CENTRO DE CIENCIAS MATEMÁTICAS

FEA of Linear Vibrations **in FEniCs**

20th July 2023

Shardul Kher

Guide: Prof. Behrooz Yousefzadeh

Concordia University

1. Installation
2. FEM Formlation
3. Simulation Workflow
4. Results
5. Future Scope
6. References

FEniCs Installation

Strong Form to Weak Form

$$\rho \ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma}, \quad \text{in } \Omega \tag{1}$$

Strong Form to Weak Form

$$\rho \ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma}, \quad \text{in } \Omega \tag{2}$$

To obtain the weak form we multiply the above equation with a test function $\boldsymbol{v}$ and integrate over the domain Ω

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\, \mathrm{dV} + \int_{\Omega} \rho \ddot{\mathbf{u}} \cdot \mathbf{v}\, \mathrm{dV} = 0, \quad \forall \mathbf{v} \text{ in } \Omega, \tag{3}$$

Strong Form to Weak Form

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Assuming $\boldsymbol{u}$ takes periodic form: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_A(\mathbf{x}) \mathrm{e}^{\mathrm{i}\omega t}$;
Substituting this **u**(**x**,t) in in the above eqn, we get:

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{dV} - \omega^2 \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} \, \mathrm{dV} = 0, \quad \forall \mathbf{v} \text{ in } \Omega, \tag{6}$$

Strong Form to Weak Form

$$\rho \ddot{\mathbf{u}} = \text{div}\, \boldsymbol{\sigma}, \quad \text{in } \Omega \tag{7}$$

To obtain the weak form we multiply the above equation with a test function $\boldsymbol{v}$ and integrate over the domain Ω

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Eigen-value problem:

$$\mathbf{Au} - \omega^2 \mathbf{Mu} = 0, \tag{10}$$

A & **M** are matrices assembled from the governing equations

Strong Form to Weak Form

$$\rho \ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma}, \quad \text{in } \Omega \tag{11}$$

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Assuming $\boldsymbol{u}$ takes periodic form: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_A(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$

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Eigen-value problem:

$$\mathbf{A}\mathbf{u} - \omega^2 \mathbf{M}\mathbf{u} = 0, \tag{14}$$

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Strong Form to Weak Form

$$\rho \ddot{\mathbf{u}} = \text{div}\, \boldsymbol{\sigma}, \quad \text{in } \Omega \tag{15}$$

To obtain the weak form we multiply the above equation with a test function $\mathbf{v}$ and integrate over the domain Ω

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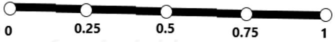
FEniCS PROJECT

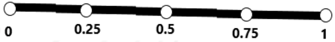
Matrix Formation Simplified

1D space Ω with a mesh of 5 equally spaced nodes:



Matrix Formation Simplified

1D space $\mathbf{\Omega}$ with a mesh of 5 equally spaced nodes:



- Linear Basis Functions: $V = fe.FunctionSpace(mesh, 1)$

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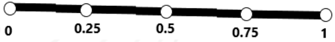

- Linear Basis Functions: $V = fe.FunctionSpace(mesh, 1)$
- Basis functions: $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)$; $\phi_i(x) = 1$ at i^{th} node & zero at others and $\phi_i(x)$ is **linearly** interpolated between neighbouring nodes

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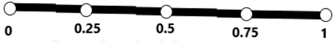

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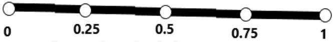

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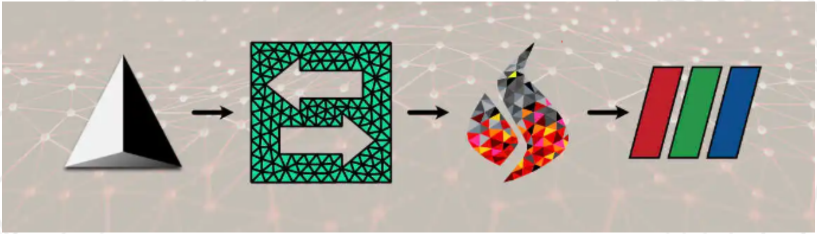
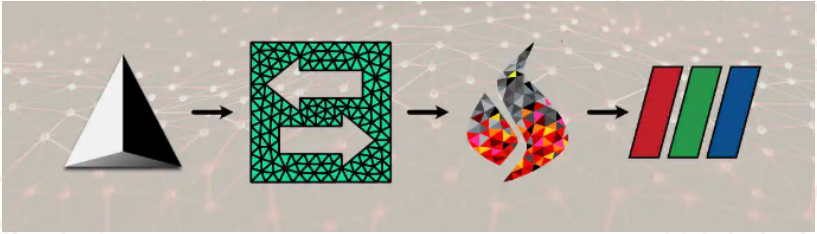
- $A = \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\, \mathrm{dx}$
- Matrix **A** = $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

Workflow

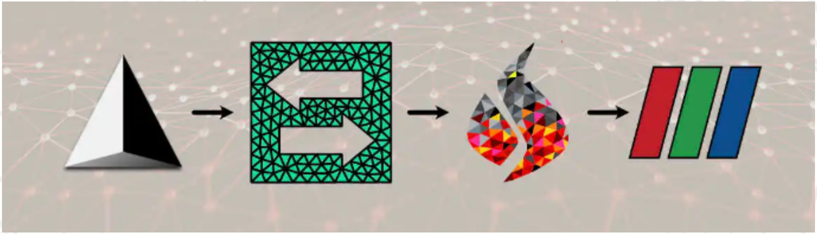


Workflow



- GMSH: Geometry creation & Meshing

Workflow



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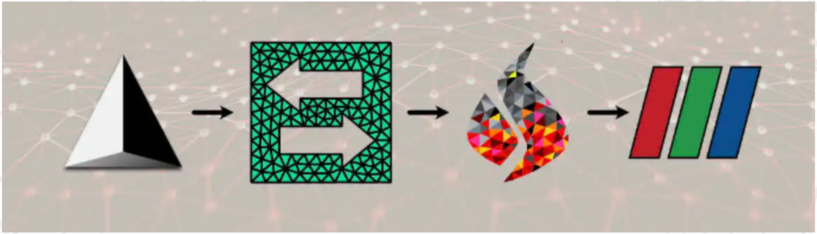

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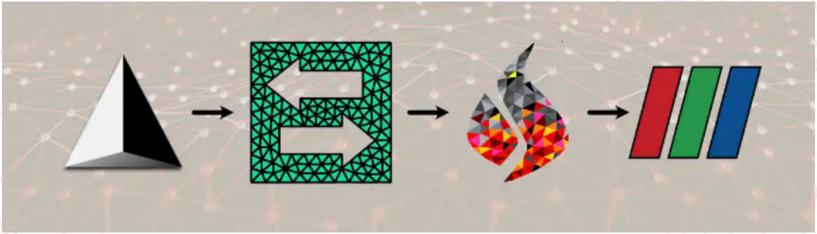

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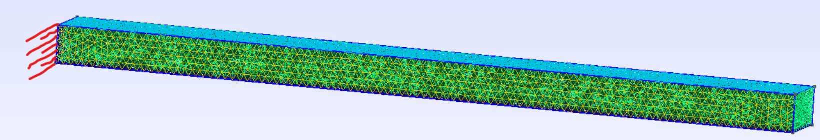
- FEniCs: PDE Solver
- Paraview: Visualisation & Post-processing

Uniform Cantilever Beam

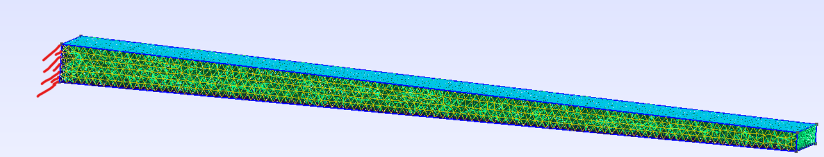


First 6 Modes:

Mode No.	ω - FEniCs [Hz]	ω - Beam theory [Hz]	Mode Participation Factor	Effective Mass	Relative Contribution
1	2.04991	2.01925	-7.83E-02	6.13E-03	61.30 %
2	4.04854	4.0385	7.88E-04	6.21E-07	0.01 %
3	12.81504	12.65443	4.34E-02	1.89E-03	18.85 %
4	25.12717	25.30886	-4.40E-04	1.94E-07	0.00 %
5	35.74168	35.43277	2.55E-02	6.49E-04	6.49 %
6	66.94816	70.86554	-3.71E-04	1.38E-07	0.00 %

Total relative mass of the first 6 modes: 86.65 %

Tapering Cantilever Beam

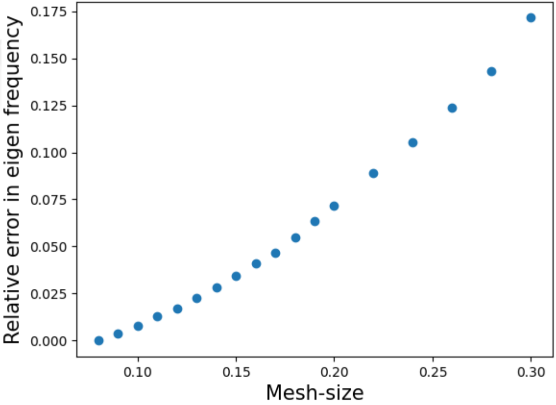


First 6 Modes:

Mode No.	ω - FEniCs [Hz]	Mode Participation Factor	Effective Mass	Relative Contribution
1	4.76505	-1.25E-01	1.57E-02	52.44 %
2	9.98477	-1.66E-05	2.76E-10	0.00 %
3	23.24731	7.78E-02	6.05E-03	20.17 %
4	52.46917	-4.62E-06	2.13E-11	0.00 %
5	59.93993	-5.02E-02	2.52E-03	8.39 %
6	76.63406	6.46E-05	4.18E-09	0.00 %

Total relative mass of the first 6 modes: 81.00 %

Tapering Beam Result Verification



Response of Mode-1 eigen frequency upon refining the mesh

- Mesh size limit (dl_o): 0.08m {Solver limit}

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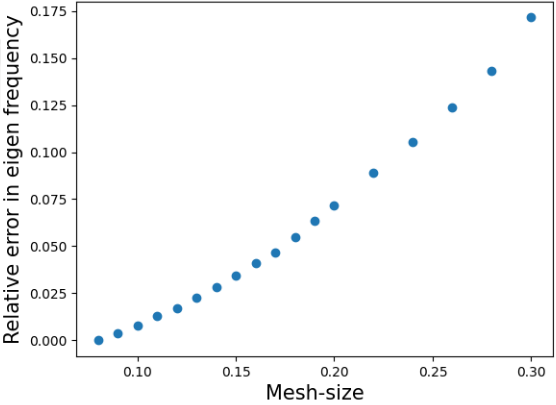

Response of Mode-1 eigen frequency upon refining the mesh

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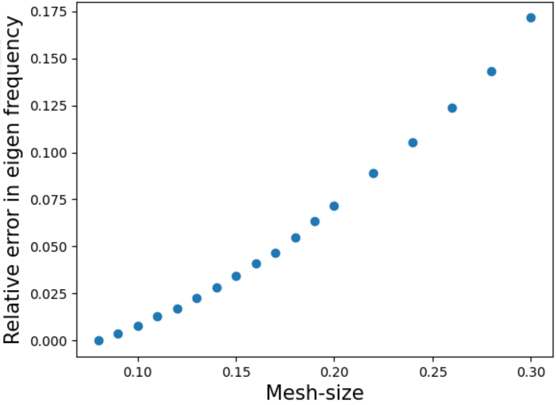
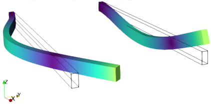


Response of Mode-1 eigen frequency upon refining the mesh

- Mesh size limit (dl_o): 0.08m {Solver limit}
- Assuming the dl_o mesh to give accurate output
- Observing depletion of error upon mesh refinement

Prospects

1. Visualisation and post-processing in Paraview [Spivak, 2018]



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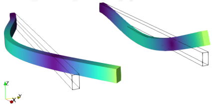
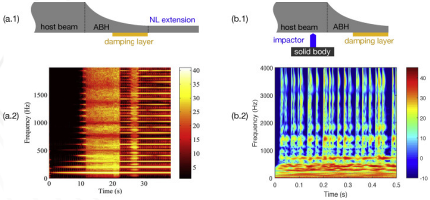


2. Simulating Acoustic Black Hole in FEniCs



[Adrien et al., 2020]

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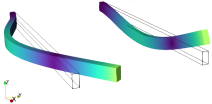

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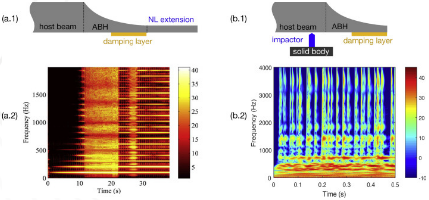

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- Power law: $h(x) = \epsilon x^{\mu} + h_{tip}$ [Hook et al., 2019]

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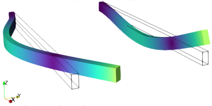

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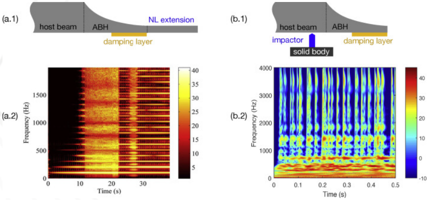

[Adrien et al., 2020]

- Power law: $h(x) = \epsilon x^{\mu} + h_{tip}$ [Hook et al., 2019]
- Geometry creation in GMSH using "BSpline" function

References

Adrien, P., François, G., Stephen, C., and Fabio, S. (2020).
The acoustic black hole: A review of theory and applications.
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THANK YOU!