Kelvin Niu

$y_i \sim N(x_i'\beta, \tau^{-1})$

$x_i \sim N_2(\mu, \Sigma)$

$\beta \sim N_2(m_b, s_b)$ $\qquad n = n_{obs} + n_{mis}$

$\tau \sim Ga(a,b)$

$\mu \sim N_2(m_v, s_v)$

$\Sigma \sim IW_{2\times 2}(3, I)$

$$\pi(\beta, \tau, \mu, \Sigma, x_{mis} \mid y, x) \propto L(y_{obs} \mid x_{obs}, \beta, \tau) L(x_{obs} \mid \mu, \Sigma) L(y_{mis} \mid x_{mis}, \beta, \tau) L(x_{mis} \mid \mu, \Sigma)$$
$$\cdot N_2(\beta; m_b, s_b) \cdot Ga(\tau; a, b) \cdot N_2(\mu; m_v, s_v) IW(\Sigma; 3, I)$$

$$\pi(x_{mis,i} \mid -) \propto L(y_{mis} \mid x_{mis}, \beta, \tau) L(x_{mis} \mid \mu, \Sigma)$$
$$\propto \left[\tau^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}\tau(y_{mis,i} - x_{mis,i}'\beta)^2\right]\right] \quad \left[|\Sigma|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(x_{mis,i} - \mu)' \Sigma^{-1}(x_{mis,i} - \mu)\right]\right]$$
$$\propto \exp\left[-\tfrac{1}{2}\tau(-2y_{mis,i} x_{mis,i}'\beta + x_{mis,i}'\beta\beta' x_{mis,i})\right] \exp\left[-\tfrac{1}{2}(x_{mis,i}'\Sigma^{-1}x_{mis,i} - 2x_{mis,i}'\Sigma^{-1}\mu)\right]$$
$$\propto \exp\left[-\tfrac{1}{2}\left[x_{mis,i}'(\tau\beta\beta' + \Sigma^{-1})x_{mis,i} - 2x_{mis,i}'(\tau\beta y_{mis,i} + \Sigma^{-1}\mu)\right]\right]$$

$$\boxed{x_{mis,i} \mid - \sim N_2\left((\tau\beta\beta' + \Sigma^{-1})^{-1}(\tau\beta y_{mis,i} + \Sigma^{-1}\mu), (\tau\beta\beta' + \Sigma^{-1})^{-1}\right)}$$

$$\pi(\beta \mid -) \propto L(y_{obs} \mid x_{obs}, \beta, \tau) L(y_{mis} \mid x_{mis}, \beta, \tau) N_2(\beta; m_b, s_b)$$
$$\propto \prod_{i=1}^{n_{obs}} \left[\tau^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}\tau(y_{obs,i} - x_{obs,i}'\beta)^2\right]\right] \prod_{i=1}^{n_{mis}} \left[\tau^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}\tau(y_{mis,i} - x_{mis,i}'\beta)^2\right]\right] |s_b|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\beta - m_b)' s_b^{-1}(\beta - m_b)\right]$$
$$\propto \exp\left[-\tfrac{1}{2}\tau\sum_{i=1}^{n_{obs}}(-2\beta' x_{obs,i} y_{obs,i} + \beta' x_{obs,i} x_{obs,i}'\beta)\right] \exp\left[-\tfrac{1}{2}\tau\sum_{i=1}^{n_{mis}}(-2\beta' x_{mis,i} y_{mis,i} + \beta' x_{mis,i} x_{mis,i}'\beta)\right]$$
$$\cdot \exp\left[-\tfrac{1}{2}(\beta' s_b^{-1}\beta - 2\beta' s_b^{-1} m_b)\right]$$
$$\propto \exp\left[-\tfrac{1}{2}\left[\beta'\left(\tau\sum_{i=1}^{n_{obs}} x_{obs,i} x_{obs,i}' + \tau\sum_{i=1}^{n_{mis}} x_{mis,i} x_{mis,i}' + s_b^{-1}\right)\beta - 2\beta'\left(\tau\sum_{i=1}^{n_{obs}} x_{obs,i} y_{obs,i} + \tau\sum_{i=1}^{n_{mis}} x_{mis,i} y_{mis,i} + s_b^{-1} m_b\right)\right]\right]$$

$$\boxed{\beta \mid - \sim N_2\left(\left(\tau\sum_{i=1}^{n_{obs}} x_{obs,i}x_{obs,i}' + \tau\sum_{i=1}^{n_{mis}} x_{mis,i}x_{mis,i}' + S_b^{-1}\right)^{-1}\left(\tau\sum_{i=1}^{n_{obs}} x_{obs,i}y_{obs,i} + \tau\sum_{i=1}^{n_{mis}} x_{mis,i}y_{mis,i} + S_b^{-1}m_b\right), \left(\tau\sum_{i=1}^{n_{obs}} x_{obs,i}x_{obs,i}' + \tau\sum_{i=1}^{n_{mis}} x_{mis,i}x_{mis,i}' + S_b^{-1}\right)^{-1}\right)}$$

$$\pi(\mu \mid -) \propto L(x_{obs} \mid \mu, \Sigma)\, L(x_{mis} \mid \mu, \Sigma)\, N_2(\mu; m_v, S_v)$$

$$\propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n_{obs}}(x_{obs,i}-\mu)'\Sigma^{-1}(x_{obs,i}-\mu)\right]\exp\left[-\frac{1}{2}\sum_{i=1}^{n_{mis}}(x_{mis,i}-\mu)'\Sigma^{-1}(x_{mis,i}-\mu)\right]\exp\left[-\frac{1}{2}(\mu-m_v)'S_v^{-1}(\mu-m_v)\right]$$

$$\propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n_{obs}}(-2\mu'\Sigma^{-1}x_{obs,i}+\mu'\Sigma^{-1}\mu)\right]\exp\left[-\frac{1}{2}\sum_{i=1}^{n_{mis}}(-2\mu'\Sigma^{-1}x_{mis,i}+\mu'\Sigma^{-1}\mu)\right]\exp\left[-\frac{1}{2}(\mu'S_v^{-1}\mu-2\mu'S_v^{-1}m_v)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\mu'(n_{obs}\Sigma^{-1}+n_{mis}\Sigma^{-1}+S_v^{-1})\mu - 2\mu'\left(\Sigma^{-1}\sum_{i=1}^{n_{obs}}x_{obs,i}+\Sigma^{-1}\sum_{i=1}^{n_{mis}}x_{mis,i}+S_v^{-1}m_v\right)\right]\right]$$

$$\boxed{\mu \mid - \sim N_2\left((n_{obs}\Sigma^{-1}+n_{mis}\Sigma^{-1}+S_v^{-1})^{-1}\left(\Sigma^{-1}\sum_{i=1}^{n_{obs}}x_{obs,i}+\Sigma^{-1}\sum_{i=1}^{n_{mis}}x_{mis,i}+S_v^{-1}m_v\right), (n_{obs}\Sigma^{-1}+n_{mis}\Sigma^{-1}+S_v^{-1})^{-1}\right)}$$

$$\pi(\Sigma \mid -) \propto L(x_{obs} \mid \mu, \Sigma)\, L(x_{mis} \mid \mu, \Sigma)\, IW(\Sigma; 3, I)$$

$$\propto |\Sigma|^{-\frac{n_{obs}}{2}}\exp\left[-\frac{1}{2}\sum_{i=1}^{n_{obs}}(x_{obs,i}-\mu)'\Sigma^{-1}(x_{obs,i}-\mu)\right]|\Sigma|^{-\frac{n_{mis}}{2}}\exp\left[-\frac{1}{2}\sum_{i=1}^{n_{mis}}(x_{mis,i}-\mu)'\Sigma^{-1}(x_{mis,i}-\mu)\right]|\Sigma|^{-3}\exp\left[-\frac{1}{2}tr(\Sigma^{-1})\right]$$

$$\propto |\Sigma|^{-\frac{n+6}{2}}\exp\left[-\frac{1}{2}tr\left(\left(\sum_{i=1}^{n_{obs}}(x_{obs,i}-\mu)(x_{obs,i}-\mu)'+\sum_{i=1}^{n_{mis}}(x_{mis,i}-\mu)(x_{mis,i}-\mu)'+I\right)\Sigma^{-1}\right)\right]$$

$$\boxed{\Sigma \mid - \sim IW\left(3+n, \left(\sum_{i=1}^{n_{obs}}(x_{obs,i}-\mu)(x_{obs,i}-\mu)'+\sum_{i=1}^{n_{mis}}(x_{mis,i}-\mu)(x_{mis,i}-\mu)'+I\right)\right)}$$

$$\pi(\tau \mid -) \propto L(y_{obs} \mid x_{obs}, \beta, \tau)\, L(y_{mis} \mid x_{mis}, \beta, \tau)\, Ga(\tau; a, b)$$

$$\propto \prod_{i=1}^{n_{obs}}\left[\tau^{\frac{1}{2}}\exp\left[-\frac{1}{2}\tau(y_{obs,i}-x_{obs,i}'\beta)^2\right]\right]\prod_{i=1}^{n_{mis}}\left[\tau^{\frac{1}{2}}\exp\left[-\frac{1}{2}\tau(y_{mis,i}-x_{mis,i}'\beta)^2\right]\right]\tau^{a-1}e^{-b\tau}$$

$$\propto \tau^{a+\frac{n}{2}-1}\exp\left[-\tau\left(\frac{1}{2}\sum_{i=1}^{n_{obs}}(y_{obs,i}-x_{obs,i}'\beta)^2+\frac{1}{2}\sum_{i=1}^{n_{mis}}(y_{mis,i}-x_{mis,i}'\beta)^2+b\right)\right]$$

$$\boxed{\tau \mid - \sim Ga\left(a+\frac{n}{2},\; b+\frac{1}{2}\sum_{i=1}^{n_{obs}}(y_{obs,i}-x_{obs,i}'\beta)^2+\frac{1}{2}\sum_{i=1}^{n_{mis}}(y_{mis,i}-x_{mis,i}'\beta)^2\right)}$$

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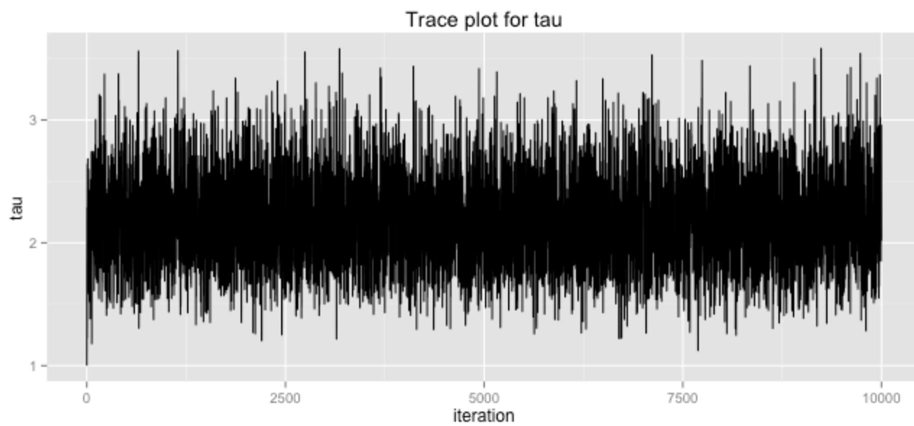
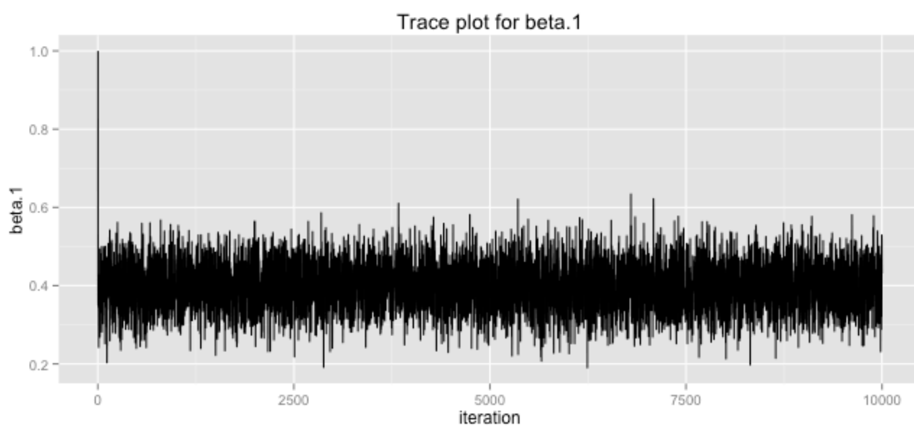
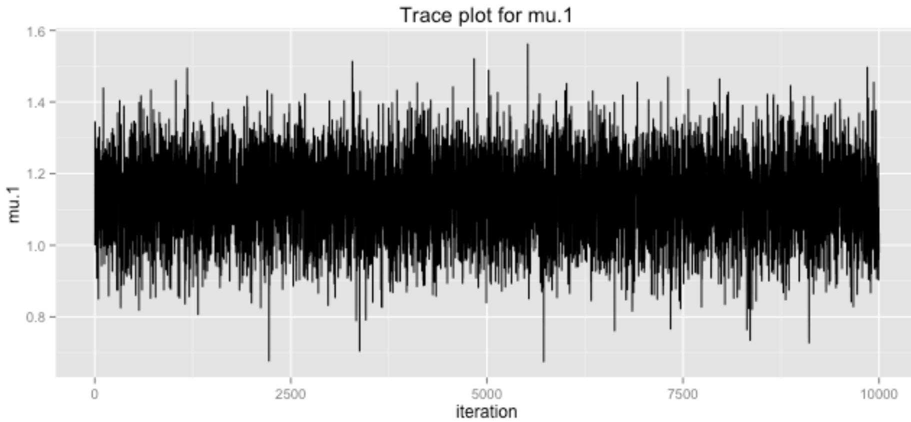
Homework 11

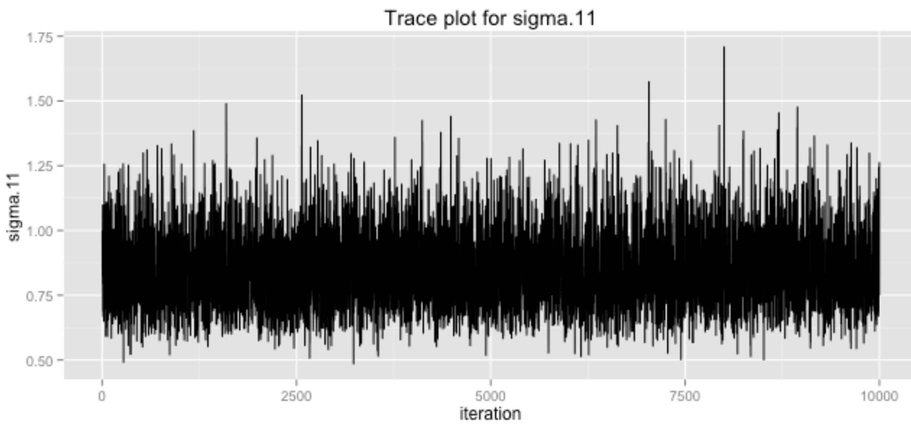
For this analysis, the y values were normalized so that the intercept term could be disregarded.

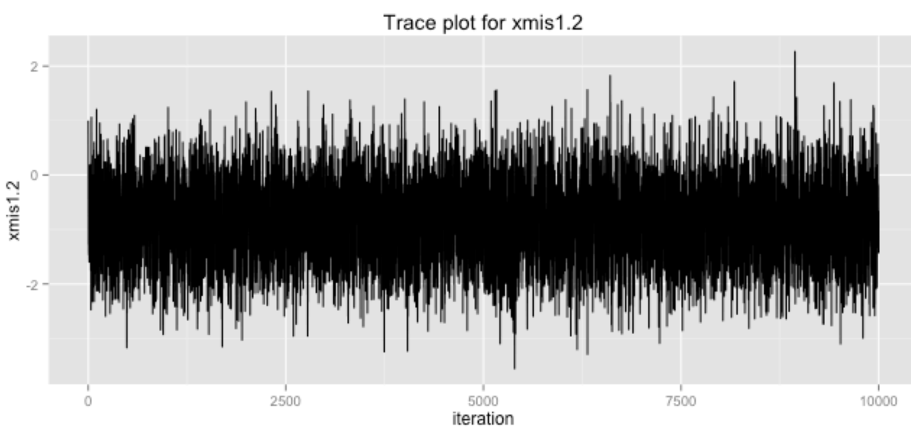
Using the full conditionals from the previous page, with $m_b = m_v = (1, 1)'$, $s_b = s_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $a = b = 0.001$, I ran 10,000 iterations of Gibbs sampling. X_{mis} values were updated by conditioning on the non-missing values. Examining a few trace plots below, it appears that the sampler has converged.







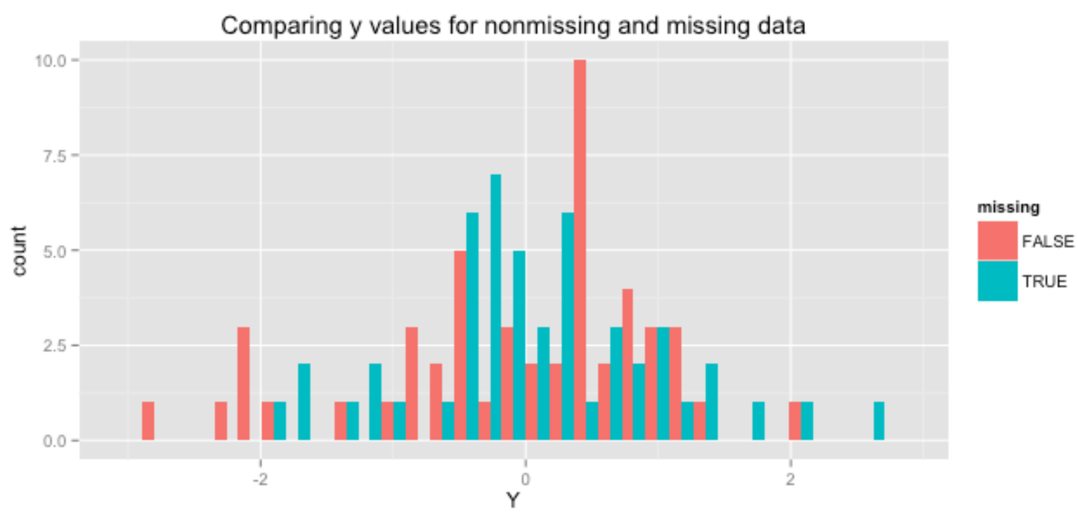




10 burn-ins were thrown out for each parameter for the analysis to follow.

The estimated values for the missing x appear to be reasonable when compared to the x values of the complete data, shown in the plot below.

There does not seem to be a drastic difference in the y values of the missing and complete data, shown in the plot below, supporting the previous observation of reasonable estimated x values.



<u>Comparing Bayesian method with MLE method</u>

The posterior mean for β is estimated to be $(0.3983, 0.5346)'$. Using the glm function and using only complete data, the estimated β is $(0.4091, 0.5400)'$. Both methods have similar MSE's as well, but the MLE method (0.5224) does slightly better than the Bayesian method (0.5226).

Table: Comparing the β estimates and MSE for Bayesian and MLE methods

	β	MSE
Bayesian	$(0.3983, 0.5346)'$	0.5226
MLE	$(0.4091, 0.5400)'$	0.5224

The following plot compares actual y values versus predicted y values from the Bayesian and MLE methods for the complete data. The predictions for both methods are very close, as seen by the blurring of the red (Bayes) and blue (MLE) points.