Kelvin Niu

Homework 9

Let
$i = 1, \dots, n$ denote the types of costumes
$j = 1, \dots, n_i$ denote the people wearing costume i
y_{ij} be the number of pieces of candy person ij acquires
t_{ij} be the time in minutes spent by person ij trick-or-treating

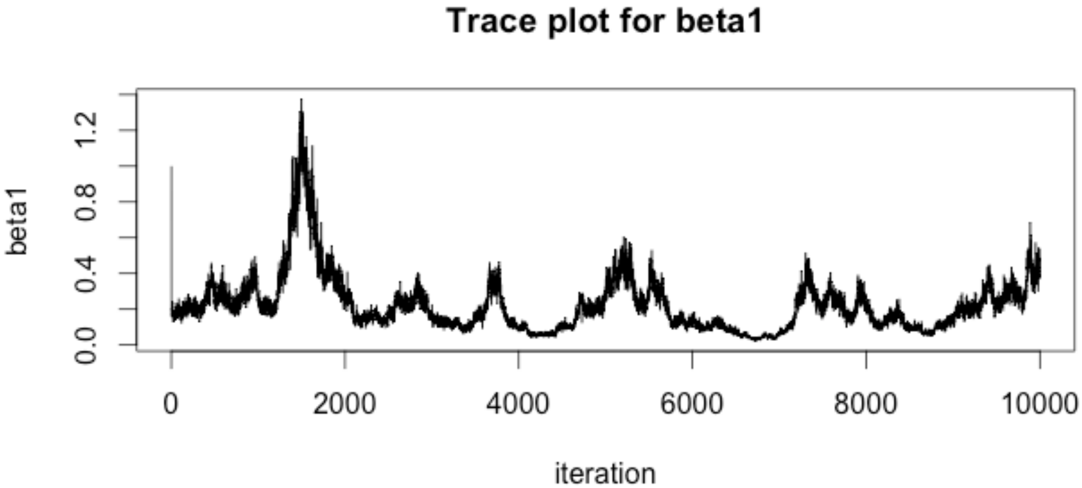
Hierarchical model
$y_{ij} \sim Poisson(t_{ij}\beta_i\lambda)$
$\beta_i \sim Gamma(\phi, \phi)$
$\lambda \sim Gamma(a, b)$
$\phi \sim Gamma(c, d)$

Simulation
To simulate data, I generated varying numbers of people (from 1 to 20) for 10 costume groups. For each person, I generated a random amount of time (from 1 to 120) they spent trick-or-treating. I then generated values for λ and ϕ and used the value of ϕ to generate 10 β_i's (from $Gamma(\phi, \phi)$). Finally I generated y_{ij}'s from $Poisson(t_{ij}\beta_i\lambda)$.

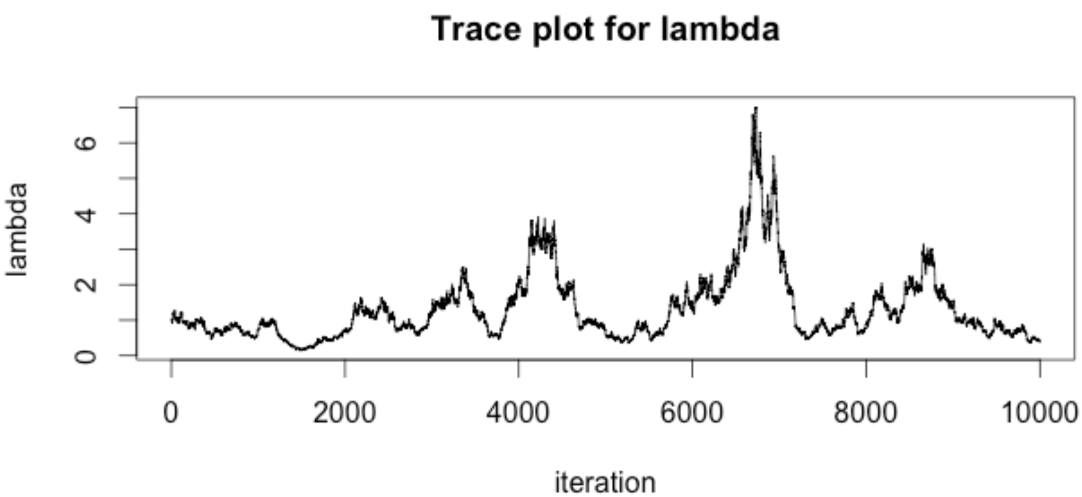
Bayes inference
[Full conditional derivations attached in the Appendix]

The full conditionals of the β_i's and λ are known distributions, so Gibbs sampling can be performed on them. However, the full conditional of ϕ is not a known distribution, so Metropolis sampling must be used on it.

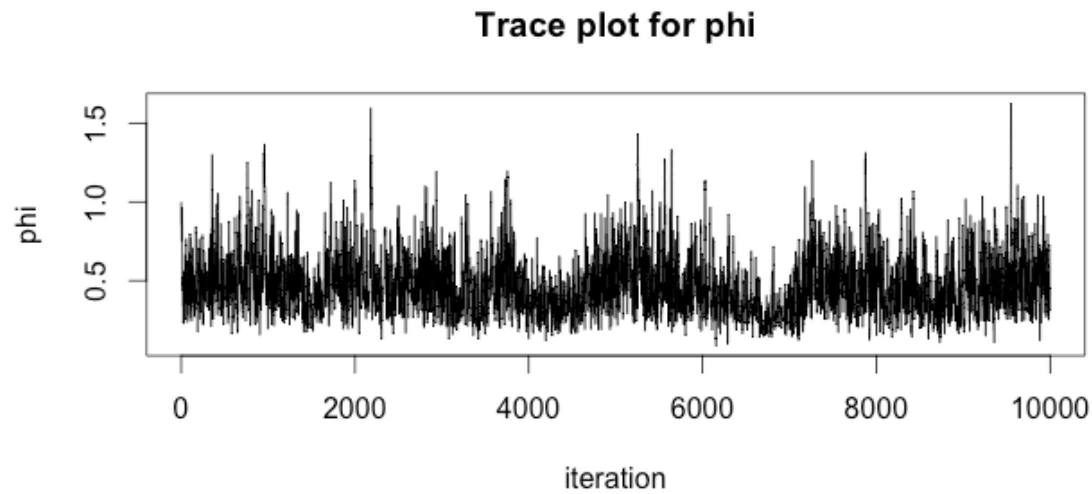
Thus, setting the prior parameters a, b, c, d as 0.001 and using an initial value of 1 for β_i's and λ and ϕ, I performed 10,000 iterations where I updated the β_i's and λ using Gibbs and ϕ using Metropolis with a normal proposal distribution. For the purpose of numerical stability, I took the logarithm of the function proportional to the posterior distribution, but I transformed back when calculating the acceptance probability. After experimenting with different variances for the proposal distribution, I settled on a variance of 0.1 that resulted in an acceptance rate of 49.40%. Below are trace plots for each of the parameters



[Trace plots for β_2, …, β_{10} omitted due to their similarity to the trace plot of β_1]
The Gibbs sampler for β_1 appears to have high autocorrelation, as it does not appear to efficiently move through the space.



The Gibbs sampler for λ also appears to have high autocorrelation, as it does not appear to efficiently move through the space.



The Metropolis sampler for ϕ appears to explore the space reasonably well.

After removing 20 burn-ins for each sampler, the below estimates of the posterior distributions were obtained.

Table: Comparison of true values and estimated marginal posterior distributions of parameters

	True value	Estimate (mean)	95% CI
β_1	0.2793	0.2223	[0.0459, 0.6926]
β_2	1.5753	1.3017	[0.2773, 4.1116]
β_3	0.0830	0.0584	[0.0118, 0.1784]
β_4	4.8815	3.7966	[0.8161, 11.8252]
β_5	0.2468	0.1829	[0.0383, 0.5597]
β_6	0.5715	0.5163	[0.1095, 1.5810]
β_7	0.1551	0.1401	[0.0257, 0.4390]
β_8	0.0289	0.0193	[0.0036, 0.0575]
β_9	0.0421	0.0345	[0.0046, 0.1105]
β_{10}	0.3537	0.3716	[0.0759, 1.1293]
λ	0.7234	1.3114	[0.2708, 4.0258]
ϕ	0.4853	0.4674	[0.1893, 0.9233]

The estimate of ϕ is rather close to the true value. Many of the β_i estimates were somewhat close to the true values with the exception of a few. However, the estimate of λ is pretty off the true value. As mentioned earlier, the trace plots suggest at a degree of autocorrelation for the β_i's and λ. Perhaps the spaces of these parameters were not explored well, and thus more iterations are needed to obtain better estimates.

$y_{ij} \sim \text{Poisson}(t_{ij}\beta_i\lambda)$

$\beta_i \sim \text{Gamma}(\phi, \phi)$

$\lambda \sim \text{Gamma}(a, b)$

$\phi \sim \text{Gamma}(c, d)$

The joint posterior is

$$\pi(\beta, \lambda, \phi \mid y, t) \propto L(y \mid \beta, \lambda, t)\,\pi(\beta \mid \phi)\,\pi(\lambda)\,\pi(\phi)$$

$$\propto \prod_{i=1}^{n}\prod_{j=1}^{n_i}\left[(t_{ij}\beta_i\lambda)^{y_{ij}} e^{-t_{ij}\beta_i\lambda}\right]\prod_{i=1}^{n}\left[\beta_i^{\phi-1}e^{-\phi\beta_i}\right]\lambda^{a-1}e^{-b\lambda}\phi^{c-1}e^{-d\phi}$$

The full conditionals are

$$\pi(\beta_i \mid \beta_{(-i)}, \lambda, \phi, y, t) \propto \beta_i^{\sum_{j=1}^{n_i} y_{ij}} \exp\left[-\beta_i\lambda\sum_{j=1}^{n_i} t_{ij}\right]\beta_i^{\phi-1}\exp[-\beta_i\phi]$$

$$\propto \beta_i^{\phi+\sum_{j=1}^{n_i} y_{ij}-1}\exp\left[-\beta_i\left(\phi+\lambda\sum_{j=1}^{n_i} t_{ij}\right)\right]$$

$$\boxed{\beta_i \mid \beta_{(-i)}, \lambda, \phi, y, t \sim \text{Gamma}\left(\phi+\sum_{j=1}^{n_i} y_{ij},\ \phi+\lambda\sum_{j=1}^{n_i} t_{ij}\right)}$$

$$\pi(\lambda \mid \beta, \phi, y, t) \propto \lambda^{\sum_{i=1}^{n}\sum_{j=1}^{n_i} y_{ij}}\exp\left[-\lambda\sum_{i=1}^{n}\sum_{j=1}^{n_i} t_{ij}\beta_i\right]\lambda^{a-1}\exp[-b\lambda]$$

$$\propto \lambda^{a+\sum_{i=1}^{n}\sum_{j=1}^{n_i} y_{ij}-1}\exp\left[-\lambda\left(b+\sum_{i=1}^{n}\sum_{j=1}^{n_i} t_{ij}\beta_i\right)\right]$$

$$\boxed{\lambda \mid \beta, \phi, y, t \sim \text{Gamma}\left(a+\sum_{i=1}^{n}\sum_{j=1}^{n_i} y_{ij},\ b+\sum_{i=1}^{n}\sum_{j=1}^{n_i} t_{ij}\beta_i\right)}$$

$$\pi(\phi \mid \beta, \lambda, y, t) \propto \prod_{i=1}^{n}\left[\beta_i^{\phi}\right]\exp\left[-\phi\sum_{i=1}^{n}\beta_i\right]\phi^{c-1}\exp[-\phi d]$$

$$\propto \prod_{i=1}^{n}\left[\beta_i^{\phi}\right]\phi^{c-1}\exp\left[-\phi\left(d+\sum_{i=1}^{n}\beta_i\right)\right]$$