Assignment 2

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1 2018-ICSE-12TH BOARD-PROBLEM

Problem 2: If the function $f(x) = \sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $(fof^{-1})(x) = x$.

Solution:

- **Verifying the invertibility of the function:**
  Given function,
  $$f(x) = \sqrt{2x-3} \tag{1.1}$$
  Domain: For the function to a well-defined function, the entity under the square root must be positive. Therefore, the possible values in the domain of $f(x)$ can be found by,
  $$2x-3 \geq 0 \tag{1.2}$$
  $$\implies x \geq \frac{3}{2} \tag{1.3}$$
  $\therefore$ Domain of $f(x)$ is $[\frac{3}{2}, \infty)$.
  We know that, for a function to be invertible, it should be an injective as well as a subjective function too.
  - **Verification for injective function:**
    We can use ”Horizontal line test” for verifying this. Plotting the graph using the python functions, we get

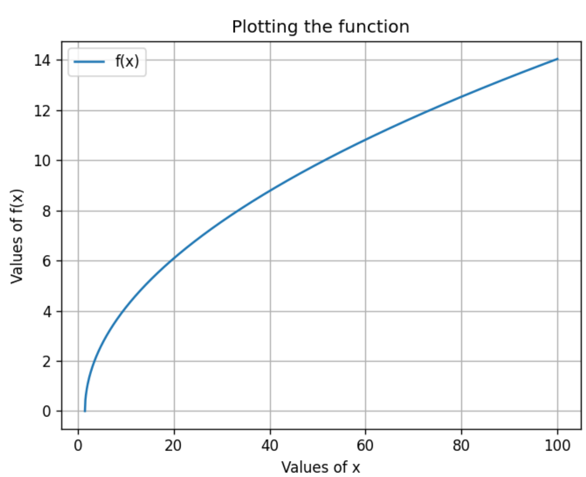

Fig. 0. Plotting the function, $f(x) = \sqrt{2x-3}$

We can clearly see, by drawing any horizontal straight line, we can't make the line touch the graph twice.
$\therefore$ $f(x)$ is an one to one, i.e., an injective function.

  - **Verification for subjective function:**
    We can verify, if the function is strictly increasing or decreasing by taking its ”Derivative test” with respect to 'x'.
    Given,
    $$f(x) = \sqrt{2x-3} \tag{1.4}$$
    Differentiating on both sides with respect to $'x'$,
    $$\frac{df(x)}{dx} = \frac{d}{dx}\sqrt{2x-3} \tag{1.5}$$
    $$\implies \frac{df(x)}{dx} = \frac{1}{2\sqrt{2x-3}}(2) \tag{1.6}$$
    $$\therefore \frac{df(x)}{dx} = \frac{1}{\sqrt{2x-3}} \tag{1.7}$$
    Since, we know that $\sqrt{2x-3}$ is always positive. Therefore, this implies that $\frac{df(x)}{dx}$ is always positive.
    $$\frac{1}{\sqrt{2x-3}} > 0 \tag{1.8}$$
    $$\therefore \frac{df(x)}{dx} > 0 \tag{1.9}$$
    Hence, we can say that the function, f(x) is always increasing. Therefore, f(x) covers all the values of its co-domain.
    $\therefore$ $f(x)$ is an onto, i.e., a subjective function.

- **Finding the inverse of the function:**
  Since, we found out that the function, f(x) is a bijective function (both injective and subjective). Therefore, its inverse exists.
  We can find the inverse of the function, by

swapping x with $f^{-1}(x)$ and $f(x)$ with x. Hence we get,

$$x = \sqrt{2(f^{-1}(x)) - 3} \tag{1.10}$$

$$\implies x^2 = 2(f^{-1}(x)) - 3 \tag{1.11}$$

$$\therefore f^{-1}(x) = \frac{x^2+3}{2} \tag{1.12}$$

We can even verify, the above obtained inverse function by plotting and checking an inverse function property.
Inverse function property: It states that, the original function($f(x)$) and its inverse($f^{-1}(x)$) are always mirror images in the line, y = x.
By plotting the equation and observing,

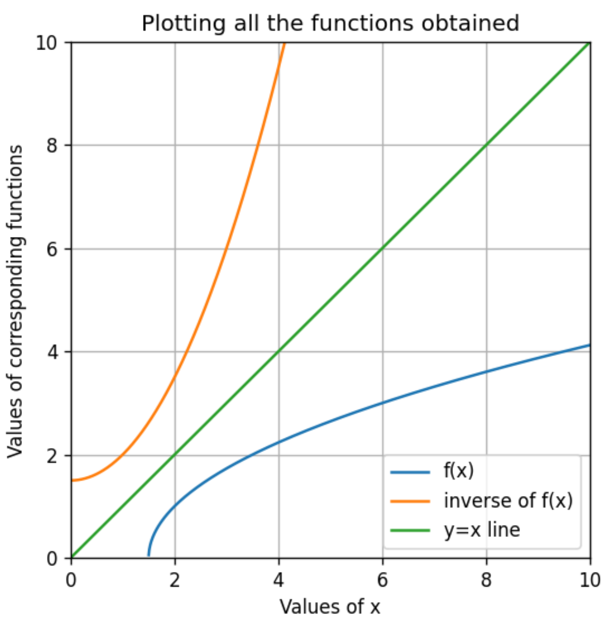

Fig. 0. Plotting the original function, the inverse function and the $y = x$ line to verify the inverse

We can clearly see that, the graph is symmetric about the line, $y = x$. Hence, we obtained that for $f(x) = \sqrt{2x-3}$, we get

$$f^{-1}(x) = \frac{x^2+3}{2} \tag{1.13}$$

- **Proving that:** $(fof^{-1})(x) = x$
  Solving R.H.S.,
  We can also write it as,

$$(fof^{-1})(x) = f(f^{-1}(x)) \tag{1.14}$$

Substituting the equation (1.13),

$$(fof^{-1})(x) = f(\frac{x^2+3}{2}) \tag{1.15}$$

Using equation (1.1),

$$(fof^{-1})(x) = \sqrt{2(\frac{x^2+3}{2}) - 3} \tag{1.16}$$

$$\implies (fof^{-1})(x) = \sqrt{(x^2+3) - 3} \tag{1.17}$$

$$\implies (fof^{-1})(x) = \sqrt{x^2} \tag{1.18}$$

$$\therefore (fof^{-1})(x) = |x| \tag{1.19}$$

As we know that the function is only defined for $x \geq \frac{3}{2}$. Therefore, here x is always positive.

$$\therefore (fof^{-1})(x) = x \tag{1.20}$$

Hence proved.