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Question 1

$y_i \sim N(\mu, \sigma^2)$ with unknown mean and variance

Choose prior

$$\pi(\mu, \sigma^{-2}) = N(\mu; \mu_0, K_0\sigma^2)\, Ga(\sigma^{-2}; a, b)$$

The joint posterior is then

$$\pi(\mu, \sigma^{-2} \mid y_1, \dots, y_n) = (2\pi K_0 \sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2K_0\sigma^2}(\mu-\mu_0)^2\right] \frac{b^a}{\Gamma(a)} (\sigma^{-2})^{a-1} e^{-b\sigma^{-2}} \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y_i-\mu)^2\right]$$

$$\propto \sigma^{-1} \exp\left[-\frac{1}{2K_0\sigma^2}(\mu-\mu_0)^2\right] \sigma^{-2a+2}\, \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right] e^{-b\sigma^{-2}}$$

$$\propto \sigma^{-2a-n+1} \exp\left[-\frac{1}{2K_0\sigma^2}(\mu-\mu_0)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2 - b\sigma^{-2}\right]$$

$$\propto \sigma^{-2a-n+1} \exp\left[-\frac{1}{2K_0\sigma^2}(\mu^2 - 2\mu\mu_0 + \mu_0^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right) - b\sigma^{-2}\right]$$

$$\propto \sigma^{-2a-n+1} \exp\left[\mu^2\left(-\frac{1}{2K_0\sigma^2} - \frac{n}{2\sigma^2}\right) + \mu\left(\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i\right) - \frac{1}{2K_0\sigma^2}\mu_0^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 - b\sigma^{-2}\right]$$

$$\propto \sigma^{-2a-n+1} \exp\left[\mu^2\left(-\frac{1}{2K_0\sigma^2} - \frac{n}{2\sigma^2}\right) + \mu\left(\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i\right)\right] \exp\left[-\frac{1}{2K_0\sigma^2}\mu_0^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 - b\sigma^{-2}\right]$$

$$\propto \sigma^{-2a-n+1} \underbrace{\exp\left[-\frac{1}{2}\left(\mu^2\left(\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}\right) - 2\mu\left(\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i\right)\right)\right]} \exp\left[-\frac{1}{\sigma^2}\left(\frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]$$

we want this in the form $\exp\left[-\frac{1}{2v}(\mu^2 - 2\mu\hat{\mu} + \hat{\mu}^2)\right]$ $\rightarrow$ so let $v = \left(\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}\right)^{-1}$

$$\propto \sigma^{-2a-n+1} \exp\left[-\frac{1}{2v}\left(\mu^2 - 2\mu v \underbrace{\left(\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i\right)}\right)\right] \exp\left[-\frac{1}{\sigma^2}\left(\frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]$$

we just need to complete the square now $\rightarrow$ so let $\hat{\mu} = \dfrac{\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i}{\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}}$

we are multiplying by a constant times σ^{-1}, so we must also multiply by σ to keep proportionality

$$\propto \sigma^{-2a-n+1} \exp\left[-\frac{1}{2v}(\mu - \hat{\mu})^2\right] \exp\left[\frac{1}{2v}\hat{\mu}^2\right] \exp\left[-\frac{1}{\sigma^2}\left(\frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]$$

$$\propto \sigma^{-2a-n+1}\, \sigma \underbrace{\sqrt{\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}}\, \exp\left[-\frac{1}{2v}(\mu - \hat{\mu})^2\right]}_{N(\mu;\, \hat{\mu},\, v)} \exp\left[\frac{1}{2v}\hat{\mu}^2 - \frac{1}{\sigma^2}\left(\frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]$$

$$\propto N(\mu; \hat{\mu}, v)\, \sigma^{-2a-n+2} \exp\left[\frac{1}{2}\hat{\mu}^2\left(\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}\right) - \frac{1}{\sigma^2}\left(\frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]$$

$$\propto N(\mu; \hat{\mu}, v)\, \underbrace{(\sigma^{-2})^{a+\frac{n}{2}-1} \exp\left[-\sigma^{-2}\left(-\frac{1}{2K_0}\hat{\mu}^2 - \frac{n}{2}\hat{\mu}^2 + \frac{\mu_0^2}{2K_0} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 + b\right)\right]}_{Ga(\tilde{a},\, \tilde{b})}$$

$Ga(\tilde{a}, \tilde{b})$ where $\tilde{a} = a + \frac{n}{2}$, $\tilde{b} = -\left(\frac{1+nK_0}{2K_0}\right)\hat{\mu}^2 + \frac{\mu_0^2 + K_0\sum_{i=1}^{n} y_i^2}{2K_0} + b$

Simplifying $\hat{\mu}, v, \tilde{a}, \tilde{b}$, we get

$$\hat{\mu} = \frac{\frac{\mu_0}{K_0\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i}{\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}} = \boxed{\frac{\mu_0 + K_0\sum_{i=1}^{n} y_i}{1 + K_0 n}}$$

$$v = \left(\frac{1}{K_0\sigma^2} + \frac{n}{\sigma^2}\right)^{-1} = \boxed{\sigma^2\left(\frac{K_0}{1+K_0 n}\right)}$$

$$\tilde{a} = \boxed{a + \frac{n}{2}} \qquad \tilde{b} = \boxed{b - \left(\frac{1+nK_0}{2K_0}\right)\hat{\mu}^2 + \frac{\mu_0^2 + K_0\sum_{i=1}^{n} y_i^2}{2K_0}}$$

Thus,

$$\boxed{\pi(\mu, \sigma^{-2} \mid y_1, \dots, y_n) = N\left(\mu; \frac{\mu_0 + k_0 \sum_{i=1}^{n} y_i}{1 + k_0 n}, \left(\frac{k_0}{1+k_0 n}\right)\sigma^2\right) Ga\left(\sigma^{-2}; a + \frac{n}{2}, b - \left(\frac{1+nk_0}{2k_0}\right)\hat{\mu}^2 + \frac{\mu_0^2 + k_0 \sum_{i=1}^{n} y_i^2}{2k_0}\right)}$$

which is conjugate to $\pi(\mu, \sigma^{-2})$

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Question 2

$$\pi(\mu, \sigma^{-2} | y_1, \dots, y_n) = N(\mu; \hat{\mu}, v)\, Ga(\sigma^{-2}; \tilde{a}, \tilde{b})$$

$$= \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{1}{2v}(\mu-\hat{\mu})^2\right] \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} (\sigma^{-2})^{\tilde{a}-1} e^{-\tilde{b}\sigma^{-2}}$$

$$P(\mu | y_1, \dots, y_n) = \int_0^\infty \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{1}{2v}(\mu-\hat{\mu})^2\right] \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} (\sigma^{-2})^{\tilde{a}-1} e^{-\tilde{b}\sigma^{-2}}\, d\sigma^{-2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \int_0^\infty v^{-\frac{1}{2}} (\sigma^{-2})^{\tilde{a}-1} \exp\left[-\frac{1}{2v}(\mu-\hat{\mu})^2 - \tilde{b}\sigma^{-2}\right] d\sigma^{-2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \int_0^\infty \left[\sigma^2\left(\frac{K_0}{1+nK_0}\right)\right]^{-\frac{1}{2}} (\sigma^{-2})^{\tilde{a}-1} \exp\left[-\frac{1}{2}\left[\sigma^{-2}\left(\frac{1+nK_0}{K_0}\right)\right](\mu-\hat{\mu})^2 - \tilde{b}\sigma^{-2}\right] d\sigma^{-2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \sqrt{\frac{1+nK_0}{K_0}} \underbrace{\int_0^\infty (\sigma^{-2})^{\tilde{a}-\frac{1}{2}} \exp\left[-\sigma^{-2}\left(\frac{(1+nK_0)(\mu-\hat{\mu})^2}{2K_0} + \tilde{b}\right)\right] d\sigma^{-2}}_{Ga\left(\sigma^{-2};\, \tilde{a}+\frac{1}{2},\, \frac{(1+nK_0)(\mu-\hat{\mu})^2}{2K_0} + \tilde{b}\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \cdot \sqrt{\frac{1+nK_0}{K_0}} \cdot \frac{\Gamma(\tilde{a}+\frac{1}{2})}{\left(\frac{1+nK_0}{2K_0}(\mu-\hat{\mu})^2 + \tilde{b}\right)^{\tilde{a}+\frac{1}{2}}}$$

$$\propto \left(\frac{1+nK_0}{2K_0}(\mu-\hat{\mu})^2 + \tilde{b}\right)^{-\frac{2\tilde{a}+1}{2}}$$

$$\propto \left(\left[\frac{1+nK_0}{2K_0}\right] \cdot \frac{2\tilde{a}}{2\tilde{a}} \cdot \frac{1}{\tilde{b}}(\mu-\hat{\mu})^2 + \tilde{b} \cdot \frac{1}{\tilde{b}}\right)^{-\frac{2\tilde{a}+1}{2}}$$

$$\propto \left(\frac{1}{2\tilde{a}} \underbrace{\left[\left(\frac{1+nK_0}{K_0}\right) \frac{\tilde{a}}{\tilde{b}} (\mu-\hat{\mu})^2\right]}_{\sqrt{\left(\frac{1+nK_0}{K_0}\right)\frac{\tilde{a}}{\tilde{b}}}(\mu-\hat{\mu}) \sim t_{2\tilde{a}}} + 1\right)^{-\frac{2\tilde{a}+1}{2}}$$

Thus the marginal posterior $\mu | y_1, \dots, y_n$ is a non-central t-distribution with $2\tilde{a}$ d.f.

$$\boxed{\sqrt{\left(\frac{1+nK_0}{K_0}\right)\frac{\tilde{a}}{\tilde{b}}}(\mu-\hat{\mu}) \sim t_{2\tilde{a}}}$$

where $\hat{\mu}, \tilde{a}, \tilde{b}$ are as defined in Question 1

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Question 3

Let $\pi(\pi, \sigma^{-2}) = \pi(\mu)\,\pi(\sigma^{-2})$ where

$\mu \sim N(\mu_0, \sigma_0^2)$

$\sigma^{-2} \sim Ga(a, b)$

The joint posterior probability is then

$$\pi(\mu, \sigma^{-2} \mid y_1, \dots, y_n) = \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right) \exp\left[-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right] \frac{b^a}{\Gamma(a)} (\sigma^{-2})^{a-1} e^{-b\sigma^{-2}} \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y_i-\mu)^2\right]$$

$$\propto \exp\left[-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right] (\sigma^{-2})^{a-1} e^{-b\sigma^{-2}} \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right]$$

$$\propto (\sigma^{-2})^{a+\frac{n}{2}-1} \exp\left[-b\sigma^{-2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right] \exp\left[-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right]$$

$$\propto (\sigma^{-2})^{a+\frac{n}{2}-1} \exp\left[-\sigma^{-2}\left(b + \frac{1}{2}\sum_{i=1}^{n} y_i^2 - \mu\sum_{i=1}^{n} y_i + \frac{1}{2}n\mu^2\right)\right] \exp\left[-\frac{1}{2\sigma_0^2}(\mu^2 - 2\mu\mu_0 + \mu_0^2)\right]$$

Looking at just the μ terms of the joint posterior, we can find the full conditional of μ

$$\pi(\mu \mid \sigma^{-2}, y_1, \dots, y_n) \propto \exp\left[\mu\sigma^{-2}\sum_{i=1}^{n} y_i - \frac{1}{2}\sigma^{-2}n\mu^2 - \frac{1}{2}\sigma_0^{-2}\mu^2 + \sigma_0^{-2}\mu_0\mu\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\mu^2 - 2\mu\left(\frac{1}{\sigma^2}\sum_{i=1}^{n} y_i + \frac{1}{\sigma_0^2}\mu_0\right)\right]\right] \rightarrow \text{let } v = \left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)^{-1}$$

$$\propto \exp\left[-\frac{1}{2v}\left(\mu^2 - 2\mu v\left(\frac{1}{\sigma^2}\sum_{i=1}^{n} y_i + \frac{1}{\sigma_0^2}\mu_0\right)\right)\right]$$

$$\propto \exp\left[-\frac{1}{2v}(\mu-\hat{\mu})^2\right] \rightarrow \text{where } \hat{\mu} = \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n} y_i + \frac{1}{\sigma_0^2}\mu_0}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}}$$

v simplified

$\hat{\mu}$ simplified

$$\boxed{\mu \mid \sigma^{-2}, y_1, \dots, y_n \sim N\left(\frac{\sigma^2\sigma_0^2}{n\sigma_0^2+\sigma^2},\ \frac{\sigma_0^2}{n\sigma_0^2+\sigma^2}\sum_{i=1}^{n} y_i + \frac{\sigma^2}{n\sigma_0^2+\sigma^2}\mu_0\right)}$$

Looking at just the σ^{-2} terms of the joint posterior, we can find the full conditional of σ^{-2}

$$\pi(\sigma^{-2} \mid \mu, y_1, \dots, y_n) \propto (\sigma^{-2})^{a+\frac{n}{2}-1} \exp\left[-\sigma^{-2}\left(b + \frac{1}{2}\sum_{i=1}^{n} y_i^2 - \mu\sum_{i=1}^{n} y_i + \frac{1}{2}n\mu^2\right)\right]$$

$$\boxed{\sigma^{-2} \mid \mu, y_1, \dots, y_n \sim Ga\left(a + \frac{n}{2},\ b + \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right)}$$

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Homework 6

Question 4

I simulated n = 30 data points y_i from a normal distribution with true mean 0 and true variance 1. I then calculated posterior summaries of the marginal posterior $\mu \mid y_1, \dots, y_n$ under the following two priors:

$\pi(\mu, \sigma^{-2}) = \pi(\mu \mid \sigma^{-2})\pi(\sigma^{-2}) = N(\mu; \mu_0, \kappa_0 \sigma^2) Ga(\sigma^{-2}; a, b)$

- In this prior μ is dependent on σ^{-2}
- We have the exact marginal posterior distribution of μ from Question 2
- For the simulation, I set $\mu_0 = 0$, $\kappa_0 = 1$, $a = 0.001$, and $b = 0.001$ as a weak prior

$\pi(\mu, \sigma^{-2}) = \pi(\mu)\pi(\sigma^{-2}) = N(\mu; \mu_0, \sigma_0^2) Ga(\sigma^{-2}; a, b)$

- In this prior μ and σ^{-2} are independent
- We can approximate the marginal posterior distribution of μ using Gibbs sampling with the full conditionals from Question 3
- For the simulation, I set $\mu_0 = 0$, $\sigma_0^2 = 100$, $a = 0.001$, and $b = 0.001$ as a weak prior and used 1000 iterations for Gibbs sampling
- Looking at the trace plot for μ, I threw away the first 10 values as burn-ins

Table: Marginal posterior summaries of $\mu \mid y_1, \dots, y_n$ for different priors

	Mean	Variance	95% CI
$\pi(\mu, \sigma^{-2}) = \pi(\mu \mid \sigma^{-2})\pi(\sigma^{-2})$	0.0727	0.0248	[-0.2307, 0.3914]
$\pi(\mu, \sigma^{-2}) = \pi(\mu)\pi(\sigma^{-2})$	0.0749	2.5959e-05	[0.0675, 0.0828]

Both priors result in a similar mean for the marginal posterior distribution of μ that is close to the 0, the true value of μ. However, the variance under the dependent prior is greater than the variance under the independent prior. Thus, this suggests that there is a tradeoff in certainty using the conjugate joint prior (which allows us to calculate the exact marginal posterior distribution of μ) for convenience. Yet the 95% credible interval of μ for the marginal posterior under the independent prior does not contain 0, so perhaps the marginal posterior variance under the independent prior is too small practically.