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Homework 8

<u>Setup</u>

$y_i \sim N(\beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{100} x_{i(100)}, \sigma^2)$
$\sigma^2 = 1$
$\beta_1, \dots, \beta_5 \sim N(0, 2)$
$\beta_6, \dots, \beta_{100} = 0$
$x_i \sim N_{100}(O, I)$
$n = 100$

For this simulation, I set 5 non-zero betas $\beta_1, \dots, \beta_5$ [each drawn from $N(0, 2)$] and 95 betas $\beta_6, \dots, \beta_{100}$ equal to zero. I then simulated 100 x_i's each with its predictors drawn independently from a standard normal $N(0, 1)$. Finally I calculated the values of y_i using the beta and x_i values plus noise distributed $N(0, 1)$.

<u>Implementation of different methods</u>

Fully Bayes ridge: I set the priors for τ and σ^{-2} to be $\tau \sim Ga(0.5, 0.5)$ and $\sigma^{-2} \sim Ga(0.001, 0.001)$. Using the full conditionals (derived in the Appendix), I performed 1000 iterations of Gibbs sampling with the initial value of the betas set to the true values of the betas. After examining the trace plots, no burn-ins were needed, perhaps because my initial values of the betas were the true values.

***t*-prior with $\boldsymbol{\nu = 1}$**: Similarly, I set the priors for τ and σ^{-2} to be $\tau \sim Ga(0.5, 0.5)$ and $\sigma^{-2} \sim Ga(0.001, 0.001)$. Using the full conditionals (derived in the Appendix), I performed 1000 iterations of Gibbs sampling with the initial value of the betas set to the true values of the betas. After examining the trace plots, no burn-ins were needed, perhaps because my initial values of the betas were the true values.

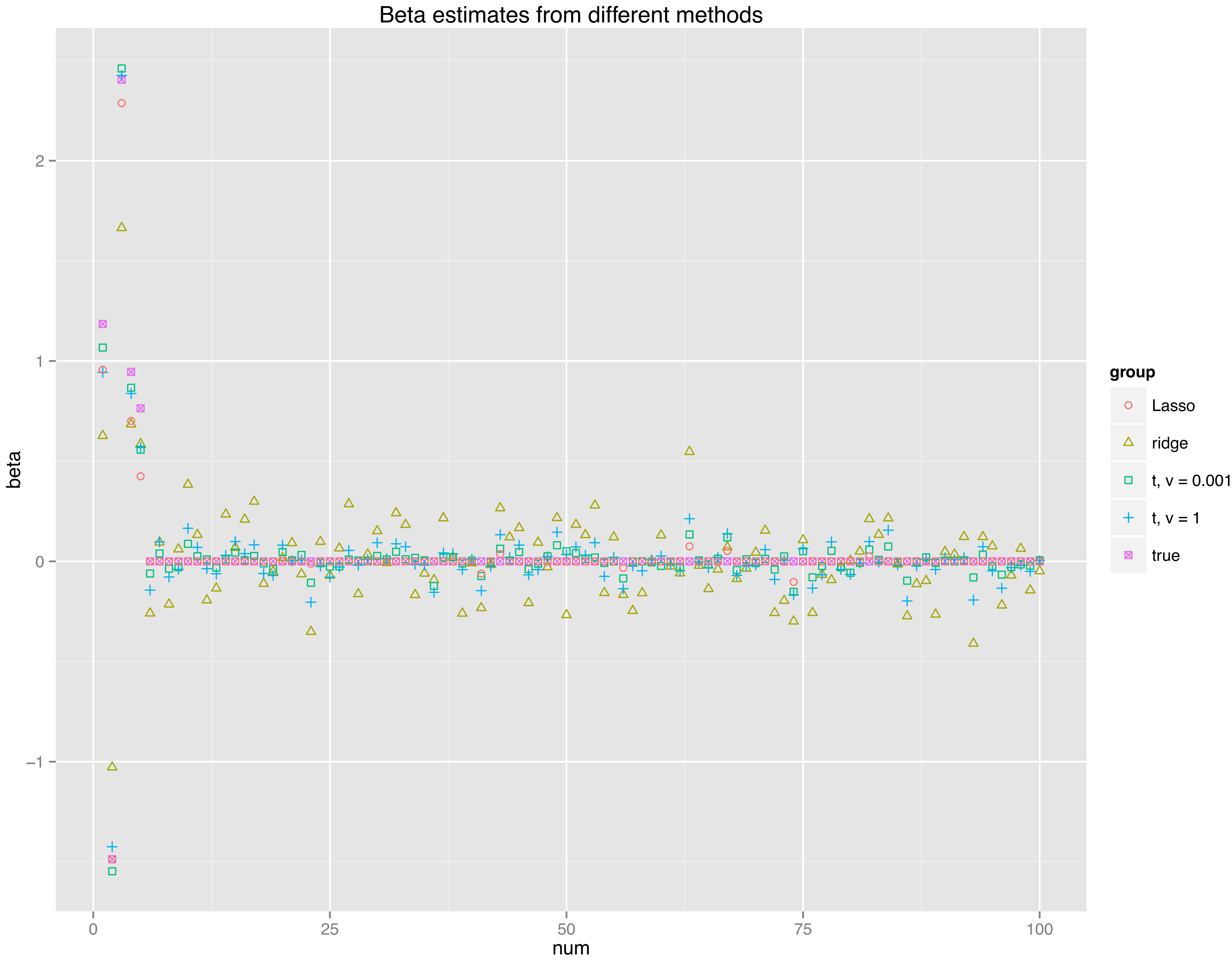
***t*-prior with $\boldsymbol{\nu = 0.001}$**: Similarly, I set the priors for τ and σ^{-2} to be $\tau \sim Ga(0.5, 0.5)$ and $\sigma^{-2} \sim Ga(0.001, 0.001)$. Using the full conditionals (derived in the Appendix), I performed 1000 iterations of Gibbs sampling with the initial value of the betas set to the true values of the betas. After examining the trace plots, no burn-ins were needed, perhaps because my initial values of the betas were the true values.

Lasso: I utilized the glmnet package in R to get Lasso estimates of the betas.

<u>Comparison among different methods</u>

The following is a plot comparing the beta estimates among the different methods.

- The fully Bayes ridge method appears to pull the estimates of the true (non-zero) betas too close to 0 while pulling the estimates of the fake betas too much out from 0
- The t-prior methods appears to estimate the true (non-zero) betas slightly better than Lasso, but Lasso estimates the fake betas better



Beta estimates from different methods
beta
num
2
1
0
−1
0
25
50
75
100
group
Lasso
ridge
t, v = 0.001
t, v = 1
true

For each method, I calculated the MSE on the β's and the predictive MSE on a newly generated test dataset of 100 observations. The results are summarized in the table below.

Table: Comparison of MSE's among different methods for estimating β's

	MSE on β's	Predictive MSE
Fully Bayes ridge	0.0411	4.4731
t-prior with $\nu = 1$	0.0074	1.7688
t-prior with $\nu = 0.001$	0.0027	1.2153
Lasso	0.0026	1.2270

- The fully Bayes ridge regression performs the worst in terms of both estimating β's and predicting new observations
- The t-prior with the smaller ν value performs better than the t-prior with the larger ν value in terms of both MSE on β's and predictive MSE
- The Bayesian method of using a t-prior with $\nu = 0.001$ performs about the same as the frequentist Lasso method in terms of both estimating β's and predicting new observations. Both of these methods do very well compared to the others.

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Fully Bayes ridge

$$y \sim N_{100}(X\beta, \sigma^2 I)$$

$$\beta \sim N_{100}(0, \tau^{-1}\sigma^2 I)$$

$$\tau \sim Ga(a, b)$$

$$\sigma^{-2} \sim Ga(c, d)$$

$$\pi(\beta, \tau, \sigma^{-2} | X, y) = L(y | X, \beta, \sigma^{-2})\, \pi(\beta | \tau, \sigma^{-2})\, \pi(\tau)\, \pi(\sigma^{-2})$$

$$\propto |\sigma^2 I|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(y - X\beta)^T(\sigma^2 I)^{-1}(y - X\beta)\right] |\tau^{-1}\sigma^2 I|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\beta^T(\tau^{-1}\sigma^2 I)^{-1}\beta\right] \tau^{a-1} e^{-b\tau} (\sigma^{-2})^{c-1} e^{-d\sigma^{-2}}$$

$$\propto (\sigma^2)^{-50} \exp\left[-\tfrac{1}{2\sigma^2}(y^Ty - y^TX\beta - \beta^TX^Ty + \beta^TX^TX\beta) - \tfrac{\tau}{2\sigma^2}\beta^T\beta\right] (\tau^{-1}\sigma^2)^{-50} \tau^{a-1} e^{-b\tau} (\sigma^{-2})^{c-1} e^{-d\sigma^{-2}}$$

$$\propto (\sigma^{-2})^{c+99} e^{-d\sigma^{-2}} \tau^{a+49} e^{-b\tau} \exp\left[-\tfrac{1}{2\sigma^2}(y^Ty - 2\beta^TX^Ty + \beta^TX^TX\beta + \tau\beta^T\beta)\right]$$

$$\pi(\beta | \tau, \sigma^{-2}, X, y) \propto \exp\left[-\tfrac{1}{2\sigma^2}(-2\beta^TX^Ty + \beta^TX^TX\beta + \tau\beta^T\beta)\right]$$

$$\propto \exp\left[-\tfrac{1}{2\sigma^2}\left[\beta^T(X^TX + \tau I)\beta - 2\beta^T(X^Ty)\right]\right]$$

$$\boxed{\beta | \tau, \sigma^{-2}, X, y \sim N_{100}\left((X^TX + \tau I)^{-1}(X^Ty), (X^TX + \tau I)^{-1}\sigma^2\right)}$$

$$\pi(\tau | \beta, \sigma^{-2}, X, y) \propto \tau^{a+49} \exp\left[-\tau(b + \tfrac{1}{2}\sigma^{-2}\beta^T\beta)\right]$$

$$\boxed{\tau | \beta, \sigma^{-2}, X, y \sim Ga(a + 50, b + \tfrac{1}{2}\sigma^{-2}\beta^T\beta)}$$

$$\pi(\sigma^{-2} | \beta, \tau, X, y) \propto (\sigma^{-2})^{c+99} \exp\left[-\sigma^{-2}(d + \tfrac{1}{2}(y - X\beta)^T(y - X\beta) + \tfrac{1}{2}\tau\beta^T\beta)\right]$$

$$\boxed{\sigma^{-2} | \beta, \tau, X, y \sim Ga(c + 100, d + \tfrac{1}{2}(y - X\beta)^T(y - X\beta) + \tfrac{1}{2}\tau\beta^T\beta)}$$

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t-prior

$$y \sim N_{100}(X\beta, \sigma^2 I)$$

$$\beta \sim N_{100}(0, \tau^{-1}L^{-1})$$ where $L = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_{100} \end{bmatrix}$

$$\lambda_j \sim Ga\left(\frac{v}{2}, \frac{v}{2}\right)$$

$$\tau \sim Ga(a, b)$$

$$\sigma^{-2} \sim Ga(c, d)$$

$$\pi(\beta, \lambda, \tau, \sigma^{-2} | x, y) = L(y|x, \beta, \sigma^{-2})\, \pi(\beta|\lambda,\tau)\, \pi(\lambda)\, \pi(\tau)\, \pi(\sigma^{-2})$$

$$\propto |\sigma^2 I|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(y - X\beta)^T(\sigma^2 I)^{-1}(y - X\beta)\right] |\tau^{-1}L^{-1}|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\beta^T(\tau^{-1}L^{-1})^{-1}\beta\right]\left[\prod_{j=1}^{100} \lambda_j^{\frac{v}{2}-1} e^{-\frac{v}{2}\lambda_j}\right] \tau^{a-1}e^{-b\tau}(\sigma^{-2})^{c-1}e^{-d\sigma^{-2}}$$

$$\propto (\sigma^2)^{-50} \exp\left[-\tfrac{1}{2\sigma^2}(y^Ty - y^TX\beta - \beta^TX^Ty + \beta^TX^TX\beta) - \tfrac{\tau}{2}\beta^TL\beta\right] \tau^{50}\left[\prod_{j=1}^{100}\lambda_j^{\frac{1}{2}}\right]\left[\prod_{j=1}^{100}\lambda_j^{\frac{v}{2}-1}\right]\exp\left[-\tfrac{v}{2}\sum_{j=1}^{100}\lambda_j\right]\tau^{a-1}e^{-b\tau}(\sigma^{-2})^{c-1}e^{-d\sigma^{-2}}$$

$$\propto (\sigma^{-2})^{c+49}e^{-d\sigma^{-2}}\tau^{a+49}e^{-b\tau}\left[\prod_{j=1}^{100}\lambda_j^{\frac{v}{2}-\frac{1}{2}}\right]\exp\left[-\tfrac{v}{2}\sum_{j=1}^{100}\lambda_j\right]\exp\left[-\tfrac{1}{2\sigma^2}(y^Ty - 2\beta^TX^Ty + \beta^TX^TX\beta) - \tfrac{\tau}{2}\beta^TL\beta\right]$$

$$\pi(\beta | \lambda, \tau, \sigma^{-2}, x, y) \propto \exp\left[-\tfrac{1}{2}(-2\sigma^{-2}\beta^TX^Ty + \sigma^{-2}\beta^TX^TX\beta + \tau\beta^TL\beta)\right]$$

$$\propto \exp\left[-\tfrac{1}{2}\left[\beta^T(\sigma^{-2}X^TX + \tau L)\beta - 2\beta^T(\sigma^{-2}X^Ty)\right]\right]$$

$$\boxed{\beta | \lambda, \tau, \sigma^{-2}, x, y \sim N_{100}\left((\sigma^{-2}X^TX + \tau L)^{-1}(\sigma^{-2}X^Ty),\ (\sigma^{-2}X^TX + \tau L)^{-1}\right) \quad \text{where } L = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_{100} \end{bmatrix}}$$

$$\pi(\lambda_j | \beta, \lambda_{(-j)}, \tau, \sigma^{-2}, x, y) \propto \lambda_j^{\frac{v}{2}-\frac{1}{2}} \exp\left[-\lambda_j \tfrac{1}{2}(\tau\beta_j^2 + v)\right]$$

$$\boxed{\lambda_j | \beta, \lambda_{(-j)}, \tau, \sigma^{-2}, x, y \sim Ga\left(\frac{v+1}{2}, \tfrac{1}{2}(\tau\beta_j^2 + v)\right)}$$

$$\pi(\tau | \beta, \lambda, \sigma^{-2}, x, y) \propto \tau^{a+49}\exp\left[-\tau(b + \tfrac{1}{2}\beta^TL\beta)\right]$$

$$\boxed{\tau | \beta, \lambda, \sigma^{-2}, x, y \sim Ga(a+50, b + \tfrac{1}{2}\beta^TL\beta)}$$

$$\pi(\sigma^{-2} | \beta, \lambda, \tau, x, y) \propto (\sigma^{-2})^{c+49}\exp\left[-\sigma^{-2}(d + \tfrac{1}{2}(y - X\beta)^T(y - X\beta))\right]$$

$$\boxed{\sigma^{-2} | \beta, \lambda, \tau, x, y \sim Ga\left(c+50, d + \tfrac{1}{2}(y - X\beta)^T(y - X\beta)\right)}$$