3.1: Derivatives of Polynomial Functions

This page is a draft and is under active development.

Learning Objectives

- State the constant, constant multiple, and Power Rules.
- Apply the Sum and Difference Rules to combine derivatives.
- Extend the Power Rule to functions with negative and rational exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.
- Use derivatives of polynomials for applications in the sciences, engineering, and business.
- Combine previous knowledge of tangent lines with derivatives of polynomials.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}(\sqrt[3]{x})$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 && \text{(Before evaluating the limit, perform any simple arithmetic. Zero over something nonzero is zero.)} \\ &= 0. \end{aligned}$$

The rule for differentiating constant functions is called the **Constant Rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Theorem 3.1.1: The Constant Rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.1.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule: $f'(8) = 0$.

Exercise 3.1.1

Find the derivative of $g(x) = -3$.

Hint

Use the preceding example as a guide

Answer

0

The Power Rule

During our exploration of the limit definition of the derivative, we showed that

$$\frac{d}{dx}(x^2) = 2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.1.2: Differentiating x^3

Find $\frac{d}{dx}(x^3)$.

Solution

$$\begin{aligned}
\frac{d}{dx}(x^3) &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} && \text{(Distribute)} \\
&= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} && \text{(Combine like terms)} \\
&= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2)}{h} && \text{(Factor)} \\
&= \lim_{h\to 0}\frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}^{1}} && \text{(Cancel like factors)} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&\overset{\text{D.S.}}{=} 3x^2
\end{aligned}$$

Exercise 3.1.2

Find $\frac{d}{dx}(x^4)$.

Hint

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

Answer

$\frac{d}{dx}(x^4) = 4x^3$

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the derivative becomes the product of the original power and x to a power one less than it was before. A lot of students think of it as follows:

When you take the derivative of x^n, you move the power to the front and subtract 1 from the exponent.

The following theorem states that the **Power Rule** holds for all positive integer powers of x.

Theorem 3.1.2: The Power Rule (positive integer exponents only)

Let n be a positive integer. If $f(x)=x^n$,then

$$f'(x)=nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}(x^n)=nx^{n-1.}$$

Proof

For $f(x)=x^n$ where n is a positive integer, we have

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}.$$

Since

$$(x+h)^n=x^n+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n,$$

we see that

$$(x+h)^n-x^n=nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n-x^n}{h}=\frac{nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n}{h}.$$

Thus,

$$\frac{(x+h)^n-x^n}{h}=nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2+\ldots+nxh^{n-2}+h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}(nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2+\ldots+nxh^{n-2}+h^{n-1}) \\ &= nx^{n-1}. \end{aligned}$$

Q.E.D.

Example 3.1.3: Applying the Power Rule

Find the derivative of the function $f(x)=x^{10}$ by applying the Power Rule.

Solution

Using the Power Rule with $n=10$, we obtain

$$f'(x)=10x^{10-1}=10x^9.$$

? Exercise 3.1.3

Find the derivative of $f(x)=x^7$.

Hint

Use the Power Rule with $n=7$.

Answer

$f'(x) = 7x^6$

This theorem actually holds for *all* powers (not just positive integers) and, despite not being able to prove it at this point in the course, we allow ourselves the luxury of using it in the meantime.

Theorem 3.1.3: The Power Rule (General Version)

Let $n \in \mathbb{R}$. If $f(x) = x^n$,then

$$\frac{d}{dx}(x^n) = nx^{n-1.}$$

Caution

As a reminder, we have not proved Theorem 3.1.3. We are allowing ourselves the ability to use it for now, but I promise this fact will be proved rigorously when we have better mathematical "technology."

Example 3.1.4: Using the Extended Power Rule

Find $\frac{d}{dx}(x^{-4})$.

Solution

By applying the extended Power Rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.1.5: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended Power Rule and the Constant Multiple Rule to find $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{6}{x^2}\right) \\ &= \frac{d}{dx}(6x^{-2}) && \left(\text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}.\right) \\ &= 6\frac{d}{dx}(x^{-2}) && (\text{Apply the Constant Multiple Rule.}) \\ &= 6(-2x^{-3}) && (\text{Use the extended Power Rule to differentiate } x^{-2}.) \\ &= -12x^{-3} && (\text{Simplify.}) \end{aligned}$$

Exercise 3.1.5

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended Power Rule.

Hint

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended Power Rule with $k = -7$.

Answer

$g'(x) = -7x^{-8}$.

Example 3.1.6: Applying the General Power Rule

Differentiate each of the following functions.

a. $f(x)=\frac{1}{x^{18}}$

b. $g(x)=\sqrt[5]{x^3}$

Solution

a. Following the Mathematical Mantra, we take care of any algebra before performing calculus. This function can be rewritten as $f(x)=x^{-18}$. Applying the General Power Rule, we get

$$\frac{d}{dx}f(x)=-18x^{-18-1}=-18x^{-19}$$

b. Again, following the Mathematical Mantra, we simplify this function first.

$$\frac{d}{dx}\left(\sqrt[5]{x^3}\right)=\frac{d}{dx}\left(x^{3/5}\right)=\frac{3}{5}x^{3/5-1}=\frac{3}{5}x^{-2/5}$$

Before stepping into the next subsection, a very important warning needs to be issued.

Caution

The Power Rule only applies to algebraic functions of the form $y=x^n$. It does not apply to functions in which a constant is raised to a variable power, such as $f(x)=3^x$. That is,

$$\frac{d}{dx}(3^x)\neq x3^{x-1}.$$

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Theorem 3.1.4: Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}\left(f(x)\right)+\frac{d}{dx}\left(g(x)\right);$$

that is,

$$\text{for } s(x)=f(x)+g(x),\quad s'(x)=f'(x)+g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x)=f(x)-g(x),\quad d'(x)=f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\left(kf(x)\right)=k\frac{d}{dx}\left(f(x)\right);$$

that is,

$$\text{for } m(x) = kf(x), \quad m'(x) = kf'(x).$$

Proof

We provide only the proof of the Sum Rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the Sum Law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

Q.E.D.

Example 3.1.7: Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the Power Rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 3.1.1.

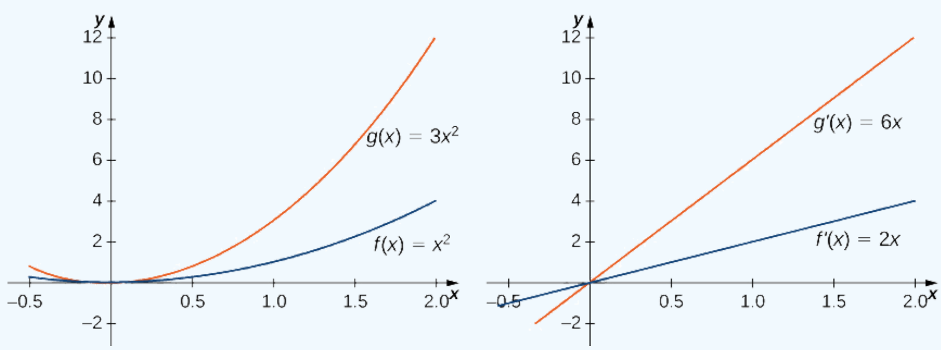

Figure 3.1.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.1.8: Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(2x^5+7\right) && \\
&= \frac{d}{dx}(2x^5)+\frac{d}{dx}(7) && \text{(Apply the Sum Rule.)} \\
&= 2\frac{d}{dx}(x^5)+\frac{d}{dx}(7) && \text{(Apply the Constant Multiple Rule.)} \\
&= 2(5x^4)+0 && \text{(Apply the Power Rule and the Constant Rule.)} \\
&= 10x^4 && \text{(Simplify.)}
\end{aligned}
$$

? Exercise 3.1.8

Find the derivative of $f(x)=2x^3-6x^2+3$.

Hint

Use the preceding example as a guide.

Answer

$f'(x)=6x^2-12x$.

The next example illustrates that we can use several of the rules within a single problem. It also emphasizes the need to pay attention to the Mathematical Mantra.

✓ Example 3.1.9

Compute the derivative of $g(x)=x^2+\frac{3x^3-4x+\sqrt{x}}{x}+x\left(x^2-1\right)^2$

Solution

$$
\begin{aligned}
\frac{d}{dx}\left(x^2+\frac{3x^3-4x+\sqrt{x}}{x}+x\left(x^2-1\right)^2\right) &= \frac{d}{dx}\left(x^2+3x^2-4+\frac{1}{\sqrt{x}}+x\left(x^4-2x^2+1\right)\right) \\
&= \frac{d}{dx}\left(x^2+3x^2-4+\frac{1}{\sqrt{x}}+x^5-2x^3+x\right) \\
&= \frac{d}{dx}\left(x^5-2x^3+4x^2+x+x^{-1/2}-4\right) \\
&= 5x^4-6x^2+8x+1-\frac{1}{2}x^{3/2}
\end{aligned}
$$

Synthesis Topic: Tangent Lines and Their Equations

✓ Example 3.1.10: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x)=x^2-4x+6$ at $x=1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1)=1^2-4(1)+6=3.$$

This gives us the point $(1,3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x)=2x-4$$

so the slope of the tangent line is $f'(1)=-2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y-3=-2(x-1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y=-2x+5.$$

Exercise 3.1.10

Find the equation of the line tangent to the graph of $f(x)=3x^2-11$ at $x=2$. Use the point-slope form.

Hint

Use the preceding example as a guide.

Answer

$y=12x-23$

Example 3.1.11: Determining Where a Function Has a Horizontal Tangent

Find the x-value(s) where the graph of $f(x)=2x^3+6x^2-90x+21$ has horizontal tangents.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x)=0$.

$$\begin{aligned} f'(x) &= 6x^2+12x-90 \\ &= 6\left(x^2+2x-15\right) \\ &= 6(x-3)(x+5) \end{aligned}$$

Hence, f has horizontal tangent lines at $x=3$ and $x=-5$.

Exercise 3.1.11

Find the values of x for which the line tangent to the graph of $f(x)=4x^2-3x+2$ has a tangent line parallel to the line $y=2x+3$.

Hint

Solve $f'(x)=2$.

Answer

$\frac{5}{8}$

Example 3.1.12

For what values a and b is the line $4x+y=b$ tangent to the curve $y=ax^2$ when $x=-4$?

Solution

Careful reading is important in Calculus. We know that, for some special values of a and b, the line $4x+y=b$ will be tangent to $y=ax^2$ at $x=-4$. This fact implies **two** important pieces of information:

1. the slope of the line and the slope of the tangent line to the quadratic function will match at $x=-4$, and
2. the line and the parabola will share a common point at $x=-4$.

Matching Slopes

To find the slope of the line, we place it in slope-intercept form.

$$y=-4x+b$$

Thus, the slope of the line is -4.

To find the slope of the tangent line to $y=ax^2$, we first compute the derivative.

$$y'=2ax$$

Evaluating this at $x=-4$, we get

$$y'(-4)=-8a.$$

Since the slope of the tangent line to the quadratic and the slope of the line are suposed to match at $x=-4$,

$$-4=-8a \implies a=\frac{1}{2}.$$

Common Point

Since the line and the quadratic meet at $x=-4$, their y-values will be the same. That is, at $x=-4$, $-4x+b=ax^2$. Therefore,

$$\begin{aligned} -4(-4)+b &= a(-4)^2 &\implies 16+b &= \frac{1}{2}(16) \\ & &\implies 16+b &= 8 \\ & &\implies b &= -8 \end{aligned}$$

Thus, for the line and quadratic to meet and be tangent at $x=-4$, we require $a=\frac{1}{2}$ and $b=-8$.

Applications of Differentiation: Physics

As we have mentioned previously, if $s(t)$ is the position of a particle at time time, then its velocity is

$$v(t)=\frac{ds}{dt}$$

and its acceleration is

$$a(t)=\frac{dv}{dt}=\frac{ds^2}{dt^2}.$$

Example 3.1.13: Ballistic Motion

A rocket is fired vertically upward. Its height in meters above the ground t seconds after launch is given by

$$h(t)=-4.9t^2+154t+388.$$

a. How high was the rocket when it was launched?
b. How high is the rocket after 6 seconds?
c. What is the velocity of the rocket after 6 seconds?
d. What is the acceleration of the rocket after 6 seconds?

Solution

a. The initial height is given by $h(0)=388$. Therefore, the initial height of the rocket is 388 meters.
b. $h(6)=1,135.6$. Therefore, the height of the rocket 6 seconds after launch is $1,135.6$ meters.
c. To compute the velocity, we need to find the derivative of $h(t)$.

$$v(t)=h'(t)=-9.8t+154$$

Evaluating h' at 6, we get

$$v(6)=h'(6)=-9.8(6)+154=95.2.$$

This means that the velocity of the rocket exactly 6 seconds after launch is 95.2 meters per second. Since this velocity is positive, the rocket is still climbing upward.

d. To compute the acceleration, we need to find the derivative of the velocity.

$$a(t)=v'(t)=-9.8$$

Hence, the acceleration is *always* -9.8 meters per second squared. That is, the rocket is *decelerating* at 9.8 meters per second squared the entire time. While this idea of "slowing down" makes sense on the rise, how can we make sense of it as the rocket begins to fall? Well, the negative acceleration implies the rocket's velocity is decreasing as the rocket rises. At some point, the velocity would have to hit 0. It is at that moment that the rocket begins its descent back to Earth. The velocity becomes negative as the distance between the rocket and the Earth decreases, and the rate of this descent increases (in the negative direction) until the rocket crashes into the ground. The negative acceleration acted to slow the ascent and speed the descent.

Applications of Differentiation: Business

Most individuals taking this course are moving into a STEM field - not business; however, it is always nice to know what the news channels imply when they mention things like revenue, cost, profit, marginal cost, and so on. To simplify the conversation, I have decided to forgo any lengthy discussion and jump right to the point.

- Revenue is the product of the price of the item and the number of items sold. Mathematically, $R(n) = p(n) \cdot n$, where $p(n)$ is the price-demand function (the price the company charges when the market demands n items) and n is the number of items sold.
- Profit is the difference between the revenue from sales and the cost of producing an item. Mathematically, $P(n) = R(n) - C(n)$, where $R(n)$ is the revenue function and $C(n)$ is the cost function.
- Marginal cost is the change in the total cost that arises when the quantity produced is incremented. That is, it is the change in cost with respect to the number of items produced. Mathematically, $M_C(n) = \frac{dC}{dn}$, where $C(n)$ is the cost function (i.e., the cost of producing n items).
- Marginal profit is the change in the total profit that arises when the quantity produced is incremented. That is, it is the change in profit with respect to the number of items produced. Mathematically, $M_P(n) = \frac{dP}{dn}$, where $P(n)$ is the profit function.
- Marginal revenue is the change in the total revenue that arises when the quantity produced is incremented. That is, it is the change in revenue with respect to the number of items produced. Mathematically, $M_R(n) = \frac{dR}{dn}$, where $R(n)$ is the revenue function.

Thus, when you hear "marginal [insert business phrasing here]," it means that you need to take the derivative of that "business phrasing." The result will represent the cost, profit, revenue, etcetra associated with one additional item being produced.

Example 3.1.14

Suppose a product's revenue function is given by $R(q) = -2q^2 + 800q$, where $R(q)$ is in dollars and q is units sold. Also, it's cost function is given by $C(q) = 153q + 40,000$, where C is in dollars and q is units produced. Find an expression for the item's marginal profit function ($MP(q)$) and use it to determine and interpret $MP(220)$.

Solution

Marginal profit is the derivative of the profit function, which they did not *explicitly* hand us; however, we know that profit is revenue minus cost. Thus,

$$P(q) = R(q) - C(q) = -2q^2 + 800q - (153q + 40,000) = -2q^2 + 647q - 40,000.$$

Therefore, the marginal profit function is

$$MP(q) = P'(q) = -4q + 647.$$

Hence,

$$MP(220) = -233.$$

This means that, if the company produces and sells 220 units, their profit will drop by \$233 if they sell another unit. This is usually tough for students to understand at first, so let's take a moment to dive a little deeper. The derivative, $\frac{dP}{dq}$, is the rate of change of profit with respect to the number of units produced (and sold). Since $\frac{dP}{dq}(220) = -233 = \frac{-233}{1}$, we can interpret this as *losing* \$233 in profit per 1 unit produced and sold when we have produced and sold *exactly* 220 units.

It's important to note that marginal profit is *not* the profit. In fact, the profit if the company produces and sells 220 units is $P(220) = 5540$. That is, the company is *still* making a profit of \$5,540 at that moment, but producing more units is going to eat into those profits.

Your Turn

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.1.3).

Figure 3.1.3: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x)=x^3+3x^2+x$ (Figure 3.1.4). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9,2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

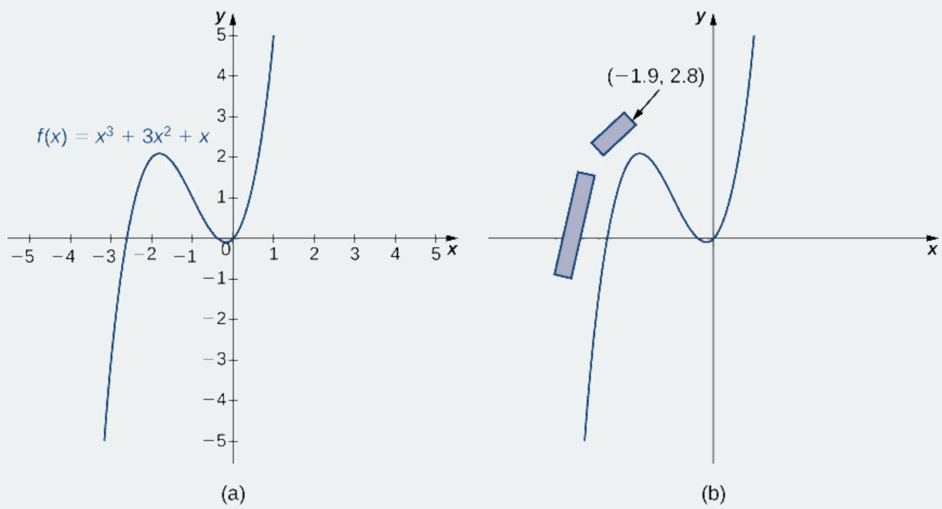

Figure 3.1.4: (a) One section of the racetrack can be modeled by the function $f(x)=x^3+3x^2+x$. (b) The front corner of the grandstand is located at $(-1.9,2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x,y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y=2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5,0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.

- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

Glossary

Constant Multiple Rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big)=cf'(x)$

Constant Rule

the derivative of a constant function is zero: $\frac{d}{dx}(c)=0$, where c is a constant

Difference Rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x)-g(x)\big)=f'(x)-g'(x)$

Power Rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}(x^n)=nx^{n-1}$

Sum Rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x)+g(x)\big)=f'(x)+g'(x)$
