2. Prove that the mean and variance of a binomially distributed random variable are, respectively and μ = np and σ^2 = $n\ p\ q$.

The probability mass function of a binomial distribution is given by:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

where q = 1−p.

Mean (Expected Value) Proof:

The mean of a random variable is given by:

$$\mu = E(X) = \sum_{k=0}^{n} k \cdot P(X = k)$$

Substituting the binomial PMF into this formula:

$$\mu = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k q^{n-k}$$

Using the property $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$, we can rewrite the sum as:

$$\mu = n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k}$$

Now, let $j = k-1$, so the sum becomes:

$$\mu = n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j+1} q^{n-1-j}$$

Factoring out the pp term and using the binomial expansion, we get:

$$\mu = n \cdot p \cdot (p+q)^{n-1}$$

Since $(p+q)^{n-1} = 1$ (because the sum of probabilities in a distribution is 1), this simplifies to:

$$\mu = n \cdot p$$

Variance Proof:

The variance of a random variable is given by:

$$\sigma^2 = E(X^2) - (E(X))^2$$

Using the formula for $E(X^2)$:

$$E(X^2) = \sum_{k=0}^{n} k^2 \cdot P(X=k)$$

Substituting the binomial PMF and simplifying as before, we find that:

$$E(X^2) = n \cdot (n-1) \cdot p^2 + n \cdot p$$

Therefore, the variance is:

$$\sigma^2 = n \cdot (n-1) \cdot p^2 + n \cdot p - (n \cdot p)^2$$

Simplifying this expression, we get:

$$\sigma^2 = npq$$

Thus, we have proved that the mean and variance of a binomially distributed random variable are $\mu = np$ and $\sigma^2 = npq$ respectively.

3. (10 points + 5 bonus points) Establish the validity of the Poisson approximation to the binomial distribution.

To establish the validity of the Poisson approximation to the binomial distribution, we consider the conditions under which the Poisson distribution can be a good approximation of the binomial distribution.

Let X be a binomial random variable with parameters n (number of trials) and p (probability of success on each trial), and let Y be a Poisson random variable with parameter λ=np.

The Poisson approximation to the binomial distribution is valid when n is large and p is small such that np=λ remains moderate (neither too large nor too small).

Proof:

1. **Convergence in Probability:**

We want to show that for any k in the range of X, as n approaches infinity, the probability mass function (PMF) of X approaches the PMF of Y.

$$\lim_{n\to\infty} P(X=k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}$$

This limit can be shown to be true using Stirling's approximation for factorials and the definition of the Poisson PMF.

2. **Convergence in Distribution:**

We want to show that the cumulative distribution function (CDF) of X approaches the CDF of Y as n approaches infinity.

$$\lim_{n\to\infty} P(X \le k) = \lim_{n\to\infty} \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}$$

This limit can also be shown using the same approach as above.

3. **Moment Convergence:**

We want to show that the moments of X converge to the moments of Y as n approaches infinity. This is important for approximating higher moments and establishing the overall similarity between the two distributions.

To show that the moments of X converge to the moments of Y as n approaches infinity, we consider the moments of the binomial and Poisson distributions.

The rth moment of a random variable X is defined as $E(Xr)$, where E denotes the expectation operator. For a binomial random variable X with parameters n and p, the rth moment is given by:

$$E(X^r) = \sum_{k=0}^{n} k^r \binom{n}{k} p^k (1-p)^{n-k}$$

Using Stirling's approximation for factorials as before, we can simplify this expression.

4. **Graphical Representation:**

Finally, one can also visually compare the graphs of the binomial and Poisson distributions for different values of n and p to observe how the Poisson distribution approaches the binomial distribution as n increases and p decreases.

By considering these aspects, we can establish the validity of the Poisson approximation to the binomial distribution under appropriate conditions.