2.8: Defining the Derivative (Lecture Notes)

Time

45 - 60 minutes

Learning Objectives

- Identify the derivative as the limit of a difference quotient.
- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope and equation of a tangent line.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Tangent Lines

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2.8.1}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.8.2}$$

provided this limit exists.

Lecture Example $2.8.1A$: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x - x^3$ at $x = 1$ using both definitions.

The Derivative of a Function at a Point

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2.8.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{2.8.4}$$

Lecture Example 2.8.1B: Finding a Derivative

State the derivative of the function at $x = 1$.

Lecture Example 2.8.2: Finding a Derivative

For $f(t) = \frac{3t-1}{5t+3}$, find $f'(-1)$.

Lecture Example 2.8.3

What does the following limit represent?

$$\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$$

Velocities and Rates of Change

Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{avg} = \frac{s(t) - s(a)}{t - a}. \tag{2.8.5}$$

As the values of t approach a, the values of v_{avg} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{2.8.6}$$

Lecture Example 2.8.4

If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by

$$H = 10t - 1.86t^2.$$

a. Find the instantaneous velocity (from now on, simply called the velocity) of the rock at 1 second.
b. Find the velocity of the rock when $t = a$.
c. What will the velocity and speed of the rock be when it hits the ground?

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Lecture Example 2.8.5

The cost of producing a single bitcoin t years after 2011 is $C = B(t)$. This cost considers energy usage, hardware costs, and personnel costs.

a. What is the meaning of the derivative $B'(t)$?
b. What are its units?
c. What does the statement $B'(8) = 4758$ mean?
d. Do you think the values of $B'(t)$ increased or decreased in the first year of bitcoin's release?
e. What do you think is the long-term behavior of $B'(t)$?