Kelvin Niu

Question 1

$X \mid (\tau^2) \sim N(0, \frac{1}{\tau^2})$

$\tau^2 \sim \text{Gamma}(\frac{r}{2}, \frac{r}{2})$

$$p(X) = \int_0^\infty \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x^2}{2}} \frac{(\frac{r}{2})^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} (\tau^2)^{\frac{r}{2}-1} e^{-\frac{r}{2}\tau^2} \, d\tau^2$$

$$= \frac{(\frac{r}{2})^{\frac{r}{2}}}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})} \int_0^\infty \underbrace{(\tau^2)^{\frac{r+1}{2}-1} e^{-(\frac{r+x^2}{2})\tau^2}}_{\text{Gamma}(\frac{r+1}{2}, \frac{x^2+r}{2}) \text{ kernel}} \, d\tau^2$$

$$= \frac{(\frac{r}{2})^{\frac{r}{2}}}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})} \cdot \frac{\Gamma(\frac{r+1}{2})}{(\frac{x^2+r}{2})^{\frac{r+1}{2}}} \cdot 1$$

$$= \frac{\Gamma(\frac{r+1}{2})\,(\frac{2}{r})^{\frac{1}{2}}}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\,(\frac{2}{r})^{\frac{r}{2}}\,(\frac{2}{r})^{\frac{1}{2}}\,(\frac{x^2+r}{2})^{\frac{r+1}{2}}}$$

$$= \boxed{\frac{\Gamma(\frac{r+1}{2})}{\sqrt{r\pi}\,\Gamma(\frac{r}{2})} \cdot \left(1+\frac{x^2}{r}\right)^{-\frac{r+1}{2}}}$$

$\rightarrow$ student's t-distribution with r degrees of freedom

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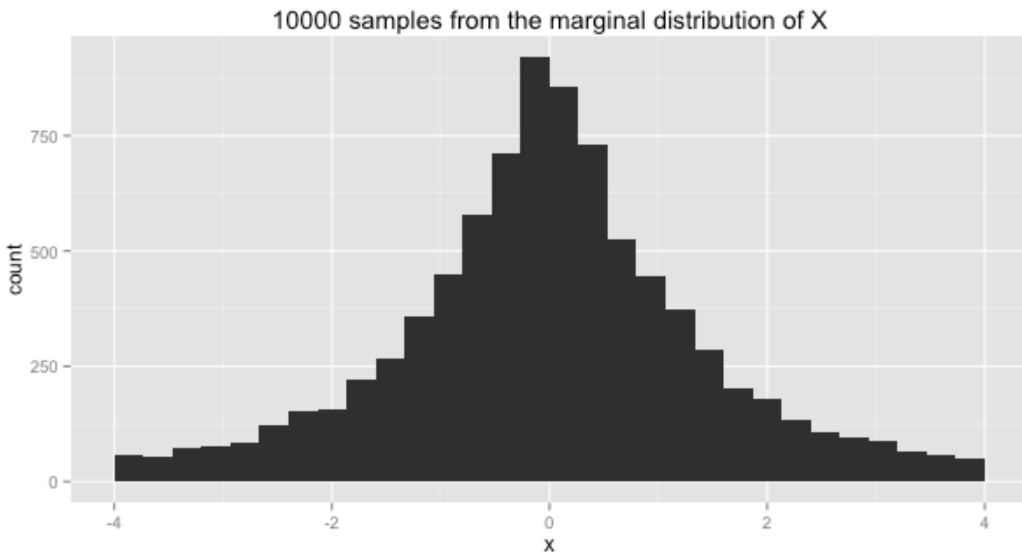
Lab 4

Question 2

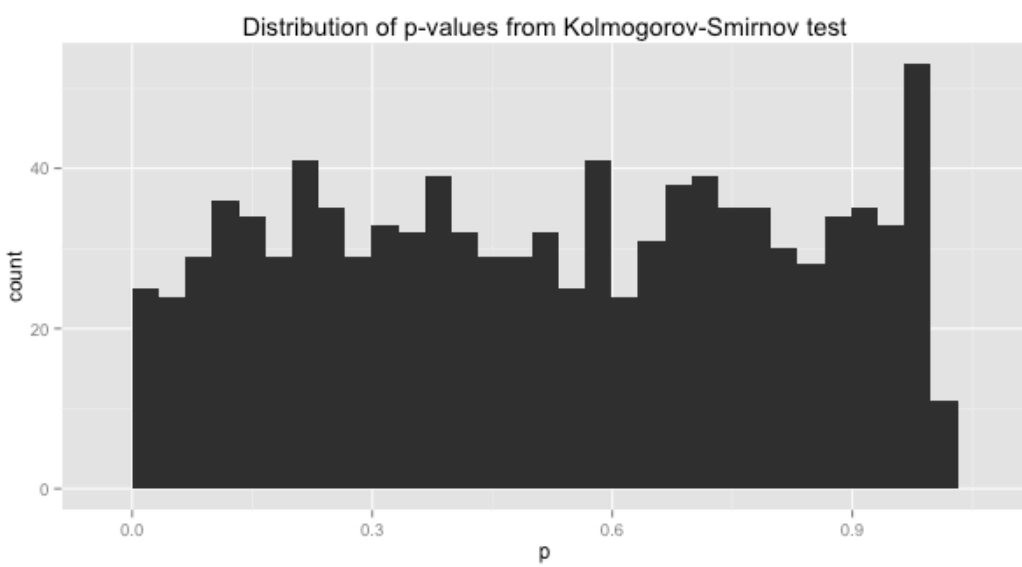


When ν = 1, the marginal distribution of X is a Student's *t*-distribution with 1 degree of freedom. This is also a Cauchy distribution with location 0 and scale 1.

Question 3

The p-value from a Kolmogorov-Smirnov test comparing my observed distribution to a *t*-distribution with 1 d.f. is **0.6656**. Thus, the test concludes that there is not enough evidence to reject the null hypothesis that my sample came from a *t*-distribution with 1 d.f.

Question 4



The distribution of p-values should be uniform Beta(1, 1) because the null hypothesis is true.

Question 5

The Central Limit Theorem does not hold for the mean of a sample from p(X) when $\nu = 1$ or $\nu = 2$ because the distribution does not have finite variance for these values of ν. However, it does hold when $\nu = 3$ because the distribution has a defined mean and finite variance under this value of ν.