

2.1: Tangent Lines and Velocity

Learning Objectives

- Relate the rate of change of a function to the slope of a secant line.
- Describe the concept and process of approximating the tangent line to a function at a given point.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Distinguish when technology is appropriate to use within a given problem.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve. Chapters two through four focus on the first of these problems. We will approach the area problem at the end of this course.

Rates of Change

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x)=-2x-3,\ g(x)=\frac{1}{2}x+1$, and h(x)=2, shown in Figure 2.1.1.

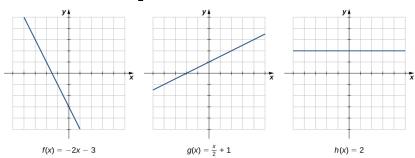


Figure 2.1.1: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of f(x) = -2x - 3, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope (-2) of the line. Similarly, the slope of 1/2 in the function g(x) tells us that for every change in x of 1 unit there is a corresponding change in y of 1/2 unit. The function h(x) = 2 has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

A Theorem 2.1.1: Lines have constant rates of change

The rate of change at any given point, (a, f(a)), of a linear function is constant and equal to the slope of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 2.1.2). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase - slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function? Since the rate of change for a nonlinear function is ever-changing, is it possible to determine the rate of change at a specific point?



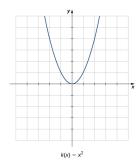


Figure 2.1.2: The function $k(x) = x^2$ does not have a constant rate of change.

Before we go any further, a word of caution is needed.

Caution: Nonlinear functions do not have slopes

Lines have slopes; however, nonlinear curves do not. That is, it's incorrect to talk about the "slope" of the nonlinear function $k(x) = x^2$.

How do we overcome this restriction in language? When referencing the "steepness" of a nonlinear function at a point (a, f(a)), we instead refer to the slope of the **tangent line** at this point.

Definition: Tangent Line (Informal)

In geometry, the **tangent line** (or simply **tangent**) to a plane curve at a given point is the straight line that "just touches" the curve at that point. Leibniz defined it as the line through a pair of infinitely close points on the curve.

As it passes through the point where the tangent line and the curve meet, called the **point of tangency**, the tangent line is "going in the same direction" as the curve, and is thus the best straight-line approximation to the curve at that point.

Therefore, if you wanted to tell someone how steep the function $k(x) = x^2$ is at the point (a, f(a)), you would reference the slope of the tangent line to the function at that point. Another phrase used for the slope of the tangent line is the **instantaneous rate of change** of f(x) at (a, f(a)).

Tangent and Secant Lines

We can approximate the instantaneous rate of change of a function f(x) at a point (a, f(a)) on its graph by taking another point (x, f(x)) on the graph of f(x), drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant line**. Figure 2.1.3 shows a secant line to a function f(x) at a point (a, f(a)).

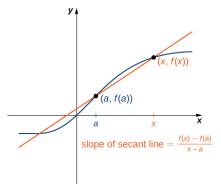


Figure 2.1.3: The slope of a secant line through a point (a, f(a)) estimates the instantaneous rate of change of the function at the point (a, f(a)).

We formally define a secant line as follows:



Definition: Secant Line

The **secant line** (also known simply as the **secant**) to the function f(x) through the points (a, f(a)) and (x, f(x)) is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$
 (2.1.1)

The accuracy of approximating the instantaneous rate of change of the function with a secant line depends on how close x is to a. As we see in Figure 2.1.4, if x is closer to a, the slope of the secant line looks to be a better measure of the instantaneous rate of change of f(x) at a. There are times when dragging (x, f(x)) closer to (a, f(a)) will not provide a better estimate of the tangent line, but we will avoid those instances for now.

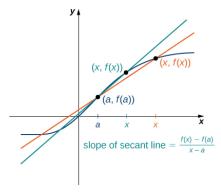


Figure 2.1.4: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function f(x) at a.

The secant lines themselves approach the tangent line to the function f(x) at a (Figure 2.1.5). As stated previously, the slope of the tangent line to the graph at a is a measure of the instantaneous rate of change of the function at a. This value is incredibly important in calculus.

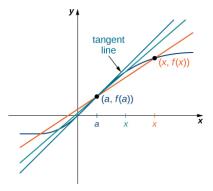


Figure 2.1.5: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Note: Anchor Points

This entire section is focused on using the slopes of the secant lines to *approximate* the slope of the tangent line to f(x) at (a, f(a)). The point (a, f(a)) is commonly referred to as the **anchor point** because it is "anchored" to a single spot while the secondary point, (x, f(x)), is moved toward it. In Figures 2.1.4 and 2.1.5, you can see the secant lines each include the anchor point (a, f(a)).

Approximating the Slope of the Tangent Line

The next example illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the instantaneous rate of change of the function at the point at which the slopes are calculated.



Example 2.1.1: Finding Slopes of Secant Lines

Estimate the instantaneous rate of change (slope of the tangent line) to $f(x) = x^2$ at x = 1 by finding slopes of secant lines through (1,1) and each of the following points on the graph of $f(x) = x^2$.

a.
$$(2,4)$$

b. $\left(\frac{3}{2},\frac{9}{4}\right)$

Solution

Use the formula for the slope of a secant line (Equation 2.1.1).

a.
$$m_{sec}=rac{4-1}{2-1}=3$$
b. $m_{sec}=rac{rac{9}{4}-1}{rac{3}{2}-1}=rac{5}{2}=2.5$

The point in part b. is closer to the point (1,1), so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure 2.1.6).

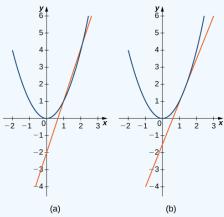


Figure 2.1.6: The secant lines to $f(x) = x^2$ at (1,1) through (a) (2,4) and (b) $(\frac{3}{2},\frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at (1,1).

? Exercise 2.1.1

Estimate the slope of the tangent line (instantaneous rate of change) to $f(x) = x^2$ at x = 1 by finding slopes of secant lines through (1,1) and the point $(\frac{5}{4},\frac{25}{16})$ on the graph of $f(x) = x^2$.

Answer

2.25

✓ Example 2.1.2: Finding Slopes of Secant Lines

Estimate the slope of the tangent line to the function $g(x) = -2\sin(\pi x) + 2$ at $x = \frac{1}{4}$ to three decimal places.

Solution



In this example, we are not given an *explicit* list of points to try. Instead, there is an *implicit* assumption that we will create our own list of x-values approaching $\frac{1}{4}$. It is important to include x-values both greater than and less than $\frac{1}{4}$ in our trials. I will demonstrate two different approaches *both of which should be read for understanding*.

In either approach, we will need to use the formula for the slope of the secant line (Equation 2.1.1); however, the implementation of the formula will differ between methods. Before going into the methods, we need to compute $g(\frac{1}{4})$.

$$g\left(\frac{1}{4}\right) = -2\sin\left(\frac{\pi}{4}\right) + 2$$
$$= -\frac{2}{\sqrt{2}} + 2$$
$$= -\sqrt{2} + 2$$
$$= 2 - \sqrt{2}$$

Method #1: Explicitly Choosing Values Near 1/4 = 0.25

The first approach is the one most students will start with because it mimics how they learned slope. We successively pick numbers closer and closer to 0.25 (from the right and left of 0.25 on the number line) to get an approximation to the slope of the tangent line.

Prior to "plugging in" values near 0.25, we write the secant slope formula, substitute any values we have been given (or have computed), and then simplify.

$$m_{sec} = rac{g(x) - g(a)}{x - a}$$

$$= rac{-2\sin(\pi x) + 2 - g\left(\frac{1}{4}\right)}{x - \frac{1}{4}}$$

$$= rac{-2\sin(\pi x) + 2 - (2 - \sqrt{2})}{x - \frac{1}{4}}$$

$$= rac{-2\sin(\pi x) + 2 - 2 + \sqrt{2}}{x - \frac{1}{4}}$$

$$= rac{-2\sin(\pi x) + \sqrt{2}}{x - \frac{1}{4}}$$

$$= rac{-8\sin(\pi x) + 4\sqrt{2}}{4x - 1}$$

Being able to do the algebraic "pre-work" for a problem is critical to success in calculus. If you were lost on any of the manipulations above, it is best to review Chapter 1 of this text, or review your College Algebra textbook.

At this point, we are ready to substitute values for x that approach 0.25. The following table contains values approaching 0.25 from the right (on the number line).

$oldsymbol{x}$	$m_{sec}=rac{-8\sin(\pi x)+4\sqrt{2}}{4x-1}$
0.35	≈ -3.6779949
0.26	≈ -4.3723692
0.251	pprox -4.4358968
0.2501	pprox -4.442185
0.25001	pprox -4.4428131
0.250001	pprox -4.442876



The next table shows similar computations for values of x approaching 0.25 from the left (on the number line).

x	$m_{sec}=rac{-8\sin(\pi x)+4\sqrt{2}}{4x-1}$
0.15	≈ -5.0623256
0.24	≈ -4.5119351
0.249	≈ -4.4498545
0.2499	pprox -4.4435808
0.24999	≈ -4.4429527
0.249999	pprox -4.4428899

In both cases, it looks like the slopes of the secant lines are approaching roughly -4.443 (when rounded to the third decimal place).

🖐 Caution: Calculus Requires Radians

Are your values not matching those in the table?

Remember, Calculus requires angles in radian mode. From this point forward, whenever you are working on a problem involving trigonometric functions and you require a calculator, be sure to set the calculator to radian mode!

Method #2: Choosing "Wiggles"

The second approach is the preferred approach for the rest of calculus. We successively pick smaller and smaller numbers to decrease the "wiggled distance" between a and x.

The "pre-work" is slightly different than what we did previously. Instead working with x, we let x = a + h and allow h to get smaller and smaller. The lower the value of h, the closer x = a + h is to a. Like before, prior to "plugging in" smaller and smaller values of h, we write the secant slope formula, substitute any values we have been given (or have computed), and then simplify.

$$m_{sec} = \frac{g(x) - g(a)}{x - a}$$

$$= \frac{g(a+h) - g(a)}{a+h-a}$$

$$= \frac{g(a+h) - g(a)}{h}$$

$$= \frac{g(0.25+h) - g(0.25)}{h}$$

$$= \frac{-2\sin(\pi(0.25+h)) + 2 - g(0.25)}{h}$$

$$= \frac{-2\sin(\pi(0.25+h)) + 2 - (2 - \sqrt{2})}{h}$$

$$= \frac{-2\sin(\pi(0.25+h)) + 2 - 2 + \sqrt{2}}{h}$$

$$= \frac{-2\sin(\pi(0.25+h)) + \sqrt{2}}{h}$$

At this point, we are ready to substitute values for h that approach 0. The following table contains values approaching 0 from the right (on the number line).

$$h = m_{sec} = rac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}$$



h	$m_{sec}=rac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}$
0.1	≈ -3.6779949
0.01	≈ -4.3723692
0.001	≈ -4.4358968
0.0001	pprox -4.442185
0.00001	≈ -4.4428131
0.000001	pprox -4.442876

The next table shows similar computations for values of x approaching 0 from the left (on the number line).

h	$m_{sec}=rac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}$
-0.1	pprox -5.0623256
-0.01	≈ -4.5119351
-0.001	pprox -4.4498545
-0.0001	pprox -4.4435808
-0.00001	≈ -4.4429527
-0.000001	pprox -4.4428899

In both cases, it looks like the slopes of the secant lines are approaching roughly -4.443 (when rounded to the third decimal place).

Result

In the end, both methods produce the same results, however, the second method demonstrates a significant building block for the study of calculus. While the "pre-work" was slightly more involved, the inputs (the values that were being substituted in for h) not only were easier to work with, but conceptually made sense. We are just decreasing the "space" between the anchor point and (x, f(x)) to something near zero.

Instantaneous Velocity

We continue our investigation by exploring a related question. Keeping in mind that **velocity** may be thought of as the rate of change of position, suppose that we have a function, s(t), that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time t = a? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let s(t) be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval [a,t] where a < t (or [t,a] if t < a) is

$$v_{avg} = \frac{s(t) - s(a)}{t - a}$$
. (2.1.2)

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the



instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition: Instantaneous Velocity

For a position function s(t), the **instantaneous velocity** at a time t = a is the value that the average velocities approach on intervals of the form [a,t] and [t,a] as the values of t become closer to a, provided such a value exists.

Example 2.1.3 illustrates this concept of limits and average velocity.

✓ Example 2.1.3: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time t = 0.5.

Solution

Substitute the data into Equation 2.1.2 for the definition of average velocity.

a.
$$v_{avg}=\frac{s(0.49)-s(0.5)}{0.49-0.5}=-15.84$$
 b.
$$v_{avg}=\frac{s(0.51)-s(0.5)}{0.51-0.5}=-16.016$$

The instantaneous velocity is somewhere between -15.84 and -16.16 ft/sec. A good guess might be -16 ft/sec.

? Exercise 2.1.3

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time t = 2 by computing its average velocity over the time interval [2, 2.001].

Hint

Use Equation 2.1.2with
$$v_{avg} = rac{s(2.001) - s(2)}{2.001 - 2}$$
 .

Answer

12.006001

Technology in Calculus

Before we leave this section, it is a good time to mention an important point about using technology in this course. Most of your calculus course should be done *without* the use of technology. As such, you might find it difficult to identify *when* you are allowed to use technology to help you with a problem. Moreover, if you are allowed technology on a given problem, it is critical to know *how* to use that technology.

When to Use Technology

An exercise or homework problem typically requires the using of technology if any of the following are true:

- you are asked to round the answer to a certain number of decimal places or significant figures
- the problem specifically states that a calculator or graphing technology must be used
- you are asked to "build a table of values" or "obtain a numerical approximation"



How to Efficiently Use Technology to Build Tables

For the material in this section, understanding how to effectively and efficiently use technology to build tables of values is critical. The following example illustrates the importance of knowing how to employ these skills, along with the significance of having a mastery of our prerequisite algebra.

\checkmark Example 2.1.2 Revisted: Efficiently Approximating the Instantaneous Rate of Change Using Algebra and Technology

Estimate the instantaneous rate of change of the function $g(x) = -2\sin(\pi x) + 2$ at $x = \frac{1}{4}$ to three decimal places.

Solution

The fact that we are being asked to approximate to three decimal places means that we can grab some type of technology to help us with the computations. Recall that, with the second method, we eventually arrived at

$$m_{sec}=rac{-2\sin\left(\pi(0.25+h)
ight)=\sqrt{2}}{h}.$$

If you go to the Desmos graphing calculator, you can insert a table by clicking on the "plus" sign and selecting "table".



You will be presented with an unfilled table where the independent variable is x_1 and the dependent variable is y_1 . Replace y_1 with $\frac{-2\sin(\pi(0.25+x_1))=\sqrt{2}}{x_1}$ (notice I have traded out h for x_1 in our secant slope formula). At this point, you can fill in the values for x_1 (which are our values of h).

	$\frac{-2\sin(\pi(0.25+x_1))+}{}$
<i>x</i> ₁	x ₁
0.1	-3.6779949
0.01	-4.3723692
0.001	-4.4358968
0.0001	-4.442185
0.00001	-4.4428131
0.000001	-4.442876
-0.1	-5.0623256
-0.01	-4.5119351
-0.001	-4.4498545
-0.0001	-4.4435808
-0.00001	-4.4429527
-0.000001	-4.4428899

The graphing calculator in Desmos is in radian mode by default, so you do not have to worry about incorrect values in a calculus course (however, the scientific calculator in Desmos is in degree mode by default).

∓ Note

This table of values also can be built in Excel, Google Sheets, and many other applications very easily.



Key Concepts

• Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the instantaneous rate of change of the function. Calculating a precise value for this instantaneous rate of change requires finding the limits of the slopes of secant lines as they progressively get closer to the tangent line.

Key Equations

• Slope of a Secant Line (also known as the Average Rate of Change of f(x) on the interval [x,a] (if x < a) or [a,x] (if a < x))

$$m_{sec} = rac{f(x) - f(a)}{x - a}$$

• Average Velocity over Interval [a,t]

$$v_{avg} = rac{s(t) - s(a)}{t - a}$$

Common Mistakes

- Calling the instantaneous rate of change of a nonlinear function "slope"
- Using degrees or degree mode while working on calculus problems

Glossary

average rate of change (also known as the rate of change)

the change in a function's values divided by the corresponding change in its arguments; this average rate of change is given by $m_{sec}=\frac{f(x)-f(a)}{x-a}$ average velocity

average velocity

the change in an object's position divided by the length of a time period; this average velocity is given by

$$v_{avg} = rac{s(t) - s(a)}{t - a}$$

instantaneous rate of change

the value that the average rates of change approach for a given function on intervals of the form [x, a] and [a, x] as the values of x move closer to a, provided such a value exists

instantaneous velocity

the value that the average velocities approach for a given position function on intervals of the form [t, a] and [a, t] as the values of t move closer to a, provided such a value exists

limit

the process of letting x or t approach a in an expression; the limit of a function f(x) as x approaches a is the value that f(x) approaches as x approaches a

secant

a line through an "anchor point" (a,f(a)) and another point on the function; the slope of the secant line is given by $m_{sec}=rac{f(x)-f(a)}{x-a}$

tangent

a line that secant lines through (a, f(a)) approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the instantaneous rate of change of the function at a

2.1: Tangent Lines and Velocity is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

• 2.1: A Preview of Calculus by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.