COL7160 : Quantum Computing

Exercise 1: Linear Algebra

Instructor: Rajendra Kumar **Scribe:** Poojan Shah

1. Read Appendix A from [dW23] and Section 2.1 from [NC10]. You should be comfortable with notions of vector spaces, linear independence, basis, eigenvalues, eigenvectors and tensor products.

2. Show that $(1,-1),(1,2)$ and $(2,1)$ are linearly dependent. Provide an orthonormal basis for the vector space spanned by these vectors. What is its dimension ?

3. (a) Consider the set of all 2×2 matrices whose entries are complex numbers, denoted by $\mathbf{M}_{2\times 2}(\mathbf{C})$. Show that $\mathbf{M}_{2\times 2}(\mathbf{C})$ is a vector field over $\mathbf{C}$. What is its dimension ? Provide a basis for it.
   (b) Consider the set $V = \{0,1\}^d$ of d bit strings. Show that it is a vector field over the field $\mathbf{F}_2 = \{0,1\}$ with addition modulo 2.
   (c) Consider the set $\mathcal{F}$ of boolean functions $f : \{0,1\}^d \to \mathbf{R}$. Show that $\mathcal{F}$ is a vector space over $\mathbf{R}$. What is its dimension ? Provide a basis for it.

4. Consider the following 3×3 complex matrices:

$$A = \begin{pmatrix} 2 & i & 0 \\ -i & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

   For each matrix:

   (a) Determine whether it is *Hermitian.*
   (b) Determine whether it is *unitary.*
   (c) Compute its eigenvalues and eigenvectors.

5. Consider the set of all 2×2 Hermitian matrices

$$\mathcal{H}_{2\times 2}(\mathbf{C}) = \{H \in \mathbf{M}_{2\times 2}(\mathbf{C}) : H = H^\dagger\}.$$

   (a) Show that $\mathcal{H}_{2\times 2}(\mathbf{C})$ is a vector space over $\mathbf{R}$. What is its dimension?
   (b) Show that the following matrices form a basis for $\mathcal{H}_{2\times 2}(\mathbf{C})$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (c) For each matrix $A \in \{\sigma_j\}_{j=0}^3$, compute its eigenvalues and corresponding eigenvectors, and hence write down its spectral decomposition

$$A = \lambda_1 |v_1\rangle \langle v_1| + \lambda_2 |v_2\rangle \langle v_2|.$$

6. (a) Prove that the eigenvalues of any Hermitian matrix are real.
   (b) Prove that two eigenvectors of a Hermitian matrix with different eigenvalues are orthogonal.
   (c) Show that the eigenvalues of a unitary matrix lie on the unit circle $\{z \in \mathbf{C} : |z| = 1\}$ and hence each eigenvalue λ of a unitary matrix U may be written as $\lambda = e^{i\theta}$ for some $\theta \in [0, 2\pi)$.
   (d) Prove that for any unitary matrix U there exists an Hermitian matrix H such that $U = \exp(iH)$. Similarly, for any Hermitian matrix H there exists a unitary matrix U such that $H = -i \log U$.
   (e) A matrix P is called a projection if $P^2 = P$. Show that all eigenvalues of a projection are either 0 or 1.

7. Define $G = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$ where $\theta \in (0, \pi/2)$.

   (a) Show that G is an orthogonal matrix and interpret its action geometrically. Compute the eigenvalues and eigenvectors of G.

   (b) Using diagonalization, derive a closed-form expression for G^k for any integer $k \geq 1$.

   (c) Let $|\psi_0\rangle = \sin\theta\,|0\rangle + \cos\theta\,|1\rangle$ and let $|\psi_k\rangle = G^k|\psi_0\rangle$ for any $k \geq 1$. Using mathematical induction show that $|\psi_k\rangle = \sin((2k+1)\theta)\,|0\rangle + \cos((2k+1)\theta)\,|1\rangle$.

8. Suppose $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$. Compute $\log A, \exp(A), \sqrt{A}, A^{-1}, A^{2026}$.

9. (a) Suppose $|\psi\rangle = \frac{\sqrt{3}}{2}\,|0\rangle + \frac{i}{2}\,|1\rangle$. Compute $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} := |\psi\rangle \otimes |\psi\rangle$.

   (b) Compute the state $|+\rangle\,|-\rangle\,|+\rangle$, where the states $|\pm\rangle$ are defined as $|\pm\rangle := \frac{1}{\sqrt{2}}\big(|0\rangle \pm |1\rangle\big)$.

   (c) Verify through direct computation that $(\sigma_0 \otimes \sigma_1)(\sigma_2 \otimes \sigma_3) = (\sigma_0\sigma_2) \otimes (\sigma_1\sigma_3)$.

   (d) Write down the 16×16 matrix represented by $\sigma_0 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_3$. What are its eigenvalues ?

References

[dW23] Ronald de Wolf. Quantum computing: Lecture notes, 2023.

[NC10] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, UK, 10th anniversary edition edition, 2010.