CSCI 416 - Intro to Machine Learning **Homework 10/21** **Kyle Chen**

(Due October 21)

Question 1

1.1) First, we will say that

$$g(z) = \frac{1}{1+e^{-z}} = (1+e^{-z})^{-1}.$$

This way, we can use the chain rule on while deriving the function. So,

$$\begin{aligned}
\frac{d}{dz}g(z) &= \frac{d}{dz}(1+e^{-z})^{-1} \\
&= (-1)(1+e^{-z})^{-2} \times \frac{d}{dz}(1+e^{-z}) \\
&= (-1)(1+e^{-z})^{-2} \times (-1)(e^{-z}) \\
&= \frac{e^{-z}}{(1+e^{-z})^2} \\
&= \frac{1}{1+e^{-z}} \times \frac{e^{-z}}{1+e^{-z}}
\end{aligned}$$

Since

$$1 - g(z) = \frac{1+e^{-z}}{1+e^{-z}} - \frac{1}{1+e^{-z}} = \frac{e^{-z}}{1+e^{-z}},$$

we have

$$\frac{d}{dz}g(z) = g(z)(1-g(z)),$$

and we are done.

1.2) Given that $1 - g(z) = \frac{e^{-z}}{1+e^{-z}}$, as shown above, and $g(-z) = \frac{1}{1+e^{z}}$ we will proceed as such.

$$\begin{aligned}
1 - g(z) &= \frac{e^{-z}}{1+e^{-z}} \\
&= \frac{1}{(1+e^{-z})(e^{z})} \\
&= \frac{1}{e^{z}+e^{-z}e^{z}} \\
&= \frac{1}{e^{z}+1} \\
&= \frac{1}{1+e^{z}}
\end{aligned}$$

Since we have already defined $g(-z) = \frac{1}{1+e^{z}}$, we have already proved that $1 - g(z) = g(-z)$, and we are done.

1.3) Given that $h_\theta(x) = g(\theta^T x) = \frac{1}{1+e^{\theta^T x}}$, we will derive the simplified cost function $J(\theta)$ as follows, assuming the base of log is the natural number e:

$$\begin{aligned}
\frac{\partial}{\partial\theta_J}J(\theta) &= \frac{\partial}{\partial\theta_J}(-y\ log\ h_\theta(x) - (1-y)\ log(1-h_\theta(x))) \\
&= -y \times \frac{1}{h_\theta(x)} \times \frac{\partial}{\partial\theta_J}(h_\theta(x)) - (1-y)\frac{1}{1-h_\theta(x)} \times \frac{\partial}{\partial\theta_J}(1-h_\theta(x)) \\
&= -y \times \frac{1}{\frac{1}{1+e^{-\theta^T x}}} \times \frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) - (1-y)\frac{1}{1-\frac{1}{1+e^{-\theta^T x}}} \times \frac{\partial}{\partial\theta_J}(1-\frac{1}{1+e^{-\theta^T x}}) \\
&= -y \times (1+e^{-\theta^T x}) \times \frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) - (1-y)\frac{1}{1-\frac{1}{1+e^{-\theta^T x}}} \times \frac{\partial}{\partial\theta_J}(1-\frac{1}{1+e^{-\theta^T x}}) \\
&= -y \times (1+e^{-\theta^T x}) \times \frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) - (1-y)\frac{1}{\frac{e^{-\theta^T x}}{1+e^{-\theta^T x}}} \times \frac{\partial}{\partial\theta_J}(1-\frac{1}{1+e^{-\theta^T x}}) \\
&= -y \times (1+e^{-\theta^T x}) \times \frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) - (1-y)\frac{1+e^{-\theta^T x}}{e^{-\theta^T x}} \times \frac{\partial}{\partial\theta_J}(1-\frac{1}{1+e^{-\theta^T x}}) \\
&= -y \times (1+e^{-\theta^T x}) \times \frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) - (1-y)\frac{1+e^{-\theta^T x}}{e^{-\theta^T x}} \times (-1)\frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) \\
&= (-y(1+e^{-\theta^T x}) - (1-y)(\frac{1+e^{-\theta^T x}}{e^{-\theta^T x}})(-1))\frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}})
\end{aligned}$$

Since we've described the derivative of $\frac{d}{d\theta}g(z) = g(z)(1-g(z))$ above, we have

$$\begin{aligned}
\frac{\partial}{\partial\theta_J}(\frac{1}{1+e^{-\theta^T x}}) &= (-1)(1+e^{-\theta^T x})^{-2} \times \frac{\partial}{\partial\theta_J}(1+e^{-\theta^T x}) \\
&= (-1)(1+e^{-\theta^T x})^{-2} \times (-x_J)(e^{-\theta^T x}) \\
&= \frac{e^{-\theta^T x}}{(1+e^{-\theta^T x})^2}(x_J) \\
&= (\frac{1}{1+e^{-\theta^T x}})(\frac{e^{-\theta^T x}}{1+e^{-\theta^T x}})(x_J)
\end{aligned}$$

So, we have:

$$
\begin{aligned}
&= (-y(1+e^{-\theta^T x}) - (1-y)(\frac{1+e^{-\theta^T x}}{e^{-\theta^T x}})(-1))(\frac{1}{1+e^{-\theta^T x}})(\frac{e^{-\theta^T x}}{1+e^{-\theta^T x}})(x_J) \\
&= (-y(\frac{1}{h_\theta(x)}) - (1-y)(\frac{1}{1-h_\theta(x)})(-1))(h_\theta(x))(1-h_\theta(x))(x_J) \\
&= (-y(\frac{1}{h_\theta(x)}) - (-1+y)(\frac{1}{1-h_\theta(x)}))(h_\theta(x))(1-h_\theta(x))(x_J) \\
&= (-y(\frac{1}{h_\theta(x)}) \times (h_\theta(x))(1-h_\theta(x)) - (-1+y)(\frac{1}{1-h_\theta(x)}) \times (h_\theta(x))(1-h_\theta(x)))(x_J) \\
&= (-y(1-h_\theta(x)) - (-1+y)(h_\theta(x)))(x_J) \\
&= (-y + y(h_\theta(x)) - (-1+y)(h_\theta(x)))(x_J) \\
&= (-y + y(h_\theta(x)) + (h_\theta(x)) - y(h_\theta(x)))(x_J) \\
&= (-y + h_\theta(x))(x_J) \\
&= (h_\theta(x) - y)(x_J)
\end{aligned}
$$

And we are done.