Místnost: D1 **Souřadnice:**

0007 *list* 1 *učo* *body* 20

Oblast strojově snímatelných informací. Své UČO vyplňte zleva dle přiloženého vzoru číslic. Jinak do této oblasti nezasahujte. 0 1 2 3 4 5 6 7 8 9

Problem 1
20 points

We denote by $G_n(a,b)$ the simple undirected n-vertex 4-regular graph defined as follows:

- $V(G_n(a,b)) = \{0,1,\ldots,n-1\}$, and
- $E(G_n(a,b)) = \{\, ij : i,j \in V(G) \wedge (\, j \equiv i + a \mod n \vee j \equiv i + b \mod n)\,\}$.

For example, $G_5(1,2) = G_5(1,3)$ is the graph K_5.

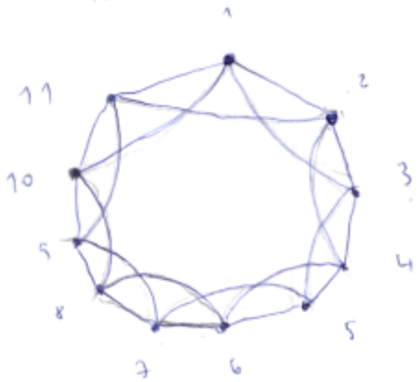
(Neformálně řečeno, graf $G_n(a,b)$ vznikne tak, že n vrcholů je nakresleno "do kruhu" a hrany se nakreslí tak, že každá hrana "přeskakuje" $a-1$ nebo $b-1$ po sobě jdoucích vrcholů.)

Among the following three graphs on 11 vertices

$$\overset{A}{G_{11}(1,2)}, \quad \overset{B}{G_{11}(2,4)}, \quad \overset{C}{G_{11}(1,3)},$$

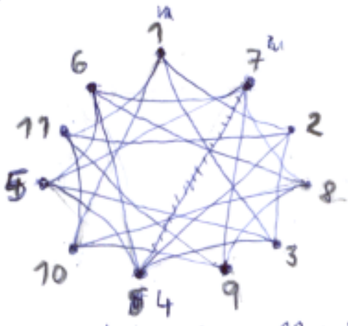
find all isomorphic pairs and prove your answer (either show an isomorphism, or argue about a difference).

$A = G_{11}(1,2)$



- contains $\triangle$ (C_3) 11×

$B = G_{11}(2,4)$



- contains C_3 11× ~~[illegible]~~

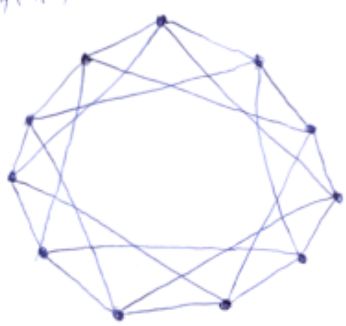
$A \not\simeq C$ because A contains C_3 and C doesn't

~~A~~C $\not\simeq$ B because B contains C_3 and C doesn't

~~[illegible]~~

$A \simeq B$ because ~~[illegible]~~ I found an isomorphic mapping of vertices from A to B.

$C = G_{11}(1,3)$



- does not contain C_3

Místnost: D1 **Souřadnice:**

0007 *list* 2 *učo* *body* 20

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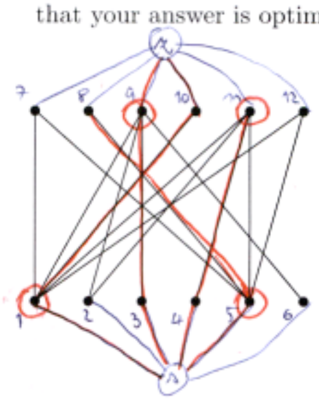
Problem 2
20 points

a) Define what is an independent set *(nezávislá množina)* and what is a vertex cover *(vrcholové pokrytí)* in a simple undirected graph. G

Independent set - is a set of vertices $X \subseteq V(G)$, such that no edge of G has both ends in X

Vertex cover - is a set $C \subseteq V(G)$, ~~every~~ such that every edge of G is incident with a vertex of C, G - C has no edges.

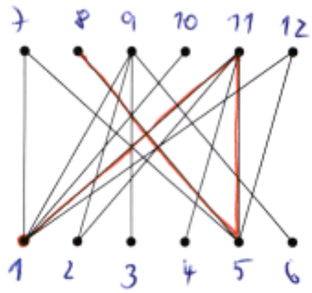
b) In the following graph on 12 vertices, find and mark a smallest vertex cover, and argue that your answer is optimal.



In bipartite graphs |maximum matching| = |vertex cover|

- The maximal matching has size 4 therefore minimum vertex cover is also size 4, as it contains vertices 1, 5, 9, 11
- Vertex cover is a complement to largest independent set which has size 8. 12 - 8 = 4

c) What is the radius *(poloměr)* of this graph? Show your answer in the picture, too.



excentricities:

1 - 3 (8)
2 - 3 (7)
3 - 5 (8)
4 - 4 (6)
5 - 4 (6)
6 - 5 (8)

7 - 3 (2)
8 - 5 (3)
9 - 4 (8)
10 - 4 (8)
11 - 3 (3)
12 - 3 (2) → vertex to which is the largest distance

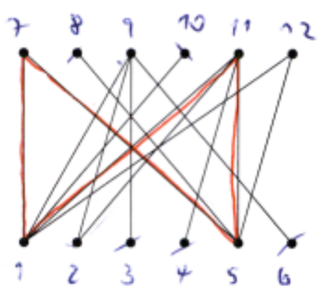
vertex in the graph

excentricity

radius = 3

I showed ~~that~~ in the graph, that ~~distance~~ excentricity of vertex 1 is 3.

d) Find and highlight in this graph a longest induced cycle *(nejdelší induk. kružnici)*:



- We can delete ~~edges~~ vertices of degree 1, they won't be a the cycle
- the cycle is 7, 1, 11, 5; we cannot add any more vertices to the cycle, because it would not be induced
- if we added 9, 3 to cycle then there must be (1, 11) edge - contradiction to induced cycle

Místnost: D1 **Souřadnice:**

0007 *list* 3 *učo* *body* 1

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Problem 3
20 points

A digraph (directed graph) D is an *orientation* of a simple graph G if

- $V(G) = V(D)$ and $E(G) = \{\{u, v\} : (u, v) \in E(D)\}$,
- there are no $u, v \in V(D)$ such that both (u, v), $(v, u) \in E(D)$.

(Informally, we give a direction to each edge of G.)
A vertex v in a digraph D is called a *sink* if the outdegree of v is 0, and v is called a *source* if the indegree of v is 0 in D.

a) Prove that in every orientation of a tree there is a sink and a source.

b) Consider any orientation of a path (of arbitrary length), and denote by c the number of sinks and by d the number of sources in this orientation. Examine how large can be the difference $|c - d|$, and prove your answer.

(V orientovaném grafu je "sink" ten vrchol, ze kterého nevycházejí šipky, a "source" ten vrchol, do kterého nepřicházejí šipky. a) Dokažte, že orientace stromu má vždy source i sink. b) Jak velký může být rozdíl (v abs. hodnotě) počtů sources a sinks na orientaci cesty?) You may write in Czech here.

Místnost: D1 Souřadnice:

0007 list 4 učo body 0

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Problem 4
20 points

For a simple (undirected) graph G, we call a *square of* G the graph G^2 defined as follows

- $V(G^2) = V(G)$, and
- $E(G^2) = \{uv : u \neq v \in V(G) \wedge d_G(u, v) \leq 2\}$.

For example, the square of P_3 is K_4 minus an edge, and the square of C_5 is K_5.

You task is to prove that there exist functions g, f such that, for every simple graph G, the following hold:

- for any integer d, if (max. deg.) $\Delta(G) \leq d$ then (chromatic n.) $\chi(G^2) \leq g(d)$;
- for any integer c, if $\chi(G^2) \leq c$ then $\Delta(G) \leq f(c)$.

(Druhou mocninou grafu G je graf G^2 takový, že hrany G^2 spojují ty vrcholy G jež byly ve vzdálenosti ≤ 2 v G. Úkolem je dokázat, neformálně řečeno, že pokud má G omezený max. stupeň, tak G^2 má omezenou barevnost a naopak.)

Only rigorous mathematical proofs will be rewarded with points. You may write in Czech here.