Assignment 2

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1 2018-ICSE-12TH BOARD-PROBLEM

Problem 2: If the function $f(x) = \sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $\left(fof^{-1}\right)(x) = x$.

Solution:

- **Verifying the invertibility of the function:**
  Given function,

$$f(x) = \sqrt{2x-3} \tag{1.1}$$

Domain: For the function to a well-defined function, the entity under the square root must be positive.

Therefore, the possible values in the domain of $f(x)$ can be found by,

$$2x - 3 \geq 0 \tag{1.2}$$

$$\implies x \geq \frac{3}{2} \tag{1.3}$$

$\therefore$ Domain of $f(x)$ is $[\frac{3}{2}, \infty)$.
We know that, for a function to be invertible, it should be an injective as well as a subjective function too.

  - **Verification for injective function:**
    We can use "Horizontal line test" for verifying this. Plotting the graph using the python functions, we get
    We can clearly see, by drawing any horizontal straight line, we can't make the line touch the graph twice.
    $\therefore$ $f(x)$ is an one to one, i.e., an injective function.
  - **Verification for subjective function:**
    We can verify, if the function is strictly increasing or decreasing by taking its "Derivative test" with respect to 'x'.
    Given,

$$f(x) = \sqrt{2x-3} \tag{1.4}$$

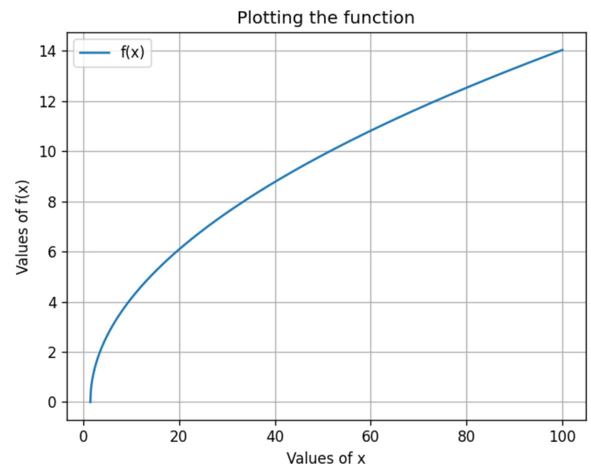

Fig. 0. Plotting the function, $f(x) = \sqrt{2x-3}$

Differentiating on both sides with respect to $'x'$,

$$\frac{df(x)}{dx} = \frac{d}{dx}\sqrt{2x-3} \tag{1.5}$$

$$= \frac{1}{2\sqrt{2x-3}}(2) \tag{1.6}$$

$$= \frac{1}{\sqrt{2x-3}} \tag{1.7}$$

Since, we know that $\sqrt{2x-3}$ is always positive. Therefore, this implies that $\frac{df(x)}{dx}$ is always positive.

$$\frac{1}{\sqrt{2x-3}} > 0 \tag{1.8}$$

$$\therefore \frac{df(x)}{dx} > 0 \tag{1.9}$$

Hence, we can say that the function, $f(x)$ is always increasing. Therefore, $f(x)$ covers all the values of its co-domain.
$\therefore$ $f(x)$ is an onto, i.e., a subjective function.

- **Finding the inverse of the function:**
  Since, we found out that the function, f(x) is a bijective function (both injective and subjective). Therefore, its inverse exists.

We can find the inverse of the function, by swapping x with $f^{-1}(x)$ and $f(x)$ with x. Hence we get,

$$x = \sqrt{2\left(f^{-1}(x)\right) - 3} \tag{1.10}$$

$$\implies x^2 = 2\left(f^{-1}(x)\right) - 3 \tag{1.11}$$

$$\therefore f^{-1}(x) = \frac{x^2+3}{2} \tag{1.12}$$

We can even verify, the above obtained inverse function by plotting and checking an inverse function property.

Inverse function property: It states that, the original function($f(x)$) and its inverse($f^{-1}(x)$) are always mirror images in the line, y = x.

By plotting the equation and observing,

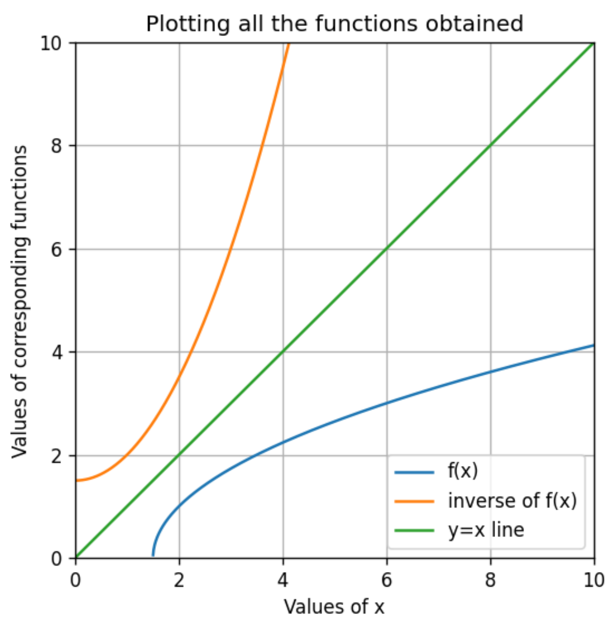

Fig. 0. Plotting the original function, the inverse function and the $y = x$ line to verify the inverse

We can clearly see that, the graph is symmetric about the line, $y = x$. Hence, we obtained that for $f(x) = \sqrt{2x-3}$, we get

$$f^{-1}(x) = \frac{x^2+3}{2} \tag{1.13}$$

- **Proving that:** $\left(fof^{-1}\right)(x) = x$
  Solving R.H.S.,
  We can also write it as,

$$\left(fof^{-1}\right)(x) = f\left(f^{-1}(x)\right) \tag{1.14}$$

Substituting the equation (1.13),

$$\left(fof^{-1}\right)(x) = f\left(\frac{x^2+3}{2}\right) \tag{1.15}$$

Using equation (1.1),

$$\left(fof^{-1}\right)(x) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} \tag{1.16}$$

$$= \sqrt{(x^2+3) - 3} \tag{1.17}$$

$$= \sqrt{x^2} \tag{1.18}$$

$$= |x| \tag{1.19}$$

As we know that the function is only defined for $x \geq \frac{3}{2}$. Therefore, here x is always positive.

$$\therefore \left(fof^{-1}\right)(x) = x \tag{1.20}$$

Hence proved.