On the commutator scaling in Hamiltonian simulation with multi-product formulas

KM, Quantum 10, 1974 (2026)

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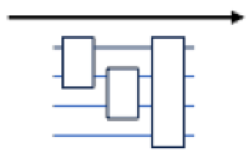
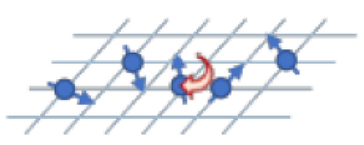
1. Intro: Hamiltonian simulation

Hamiltonian simulation

Solve Schrodinger equation with quantum computers

$|\psi\rangle$ → $e^{-iHt}|\psi\rangle + \mathcal{O}(\varepsilon)$





N-qubit state — Solution at time t — Small error

- Potential exp. speedup
- Application to physics & chemistry

Goal: What is the **fastest** quantum algorithm
achieving the best scaling in size N, time t, and error ε ?

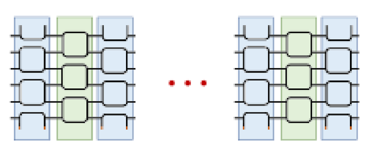
Various quantum algorithms

[Product formula (PF)] or Trotterization

$$T_p(\tau) = e^{-iH\tau} + \mathcal{O}(\tau^{p+1})$$

e.g. $T_1(\tau) = e^{-iH_2\tau}e^{-iH_1\tau}$

$T_2(\tau) = e^{-iH_1\tau/2}e^{-iH_2\tau}e^{-iH_1\tau/2}$

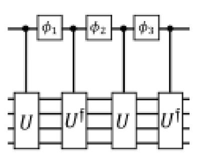


S. Lloyd, Science 273, 1073 (1996)

[Linear combination of unitaries (LCU)]
/ [Quantum singular value transform (QSVT)]

$$e^{-iHt} = \sum_{n=0}^{q} \frac{(-iHt)^q}{n!} + \mathcal{O}\left(\frac{(\|H\|t)^n}{n!}\right)$$

D. W. Berry, et al., PRL 114, 090502 (2015).
A. Gilyén, et al., STOC 2019



2. Intro: Multi-product formula (MPF)

Cost of existing algorithms

	pth-order PF	LCU	QSVT
Pros & Cons	Good scaling in N / Bad scaling in ε	Exponentially good scaling in ε Good scaling in t / Bad scaling in N	
Origin	Commutator scaling A. M. Childs, et al., PRX 11, 011020 (2021)	Rapid convergence of polynomial approximation	

(Well-conditioned) MPF

G. H. Low, et al., arXiv:1907.11679 (2019)

→ Promising candidate simultaneously efficient in N, t, ε

$$M(\tau) = \underbrace{\sum_{j=1}^{J} c_j}_{\text{Linear combination of}} \underbrace{[T_p(\tau/k_j)]^{k_j}}_{p\text{th-order PF}} = e^{-iH\tau} + \mathcal{O}(\tau^{m+1}).$$

Smaller than $\mathcal{O}(\tau^{p+1})$ by Richardson extrapolation

Suppose we have 2 approximations, $f_j(\tau) = f_{\mathrm{true}} + a_j\tau^2 + \mathcal{O}(\tau^3)$.
Then, we obtain better approximation $c_1 f_1(\tau) + c_2 f_2(\tau) = f_{\mathrm{true}} + \mathcal{O}(\tau^3)$ with c_j s.t. $c_1 + c_2 = 1$, $c_1 a_1 + c_2 a_2 = 0$.

Open problem

Does the cost of MPF has good scaling in any of N, t, ε ?

3. Results: Simultaneous efficiency of MPF in size, time, accuracy

Setup

Generic k-local Hamiltonian $H = \sum_{\gamma=1}^{\Gamma} H_\gamma, \quad H_\gamma = \sum_{X\subset\Lambda} h_X^\gamma$

Extensiveness: $g = \max_{i\in\Lambda}\left(\sum_\gamma \sum_{X\ni i} \|h_X^\gamma\|\right)$

$$g \in \begin{cases} \mathcal{O}(1) & \text{(finite-range interaction)} \\ \mathcal{O}(1) & \text{(long-range int., } \nu > d) \\ \mathcal{O}(\log N) & \text{(long-range int., } \nu = d) \\ \mathcal{O}(N^{1-\nu/d}) & \text{(long-range int., } \nu < d) \end{cases} \qquad U(r)\sim r^{-\nu}$$

Nested commutators

$$\alpha_{\mathrm{com},q} = \sum_{\gamma_0,\cdots,\gamma_q=1}^{\Gamma} [H_{\gamma_q},\cdots,[H_{\gamma_1},H_{\gamma_0}]] \xrightarrow{\text{Locality of Hamiltonian}} \alpha_{\mathrm{com},q} \leq q!(2kg)^q Ng$$

c.f. PF error with commutator scaling: $\|e^{-iH\tau} - T_p(\tau)\| \leq \mathrm{Const.} \times \alpha_{\mathrm{com},p}\tau^{p+1} \in \mathcal{O}((N^{\frac{1}{p+1}}g\tau)^{p+1})$

Cost

	Number of $\mathcal{O}(1)$-qubit gates for finite-range int.	for long-range int.	Ancilla qubits
pth-order PF A. Childs, et al. (2021)	$Ng\left(\frac{Ngt}{\varepsilon}\right)^{\frac{1}{p}}$	$N^k g\left(\frac{Ngt}{\varepsilon}\right)^{\frac{1}{p}}$	0
LCU/QSVT A. Gilyén, et al. (2019)	$N(Ngt + \log(1/\varepsilon))$	$N^k(Ngt + \log(1/\varepsilon))$	$\log N$
HHKL J. Haah, et al. (2021)	$Ngt \times \mathrm{polylog}(Ngt/\varepsilon)$ ← Needs Lieb-Robinson bound	$Ngt\left(\frac{Ngt}{\varepsilon}\right)^{\frac{2d}{\nu-d}}$ Only for $\nu > 2d$	$\log\log(Ngt/\varepsilon)$
pth-order MPF (this work)	$N^{1+\frac{1}{p+1}}gt \times \mathrm{polylog}(Ngt/\varepsilon)$	$N^{k+\frac{1}{p+1}}gt \times \mathrm{polylog}(Ngt/\varepsilon)$	$\log\log(Ngt/\varepsilon)$

Good N-scaling like PF & Good t-, ε-scaling like LCU/QSVT & Versatility to generic local Hamiltonians !

4. Results & Derivation

MPF error with commutator scaling

Commutator factor for MPF: $\mu_{p,m}[p_0] = \max_{q: p\leq q\leq p_0}\left((\alpha_{\mathrm{com},q})^{\frac{1}{q+1}}\right)$

Theorem 1

$\epsilon \in (0,1)$: arbitrary fixed value, $p_0(N,\epsilon) = \lceil \log(3N/\epsilon)\rceil$: truncation order

When the time τ is small enough to satisfy $|\tau| \leq \frac{\mathrm{Const.}}{\mu_{p,m}[p_0(N,\epsilon)]}$, the MPF error is bounded by

$$\|e^{-iH\tau} - M(\tau)\| \leq \mathrm{Const.} \times \|c\|_1(\mu_{p,m}[p_0(N,\epsilon)]\tau)^{m+1} + \|c\|_1\|k\|_1\epsilon$$

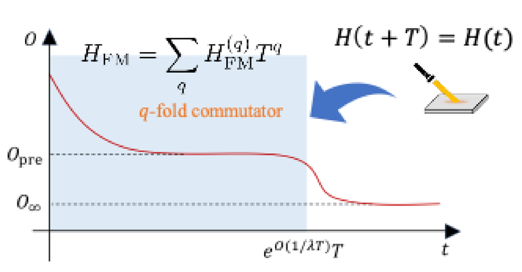
Theorem 2

$$\mu_{p,m}[p_0(N,\epsilon)] \leq \mathrm{Const.} \times (N^{\frac{1}{p+1}}g + g\log(N/\epsilon))$$

Derivation of the error bound

Theorem 1 Inspired by Floquet prethermalization T. Kuwahara, T. Mori, K. Saito, Ann. Phys. 367, 96 (2016)

$$\|e^{-iH\tau} - M(\tau)\| \leq \left(\sum_{\{q_i\}}\prod_i \alpha_{\mathrm{com},q_i}\right) t^{p+J} + \underbrace{E^{(p+J+1)}(\{\alpha_{\mathrm{com},q_i}\})}_{q_i\text{-fold commutator}} t^{p+J+1} + \cdots + E^{(q)}(\{\alpha_{\mathrm{com},q_i}\})t^q + \cdots$$



[$q \to \infty$] Divergent [Consider $q \leq \log(N/\epsilon)$] Convergence within error ϵ

Theorem 2 Use the bound on the nested commutator $\alpha_{\mathrm{com},q}$ with the locality

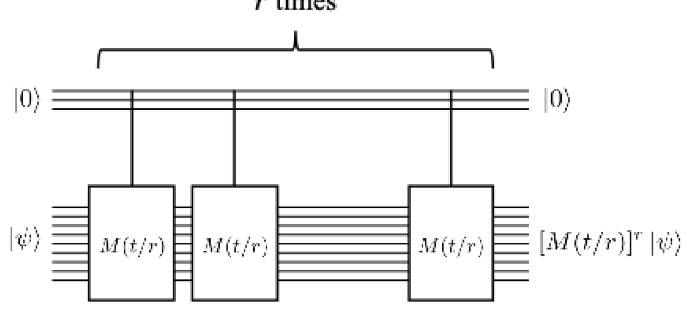
Derivation of the computational cost

Find Trotter number $r = t/\tau$ such that

$$r \times \|e^{-iHt/r} - M(t/r)\| \in \mathcal{O}\left(r\left(N^{\frac{1}{p+1}}g\frac{t}{r}\right)^{m+1}\right) \leq \varepsilon$$

with setting $m, J \in \mathcal{O}(\log(Ngt/\varepsilon)), \quad \epsilon \in \tilde{\mathcal{O}}(\varepsilon/r)$

[Gate counts] $\|c\|_1\|k\|_1\, r \times (\text{Number of terms in } H)$



LCU + amplitude amplification

5. Comparison with previous results

[Low-Kliuchnikov-Wiebe 2019]

G. H. Low, V. Kliuchnikov, and N. Wiebe, arXiv:1907.11679 (2019)

- Prove the good scaling in t & ε by the well-conditioned solution $\|c\|_1, \|k\|_1 \in \mathrm{poly}(J)$
- Prove that the error is given by commutators but NOT provide its explicit form

→ The good scaling in size N is not proven

[Aftab-An-Trivisa 2024]

J. Aftab, D. An, and K. Trivisa, arXiv:2403.08922 (2024)

- Prove an explicit error bound of the MPF composed of nested commutators

(Simplified version)

$$\|e^{-iH\tau} - M(\tau)\| \leq \mathrm{Const.} \times \|c\|_1(\mu_{p,m}[\infty]\tau)^{m+1} \quad \text{if} \quad |\tau| \leq \frac{\mathrm{Const.}}{\mu_{p,m}[\infty]}$$

- However, it involves q-fold nested commutators $\alpha_{\mathrm{com},q}$ with arbitrarily large q

→ Locality gives no meaningful upper bound on $\mu_{p,m}[\infty]$ due to $[q!(2kg)^q Ng]^{\frac{1}{q+1}} \to \infty$

We can only confirm $\mu_{p,m}[\infty] \leq \mathrm{Const.} \times Ng$ in general

→ The cost results in $r \in \tilde{\Theta}(Ngt)$ for generic local Hamiltonians (1-norm scaling)

Our result with commutator scaling: $r \in \tilde{\Theta}(N^{\frac{1}{p+1}}gt)$

Our contribution

Prove the MPF error composed of a finite number of nested commutators
& Complete the proof of the efficiency in N, t, ε for generic local Hamiltonians !

6. Summary & Future direction

✓ Complete proof of the MPF cost for generic local Hamiltonians

✓ Truncation order $p_0 \in \mathcal{O}(\log(N/\epsilon))$ for nested commutators plays a central role in reducing N-scaling, i.e., reflecting the locality

✓ Application: MPF algorithm for time-dependent $H(t)$

K. Mizuta, T. N. Ikeda, and K. Fujii, arXiv:2410.14243 (2024) [QIP2025] ver. 2.

Other potential applications: Trotter extrapolation or interpolation

J. D. Watson, et al., PRX Quantum 6, 030325 (2025), etc.