Numerical Analysis Equations

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1. Absolute Error
$$x^* = x \pm \epsilon_x \tag{1}$$
$$\Delta_{f(x)} = |f'(x*)| \, \Delta_x \tag{2}$$

2. Relative Error
$$x^* = x(1 \pm \eta_x) \tag{3}$$

3. Taylor's Error Bound
$$\Delta_x \leq \frac{M \, |x - x^*|^{n+1}}{(n+1)!} \tag{4}$$
Where $M \geq max\{\left|f^{n+1}(\xi)\right|\}$ in the range $[min(x, x*), max(x, x*)]$

4. First Derivative Forward Approximation
$$D_f^+ = \frac{f(x+h) - f(x)}{h} \tag{5}$$
Backward Approximation
$$D_f^- = \frac{f(x) - f(x-h)}{h} \tag{6}$$
Central Approximation
$$D_f^0 = \frac{f(x+h) - f(x-h)}{2h} \tag{7}$$

5. Truncation Errors Forward Approximation
$$\frac{-hf''(x)}{2} \tag{8}$$
Central Approximation
$$\frac{h^2 f'''(x)}{3!} \tag{9}$$

6. Integration

   Right Riemann Approximation

$$A = \frac{b-a}{h}(f(x_1) + f(x_2) + \cdots + f(x_n)) \tag{10}$$

   Left Riemann Approximation

$$A = \frac{b-a}{h}(f(x_0) + f(x_1) + \cdots + f(x_{n-1})) \tag{11}$$

   Midpoint Riemann Approximation

$$A = \frac{b-a}{h}(f(\frac{x_1 - x_0}{2}) + f(\frac{x_2 - x_1}{2}) + \cdots + f(\frac{x_n - x_{n-1}}{2})) \tag{12}$$

   Trapezium

$$A = \frac{b-a}{2h}(f(x_0) + 2(f(x_1) + \cdots + f(x_{n-1})) + f(x_n)) \tag{13}$$

   Simpson
   (n must be even)

$$A = \frac{b-a}{3h}(f(x_0) + 2(f_{evens}) + 4(f_{odds}) + f(n)) \tag{14}$$

7. Newton Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{15}$$

   Error:

$$\epsilon_{n+1} \approx \epsilon_n^2 \frac{f''(x_n)}{2f'(r)} \tag{16}$$

8. Mean Value Theorem For a continuously differentiable function f in interval $[a, b]$. For some c in $[a, b]$

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{17}$$

9. Secant Method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \tag{18}$$

   Error:

$$\epsilon_{n-1}^{p^2-p-1} \approx 1 \tag{19}$$
$$p^2 - p - 1 = 0 \tag{20}$$
$$p = \phi \tag{21}$$

10. Gradient Descent

$$x_{n+1} = x_n - \gamma f'(x) \tag{22}$$

Until $|f'(x_n)| \leq \epsilon$

11. Gaussian Elimination

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$R_3 = R_3 - 7R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -13 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -7 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

12. LU factorisation Same as Gaussian but keep track of the factors used to modify the matrix in the L matrix.

Example:

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix}$$

$R_3 = R_3 + 2R_2$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

13. Cholesky

$$A = LL^T \tag{23}$$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \tag{24}$$

$$A = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{bmatrix} \tag{25}$$

14. Cholesky Diagnolisation

$$A = LDL^T \tag{26}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

$$\tag{28}$$

15. Condition Number

$$K(x) = \left| \frac{x f'(x)}{f(x)} \right| \tag{29}$$

16. QR Factorisation

$$A = QR \tag{30}$$

$$\begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_0 & q_1 & q_2 \end{bmatrix} \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ 0 & r_{11} & r_{12} \\ 0 & 0 & r_{22} \end{bmatrix} \tag{31}$$

- $r_{00} = |a_0|$
- $q_0 = \frac{a_0}{r_{00}}$
- $r_{01} = q_0 \cdot a_1$

- $\bar{q_1} = a_1 - r_{01}(q_0)$
- $r_{11} = |\bar{q_1}|$
- $q_1 = \frac{\bar{q_1}}{|\bar{q_1}|}$
- $r_{02} = q_0 \cdot a_2$
- $r_{12} = q_1 \cdot a_2$
- $\bar{q_2} = a_2 - r_{02}(q_0) - r_{12}(q_1)$
- $r_{22} = |\bar{q_2}|$
- $q_2 = \frac{\bar{q_2}}{|\bar{q_2}|}$

17. Linear Least Squares

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \tag{32}$$
$$r_i = y_i - \beta_0 - \beta_1 x_i \tag{33}$$
$$S(\beta_0, \beta_1) = \sum_i (r_i)^2 \tag{34}$$
$$\sum_i (r_i)^2 = \sum_i (y_i - \beta_0 - \beta_1 x_i)^2 \tag{35}$$
$$\hat{\beta_0} = \sum_i y_i - \beta_1 \frac{1}{n} \sum_i x_i \tag{36}$$
$$\hat{\beta_1} = \frac{\sum_i x_i y_i - \frac{1}{n} \sum_i x_i \sum_i y_i}{\sum_i (x_i)^2 - \frac{1}{n} (\sum_i x_i)^2} \tag{37}$$

18. Least Sqaures Matrices

$$\hat{\beta} = (X^T X)^{-1} X^T y \tag{38}$$