Agenda

- TD (bootstrapping) with approximation
- Off-policy with approximation

Value function approximation II

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Recap

Approximate value, approximate policy

We can show that:

$$v_\pi(s) \geq v^*(s) - \frac{2\epsilon}{1-\gamma}$$

- $v^*(s)$ is the optimal policy performance
- $v_\pi(s)$ is our policy (using $q_\theta(s,a)$)
- ϵ the maximum error between $q_\theta(s,a)$ and $q^*(s,a)$
- Our policy has a lower bound depending on the error!

Intuitive interpretation

$$v_\pi(s) \geq v^*(s) - \frac{2\epsilon}{1-\gamma}$$

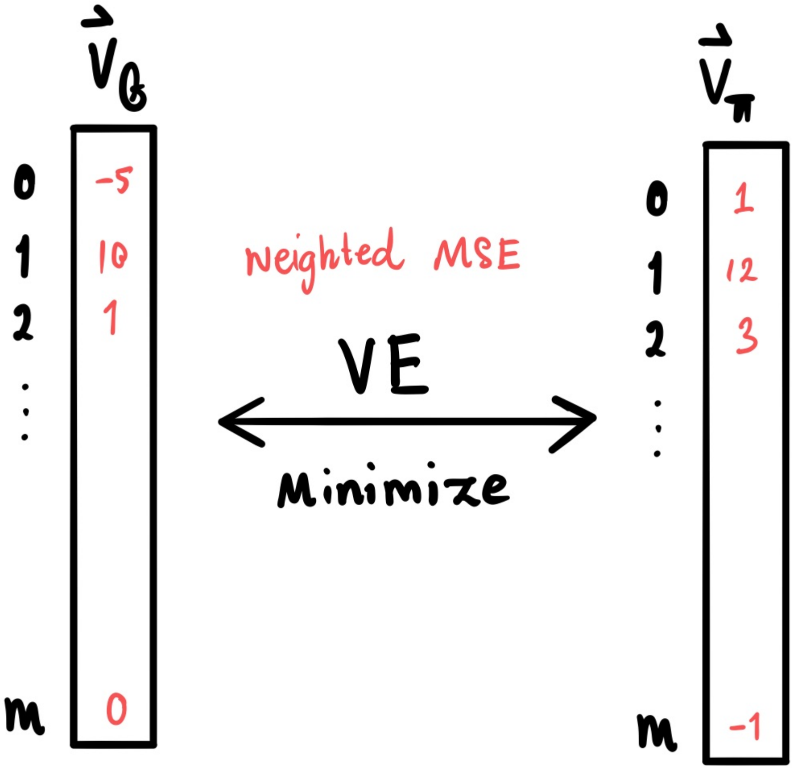
- Per-step error:

$$v^*(s) - q^*(s, \pi(s)) \leq 2\epsilon$$

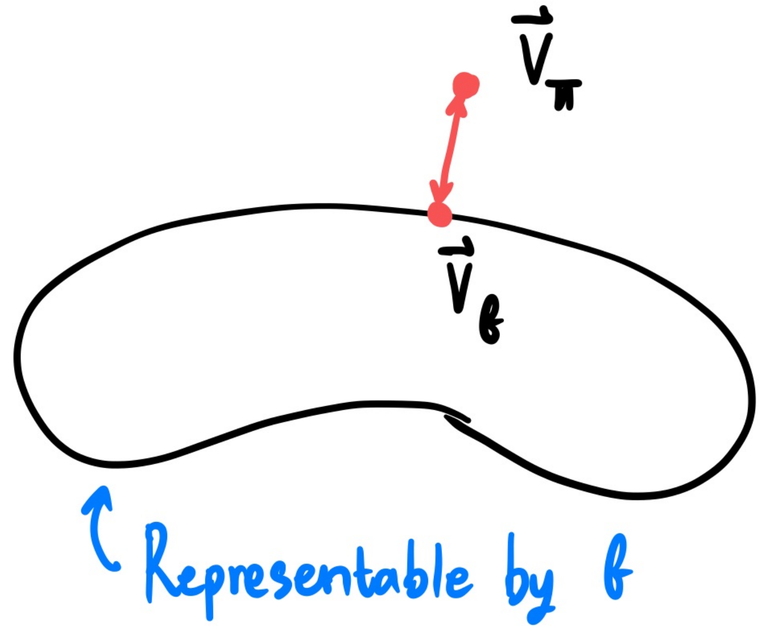
- Trajectory error:

$$v_\pi(s) \geq v^*(s) - \frac{2\epsilon}{1-\gamma}$$

Views of approximation



$$\underset{\theta}{\arg\min} \; \| \vec{V}_\pi - \vec{V}_\theta \|^2_{p^\pi}$$



Value Error + SGD

$$\mathcal{L}(\theta) = \mathrm{VE}(\theta) = \sum_s \mathrm{P}^\pi(s) \left[\frac{1}{2} (G(s) - v_\theta(s))^2 \right]$$

$$\nabla_\theta \mathcal{L}(\theta) = -(G - v_\theta(s)) \nabla_\theta v_\theta(s)$$

$$S_0 \sim \mathrm{P}_{s_0}, S \sim \pi | S_0, G \sim \pi | S$$

$$\theta \leftarrow \theta + \alpha \left(G - v_\theta(S) \right) \nabla_\theta v_\theta(S)$$

Convergence and fixed point

Convergence

- A training process converges to a fixed point where there is no further progress

Fixed point

- The solution when the training process converges

TD with approximation

Prediction and control

Prediction

- Get V (estimate of the policy)
- A single policy in concern

Control

- Prediction + improvement
- A series of policies

Linear vs non-linear

- Linear

$$v_\theta(s) = \theta^T \phi(s)$$

$$\nabla_\theta v_\theta(s) = \phi(s)$$

- Non-linear

$$v_\theta(s) = f_\theta(\phi(s))$$

TD Prediction with approximation

Semi-gradient one-step TD

$$\mathrm{VE}(\theta) = \mathbb{E}_{S\sim \mathrm{P}^{\pi}(s)}\left[\frac{1}{2}\left(v_{\pi}(S) - v_{\theta}(S)\right)^2\right]$$

$$\theta \leftarrow \theta + \alpha\left(v_{\pi}(s_t) - v_{\theta}(s_t)\right)\nabla_{\theta} v_{\theta}(s_t)$$

We don't have a return, we use "bootstrapping" target instead

$$\theta \leftarrow \theta + \alpha\left(r_{t+1} + \gamma v_{\theta}(s_{t+1}) - v_{\theta}(s_t)\right)\nabla_{\theta} v_{\theta}(s_t)$$

Why we call “semi-gradient”?

$$\theta \leftarrow \theta + \alpha \left(R + \gamma v_\theta(S') - v_\theta(S)\right) \nabla_\theta v_\theta(S)$$

- The gradient is “incomplete”
- We assume that v(s’) is “independent” from theta
- This is a false assumption
- **Semi-gradient doesn’t share a usual SGD convergence guarantee**

Semi-gradient TD fixed point

$$\theta \leftarrow \theta + \alpha \left(R + \gamma v_\theta(S') - v_\theta(S) \right) \nabla_\theta v_\theta(S)$$

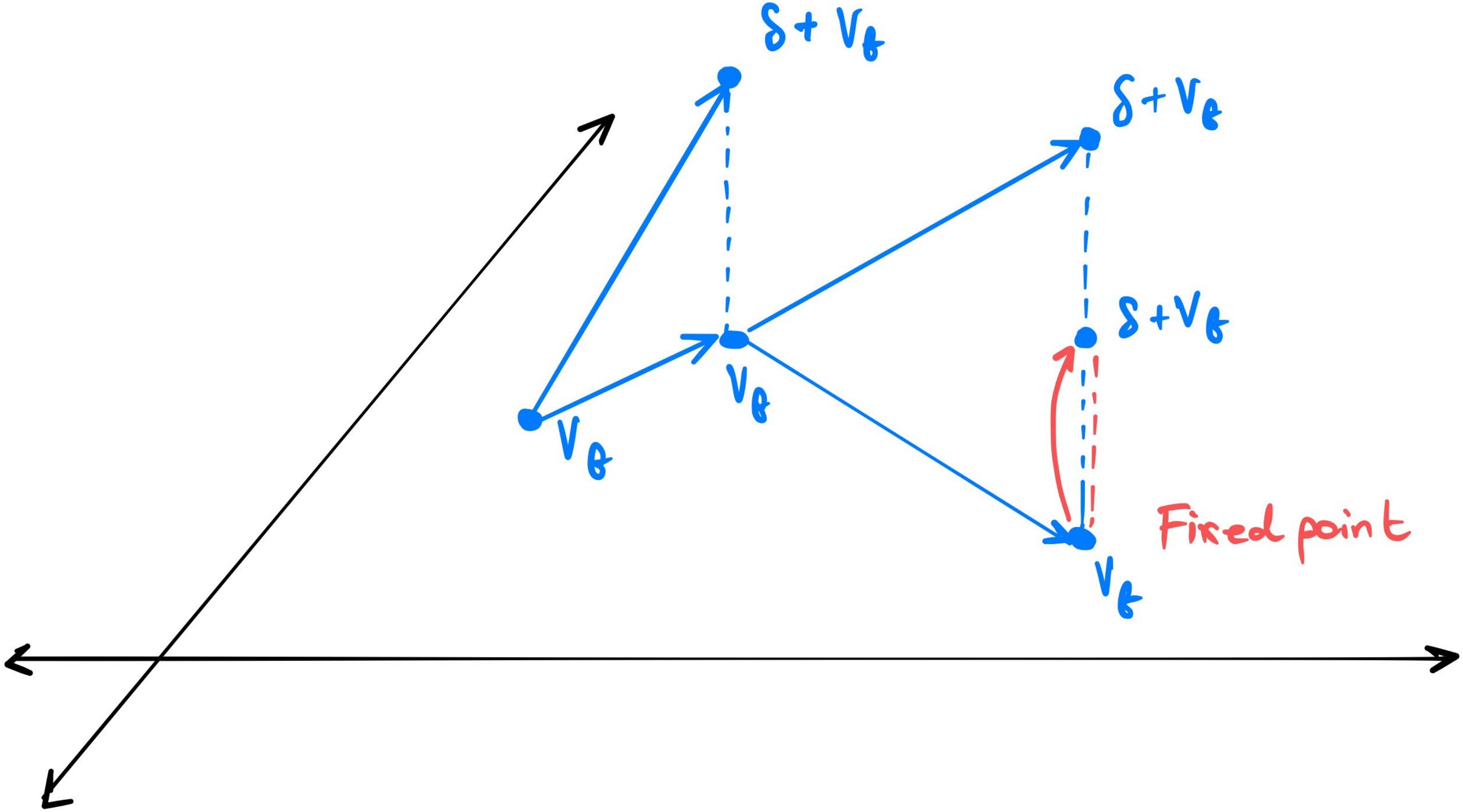
- No further update

$$\theta = \theta + \alpha \mathbb{E} \left[\delta \nabla_\theta v_\theta(S) \right]$$

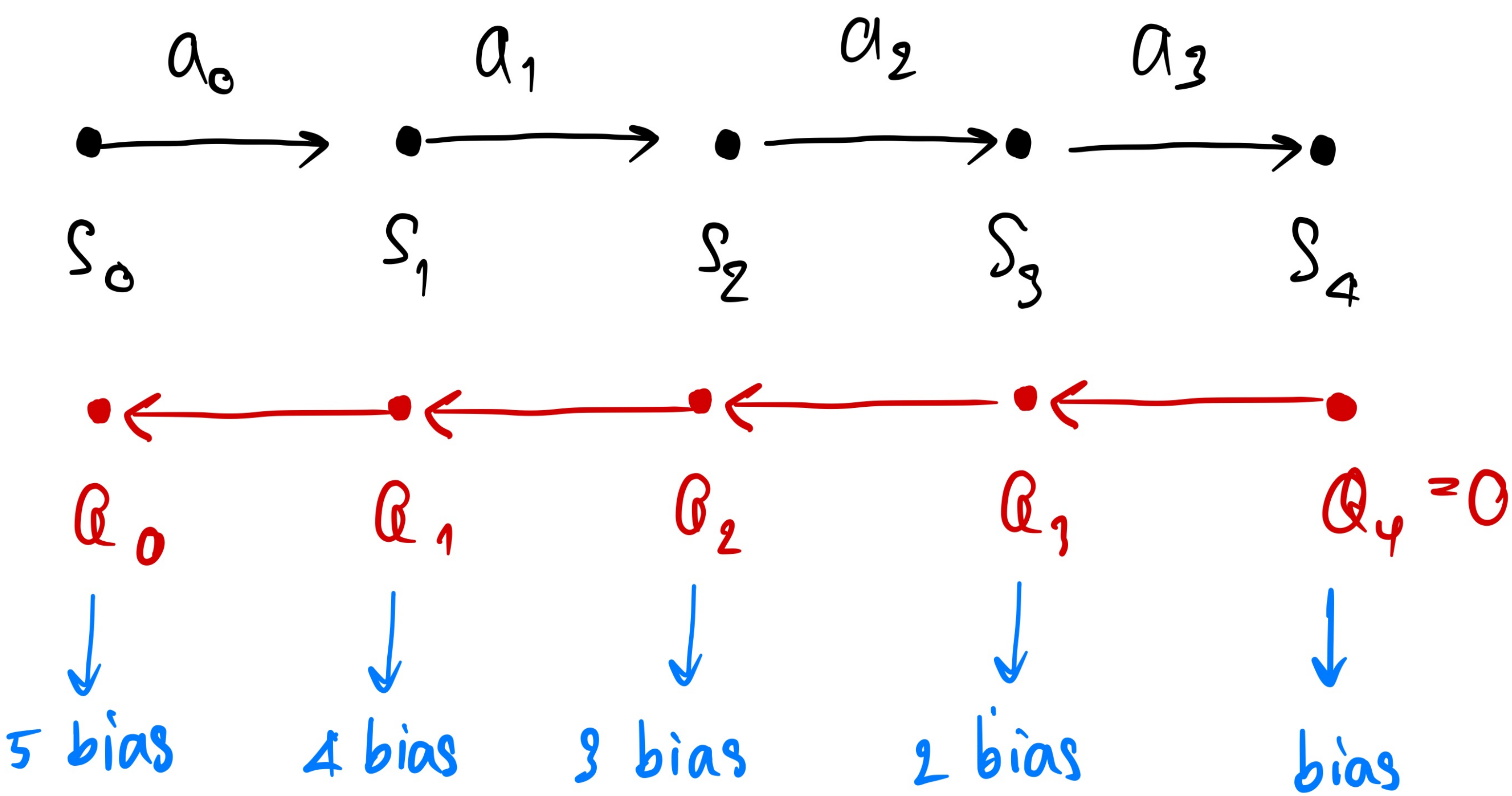
$$0 = \mathbb{E} \left[\delta \nabla_\theta v_\theta(S) \right]$$

- Not much could be said…

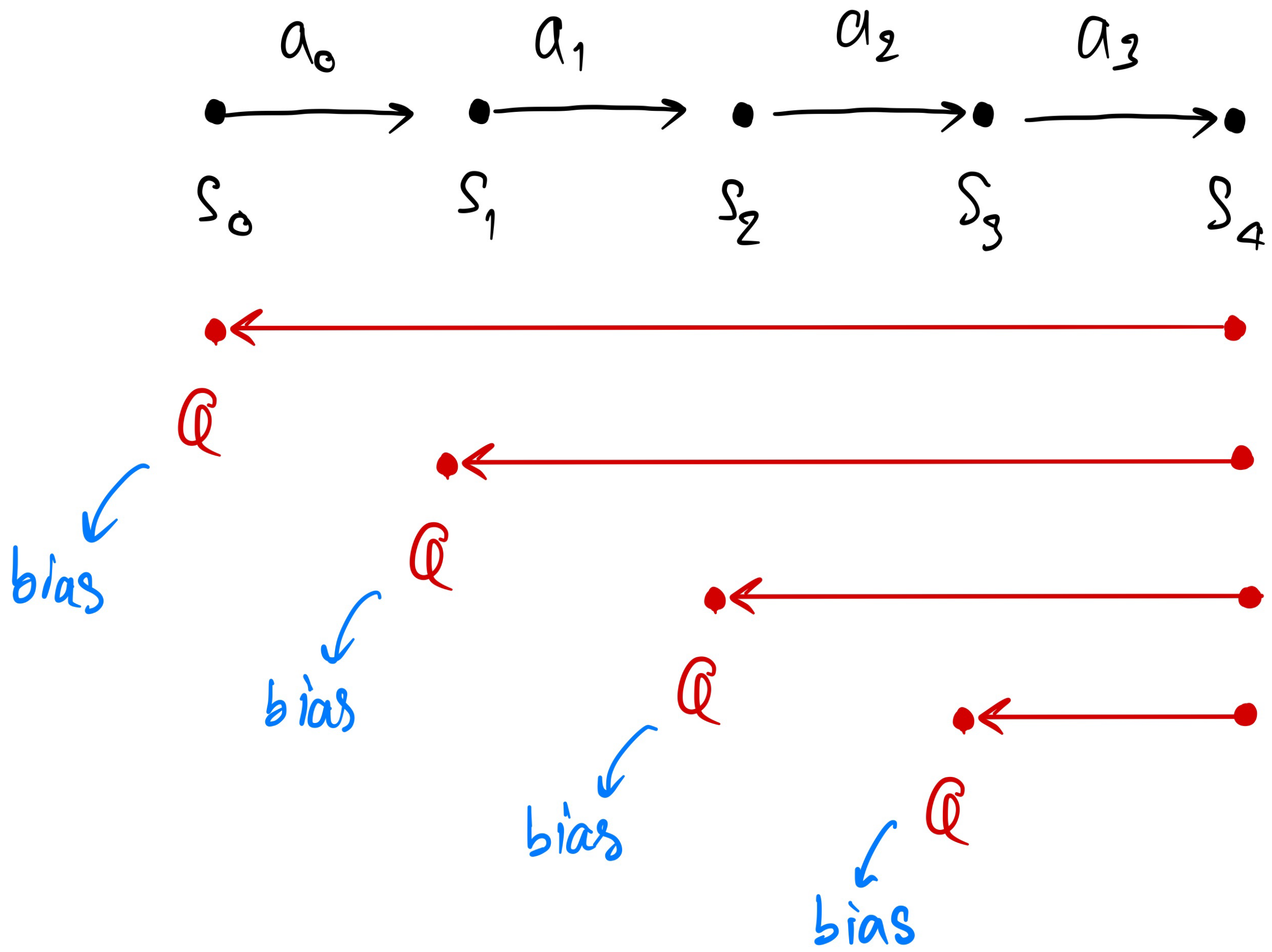
Semi-gradient visualized



Solution of semi-gradient TD

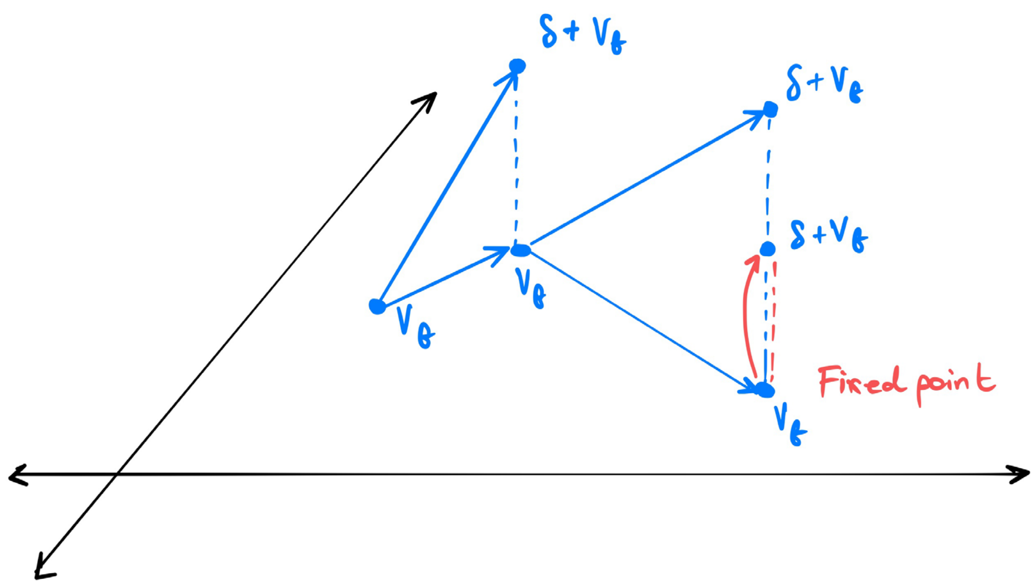


Solution of VE



Solution of semi-gradient TD

- Poorer solution that VE
- Propagation of errors
  - Each step incurs some error, many steps large error
  - Due to projection onto representable space



N-step semi-gradient

$$\theta \leftarrow \theta + \alpha \left(r_{t+1} + \gamma v_\theta(s_{t+1}) - v_\theta(s_t) \right) \nabla_\theta v_\theta(s_t)$$

$$\theta \leftarrow \theta + \alpha \left(g_{t:t+n} - v_\theta(s_t) \right) \nabla_\theta v_\theta(s_t)$$

- Replace the target with an n-step return
- You could use any kind of target here
- All returns are semi-gradients (except full-return)

Stability and convergence

Stability

- Weights don’t explode to **infinity**
- Proof is easier

Convergence

- Weights converge to a fixed point where objective function is minimized
- Convergence != good fixed point

Stability of semi-gradient linear TD(0)

- We describe the update in terms of "matrix" multiplication

$$\theta_{t+1} = M\theta_t + c$$

- We show that the matrix is a **"contraction"** mapping
  - Output is smaller than the input vector

$$\|M\theta\| < \|\theta\|$$

Stability of semi-gradient linear TD(0)

$$\theta \leftarrow \theta + \alpha \left(r_{t+1} + \gamma v_\theta(s_{t+1}) - v_\theta(s_t) \right) \nabla_\theta v_\theta(s_t)$$

$$\theta_{t+1} = \theta_t + \alpha \left(r_{t+1} + \gamma \theta_t^T x_{t+1} - \theta_t^T x_t \right) x_t$$

Goal:

$$\theta_{t+1} = M\theta_t + c$$

$$\|M\theta\| < \|\theta\|$$

Stability of semi-gradient linear TD(0)

$$\theta_{t+1} = M\theta_t + c$$

$$\theta_{t+1} = \theta_t + \alpha \left(r_{t+1} + \gamma \theta_t^T x_{t+1} - \theta_t^T x_t \right) x_t$$

Stability of semi-gradient linear TD(0)

$$\theta_{t+1} = M\theta_t + c$$

$$\theta_{t+1} = \theta_t + \alpha \left(r_{t+1} x_t - x_t (x_t - \gamma x_{t+1})^T \theta_t \right)$$

Stability of semi-gradient linear TD(0)

$$\mathbb{E}\theta_{t+1} = (I - \alpha A)\theta_t + \alpha b \qquad b = \mathbb{E}\left[r_{t+1} x_t\right]$$

$$A = \mathbb{E}\left[x_t (x_t - \gamma x_{t+1})^T\right]$$

- $I - \alpha A$ is a contraction
- By showing that $I - \alpha A$ has eigenvalues between 0 and 1
- By showing that A is positive-definite matrix
- **Under on-policy assumption**

Semi-gradient linear TD fixed-point

$$\mathbb{E}\theta_{t+1} = \theta_t + \alpha b - \alpha A \theta_t \qquad \mathbb{E} = \underbrace{x_t, x_{t+1} \sim \mathbb{P}^\pi, R \sim \pi}_{\text{signs of on-policy}}$$

$$b = \mathbb{E}\left[r_{t+1} x_t\right]$$

$$A = \mathbb{E}\left[x_t (x_t - \gamma x_{t+1})^T\right]$$

No further update

$$\theta = \theta + \alpha b - \alpha A \theta \qquad b = A\theta$$

$$0 = \alpha b - \alpha A \theta \qquad A^{-1} b = \theta$$

$$0 = b - A\theta$$

Semi-gradient TD property

$$\mathbb{E}\theta_{t+1} = \theta_t + \alpha \mathbb{E}\left[\delta_t \nabla_\theta v_\theta(s_t)\right]$$

- It converges to **TD fixed point** in linear case with on-policy

$$\theta = A^{-1}b$$

$$b = \mathbb{E}\left[r_{t+1} x_t\right]$$

$$A = \mathbb{E}\left[x_t (x_t - \gamma x_{t+1})^T\right]$$

- **TD fixed point is considered to be a "good" fixed point**
- It **doesn't** converge in non-linear case even with on-policy

Summary

- VE update

$$\theta \leftarrow \theta + \alpha \left(v_\pi(s_t) - v_\theta(s_t)\right) \nabla_\theta v_\theta(s_t)$$

- Semi-gradient TD update

$$\theta \leftarrow \theta + \alpha \left(r_{t+1} + \gamma v_\theta(s_{t+1}) - v_\theta(s_t)\right) \nabla_\theta v_\theta(s_t)$$

- Converges to TD fixed point with on-policy (only linear case)

TD control with approximation

SARSA with approximation

$$\theta \leftarrow \theta + \alpha \left(r_{t+1} + \gamma v_\theta(s_{t+1}) - v_\theta(s_t)\right) \nabla_\theta v_\theta(s_t)$$

for until v is stable **do**

  take action according to $q_\theta(s, a)$ epsilon greedy

  collect (s, a, r, s', a')

  $\delta = r + \gamma q_\theta(s', a') - q_\theta(s, a)$

  $\theta \leftarrow \theta + \alpha\delta\nabla_\theta q_\theta(s, a)$

end for

- We need to approximate $q_\theta(s, a)$

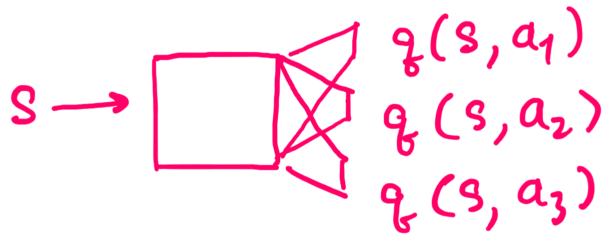
Implementation considerations

Implementation considerations

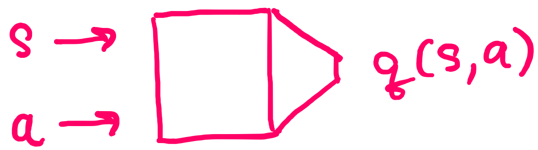
- Architecture design
- Co-adaptation nature of on-policy learning
- Sample correlation
  - Forgetting problems

Architecture design

- Layer type
  - Conv layers for image inputs
- Prediction head for Q function
  - Discrete actions = multi-head



  - Continuous action = single-head



Co-adaptation nature

Supervised learning formulation

$$\underset{\theta}{\text{minimize}} \quad \mathbb{E}_{(x,y)\sim D}\left[\mathcal{L}(x, y; \theta) \right]$$

Reinforcement learning formulation

$$\underset{\theta}{\text{minimize}} \quad \mathbb{E}_{(s,a,r,s')\sim D_\theta}\left[\mathcal{L}(\ldots; \theta) \right]$$

$$\theta \rightarrow \pi \rightarrow \theta \rightarrow \pi$$

Co-adaptation nature

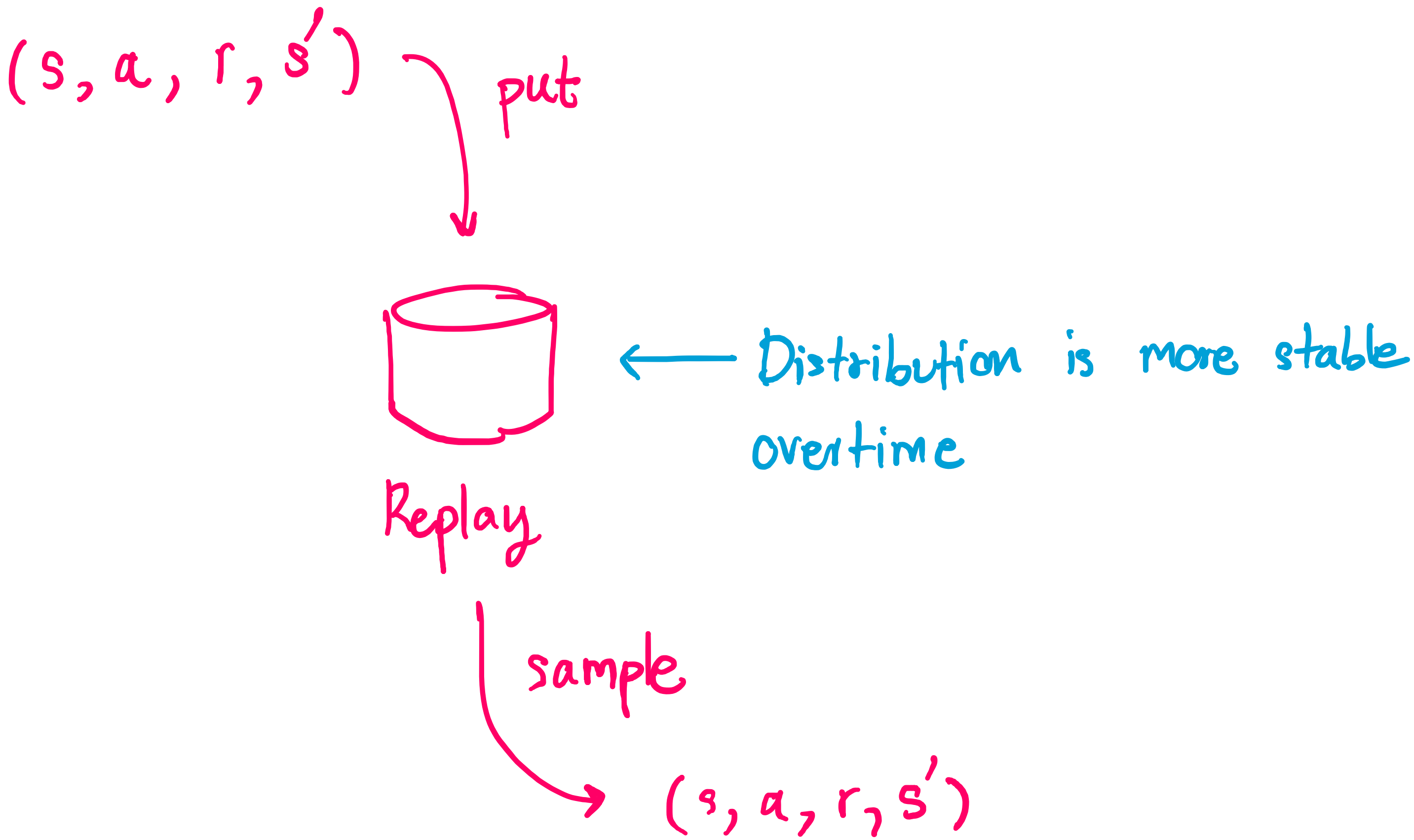
- The data distribution is constantly changing
  - Because the on-policy is constantly changing
- This could lead to unstable learning loop

$V_\theta \longrightarrow \pi \longrightarrow$ Bad data

Bad estimate   Worse policy

- Off-policy with more stable data distribution helps

Off policy with replay



Sample correlation

- On-policy sample is highly correlated

$(s_0, s_1, s_2, \dots)$

$$\left.\begin{matrix}(s_0, s_1) \\ (s_1, s_2)\end{matrix}\right\} \text{highly correlated}$$

- SGD with independent assumption doesn't work very well

$$\underset{\theta}{\text{minimize}} \; \underset{\underset{\text{i.i.d.}}{x \sim D}}{\mathbb{E}} \left[l(x;\theta) \right]$$

  - It might converge to sub-optimal minima
  - Very low learning rate is needed otherwise

Forgetting

- If the gradient is not representative (correlated gradients)

$$
\begin{tikzcd}
S_0 \arrow[r] & S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & S_4 \arrow[d] \\
\text{Forget} \arrow[u] & & & & \nabla L
\end{tikzcd}
$$

- To reduce we might need very small learning rate