Natural Science Maths - Michaelmas

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Contents

1 Vectors

1.1 Scalar Product

- $\underline{a} \cdot \underline{b} \equiv |\underline{a}|\, |\underline{b}| \cos(\theta)$
- The scalar product is commutative
- Note: $\cos(2\pi - \theta) = \cos(\theta)$
- $\underline{a} \cdot \underline{a} \equiv |\underline{a}|^2$
- Resolving vector $\underline{a}$, with respect to unit vector $\underline{\hat{a}}$:
  - Parallel: $(\underline{a} \cdot \underline{\hat{n}})\underline{\hat{n}}$
  - Perpendicular: $\underline{a} - (\underline{a} \cdot \underline{\hat{n}})\underline{\hat{n}}$
- Properties:
  - Commutative
  - Multiplication by scalar can be done at any point
  - Distributive over addition - e.g. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ - can be proved using diagram

1.2 Vector Product

- $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\sin(\theta)\underline{\hat{n}} = -\underline{b} \times \underline{a}$ **anti-commutative**
- If the vector product equals 0 then either the vectors are parallel or at least one is 0
- $\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$ as $\underline{a} \times \underline{b}$ is perpendicular to $\underline{a}$
- $|\underline{a} \times \underline{b}|$ is the area of a parallelogram formed by $\underline{a}$ and $\underline{b}$
- Properties:
  - anti-commutative
  - Multiplication by scalar can be done at any point
  - Distributive over addition - can be proved using diagram or using the distributive dot product property
  - **Non-associative** - intuitively, $\underline{a} \times (\underline{b} \times \underline{c})$ means that $\underline{a}$ is $\perp$ to the vector that is $\perp$ to $\underline{b}$ and $\underline{c}$, and therefore that vector is back in the $\underline{b}$ and $\underline{c}$ plane. Whereas for $(\underline{a} \times \underline{b}) \times \underline{c}$ this time $\underline{c}$ is $\perp$ to the vector $\perp$ to $\underline{a}$ and $\underline{b}$ therefore the vector is back in the $\underline{a}$ and $\underline{b}$ plane.

1.3 Vector Area

- The vector area of a finite plane surface has a magnitude equal to the area of the plane, and direction equal to the normal of the plane, denoted by $\underline{S}$
- Projecting the area onto the $x - y$ plane: $S\cos(\theta) = \underline{S} \cdot \underline{\hat{z}}$
- For non-planar surfaces:
  - Area projections of the surface only depend on the rim of the surface
  - Total vector surface area is defined as the sum of individual surface areas

- All surfaces spanning a give rim have the same vector area - e.g. the size of the dome on a circle doesnt affect the vector surface area
- Closed surface has $S = 0$ as there only exists a vector area if there is a rim, and a closed surface has no rim and therefore no vector area (all the vector directions cancel each other)

- The vector area can be used to prove the distributive vector product law - shown earlier

1.4 Triple Products

1.4.1 Scalar Triple Product

- $[\underline{a}, \underline{b}, \underline{c}] \equiv \underline{a} \cdot (\underline{b} \times \underline{c})$
- $[\underline{a}, \underline{b}, \underline{c}] = [\underline{b}, \underline{c}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}]$
- However, $[\underline{a}, \underline{b}, \underline{c}] = -[\underline{b}, \underline{a}, \underline{c}]$
- $[\underline{a}, \underline{b}, \underline{c}]$ defines the area of the parallelepiped formed by $\underline{a}, \underline{b}, \underline{c}$
- If any two vectors are the same, or **coplanar** then the scalar product equals 0 - visually there would be no parallelipied and therefore there would be no volume

1.4.2 Vector Triple Product

- $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$
- Remember: BAC-CAB
- Not associative - $\underline{a} \times (\underline{b} \times \underline{c}) = -\underline{c} \times (\underline{a} \times \underline{b})$

1.5 Lines

- $\underline{r} = \underline{a} + \lambda\hat{\underline{l}}$
- Therefore $(\underline{r} - \underline{a}) \parallel \hat{\underline{l}}$
- $\underline{r} \times \hat{\underline{l}} = (\underline{a} + \lambda\hat{\underline{l}}) \times \hat{\underline{l}}$
  $\underline{r} \times \hat{\underline{l}} = (\underline{a} \times \hat{\underline{l}}) + (\lambda\hat{\underline{l}} \times \hat{\underline{l}})$
  $\underline{r} \times \hat{\underline{l}} = \underline{a} \times \hat{\underline{l}}$
- Equation of line connecting two points, $\underline{\mathrm{a}}$ amd $\underline{\mathrm{b}}$: $\underline{r} = \underline{a} + \lambda\frac{(\underline{b}-\underline{a})}{|\underline{b}-\underline{a}|}$ - where λ gives the distance between the two points, as diving by the modulus means that the direction vector has magnitude 1

1.6 Planes

- $\underline{r} = \lambda\underline{p} + \mu\underline{q}$
- For plane containing point $\underline{\mathrm{a}}$ $\underline{r} = \underline{a} + \lambda\underline{p} + \mu\underline{q}$
- Given three points that lie in a plane, $\underline{\mathrm{a}}$, $\underline{\mathrm{b}}$, $\underline{\mathrm{c}}$: $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$

- Alternate form:
$\underline{r} \cdot \underline{\hat{n}} = \underline{a} \cdot \underline{\hat{n}} + \lambda \underline{p} \cdot \underline{\hat{n}} + \mu \underline{q} \cdot \underline{\hat{n}}$
$\underline{r} \cdot \underline{\hat{n}} = \underline{a} \cdot \underline{\hat{n}} = \underline{d}$
(as $\underline{\hat{n}}$ is perpendicular to both $\underline{\mathrm{p}}$ and $\underline{\mathrm{q}}$ and therefore dotting gives 0)
  - $\underline{r} \cdot \underline{\hat{n}} = \underline{d}$
    $|\underline{r}|\,|\underline{\hat{n}}| \cos(\theta) = \underline{d}$
    As $|\underline{\hat{n}}| = 1$
    $|\underline{r}| \cos(\theta) = \underline{d}$
    This means that $|\underline{d}|$ is the distance of the plane from the origin as $|\underline{r}| \cos(\theta)$ is the projection of $\underline{\mathrm{r}}$ in the normal direction.

1.7 Orthogonal Bases

- Three non-planar vectors can form a basis
- They can be used to specify every point in space
- $\underline{r} = \lambda \underline{a} + \mu \underline{b} + \gamma \underline{c}$ - for unique (λ, μ, γ)
- To get these component we have to use reciprocal basis

  $\underline{A} \equiv \frac{\underline{b} \times \underline{c}}{[\underline{a}, \underline{b}, \underline{c}]}$

  $\underline{B} \equiv \frac{\underline{c} \times \underline{a}}{[\underline{a}, \underline{b}, \underline{c}]}$

  $\underline{C} \equiv \frac{\underline{a} \times \underline{b}}{[\underline{a}, \underline{b}, \underline{c}]}$

- The reciprocal basis follow the rules:

  $\underline{a} \cdot \underline{A} = \underline{a} \cdot \frac{\underline{b} \times \underline{c}}{[\underline{a}, \underline{b}, \underline{c}]} = \frac{\underline{a} \cdot (\underline{b} \times \underline{c})}{\underline{a} \cdot (\underline{b} \times \underline{c})} = 1$

  $\underline{a} \cdot \underline{B} = \underline{a} \cdot \frac{\underline{c} \times \underline{a}}{[\underline{a}, \underline{b}, \underline{c}]} = \frac{\underline{a} \cdot (\underline{c} \times \underline{a})}{\underline{a} \cdot (\underline{b} \times \underline{c})} = 0$

  $\underline{a} \cdot \underline{C} = \underline{a} \cdot \frac{\underline{a} \times \underline{b}}{[\underline{a}, \underline{b}, \underline{c}]} = \frac{\underline{a} \cdot (\underline{a} \times \underline{b})}{\underline{a} \cdot (\underline{b} \times \underline{c})} = 0$

- Therefore to get each component dot $\underline{\mathrm{r}}$ with appropraite reciprocal basis:

  $\underline{A} \cdot \underline{r} = \underline{A} \cdot \lambda \underline{a} + \underline{A} \cdot \mu \underline{b} + \underline{A} \cdot \gamma \underline{c} = \lambda \underline{A} \cdot \underline{a} = \lambda$

  $\underline{B} \cdot \underline{r} = \underline{B} \cdot \lambda \underline{a} + \underline{B} \cdot \mu \underline{b} + \underline{B} \cdot \gamma \underline{c} = \mu \underline{B} \cdot \underline{b} = \mu$

  $\underline{C} \cdot \underline{r} = \underline{C} \cdot \lambda \underline{a} + \underline{C} \cdot \mu \underline{b} + \underline{C} \cdot \gamma \underline{c} = \gamma \underline{C} \cdot \underline{c} = \gamma$

- **Orthonormal Basis** - when $|\underline{a}| = |\underline{b}| = |\underline{c}| = 1$

  Reciprocal basis is the same as normal basis

  E.g. Basis vectors $\underline{i}, \underline{j}, \underline{k}$ along the Cartesian axes

  Where:

  $\underline{i} \cdot \underline{i} = 1$, $\underline{j} \cdot \underline{j} = 1$ $\underline{k} \cdot \underline{k} = 1$ and other combinations $= 0$

  $[\underline{i}, \underline{j}, \underline{k}] = 1$

  $\underline{i} \times \underline{j} = \underline{k} etc.$

  $\underline{j} \times \underline{i} = -\underline{k} etc.$

1.7.1 Direction Cosines

A direction cosine is the cosine of the angle between a vector and the axes.

- For point $\underline{a}$, we can write this as $\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$, where $\underline{\text{i}}$, $\underline{\text{j}}$, $\underline{\text{k}}$ form a orthonormal basis.
- $a_x = \underline{a} \cdot \underline{i} = |\underline{a}|\,|\underline{i}|\cos(\theta_x) = |\underline{a}|\cos(\theta_x)$
- $a_y = \underline{a} \cdot \underline{j} = |\underline{a}|\,\left|\underline{j}\right|\cos(\theta_y) = |\underline{a}|\cos(\theta_y)$
- $a_z 6 = \underline{a} \cdot \underline{k} = |\underline{a}|\,|\underline{k}|\cos(\theta_z) = |\underline{a}|\cos(\theta_z)$
- $\cos(\alpha) = \frac{a_x}{|\underline{a}|}$
- $\cos(\beta) = \frac{a_y}{|\underline{a}|}$
- $\cos(\gamma) = \frac{a_z}{|\underline{a}|}$
- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

1.8 Using Components

- $\underline{a} \times \underline{b} = a_xb_x + a_yb_y + a_zb_z$
- $\underline{a}^2 = a_x^2 + a_y^2 + a_z^2$
- $\underline{a} \times \underline{b} = (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) \times (b_x\underline{i} + b_y\underline{j} + b_z\underline{k})$
  ...
  $= (a_yb_z - a_zb_y)\underline{i} + (a_zb_x - a_xb_z)\underline{j} + (a_xb_y - a_yb_x)\underline{k}$

1.9 Determinants

- The cross product can also be represented by a determinant:
  $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- The scalar triple product can be written as:
  $\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$
- The vector triple product can be written as:
  $\underline{a}\times(\underline{b}\times\underline{c}) = \begin{vmatrix} \underline{i} & \underline{j}\underline{k} & \\ a_x & a_y & a_z \\ (b_yc_z - b_zc_y) & (b_zc_x - b_xc_z) & (b_xc_y - b_yc_x) \end{vmatrix} = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

1.10 Coordinates

1.10.1 Plane Polar Coordinates

Point specified by (r, θ)

- Polar to Cartesian:

  $x = r\cos\theta$

$y = r \sin \theta$

- Cartesian to Polar :

  $r = \sqrt{x^2 + y^2}$

  $\theta = \tan^{-1} (\frac{y}{x})$

- Examples:

  $r = a$ is a sphere

  Shortest distance from origin to a line angle α to the y-axis is $d = r \cos (\theta - \alpha)$

- **Area Element** $= r dr d\theta$

1.10.2 Cyclindrical Polar Coordinates

Point is written as (r, ϕ, z), where r and ϕ are in the $x - y$ plane, like the polar coordinates, but the z component gives height

- Cylindrical to Cartesian:

  $x = r \cos \phi$

  $y = r \sin \phi$

  $z = z$

- Cartesian to Cylindrical:

  $r = \sqrt{x^2 + y^2}$

  $\theta = \tan^{-1} (\frac{y}{x})$

  $z = z$

- **Volume Element** $= r dr d\phi dz$

- Examples:

  $z = a$ is a plane perpendicular to the z axis and distance a from the origin

  $r = a$ is a cylinder radius a, axis through origin and along the z axis

  $r = a$ and $z = \phi$ specifies a helix about z axis and pitch angle $\tan^{-1}(\frac{1}{a})$

1.10.3 Spherical Polar Coordinates

Point is written as (r, θ, ϕ), where r is the distance from the origin, $0 <= \theta <= \pi$ specifies angle between z axis and point, $0 <= \phi <= 2\pi$ specifies angle on the $x - y$ plane

- Spherical to Cartesian:

  $x = r \sin \theta \cos \phi$

  $y = r \sin \theta \sin \phi$

  $z = r \cos \theta$

- Cartesian to Spherical:

  $r = \sqrt{x^2 + y^2 + z^2}$

  $\theta = \cos^{-1}\left(\frac{z}{r}\right)$

  $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

- An orthogonal coordinate system
- Examples:

  $\theta = a$ is a half cone with apex at the origin, opening angle a

  $\phi = a$ is a vertical half-plane, angle a to the x axis

  $r = a$ sphere, centre origin, radius a

  $r = \theta = a$, horizontalal circle on the z axis, height $a\cos(\theta)$, radius $a\sin(\theta)$

- **Volume Element** $= r^2 \sin\theta dr d\theta d\phi$

1.11 Other

- Distance from point $\underline{b}$ to line, direction $\hat{\underline{l}}$ passing through point $\underline{a}$:
  $d = \left|\hat{\underline{l}} \times (\underline{b} - \underline{a})\right|$
- Distance from point to plane:
  $d = \frac{|\underline{r} \cdot \underline{n}|}{|\underline{n}|}$
- Distance between two skew lines where $l_1 = a + \lambda s$, $l_2 = b + \gamma t$, $c = a - b$
  $d = \frac{\underline{c} \cdot (\underline{\lambda} \times \underline{\gamma})}{|\underline{\lambda} \times \underline{\gamma}|}$
- Given $|\underline{a} \cdot \underline{b} \leq |a|\,|b||$, we can deduce $|\underline{a} + \underline{b}| \leq |a| + |b|$

  $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = |\underline{a} + \underline{b}|^2$
  $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = \underline{a} \cdot \underline{a} + 2(\underline{a} \cdot \underline{b}) + \underline{b} \cdot \underline{b}$
  $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = |\underline{a}|^2 + 2(\underline{a} \cdot \underline{b}) + |\underline{b}|^2$
  $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) \leq |\underline{a}|^2 + 2\,|a|\,|b| + |\underline{b}|^2$
  $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) \leq (|a| + |b|)^2$

  $|\underline{a} + \underline{b}|^2 \leq (|a| + |b|)^2$
  $|\underline{a} + \underline{b}| \leq (|a| + |b|)$

- Surfaces:

  $|r| = k$ - sphere

  $\underline{r} \times \underline{u} = m\,|r| - cone$

  $|\underline{r} - (\underline{r} \cdot \underline{u})\underline{u}| = n$ - cylinder

- Centroid of triangle $= \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$

2 Complex Numbers

- Number represented by $z = a + bi$, where $i^2 = -1$
- Complex conjugate: $z^* \equiv a - bi$
- $zz^* = (a+bi)(a-bi) = a^2 + b^2 + i(-ab+ab) = a^2 + b^2$
- $|z| = \sqrt{zz^*} = \sqrt{a^2+b^2}$

2.1 Polar Form

- $z = x + iy = r(\cos\theta + i\sin\theta)$
- $z_1 z_2 = |z_1|\,|z_2|\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$
- Multiplying by a complex number, z with $|z| = 1$ is the same as rotation anticlockwise by $arg(z)$

2.2 De Moivre's theorem

- $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- $\cos n\theta = \Re(\cos\theta + i\sin\theta)^n$

  e.g. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

  e.g. $\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$
- $z + z^{-1} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$
  $z - z^{-1} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2\sin\theta$
  Therefore it follows:
  - $\cos^n\theta = \frac{1}{2^n}(z+z^{-1})^n$
  - $\sin^n\theta = \frac{1}{2^n}(z-z^{-1})^n$
  - e.g. $\cos^3\theta = \frac{1}{8}(2\cos 3\theta + 6\cos\theta)$

2.3 Exponential Form

By using the power series, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ by setting $x = i\theta$, we get:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + ... \tag{1}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right) \tag{2}$$

$$= \cos\theta + i\sin\theta \tag{3}$$

- **Euler's formula:** $e^{i\theta} = \cos\theta + i\sin\theta$
- $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$
- $(e^{i\theta})^n = e^{in\theta}$

2.4 Roots of Unity

nth roots of unity solve $z^n = 1$

- $|z| = 1$ therefore $z = e^{i\theta}$
- $e^{in\theta} = \cos n\theta + i \sin n\theta = 1 \implies \theta = \frac{2\pi k}{n}, \qquad k = 0, 1, 2, ...n-1$
- There are n, nth roots of unity
- Roots of unity come in complex-conjugate pairs
- $\omega = e^{\frac{2\pi i}{n}}$, then roots are $1, \omega, \omega^2, ...\omega^{n-1}$
- cube roots of unity are $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$

2.5 Complex Powers and Logarithms

- The natural logarithm $\omega = \ln z$ are the solutions of $e^\omega = z$
- $\ln z$ is **multi-valued**
- $\ln z = \ln re^{i\theta} = \ln r + i(\theta + 2n\pi) \qquad n = 0, \pm 1, \pm 2, ...$
- Principle value is when $0 \leq \theta \leq 2\pi$
- $z_1^{z_2} \equiv (e^{\ln z_1})^{z_2} \equiv e^{z_2 \ln z_1}$

2.6 Fundamental Theorem of Algebra

$a_n z^n + a_n - 1z^{n-1} + ... + a_1 z + a_0 = 0 \qquad a_n \neq 0$ has n roots
Alternatively, can say that the equation must have at least one root:
If z_1 is a root, then the equation can be written as $(z - z_1)Q(z) = 0$, where $Q(z)$ is of order $n - 1$. $Q(z)$ must have at least one root etc.

2.7 Hyperbolics and Trig

- Define:
  $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
- When z is real $z = x$:
  $\frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$ and $\frac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$
- When z is imaginary $z = iy$:
  NOT SURE ABOUT SINH
  $$\cos iy = \frac{1}{2}(e^{i(iy)} + e^{-i(iy)}) = \frac{1}{2}(e^{-y} + e^{y}) = \cosh y$$
  $$\sin iy = \frac{1}{2i}(e^{i(iy)} - e^{-i(iy)}) = \frac{1}{2}(e^{-y} - e^{y}) = i \sinh y$$
- Identities:
  $$\cosh^2 y - \sinh^2 y = 1$$
  $$\cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b$$
  $$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

- Inverse Hyperbolics:

  $\cosh^{-1} x = \pm \ln(x + \sqrt{x^2 - 1})$

  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

  $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

2.8 Other

- Sets of points:
  -
    $$\begin{aligned} |z-2| &= |z^* + i| \\ ((x-2)+iy)((x-2)-iy) &= (x+i(y-1))(x-i(y-1)) \\ (x-2)^2 + y^2 &= x^2 + (y-1)^2 \\ -4x + 4 &= -2y + 1 \\ y &= 2x - \frac{3}{2} \end{aligned}$$

    Perpendicular bisector of point $z = i$ and $z = 2$
  - $arg(z^*) = \frac{\pi}{4}$ is the line segment $y = -x$ for $x \geq 0$
  - $arg(z) = |z|$, if we convert it into the polar coordinate system it is the same as $\theta = r$ which is a spiral
- General form for an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

  if $a > b$ then horizontal major axis

  if $a < b$ then vertical major axis
- $(x, y) = (a\cos\theta, b\sin\theta)$

  $$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \cos^2\theta &= \frac{x^2}{a^2} \\ \sin^2\theta &= \frac{y^2}{b^2} \\ \therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$

- General form for a hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- $(x, y) = (a\cosh\theta, b\sinh\theta) \implies \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- Logarithmic form of $\tanh^{-1} x$

$$\begin{aligned}
\tanh^{-1} x &= y \\
\tanh y &= x \\
x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
&= \frac{e^y - e^{-y}}{e^{-y}(e^{2y} + 1)} \\
&= \frac{e^{2y} - 1}{e^{2y} + 1} \\
xe^{2y} + x &= e^{2y} - 1 \\
(x-1)e^{2y} &= -1 - x \\
e^{2y} &= \frac{-(1+x)}{-(1-x)} \\
y &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)
\end{aligned}$$

- $\sin(a + bi) = \sin a \cosh b + i \cos a \sinh b$
- $\cos(a + bi) = \cos a \cosh b - i \sin a \sinh b$
- $\sinh(a + bi) = \sinh a \cos b + i \cosh a \sin b$
- $\cosh(a + bi) = \cosh a \cos b + i \sinh a \sin b$

3 Differentiation

- Defined as: $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}$
- For a function $f(x)$ to be **differentiable** at x:
  - function must be continuous
  - derivative exists $\implies$ left-handed and right-handed limits must be the same
- Chain rule: $\frac{d}{dx}(y(u(x))) = \frac{dy}{du}\frac{du}{dx}$
- Product rule: $\frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f$
- Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- Reciprocal rule: $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

3.1 Hyperbolics

- $\frac{d}{dx} \cosh x = \sinh x$
- $\frac{d}{dx} \sinh x = \cosh x$
- $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x}$

3.2 Implicit Differentiation

To differentiate with respect to x an equation of the form $g(y) = f(x)$

$$\frac{dg(y)}{dx} = \frac{dg(y)}{dy}\frac{dy}{dx}$$
$$\therefore \frac{dg(y)}{dy}\frac{dy}{dx} = \frac{df}{dx}$$

3.3 Leibnitz's Formula

This formula gives the nth derivative of $y(x) = f(x)g(x)$

$$\frac{d^n(fg)}{dx^n} = \sum_{m=0}^{n}\binom{n}{m} f^{(n-m)}g^{(m)} = f^{(n)}g^{(0)} + nf^{(n-1)}g^{(1)} + \frac{n(n-1)}{2!}f^{(n-2)}g^{(2)} + \cdots + f^{(0)}g^{(n)}$$

Note:

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$$

Proof done using induction

4 Curve Sketching

5 Elementary Analysis

5.1 Limits

$\lim_{x \to x_0} f(x) = K$ means:

$$\text{for any } \epsilon > 0,\ \exists \delta > 0 \text{ such that } |f(x) - K| < \epsilon \,\forall\, 0 < |x - x_0| < \delta$$

Define limit at infinity $\lim_{x \to \infty} f(x) = K$ as:

$$\text{for any } \epsilon > 0,\ \exists X > 0 \text{ such that } |f(x) - K| < \epsilon \,\forall\, x > X$$

5.2 L'Hopital's Rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

- Used for when limits of a quotient are indeterminable e.g. when both 0 or ∞
- e.g. $\lim_{x \to 0} \frac{\sin x}{x}$:

$$\lim_{x \to x_0} \frac{\sin x}{x} = \lim_{x \to x_0} \frac{\cos x}{1} = 1$$

5.3 Order of Magnitude

- $\mathcal{O}$ notation expresses how a function behaves as it approaches a point (or ∞)
- Given two real functions $f(x)$ and $g(x)$, then $f(x) = \mathcal{O}(g(x))$ as $x \to a$ iff:

$$\exists \text{ constants } \epsilon \text{ and } K > 0 \text{ such that } |f(x)| \le K\,|g(x)| \quad \forall\, |x - a| < \epsilon$$

- Given two real functions $f(x)$ and $g(x)$, then $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ iff:

$$\exists \text{ constants } X \text{ and } K > 0 \text{ such that } |f(x)| \le K\,|g(x)| \quad \forall\, x > X$$

- e.g. $x^2 + x = \mathcal{O}(x^2)$ as $x \to \infty$ since:

$$\left|x^2 + x\right| \le 2x^2 \quad \forall\, x > 1$$
$$\therefore \mathcal{O}(x^2)$$

5.4 Continuity

- A real function $f(x)$ is continuous at $x = a$ if:

  $f(a)$ exists

  $\lim_{x \to a} f(x) = f(a)$

- In $\epsilon - \delta$ form:

$$\text{for any } \epsilon > 0,\ \exists \delta > 0 \text{ such that } |f(x) - K| < \epsilon \quad \forall\, |x - x_0| < \delta$$

6 Infinite Series

- $S_n \equiv \sum_{k=0}^{n} u_k$ is the nth partial sum of an infinite series
- As $n \to \infty$, if $S_n \to$ a finite limit, then we say the sequence converges.

  Formally, $\lim_{n\to\infty} S_n = S$ if:

  For any $\epsilon > 0$, $\exists N$ such that $|S - S_n| < \epsilon \quad \forall n > N$
- Else if $S_n \to \pm\infty$ the we say it diverges.
- A sequence can also oscillate

6.1 Absolute Convergence

- If $\sum_{k=0}^{\infty} |u_k|$ converges, then the series is **absolutely convergent**

  Order of terms doesnt matter
- If $\sum_{k=0}^{\infty} u_k$ converges, but $\sum_{k=0}^{\infty} |u_k|$ doesnt, then the series is **conditionally convergent**

  Order of terms can give different results

  e.g. $1 - \frac{1}{2} + \frac{1}{3} - ...$

  $1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5}) - ... < 1$

  $(1 + \frac{1}{3} + \frac{1}{5}) - \frac{1}{2} + (\frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}) - \frac{1}{4} + ... = \frac{3}{2}$

6.2 Geometric Progression

$$S_n \equiv \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

- Series absolutely converges when $|r| < 1$

  $\lim_{n\to\infty} |r|^{n+1} = 0$ as, for any $\epsilon > 0$, $|r|^{n+1} < \epsilon \quad \forall n + 1 > \frac{\ln \epsilon}{\ln r}$
- $$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \qquad |r| < 1$$

6.3 Convergence Tests

6.3.1 Comparison

- For positive terms
- Compares unknown $\sum_k u_k$ to known $\sum_k v_k$
- If $u_k \leq v_k \quad \forall k \geq K$, then if $\sum_{k=0}^{\infty} v_k$ converges, $\sum_{k=0}^{\infty} u_k$ must converge as well
- If $u_k \geq v_k \quad \forall k \geq K$, then if $\sum_{k=0}^{\infty} v_k$ diverges, $\sum_{k=0}^{\infty} u_k$ must diverge as well
- Examples:

- The Harmonic Series: $\sum_{k=0}^{\infty} \frac{1}{k}$

  If we regroup as:

$$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ...$$

  we can see that each bracketed term added is greater than $\frac{1}{2}$ and therefore the series is actually diverging as $\sum_k \frac{1}{k} > \sum_k \frac{1}{2}$

- $\sum_{k=1}^{\infty} \frac{1}{k}(\frac{1}{2})^k < \sum_{k=1}^{\infty}(\frac{1}{2})^k$ therefore the first series converges

6.3.2 Ratio Test

For a positive series $\sum u_k$ if:

$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} < 1, \qquad \sum u_k$ converges
$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} > 1, \qquad \sum u_k$ diverges
$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = 1, \qquad \sum u_k$ may converge

- Examples:

  $\sum_k \frac{k}{2^k}$

$$\frac{u_{k+1}}{u_k} = \frac{k+1}{2^{k+1}} \frac{2^k}{k} = \frac{1}{2}\left(1 + \frac{1}{k}\right)$$
$$\implies \lim_{k\to\infty} = \frac{1}{2}$$

6.3.3 Alternating Series

This is also known as Liebniz's criterion:

> For an alternating sequence, $\sum_k (-1)^{k+1} a_k$, with $a_k > 0$ converges if a_k *monotonically decreases* for large enough k, and $\lim_{k\to\infty} a_k = 0$

6.4 Power Series

Infinite series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a + 1x + a_2 x^2 + \dots$$

- Converges if $x = 0$
- Absolutely converges for finite x
- Absolutely converges for $|x| < R$, for some R, diverges for $|x| > R$ and might converge for $|x| = R$
- Ratio test:
$$\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = |x| \left|\frac{a_{k+1}}{a_k}\right|$$
  - $L \equiv \lim_{k=0\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$
  - Converges if $\qquad |x| < \frac{1}{L}$

– Diverges if $|x| > \frac{1}{L}$

– Indeterminent if $|x| = \frac{1}{L}$

6.5 Taylor Series

- Maclaurin Series:

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \dots$$

  Dervied by setting $x = 0$ and repeatedly differentiating to get coefficients: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

- Above is a special case of the Taylor Series:

$$f(x) = f(a)+(x-a)f^{(1)}(a)+\frac{(x-a)^2}{2!}f^{(2)}(a)+\frac{(x-a)^3}{3!}f^{(3)}(a)+\dots+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_{n+1}$$

  where R_{n+1} is the remainder term.

- Alternatively can write the Taylor Series as:

$$f(a+h) = f(a)+hf^{(1)}(a)+\frac{h^2}{2!}f^{(2)}(a)+\frac{h^3}{3!}f^{(3)}(a)+\dots+\frac{h^n}{n!}f^{(n)}(a)+R_{n+1}$$

- Taylor's Theorem states that for $x = \zeta$ for $a < \zeta < a + h$:

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\zeta)$$

- Taylor Series to Learn

  –
$$exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

  –
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

  –
$$\cos(x) = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

  –
$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

  –
$$\ln(x+1) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

7 Integration

$$\int_a^b f(x)dx \equiv \lim_{N\to\infty} \sum_{i=0}^{N-1} f(\zeta_i)(x_{i+1} - x_i)$$

- Infinite integrals:

$$\int_a^\infty f(x)dx \equiv \lim_{b\to\infty} \int_a^b f(x)dx$$

- Improper Integrals: If the intrgrand is singular in the range of integration, we exclude a small region around singularity and then take a limit.
e.g. To calculate $\int_0^1 \frac{1}{\sqrt{x}}dx$, consider:

$$\int_\epsilon^1 \frac{1}{\sqrt{x}}dx = 2\left|\sqrt{x}\right|_\epsilon^1 = 2 - 2\sqrt{\epsilon} \qquad \lim_{\epsilon\to 0} 2 - 2\sqrt{\epsilon} = 2$$

7.1 Trig Substitutions

- If integral involves $\sqrt{a^2 - x^2}$ then use $x = a\sin\theta$

  as $dx = a\cos\theta d\theta = a\sqrt{1 - sin^2\theta}d\theta = \sqrt{a^2 - x^2}d\theta$

  e.g.

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = sin^{-1}\left(\frac{x}{a}\right) + c$$

- If integral involves $x^2 + a^2$ use $x = a\tan\theta$

  as $dx = a\sec^2\theta d\theta = \frac{1}{a}(1 + \tan^2\theta)$

  e.g.

$$\int \frac{1}{a^2 + x^2} = \frac{1}{a}tan^{-1}(\frac{x}{a})$$

- Completing the square is useful when the integrand is of the form $\frac{1}{Q(x)}$, $\frac{1}{\sqrt{Q(x)}}$ where $Q(x)$ is a quadratic, as then a trig substitution can be used.

7.2 Integration by Parts

$$\int fg'dx = fg - \int gf'dx$$

e.g.

$$\int \ln x\, dx = x\ln x - \int (\frac{x}{x})dx = x\ln x - x + c$$

7.3 Differentiation of Integrals w.r.t. Parameters

$$I(\alpha) \equiv \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\, dx \qquad \frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x;\alpha a)}{\partial\alpha}\, dx + \frac{db}{d\alpha}f(b;\alpha) - \frac{da}{d\alpha}f(a;\alpha)$$

Note:

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

7.4 Sums as Integrals

- Stirling's Approximation: Starts with noticing:
  $\ln n! = \ln(1 \times 2 \times 3 \times \cdots \times n) = \ln 1 + \ln 2 + \ln 3 + \ldots \ln n = \sum_{k=1}^{n} \ln k$ If we plot x against $\ln x$ and draw two sets of rectangles to put bounds on the curve, we get:
  $$\ln n! \approx n \ln n - n$$
- Better approximation:
  $$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

7.5 Schwarz's Inequality

$\underline{a} \cdot \underline{b} = |a|\,|b| \cos\theta$
$|\underline{a} \cdot \underline{b}| \leq |a|\,|b|$
$(\underline{a} \cdot \underline{b})^2 \leq |a|^2\,|b|^2$
$(a_x b_x + a_y b_y + a_z b_y)^2 \leq (a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2)$
Extending this to N dimensions:

$$\left(\sum_{i=1}^{N} a_i b_i\right) \leq \left(\sum_{i=1}^{N} a_i^2\right)\left(\sum_{i=1}^{N} b_i^2\right)$$

By realising an integral is the limit of of a sum, as $N \to \infty$, we get *Schwarz's Inequality*

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right)$$

7.6 Multiple Integrals

Extends Riemann's definition for one variable into several variables. E.g.

$$\iint_S f(x,y)\,dxdy \equiv \lim_{\substack{\delta x \to \infty \\ \delta y \to \infty}} \sum_i \sum_j f(x_i, y_i)\delta x \delta y \lim_{\substack{\delta x \to \infty \\ \delta y \to \infty}}$$

e.g.

$$\int_{x=0}^{2} \int_{y=\frac{x}{2}}^{1} 2xy^2 dydx \equiv \int_{y=0}^{1} \int_{x=0}^{2y} 2xy^2 dxdy \equiv \frac{4}{5}$$

In plane-polar coordinates:

$$\iint_S f(r,\phi)\,rdrd\phi$$

Integrands that are products can be simplified as following

$$\begin{aligned}\iint_S f(x)g(x)\,dxdy &= \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x)g(x)dx\right)dy \\ &= \int_{y=c}^{y=d} g(y)\left(\int_{x=a}^{x=b} f(x)dx\right)dy \\ &= \left(\int_c^d y(x)dy\right)\left(\int_a^b f(x)dx\right)\end{aligned}$$

7.7 Volume Integrals

Cartesian

$$dV = dxdydz$$

Cylindrical Polar

$$dV = r\, drd\phi dz$$

Spherical Poalr

$$dV = r^2 \sin\theta\, drd\theta d\phi$$

7.8 Gaussian Integrals

$$I = lim_{a\to\infty} I_a \quad \text{where} \quad I_a \equiv \int_{-a}^{a} e^{-x^2}\, dx$$

If we consider I_a^2

$$\begin{aligned} I_a^2 &= \left(\int_{-a}^{a} e^{-x^2}\, dx\right)\left(\int_{-a}^{a} e^{-y^2}\, dx\right) \\ &= \iint_S e^{-(x^2+y^2)}\, dxdy \end{aligned}$$

where S is a square of side $2a$ centred on the origin

Switch to plane-polar coordinates and establish

$$\iint_{S-} e^{-r^2} rdrd\phi < I_a^2 < \iint_{S+} e^{-r^2} rdrd\phi$$

where S_+ is the outcircle of S, with a radius of $\sqrt{2}a$, and S_- is the incircle of S, with a radius a

$$\implies \qquad \pi\left(1 - e^{-a^2}\right) < I_a^2 < \pi\left(1 - e^{-2a^2}\right)$$

as $a \to \infty$ both bounds is π, therefore:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

7.8.1 Error Function

- For a Gaussian distribution, the area that lies between $\pm a\sigma$ is given by the integral:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-a\sigma}^{a\sigma} e^{\frac{-x^2}{2\sigma^2}\, dx} = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2}}} e^{-u^2}\, du$$

- Error function: (shows the standard deviation):

$$erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

- $\text{erf}(\infty) = 1$