

2.7: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- Evaluate limits using the Generalized Direct Substitution Property.
- State the theorem for limits of composite functions.
- Understand, and investigate uses of, the Intermediate Value Theorem.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point; however, as often is the case, intuition has its flaws.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 2.7.1. We see that the graph of $f(x)$ has a hole at a . In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a , we need the following condition:

i. $f(a)$ is defined

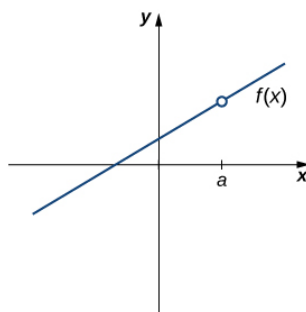


Figure 2.7.1: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure 2.7.2, this condition alone is insufficient to guarantee continuity at the point a . Although $f(a)$ is defined, the function has a gap at a . In this example, the gap exists because $\lim_{x \rightarrow a} f(x)$ does not exist. We must add another condition for continuity at a - namely,

ii. $\lim_{x \rightarrow a} f(x)$ exists

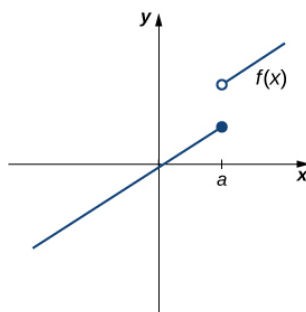


Figure 2.7.2: The function $f(x)$ is not continuous at a because $\lim_{x \rightarrow a} f(x)$ does not exist.

However, as we see in Figure 2.7.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a . We must add a third condition to our list:

iii. $\lim_{x \rightarrow a} f(x) = f(a)$

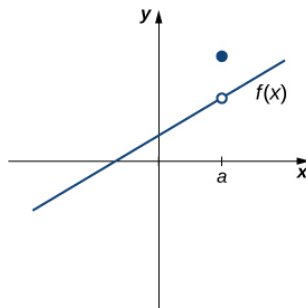


Figure 2.7.3: The function $f(x)$ is not continuous at a because $\lim_{x \rightarrow a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is said to be **continuous at a point a** if each of the following three conditions is satisfied:

- i. $f(a)$ is defined
- ii. $\lim_{x \rightarrow a} f(x)$ exists
- iii. $\lim_{x \rightarrow a} f(x) = f(a)$

A function is **discontinuous at a point a** if it fails to be continuous at a .

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

✓ Example 2.7.1A: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \frac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = \frac{4-4}{2-2}$, which is undefined. Therefore, $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.7.4.

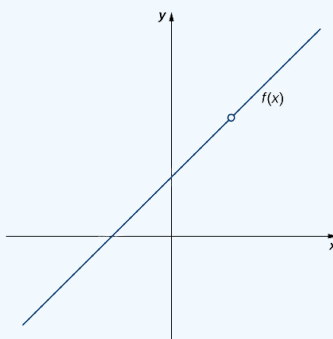


Figure 2.7.4: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

✓ Example 2.7.1B: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5 .$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \rightarrow 3} f(x)$. To do this, we must compute $\lim_{x \rightarrow 3^-} f(x)$ and $\lim_{x \rightarrow 3^+} f(x)$:

$$\lim_{x \rightarrow 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \rightarrow 3^+} f(x) = 4(3) - 8 = 4 .$$

Therefore, $\lim_{x \rightarrow 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure 2.7.5.

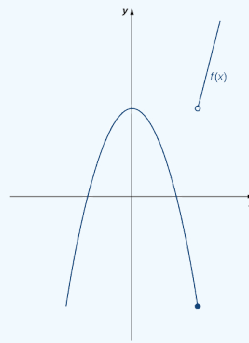


Figure 2.7.5: The function $f(x)$ is not continuous at 3 because $\lim_{x \rightarrow 3} f(x)$ does not exist.

✓ Example 2.7.1C: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1$$

Next,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 .$$

Last, compare $f(0)$ and $\lim_{x \rightarrow 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x \rightarrow 0} f(x) .$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

? Exercise 2.7.1

Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x + 4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint

Check each condition of the definition.

Answer

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \rightarrow 1} f(x)$.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.7.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \rightarrow a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

Q.E.D.

We now apply Theorem 2.7.1 to determine the points at which a given rational function is continuous.

✓ Example 2.7.2: Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

? Exercise 2.7.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint

Use the Continuity of Polynomials and Rational Functions stated above.

Answer

$f(x)$ is continuous at every real number.

Types of Discontinuities

As we have seen in Example 2.7.1A and Example 2.7.1B, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.7.6 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

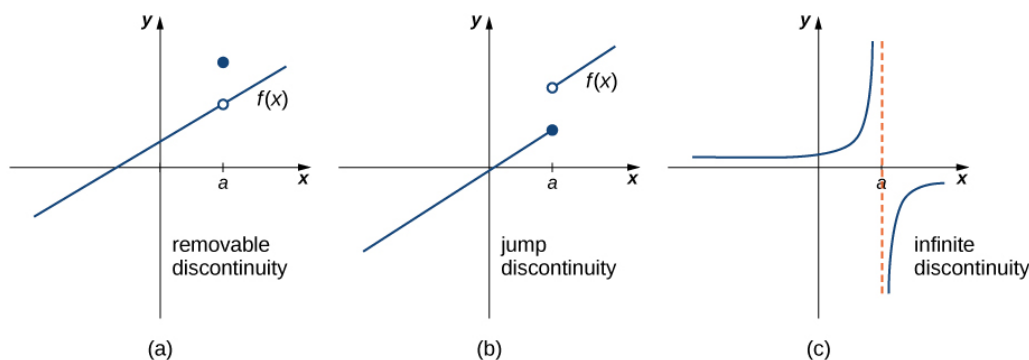


Figure 2.7.6: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition: Types of Discontinuities

If $f(x)$ is discontinuous at a , then

1. f has a **removable discontinuity** at a if $\lim_{x \rightarrow a} f(x)$ exists. (Note: When we state that $\lim_{x \rightarrow a} f(x)$ exists, we mean that $\lim_{x \rightarrow a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, but $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$. (Note: When we state that $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$.

✓ Example 2.7.3: Classifying a Discontinuity

In Example 2.7.1A, we showed that $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim_{x \rightarrow 2} f(x)$:

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x \rightarrow 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

✓ Example 2.7.4: Classifying a Discontinuity

In Example 2.7.1B, we showed that $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x \rightarrow 3} f(x)$ does not exist. However, since $\lim_{x \rightarrow 3^-} f(x) = -5$ and $\lim_{x \rightarrow 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

✓ Example 2.7.5: Classifying a Discontinuity

Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at -1 . If the function is discontinuous at -1 , classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1 . To determine the type of discontinuity, we must determine the limit at -1 . We see that $\lim_{x \rightarrow -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim_{x \rightarrow -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1 .

? Exercise 2.7.5

For $f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint

Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim_{x \rightarrow 1} f(x)$

Answer

Discontinuous at 1; removable

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive (but flawed) idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.

Definition: Continuity over Intervals

A function is **continuous over an open interval** if it is continuous at every point in the interval. A function $f(x)$ is **continuous over a closed interval** of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b . Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b . Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \rightarrow a^+} f(x) = f(a)$ and $\lim_{x \rightarrow b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \rightarrow a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

✓ Example 2.7.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, +\infty)$.

✓ Example 2.7.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

Solution

From the Limit Laws, we know that $\lim_{x \rightarrow a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

? Exercise 2.7.7

State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

Hint

Use Example 2.7.7 as a guide.

Answer

$[-3, +\infty)$

Continuity and Compositions

Theorem 2.7.2: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \rightarrow a} g(x) = L$, then

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(L).$$

Theorem 2.7.2 allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Before we move on to Example 2.7.8, recall that earlier, in the section on Limit Laws, we showed $\lim_{x \rightarrow 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 2.7.8, we see how to combine this result with the Composite Function Theorem.

✓ Example 2.7.8: Limit of a Composite Cosine Function

Evaluate $\lim_{x \rightarrow \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x \rightarrow \pi/2} \left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the Composite Function Theorem. Thus,

$$\lim_{x \rightarrow \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \rightarrow \pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

? Exercise 2.7.8:

Evaluate $\lim_{x \rightarrow \pi} \sin(x - \pi)$.

Hint

$f(x) = \sin x$ is continuous at 0. Use Example 2.7.8 as a guide.

Answer

0

The proof of the next theorem uses the Composite Function Theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.7.3: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \rightarrow a} \cos x = \cos a$ for all values of a .

$$\lim_{x \rightarrow a} \cos x = \lim_{x \rightarrow a} \cos((x - a) + a)$$

Rewrite $x = x - a + a$.

$$= \lim_{x \rightarrow a} (\cos(x - a) \cos a - \sin(x - a) \sin a)$$

Apply the identity for the cosine of the sum of two angles.

$$= \cos(\lim_{x \rightarrow a} (x - a)) \cos a - \sin(\lim_{x \rightarrow a} (x - a)) \sin a$$

Since $\lim_{x \rightarrow a} (x - a) = 0$, and $\sin x$ and $\cos x$ are continuous at 0.

$$= \cos(0) \cos a - \sin(0) \sin a$$

Evaluate $\cos(0)$ and $\sin(0)$ and simplify.

$$= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

Q.E.D.

As you can see, the Composite Function Theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Theorem 2.7.4: Intermediate Value Theorem (IVT)

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

See Figure 2.7.7 for a visualization of the IVT.

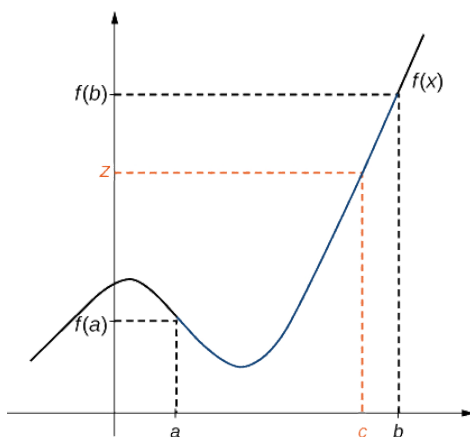


Figure 2.7.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

✓ Example 2.7.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos \frac{\pi}{2} = \frac{\pi}{2} > 0 .$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

✓ Example 2.7.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0$, $f(2) = 1 > 0$, and $f(1) = 0$.

✓ Example 2.7.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

? Exercise 2.7.11

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Answer

$f(0) = 1 > 0$, $f(1) = -2 < 0$; $f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The Composite Function Theorem states: If $f(x)$ is continuous at L and $\lim_{x \rightarrow a} g(x) = L$, then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \rightarrow a} f(x)$ exists, and (3) $\lim_{x \rightarrow a} f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim_{x \rightarrow b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) , and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, but $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a , but $\lim_{x \rightarrow a} f(x)$ exists

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