Jméno: **Místnost:** **Souřadnice:**

list *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte.

0 1 2 3 4 5 6 7 8 9

Problem 1
20 points

A *black-white* graph is a simple graph such that each of its vertices is either black or white (this is not a proper colouring, two black or two white vertices easily may be neighbours). Two black-white graphs are isomorphic if there is a graph isomorphism between them which takes white vertices onto white vertices and black vertices onto black vertices.

Find all pairwise non-isomorphic black-white cycles of length 5. To receive points, your solution must be systematic, so that it can be easily seen how you have constructed the graphs and why your list is complete with no repetitions.

<u>*Solution*</u>:
This turn out to be quite easy problem, most students succeeded with it.
The problem is symmetric in the colours, and so we just count 0,1, or 2 black vertices, and then multiply the result by 2. There is one 5-cycle with only white vertices, and one with single black. For two black vertices, the two nonisomorphic possibilitites are "neighbours" and "at distance two". Altogether 4 choices, and total $4 \cdot 2 = 8$ solutions.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 2 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0123456789

Problem 2
20 points

Your task is to construct several graphs having properties specified below. You should draw your constructed graphs, and shortly and clearly argue why they possess the required properties.

a) Construct two non-isomorphic trees on 4 vertices.

b) Construct two planar graphs having the same number of vertices, but different numbers of faces.

c) Construct a planar graph on 10 vertices such that every two of its vertices have distance at most 2, and that this graph has no triangles.

d) Construct a 3-connected graph with no triangles which does not contain a Hamiltonian cycle (i.e. no cycle through all the vertices as a subgraph).

Solution:

a) Really trivial...

b) Again very easy, many possible solutions such as, for instance, C_4 and K_4.

c) Simply draw the "star" $K_{1,9}$.

d) Say, the graph $K_{3,4}$. Draw this yourself, and think about a proof of the claimed property.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 3 *učo* *body*

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Problem 3
20 points

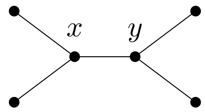
A *dominating set* in a graph G is a set $D \subseteq V(G)$ such that every vertex of G outside of D has a neighbour in D. An *independent set* $I \subseteq V(G)$ is such that no two vertices of I are neighbours. You have to answer the following questions, and provide mathematical proofs for the answers (no points will be given without proofs).

a) (*5 points only*) Construct (and draw) an example of a graph H such that H contains a dominating set smaller than the smallest independent dominating set in H (i.e. a set which is dominating and independent at the same time).

b) Decide (and prove!) whether the following claim is true: Every simple graph contains some independent dominating set.

<u>*Solution*</u>:

a) Join two paths of length 2 by an edge connecting their middle vertices.



Then $D = \{x, y\}$ is dominating, but not independent. The smallest independent dominating set S must not contain both x, y, and so (up to symmetry) $x \notin S$. Then both two leaves adjacent to x must be in S, and S has at lest 3 elements – more than D (or at lest 4 elements in case $x, y \notin S$).

However, many of you lost 1 point for not proving that an independent dominating set must be larger than your D. The solution is not complete without this argument, and loss of only 1 point is quite gentle grading…

b) This is true. A proof is very short – take any maximal (by inclusion or by cardinality) independent set I. Then by the assumed maximality, every vertex x not in I has an edge with some vertex in I (otherwise x could be added to I). The latter claim is actually the definition that I is also dominating.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 4 *učo* *body*

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0123456789

Problem 4
20 points

a) Prove that every simple graph G contains a vertex set $X \subseteq V(G)$ such that, denoting by $F \subseteq E(G)$ the set of all edges of G having one end in X and the other in $V(G) \setminus X$, the following holds

$$|F| \geq \frac{1}{2}|E(G)|\,. \tag{1}$$

b) Decide (and prove) whether the factor $\frac{1}{2}$ from (1) can be improved to $\frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$ (i.e. to claim stronger $|F| \geq (\frac{1}{2} + \varepsilon) \cdot |E(G)|$ in the statement a).

<u>*Solution*</u>:

a) As usual with my "proof problems", a correct solution is very short, but this time also very tricky:

Choose a set X randomly, taking every vertex $v \in V(G)$ with probability $\frac{1}{2}$ ("toss a coin"!). The probability that any particular edge $e = uv \in E(G)$ falls into F is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Hence the expected (average) number of edges falling into F by our random choice of X is exactly $\frac{1}{2}|E(G)|$, and so there must be a (random) instance of F such that $|F| \geq \frac{1}{2}|E(G)|$.

Explicit construction of F is not required…

b) For this part, it is enough to take the complete graph K_{2n} on an even number of vertices, and count that the largest F for it has exactly n^2 edges, as opposite to $n(2n-1) = 2n^2 - n$ edges of the whole graph. The limit of their fraction is $\frac{1}{2}$, and so no constant higher than $\frac{1}{2}$ could be true (1) for all graphs.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 1 *učo* *body*

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Problem 1
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Find all pairwise non-isomorphic black-white paths of length 3. To receive points, your solution must be systematic, so that it can be easily seen how you have constructed the graphs and why your list is complete with no repetitions.

<u>*Solution*</u>:
This turn out to be quite easy problem, most students succeeded with it.
There is one 3-path with only white vertices, and two 3-paths with single black vertex – “end” and “middle”. For 3-paths with two white and two black vertices, the choices are 4 as can be checked in a picture using obvious symmetries. Then the cases of no or single white vertex are analogous, and the total number of solutions is $1 + 2 + 4 + 2 + 1 = 10$.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 2 *učo* *body*

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Problem 2
20 points

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a) Construct two non-isomorphic trees on 4 vertices.

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c) Construct a planar graph on 10 vertices such that every two of its vertices have distance at most 2, and that this graph is 3-connected.

d) Construct a connected 3-regular graph which does not contain a Hamiltonian cycle (i.e. no cycle through all the vertices as a subgraph).

Solution:

a) Really trivial. . .

b) Again very easy, many possible solutions such as, for instance, C_3 and C_4.

c) Simply draw the "wheel" W_9 – a 9-cycle with an extra vertex adjacent to everything.

d) Say, the graph obtained from two copies of the "cube" by subdividing one edge of each, and then joining the two subdivision vertices. Draw this yourself, and think about a proof of the claimed property.

Jméno: **Místnost:** **Souřadnice:**

0007 *list* 3 *učo* *body*

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Problem 3
20 points

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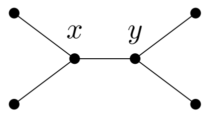

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Jméno: **Místnost:** **Souřadnice:**

0007 *list* 4 *učo* *body*

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0 1 2 3 4 5 6 7 8 9

Problem 4
20 points

a) Prove that every simple graph G contains a vertex set $X \subseteq V(G)$ such that, denoting by $F \subseteq E(G)$ the set of all edges of G having one end in X and the other in $V(G) \setminus X$, the following holds

$$|F| \geq \frac{1}{2}|E(G)|\,. \tag{2}$$

b) Decide (and prove) whether the factor $\frac{1}{2}$ from (2) can be improved to $\frac{1}{2}+\varepsilon$ for any $\varepsilon > 0$ (i.e. to claim stronger $|F| \geq (\frac{1}{2}+\varepsilon)\cdot|E(G)|$ in the statement a).

<u>*Solution*</u>:

a) As usual with my "proof problems", a correct solution is very short, but this time also very tricky:

Choose a set X randomly, taking every vertex $v \in V(G)$ with probability $\frac{1}{2}$ ("toss a coin"!). The probability that any particular edge $e = uv \in E(G)$ falls into F is $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Hence the expected (average) number of edges falling into F by our random choice of X is exactly $\frac{1}{2}|E(G)|$, and so there must be a (random) instance of F such that $|F| \geq \frac{1}{2}|E(G)|$.

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