3.2: Differentiation Techniques - The Product and Quotient Rules

Learning Objectives

- Use the Product Rule for finding the derivative of a product of functions.
- Use the Quotient Rule for finding the derivative of a quotient of functions.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and Difference Rules, the **Product Rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x)=x^2$, whose derivative is $f'(x)=2x$ and not $\frac{d}{dx}(x)\cdot\frac{d}{dx}(x)=1\cdot 1=1.$

Theorem 3.2.1: The Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x))=\frac{d}{dx}(f(x))\cdot g(x)+\frac{d}{dx}(g(x))\cdot f(x).$$

That is,

$$\text{if } p(x)=f(x)g(x), \quad \text{then } p'(x)=f'(x)g(x)+g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim\limits_{h\to 0} g(x+h)=g(x)$.

By applying the limit definition of the derivative to $p(x)=f(x)g(x),$ we obtain

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the Sum Law for limits, the derivative becomes

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\cdot f(x)\right) \\ &= \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\cdot\left(\lim_{h\to 0}\ g(x+h)\right)+\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the Limit Laws, we arrive at the Product Rule,

$$p'(x)=f'(x)g(x)+g'(x)f(x).$$

Q.E.D.

Example 3.2.1: Applying the Product Rule to Constant Functions

For $p(x)=f(x)g(x)$, use the Product Rule to find $p'(2)$ if $f(2)=3,\ f'(2)=-4,\ g(2)=1$, and $g'(2)=6$.

Solution

Since $p(x)=f(x)g(x)$, $p'(x)=f'(x)g(x)+g'(x)f(x)$, and hence

$$p'(2)=f'(2)g(2)+g'(2)f(2)=(-4)(1)+(6)(3)=14.$$

Example 3.2.2: Applying the Product Rule to Binomials

For $p(x)=(x^2+2)(3x^3-5x)$, find $p'(x)$ by applying the Product Rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x)=x^2+2$ and $g(x)=3x^3-5x$, then $f'(x)=2x$ and $g'(x)=9x^2-5$. Thus,

$$p'(x)=f'(x)g(x)+g'(x)f(x)=(2x)(3x^3-5x)+(9x^2-5)(x^2+2).$$

Simplifying, we have

$$p'(x)=15x^4+3x^2-10.$$

To check, we see that $p(x)=3x^5+x^3-10x$ and, consequently, $p'(x)=15x^4+3x^2-10$.

Exercise 3.2.2

Use the Product Rule to obtain the derivative of $p(x)=2x^5(4x^2+x)$.

Hint

Set $f(x)=2x^5$ and $g(x)=4x^2+x$ and use the preceding example as a guide.

Answer

$p'(x)=10x^4(4x^2+x)+(8x+1)(2x^5)=56x^6+12x^5.$

Before we move on, it is important to repeat a previous warning.

Caution

$$\frac{d}{dx}(f(x)\cdot g(x))\neq\frac{d}{dx}(f(x))\cdot\frac{d}{dx}(g(x))$$

The Quotient Rule

Having developed and practiced the Product Rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Theorem 3.2.2: The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{\left(g(x)\right)^2}.$$

That is, if

$$q(x) = \frac{f(x)}{g(x)}$$

then

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}.$$

The proof of the Quotient Rule is very similar to the proof of the Product Rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.2.3: Applying the Quotient Rule

Find the derivative of each function.

a. $q(x) = \frac{5x^2}{4x+3}$

b. $s(t) = \frac{\sqrt[3]{t}}{t-5}$

Solution

a. Let $f(x) = 5x^2$ and $g(x) = 4x+3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the Quotient Rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x+3)^2}$$

b. It's best to rewrite s using rational exponents, where possible.

$$s(t) = \frac{t^{1/3}}{t-5}$$

Now we can start to take the derivative of this quotient using the Quotient Rule. For reference, if we let $N(t)$ and $D(t)$ be the numerator and denominator, respectively, then

$$\begin{array}{rclcrcl} N(t) & = & t^{1/3} & \text{and} & D(t) & = & t-5 \\ N'(t) & = & \frac{1}{3}t^{-2/3} & \text{and} & D'(t) & = & 1 \end{array}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}(s(t)) &= \frac{d}{dt}\left(\frac{N(t)}{D(t)}\right) \\ &= \frac{N'(t)D(t)-N(t)D'(t)}{[D(t)]^2} && \text{(Calculus: Quotient Rule)} \\ &= \frac{\left(\frac{1}{3}t^{-2/3}\right)(t-5)-\left(t^{1/3}\right)(1)}{(t-5)^2} \\ &= \frac{(t-5)-3t}{3t^{2/3}(t-5)^2} \\ &= \frac{-2t-5}{3t^{2/3}(t-5)^2}\end{aligned}$$

The previous example demonstrates the need for us to use our prerequisite algebra skills often when using the Quotient Rule. We will commonly have to factor, simplify, and multiply by fractions equivalent to 1 when using this rule.

Exercise 3.2.3

Find the derivative of $h(x)=\frac{3x+1}{4x-3}$.

Hint

Apply the Quotient Rule with $f(x)=3x+1$ and $g(x)=4x-3$.

Answer

$$h'(x)=-\frac{13}{(4x-3)^2}.$$

As with the last topic, a word of caution is merited here.

Caution

$$\frac{d}{dx}\left(\frac{N(x)}{D(x)}\right) \neq \frac{\frac{d}{dx}(N(x))}{\frac{d}{dx}(D(x))}$$

Revisiting the Power Rule

It is now possible to use the Quotient Rule to partially prove the General Power Rule. Recall that in the previous section we proved the Power Rule for *positive integers* but stated that we had to wait until later in the course to prove the Power Rule for the general case when the exponent is any real number. It is at this point that we can add to our previous proof by extending the Power Rule to negative integers as well. This proof is provided for completeness.

Proof

Let $f(x)=x^k$. If k is a negative integer, we may set $n=-k$, so that n is a positive integer with $k=-n$. Since for each positive integer n,$x^{-n}=\frac{1}{x^n}$, we may now apply the Quotient Rule by setting $f(x)=1$ and $g(x)=x^n$. In this case, $f'(x)=0$ and $g'(x)=nx^{n-1}$. Thus,

$$\frac{d}{dx}\left(x^{-n}\right)=\frac{0(x^n)-1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned}\frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}.\end{aligned}$$

Finally, observe that since $k=-n$, by substituting we have

$$\frac{d}{dx}(x^k)=kx^{k-1}.$$

Q.E.D.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.2.4: Combining Differentiation Rules

For $k(x)=3h(x)+x^2g(x)$, find $k'(x)$.

Solution

Finding this derivative requires the Sum Rule, the Constant Multiple Rule, and the Product Rule.

$$\begin{aligned}k'(x) &= \frac{d}{dx}\big(3h(x)+x^2g(x)\big) \\ &= \frac{d}{dx}\big(3h(x)\big)+\frac{d}{dx}\big(x^2g(x)\big) && \text{(Apply the Sum Rule.)} \\ &= 3\frac{d}{dx}\big(h(x)\big)+\left(\frac{d}{dx}(x^2)g(x)+\frac{d}{dx}(g(x))x^2\right) && \text{(Apply the Constant Multiple and Product Rules.)} \\ &= 3h'(x)+2xg(x)+g'(x)x^2\end{aligned}$$

Example 3.2.5: Extending the Product Rule

For $k(x)=f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x)=(f(x)g(x))\cdot h(x)$. Thus,

$$\begin{aligned}k'(x) &= \frac{d}{dx}\big(f(x)g(x)\big)\cdot h(x)+\frac{d}{dx}\big(h(x)\big)\cdot\big(f(x)g(x)\big) && \text{(Apply the Product Rule.)} \\ &= \big(f'(x)g(x)+g'(x)f(x)\big)h(x)+h'(x)f(x)g(x) && \text{(Apply the Product Rule.)} \\ &= f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)\end{aligned}$$

Example 3.2.6: Combining the Quotient Rule and the Product Rule

For $h(x)=\frac{2x^3k(x)}{3x+2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}(2x^3k(x))\cdot(3x+2)-\frac{d}{dx}(3x+2)\cdot(2x^3k(x))}{(3x+2)^2} && \text{(Apply the Quotient Rule.)} \\ &= \frac{(6x^2k(x)+k'(x)\cdot 2x^3)(3x+2)-3(2x^3k(x))}{(3x+2)^2} && \text{(Apply the Product Rule.)} \\ &= \frac{-6x^3k(x)+18x^3k(x)+12x^2k(x)+6x^4k'(x)+4x^3k'(x)}{(3x+2)^2} \end{aligned}$$

Key Concepts

- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.

Glossary

Product Rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\dfrac{d}{dx}\big(f(x)g(x)\big)=f'(x)g(x)+g'(x)f(x)$

Quotient Rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: \(\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{f′(x)g(x)−g′(x)f(x)}{\big(g(x)\big)^2}\