LibreTexts

1.1: Algebraic Manipulations Critical for Calculus

Learning Objectives

- State the Mathematical Mantra.
- Briefly review the following topics from Algebra:
  - nontrivial factoring techniques
  - understand when to distribute and, just as important, when not to distribute
  - simplifying compound fractions
  - rationalizing numerators and denominators
  - using the piecewise definition of the absolute value

The material in this section was included from years of experience teaching Calculus and seeing students, semester after semester, struggle to succeed in the subject due to holes in their prerequisite Algebra skills.

It is *incredibly* important to note that the material reviewed in this section is *not* instruction. If you have never seen some of the topics being covered here, you should speak with your instructor about course alternatives that will better set you up for success in Calculus.[1]

Subsection Footnotes

[1] Trust me when I say it is better to step back to a previous course to fill in knowledge gaps than start a cycle of failure in Calculus.

The Mathematical Mantra

The following "mantra" is very helpful throughout all levels of mathematics, and significantly more so in Calculus.

> *As a rule of thumb,[2] perform mathematics in the order you learned mathematics. That is, perform Arithmetic before Algebra, Algebra before Trigonometry, and Trigonometry before Calculus.*

We will reference this Mathematical Mantra continuously throughout this course.

Subsection Footnotes

[2] The Mathematical Mantra is a "rule of thumb" and, as such, there are times when it is not applicable. In such cases, I will endeavor to point out why it does not apply.

Nontrivial Factoring Techniques

Each of the following factoring techniques *will* be required at many different points throughout this Calculus course. They will often be needed as required peripheral skills while working on your homework, quizzes, and exams. As with all material in this section, it is assumed you have mastered these skills in your prerequisite courses; however, it is presented here as a gentle reminder for those who might have been away from their studies in mathematics for a while.

Sums and Differences of Cubes

Theorem 1.1.1: Sums and Differences of Cubes

$$F^3 \pm L^3 = (F \pm L)(F^2 \mp FL + L^2) \tag{1.1.1}$$

Example 1.1.1: Factor a Difference of Cubes

You are working on a complex Calculus problem during an exam and arrive at the following expression.

$$\frac{e^{6x}-8}{e^{2x}-2}$$

You are told that $x \neq \ln\left(2^{1/2}\right)$.

You cannot move forward on the problem without somehow removing that denominator (this is something that will be necessary, for reasons to be revealed later in the course). Simplify the expression to an equivalent expression not involving the denominator $e^{2x}-2$.

Solution

Note that[3]

$$\begin{aligned} x \neq \ln\left(2^{1/2}\right) &\implies & x &\neq \frac{1}{2}\ln(2) && \text{(Algebra: Laws of Logarithms)} \\ &\implies & 2x &\neq \ln(2) && \text{(Algebra: Multiply both sides of an equation by a nonzero constant)} \\ &\implies & e^{2x} &\neq 2 && \text{(Algebra: Logarithms and exponentials are inverses)} \\ &\implies & e^{2x}-2 &\neq 0 && \text{(Algebra: Subtract a constant from both sides of an equation)} \end{aligned}$$

Therefore, we know the denominator of

$$\frac{e^{6x}-8}{e^{2x}-2}$$

is nonzero. Moreover, we see that the numerator is a difference of cubes. Thus, using the difference of cubes formula (Equation 1.1.1) we get

$$\begin{aligned}
\frac{e^{6x}-8}{e^{2x}-2} &= \frac{\left(e^{2x}\right)^3-(2)^3}{e^{2x}-2} && \text{(Algebra: Laws of Exponents)}\\
&= \frac{\left(e^{2x}-2\right)\left(\left(e^{2x}\right)^2+2e^{2x}+(2)^2\right)}{e^{2x}-2} && \text{(Algebra: Difference of Cubes)}\\
&= \frac{\left(e^{2x}-2\right)\left(e^{4x}+2e^{2x}+4\right)}{e^{2x}-2} && \text{(Algebra: Laws of Exponents)}\\
&= \frac{\cancel{\left(e^{2x}-2\right)}\left(e^{4x}+2e^{2x}+4\right)}{\cancel{\left(e^{2x}-2\right)}} && \text{(Algebra: Cancel like factors)}\\
&= e^{4x}+2e^{2x}+4
\end{aligned}$$

Factoring the GCF Out of Expressions Involving Negative Rational Exponents

One of the more common factoring techniques required in Calculus comes up when the result of some Calculus manipulations leads to an expression of the form

$$3(1+x)^{1/3}-x(1+x)^{-2/3}.$$

It will be *necessary* to factor this expression. But how?

When all terms of an algebraic expression share a common factor, but the common factor has different powers in each term, factor out the *smallest-powered* version of the shared factor.

Example 1.1.2: Factor Out a Common Expression Having a Negative Rational Exponent

It's the third exam in your Differential Calculus[4] course and you are trying to use Calculus to graph the function

$$f(x)=\frac{3}{2}(1+x)^{4/3}+3(1+x)^{1/3}+2.$$

You have done some great work so far, but at some point you arrive at the expression

$$3(1+x)^{1/3}-x(1+x)^{-2/3},$$

and you *need* to factor this to make the rest of your mathematics easy.

Solution

Both terms in the expression share the factor $(1+x)$; however, the second term has the smallest-powered version of this factor. Therefore, we factor out $(1+x)^{-2/3}$.

$$\begin{aligned}
3(1+x)^{1/3}-x(1+x)^{-2/3} &= (1+x)^{-2/3}\left(\frac{3(1+x)^{1/3}}{(1+x)^{-2/3}}-\frac{x(1+x)^{-2/3}}{(1+x)^{-2/3}}\right) && \text{(Algebra: Factoring out the GCF is division)}\\
&= (1+x)^{-2/3}\left(3(1+x)^{1/3+2/3}-x(1+x)^{-2/3+2/3}\right) && \text{(Algebra: Laws of Exponents)}\\
&= (1+x)^{-2/3}\left(3(1+x)-x(1+x)^{0}\right) && \text{(Arithmetic)}\\
&= (1+x)^{-2/3}(3(1+x)-x) && \text{(Algebra: Laws of Exponents)}\\
&= (1+x)^{-2/3}(3+2x) && \text{(Algebra: Distribution and combining like terms)}
\end{aligned}$$

Subsection Foonotes

[3] All examples in this section will have step-by-step justifications; however, as we move forward in the text, the justifications occur less frequently. At some point, several elementary steps will be skipped between steps. As you can see by the justifications listed here, your success in Calculus is built on a solid foundation of Arithmetic, Algebra, and Trigonometry.

[4] "Differential Calculus" is another name for "Calculus I."

Knowing When to Distribute

In the past, when we were allowed to offer Algebra to help students remediate lost or broken skills,[5] I would make a point to my Algebra students that factoring is the ultimate end game for a lot of mathematics. Are there times when you don't want or need to factor? Sure! However, if you have an expression that is already factored, then distributing things out is *almost* always a bad idea.

In Calculus, there are going to be many times when you are going to multiply the numerator and denominator of a given expression by a fraction equivalent to 1. You might be multiplying both the numerator and denominator by the LCD of all fractions within a compound fraction,[6] or you might be multiplying the numerator and denominator by the conjugate of the numerator, for example.

In these cases, you are faced with a simple decision - do you distribute *everything* out? Do you only distribute out the numerator? Do you only distribute out the denominator? Do you leave things factored?

The following is a good rule of thumb to follow:

Rule of Thumb

When you have no choice but to multiply the numerator and denominator of a rational expression by either

a. a conjugate (to clear radicals), or
b. an LCD (to clear a compound fraction),

only perform distribution on the part (numerator or denominator) that caused the need for such an operation. That is, do not use distribution on the non-offending piece.

That Rule of Thumb probably doesn't make sense, so we will clarify it in Example 1.1.3.

Subsection Footnotes

[5] This is no longer a reality due to a horrible bill, AB 705, which essentially closes the door to opportunity in STEM for an entire generation of students.

[6] Also known as a complex fraction.

Simplifying Compound Fractions

The first half of this course will be littered with "fractions containing fractions." These are commonly known as **compound fractions**. For example, it is inevitable that you will have to deal with a compound fraction of the form

$$\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h}.$$

When faced with such an expression, you need to find the LCD of *all* fractions in the numerator and denominator. If you multiply both numerator and denominator of the compound fraction by this LCD, the entire compound fraction will unravel.

Example 1.1.3: Simplify a Compound Rational Expression

On your first Calculus exam, you end up having to work with the following rational expression.

$$\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h}$$

However, you cannot move past the first step of the given exam problem without having to simplify the compound rational expression (a fact that will *not* be explicitly stated).

Solution

$$\begin{aligned}
\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h} &= \frac{\frac{1}{1+x+h}-\frac{1}{1+x}}{h} && \text{(Algebra: Laws of Exponents)} \\
&= \frac{\left(\frac{1}{1+x+h}-\frac{1}{1+x}\right)}{h}\cdot\frac{(1+x+h)(1+x)}{(1+x+h)(1+x)} && \text{(Algebra: Multiply numerator and denominator by the LCD)} \\
&= \frac{\frac{(1+x+h)(1+x)}{1+x+h}-\frac{(1+x+h)(1+x)}{1+x}}{h(1+x+h)(1+x)} && \left(\text{Algebra: Distribute}^{7}\right) \\
&= \frac{\frac{\cancel{(1+x+h)}(1+x)}{\cancel{(1+x+h)}}-\frac{(1+x+h)\ \cancel{(1+x)}}{\cancel{(1+x)}}}{h(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)} \\
&= \frac{(1+x)-(1+x+h)}{h(1+x+h)(1+x)} \\
&= \frac{1+x-1-x-h}{h(1+x+h)(1+x)} && \text{(Algebra: Distribution)} \\
&= \frac{-h}{h(1+x+h)(1+x)} && \text{(Algebra: Combine like terms)} \\
&= \frac{-\cancel{h}}{\cancel{h}(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)} \\
&= \frac{-1}{(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)}
\end{aligned}$$

Subsection Footnotes

[7] In the distribution step (third line), notice that we did *not* distribute the denominator. This was because the entire reason we were forced to multiply by the LCD was the fractions in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

Rationalizing Numerators and Denominators

As with simplifying compound fractions, you will need to instinctively know when to rationalize a numerator or denominator at many points throughout this course (without explicit instruction to do so from your instructor). My Rule of Thumb on knowing when to distribute and when not to distribute will be helpful here.

Example 1.1.4: Rationalize to Avoid Division By Zero

Early on in this course you will be working with expressions like

$$-\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}}{h}.$$

You must immediately identify the need to simplify the compound fraction. Moreover, the h in the denominator is going to be a problem (the reason for which will be discussed in Chapter 2). Simplify this expression and try to remove h from the denominator.

Solution

The first step to simplifying this expression is to simplify the compound fraction.

$$\begin{aligned}
-\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}}{h} &= -\frac{\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}\right)}{h}\cdot\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} && (\text{Algebra: Multiply numerator and denominator by the LCD})\\
&= -\frac{\sqrt{x+h}-\sqrt{x}}{h\sqrt{x}\sqrt{x+h}} && \left(\text{Algebra: Distribute}^{8}\right)\\
&= \frac{-\sqrt{x+h}+\sqrt{x}}{h\sqrt{x}\sqrt{x+h}} && (\text{Algebra: Distribute})\\
&= \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} && (\text{Arithmetic: Commutative property of addition})
\end{aligned}$$

Now that the compound fraction has been simplified, we are faced with an expression that *still* has the h in the denominator. Remember, there is a hidden, underlying reason that will compel us to somehow remove that through creative uses of Algebra. One such use is conjugate multiplication!

$$\begin{aligned}
\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} &= \frac{(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})} && (\text{Algebra: Multiply numerator and denominator by the conjugate of the numerator})\\
&= \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \left(\text{Algebra: Distribute}^{9}\right)\\
&= \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && (\text{Algebra: Distribute})\\
&= \frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && (\text{Algebra: Cancel like factors})\\
&= \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}
\end{aligned}$$

Our final expression is no longer a compound fraction nor does it have that pesky h in the denominator!

Subsection Footnotes

[8] In this distribution step, notice that we did *not* (and, technically, could not) distribute the denominator. The entire reason we were forced to multiply by the LCD was the fractions in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

[9] In this distribution step, notice that we did *not* distribute the denominator. The entire reason we were forced to multiply by the conjugate was the difference of radicals in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

Using the Piecewise Definition of the Absolute Value

This topic is so important that it is covered *twice* in this chapter. When we first learn of the absolute value, it is often from a flawed perspective. We are told, "The absolute value makes things positive," and, in essence, it does. It forces -3 to become 3 and π to stay π, but what about algebraic expressions like $-3x+\pi$? Since this expression has an unknown, there is no way of knowing beforehand whether or not this expression is negative. This is where knowledge of the **piecewise definition of the absolute value** (the *true* definition) comes into play!

$$|\square| = \begin{cases} \square & \text{if } \square \geq 0 \\ -\square & \text{if } \square < 0 \end{cases}$$

In language, this is saying that the absolute value of $\square$ has two possibilities:

1. If $\square$ is already non-negative, then there is no need for the absolute values.
2. If $\square$ is negative, then the absolute value will force it to become positive by returning the *opposite* of $\square$.

The part people trip up on constantly are the inequalities. They assume the absolute value returns $\square$ if $x \geq 0$ and $-\square$ if $x < 0$. **This is not true!**

Example 1.1.5: Using the Piecewise Definition of the Absolute Value

You are diligently working through your first exam and are given a problem where you know that $x < 2$ and you need to simplify

$$\frac{x^2+x-6}{|x-2|}.$$

Solution

$$\frac{x^2+x-6}{|x-2|} = \frac{(x+3)(x-2)}{|x-2|} \qquad \text{(Algebra: Factoring)}$$

At this point, canceling $(x-2)$ and $|x-2|$ is impossible - you can only cancel like factors and these expressions are not exactly the same. Using the piecewise definition of the absolute value, we see

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 & \implies & x \geq 2 \\ -(x-2) & \text{if } x-2 < 0 & \implies & x < 2 \end{cases}$$

Luckily, we know that $x < 2$, so we can replace $|x-2|$ with $-(x-2)$. Therefore,

$$\begin{aligned} \frac{x^2+x-6}{|x-2|} &= \frac{(x+3)(x-2)}{|x-2|} && \text{(Algebra: Factoring)} \\ &= \frac{(x+3)(x-2)}{-(x-2)} && \text{(Algebra: Piecewise definition of the absolute value)} \\ &= \frac{(x+3)\ \cancel{(x-2)}}{-\ \cancel{(x-2)}} && \text{(Algebra: Cancel like factors)} \\ &= \frac{(x+3)}{-1} \\ &= -(x+3) \end{aligned}$$

Caution

$$|\square| \text{ is not necessarily equal to } \square$$

Therefore,

$$\frac{|\square|}{\square} \text{ is not necessarily } 1.$$