# 1.4: Inverse Functions

## Learning Objectives

- Determine the conditions for when a function has an inverse.
- Use the Horizontal Line Test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse.

## Existence of an Inverse Function

We begin with an example. Given a function $f$ and an output $y=f(x)$, we are often interested in finding what value or values $x$ were mapped to $y$ by $f$. For example, consider the function $f(x)=x^3+4$. Since any output $y=x^3+4$, we can solve this equation for $x$ to find that the input is $x=\sqrt[3]{y-4}$. This equation defines $x$ as a function of $y$. Denoting this function as $f^{-1}$, and writing $x=f^{-1}(y)=\sqrt[3]{y-4}$, we see that for any $x$ in the domain of $f, f^{-1}(f(x))=f^{-1}(x^3+4)=x$. Thus, this new function, $f^{-1}$, "undid" what the original function $f$ did. A function with this property is called the **inverse function** of the original function.

### Definition: Inverse Functions

Given a function $f$ with domain $D$ and range $R$, its **inverse function** (if it exists) is the function $f^{-1}$ with domain $R$ and range $D$ such that $f^{-1}(y)=x$ if and only if $f(x)=y$. In other words, for a function $f$ and its inverse $f^{-1}$,

$$f^{-1}(f(x))=x$$

for all $x$ in $D$ and

$$f(f^{-1}(y))=y$$

for all $y$ in $R$.

Note that $f^{-1}$ is read as "$f$ inverse." Here, the $-1$ is not used as an exponent so

$$f^{-1}(x)\neq\frac{1}{f(x)}.$$

Figure $1.4.1$shows the relationship between the domain and range of $f$ and the domain and range of $f^{-1}$.

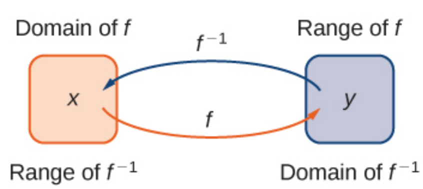



Figure 1.4.1: Given a function $f$ and its inverse $f^{-1}, f^{-1}(y)=x$ if and only if $f(x)=y$. The range of $f$ becomes the domain of $f^{-1}$ and the domain of $f$ becomes the range of $f^{-1}$.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let’s try to find the inverse function for $f(x)=x^2$. Solving the equation $y=x^2$ for $x$, we arrive at the equation $x=\pm\sqrt{y}$. This equation does not describe $x$ as a function of $y$ because there are two solutions to this equation for every $y>0$. The problem with trying to find an inverse function for $f(x)=x^2$ is that two inputs are sent to the same output for each output $y>0$. The function $f(x)=x^3+4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a **one-to-one function**.

**Definition: One-to-One functions**

We say a function $f$ is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$ .

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the $xy$-plane, according to the **Horizontal Line Test**, it cannot intersect the graph more than once. We note that the Horizontal Line Test is different from the Vertical Line Test. The Vertical Line Test determines whether a graph is the graph of a function. The Horizontal Line Test determines whether a function is one-to-one (Figure $1.4.2$).

**Theorem 1.4.1: Horizontal Line Test**

A function $f$ is one-to-one if and only if every horizontal line intersects the graph of $f$ no more than once.

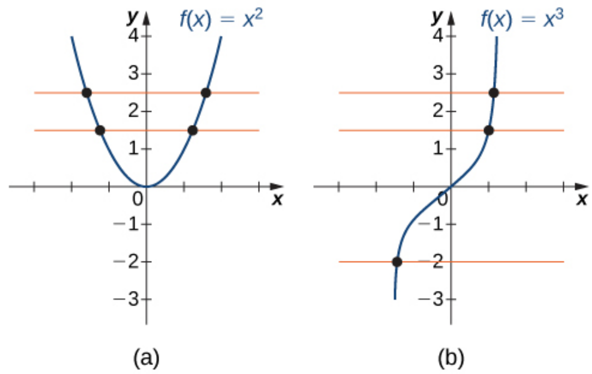
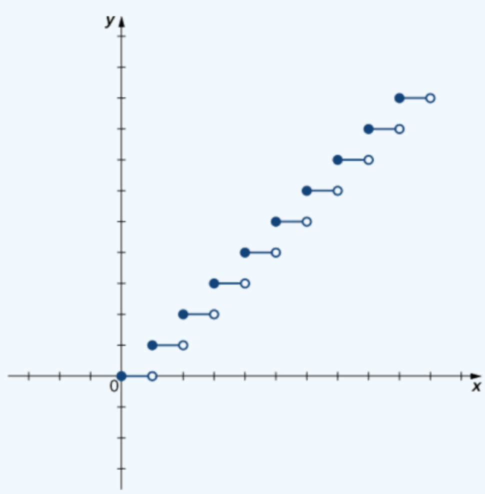



Figure $1.4.2$: (a) The function $f(x) = x^2$ is not one-to-one because it fails the Horizontal Line Test. (b) The function $f(x) = x^3$ is one-to-one because it passes the Horizontal Line Test.

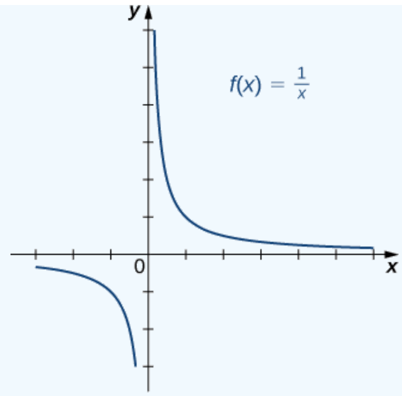
**Example 1.4.1: Determining Whether a Function Is One-to-One**

For each of the following functions, use the Horizontal Line Test to determine whether it is one-to-one.

a)

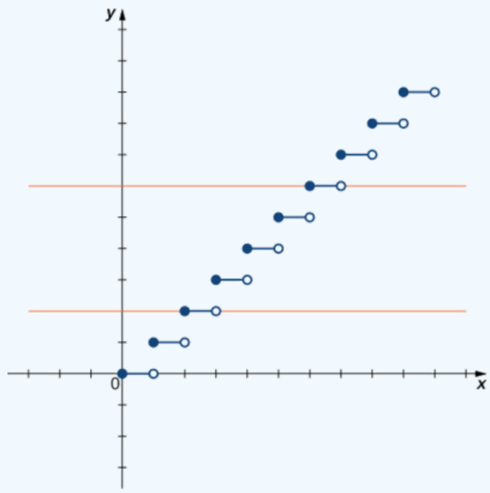



b)

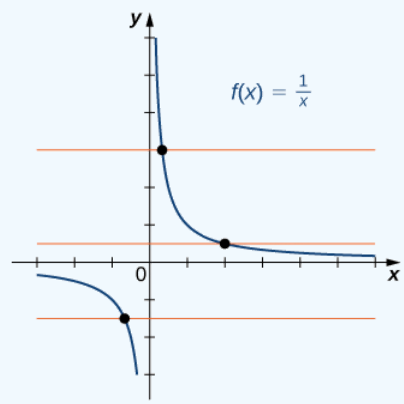



## Solution

a) Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.

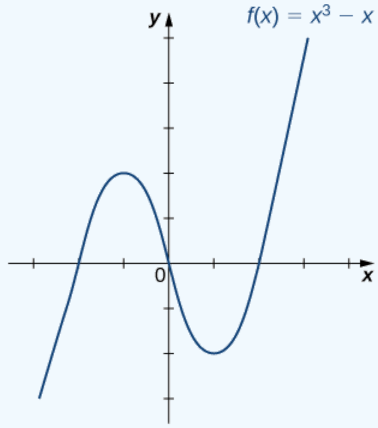



b) Since every horizontal line intersects the graph once (at most), this function is one-to-one.




## ? Exercise 1.4.1

Is the function $f$ graphed in the following image one-to-one?




**Solution**

Use the Horizontal Line Test.

**Answer**

No

## Finding the Inverse of a Function

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of $f$ to elements in the range of $f$. The inverse function maps each element from the range of $f$ back to its corresponding element from the domain of $f$. Therefore, to find the inverse function of a one-to-one function $f$, given any $y$ in the range of $f$, we need to determine which $x$ in the domain of $f$ satisfies $f(x)=y$. Since $f$ is one-to-one, there is exactly one such value $x$. We can find that value $x$ by solving the equation $f(x)=y$ for $x$. Doing so, we are able to write $x$ as a function of $y$ where the domain of this function is the range of $f$ and the range of this new function is the domain of $f$. Consequently, this function is the inverse of $f$, and we write $x=f^{-1}(y)$. Since we typically use the variable $x$ to denote the independent variable and y to denote the dependent variable, we often interchange the roles of $x$ and $y$, and write $y=f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function $f$ and its inverse $f^{-1}$ on the same axes.

### Example 1.4.2: Finding an Inverse Function

Find the inverse for the function $f(x)=3x-4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x))=x$.

**Solution**

If $y=3x-4$, then $3x=y+4$ and $x=\frac{1}{3}y+\frac{4}{3}$.

Rewrite as $y=\frac{1}{3}x+\frac{4}{3}$ and let $y=f^{-1}(x)$.Therefore, $f^{-1}(x)=\frac{1}{3}x+\frac{4}{3}$ .

Since the domain of $f$ is $(-\infty,\infty)$, the range of $f^{-1}$ is $(-\infty,\infty)$. Since the range of $f$ is $(-\infty,\infty)$, the domain of $f^{-1}$ is $(-\infty,\infty)$.

You can verify that $f^{-1}(f(x))=x$ by writing

$$f^{-1}(f(x))=f^{-1}(3x-4)=\frac{1}{3}(3x-4)+\frac{4}{3}=x-\frac{4}{3}+\frac{4}{3}=x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$ for all $x$ in the domain of the inside function.

## ? Exercise 1.4.2

Find the inverse of the function $f(x) = 3x/(x-2)$. State the domain and range of the inverse function.

**Answer**

$f^{-1}(x) = \dfrac{2x}{x-3}$ . The domain of $f^{-1}$ is $\{x \mid x \neq 3\}$. The range of $f^{-1}$ is $\{y \mid y \neq 2\}$.

## Graphing Inverse Functions

Let's consider the relationship between the graph of a function $f$ and the graph of its inverse. Consider the graph of $f$ shown in Figure 1.4.3 and a point $(a, b)$ on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph $f^{-1}$, the point $(b, a)$ is on the graph. As a result, the graph of $f^{-1}$ is a reflection of the graph of $f$ about the line $y = x$.

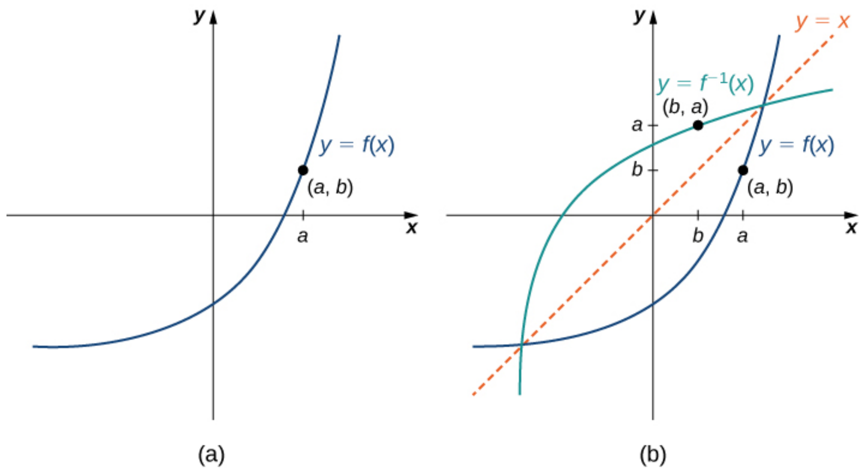
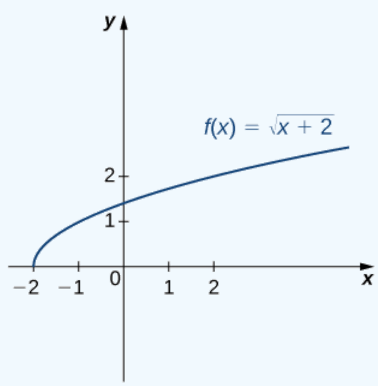



Figure 1.4.3: (a) The graph of this function $f$ shows point $(a, b)$ on the graph of $f$. (b) Since $(a, b)$ is on the graph of $f$, the point $(b, a)$ is on the graph of $f^{-1}$. The graph of $f^{-1}$ is a reflection of the graph of $f$ about the line $y = x$.

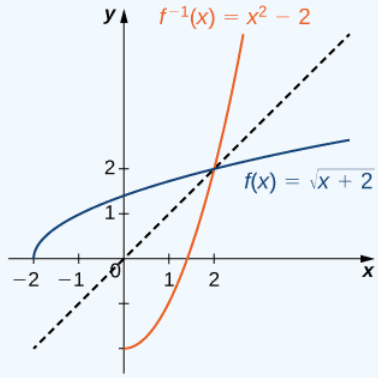
## ✓ Example 1.4.3: Sketching Graphs of Inverse Functions

For the graph of $f$ in the following image, sketch a graph of $f^{-1}$ by sketching the line $y = x$ and using symmetry. Identify the domain and range of $f^{-1}$.




### Solution

Reflect the graph about the line $y = x$. The domain of $f^{-1}$ is $[0, \infty)$. The range of $f^{-1}$ is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$ , as shown in the graph.



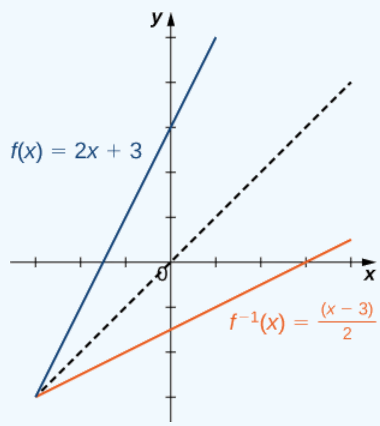

## ? Exercise 1.4.3

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

**Hint**

The graphs are symmetric about the line $y = x$

**Answer**




## Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of $f$ such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of $f$, we can define a new function $g$ such that the domain of $g$ is the restricted domain of $f$ and $g(x) = f(x)$ for all $x$ in the domain of $g$. Then we can define an inverse function for $g$ on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function $g$ such that the domain of $g$ is $[0, \infty)$ and $g(x) = x^2$ for all $x$ in its domain. Since $g$ is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function $h$ such that the domain of $h$ is $(-\infty, 0]$ and $h(x) = x^2$ for all $x$ in the domain of $h$. Then $h$ is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (Figure 1.4.4).

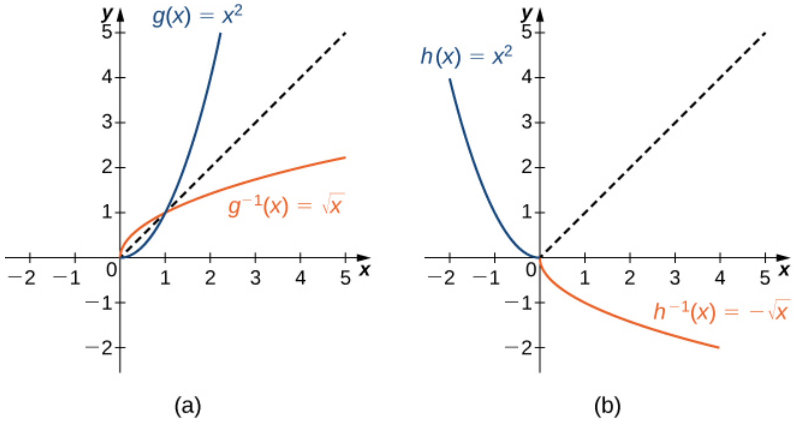



Figure 1.4.4: (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

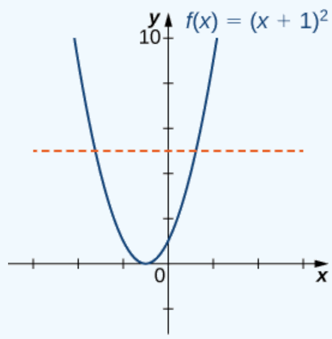
**Example 1.4.4: Restricting the Domain**

Consider the function $f(x) = (x+1)^2$.

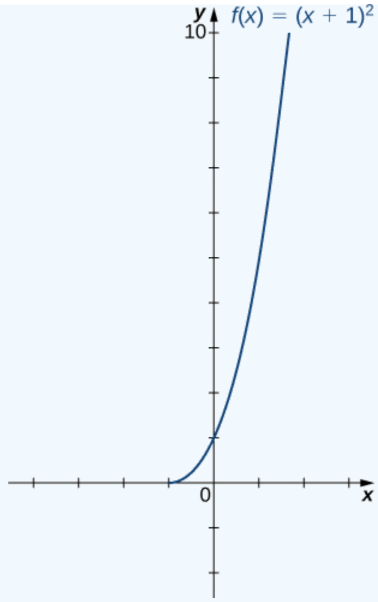
a. Sketch the graph of $f$ and use the Horizontal Line Test to show that $f$ is not one-to-one.
b. Show that $f$ is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of $f$ on this restricted domain and find a formula for $f^{-1}$.

## Solution

a) The graph of $f$ is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, $f$ is not one-to-one.




b) On the interval $[-1, \infty)$, $f$ is one-to-one.




The domain and range of $f^{-1}$ are given by the range and domain of $f$, respectively. Therefore, the domain of $f^{-1}$ is $[0, \infty)$ and the range of $f^{-1}$ is $[-1, \infty)$. To find a formula for $f^{-1}$, solve the equation $y=(x+1)^2$ for $x$. If $y=(x+1)^2$, then $x=-1\pm\sqrt{y}$. Since we are restricting the domain to the interval where $x\geq -1$, we need $\pm\sqrt{y}\geq 0$. Therefore, $x=-1+\sqrt{y}$. Interchanging $x$ and $y$, we write $y=-1+\sqrt{x}$ and conclude that $f^{-1}(x)=-1+\sqrt{x}$.

**? Exercise 1.4.4**

Consider $f(x)=1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that $f$ is one-to-one on this domain. Determine the domain and range of the inverse of $f$ and find a formula for $f^{-1}$.

**Hint**

The domain and range of $f^{-1}$ is given by the range and domain of $f$, respectively. To find $f^{-1}$, solve $y=1/x^2$ for $x$.

**Answer**

The domain of $f^{-1}$ is $(0, \infty)$. The range of $f^{-1}$ is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x)=-1/\sqrt{x}$.

## Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the Horizontal Line Test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function $f$ and its inverse $f^{-1}$, $f(f^{-1}(x))=x$ for all $x$ in the domain of $f^{-1}$ and $f^{-1}(f(x))=x$ for all $x$ in the domain of $f$.
- The graph of a function $f$ and its inverse $f^{-1}$ are symmetric about the line $y=x$.

## Key Equations

- **Inverse function**

  $f^{-1}(f(x))=x$ for all $x$ in $D$, and $f(f^{-1}(y))=y$ for all $y$ in $R$.

## Glossary

**Horizontal Line Test**

a function $f$ is one-to-one if and only if every horizontal line intersects the graph of $f$, at most, once

**inverse function**

for a function $f$, the inverse function $f^{-1}$ satisfies $f^{-1}(y)=x$ if $f(x)=y$

**one-to-one function**

a function $f$ is one-to-one if $f(x_1)\neq f(x_2)$ if $x_1\neq x_2$

**restricted domain**

a subset of the domain of a function $f$

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