Seminar Problems & Solutions

Woodson High School Math Club

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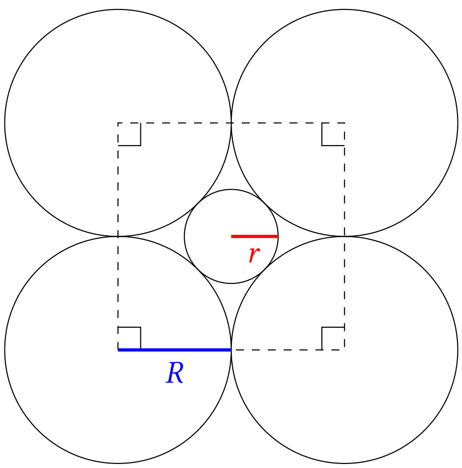
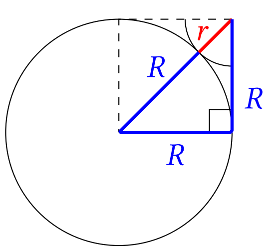
October 4, 2023

1. Find r in terms of R in the geometric figure below.

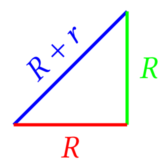


Solution: $\boxed{r = R(\sqrt{2} - 1)}$

If you zoom in to the bottom left circle, you can manipulate the radii to observe the following:



You may notice we get a right triangle of the form:



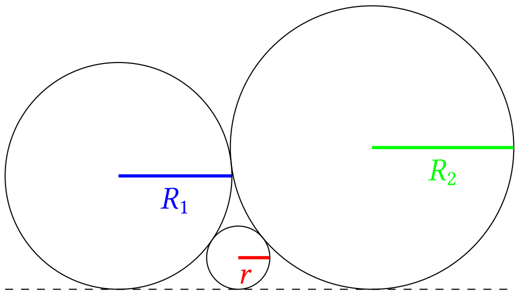
Using the Pythagorean Theorem, we have

$$(R + r)^2 = R^2 + R^2.$$

Algebraic manipulation yields the solution.

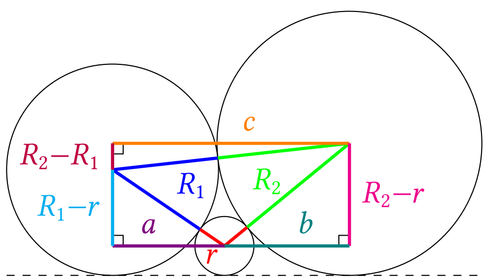
$$\begin{aligned} (R + r)^2 &= R^2 + R^2 \\ &= 2R^2 \\ R + r &= \sqrt{2}R \\ r &= \sqrt{2}R - R \\ &= R(\sqrt{2} - 1) \end{aligned}$$

2. Find r in terms of R_1 and R_2.



Solution: $\boxed{r = \left(\frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}}\right)^{-2}}$

Firstly, connect the radii to form the following triangle inscribed in a rectangle:



We then notice that we need to solve for a, b, and c to complete the rectangle. Since the corners of the rectangle form right triangles, we can use the Pythagorean Theorem.

$$\begin{aligned} (R_1 + r)^2 &= a^2 + (R_1 - r)^2 \\ a &= \sqrt{(R_1 + r)^2 - (R_1 - r)^2} \\ &= 2\sqrt{R_1 r} \\ b &= 2\sqrt{R_2 r} \\ c &= 2\sqrt{R_1 R_2} \end{aligned}$$

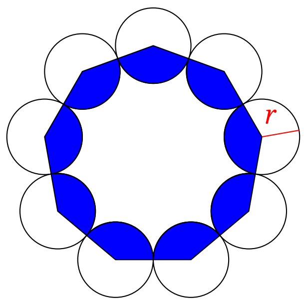
We also notice that

$$c = a + b.$$

If we plug in our previous values, then we get our answer after doing some algebra:

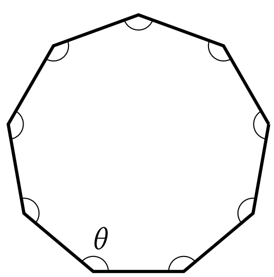
$$\begin{aligned} 2\sqrt{R_1 R_2} &= 2\sqrt{R_1 r} + 2\sqrt{R_2 r} \\ \frac{2\sqrt{R_1 R_2}}{2\sqrt{R_1 R_2 r}} &= \frac{2\sqrt{R_1 r}}{2\sqrt{R_1 R_2 r}} + \frac{2\sqrt{R_2 r}}{2\sqrt{R_1 R_2 r}} \\ \frac{1}{\sqrt{r}} &= \frac{1}{\sqrt{R_2}} + \frac{1}{\sqrt{R_1}} \\ \frac{1}{r} &= \left(\frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}}\right)^2 \\ r &= \left(\frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}}\right)^{-2} \end{aligned}$$

3. Consider the closed chain of 9 circles with equal radii below. The area shaded in blue is formed by connecting the circles' centers. What is the difference of the blue area from the outer white area in terms of a single circle's radius r?

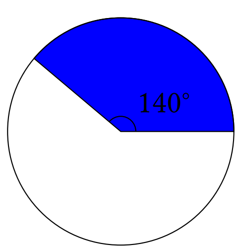


Solution: $\boxed{Area\,\square - Area\,\blacksquare = 2\pi r^2}$

First, notice that the shaded area plus the negative space inside forms a nanogan.



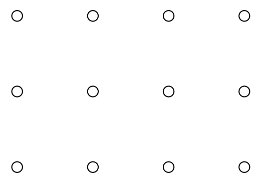
In order to find each of the angles of a given regular polygon, you can use the formula $\theta = 180°(n-2)/n$, where n is the number of sides in the polygon. In our case $n = 9$, so $\theta = 140°$.



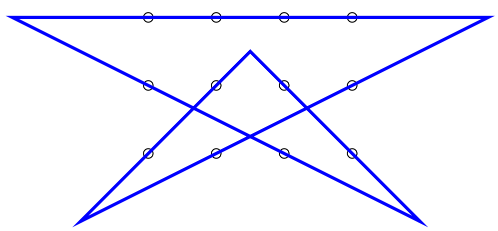
The shaded area of one of the circles is merely the area of a sector, or $S = \theta/360°\pi r^2$. Also notice that each of the sectors is $\frac{1}{9}$ of the total blue area, $Area\,\blacksquare$. Using the fact that $\theta = 140°$ as determined earlier and $r = r$, we find $\frac{1}{9}Area\,\blacksquare = \frac{7}{18}\pi r^2$. The rest of the area of the circle is just $\frac{1}{9}Area\,\square$ which is $\pi r^2 - \frac{1}{9}Area\,\blacksquare$, which simplifies to $\frac{11}{18}\pi r^2$.

$$\begin{aligned} Area\,\square - Area\,\blacksquare &= 9\left(\tfrac{11}{18}\pi r^2 - \tfrac{7}{18}\pi r^2\right) \\ &= 2\pi r^2 \end{aligned}$$

4. Draw 5 straight line segments forming a closed loop that pass through all the points below.



Solution:

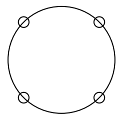


5. Each of the letters in the base-10 addition below represent a digit. What is $A+W+A+Y$?

$$\begin{array}{r} 8\,7\,5 \\ +\,8\,7\,5 \\ +\,8\,7\,5 \\ \hline 2\,6\,2\,5 \end{array}$$

Solution: To be written.

6. A circle covers 4 points in a 2×2 square:



Cover the 25 points in the 5×5 square below with 5 circles.

Solution: To be written.

7. There were n people in a room, with names $A_1, A_2, \dots, A_n$, where each person A_m said at least m of them were lying:

A_1: At least 1 of us are lying.
A_2: At least 2 of us are lying.
$\vdots$ $\vdots$
A_{n-1}: At least $n-1$ of us are lying.
A_n: At least n of us are lying.

Eve, Felix, Glena and Henry each made a statement:

Eve: There were 4 people in the room.
Felix: There were 5 people in the room.
Glena: There were 9 people in the room.
Henry: There were 14 people in the room.

Who are lying?

Solution: To be written.