2.2: The Limit of a Function - A Numerical and Graphical Investigation

Learning Objectives

- Describe the limit of a function and use proper limit notation.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define and investigate one-sided limits.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote in terms of limits.
- Determine the value of a limit near a vertical asymptote conceptually.
- Explain why investigating limits using technology is inherently flawed.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x)=\dfrac{x^2-4}{x-2}$,
- $g(x)=\dfrac{|x-2|}{x-2}$, and
- $h(x)=\dfrac{1}{(x-2)^2}$,

which are shown in Figure $2.2.1$. In particular, let's focus our attention on the behavior of each graph at and around $x=2$.

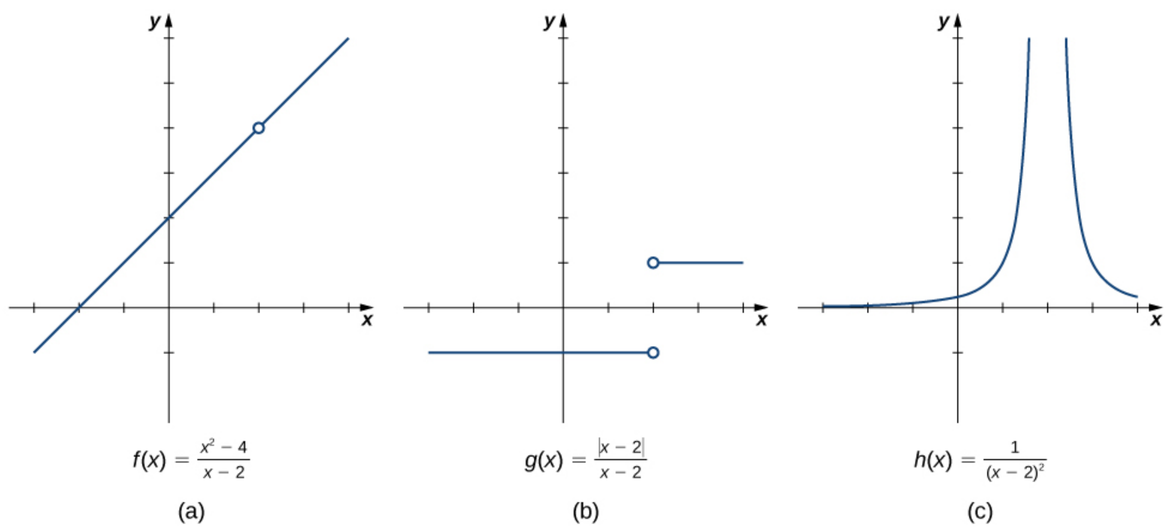

Figure $2.2.1$: These graphs show the behavior of three different functions around $x=2$.

Each of the three functions is undefined at $x=2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x=2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Finite Limit at a Finite Number

Let's first take a closer look at how the function $f(x)=(x^2-4)/(x-2)$ behaves around $x=2$ in Figure $2.2.1$. As the values of x approach 2 from either side of 2, the values of $y=f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x

approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x\to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the finite limit at a finite number*. We can think of the limit of a function at a *finite number* a as being the one real finite number L (the *finite limit*) that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Finite Limit at a Finite Number

Let $f(x)$ be a function defined at all values in an open interval containing a finite number a, with the possible exception of a itself, and let L be a finite real number. If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the **finite limit** of $f(x)$ as x approaches the **finite number** a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x\to a} f(x) = L. \tag{2.2.1}$$

Moreover, most people call this a **limit** (rather than a finite limit at a finite number).

Note: $x \neq a$

The intuitive definition of a finite limit at a finite number states that x need not equal a. This is critical to understand. The x-values are allowed to get very close to a, but will never "touch" a. In a very real sense, the limit

$$\lim_{x\to a} f(x) = L$$

is concerned with the function values as x gets closer and closer to a; however, it is *not* asking, "What is $f(a)$?" In fact, it could be the case that $f(a)$ does exist or $L \neq f(a)$.

In this section, we explore how to *estimate* the value of a limit using two methods - tables and graphs. While these methods are introduced to gain an intuitive understanding of limits, it is important to state that both of these methods are not allowable to investigate limits outside of this section.

Using Tables to Estimate Limits

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x\to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table $2.2.1$ demonstrates what your tables *might* look like.

Table 2.2.1

x	$f(x)$	x	$f(x)$
$a-0.1$	$f(a-0.1)$	$a+0.1$	$f(a+0.1)$
$a-0.01$	$f(a-0.01)$	$a+0.01$	$f(a+0.01)$
$a-0.001$	$f(a-0.001)$	$a+0.001$	$f(a+0.001)$
$a-0.0001$	$f(a-0.0001)$	$a+0.0001$	$f(a+0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on.

Caution: Be prepared to try "wiggle" sizes other than powers of ten

Although we have chosen the x-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.

3. If both columns approach a common y-value L, we state (with false confidence) $\lim\limits_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using technology that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. If the technology allows it, we can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim\limits_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples $2.2.1A$ and $2.2.1B$.

Example $2.2.1A$: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim\limits_{t \to 0} \dfrac{\sinh(t)}{t}$ using a table of functional values.

Solution

We have calculated the values of $f(t) = \dfrac{\sinh(t)}{t}$ for the values of t listed in Table $2.2.2$.

Table 2.2.2

t	$\frac{\sinh(t)}{t}$	t	$\frac{\sinh(t)}{t}$
-0.1	1.0016675	0.1	1.0016675
-0.01	1.000016667	0.01	1.000016667
-0.001	1.000000167	0.001	1.000000167
-0.0001	1.000000002	0.0001	1.000000002

Note: The values in this table were obtained using Desmos and using all the places given by Desmos.

As we read down each $\dfrac{\sinh(t)}{t}$ column, we see that the values in each column appear to be approaching one. Thus, it is *somewhat* reasonable to conclude that $\lim\limits_{t \to 0} \dfrac{\sinh(t)}{t} = 1$. A Desmos-generated graph of $f(t) = \dfrac{\sinh(t)}{t}$ would be similar to that shown in Figure $2.2.2$, and it confirms our estimate.

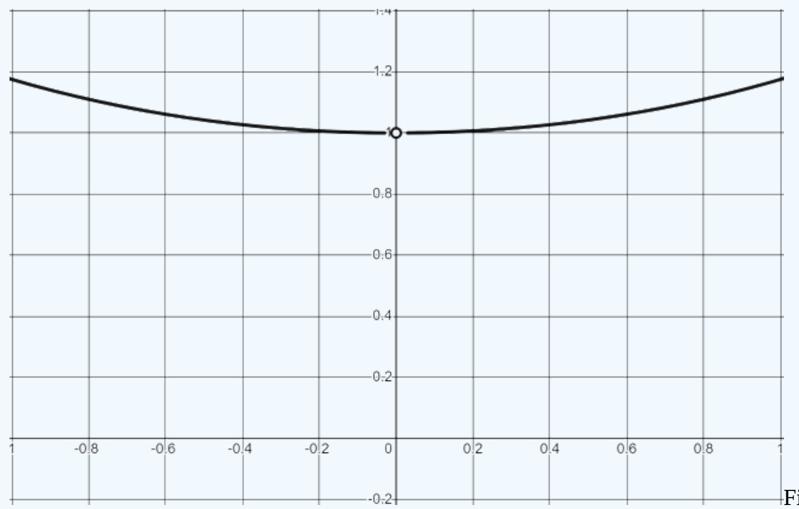

Figure 2.2.2: The graph of $f(t) = \sinh(t)/t$ confirms the estimate from Table 2.2.2.

Example 2.2.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 2.2.3—to list the values of the function for the given values of x.

Table 2.2.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{x-4} = 0.25$. We confirm this estimate using the graph of $f(x) = \dfrac{\sqrt{x}-2}{x-4}$ shown in Figure 2.2.3.

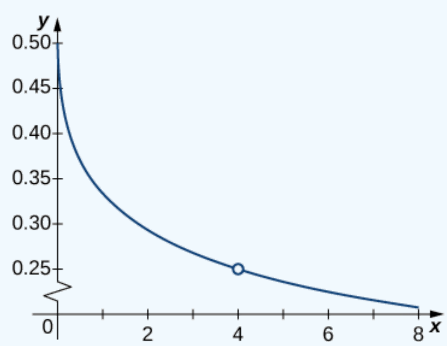

Figure 2.2.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 2.2.3.

? Exercise 2.2.1

Estimate $\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

Answer

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=-1$$

Using Graphs to Estimate Limits

At this point, we see from Examples $2.2.1A$ and $2.2.1b$ that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example $2.2.2$, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

✔ Example 2.2.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure $2.2.4$, evaluate $\lim_{x\to -1} g(x)$.

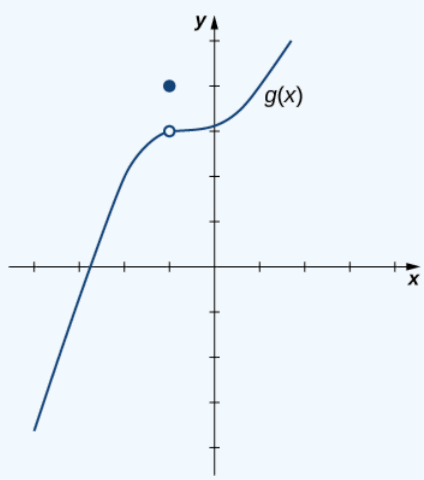

Figure $2.2.4$: The graph of $g(x)$ includes one value not on a smooth curve.

Solution

Despite the fact that $g(-1)=4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x\to -1} g(x)=3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example $2.2.2$, we make the following observation:

> It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

? Exercise 2.2.2

Use the graph of $h(x)$ in Figure $2.2.5$ to evaluate $\lim_{x\to 2} h(x)$, if possible.

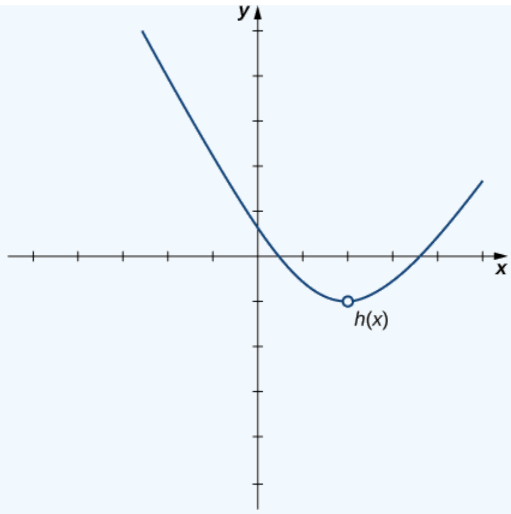

Figure 2.2.5

Solution

Hint

What y-value does the function approach as the x-values approach 2?

$\lim_{x \to 2} h(x) = -1.$

Two Important Limits

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim_{x \to a} x = a$
ii. $\lim_{x \to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.
ii. For the second limit, consider Table $2.2.4$.

Table 2.2.4

x	$f(x) = c$	x	$f(x) = c$
$a - 0.1$	c	$a + 0.1$	c
$a - 0.01$	c	$a + 0.01$	c
$a - 0.001$	c	$a + 0.001$	c
$a - 0.0001$	c	$a + 0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.2.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim\limits_{x \to 0} \sin(1/x)$ using a table of values.

Solution

Table $2.2.5$ lists values for the function $\sin(1/x)$ for the given values of x.

Table 2.2.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let’s take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim\limits_{x \to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate “does not exist” as DNE. Thus, we would write $\lim\limits_{x \to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure $2.2.6$ and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

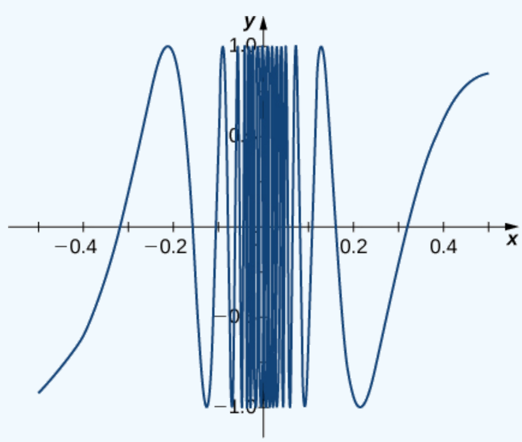

Figure $2.2.6$: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

Exercise 2.2.3

Use a table of functional values to evaluate $\lim\limits_{x \to 2} \frac{|x^2-4|}{x-2}$, if possible.

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

Answer

$\lim\limits_{x \to 2} \frac{|x^2-4|}{x-2}$ does not exist.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x)=|x-2|/(x-2)$ introduced at the beginning of the section (see Figure $2.2.1$(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim\limits_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x)=-1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x)=-1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x)=1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z,a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x<a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x)=L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a,c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x>a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x)=L.$$

Example 2.2.4: Evaluating One-Sided Limits

For the function $f(x)=\begin{cases} x+1, & \text{if } x<2 \\ x^2-4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim\limits_{x \to 2^-} f(x)$
b. $\lim\limits_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table $2.2.6$ that for values of x less than 2, we use $f(x)=x+1$ and for values of x greater than 2, we use $f(x)=x^2-4$.

Table 2.2.6

x	$f(x)=x+1$	x	$f(x)=x^2-4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim\limits_{x\to 2^-} f(x)=3$ and b. $\lim\limits_{x\to 2^+} f(x)=0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x=2$. Figure $2.2.7$ shows a graph of $f(x)$ and reinforces our conclusion about these limits.

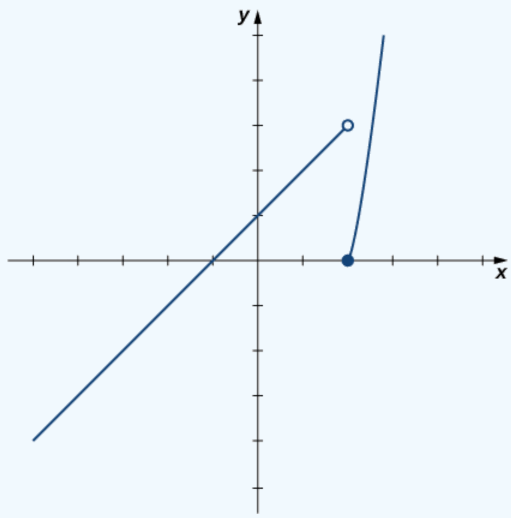

Figure $2.2.7$: The graph of $f(x)=\begin{cases} x+1, & \text{if } x<2 \\ x^2-4, & \text{if } x\geq 2\end{cases}$ has a break at $x=2$.

? Exercise $2.2.4$

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x\to 2^-}\dfrac{|x^2-4|}{x-2}$

b. $\lim\limits_{x\to 2^+}\dfrac{|x^2-4|}{x-2}$

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim\limits_{x\to 2^-}\dfrac{|x^2-4|}{x-2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim\limits_{x\to 2^+}\dfrac{|x^2-4|}{x-2}$.

(These tables are available from a previous Checkpoint problem.)

Solution a

a. $\lim_{x \to 2^-} \frac{|x^2-4|}{x-2} = -4$

Solution b

$$\lim_{x \to 2^+} \frac{|x^2-4|}{x-2} = 4$$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in the following theorem.

Theorem 2.2.1: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number, ∞, or $-\infty$. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limit at a Finite Number

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure $2.2.1$(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define an **infinite limit at a finite number** as follows:

Definitions (Intuitive): Infinite Limits at Finite Numbers

We define three types of **infinite limits at finite numbers**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim\limits_{x \to a} f(x) = +\infty$, we always write $\lim\limits_{x \to a} f(x) = +\infty$ rather than $\lim\limits_{x \to a} f(x)$ DNE.

✔ Example 2.2.5: Recognizing an Infinite Limit at a Finite Number

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \frac{1}{x}$
b. $\lim\limits_{x \to 0^+} \frac{1}{x}$
c. $\lim\limits_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 2.2.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure $2.2.8$ confirms these conclusions.

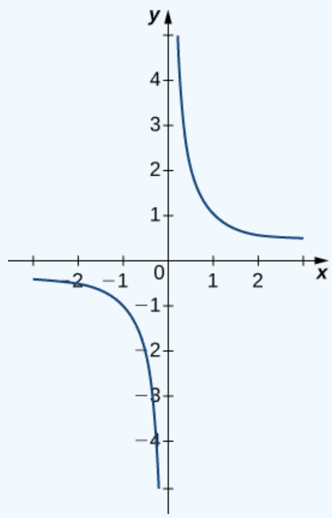

Figure $2.2.8$: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

The definition of an infinite limit at a finite number, along with the Figure $2.2.8$, allows us to formalize a concept from past courses.

Definition: Vertical Asymptote

The function $f(x)$ is said to have a **vertical asymptote** at $x = a$ if any of the following are true.

$$\lim_{x \to a^-} f(x) = +\infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

Exercise $2.2.5$

Evaluate each of the following limits, if possible, and state whether the function has a vertical asymptote at $x = 0$. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \dfrac{1}{x^2}$
b. $\lim\limits_{x \to 0^+} \dfrac{1}{x^2}$
c. $\lim\limits_{x \to 0} \dfrac{1}{x^2}$

Hint

Follow the procedures from Example $2.2.5$

Answer

a. $\lim\limits_{x \to 0^-} \dfrac{1}{x^2} = +\infty$;

b. $\lim\limits_{x \to 0^+} \dfrac{1}{x^2} = +\infty$;

c. $\lim\limits_{x \to 0} \dfrac{1}{x^2} = +\infty$

From the fact that *any* of these limits (at the finite number $x = 0$) is infinite, we can state that $f(x)$ has a vertical asymptote at $x = 0$.

Conceptual Investigation of an Infinite Limit

When given a function that has a vertical asymptote at $x = a$, a conceptual investigation of the limit as x approaches a can often be done. This type of investigation relies *heavily* on your mastery of the prerequisite knowledge to calculus - specifically, domains of functions, vertical asymptotes, and the graphs of logarithmic and trigonometric functions.

Before get too deep into conceptual investigations of inifinite limits, we need to understand the nature of a ratio when its denominator is approaching zero.

Table $2.2.8$: A comparison of ratios with denominators approaching zero

$f(x) = \frac{1}{x}$	$g(x) = \frac{13}{x}$	$h(x) = \frac{-2}{x^2}$
$\frac{1}{0.1} = 10$	$\frac{13}{-0.1} = -130$	$\frac{-2}{(-0.1)^2} = -200$
$\frac{1}{0.01} = 100$	$\frac{13}{-0.01} = -1300$	$\frac{-2}{(-0.01)^2} = -20000$
$\frac{1}{0.001} = 1000$	$\frac{13}{-0.001} = -13000$	$\frac{-2}{(-0.001)^2} = -2000000$
$\frac{1}{0.0001} = 10000$	$\frac{13}{-0.0001} = -130000$	$\frac{-2}{(-0.0001)^2} = -200000000$

In all three functions from Table $2.2.8$, the denominators are approaching zero. In the first and third columns, the denominators approach 0^+ (zero from the right). In the second column, the denominators approach 0^- (zero from the left). Notice that, in all three cases, the smaller the denominator gets, the larger the value of the fraction becomes. Understanding this property is a key to conceptually investigating infinite limits at finite numbers. I have summarized this result below.

If the numerator is approaching a **nonzero** constant and the denominator is approaching 0, the ratio $\frac{N(x)}{D(x)}$ approaches either $+\infty$ or $-\infty$. You determine the sign of this level of infinity by comparing the signs of the numerator and denominator of the ratio of functions as x is approaching a.

Let's take a look at how this conceptual investigation should be done in practice.

✓ Example 2.2.6: Conceptually Investigating an Infinite Limit at a Finite Number

Evaluate each of the following limits.

a. $\lim\limits_{x\to7^+}\dfrac{2}{x-7}$

b. $\lim\limits_{x\to7^-}\dfrac{2}{x-7}$

c. $\lim\limits_{x\to7}\dfrac{2}{x-7}$

d. $\lim\limits_{x\to7^-}\dfrac{5-x}{(x-7)^2}$

e. $\lim\limits_{x\to(\pi/2)^-}\ln(\cot(x))$

Solution

In each of these solutions, pay close attention to how things are justified. You need to, at minimum, show this level of rigor in your classwork.

a.

Numerator: As $x\to7^+$, $x=2$ (the numerator is *always* 2).

Denominator: As $x\to7^+$, $(x-7)\to0^+$.

Thus,

$$\begin{array}{ccc} & 2 & \\ & \nearrow & \\ \lim\limits_{x\to7^+}\dfrac{2}{x-7} & = & \infty \\ & \searrow & \\ & 0^+ & \end{array}$$

Note: The numerator is approaching a *positive* constant and the denominator is approaching 0 from the *positive* side. The ratio of positive numbers is positive. This is why the result is *positive* ∞.

b.

Numerator: As $x\to7^+$, $x=2$ (the numerator is *always* 2).

Denominator: As $x\to7^-$, $(x-7)\to0^-$.

Thus,

$$\begin{array}{ccc} & 2 & \\ & \nearrow & \\ \lim\limits_{x\to7^-}\dfrac{2}{x-7} & = & -\infty \\ & \searrow & \\ & 0^- & \end{array}$$

Note: The numerator is approaching a *positive* constant and the denominator is approaching 0 from the *negative* side. The ratio of a positive number and a negative number is negative. This is why the result is *negative* ∞.

c. Since $\lim\limits_{x\to7^+}\dfrac{2}{x-7}\neq\lim\limits_{x\to7^+}\dfrac{2}{x-7}$, the *overall limit* $\lim\limits_{x\to7}\dfrac{2}{x-7}$ DNE (does not exist) by Theorem $2.2.1$.

d.

Numerator: As $x \to 7^-$, $(5-x) \to -2^+$.

Denominator: As $x \to 7^-$, $(x-7) \to 0^-$, but when we square this, it approaches 0^+.

Thus,

$$\lim_{x \to 7^-} \frac{\overset{\nearrow\, -2}{5-x}}{\underset{\searrow\, 0^+}{(x-7)^2}} = -\infty$$

Note: The numerator is approaching a *negative* constant and the denominator is approaching 0 from the *positive* side. The ratio of a negative number and a poitive number is negative. This is why the result is *negative* ∞.

e.

Innermost function: From our prerequisite trigonometry course, we know that as $x \to (\pi/2)^-$, $\cot(x) \to 0^+$.

Outermost function: From our prerequisite algebra courses, as $\cot(x) \to 0^+$, $\ln(\cot(x)) \to -\infty$.

Thus,

$$\lim_{x \to (\pi/2)^-} \ln(\cot(x)) = -\infty$$

The last part of Example $2.2.6$ is often very challenging for students. This is because it requires a lot of visualization from your prerequisite courses. If you have a hard time visualizing the graphs of the trigonometric functions or the logarithmic function, it is best that you review that material from Chapter 1 of this textbook. Understanding and quickly recalling the domains and graphs of those functions is a fundamental skill needed to survive calculus.

Exercise $2.2.6$

Evaluate each of the following limits.

a. $\displaystyle\lim_{x \to 2^-} \frac{1}{(x-2)^3}$
b. $\displaystyle\lim_{x \to 2^+} \frac{1}{(x-2)^3}$
c. $\displaystyle\lim_{x \to 2} \frac{1}{(x-2)^3}$
d. $\displaystyle\lim_{x \to (\pi/2)^+} \sec(x)$

Answer a

$$\lim_{x \to 2^-} \frac{\overset{\nearrow\, 1}{1}}{\underset{\searrow\, 0^-}{(x-2)^3}} = -\infty$$

Answer b

$$\begin{array}{ccc} & & 1 \\ & \nearrow & \\ \lim_{x\to 2^+}\frac{1}{(x-2)^3} & = & \infty \\ & \searrow & \\ & & 0^+ \end{array}$$

Answer c

$\lim_{x\to 2}\frac{1}{(x-2)^3}$ DNE (by Theorem $2.2.1$).

Answer d

$$\begin{array}{ccccc} & & & & 1 \\ & & & \nearrow & \\ \lim_{x\to(\pi/2)^+}\sec(x) & = & \lim_{x\to(\pi/2)^+}\frac{1}{\cos(x)} & = & \infty \\ & & & \searrow & \\ & & & & 0^+ \end{array}$$

Example 2.2.7: Einstein's Equation

Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

what is the value of this bound?

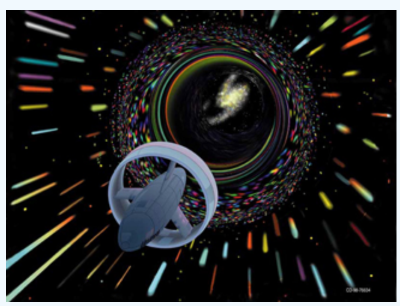

Figure $2.2.9$. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we start evaluating

$$\lim_{v\to c^-} m=\lim_{v\to c^-}\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Note that we are considering the velocity as it approaches the speed of light *from below*. This is because nothing can move faster than the speed of light. This one-sidedness of the limit is *incredibly* important.

As $v \to c^-$, $v^2 \to (c^2)^-$.

However, as $v^2 \to (c^2)^-$, $\frac{v^2}{c^2} \to \frac{(c^2)^-}{c^2} \to 1^-$.

But as $\frac{v^2}{c^2} \to 1^-$, $1 - \frac{v^2}{c^2} \to 1 - 1^- \to 0^+$.

This is important because, as $1 - \frac{v^2}{c^2} \to 0^+$, $\sqrt{1 - \frac{v^2}{c^2}} \to 0^+$. Had we allowed $v \to c^+$, we would have had a negative under the square root!

Finally, as $\sqrt{1 - \frac{v^2}{c^2}} \to 0^+$, $\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \to \infty$.

Thus, as an object's velocity approaches the speed of light, it becomes infinitely massive. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 2.2.9

v/c	$\sqrt{1 - \frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Revisiting Evaluation of Limits from Graphs

Let's finish this section by revisiting the graphical investigation of limits.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.2.8: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure $2.2.10$ to determine each of the following values:

a. $\lim_{x \to -4^-} f(x)$; $\lim_{x \to -4^+} f(x)$; $\lim_{x \to -4} f(x)$; $f(-4)$
b. $\lim_{x \to -2^-} f(x)$; $\lim_{x \to -2^+} f(x)$; $\lim_{x \to -2} f(x)$; $f(-2)$
c. $\lim_{x \to 1^-} f(x)$; $\lim_{x \to 1^+} f(x)$; $\lim_{x \to 1} f(x)$; $f(1)$
d. $\lim_{x \to 3^-} f(x)$; $\lim_{x \to 3^+} f(x)$; $\lim_{x \to 3} f(x)$; $f(3)$

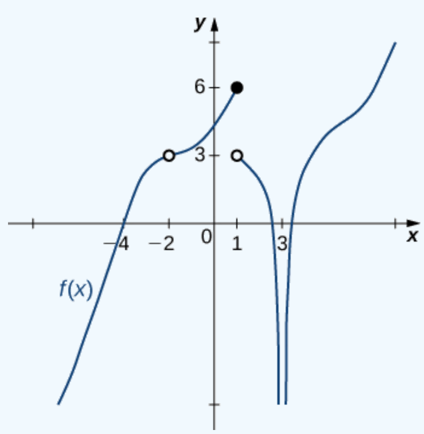

Figure 2.2.10: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim_{x \to -4^-} f(x) = 0$; $\lim_{x \to -4^+} f(x) = 0$; $\lim_{x \to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim_{x \to -2^-} f(x) = 3$; $\lim_{x \to -2^+} f(x) = 3$; $\lim_{x \to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim_{x \to 1^-} f(x) = 6$; $\lim_{x \to 1^+} f(x) = 3$; $\lim_{x \to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim_{x \to 3^-} f(x) = -\infty$; $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 2.2.8

Evaluate $\lim_{x \to 1} f(x)$ for $f(x)$ shown here:

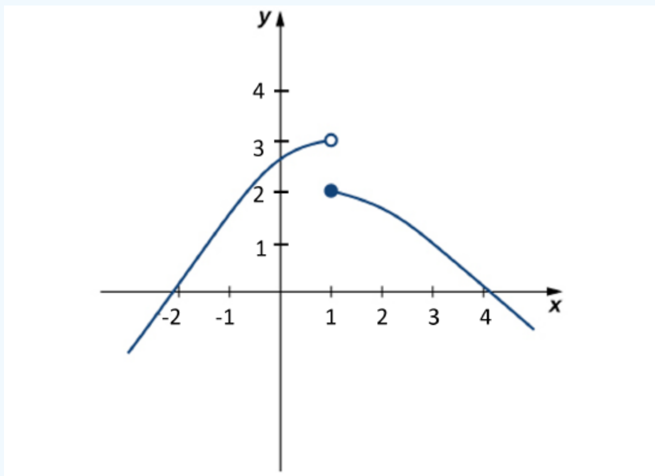

Figure 2.2.11. The graph of a piecewise function f.

Hint

Compare the limit from the right with the limit from the left.

Answer

$\lim_{x \to 1} f(x)$ does not exist

A Warning

Up to this point, the only way you know how to evaluate the limit of a function is by either (a) building a table of values to estimate the limit, or (b) using a given graph of the function to "eyeball" the value of the limit. Although it will happen during the homework in this section, it is not common to be handed the graph of a function and to be asked for the values of some of its limits. Moreover, consider the following warning:

Caution: Using Tables for Limits is a Terrible Idea

Outside of the homework in this section (that is meant to get you used to the idea of a limit), attempting to estimate a limit using a table of values is horrendously bad. Again, read the following sentence carefully:

Beyond this section and unless specifically asked to do so, you are to never use a table of values to estimate a limit.

To illustrate why this warning is so important, consider the following example.

Example 2.2.9: The Issue with Using Tables

Evaluate $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ using a table of functional values.

Solution

We have calculated the values of $f(x) = \sin\left(\frac{\pi}{x}\right)$ for the values of x listed in Table $2.2.10$.

Table 2.2.10

x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$
0.1	0	0.3	-0.8660254	0.6	-0.8660254	0.7	-0.97492791
0.01	0	0.03	-0.8660254	0.06	0.8660254	0.07	0.78183148
0.001	0	0.003	-0.8660254	0.006	0.8660254	0.007	0.43388374
0.0001	0	0.0003	-0.8660254	0.0006	0.8660254	0.0007	0.97492791

The values chosen for x in each column approach 0; however, having the last digit as 1 tempts us into thinking the limit of this function is 0. Having the final digit as 3, makes -0.8660254 seem like the limiting value. With a 6 as the final digit, we *think* the limit is 0.8660254 Finally, the 7 as the final digit leaves us completely confused. Is there an actual value for the limit?

Looking at a graph of this function, we can see that as x approaches 0, it looks like the function bounces around infinitely many times between -1 and 1.

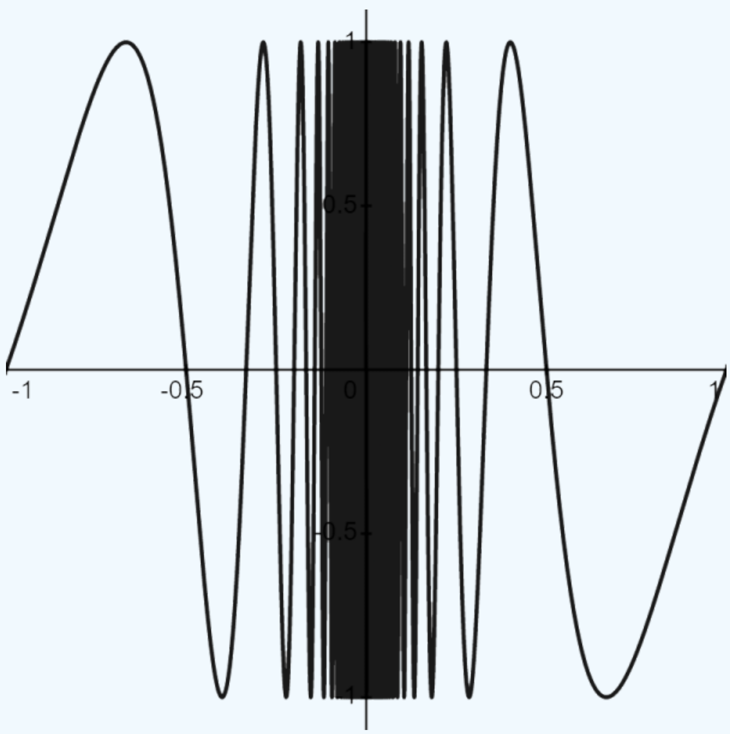

Figure 2.2.12: The graph of $f(x) = \sin(\pi/x)$shows a that the values of f varying wildly as x gets close to 0.

If estimating a limit by building a table is not allowed (beyond this section), then how can we investigate the limit of a function without being given its graph? I am glad you asked!

Key Concepts

- A table of values may be used to estimate a limit.
- A graph may be used to estimate a limit, but it is rare that a graph will be given. Therefore, we need better methods to investigate limits.
- If the limit of a function at a number does not exist, it is still possible that the limits from the left and right at that number may exist.

- If the limits of a function from the left and right exist and are equal, then the overall limit of the function exists and is that common value.
- We may use limits to describe infinite behavior of a function at a number.
- Conceptually investigating an infinite limit can be done by (a) understanding that nonzero constants divided by numbers approaching zero leads to infinities, and (b) knowing the domains and graphs of functions that commonly have domain restrictions (e.g., logarithms and trigonometric functions).

Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x \to a} f(x) = +\infty\text{: } \lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty\text{: } \lim_{x \to a^-} f(x) = -\infty \text{ and } \lim_{x \to a^+} f(x) = -\infty$$

Common Mistakes

- Using a table to estimate the value of a limit outside of this section
- Always using powers of ten when building tables. That is, using $\pm 0.1, \pm 0.01, \pm 0.001, \ldots$ when investigating $\lim_{x \to 0} \sin \frac{\pi}{x}$.
- Not understanding that when the denominator within a limit is *approaching* zero, it doesn't *equal* zero. Division by zero does not occur in calculus; however, division by smaller and smaller values approaching zero occurs very often. Analyzing the ramification of those denominators approaching zero is *important* in calculus.

Glossary

infinite limit at a finite number

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

finite limit at a finite number

If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite