

Lab 5

Question 1

$$(y | N, \beta) \sim \text{Binomial}(N, \beta) \rightarrow P(y | N, \beta) = \binom{N}{y} \beta^y (1-\beta)^{N-y}$$

$$\beta \sim \text{Uniform}(0, 1) \rightarrow P(\beta) = \mathbb{1}(\beta \in [0, 1])$$

$$N \sim \text{Poisson}(25) \rightarrow P(N) = \frac{25^N e^{-25}}{N!}$$

$$\begin{aligned} P(y, N, \beta) &= P(y | N, \beta) P(\beta) P(N) \\ &= \binom{N}{y} \beta^y (1-\beta)^{N-y} \cdot 1 \cdot \frac{25^N e^{-25}}{N!} \\ &= \boxed{\frac{1}{y!(N-y)!} \beta^y (1-\beta)^{N-y} 25^N e^{-25}} \end{aligned}$$

Question 2

To calculate the distribution of $\beta | N, y$, we can look at the β terms in $P(y, N, \beta)$

$$P(\beta | N, y) \propto \beta^y (1-\beta)^{N-y} \sim \boxed{\text{Beta}(y+1, N-y+1)}$$

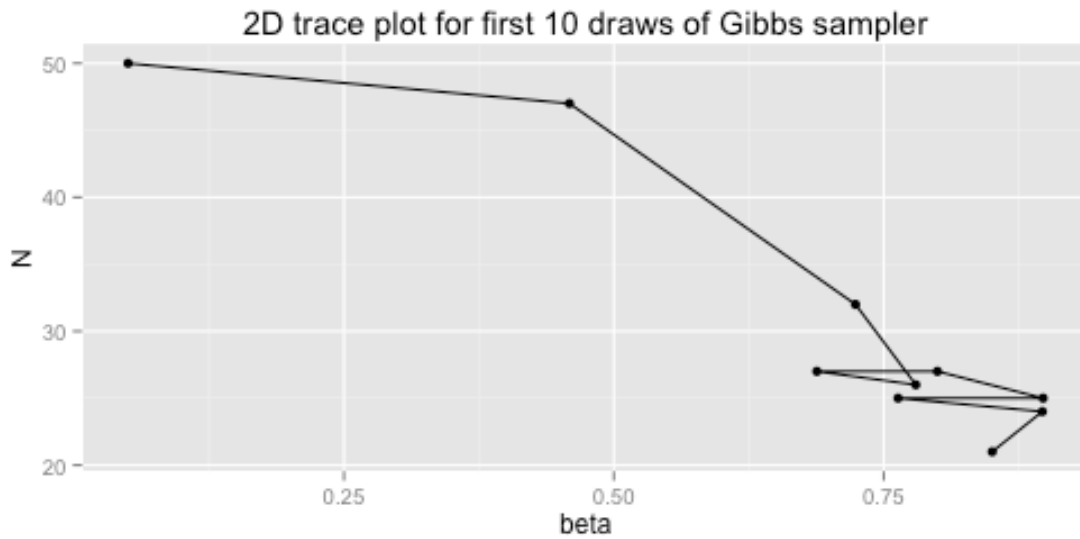
To calculate the distribution of $N | \beta, y$, we can look at the N terms in $P(y, N, \beta)$

$$\begin{aligned} P(N | \beta, y) &\propto \frac{1}{(N-y)!} (1-\beta)^{N-y} 25^N \\ &\propto \frac{1}{(N-y)!} (1-\beta)^{N-y} 25^N \cdot \frac{25^{-y}}{25^{-y}} \rightarrow \text{multiplying by 1} \\ &\propto \frac{[25(1-\beta)]^{N-y}}{(N-y)!} \rightarrow N-y \sim \text{Poisson}(25(1-\beta)), \text{ so we can just add } y \text{ to} \\ &\quad \text{get the full conditional of } N \\ &\sim \boxed{\text{Poisson}(25(1-\beta)) + y} \end{aligned}$$

Question 3

For the rest of the questions, I performed Gibbs sampling with 10,010 iterations and set $\beta^{(0)} = 0.05$. I used the value of $\beta^{(t-1)}$ to draw $N^{(t)}$ from $(N | \beta, y)$ and then used this value of $N^{(t)}$ to draw $\beta^{(t)}$ from $(\beta | N, y)$.

Question 4



The 2D trace plot of the first 10 draws of the Gibbs sampler appears to stabilize in the later draws. This stabilization appears to occur as early as the 5th sample, but this changes slightly as I rerun my code, so as a precaution, I threw out the first 10 values of β and N as burn-in's.

Question 5

90% posterior credible interval for β : **[0.55, 0.97]**

Question 6

$P(N = 20 \mid y) \approx \mathbf{0.075}$