Natural Science Maths - Lent

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April 4, 2019

Contents

1 Differential Equations

- ODE: one independent variable
- First Order ODE:

$$\hat{F}(\frac{dy}{dx}, y, x) = 0 \qquad \frac{dy}{dx} = F(y, x)$$

- Second Order ODE:

$$\frac{d^2y}{dx^2} = F(\frac{dy}{dx}, y, x)$$

1.1 First Order Separable

Equations are in the form:

$$\frac{dy}{dx} = F(y, x) = \frac{g(x)}{h(y)}$$

We can then solve by:

$$\int h(y)\,dy = \int g(x)\,dx$$

e.g.

$$\frac{dy}{dx} - \frac{x+1}{y-1} = 0$$

1.2 First Order Linear

$$\frac{dy}{dx} + p(x)y = f(x)$$

Homogeneous - when $f(x) = 0$ Inhomogenerous - when $f(x) \neq 0$

In order to solve this form of equation, we use the integrating factor method:

$$I(x) = e^{\int p(x)\,dx}$$

$$I(x)(\frac{dy}{dx} + p(x)y) = I(x)f(x)$$

$$I(x)y = \int I(x)f(x)\,dx$$

e.g.

$$\frac{dy}{dx} = e^{-2x} - 3y$$

1.3 Substitution

Equations can also be considered homogeneous when the x, y can be replaced with $\lambda x, \lambda y$ and the equation remains unchanged.

$$\frac{dy}{dx} = H(\frac{y}{x})$$

Use the substitution $y = ux$
e.g.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Other substitutions to consider:

- $u = x \pm y + c$
- $u = f(x, y)$

1.3.1 Bernoulli

If the equation is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

Then we use the substitution $z = y^{1-n}$

1.4 Second Order Linear Constant Coefficients

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$$

When $f(x) = 0$ the equation is homogeneous. The solution is of the form:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

λ_1, λ_2 are found by solving:

$$\lambda^2 + p\lambda + q = 0$$

Case: $\lambda_1 = \lambda_2$
Then the solution is of the form

$$y = (Ax + B)e^{\lambda}$$

Case 2: λ_1, λ_2 are complex
This means that $\lambda_{1,2} = a \pm bi$ as complex roots are conjugates. Therefore the form of the solution is:

$$y = e^{ax}(A\cos(bx) + B\sin(bx))$$

All the above give the complementary integral. We also need to get particular. This can be obtained by substituting a trial equation into the differential equation and then comparing coefficients.

e.g.

$$\text{Equation: } \frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos(t)$$
$$\text{Auxillary Equation: } \lambda^2 + 1 = 0 \implies \lambda = \pm i$$
$$\text{Complementary Function: } A\cos(t) + B\sin(t)$$
$$\text{Trial Function: } a\sin(t) + b\cos(t)$$
$$\text{Substitution: } (2a\cos(t) - at\sin(t) - 2b\sin(t) - bt\cos(t)) + (at\sin(t) + bt\cos(t)) = \cos(t)$$
$$\text{After equating coefficients, we get}$$
$$y = A\sin(t) + B\cos(t) + \frac{1}{2}t\sin(t)$$

1.5 Second Order Linear Non-Constant Coefficients

If the equation is of the form:

$$y'' = F(x, y')$$

Then let $z = \frac{dy}{dx}$. Therefore:

$$z' = F(x, z)$$

e.g.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = x$$
$$z = \frac{dy}{dx}$$
$$\frac{dz}{dx} + \frac{1}{x}z = x$$
$$\text{(solve using integrating factor)}$$

2 Partial Derivatives

- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$
- gradient vector for $f(x, y)$ is $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$
- $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$
- $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
- Cartesian to Polar:

$$\left(\frac{\partial f}{\partial r}\right)_\theta = \left(\frac{\partial f}{\partial x}\right)_y (\cos(\theta)) + \left(\frac{\partial f}{\partial y}\right)_x (\sin(\theta))$$

$$\left(\frac{\partial f}{\partial \theta}\right)_r = \left(\frac{\partial f}{\partial x}\right)_y (-r\sin\theta) + \left(\frac{\partial f}{\partial y}\right)_x (r\cos\theta)$$

- Polar to Cartesian:

$$\left(\frac{\partial g}{\partial x}\right)_y = \left(\frac{\partial g}{\partial r}\right)_\theta \cos\theta + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{-\sin\theta}{r}$$

$$\left(\frac{\partial g}{\partial y}\right)_x = \left(\frac{\partial g}{\partial r}\right)_\theta \sin\theta + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{\cos\theta}{r}$$

- Reciprocity Relation: if $z = z(x,y)$, then $x = x(y,z), y = y(x,z)$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

- Cyclic Relation: if $z = z(x,y)$, then $x = x(y,z), y = y(x,z)$

$$\left(\frac{\partial x}{\partial z}\right)_y = \frac{-\left(\frac{\partial y}{\partial z}\right)_x}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1 = \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y$$

2.1 Exact Differentials

Consider a differential equation of the form:

$$P(x,y)dx + Q(x,y)dy$$

It is exact if and only if:

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

This means that there exists $f(x,y)$ such that:

$$df = P(x,y)dx + Q(x,y)dy$$

e.g.
$y\,dx - x\,dy$ is NOT an exact differential.
$y\,dx + x\,dy$ is an exact differential.

Method 1 of Solving:

$$P(x,y)dx + Q(x,y)dy = 0$$
$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

Method 2 of Solving:

$$\frac{\partial f}{\partial x} = P(x,y) \implies f = \int P(x,y)dx + h(y)$$
$$\frac{\partial f}{\partial y} = Q(x,y) \implies f = \int Q(x,y)dy + k(x)$$

We know these two equations are equivalent and therefore

$$f = \int P(x,y)dx + h(y) = \int Q(x,y)dy + k(x)$$

2.2 Non-exact Differentials

Integrating Factor Method:
We want to transform $P(x,y)dx + Q(x,y)dy$ into $\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy$ so that the transformed equation is an exact differential. This requires:

$$\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$$
$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\left(\frac{\partial \mu}{\partial y}\right) - Q\left(\frac{\partial \mu}{\partial x}\right) = 0$$

Suppose $\mu = \mu(x)$. Then $\frac{\partial \mu}{\partial y} = 0$

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\left(\frac{d\mu}{dx}\right)$$
$$\frac{1}{\mu}\left(\frac{d\mu}{dx}\right) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

Likewise, if we suppose $\mu = \mu(y)$. Then $\frac{\partial \mu}{\partial x} = 0$

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -P\left(\frac{d\mu}{dy}\right)$$
$$\frac{1}{\mu}\left(\frac{d\mu}{dy}\right) = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

Therefore, if $\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\,h(x)$ then $\frac{1}{\mu}\frac{d\mu}{dx} = h(x)$
Else, if $\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = P\,h(y)$ then $\frac{1}{\mu}\frac{d\mu}{dy} = -h(y)$
Note: It is not always possible to find an integrating factor of this form

e.g.

$$[\cot(x)\sin(x+y)+\cos(x+y)]dx+\cos(x+y)dy$$
$$\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\cot(x)\cos(x+y)=Q\cot(x)$$
$$\therefore \frac{1}{\mu}\frac{d\mu}{dx}=\cot(x)$$
$$\mu=\sin(x)$$

2.3 Stationary Points

We can determine the stationary points of multivariable functions by knowing that $\nabla f=\mathbf{0}$
If we do a Taylor Expansion of $f(x,y)$ at (x_0,y_0), and taking $(x-x_0)=\delta x$, $(y-y_0)=\delta y$

$$f(x,y)\approx f(x_0,y_0)+\delta x\frac{\partial f}{\partial x}(x_0,y_0)+\delta y\frac{\partial f}{\partial y}(x_0,y_0)+\frac{1}{2}\delta x^2\frac{\partial^2 f}{\partial x^2\partial}+\delta x\delta y\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 f}{\partial y^2\partial}$$

At the stationary point, $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ disappear and therefore we are left with:

$$f(x,y)=f(x_0,y_0)+\frac{1}{2}\delta x^2\frac{\partial^2 f}{\partial x^2\partial}+\delta x\delta y\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 f}{\partial y^2\partial}$$

This gives the following conditions:

- Saddle Point: $f_{xx}f_{yy}-f_{xy}^2<0$
- Min Point: $f_{xx}>0, f_{yy}>0, f_{xx}f_{yy}-f_{xy}^2>0$
- Max Point: $f_{xx}<0, f_{yy}<0, f_{xx}f_{yy}-f_{xy}^2>0$
- $f_{xx}f_{yy}-f_{xy}^2=0$ - cannot classify with this method

2.4 Laplace and Poisson

When an equation is of the following form it is called Poisson's equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=s$$

In the special case when $s=0$, the equation is known as Laplace's Equation

3 Scalar and Vector Fields

Scalar Field - a scalar is assigned to each point in space. e.g. temperature
Vector Field - a vector is assigned to each point in space. e.g. fluid velocity

A scalar field can be written as $\sigma(x,y,z)$ and a vector field can be written as $\vec{F}(x,y,z)$. Equivalently, if $\mathbf{x}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ then $\sigma(\mathbf{x})$ and $\mathbf{F(x)}$ are also valid notations for scalar and vector fields.

3.1 Grad

The grad of a scalar field, is a vector field. It represents the rate of change of $\sigma(\mathbf{x})$ with respect to position $\mathbf{x}$.

$$\operatorname{grad}\sigma = \nabla\sigma = \frac{\partial\sigma}{\partial x}\mathbf{i} + \frac{\partial\sigma}{\partial y}\mathbf{j} + \frac{\partial\sigma}{\partial z}\mathbf{k}$$

The change in $\sigma(\mathbf{x})$ between $x, x + \delta x$ can be written using the Taylor Series:

$$\begin{aligned}\delta\sigma &\approx \frac{\partial\sigma}{\partial x}\delta x + \frac{\partial\sigma}{\partial y}\delta y + \frac{\partial\sigma}{\partial z}\delta z \\ &= \nabla\,\sigma\cdot\delta\,x\end{aligned}$$

3.2 Normal

$\mathbf{n}$ is unit normal to the surface of constant σ. It is calculate by:

$$\mathbf{n} = \frac{\nabla\,\sigma}{|\nabla\,\sigma|}$$

e.g. If $\sigma(x, y, z) = x^2 + y^2 + z^2$ then

$$\begin{aligned}\mathbf{n} &= \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{(4x^2 + 46^2 + 4z^2)^{\frac{1}{2}}} \\ &= \frac{\mathbf{x}}{|\mathbf{x}|}\end{aligned}$$

grad can also be thought of as a vector of differentials operator:

$$\begin{aligned}\nabla\,\sigma &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\sigma \\ &= \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\sigma\end{aligned}$$

3.3 Divergence

div operates on a vector field. Let $\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

$$\begin{aligned}div(\mathbf{U}) = \nabla\,\mathbf{U} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix}u\\ v\\ w\end{pmatrix} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\end{aligned}$$

e.g. Suppose $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$

$$\operatorname{div}(\mathbf{F}) = a + b + c$$

In the case when $a = b = c = 1$, $\mathbf{F} = \mathbf{x}$. Therefore $\operatorname{div}(\mathbf{F}) = 3$

3.4 Curl

It is defined as the cross product of ∇ and a vector field. Therefore it returns a vector.

$$\mathrm{curl}\,\mathbf{U} = \nabla \times \mathbf{U}$$

If $\mathbf{F} = \nabla\omega$ then

$$\mathrm{curl}\,(\mathbf{F}) = 0$$
$$\mathrm{div}\,(\mathrm{curl}\,(\mathbf{F})) = 0$$

4 Line Integrals

4.1 Scalar Field

Lawnmower example - line integral is the integral of the grass length over the distance of the path being mowed.

In a general form:

$$\Gamma = \text{path - parameterised e.g. } \mathbf{x}(t)$$
$$\text{Scalar Field - } f(x,y)$$
$$\text{Line integral - } \int_\Gamma f(\mathbf{x}(t))\,dt$$

e.g.

$$\Gamma = \mathbf{x}(t) = t\mathbf{i} + (t-1)^2\mathbf{j}\,\text{for }\; 0 \le t \le 2$$
$$f(x,y) = x + y$$
$$\text{Line integral - } \int_0^2 f(t,(t-1)^2)\,dt = \frac{8}{3}$$

4.2 Vector Field

Note: $d\mathbf{x} = \frac{d\mathbf{x}}{ds}ds = \frac{d\mathbf{x}}{dt}dt$
Need to calculate:

$$\int f(\mathbf{x}(s))\,ds = \int \mathbf{F}(\mathbf{x}(s)) \cdot \frac{d\mathbf{x}}{ds}ds$$
$$\int \mathbf{F}(\mathbf{x}(s)) \cdot \frac{d\mathbf{x}}{dt}dt = \int_\Gamma \mathbf{F} \cdot d\mathbf{x}$$

e.g. Evaluate $I = \int_\Gamma \mathbf{F} \cdot d\mathbf{x}$ for the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$ where the path Γ is the semicricle $x^2 + y^2 = 1$, for positive y

$$\text{Parameterise the semicircle: } \mathbf{x}(\theta) = -\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \quad \text{for } \; 0 \le \theta \le \pi$$
$$\frac{d\mathbf{x}}{\theta} = \sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$$
$$F = r^2 sin^2(\theta)\mathbf{i} + r^2 cos^2(\theta)\mathbf{j} = sin^2(\theta)\mathbf{i} + cos^2(\theta)\mathbf{j} \quad (r = 1)$$
$$I = \int_\Gamma \mathbf{F} \cdot d\mathbf{x} = \int_0^\pi sin^3(\theta) + cos^3(\theta) = \frac{4}{3}$$

4.3 Conservative Vector Fields

- Vector field that is the gradient of some function - $\mathbf{F} = \nabla\omega$
- Line integrals are path independent

  $\oint \mathbf{F} - \Gamma \cdot d\mathbf{x} = 0$
- Test if conservative:

  We know $\nabla \times \nabla\omega = 0$

  Therefore $\nabla \times \mathbf{F} = 0$

  If for all closed curves $\oint \mathbf{F} - \Gamma \cdot d\mathbf{x} = 0$ is true, then $\mathbf{F}$ is conservative

4.4 Surface Integrals

Vector Area: $\mathbf{dS} = \mathbf{n}dS$ - represents small region on the surface, tends to 0
To get the area we integrate: $\int_S \mathbf{dS} = \int_S \mathbf{n}dS = \mathbf{n}\int_S dS = A\mathbf{n}$

For any closed surface this equals to 0, as the opposite sides cancel.

e.g. Vector area of a hemisphere surface S given by $x^2 + y^2 + z^2 = a^2 \qquad z > 0$

In spherical polar coordinates, we can write this as $r = a, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad -\pi \leq \theta \leq \pi$
Therefore we can parameterise the surface as:

$$\mathbf{x} = \mathbf{x}(\theta, \phi) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\sin(\theta)\cos(\phi)\mathbf{i} + a\sin(\theta)\sin(\phi)\mathbf{j} + a\cos(\theta)\mathbf{k}$$

For small variations in θ, ϕ we have

$$dS = a^2 \sin(\theta) d\theta d\phi$$

The normal vector can be calculated as:

$$\mathbf{n} = \frac{\mathbf{x}}{|x|} = \sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k}$$

We then need to calculate:

$$\begin{aligned}\int_S \mathbf{dS} &= \int_S \mathbf{n}dS \\ &= \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi} (\sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k})\, a^2 \sin(\theta) d\theta d\phi \\ &= \pi a^2 \mathbf{k}\end{aligned}$$

4.5 Flux

The flux of a vector field across a surface element with vector area $\mathbf{dS}$ is defined as:

$$\mathbf{F} \cdot \mathbf{dS}$$

Therefore the total flux over a surface is defined as:

$$\int_S \mathbf{F} \cdot \mathbf{dS} \equiv \int_S \mathbf{F} \cdot \mathbf{n} \, dS$$

For the earlier example, for the hemisphere, we can calculate the flux as:

$$\begin{aligned}
\mathbf{F} &= \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \\
\mathbf{F} \cdot \mathbf{dS} &= a^2(\alpha\sin(\theta)\cos(\phi) + \beta\sin(\theta)\sin(\phi) + \gamma\cos(\theta))\sin(\theta)d\theta d\phi \\
\int_S \mathbf{F} \cdot \mathbf{dS} &= \int_{\theta=0}^{\frac{\pi}{2}} a^2 \sin\theta \left[\int_{\phi=-\pi}^{\pi} (\alpha\sin(\theta)\cos(\phi) + \beta\sin(\theta)\sin(\phi) + \gamma\cos(\theta))\sin(\theta)d\phi\right] d\theta \\
&= \gamma\pi a^2
\end{aligned}$$

5 Fourier Series

5.1 Orthogonal Functions

- A set of functions $\{\phi_1(x), \phi_2(x), \dots\}$ is an orthogonal set on the interval $[a, b]$ if any two functions in the set are orthogonal to each other:

$$(\phi_n, \phi_m) = \int_a^b \phi_n(x)\phi_m(x)dx = 0 \quad (n \neq m) \tag{1}$$

5.2 Fourier Series

- Mutually Orthogonal Sets on $-L \leq x \leq L$

$$\left\{cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty} \tag{2}$$

and

$$\left\{sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty} \tag{3}$$