Kelvin Niu

Homework 10

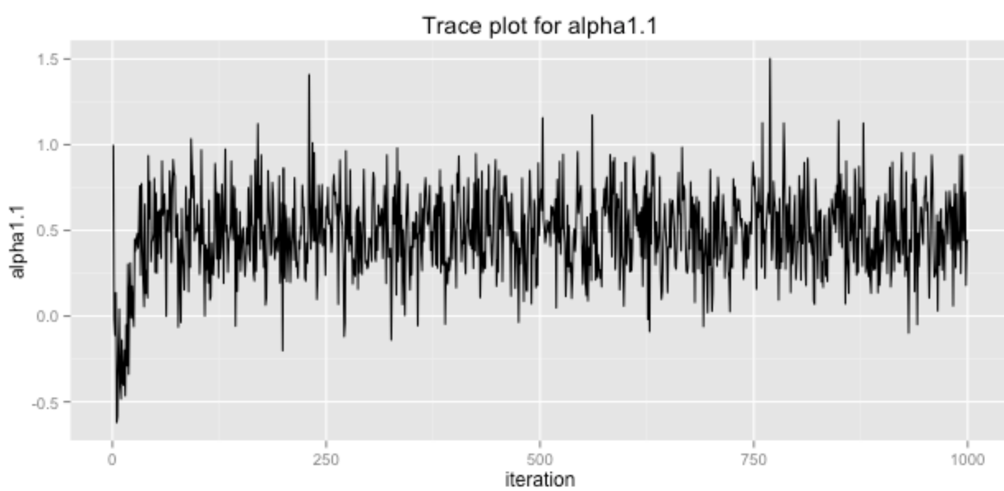
Setup

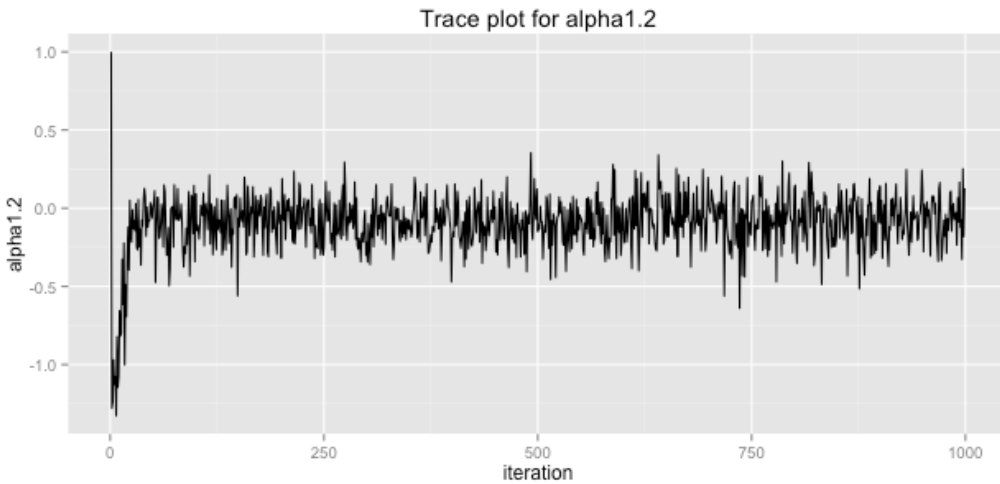
First I calculated the response y_{ij} for each person as BP_syst/BP_dia. I then normalized both y_{ij} and x_{ij} (age). Finally, since some ids were skipped (for example, there are no people with id 30), I mapped people to a new id group without skips in order to make the later calculations more convenient.

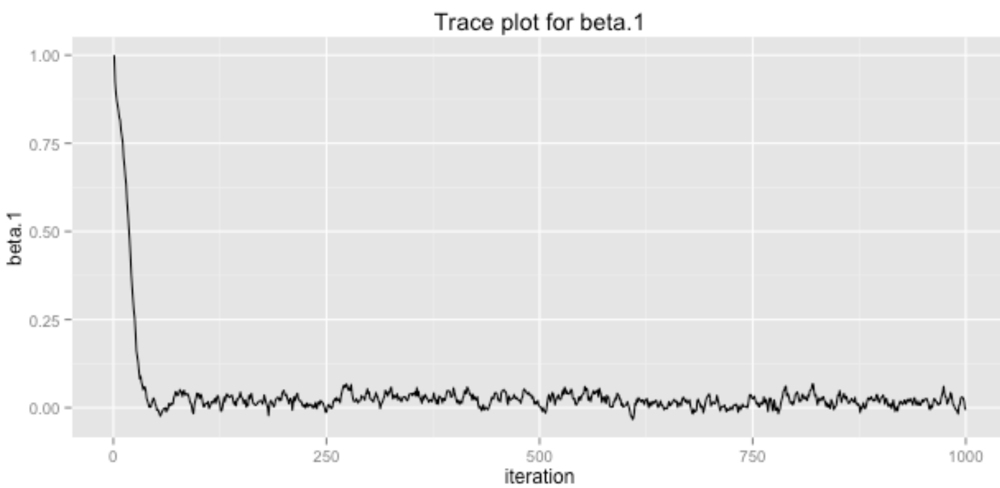
Approach (a)

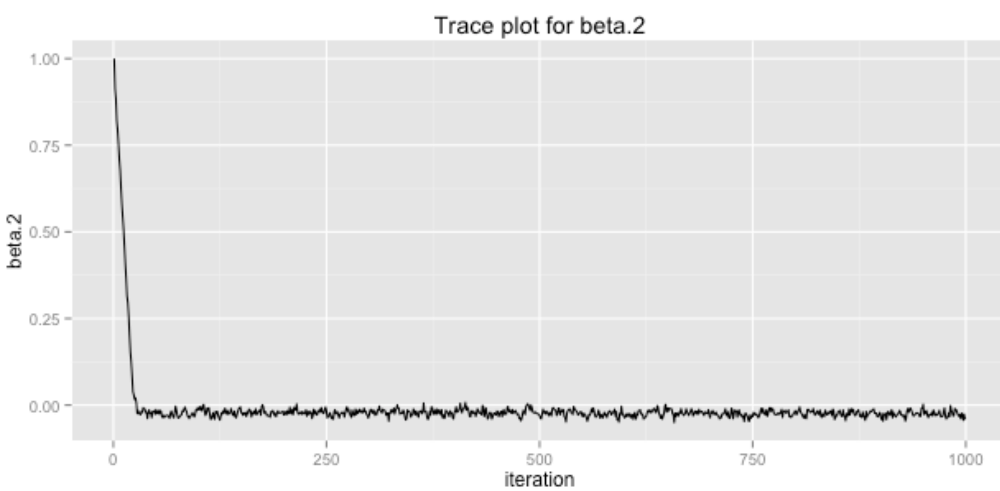
$y_{ij} \sim N\left(x_{ij}'\beta + z_{ij}'\alpha_i, \sigma^2\right)$
$x_{ij} = z_{ij} = \left(1, age_{ij}\right)'$
$\alpha_i \sim N_2(O, \Omega)$
$\beta \sim N_2(O, I)$
$\sigma^{-2} \sim Ga(1,1)$
$\Omega \sim IW(3, I)$

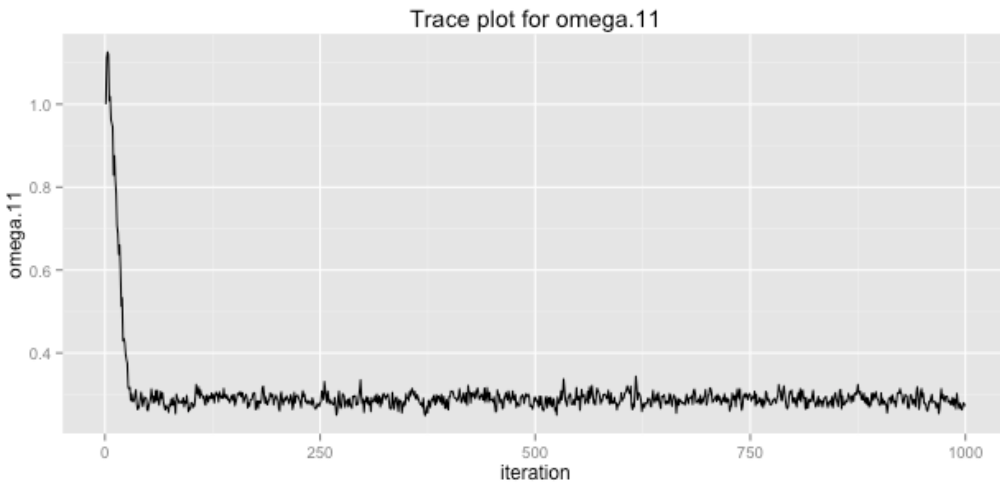
Using the full conditionals (derived in the Appendix), I ran 1000 iterations of Gibbs sampling on the blood pressure data to obtain posterior estimates for the parameters. Examining the trace plots (broken down by elements for matrices), we can see that the samplers appear to converge and throw out burn-ins.

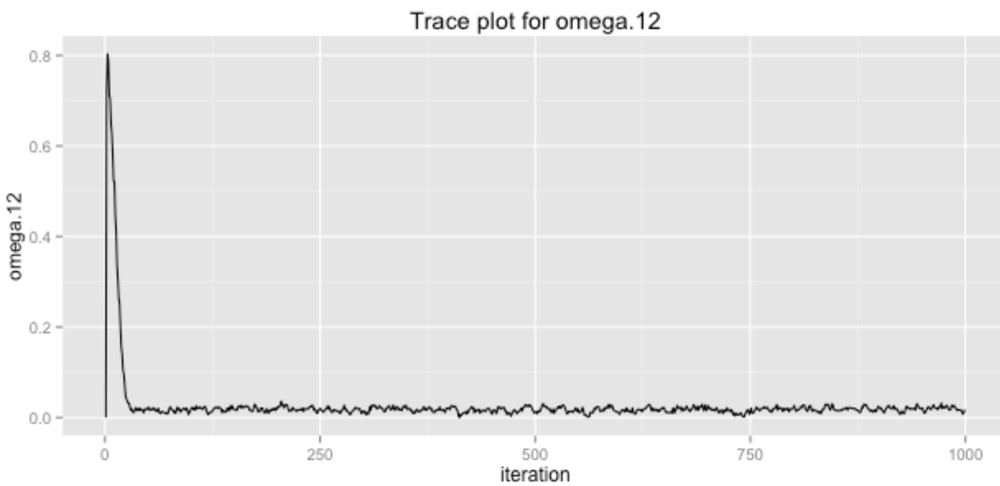


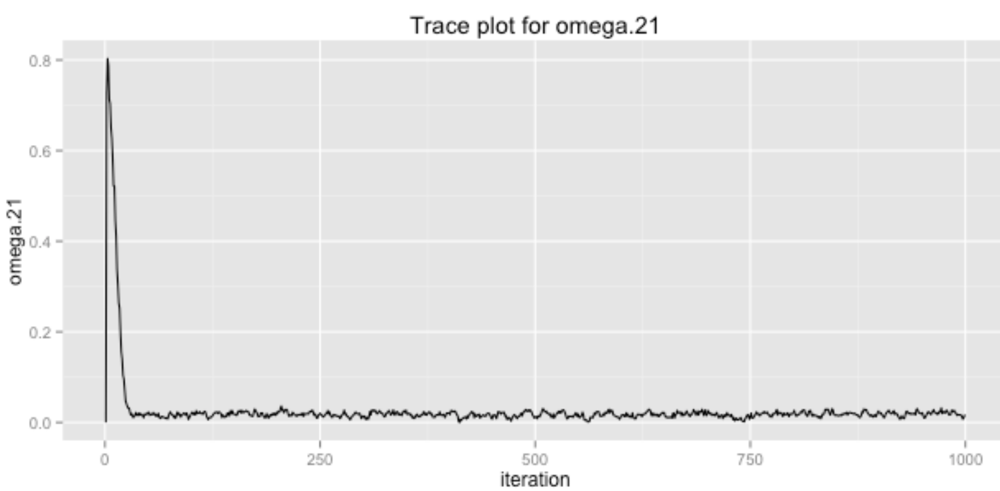

Trace plot for alpha1.2
alpha1.2
1.0
0.5
0.0
-0.5
-1.0
0
250
500
750
1000
iteration

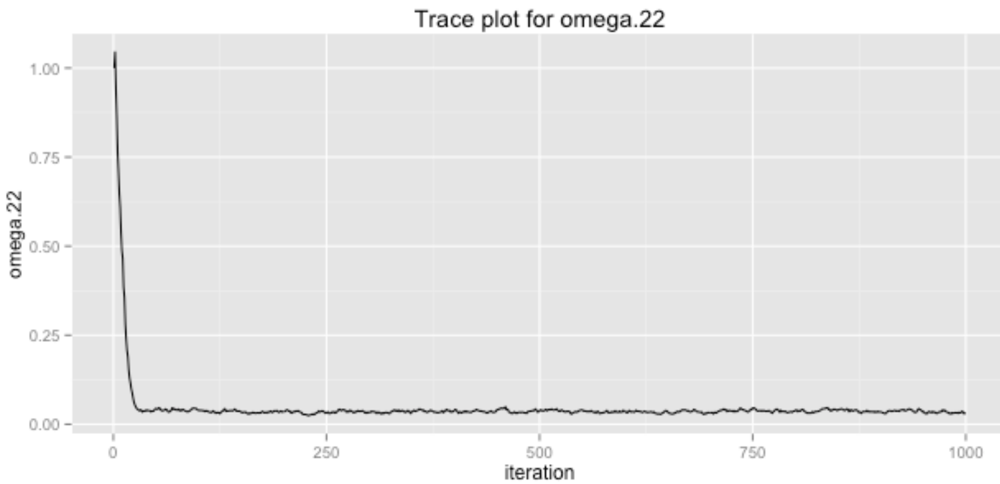

Trace plot for beta.1
beta.1
1.00
0.75
0.50
0.25
0.00
0
250
500
750
1000
iteration

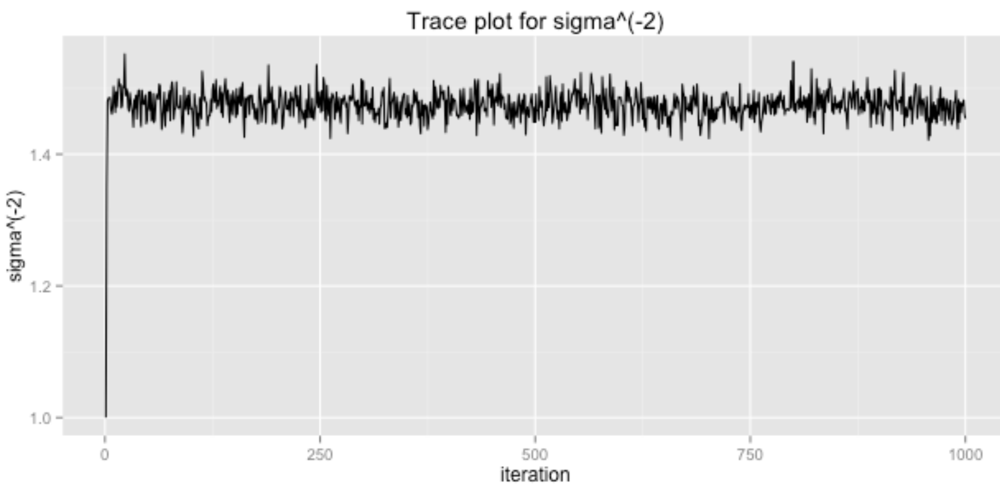

Trace plot for beta.2
beta.2
1.00
0.75
0.50
0.25
0.00
0
250
500
750
1000
iteration


Trace plot for omega.11
omega.11
1.0
0.8
0.6
0.4
0
250
500
750
1000
iteration


Trace plot for omega.12
omega.12
0.8
0.6
0.4
0.2
0.0
0
250
500
750
1000
iteration


Trace plot for omega.21
omega.21
0.8
0.6
0.4
0.2
0.0
0
250
500
750
1000
iteration



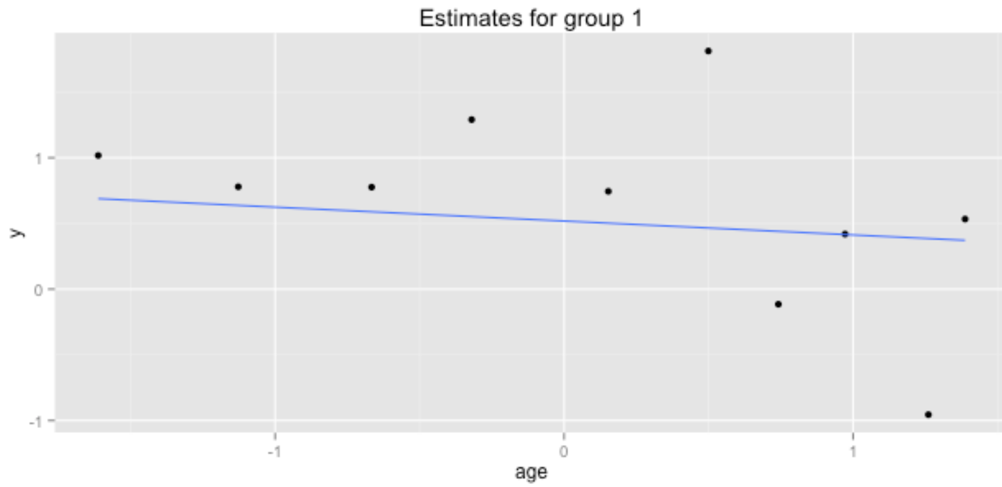


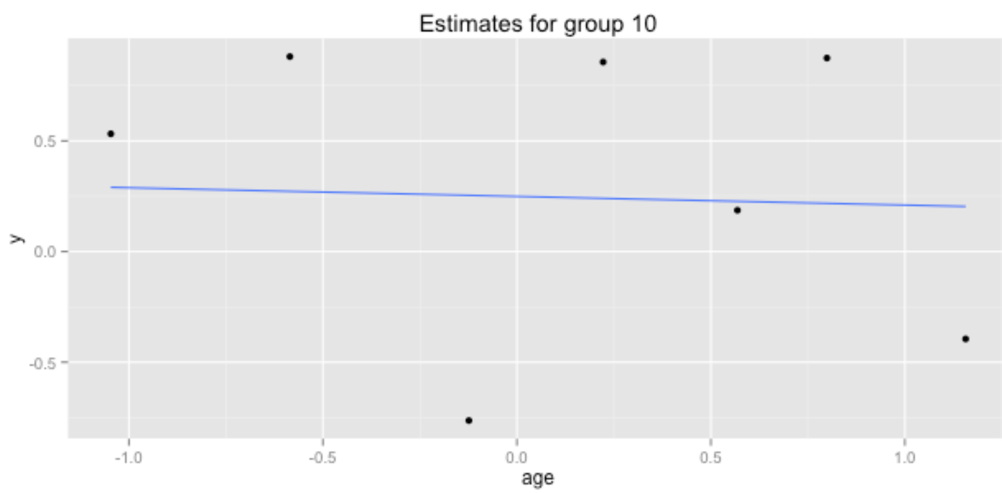
Below are summaries for the posterior estimates of the parameters. (Because there were so many α_i's, I chose to report only a couple of them for convenience)

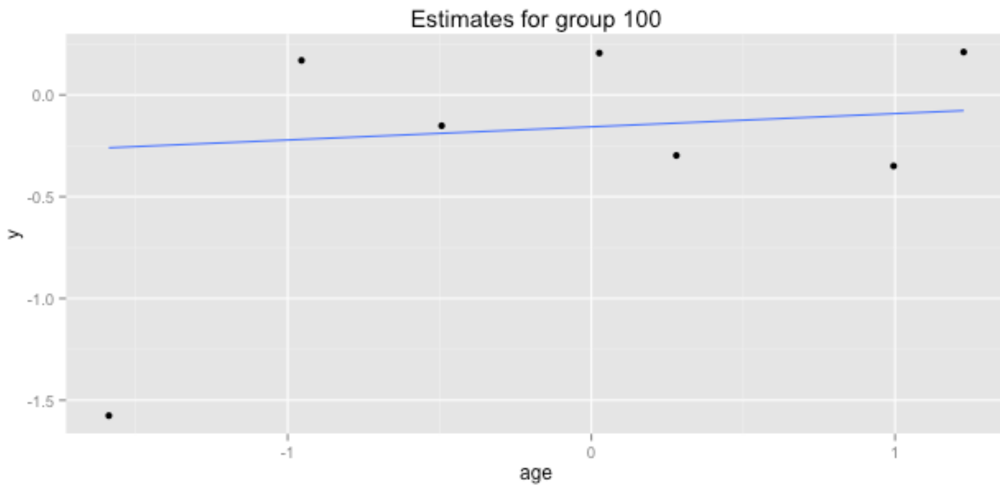
Table: Summary of posterior estimates of parameters

	Mean	95% CI
α_1	$\begin{bmatrix} 0.4816 \\ -0.0949 \end{bmatrix}$	$\begin{bmatrix} [-0.0471, 0.9406] \\ [-0.4552, 0.1971] \end{bmatrix}$
α_{10}	$\begin{bmatrix} 0.2123 \\ -0.0284 \end{bmatrix}$	$\begin{bmatrix} [-0.4046, 0.7651] \\ [-0.4255, 0.3243] \end{bmatrix}$
α_{100}	$\begin{bmatrix} -0.1936 \\ 0.0756 \end{bmatrix}$	$\begin{bmatrix} [-0.7804, 0.3330] \\ [-0.2680, 0.4112] \end{bmatrix}$
β	$\begin{bmatrix} 0.0369 \\ -0.0108 \end{bmatrix}$	$\begin{bmatrix} [-0.0110, 0.2144] \\ [-0.0403, 0.0167] \end{bmatrix}$
σ^{-2}	1.4739	[1.4373, 1.5131]
Ω	$\begin{bmatrix} 0.2997 & 0.0267 \\ 0.0267 & 0.0458 \end{bmatrix}$	$\begin{bmatrix} [0.2612, 0.3756] & [0.0067, 0.0356] \\ [0.0067, 0.0356] & [0.0291, 0.0506] \end{bmatrix}$

The model appears to fit the data decently, which can be visualized in the below plots for a couple groups.







The within-subjects variance σ^2 is estimated to be 0.6785. The between-subjects variance $z_{ij}'\Omega z_{ij}$ for a couple people is estimated to be:

	Between-subjects variance
y_{11}	0.3326
y_{12}	0.2977
y_{21}	0.3529
y_{22}	0.3108

The within-subjects variance appears to be generally greater than the between-subject variances. This makes sense because people in the same group are expected to be more similar with less variation.

<u>Approach (b)</u>

$y_{ij} \sim N\left(x_{ij}'\beta + z_{ij}'(\Lambda \otimes \alpha_i^*), \sigma^2\right)$

$x_{ij} = z_{ij} = \left(1, age_{ij}\right)'$

$\alpha_i^* \sim N_2(O, I)$

$\Lambda \sim N_2(O, I)$

$\beta \sim N_2(O, I)$

$\sigma^{-2} \sim Ga(1, 1)$

This setup induces conjugacy (the full conditionals are calculated in the Appendix) and the code for performing Gibbs sampling is attached in HW 10.R. However, examining the full conditional for Λ, we see that we must sum over a function of α_i^* over ALL GROUPS at EACH ITERATION. Thus the Gibbs sampler is extremely slow and cannot run in a reasonable time (I tried leaving it on for a several hours).

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Approach (a)

$y_{ij} \sim N(x_{ij}'\beta + z_{ij}'\alpha_i, \sigma^2)$

$\alpha_i \sim N_2(0, \Omega)$

$\beta \sim N_2(0, I)$

$\sigma^{-2} \sim Ga(1,1)$

$\Omega \sim IW(3, I)$

$$\pi(\alpha, \beta, \sigma^{-2}, \Omega | y) \propto L(y|\alpha,\beta,\sigma^{-2},\Omega)\pi(\alpha|\Omega)\pi(\beta)\pi(\sigma^{-2})\pi(\Omega)$$

$$\propto \prod_{i=1}^{n}\prod_{j=1}^{n_i}\left[\frac{1}{\sigma}\exp\left[-\frac{1}{2\sigma^2}(y_{ij}-(x_{ij}'\beta+z_{ij}'\alpha_i))^2\right]\right]\cdot\prod_{i=1}^{n}\left[|\Omega|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\alpha_i'\Omega^{-1}\alpha_i\right]\right]$$

$$\cdot\exp\left[-\frac{1}{2}\beta'\beta\right]\cdot e^{-\sigma^{-2}}\cdot|\Omega|^{-\frac{3+2+1}{2}}\exp\left[-\frac{1}{2}tr(\Omega^{-1})\right]$$

$$\pi(\alpha_i|\alpha_{(-i)},\beta,\sigma^{-2},\Omega,y) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{n_i}(-2y_{ij}z_{ij}'\alpha_i + 2x_{ij}'\beta z_{ij}'\alpha_i + (z_{ij}'\alpha_i)^2)\right]\exp\left[-\frac{1}{2}\alpha_i'\Omega^{-1}\alpha_i\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(-2\sigma^{-2}\sum_{j=1}^{n_i}y_{ij}z_{ij}'\alpha_i + 2\sigma^{-2}\sum_{j=1}^{n_i}x_{ij}'\beta z_{ij}'\alpha_i + \sigma^{-2}\sum_{j=1}^{n_i}\alpha_i'z_{ij}z_{ij}'\alpha_i + \alpha_i'\Omega^{-1}\alpha_i\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\alpha_i'\left(\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}z_{ij}'+\Omega^{-1}\right)\alpha_i - 2\alpha_i'\left(\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}y_{ij} - \sigma^{-2}\sum_{j=1}^{n_i}z_{ij}\beta'x_{ij}\right)\right]\right]$$

$$\boxed{\alpha_i|\alpha_{(-i)},\beta,\sigma^{-2},\Omega,y \sim N_2\left(\left(\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}z_{ij}'+\Omega^{-1}\right)^{-1}\left(\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}y_{ij}-\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}\beta'x_{ij}\right), \left(\sigma^{-2}\sum_{j=1}^{n_i}z_{ij}z_{ij}'+\Omega^{-1}\right)^{-1}\right)}$$

$$\pi(\beta|\alpha,\sigma^{-2},\Omega,y) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(-2y_{ij}x_{ij}'\beta + \beta'x_{ij}x_{ij}'\beta + 2z_{ij}'\alpha_i x_{ij}'\beta)\right]\exp\left[-\frac{1}{2}\beta'\beta\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\beta'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}x_{ij}'+I\right)\beta - 2\beta'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}y_{ij} - \sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}\alpha_i'z_{ij}\right)\right]\right]$$

$$\boxed{\beta|\alpha,\sigma^{-2},\Omega,y \sim N_2\left(\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}x_{ij}'+I\right)^{-1}\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}y_{ij}-\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}\alpha_i'z_{ij}\right), \left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}x_{ij}x_{ij}'+I\right)^{-1}\right)}$$

$$\pi(\sigma^{-2}|\alpha,\beta,\Omega,y) \propto \sigma^{-N}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(y_{ij}-(x_{ij}'\beta+z_{ij}'\alpha_i))^2\right]e^{-\sigma^{-2}}$$

$$\propto (\sigma^{-2})^{\frac{N}{2}}\exp\left[-\sigma^{-2}\left[1+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(y_{ij}-(x_{ij}'\beta+z_{ij}'\alpha_i))^2\right]\right]$$

$$\boxed{\sigma^{-2}|\alpha,\beta,\Omega,y \sim Ga\left(\frac{N}{2}+1, \ 1+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(y_{ij}-(x_{ij}'\beta+z_{ij}'\alpha_i))^2\right) \quad \text{where } N=\sum_{i=1}^{n}\sum_{j=1}^{n_i}1}$$

$$\pi(\Omega \mid \alpha, \beta, \sigma^{-2}, y) \propto |\Omega|^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \alpha_i' \Omega^{-1} \alpha_i\right] |\Omega|^{-3} \exp\left[-\frac{1}{2}tr(\Omega^{-1})\right]$$

$$\propto |\Omega|^{-\frac{n+6}{2}} \exp\left[-\frac{1}{2} tr\left(\left(\sum_{i=1}^{n} \alpha_i \alpha_i'\right)\Omega^{-1}\right)\right] \exp\left[-\frac{1}{2}tr(\Omega^{-1})\right]$$

$$\propto |\Omega|^{-\frac{n+6}{2}} \exp\left[-\frac{1}{2} tr\left[\left(I + \sum_{i=1}^{n} \alpha_i \alpha_i'\right)\Omega^{-1}\right]\right]$$

$$\boxed{\Omega \mid \alpha, \beta, \sigma^{-2}, y \sim IW\left(3+n, \left(I + \sum_{i=1}^{n} \alpha_i \alpha_i'\right)\right)}$$

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Approach (b)

$y_{ij} \sim N(x_{ij}'\beta + z_{ij}'(\Lambda \otimes \alpha_i^*), \sigma^2)$

$\alpha_i^* \sim N_2(0, I)$

$\Lambda \sim N_2(0, I)$

$\beta \sim N_2(0, I)$

$\sigma^{-2} \sim Ga(1,1)$

Properties of $\otimes$ → at least for vectors

$a \otimes b = b \otimes a$

$a \otimes b = \text{Diag}(a)\, b = a\, \text{Diag}(b)$

↓

diagonal matrix with elements of b

$$\pi(\alpha^*, \Lambda, \beta, \sigma^{-2} | y) \propto L(y|x, \alpha^*, \Lambda, \beta, \sigma^{-2})\, \pi(\alpha^*)\, \pi(\Lambda)\, \pi(\beta)\, \pi(\sigma^{-2})$$

$$\propto \prod_{i=1}^{n} \prod_{j=1}^{n_i} \left[\frac{1}{\sigma} \exp\left[-\frac{1}{2\sigma^2}(y_{ij} - (x_{ij}'\beta + z_{ij}'\Lambda \otimes \alpha_i^*))^2\right]\right] \cdot \prod_{i=1}^{n} \left[\exp\left[-\frac{1}{2}\alpha_i^{*\prime}\alpha_i^*\right]\right] \cdot \exp\left[-\frac{1}{2}\Lambda'\Lambda\right]$$

$$\cdot \exp\left[-\frac{1}{2}\beta'\beta\right] \cdot e^{-\sigma^{-2}}$$

$$\pi(\alpha_i^* | \alpha_{(-i)}^*, \Lambda, \beta, \sigma^{-2} | y) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{n_i}(-2\alpha_i^{*\prime} \otimes \Lambda' z_{ij} y_{ij} + 2\alpha_i^{*\prime} \otimes \Lambda' z_{ij} x_{ij}'\beta + \alpha_i^{*\prime} \otimes \Lambda' z_{ij} z_{ij}' \Lambda \otimes \alpha_i^*)\right]$$

$$\cdot \exp\left[-\frac{1}{2}\alpha_i^{*\prime}\alpha_i^*\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\alpha_i^{*\prime}\left(\sigma^{-2}\,\text{Diag}(\Lambda)\left(\sum_{j=1}^{n_i} z_{ij}z_{ij}'\right)\text{Diag}(\Lambda) + I\right)\alpha_i^* - 2\alpha_i^{*\prime}\left(\sigma^{-2}\,\text{Diag}(\Lambda)\sum_{j=1}^{n_i} z_{ij}y_{ij} - \sigma^{-2}\,\text{Diag}(\Lambda) \cdot \left(\sum_{j=1}^{n_i} z_{ij}x_{ij}'\right)\beta\right)\right]\right]$$

$$\boxed{\begin{aligned}\alpha_i^* | \alpha_{(-i)}^*, \Lambda, \beta, \sigma^{-2} | y \sim N_2\Bigg(&\left(\sigma^{-2}\text{Diag}(\Lambda)\left(\sum_{j=1}^{n_i} z_{ij}z_{ij}'\right)\text{Diag}(\Lambda) + I\right)^{-1}\left(\sigma^{-2}\text{Diag}(\Lambda)\sum_{j=1}^{n_i} z_{ij}y_{ij} - \sigma^{-2}\,\text{Diag}(\Lambda)\left(\sum_{j=1}^{n_i} z_{ij}x_{ij}'\right)\beta\right),\\ &\left(\sigma^{-2}\text{Diag}(\Lambda)\left(\sum_{j=1}^{n_i} z_{ij}z_{ij}'\right)\text{Diag}(\Lambda) + I\right)^{-1}\Bigg)\end{aligned}}$$

$$\pi(\Lambda | \alpha^*, \beta, \sigma^{-2}, y) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(-2\Lambda' \otimes \alpha_i^{*\prime} z_{ij} y_{ij} + 2\Lambda' \otimes \alpha_i^{*\prime} z_{ij} x_{ij}'\beta + \Lambda' \otimes \alpha_i^{*\prime} z_{ij} z_{ij}' \alpha_i^* \otimes \Lambda)\right]\exp\left[-\frac{1}{2}\Lambda'\Lambda\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\Lambda'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}z_{ij}'\,\text{Diag}(\alpha_i^*) + I\right)\Lambda - 2\Lambda'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}y_{ij} - \sigma^{-2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}x_{ij}'\right)\beta\right)\right]\right]$$

$$\boxed{\begin{aligned}\Lambda | \alpha^*, \beta, \sigma^{-2}, y \sim N_2\Bigg(&\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}z_{ij}'\,\text{Diag}(\alpha_i^*) + I\right)^{-1}\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*)\, z_{ij}y_{ij} - \sigma^{-2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}x_{ij}'\right)\beta\right),\\ &\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\text{Diag}(\alpha_i^*) z_{ij}z_{ij}'\,\text{Diag}(\alpha_i^*) + I\right)^{-1}\Bigg)\end{aligned}}$$

$$\pi(\beta \mid \alpha^*, \Lambda, \sigma^{-2}, y) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(-2\beta' x_{ij} y_{ij} + 2\beta' x_{ij} z_{ij}' \Lambda \otimes \alpha_i^* + \beta' x_{ij} x_{ij}' \beta\right)\right] \exp\left[-\frac{1}{2}\beta'\beta\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\beta'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} x_{ij}' + I\right)\beta - 2\beta'\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} y_{ij} - \sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} z_{ij}' \Lambda \otimes \alpha_i^*\right)\right]\right]$$

$$\boxed{\beta \mid \alpha^*, \Lambda, \sigma^{-2}, y \sim N_2\left(\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} x_{ij}' + I\right)^{-1}\left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} y_{ij} - \sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} z_{ij}' \Lambda \otimes \alpha_i^*\right), \left(\sigma^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} x_{ij} x_{ij}' + I\right)^{-1}\right)}$$

$$\pi(\sigma^{-2} \mid \alpha^*, \Lambda, \beta, y) \propto \sigma^{-N} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(y_{ij} - (x_{ij}'\beta + z_{ij}'\Lambda \otimes \alpha_i^*)\right)^2\right] e^{-\sigma^{-2}}$$

$$\propto (\sigma^{-2})^{\frac{N}{2}} \exp\left[-\sigma^{-2}\left(1 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(y_{ij} - (x_{ij}'\beta + z_{ij}'\Lambda \otimes \alpha_i^*)\right)^2\right)\right]$$

$$\boxed{\sigma^{-2} \mid \alpha^*, \Lambda, \beta, y \sim Ga\left(1 + \frac{N}{2},\ 1 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(y_{ij} - (x_{ij}'\beta + z_{ij}'\Lambda \otimes \alpha_i^*)\right)^2\right)}$$