Jméno: **Místnost:** **Souřadnice:**

0007 *list* 1 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0 1 2 3 4 5 6 7 8 9

Problem 1
20 points

Your task is to find all nonisomorphic simple planar graphs that have exactly 5 faces and minimum degree 3. Draw the pictures of these graphs, and **clearly argue** why your list is complete. (Just finding these graphs will be rewarded by at most half of the points, the other half is for the completeness argumentation.)

Solution:
The main tool for this problem is Euler's formula. For our graphs G, it is $|E(G)|-|V(G)| = 5-2 = 3$. To fulfill the property of minimum degree 3, i.e. $|E(G)| \geq 3|V(G)/2$, we see that $|V(G)| \in \{5, 6\}$. For each of these two possibilities we construct one solution – the 4-sided pyramid, and the 3-sided prism.

There were a lot of mistakes in this problem solutions. First, you cannot count faces if you draw a graph with crossings! Also, the outer face is a face too – several students thought otherwise. Perhaps the strangest recurring mistake was the claim that all vertices of planar graphs must have degree at most 5 – the correct statement is that every planar graph contains at least one vertex of degree at most 5.

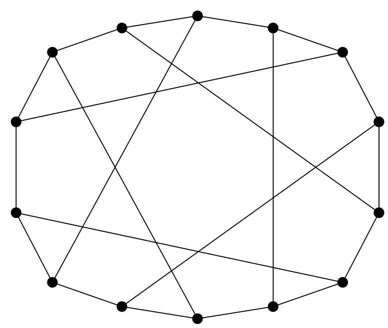
Jméno: **Místnost:** **Souřadnice:**

0007 *list* 2 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0 1 2 3 4 5 6 7 8 9

Problem 2
20 points

Given is the following graph G on 14 vertices:



Your task is to correctly answer the following three questions about its properties. It is not enough to write down the answer itself, but you also **have to (shortly) argue** why your answer is correct.

a) Find all integers k such that G contains a cycle of length k as a subgraph.

b) Find a minimum cardinality vertex cover in this graph G, and depict it below. A *vertex cover* is a set $C \subseteq V(G)$ such that every edge of G has some end in C. Do not forget to argue about optimality.

c) Find a minimum cardinality dominating set in this graph G, and depict it below. A *dominating set* is a set $D \subseteq V(G)$ such that every vertex not in D has some neighbour in D. Argue about optimality.

<u>*Solution*</u>:

a) $k = 6, 8, 10, 12, 14$. Since the graph is 2-colourable, there are no odd cycles there. 4-cycle is also impossible.

b) A minimum vertex cover is a complement of a maximum independent set which is of size 7, and so the answer is also $14 - 7 = 7$.

c) A vertex in our graph dominates itself and at most 3 neighbours. Hence we need at least $\lceil 14/4 \rceil = 4$ vertices in a dominating set, and such an example is easy to find.

Most of you solved this without problems. While in variant B it was somehow easier to find all the cycle lengths, variant A had slightly easier parts (b),(c).

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 3 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0 1 2 3 4 5 6 7 8 9

Problem 3
20 points

Proper edge-colouring of a graph assigns colours to the edges such that no two edges sharing the same vertex get the same colour.

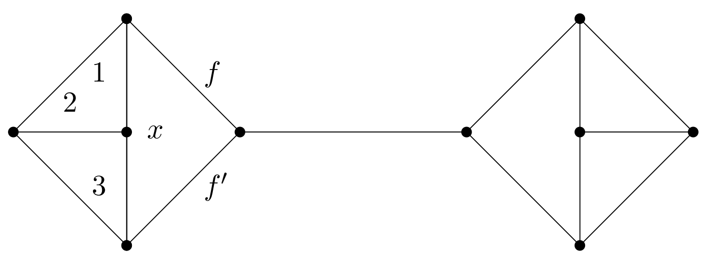
a) Construct (and draw a picture of it) any 3-regular simple graph G such that G is not 3-edge-colourable.

b) Then make a graph G' by subdividing an arbitrary edge of your G, and answer correctly whether this G' is 3-edge-colourable.

c) Decide whether, for some other graph G as in (a), the correct answer to part (b) could be different from your current (assuming correct again) answer in (b).

You must provide mathematical proofs of all your answers to receive points.

<u>*Solution*</u>:
a) One possible solution is:



To prove that this graph is not 3-edge-colourable, notice that up to symmetry, the edges incident with x are coloured $1, 2, 3$. If only three colours were used in the whole graph, then, necessarily, both edges f, f' would have to receive colour 2, a contradiction.

b) No. This is better answered by (c).

c) Nobody was able to solve this part, even though it is not difficult. We actually prove a stronger statement. Let G be any 3-regular graph, then G has an even number $2k$ of vertices, and $3k$ edges. If G' results from G by subdividing an arbitrary (one) edge, then G' has $2k+1$ vertices and $3k+1$ edges. However, at most $\lfloor (2k+1)/2 \rfloor = k$ edges of G' may receive the same colour, which totals to $3k < 3k+1$ edges. Therefore, G' cannot be 3-edge-colourable.

The points were given at most $7+6+7$, but nobody received points for part (c). Parts (a) and (b) were not so bad.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 4 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte.

0 1 2 3 4 5 6 7 8 9

Problem 4
20 points

An edge e is a *bridge* in a connected graph G if $G - e$ (removal of e) is not connected.

a) Prove that any connected graph on n vertices has at most $n - 1$ bridges.

b) Can a connected simple graph on n vertices have exactly $n - 2$ bridges? Prove it.

<u>*Solution*</u>:
The both statements are quite obviuous (in (b) the obvious answer is NO), and so the main task in this problem is to formulate rigorous mathematical proofs. There are two common ways to approach this task, both have appeared frequently among the solutions.

a) Since G is connected, there is a spanning tree $T \subseteq G$. Every edge $e \in E(G) \setminus E(T)$ makes a cycle with T, and hence e cannot be a bridge. Therefore, the maximum number of bridges can be $|E(T)| = n - 1$, and equality occurs iff $G = T$.
Another approach: Let $X \subseteq E(G)$ be the set of all bridges in G. In the subgraph $G - X$, we "shrink" every connected component (2-connected, btw.) into a new vertex, removing all edges not in X. The resulting connected graph G' has $\leq n$ vertices and X as the edge set, and moreover, G' has no cycles since otherwise the edges of a cycle would not be bridges. Therefore, G' is a tree and $|X| = |V(G')| - 1 \leq n - 1$.

b) Considering the second approach, with G', we see that $|X| = n - 2$ implies $|V(G')| = |V(G)| - 1$, and so exactly one component C of $G - X$ has two vertices (others have one). Since G is simple, however, $C \simeq K_2$, and the only edge of C is a bridge, too. This contradicts the assumption that all bridges are in X.

Besider several students who did not understand the definition of a bridge, the main problem of many solutions was the following: Students might have had a correct idea (similar to one of the above two) in their minds, but they were not able to formulate it on the paper. Since the points were given for mathematical proofs, such problematic solutions received usually only a few points.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 1 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0123456789

Problem 1
20 points

Your task is to find all nonisomorphic simple planar graphs that have exactly 6 vertices and minimum degree 4. Draw the pictures of these graphs, and **clearly argue** why your list is complete. (Just finding these graphs will be rewarded by at most half of the points, the other half is for the completeness argumentation.)

<u>*Solution*</u>:
The main tool for this problem is Euler's formula. With 6 vertices and minimum degree 4 we must have at least $6 \cdot 4/2 = 12$ edges, but the maximum for a planar graph is $3 \cdot 6 - 6 = 12$. Hence our graph must have 3 less edges that the complete graph K_6, and we can obviously construct it by removing these 3 edges from K_6. However, we cannot remove two edges incident with the same vertex of K_6 since the resulting degree would be 3. Therefore, there is only one possibility – to remove a 3-matching from K_6. The resulting planar graph is the octahedron.

There were a lot of mistakes in this problem solutions. First, you cannot count faces if you draw a graph with crossings! Also, the outer face is a face too – several students thought otherwise. Perhaps the strangest recurring mistake was the claim that all vertices of planar graphs must have degree at most 5 – the correct statement is that every planar graph contains at least one vertex of degree at most 5.

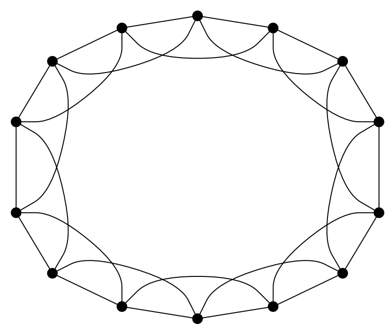
Jméno: **Místnost:** **Souřadnice:**

0007 *list* 2 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0123456789

Problem 2
20 points

Given is the following graph G on 14 vertices:



Your task is to correctly answer the following three questions about its properties. It is not enough to write down the answer itself, but you also **have to (shortly) argue** why your answer is correct.

a) Find all integers k such that G contains a cycle of length k as a subgraph.

b) Find a minimum cardinality vertex cover in this graph G, and depict it below. A *vertex cover* is a set $C \subseteq V(G)$ such that every edge of G has some end in C. Do not forget to argue about optimality.

c) Find a minimum cardinality dominating set in this graph G, and depict it below. A *dominating set* is a set $D \subseteq V(G)$ such that every vertex not in D has some neighbour in D. Argue about optimality.

Solution:

a) $k = 3, 4, 5, \ldots, 14$.

b) A minimum vertex cover is a complement of a maximum independent set which is of size 4, and so the answer is also $14 - 4 = 10$.

c) A vertex in our graph dominates itself and at most 4 neighbours. Hence we need at least $\lceil 14/5 \rceil = 3$ vertices in a dominating set, and such an example is easy to find.

Most of you solved this without problems. While in variant B it was somehow easier to find all the cycle lengths, variant A had slightly easier parts (b),(c).

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 3 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0123456789

Problem 3
20 points

Proper edge-colouring of a graph assigns colours to the edges such that no two edges sharing the same vertex get the same colour.

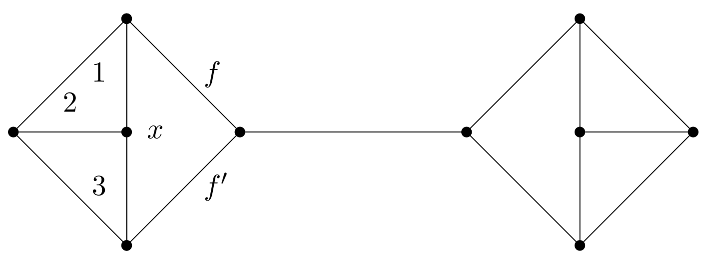
a) Construct (and draw a picture of it) any 3-regular simple graph G such that G is not 3-edge-colourable.

b) Then make a graph G' by subdividing an arbitrary edge of your G, and answer correctly whether this G' is 3-edge-colourable.

c) Decide whether, for some other graph G as in (a), the correct answer to part (b) could be different from your current (assuming correct again) answer in (b).

You must provide mathematical proofs of all your answers to receive points.

Solution:
a) One possible solution is:



To prove that this graph is not 3-edge-colourable, notice that up to symmetry, the edges incident with x are coloured $1, 2, 3$. If only three colours were used in the whole graph, then, necessarily, both edges f, f' would have to receive colour 2, a contradiction.

b) No. This is better answered by (c).

c) Nobody was able to solve this part, even though it is not difficult. We actually prove a stronger statement. Let G be any 3-regular graph, then G has an even number $2k$ of vertices, and $3k$ edges. If G' results from G by subdividing an arbitrary (one) edge, then G' has $2k+1$ vertices and $3k+1$ edges. However, at most $\lfloor (2k+1)/2 \rfloor = k$ edges of G' may receive the same colour, which totals to $3k < 3k+1$ edges. Therefore, G' cannot be 3-edge-colourable.

The points were given at most $7+6+7$, but nobody received points for part (c). Parts (a) and (b) were not so bad.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 4 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0 1 2 3 4 5 6 7 8 9

Problem 4
20 points

An edge e is a *bridge* in a connected graph G if $G - e$ (removal of e) is not connected.

a) Prove that any connected graph on n vertices has at most $n - 1$ bridges.

b) Can a connected simple graph on n vertices have exactly $n - 2$ bridges? Prove it.

<u>*Solution*</u>:
The both statements are quite obviuous (in (b) the obvious answer is NO), and so the main task in this problem is to formulate rigorous mathematical proofs. There are two common ways to approach this task, both have appeared frequently among the solutions.

a) Since G is connected, there is a spanning tree $T \subseteq G$. Every edge $e \in E(G) \setminus E(T)$ makes a cycle with T, and hence e cannot be a bridge. Therefore, the maximum number of bridges can be $|E(T)| = n - 1$, and equality occurs iff $G = T$.
Another approach: Let $X \subseteq E(G)$ be the set of all bridges in G. In the subgraph $G - X$, we "shrink" every connected component (2-connected, btw.) into a new vertex, removing all edges not in X. The resulting connected graph G' has $\leq n$ vertices and X as the edge set, and moreover, G' has no cycles since otherwise the edges of a cycle would not be bridges. Therefore, G' is a tree and $|X| = |V(G')| - 1 \leq n - 1$.

b) Considering the second approach, with G', we see that $|X| = n - 2$ implies $|V(G')| = |V(G)| - 1$, and so exactly one component C of $G - X$ has two vertices (others have one). Since G is simple, however, $C \simeq K_2$, and the only edge of C is a bridge, too. This contradicts the assumption that all bridges are in X.

Besider several students who did not understand the definition of a bridge, the main problem of many solutions was the following: Students might have had a correct idea (similar to one of the above two) in their minds, but they were not able to formulate it on the paper. Since the points were given for mathematical proofs, such problematic solutions received usually only a few points.