2.6: The Precise Definitions of Infinite Limits and Limits at Infinity

The previous section walked us slowly through the theory of finite limits at finite numbers. Instead of *evaluating* limits, we *proved* limits using $\epsilon-\delta$ proofs. We now turn our attention to limits involving infinity. There are three such limits:

1. infinite limits at finite numbers,
2. finite limits at infinity, and
3. infinite limits at infinity.

We treat each of these separately and pull all limit proof styles into a unified theory at the end of the section.

Infinite Limits at Finite Numbers

In Section 2.2, we learned how to *conceptually* investigate limits of the form

$$\lim_{x\to 1}\frac{1}{(x-1)^2}.$$

These types of limits are called **infinite limits at finite numbers** because their limits approach infinity as x approaches a finite number.

To investigate this limit, we stated that $(x-1)^2\to 0^+$ as $x\to 1$. Thus, $\dfrac{1}{(x-1)^2}\to\dfrac{1}{0^+}\to\infty$.

At the time, these arguments were sufficient, if not precise; however, we now turn our attention to *proving*

$$\lim_{x\to 1}\frac{1}{(x-1)^2}=\infty. \tag{2.6.1}$$

To understand the structure of the proof for infinite limits at finite numbers, we need to discuss the structure of the traditional $\epsilon-\delta$ proof.

Given $\lim\limits_{x\to a}(f(x))=L$, where a and L are finite, it should now make sense that we *require* an ϵ-nieghborhood about the *finite* number L, and a δ-neighborhood about the *finite* number a; however, in the limit 2.6.1, $L=\infty$ is *not* finite. Therefore, creating a small ϵ-neighborhood about L makes no sense. There is no possibility for us to bound ∞ in a small ϵ-neighborhood. Thus, our precise definition needs a bit of an adjustment for this type of limit.

Figure $2.6.1$ shows the graph of $f(x)=\dfrac{1}{(x-1)^2}$.

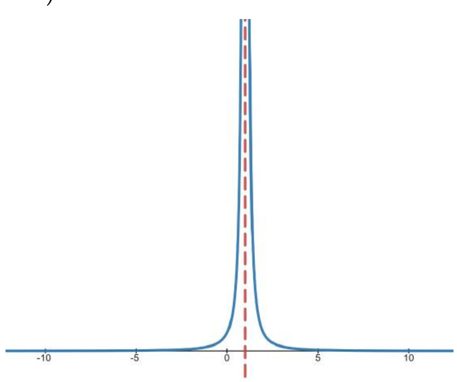


Figure $2.6.1$: $f(x)=\dfrac{1}{(x-1)^2}$

While it's easy to *see* that the function values, $f(x)$, tend to ∞ as our δ-neighborhood centered at $x=1$ gets smaller and smaller, this is *not* a proof. What we need is for someone to challenge us (as they did with the ϵ-gauntlet in Section 2.5) and say, "I bet you cannot find a small enough δ-neighborhood centered at $x=1$ so that all the corresponding function values are above... oh, let's say a billion!" **And this is the key to the structure of the precise definition of an infinite limit at a finite point.** Instead of "for any given $\epsilon>0$," we are faced with "for any given $N\ggg 0$," where the notation $\ggg$ means "much greater than." And instead of a small ϵ-neighborhood around L, we are challenged with getting the function values *above* (or below, if the limit is tending to $-\infty$) some massively large number N. Visually, it looks a bit like Figure $2.6.2$.

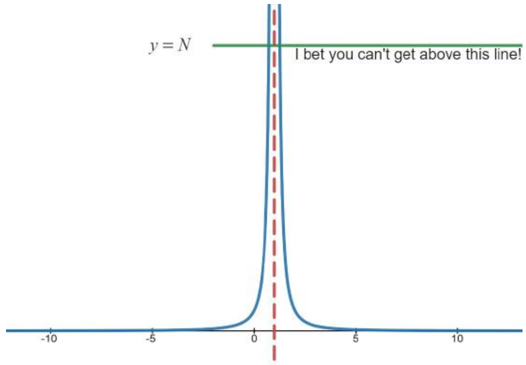


Figure $2.6.2$

We can now formalize the precise definition of an infinite limit at a finite number.

Definition: Infinite Limit at a Finite Number (Precise Definition)

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Then we say

$$\lim_{x \to a} f(x) = \infty$$

if for every $N \ggg 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) > N$.

Stated with symbolic logic,

$$\lim_{x \to a} f(x) = \infty$$

means

$$\forall N \ggg 0, \quad \exists \delta > 0 \ni$$

$$0 < |x - a| < \delta \implies f(x) > N.$$

Additionally, we say

$$\lim_{x \to a} f(x) = -\infty$$

if for every $N \lll 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) < N$.

Stated with symbolic logic,

$$\lim_{x \to a} f(x) = -\infty$$

means

$$\forall N \lll 0, \quad \exists \delta > 0 \ni$$

$$0 < |x - a| < \delta \implies f(x) < N.$$

Let's see this in an example.

Example $2.6.1$

Prove

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty.$$

Solution

PROOF	SCRATCH WORK
For $N \ggg 0$, let $\delta = \frac{1}{\sqrt{N}}$. If $0 < \lvert x-1 \rvert < \delta$, then $$\begin{aligned} \lvert x-1\rvert &< \frac{1}{\sqrt{N}} \\ \implies (x-1)^2 &< \frac{1}{N} \\ \implies \frac{1}{(x-1)^2} &> N \\ \implies f(x) &> N \end{aligned}$$ Thus, we have proved the limit $$\lim_{x\to 1}\frac{1}{(x-1)^2}=\infty.$$	We want to arrive at $$\frac{1}{(x-1)^2} > N \quad \text{(consequent)}$$ given that $$0 < \lvert x-1\rvert < \delta. \quad \text{(antecedent)}$$ Working backwards from the consequent, and trying to arrive at the antecedent, we get $$\begin{aligned} \frac{1}{(x-1)^2} > N &\implies (x-1)^2 < \frac{1}{N} \\ &\implies \lvert x-1\rvert < \frac{1}{\sqrt{N}} \\ &\implies \text{Let } \delta = \frac{1}{\sqrt{N}} \end{aligned}$$

There are a few tripping points that commonly spring up while working with this material. It's important to remember the following from your prerequisite courses:

1. An inequality switches directions if you multiply or divide by a negative.
2. Applying an even power to an expression results in a positive value (this can be seen between the first and second steps of the proof in Example $2.6.1$).
3. $\sqrt{(\blacksquare)^2} = \lvert\blacksquare\rvert$. This fact is used in the second-to-last step of the Scratch Work.

We will add to this list as we move forward, but if you have solid prerequisites, you have nothing to worry about.

Vertical Asymptotes

You have encountered vertical asymptotes in your algebra courses as part of the material on rational functions and logarithmic functions. Moreover, you worked with them in trigonometry. We now formalize their definition using calculus.

Definition: Vertical Asymptote

The vertical line $x=a$ is called a **vertical asymptote** of the function $f(x)$ if at least one of the following is true:

$$\begin{array}{ccc} \lim\limits_{x\to a} f(x)=\infty & \lim\limits_{x\to a^-} f(x)=\infty & \lim\limits_{x\to a^+} f(x)=\infty \\ \lim\limits_{x\to a} f(x)=-\infty & \lim\limits_{x\to a^-} f(x)=-\infty & \lim\limits_{x\to a^+} f(x)=-\infty \end{array}$$

Example $2.6.2$

Prove that $x=-2$ is a vertical asymptote of $f(x)=\ln(x+2)$.

Solution

From algebra, we know the domain of $f(x)=\ln(x+2)$ is $(-2,\infty)$. Therefore, to prove $x=-2$ is a vertical asymptote, we only need concern ourselves with the one-sided limit

$$\lim_{x\to -2^+} \ln(x+2).$$

We know that the graph of $\ln(x+2)$ tends to $-\infty$ as x approaches -2. Now we are asked to *prove* it. That is, we want to prove

$$\lim_{x\to -2^+} \ln(x+2) = -\infty.$$

PROOF	SCRATCH WORK
For $N \lll 0$, let $\delta = e^N$. If $0 < x-(-2) < \delta$, then $$\begin{aligned} x+2 &< e^N \\ \implies \ln(x+2) &< N \\ \implies f(x) &< N \end{aligned}$$ Thus, we have proved the limit $$\lim_{x\to -2} \ln(x+2) = -\infty.$$ Therefore, $x=-2$ is a vertical asymptote of $f(x) = \ln(x+2)$.	We want to arrive at $$\ln(x+2) < N \quad \text{(consequent)}$$ given that $$0 < x-(-2) = x+2 < \delta. \quad \text{(antecedent)}$$ (Since $x \to -2^+$, $x > -2 \implies x+2 > 0$.) Working backwards from the consequent, and trying to arrive at the antecedent, we get $$\begin{aligned} \ln(x+2) < N &\implies x+2 < e^N \\ &\implies \text{Let } \delta = e^N \end{aligned}$$

Adding to that list of tripping points:

4. If $x < y$, then $e^x < e^y$ (this is a direct result of the fact that $y = e^x$ is an increasing function). This allows us to rewrite inequalities like $\ln(x+2) < N$ as $e^{\ln(x+2)} < e^N \implies x+2 < e^N$.
5. Stating that $\delta = e^N$ might *seem* wrong to you because δ is typically *very* small; however, pay close attention to the value of N. In Example $2.6.2$, $N \lll 0$. Therefore, e^N *is* very small.

Finite Limits at Infinity

Consider

$$\lim_{x\to\infty} \tanh(x).$$

To evaluate this limit, we take advantage of the Ratio Identities for hyperbolic functions and the exponential forms of the hyperbolic sine and cosines.

$$\begin{aligned}
\lim_{x\to\infty} \tanh(x) &= \lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)} && (\text{Calculus: Ratio Identities for hyperbolic functions}) \\
&= \lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} && (\text{Calculus: Exponential form of hyperbolic functions}) \\
&= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} && \left(\text{Algebra: Multiply by } \frac{1/e^x}{1/e^x}\right) \\
&= \lim_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} \\
&= \frac{1-0}{1+0} && \left(\text{Calculus: } \frac{1}{e^{2x}} \to 0 \text{ as } x \to \infty\right) \\
&= 1
\end{aligned}$$

The second-to-last step in this derivation relied on *assumptions* that $\frac{1}{e^{2x}} \to 0$ as $x \to \infty$. In previous sections, we bolstered this assumption with a "hand-waving" argument, saying, "Well it's *obvious* that the denominator of $\frac{1}{e^{2x}}$ goes to infinity as x goes to infinity. Therefore, 1 over something going to infinity will shrink to 0." However, this is not a rigorous mathematical justification.

So, how can we *prove*

$$\lim_{x\to\infty} \frac{1}{e^{kx}} = 0 \ \ (\text{for } k > 0), \tag{2.6.2}$$

which in turn will *prove*

$$\lim_{x\to\infty} \tanh(x) = 1? \tag{2.6.3}$$

Limits like 2.6.2 and 2.6.3 are called **finite limits at infinity** because the limits become finite (0 in 2.6.2 and 1 in 2.6.3) as x approaches infinity.

To understand the structure of the proof for finite limits at infinity, we again need to modify the traditional $\epsilon-\delta$ proof. In 2.6.2, $L=0$ is finite, but $a=\infty$ is *not* finite. Therefore, creating a small δ-neighborhood about a makes no sense. Just as before, there is no possibility for us to bound ∞ in a small δ-neighborhood.

Figure $2.6.3$ shows the graph of $f(x) = \dfrac{1}{e^{2x}}$ (we will worry about $\dfrac{1}{e^{kx}}$ later).

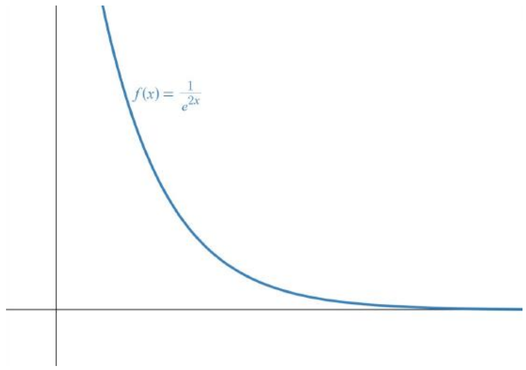


Figure $2.6.3$: $f(x) = \dfrac{1}{e^{2x}}$

If the claim is that

$$\lim_{x\to\infty} \frac{1}{e^{2x}} = 0,$$

then someone is going to throw the ϵ-gauntlet to challenge us to force the function to be bounded within the interval $(0-\epsilon, 0+\epsilon)$, as in Figure $2.6.4$.

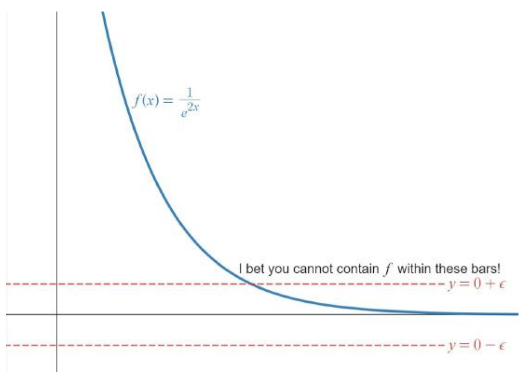


Figure $2.6.4$

It's apparent from Figure $2.6.4$ that, as long as x is large enough, $-\epsilon < f(x) < \epsilon$. In fact, we see in Figure $2.6.5$, as long as $x > M$, $|f(x) - 0| < \epsilon$.

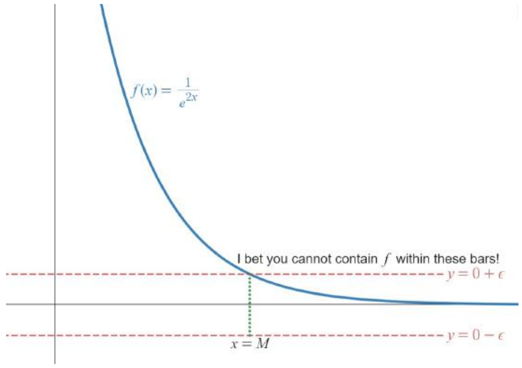


Figure 2.6.5

This gives us great insight into the formal definition for finite limits at infinity.

Definition: Finite Limit at Infinity (Precise Definition)

Let $f(x)$ be defined for all $x > a$. Then we say

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a number $M > 0$, such that if $x > M$, then $|f(x) - L| < \epsilon$.

Stated with symbolic logic,

$$\lim_{x\to\infty} f(x) = L$$

means

$$\forall \epsilon > 0, \quad \exists M > 0 \ni$$

$$x > M \implies |f(x) - L| < \epsilon.$$

Additionally, we say

$$\lim_{x\to-\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a number $M < 0$, such that if $x < M$, then $|f(x) - L| < \epsilon$.

Stated with symbolic logic,

$$\lim_{x\to-\infty} f(x) = L$$

means

$$\forall \epsilon > 0, \quad \exists M < 0 \ni$$

$$x < M \implies |f(x) - L| < \epsilon.$$

We now introduce two very powerful theorems. The proofs of each of these theorems showcase how to use the precise definition of a finite limit at infinity.

Theorem 2.6.1

$$\lim_{x\to\infty} e^{-x} = 0$$

Proof

PROOF	SCRATCH WORK
For $\epsilon > 0$, let $M = -\ln(\epsilon)$. If $x > M$, then $$\begin{array}{rrcl} & x & > & -\ln(\epsilon) \\ \Longrightarrow & -x & < & \ln(\epsilon) \\ \Longrightarrow & e^{-x} & < & \epsilon \\ \Longrightarrow & \lvert e^{-x}-0\rvert & < & \epsilon \\ \Longrightarrow & \lvert f(x)-L\rvert & < & \epsilon \end{array}$$ Thus, we have proved the limit $$\lim_{x\to\infty} e^{-x} = 0.$$	We want to arrive at $$\lvert e^{-x}-0\rvert < \epsilon \quad \text{(consequent)}$$ given that $$x > M. \quad \text{(antecedent)}$$ (Since $x\to\infty$, $x > M$.) Working backwards from the consequent, and trying to arrive at the antecedent, we get $$\begin{array}{rcl} \lvert e^{-x}-0\rvert < \epsilon & \Longrightarrow & e^{-x} < \epsilon \\ & \Longrightarrow & -x < \ln(\epsilon) \\ & \Longrightarrow & x > -\ln(\epsilon) \\ & \Longrightarrow & \text{Let } M = -\ln(\epsilon) \end{array}$$

Again, adding to our list of tripping points:

6. $\lvert e^{-x}\rvert = e^{-x}$ because $e^{-x} > 0$.
7. In the proof of Theorem $2.6.1$, we found $M = -\ln(\epsilon)$; however, we also stated that $M > 0$. How can this be? Don't be fooled into thinking an "opposite sign" $(-)$ means that you're dealing with a negative number. In reality, ϵ is a *very* small positive number. Therefore, $\ln(\epsilon)$ is negative. Hence, $M = -\ln(\epsilon)$ is positive.

Theorem $2.6.1$ begets a simple corollary.

Corollary $2.6.1$

If $k > 0$,

$$\lim_{x\to\infty} e^{-kx} = 0$$

Proof

$$\begin{array}{rcll} \lim_{x\to\infty} e^{-kx} & = & \lim_{x\to\infty} \left(e^{-x}\right)^k & \\ & = & \left(\lim_{x\to\infty} e^{-x}\right)^k & \text{(Calculus: Power Law of Limits)} \\ & = & (0)^k & \text{(Calculus: Theorem 2.6.1)} \\ & = & 0 & \end{array}$$

The following theorem will be used throughout calculus.

Theorem $2.6.2$

If $p > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^p} = 0.$$

If $p > 0$ is a rational number such that x^p is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^p} = 0.$$

Proof

PROOF (for $\lim_{x\to\infty}\frac{1}{x^p}=0$)

For $\epsilon>0$, let $M=\frac{1}{\epsilon^{1/p}}$. If $x>M$, then

$$\begin{aligned} & x &>& \frac{1}{\epsilon^{1/p}} \\ \Longrightarrow\quad & x^p &>& \frac{1}{\epsilon} \\ \Longrightarrow\quad & \frac{1}{x^p} &<& \epsilon \\ \Longrightarrow\quad & \left|\frac{1}{x^p}-0\right| &<& \epsilon \\ \Longrightarrow\quad & |f(x)-L| &<& \epsilon \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x\to\infty}\frac{1}{x^p}=0.$$

SCRATCH WORK (for $\lim_{x\to\infty}\frac{1}{x^p}=0$)

We want to arrive at

$$\left|\frac{1}{x^p}-0\right|<\epsilon \qquad \text{(consequent)}$$

given that

$$x>M. \qquad \text{(antecedent)}$$

Working backwards from the consequent, and trying to arrive at the antecedent, we get

$$\begin{aligned} \left|\frac{1}{x^p}-0\right|<\epsilon \quad & \Longrightarrow \quad \frac{1}{x^p}<\epsilon \\ & \Longrightarrow \quad x^p>\frac{1}{\epsilon} \\ & \Longrightarrow \quad x>\frac{1}{\epsilon^{1/p}} \\ & \Longrightarrow \quad \text{Let } M=\frac{1}{\epsilon^{1/p}} \end{aligned}$$

The proof of the second half of Theorem $2.6.2$ is left as a homework exercise.

As before, the proof of Theorem $2.6.2$ adds to our list of tripping points.

8. If $x>0$, then $\left|\frac{1}{x^p}-0\right|=\frac{1}{x^p}$.

Horizontal Asymptotes

We have already introduced the definition of a horizontal asymptote; however, we repeat it here due to the connection with the finite limit at infinity.

Definition: Horizontal Asymptote

The vertical line $y=L$ is called a **horizontal asymptote** of the function $f(x)$ if at least one of the following is true:

$$\lim_{x\to\infty} f(x)=L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x)=L$$

Infinite Limits at Infinity

Up to this point, we have allowed ourselves the luxury of *assuming*

$$\lim_{x\to\infty} 3^x=\infty; \tag{2.6.4}$$

however, it is at this moment that we need to *prove* this assumption is true. Limits of the form 2.6.4 are called **infinite limits at infinity** because the function tends to infinity (or negative infinity) and x tends to infinity (or negative infinity).

As with all our work in this section, developing the precise definition of an infinite limit at infinity requires adjusting the traditional $\epsilon-\delta$ definition of a limit.

Since the both x and the function are approaching infinity in 2.6.4, it is impossible for us to say, "For any given $\epsilon>0$, there exists a $\delta>0$," because there is no way for us to contain infinity in a small interval (whether an ϵ- or δ-neighborhood).

So, how do we adjust the definition? Consider the following figure.

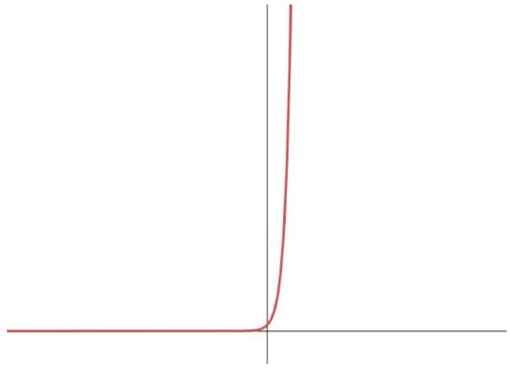


Figure $2.6.6$: $f(x) = 3^x$

Instead of someone challenging us by throwing down an ϵ-gauntlet, they are going to challenge us to get our function values *above* a very large number (see Figure $2.6.7$.

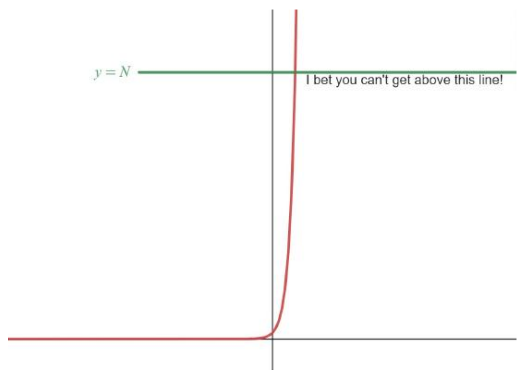
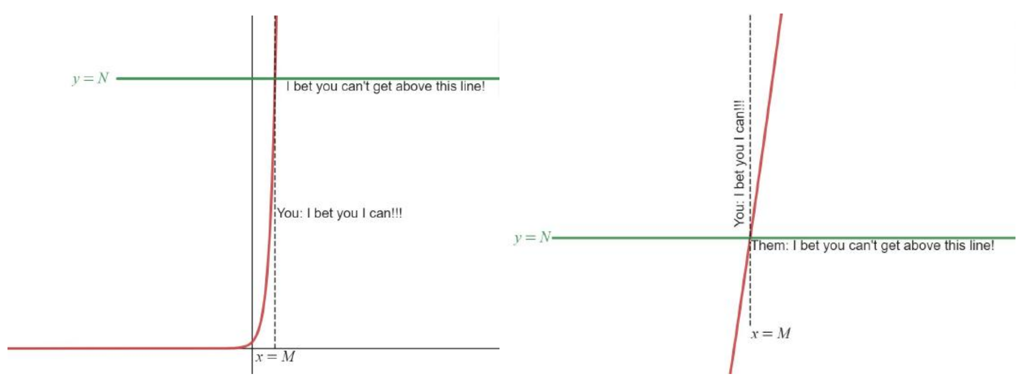


Figure $2.6.7$

That is, someone is going to hand us $N \ggg 0$. Our job is to somehow find x-values whose function values are *above* the line $y = N$. Figure $2.6.8A$ shows graphically how this is done.



Figures $2.6.8A$ (left) and $2.6.8B$ (right)

Figure $2.6.8B$ shows a zoomed-in view of Figure $2.6.8A$. As long as $x > M$, it looks as though $3^x > N$.

Using this same idea, let's write our final set of precise definitions for limits.

Definition: Infinite Limit at Infinity (Precise Definition)

As $x \to \infty$...	As $x \to -\infty$...
Let $f(x)$ be defined for all $x > a$. Then we say $$\lim_{x\to\infty} f(x) = \infty$$ if for every $N \ggg 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) > N$. Stated with symbolic logic, $$\lim_{x\to\infty} f(x) = \infty$$ means $$\forall N \ggg 0, \quad \exists M > 0 \ni$$ $$x > M \implies f(x) > N.$$ Additionally, we say $$\lim_{x\to\infty} f(x) = -\infty$$ if for every $N \lll 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) < N$. Stated with symbolic logic, $$\lim_{x\to\infty} f(x) = -\infty$$ means $$\forall N \lll 0, \quad \exists M > 0 \ni$$ $$x > M \implies f(x) < N.$$	Let $f(x)$ be defined for all $x < a$. Then we say $$\lim_{x\to-\infty} f(x) = \infty$$ if for every $N \ggg 0$, there exists a number $M < 0$, such that if $x < M$, then $f(x) > N$. Stated with symbolic logic, $$\lim_{x\to-\infty} f(x) = \infty$$ means $$\forall N \ggg 0, \quad \exists M < 0 \ni$$ $$x < M \implies f(x) > N.$$ Additionally, we say $$\lim_{x\to-\infty} f(x) = -\infty$$ if for every $N \lll 0$, there exists a number $M < 0$, such that if $x < M$, then $f(x) < N$. Stated with symbolic logic, $$\lim_{x\to-\infty} f(x) = -\infty$$ means $$\forall N \lll 0, \quad \exists M < 0 \ni$$ $$x < M \implies f(x) < N.$$

Example 2.6.3

Prove

$$\lim_{x\to-\infty} x^3 = -\infty.$$

Solution

PROOF	SCRATCH WORK
For $N \lll 0$, let $M = \sqrt[3]{N}$. If $x < M$, then $$\begin{array}{rrcl} & x & < & \sqrt[3]{N} \\ \implies & x^3 & < & N \\ \implies & f(x) & < & N \end{array}$$ Thus, we have proved the limit $$\lim_{x\to-\infty} x^3 = -\infty.$$	We want to arrive at $$x^3 < N \quad \text{(consequent)}$$ given that $$x < M. \quad \text{(antecedent)}$$ Working backwards from the consequent, and trying to arrive at the antecedent, we get $$\begin{array}{rcl} x^3 < N & \implies & x < \sqrt[3]{N} \\ & \implies & \text{Let } \delta = \sqrt[3]{N} \end{array}$$

Exercise 2.6.3

Use the formal definition of infinite limit at infinity to prove that $\lim\limits_{x\to\infty} 3x^2 = \infty$.

Hint

Let $N=\sqrt{\frac{M}{3}}$.

Answer

Let $M>0$. Let $N=\sqrt{\frac{M}{3}}$. Then, for all $x>N$, we have

$3x^2>3N^2=3\left(\sqrt{\frac{M}{3}}\right)^2=\frac{3M}{3}=M$

Unifying the Precise Definitions of Limits

The following table *might* help you understand when to use ϵ, δ, N, or M in your proofs.

	Finite Limit at a Finite Number	Finite Limit at Infinity	Infinite Limit at a Finite Number	Infinite Limit at Infinity
Limit	$\lim_{x\to a} f(x)=L$	$\lim_{x\to\pm\infty} f(x)=L$	$\lim_{x\to a} f(x)=\pm\infty$	$\lim_{x\to\pm\infty} f(x)=\pm\infty$
Universal Quantifier	$\forall\epsilon>0$	$\forall\epsilon>0$	$f(x)\to+\infty \implies \forall N\ggg 0$ $f(x)\to-\infty \implies \forall N\lll 0$	$f(x)\to+\infty \implies \forall N\ggg 0$ $f(x)\to-\infty \implies \forall N\lll 0$
Existential Quantifier	$\exists\delta>0$	$x\to+\infty \implies \exists M>0$ $x\to-\infty \implies \exists M<0$	$\exists\delta>0$	$x\to+\infty \implies \exists M>0$ $x\to-\infty \implies \exists M<0$

I did not include the conditional statements in this list because each conditional statement depends heavily on the type of limit in consideration. They each have the *basic* form

$$[\text{Universal Quantifier Statement}],\ [\text{Existential Quantifier Statement}]\ \ni$$

$$[\text{inequality involving } x \text{ and either } \delta \text{ or } M] \implies [\text{inequality involving } y \text{ and either } \epsilon \text{ or } N]$$