I_1,\lambda_1)$ and $(I_1,\lambda_2)$ and between $(I_2,\lambda_1)$ and $(I_2,\lambda_2)$. This deviation causes a loss.

The optimal schedule above has some prominent characteristics. First, it presents a generalization of the result in the previous section with identical income,which is revealed by assuming $I_1=I_2=\cdots=I_m$ in the plan for $(s,t)$. Second, pension return is equivalent for two consumers if, and only if, an equivalent degree of temptation exists. Determining the amount of return depends only on the degree of temptation and not income. Third, in contrast, payment

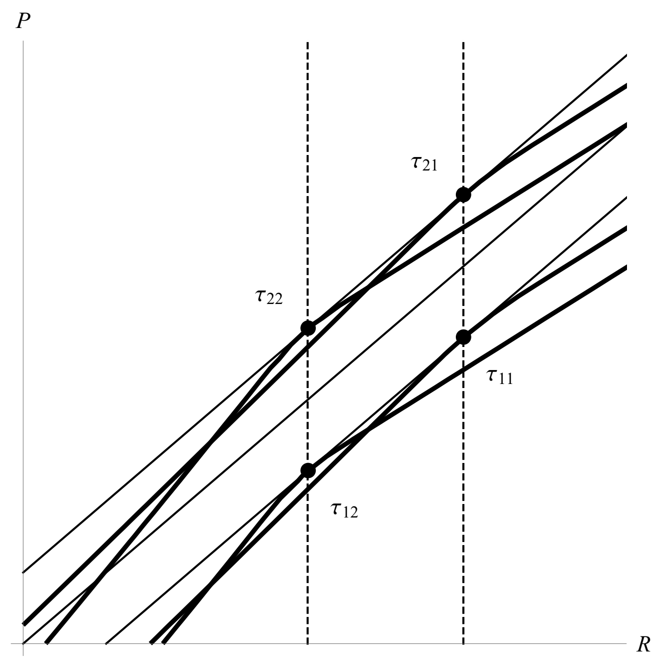


Figure 5: An optimal schedule under income differences. Income transfers between consumers with different incomes occur, but income transfers between consumers with different levels of self-control do not occur.

for the pension fund depends on both of income and the degree of temptation. Consumers with relatively higher incomes pay more because the summation in the numerator of $P_{st}$ becomes larger when $I_s$ is high, implying a monetary transfer from high-income consumers to low-income consumers through the differences in payments.

# 6 Conclusion

This study considers optimally-funded pensions for consumers facing a conflict between temptation and self-control. As funded pensions tighten consumers' budgets, they can serve as commitment devices to avoid overconsumption. We found that lower pension premiums and payouts are preferable for higher temptation consumers. An increase in the premium leads to higher pension income, augmenting the possibility of debt. The effect is greater than the benefit of tightening the budget set. As a main result, we considered an optimal pension schedule with two or more types in the economy, presenting the necessary conditions for an optimal schedule. Also, the money transfer will not occur among consumers with differing levels of temptation if the utility of consumption is $\log(c)$. The result that the optimal schedule does not depend on the distribution of types suggests that the government can make pension policy easier.

The optimal pension scheme is one in which income redistribution occurs. Future research will extend the model to overlapping generations to investigate the relation between self-control cost and intergenerational redistribution to assess the long-term effect.

# APPENDIX

## A Proofs

### A.1 Proof of lemma 3.1

*Proof of property (i).* Assume that $c_1(\tau) < I - P$, resulting in

$$(1+\lambda)u'(c_1(\tau)) = (1+r)\delta u'(c_2(\tau)),$$

$$c_2(\tau) = (1+r)(I - P - c_1(\tau)) + R,$$
$$\frac{\partial c_2(\tau)}{\partial R} = -(1+r)\frac{\partial c_1(\tau)}{\partial R} + 1.$$

The first and second equations follow from the condition for optimal consumption. The marginal welfare of $R$ is

$$\begin{aligned}\frac{\partial W(\tau;\lambda)}{\partial R} &= (1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial R} + \delta u'\big(c_2(\tau)\big)\big)\frac{\partial c_2(\tau)}{\partial R} - \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right) \\ &= \Big\{(1+\lambda)u'(c_1(\tau)) - (1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial R} \\ &+ \delta u'(c_2(\tau)) - \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right) \\ &= \delta u'(c_2(\tau)) - \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right) \\ &= \frac{1+\lambda}{1+r}u'(c_1(\tau)) - \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right).\end{aligned}$$

This is strictly positive because $c_1(\tau) \leqslant I-P+\frac{R}{1+\rho}$ and $u''(c) < 0$; hence $u'(c_1(\tau)) \geqslant u'\left(I-P+\frac{R}{1+\rho}\right)$. By $\rho > r$ and $\lambda \geqslant 0$, we have $\frac{\partial W(\tau;\lambda)}{\partial R} > 0$.

Next, the marginal welfare of $P$ is

$$\begin{aligned}\frac{\partial W(\tau;\lambda)}{\partial P} &= (1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial P} - \delta u'(c_2(\tau))(1+r)\left(1+\frac{\partial c_1(\tau)}{\partial P}\right) \\ &+ \lambda u'\left(I-P+\frac{R}{1+\rho}\right) \\ &= \Big\{(1+\lambda)u'(c_1(\tau)) - (1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial P} \\ &- (1+r)\delta u'(c_2(\tau)) + \lambda u'\left(I-P+\frac{R}{1+\rho}\right) \\ &= -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I-P+\frac{R}{1+\rho}\right).\end{aligned}$$

In the same manner, the equalities follow, and the last line is strictly negative.

Thus, $\sigma_\lambda(\tau)$ is

$$\sigma_\lambda(\tau) = -\frac{\frac{\partial W(\tau;\lambda)}{\partial R}}{\frac{\partial W(\tau;\lambda)}{\partial P}} = \frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I-P+\frac{R}{1+\rho}\right)}.$$

We show that $\sigma_\lambda(\tau) \geqslant \frac{1}{1+r}$. Since $\rho > r$,

$$\begin{aligned}
&-(1+\lambda)u'(c_1(\tau)) + \frac{1+r}{1+\rho}\lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\
&\qquad \leqslant -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right),
\end{aligned}$$

where equality holds if $\lambda = 0$. Furthermore, because $u'\left(I - P + \frac{R}{1+\rho}\right) > 0$ it holds only if $\lambda = 0$. Using the first-order condition,

$$\begin{aligned}
&-(1+r)\delta u'(c_2(\tau)) + \frac{1+r}{1+\rho}\lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\
&\qquad \leqslant -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right).
\end{aligned}$$

Note that the right-hand side is strictly negative; thus, this inequality is rewritten as

$$\begin{aligned}
&\frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \geqslant \frac{1}{1+r} \geqslant \frac{1}{1+r} \\
\Leftrightarrow &\sigma_\lambda(\tau) \geqslant \frac{1}{1+r}.
\end{aligned}$$

□

*Proof of Property (ii).* $\sigma_\lambda(\tau)$ for $\tau$ such that $c_1(\tau) > I - P$ is

$$\begin{aligned}
\sigma_\lambda(\tau) &= \frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
&= \frac{-(1+\rho)\delta u'(c_2(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\rho)\left\{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)\right\}}.
\end{aligned}$$

By the optimality of the consumption,

$$(1+\rho)\delta u'(c_2(\tau)) = (1+\lambda)u'(c_1(\tau)).$$

Therefore, $\sigma_\lambda(\tau) = \frac{1}{1+\rho} < \frac{1}{1+r}$. □

*Proof of Property (iii).* To prove this property, consumption function $c$ is continuous. If so, the value of $u(c)$ is continuous and $u$ is differentiable by assumption 2.1 (i). Then $W$ is also continuous and differentiable, as it is defined by using $u$.

$c$ depends on a tuple of parameters $a = (I, r, \rho, P, R)$. Note that $P$ and $R$ are parameters when a consumer chooses consumption, and are endogenous variables when a pension plan is chosen.

**Lemma A.1.** *Budget correspondence is hemicontinuous.*

**Lemma A.2.** *Budget correspondence is upper hemicontinuous.*

*Proof.* Fix a tuple of parameters $a \in A$ and choose any sequence $(a^i)_{i=1}^{\infty} \subset A$ that converges to $a$. Assume that $c^i \in B(a^i)$ for all $i$ and that $c^i \to c$ for some $c \in \mathbb{R}^2_+$. We show that $c \in B(a)$. Suppose that $c \notin B(a)$. This implies that either $c_2 > (1+r)(I-P-c_1)+R$ or $c_2 > (1+\rho)(I-P-c_1)+R$. This proof considers the former case. Since $c^i \to c$ and $a^i \to a$, $k \in \mathbb{N}$ exists, satisfying $c_2^k > (1+r^k)(I^k - P^k - c_1^k) + R^k$. Hence $c^k \notin B(a^k)$, contradicting an assumption that $c^i \in B(a^i)$ for all $i$. □

**Lemma A.3.** *Budget correspondence is lower hemicontinuous.*

*Proof.* Let $c^0 \in B(a^0)$. Assume that $c_1^0 \leqslant I^0 - P^0$. The following proof works similarly by substituting $r^0$ to $\rho^0$ when $c_1^0 > I^0 - P^0$. A sequence $\{c^i\}_{i=1}^{\infty}$ that converges to $c^0$ and satisfies $c^i \in B(a^i)$ is constructed. The sign of the amount of savings affects the interest rate. In this proof, $\tilde{r}^i$ denotes an applied interest rate, that is, $\tilde{r}^i = r^i$ if $c^i \leqslant I^i - P^i$ and $\tilde{r}^i = \rho^i$ if $c^i > I^i - P^i$. Only cases in which $(1+r^0)(I^0 - P^0) + R^0 > 0$ require consideration. Otherwise, $c^0 \in B(a^0)$ implies that $c^0 = (0,0)$, then the sequence $c^i = (0,0)$ for all $i$ will work. Therefore, we assume $(1+r^0)(I^0 - P^0) + R^0 > 0$. For any $i = 1, 2, \ldots$, let $c^i$ be $c^i = \alpha^i c^0$ where

$$\alpha^i \equiv \frac{(1+\tilde{r}^i)(I^i - P^i) + R^i}{|(\tilde{r}^i - r^0)c_1^0| + (1+r^0)(I^0 - P^0) + R^0}.$$

Note that $\alpha^i \geqslant 0$ and $\alpha^i \to 1$. Hence we have $c^i \in \mathbb{R}^2_+$ and $c^i \to c^0$. Moreover,

$$\begin{aligned}
&(1+\tilde{r}^i)c^i + c_2^i \\
&= (1+r^i)\alpha^i c_1^0 + \alpha^i c_2^0 = \alpha^i \left[((1+\tilde{r}^i) - (1+r^0))c_1^0 + (1+r^0)c_1^0 + c_2^0\right] \\
&\leqslant \alpha^i \left[(\tilde{r}^i - r^0)c_1^0 + (1+r^0)(I^0 - P^0) + R^0\right] \\
&\leqslant \alpha^i \left[|(\tilde{r}^i - r^0)c_1^0| + (1+r^0)(I^0 - P^0) + R^0\right] = (1+\tilde{r}^i)(I^i - P^i) + R^i.
\end{aligned}$$

Therefore $c^i \in B(a^i)$ for all $i = 1, 2, \ldots$. □

In lemma A.2 and A.3, a budget set is hemicontinuous. We have the following lemma.

**Lemma A.4.** *Consumption function $c$ is continuous.*

*Proof.* A budget set is non-empty because $(0,0) \in B(a)$ for all $a$. $u$ is continuous in the assumption. A budget set can be shown as bounded and closed in the usual way. Thus Berge's maximum theorem conditions are satisfied by lemma A.1, so $c(a)$ is continuous. □

Therefore $W$ is continuous and differentiable. So the indifference curve of $W$ is also continuous and differentiable. □

## A.2 Proof of lemma 3.2

*Proof.* Without loss of generality, the schedule that satisfies (2) is feasible. Suppose that $\{\hat{\tau}_L, \hat{\tau}_H\}$ is feasible; that is,

$$\begin{aligned}
&(1+r)(n_L\hat{P}_L + n_H\hat{P}_H) \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\
\Leftrightarrow &(1+r)n_L\hat{P}_L \geqslant -(1+r)n_H\hat{P}_H + n_L\hat{R}_L + n_H\hat{R}_H.
\end{aligned} \tag{6}$$

Choosing a schedule such that $P_L \geqslant \frac{R_L-\hat{R}_L}{1+r} + \hat{P}_L, \tau_H = \hat{\tau}_H$ arbitrarily, it follows that

$$\begin{aligned}
&P_L \geqslant \frac{R_L - \hat{R}_L}{1+r} + \hat{P}_L \\
\Leftrightarrow &(1+r)n_L P_L - n_L(R_L - \hat{R}_L) \geqslant (1+r)n_L\hat{P}_L.
\end{aligned}$$

Together with (6), we have

$$\begin{aligned}
&(1+r)n_L P_L - n_L(R_L - \hat{R}_L) + (1+r)n_H\hat{P}_H \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\
\Leftrightarrow &(1+r)(n_L P_L + n_H\hat{P}_H) \geqslant n_L R_L + n_H\hat{R}_H.
\end{aligned}$$

By $\tau_H = \hat{\tau}_H$, it follows that

$$(1+r)(n_L P_L + n_H P_H) \geqslant n_L R_L + n_H R_H.$$

Thus, the schedule $\{\tau_L, \tau_H\}$ is feasible. The feasibility of the schedule that satisfies (3) similarly can be shown. □

## A.3 Proof of lemma 3.3

*Proof.* Fix arbitrary $\lambda \in \Lambda \subseteq \mathbb{R}_+$ and choose $\tau$ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$. Note that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ implies $\tau \in T_b$. $\sigma_\lambda(\tau)$ is equal to

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}.$$

Then, by assumption 2.1 (iii),

$$\begin{aligned}
&\lim_{R\to 0} \frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
=&\frac{-\lim_{R\to 0} \delta u'(R) + \frac{\lambda}{1+\rho} u'(I-P)}{-u'(I-P)} \\
=&\frac{-\infty}{-u'(I-P)} = \infty.
\end{aligned}$$

Because $\sigma_\lambda(\tau)$ is decreasing in $R$, $R > 0$ for $\tau$ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$. □

## A.4 Proof of lemma 3.4

*Proof.* Suppose that either $\sigma_L(\tau_L)$ or $\sigma_H(\tau_H)$ is not equal to $\frac{1}{1+r}$. Without loss of generality, suppose that $\sigma_L(\lambda_L) \neq \frac{1}{1+r}$.

(i) $\sigma_L(\tau_L) > \frac{1}{1+r}$

For arbitrary $\varepsilon_1 > 0$, let $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For $\varepsilon_1$ and $\varepsilon_2$, let $R' \equiv R_L + \varepsilon_1$ and $P' \equiv P_L + \varepsilon_2$. It follows that

$$\begin{aligned}
&\varepsilon_2 = \frac{\varepsilon_1}{1+r} \\
\Leftrightarrow &P_L + \varepsilon_2 = \frac{R_L + \varepsilon_1 - R_L}{1+r} + P_L \\
\Leftrightarrow &P' = \frac{R' - R_L}{1+r} + P_L,
\end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by lemma 3.2. The line represented by this formula, which passes through $(R_L, P_L)$, has the gradient of $\frac{1}{1+r}$ on $R$-$P$ plane and a marginal rate of substitution at $\tau_L$ is $\sigma_L(\tau_L) > \frac{1}{1+r}$. Thus, a sufficiently small $\varepsilon_1$ and $\varepsilon_2$, $\tau'$ improves individual welfare. As another schedule is feasible and better, $\{\tau_L, \tau_H\}$ is not optimal.

(ii) $\sigma_L(\tau_L) < \frac{1}{1+r}$

Consider $\varepsilon_1$, $\varepsilon_2 > 0$, which satisfies that $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For such $\varepsilon_1$ and $\varepsilon_2$, let $R' \equiv R_L - \varepsilon_1$ and $P' \equiv P_L - \varepsilon_2$. Here, for $\tau_L$ such that $\sigma_L(\tau_L) < \frac{1}{1+r}$, it is satisfied that $R > 0$ by the lemma 3.3. Then, similar to the previous case, it is satisfied that

$$\begin{aligned}
&\varepsilon_2 = \frac{\varepsilon_1}{1+r} \\
\Leftrightarrow &P_L - \varepsilon_2 = \frac{R_L - \varepsilon_1 - R_L}{1+r} + P_L \\
\Leftrightarrow &P' = \frac{R' - R_L}{1+r} + P_L,
\end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by lemma 3.2. This line passes through $(R_L, P_L)$ with a gradient of $\frac{1}{1+r}$. Furthermore, the marginal rate of substitution at $\tau_L$ is $\sigma_L(\tau_L) < \frac{1}{1+r}$. Thus, for sufficiently small $\varepsilon_1$ and $\varepsilon_2$, $\tau'$ improves individual welfare. As another schedule is feasible and better, $\{\tau_L, \tau_H\}$ is not optimal. $\square$

## A.5 Proof of proposition 4.1

*Proof.* Lemma 3.4 also applies in this model. Substituting $P_i = I - K_i R_i$ in the problem yields

$$\max_{S} \sum_{i=1}^{m} n_i \left\{ (1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\}$$
$$\text{s.t. } (1+r)\sum_{i=1}^{m} n_i (I - K_i R_i) \geqslant \sum_{i=1}^{m} n_i R_i.$$

The associated Lagrangian[3] is

$$\begin{aligned}\mathscr{L} \equiv& \mu_0 \left[\sum_{i=1}^{m} n_i \left\{ (1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\}\right] \\ &+ \mu_1 \left[(1+r)\sum_{i=1}^{m} n_i (I - K_i R_i) - \sum_{i=1}^{m} n_i R_i\right].\end{aligned}$$

The necessary condition is

$$\frac{\partial \mathscr{L}}{\partial R_j} = \mu_0 \left[n_j \left\{\frac{1+\lambda_i}{R_j} + \frac{\delta}{R_j} - \frac{\lambda_j}{R_j}\right\}\right] + \mu_1[-(1+r)n_j K_j - n_j] = 0, \ \forall \lambda \in \Lambda$$

$$(\mu_0, \mu_1) \geqslant 0$$

$$(\mu_0, \mu_1) \neq 0$$

$$\mu_1 \left[(1+r)\sum_{i=1}^{m} n_i (I - K_i R_i) - \sum_{i=1}^{m} n_i R_i\right] = 0.$$

Note that $[-(1+r)n_j K_j - n_j] < 0$. If $\mu_0 = 0$, the first-order condition is satisfied only if $\mu_1 = 0$. This violates the non-zero condition of Lagrange multipliers. Thus $\mu_0 > 0$ and we can be standardized this as $\mu_0 = 1$. First-order conditions are rearranged as

$$n_j \frac{1+\delta}{R_j} - \mu_1[(1+r)n_j K_j - n_j] = 0, \ \forall \lambda \in \Lambda. \tag{7}$$

[3]Fritz-John (FJ) conditions are used to solve this problem.

$\mu_1$ is not zero because the first term is strictly positive; thus, the feasibility condition is satisfied with equality. By (7), we have

$$\mu_1 = \frac{1}{(1+r)K_j + 1}\left(\frac{1+\delta}{R_j}\right).$$

For any $\lambda_j \in \Lambda$ and $\lambda_k \in \Lambda$, it follows that

$$\begin{aligned} &\frac{1}{(1+r)K_j + 1}\left(\frac{1+\delta}{R_j}\right) = \frac{1}{(1+r)K_k + 1}\left(\frac{1+\delta}{R_k}\right) \\ \Leftrightarrow &[(1+r)K_j + 1]R_j = [(1+r)K_k + 1]R_k. \end{aligned} \tag{8}$$

By the feasibility condition with equality, we have

$$\begin{aligned} &(1+r)\sum_{i=1}^{m} n_i(I - K_iR_i) - \sum_{i=1}^{m} n_iR_i = 0 \\ \Leftrightarrow &\sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = (1+r)I\sum_{i=1}^{m} n_i = (1+r)I. \end{aligned}$$

Fixing $\lambda_j \in \Lambda$ arbitrarily and using (8), it follows that

$$\begin{aligned} (1+r)I &= \sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = ((1+r)K_j + 1)R_j\sum_{i=1}^{m} n_i \\ &= ((1+r)K_j + 1)R_j \\ \Leftrightarrow R_j &= \frac{(1+r)I}{(1+r)K_j + 1}. \end{aligned}$$

Substituting this to $P_j = I - K_jR_j$, we have

$$P_j = I - K_j\frac{(1+r)I}{(1+r)K_j + 1} = \frac{((1+r)K_j + 1)I - (1+r)K_jI}{(1+r)K_j + 1} = \frac{I}{(1+r)K_j + 1}.$$

□

## A.6 Proof of theorem 5.1

*Proof.* The solution satisfies the following property.

**Lemma A.5.** *If a pension schedule is optimal, it holds that* $R_{st} > 0$ *for all* $s$ *and* $t$.

*Proof.* At first, at least one type exists wherein pension return is strictly positive. Suppose a consumer with $(I_s, \lambda_t)$ whose pension plan is $R_{st} = 0$ exists. Then the welfare of the consumer is

$$(1+\lambda_t)\log(I_s - P_{st}) + \delta \log R_{st} - \lambda_t \log\left(I_s - P_{st} + \frac{R_{st}}{1+\rho}\right)$$
$$= (1+\lambda_t)\log(I_s - P_{st}) + \delta\log 0 - \lambda_t \log(I_s - P_{st}) = \log(I_s - P_{st}) + \delta \log 0 = -\infty.$$

Social welfare in this case is $-\infty$ because individual welfare has upper bound. In contrast, let $(R'_{st}, P'_{st}) = (\varepsilon, \varepsilon/(1+r))$ for all $s$ and $t$ with $\varepsilon > 0$ that satisfies $\varepsilon/(1+r) < I_1$. A pension schedule composed of this plan is feasible, holding that

$$\sum_{(s,t)\in\Theta} n_{st}\left\{(1+r)P'_{st} - R_{st}\right\} = \sum_{(s,t)\in\Theta} n_{st}\left\{\varepsilon - \varepsilon\right\} = 0.$$

The social welfare of this schedule is

$$\sum_{(s,t)\in\Theta} n_{st}\left\{(1+\lambda_t)\log\left(I_s - \frac{\varepsilon}{1+r}\right) + \delta\log\varepsilon - \lambda_t\log\left(I_s - \frac{\varepsilon}{1+r} + \frac{\varepsilon}{1+\rho}\right)\right\}$$

This is bounded below because the antilogarithms of the first and the second terms are greater than 0 and the third term is bounded above. Therefore, it is not optimal to assign $R_{st} = 0$ for all consumers.

Next suppose that $R_{st} = 0$ for some types $(s,t)$ in the optimal schedule. Let $Z$ be the set of these types. As noted, at least one type of consumer receives positive return. Let $(\tilde{s}, \tilde{t})$ represent this type. As presented, social welfare is $-\infty$ when there are consumers whose pension returns are 0. Then, we can achieve larger social welfare by lowering the amount of return for $(\tilde{s}, \tilde{t})$ slightly and transferring it to $(s, t) \in Z$.

As in the discussion above, the social welfare of the revised schedule is strictly larger than $-\infty$. Therefore, the schedule that assigns zero return to more than one consumer is not optimal. $\square$

The associated Lagrangian for the government's problem is

$$\begin{aligned}
\mathcal{L} = {} & \mu_0 \sum_{(s,t)\in\Theta} n_{st} \left[ (1+\lambda_t) \log K_t R_{st} + \delta \log R_{st} - \lambda_t \log \left( K_t + \frac{1}{1+\rho} \right) R_{st} \right] \\
& + \mu_1 \sum_{(s,t)\in\Theta} n_{st} \left[ (1+r) I_s - (1 + (1+r) K_t) R_{st} \right] \\
& + \sum_{(s,t)\in\Theta} \mu_{2st} R_{st} + \sum_{(s,t)\in\Theta} \mu_{3st} \left( \frac{I_s}{K_t} - R_{st} \right),
\end{aligned}$$

where $\mu_0$, $\mu_1$, $\mu_{2st}$ and $\mu_{3st}$ are Lagrange multipliers. Note that $\mu_{2st} = 0$ for all $(s, t)$ for the solution because of complementary slackness conditions for the constraint $R_{st} \geqslant 0$ and lemma A.5. Then the first-order conditions for $R_{st}$ are

$$\frac{\partial \mathcal{L}}{\partial R_{st}} = \mu_0 n_{st} \frac{1+\delta}{R_{st}} - \mu_1 n_{st} (1 + (1+r) K_t) - \mu_{3st} = 0. \tag{9}$$

$\mu_0$ is strictly positive for the solution. Suppose that $\mu_0 = 0$. Then $\mu_1 = 0$ and $\mu_{3st} = 0$ must hold for all $(s, t)$ by (9). However, this violates the non-zero condition of Lagrange multipliers. Hence we can standardize $\mu_0$ as 1. Then (9) is rewritten as

$$\mu_1 = \left( \frac{1+\delta}{R_{st}} - \frac{\mu_{3st}}{n_{st}} \right) \frac{1}{1 + (1+r) K_t}. \tag{10}$$

The left-hand side is independent of $(s, t)$, the right-hand side is equivalent for all $(s,t)$.

A schedule with $R_{st} = I_s/K_t$ can be excluded for all $(s,t)$ as a solution candidate because this schedule is infeasible. In addition, the following lemma is demonstrated.

**Lemma A.6.** *For all $(s,t) \in \Theta$, $R_{st} < I_s/K_t$.*

*Proof.* To show this, $R_{st} = I_s/K_t$ is assumed for some $(s,t) \in \Theta$ and optimal plans are derived for $(\tilde{s},\tilde{t}) \neq (s,t)$. A contradiction between $R_{\tilde{s}\tilde{t}} > 0$ and $\mu_{3st} \geqslant 0$ is proposed. Define $C$ to be $C \equiv \{(s,t) \in \Theta\colon R_{st} = I_s/K_t\}$. By the definition and the complementary slackness, $\mu_{3s't'} = 0$ for $(s',t') \notin C$ implies two points. First, $\mu_1 > 0$ by (10). First, the feasibility condition is satisfied with equality. Second, again by (10), it holds for any $(s,t)$, $(s',t') \notin C$ that

$$\begin{aligned} &\frac{1+\delta}{R_{st}} \frac{1}{1+(1+r)K_t} = \frac{1+\delta}{R_{s't'}} \frac{1}{1+(1+r)K_{t'}} \\ \iff &(1+(1+r)K_t)R_{st} = (1+(1+r)K_{t'})R_{s't'}. \end{aligned} \tag{11}$$

Using this relation, we have

$$\begin{aligned} &\sum_{(s,t)\in\Theta} n_{st}\left[(1+r)I_s - (1+(1+r)K_t)R_{st}\right] \\ &= \sum_{(s,t)\notin C} n_{st}\left[(1+r)I_s - (1+(1+r)K_t)R_{st}\right] \\ &\qquad + \sum_{(s,t)\in C} n_{st}\left[\frac{(1+r)K_t I_s - (1+(1+r)K_t)I_s}{K_t}\right] = 0 \\ \iff &(1+(1+r)K_{\tilde{t}})R_{\tilde{s}\tilde{t}} \sum_{(x,t)\notin C} n_{st} = (1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C} \frac{n_{st}I_s}{K_t} \\ \iff &R_{\tilde{s}\tilde{t}} = \frac{1}{(1+(1+r)K_{\tilde{t}})\sum_{(s,t)\notin C} n_{st}} \left[(1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C} \frac{n_{st}I_s}{K_t}\right] \end{aligned}$$

for any $(\tilde{s},\tilde{t}) \notin C$. This is positive if, and only if,

$$(1+r)\sum_{(s,t)\notin C} n_{st}I_s > \sum_{(s,t)\in C} \frac{n_{st}I_s}{K_t}. \tag{12}$$

Conversely, it must hold that $\mu_{3st} \geqslant 0$ for $(s,t) \in C$. By (10), for any $(s,t) \in C$ and $(s',t') \notin C$, we have the following:

$$\begin{aligned}
&\frac{1+\delta}{R_{s't'}}\frac{1}{1+(1+r)K_{t'}} = \left(\frac{1+\delta}{R_{st}} - \frac{\mu_{3st}}{n_{st}}\right)\frac{1}{1+(1+r)K_t} \\
\Rightarrow &\mu_{3st} = n_{st}(1+\delta)\left(\frac{K_t}{I_s} - \frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_{st}}{(1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)}\right).
\end{aligned}$$

A sufficient and necessary condition for $\mu_{3st} \geqslant 0$ is

$$\begin{aligned}
&\frac{K_t}{I_s} \geqslant \frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_{st}}{(1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)} \\
\Longleftrightarrow &\frac{I_s}{K_t}\frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_{st}}{(1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)} \leqslant 0 \\
\Longleftrightarrow &(1+r)\sum_{(s,t)\notin C} n_{st}I_s < \sum_{(s,t)\in C} \frac{n_{st}I_s}{K_t} \qquad (13)
\end{aligned}$$

Thus, (13) contradicts (12). □

We can focus on an inner solution by this lemma. Then, with the complementary slackness conditions, $\mu_{3st} = 0$ for all $(s,t) \in \Theta$, we can derive (11) for any pair $(s,t)$, $(s',t') \in \Theta$. Moreover, the feasibility condition is satisfied with equality by (10) and $\mu_{3st} = 0$. Substituting (11) for the feasibility condition, we have

$$R_{st} = \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}}I_{\tilde{s}}}{1+(1+r)K_t}$$

for all $(s,t) \in \Theta$. To derive $P_{st}$, substitute $R_{st}$ to $P_{st} = I_s - K_t R_{st}$. Then

$$\begin{aligned}
P_{st} &= I_s - K_t \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t} \\
&= \frac{[1+(1+r)K_t]I_s - K_t(1+r)n_{st}I_s - K_t(1+r)\sum_{(s',t')\neq(s,t)} n_{s't'}I_{s'}}{1+(1+r)K_t} \\
&= \frac{I_s + (1+r)K_t \sum_{(s',t')\neq(s,t)} n_{s't'}I_s - (1+r)K_t \sum_{(s',t')\neq(s,t)} n_{s't'}I_{s'}}{1+(1+r)K_t} \\
&= \frac{I_s + (1+r)K_t \sum_{(s',t')\neq(s,t)} n_{s't'}(I_s - I_{s'})}{1+(1+r)K_t}.
\end{aligned}$$

Thus we have solution to the problem. □

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