Kelvin Niu

Question 1

From the class notes, if we choose priors

Under H_0: $\theta \sim \text{beta}(a,b)$

Under H_1: $\theta_1 \sim \text{beta}(c,d)$, $\theta_2 \sim \text{beta}(e,f)$

then we have

$$L(\text{data} \mid M=0) = \frac{c(a,b)}{c(\underbrace{a+2}_{\text{\# total animals with tumors}}, \underbrace{b+58}_{\text{\# total animals without tumors}})}$$

and

$$L(\text{data} \mid M=1) = \frac{c(c,d)}{c(\underbrace{c+0}_{\text{\# animals with tumors in control}}, \underbrace{d+30}_{\text{\# animals without tumors in control}})} \cdot \frac{c(e,f)}{c(\underbrace{e+2}_{\text{\# animals with tumors in exposed}}, \underbrace{f+28}_{\text{\# animals without tumors in exposed}})}$$

where $c(x,y) = \dfrac{1}{\underbrace{B(x,y)}_{\text{beta function}}}$

Thus, the Bayes factor in favor of H_0 over H_1 is

$$BF = \frac{L(\text{data} \mid M=0)}{L(\text{data} \mid M=1)} = \frac{c(a,b)}{c(a+2,b+58)} \cdot \frac{c(c,d+30)\,c(e+2,f+28)}{c(c,d)\,c(e,f)}$$

$$= \frac{B(a+2,b+58)\,B(c,d)\,B(e,f)}{B(a,b)\,B(c,d+30)\,B(e+2,f+28)} \qquad \text{since } B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{gamma function}$$

$$= \frac{\Gamma(a+2)\Gamma(b+58)}{\Gamma(a+b+60)} \cdot \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)} \cdot \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(c+d+30)}{\Gamma(c)\Gamma(d+30)} \cdot \frac{\Gamma(e+f+30)}{\Gamma(e+2)\Gamma(f+28)}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)} \cdot \frac{\Gamma(b+58)}{\Gamma(b)} \cdot \frac{\Gamma(c+d+30)}{\Gamma(c+d)} \cdot \frac{\Gamma(e+f+30)}{\Gamma(e+f)} \cdot \frac{\Gamma(a+b)}{\Gamma(a+b+60)} \cdot \frac{\Gamma(d)}{\Gamma(d+30)} \cdot \frac{\Gamma(e)}{\Gamma(e+2)} \cdot \frac{\Gamma(f)}{\Gamma(f+28)}$$

$$= \boxed{\frac{\prod_{i=1}^{2}(a+2-i)\prod_{i=1}^{58}(b+58-i)\prod_{i=1}^{30}(c+d+30-i)\prod_{i=1}^{30}(e+f+30-i)}{\prod_{i=1}^{60}(a+b+60-i)\prod_{i=1}^{30}(d+30-i)\prod_{i=1}^{2}(e+2-i)\prod_{i=1}^{28}(f+28-i)}}$$

since $\dfrac{\Gamma(x+K)}{\Gamma(x)} = \prod_{i=1}^{K}(x+K-i)$

using $\Gamma(x) = x\,\Gamma(x-1)$

From the class notes,

$$P(M=1 \mid \text{data}) = \frac{1}{1+BF}$$

$$= \boxed{\frac{1}{1 + \dfrac{\prod_{i=1}^{2}(a+2-i)\prod_{i=1}^{58}(b+58-i)\prod_{i=1}^{30}(c+d+30-i)\prod_{i=1}^{30}(e+f+30-i)}{\prod_{i=1}^{60}(a+b+60-i)\prod_{i=1}^{30}(d+30-i)\prod_{i=1}^{2}(e+2-i)\prod_{i=1}^{28}(f+28-i)}}}$$

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Homework 3

<u>Question 2</u>

I compared the following 3 choices of priors (where θ_0 is the mean under H_0 and θ_1 and θ_2 are the control and exposed means, respectively, under H_1):

- Prior 1: $\theta_0 \sim \text{beta}(1, 299)$, $\theta_1 \sim \text{beta}(1, 299)$, $\theta_2 \sim \text{beta}(0.5, 0.5)$ – This combination of priors puts historical information on the control group but not the exposed group. It also applies the same historical information to rats in general if there is no difference in tumor risk between the control and exposed group.
- Prior 2: $\theta_0 \sim \text{beta}(0.5, 0.5)$, $\theta_1 \sim \text{beta}(0.5, 0.5)$, $\theta_2 \sim \text{beta}(0.5, 0.5)$ – This combination of priors is uninformative as it does not include any historical information.
- Prior 3: $\theta_0 \sim \text{beta}(1, 2499)$, $\theta_1 \sim \text{beta}(1, 2499)$, $\theta_2 \sim \text{beta}(2, 28)$ – This combination of priors strongly favors the alternative hypothesis as well as placing the probability of an exposed rat growing a tumor much more likely than that of a control rat growing a tumor.

Using these priors, I calculated the Bayes factor in favor of H_0 over H_1 and the posterior probability of the alternative hypothesis being true for each prior for the perchlorate example results. I also calculated a p-value from Fisher's exact test.

Table: Bayes factors and posterior probabilities for different choices of priors

Priors	Bayes Factor	Pr(M = 1 | data)
$\theta_0 \sim \text{beta}(1, 299)$ $\theta_1 \sim \text{beta}(1, 299)$ $\theta_2 \sim \text{beta}(0.5, 0.5)$	0.1567	0.8645
$\theta_0 \sim \text{beta}(0.5, 0.5)$ $\theta_1 \sim \text{beta}(0.5, 0.5)$ $\theta_2 \sim \text{beta}(0.5, 0.5)$	1.6685	0.3747
$\theta_0 \sim \text{beta}(1, 2499)$ $\theta_1 \sim \text{beta}(1, 2499)$ $\theta_2 \sim \text{beta}(2, 28)$	0.0007	0.9993

Fisher's exact test p-value = 0.4915

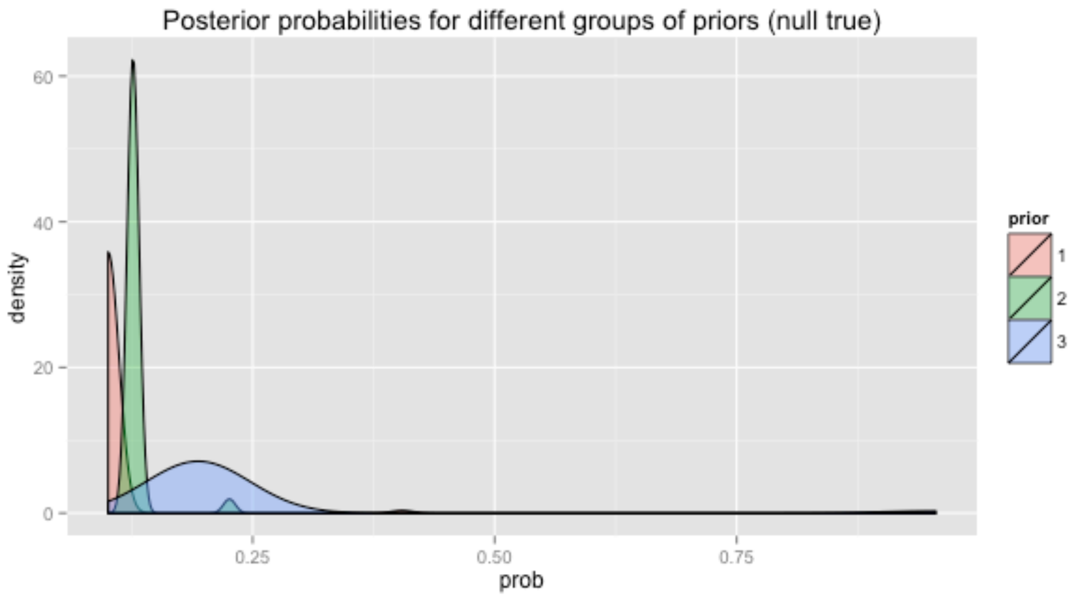
Fisher's exact test is more conservative than the Bayesian method of posterior probabilities. Because data supports the direction of priors 1 and 3, their posterior probabilities support the alternative hypothesis, with prior 3 having a greater posterior probability because it is stronger. Although the posterior probability of posterior 2 (0.3747) does not support the alternative hypothesis, it is closer to the standard cutoff of 0.5 when compared to Fisher's exact test's p-value (0.4915) and the standard cutoff of 0.05.

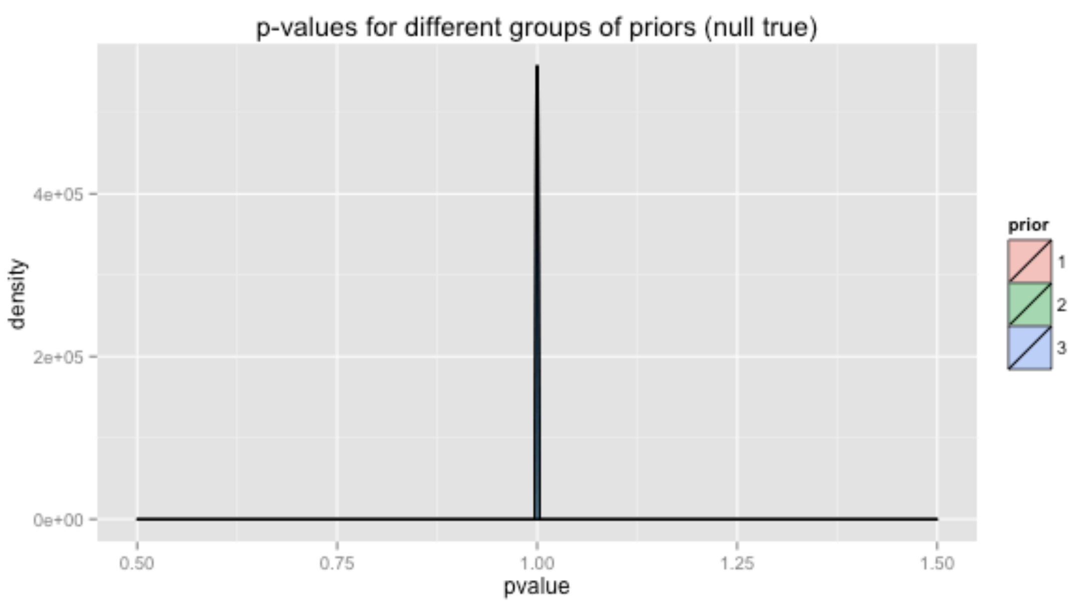
<u>Question 3</u>

For this question, I used the same priors I picked in Question 2.

To simulate this scenario, I ran 100 trials of 30 rats in each group under the null hypothesis being true and 100 trials of 30 rats in each group under the alternative hypothesis being true. When the null hypothesis was true, I set $\theta_0 = 1/2500$. When the alternative hypothesis was true, I set $\theta_1 = 1/2500$ and $\theta_2 = 2/30$. Next, I plotted the density of the posterior probabilities Pr(M = 1 | data) and p-values under Fisher's exact test for each prior.

Null hypothesis true

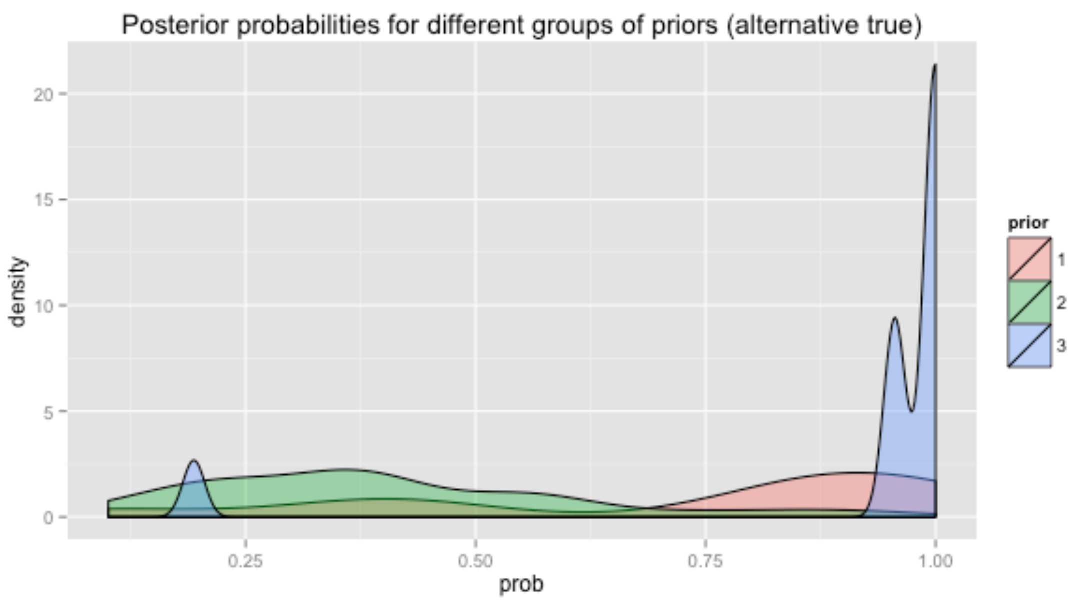


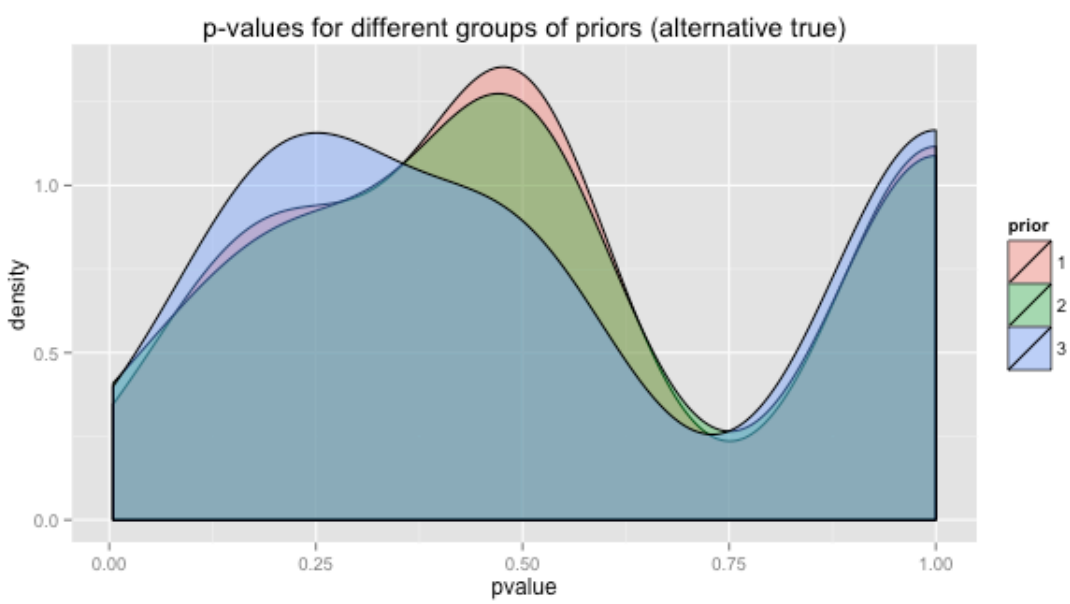


When the null hypothesis is true, all three priors generally place low posterior probabilities on the alternative being true. However, the stronger priors that favor the alternative have higher probabilities. For example, prior 3, the strongest favoring the alternative, has the highest posterior probabilities. Thus, priors can actually hurt predictions if they strongly favor the wrong hypothesis.

When Fisher's exact test, a frequentist method, is used, the p-value is 1 for all trials. This makes sense because with θ_0 set at 1/2500, it is very hard for any trial of size 30 in each group to have any rats that develop a tumor. This suggests that Fisher's exact test is not too helpful when probabilities are low and the sample size is small.

<u>Alternative hypothesis true</u>





When the alternative hypothesis is true, the informative priors 1 and 3 generally have large posterior probabilities, while the uninformative prior 2 has smaller posterior probabilities. Prior 3 has large probabilities near 1 and greater than those of prior 1 because it is stronger in the direction of the truth. Thus, if priors are chosen adeptly, they can really benefit predictions.

Fisher's exact test is more conservative with most trials having p-values above the standard significance cutoff of 0.05. Again, with small probabilities and a small sample size of 30 in each group for each trial, Fisher's exact test is not as helpful and is more conservative than the Bayesian method as it does not leverage prior information. The distribution for p-values is very similar for each prior because Fisher's exact test is a frequentist method and does not depend on any prior.

Below is a summary of the results stated above in a different format.

Table: Number of rejections of the null hypothesis (> 0.5 for Bayesians, < 0.05 for frequentist) for 100 trials

	Prior 1	Prior 2	Prior 3	Fisher's exact test
Null True: $\theta_0 = 1/2500$	0/100	0/100	4/100	0/100 0/100 0/100
Alternative True: $\theta_1 = 1/2500$ $\theta_2 = 2/30$	67/100	30/100	92/100	0/100 3/100 1/100