Universidade do Minho MIECiv

Departamento de Matemática

# Análise Matemática para Engenharia

folha de exercícios 3 — 2021/2022

**Queiram por favor informar-me caso detetem algum erro ou gralha.**

- **Derivadas parciais**

**1.** Usando a definição de derivada parcial, determine

(a) Seja $f(x,y) = x^2y$. Então

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0,$$
$$f_y(1,2) = \lim_{h\to 0} \frac{f(1,2+h)-f(1,2)}{h} = \lim_{h\to 0} \frac{2+h-2}{h} = \lim_{h\to 0} 1 = 1.$$

(b) Seja $f(x,y) = \begin{cases} \frac{x^3y}{x^2+y^2} & \text{se} \quad (x,y)\neq(0,0) \\ 0 & \text{se} \quad (x,y)=(0,0) \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3\times 0}{h^2+0^2}-0}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^3\times h}{0^2+h^2}-0}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

(c) Seja $f(x,y) = \begin{cases} \frac{xy}{x+y} & \text{se} \quad x+y\neq 0 \\ x & \text{se} \quad x+y=0 \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\times 0}{h+0}-0}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0, \qquad f(h,0) = \frac{h\times 0}{h+0}\text{pq}h+0\neq 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0\times h}{0+h}-0}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0, \qquad f(0,h) = \frac{0\times h}{0+h}\text{pq}0+h\neq 0.$$

**2.** Determine as derivadas parciais de primeira ordem de $f : D\subset\mathbb{R}^n \longrightarrow \mathbb{R}$ dadas por:

(a) $f(x,y) = x^3y+7x^2-2y^3-1$;

$$f_x(x,y) = 3x^2y+14x$$
$$f_y(x,y) = x^3-6y^2$$

(b) $f(x,y) = \dfrac{3x+y^2}{7x+y}$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{3x+y^2}{7x+y}\right) = \frac{\frac{\partial}{\partial x}\left(3x+y^2\right)(7x+y)-(3x+y^2)\frac{\partial}{\partial x}(7x+y)}{(7x+y)^2} = \frac{3(7x+y)-(3x+y^2)7}{(7x+y)^2}$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{3x+y^2}{7x+y}\right) = \frac{\frac{\partial}{\partial y}\left(3x+y^2\right)(7x+y)-(3x+y^2)\frac{\partial}{\partial y}(7x+y)}{(7x+y)^2} = \frac{2y(7x+y)-(3x+y^2)1}{(7x+y)^2}$$

(c) $f(x,y) = \sin\left(1+e^{xy}\right)$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\sin\left(1+e^{xy}\right)\right) = \frac{\partial}{\partial x}\left(1+e^{xy}\right)\cos\left(1+e^{xy}\right) = 0+\frac{\partial}{\partial x}(xy)\,e^{xy}\cos\left(1+e^{xy}\right) = ye^{xy}\cos\left(1+e^{xy}\right)$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(\sin\left(1+e^{xy}\right)\right) = \frac{\partial}{\partial y}\left(1+e^{xy}\right)\cos\left(1+e^{xy}\right) = 0+\frac{\partial}{\partial y}(xy)\,e^{xy}\cos\left(1+e^{xy}\right) = xe^{xy}\cos\left(1+e^{xy}\right)$$

(d) $f(x,y) = (x^3 - y^2)^2$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left((x^3-y^2)^2\right) = 2(x^3-y^2)\frac{\partial}{\partial x}\left(x^3-y^2\right) = 2(x^3-y^2)\frac{\partial}{\partial x}\left(3x^2\right)$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left((x^3-y^2)^2\right) = 2(x^3-y^2)\frac{\partial}{\partial y}\left(x^3-y^2\right) = 2(x^3-y^2)\frac{\partial}{\partial x}\left(-2y\right)$$

(e) $f(x,y) = xe^y + y \operatorname{sen} x$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(xe^y + y\operatorname{sen} x\right) = \frac{\partial}{\partial x}\left(xe^y\right) + \frac{\partial}{\partial x}\left(y\operatorname{sen} x\right) = e^y\frac{\partial}{\partial x}\left(x\right) + y\frac{\partial}{\partial x}\left(\operatorname{sen} x\right) = e^y + y\cos x$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(xe^y + y\operatorname{sen} x\right) = \frac{\partial}{\partial y}\left(xe^y\right) + \frac{\partial}{\partial y}\left(y\operatorname{sen} x\right) = x\frac{\partial}{\partial y}\left(e^y\right) + \operatorname{sen} x\frac{\partial}{\partial y}\left(y\right) = xe^y + \operatorname{sen} x$$

(f) $f(s,t) = e^s \ln(st)$;

$$f_s(s,t) = \frac{\partial}{\partial s}\left(e^s\ln(st)\right) = \frac{\partial}{\partial s}\left(e^s\right)\ln(st) + e^s\frac{\partial}{\partial s}\left(\ln(st)\right) = e^s\ln(st) + e^s\frac{\frac{\partial}{\partial s}\left(st\right)}{st} = e^s\ln(st) + e^s\frac{t}{st}$$
$$f_t(s,t) = \frac{\partial}{\partial t}\left(e^s\ln(st)\right) = e^s\frac{\partial}{\partial t}\left(\ln(st)\right) = e^s\frac{\frac{\partial}{\partial t}\left(st\right)}{st} = e^s\frac{s}{st}$$

(g) $f(x,y) = x\cos\frac{x}{y}$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x\cos\frac{x}{y}\right) = \frac{\partial}{\partial x}\left(x\right)\cos\frac{x}{y} + x\frac{\partial}{\partial x}\left(\cos\frac{x}{y}\right) = 1\cos\frac{x}{y} + x\frac{\partial}{\partial x}\left(\frac{x}{y}\right)\sin\frac{x}{y} = 1\cos\frac{x}{y} + x\frac{1}{y}\sin\frac{x}{y}$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(x\cos\frac{x}{y}\right) = x\frac{\partial}{\partial y}\left(\cos\frac{x}{y}\right) = x\frac{\partial}{\partial y}\left(\frac{x}{y}\right)\sin\frac{x}{y} = x\left(-\frac{x}{y^2}\right)\sin\frac{x}{y}$$

(h) $f(x,y) = e^{2xy^3}$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(e^{2xy^3}\right) = \frac{\partial}{\partial x}\left(2xy^3\right)e^{2xy^3} = 2y^3e^{2xy^3}$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(e^{2xy^3}\right) = \frac{\partial}{\partial y}\left(2xy^3\right)e^{2xy^3} = 6xy^2e^{2xy^3}$$

(i) $f(x,y) = xe^{\sqrt{xy}}$;

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\left(xe^{\sqrt{xy}}\right) = \frac{\partial}{\partial x}\left(x\right)e^{\sqrt{xy}} + x\frac{\partial}{\partial x}\left(e^{\sqrt{xy}}\right) = 1e^{\sqrt{xy}} + x\frac{\partial}{\partial x}\left(\sqrt{xy}\right)e^{\sqrt{xy}} \\ &= e^{\sqrt{xy}} + x\frac{1}{2}(xy)^{\frac{1}{2}-1}\frac{\partial}{\partial x}\left(xy\right)e^{\sqrt{xy}} = e^{\sqrt{xy}} + x\frac{1}{2}(xy)^{\frac{1}{2}-1}ye^{\sqrt{xy}} = e^{\sqrt{xy}} + \frac{1}{2}(xy)^{\frac{1}{2}}e^{\sqrt{xy}} \\ f_y(x,y) &= \frac{\partial}{\partial y}\left(xe^{\sqrt{xy}}\right) = x\frac{\partial}{\partial y}\left(e^{\sqrt{xy}}\right) = x\frac{\partial}{\partial y}\left(\sqrt{xy}\right)e^{\sqrt{xy}} \\ &= x\frac{1}{2}(xy)^{\frac{1}{2}-1}\frac{\partial}{\partial y}\left(xy\right)e^{\sqrt{xy}} = x\frac{1}{2}(xy)^{\frac{1}{2}-1}xe^{\sqrt{xy}} = \frac{1}{2}x^2(xy)^{\frac{-1}{2}}e^{\sqrt{xy}} \end{aligned}$$

(j) $f(x,y) = x^y$;

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^y\right) = yx^{y-1}\frac{\partial}{\partial x}\left(x\right) = yx^{y-1}$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^y\right) = \frac{\partial}{\partial y}\left(y\right)x^y\ln x = x^y\ln x$$

(k) $f(x,y,z) = xe^{xy}\,\mathsf{sin}(yz)$;

$$
\begin{aligned}
f_x(x,y,z) &= \frac{\partial}{\partial x}\left(xe^{xy}\,\mathsf{sin}(yz)\right) = \mathsf{sin}(yz)\frac{\partial}{\partial x}\left(xe^{xy}\right)\mathsf{sin}(yz)\left[\frac{\partial}{\partial x}\left(x\right)e^{xy} + x\frac{\partial}{\partial x}\left(e^{xy}\right)\right] \\
&= \mathsf{sin}(yz)\left[\frac{\partial}{\partial x}\left(x\right)e^{xy} + x\frac{\partial}{\partial x}\left(xy\right)e^{xy}\right] = \mathsf{sin}(yz)\left[\frac{\partial}{\partial x}\left(x\right)e^{xy} + xye^{xy}\right] \\
f_y(x,y,z) &= \frac{\partial}{\partial y}\left(xe^{xy}\,\mathsf{sin}(yz)\right) = x\frac{\partial}{\partial y}\left(\mathsf{sin}(yz)e^{xy}\right) = x\left[\frac{\partial}{\partial y}\left(\mathsf{sin}(yz)\right)e^{xy} + \mathsf{sin}(yz)\frac{\partial}{\partial y}\left(e^{xy}\right)\right] \\
&= x\left[\frac{\partial}{\partial y}\left(yz\right)\mathsf{cos}(yz)e^{xy} + \mathsf{sin}(yz)\frac{\partial}{\partial y}\left(xy\right)e^{xy}\right] = x\left[z\,\mathsf{cos}(yz)e^{xy} + \mathsf{sin}(yz)xe^{xy}\right] \\
f_z(x,y,z) &= \frac{\partial}{\partial z}\left(xe^{xy}\,\mathsf{sin}(yz)\right) = xe^{xy}\frac{\partial}{\partial z}\left(\mathsf{sin}(yz)\right) \\
&= xe^{xy}\frac{\partial}{\partial z}\left(yz\right)\mathsf{cos}(yz) = xe^{xy}y\,\mathsf{cos}(yz)
\end{aligned}
$$

(l) $f(x,y,z) = xyze^{xyz}$;

$$
\begin{aligned}
f_x(x,y,z) &= \frac{\partial}{\partial x}\left(xyze^{xyz}\right) = yz\frac{\partial}{\partial x}\left(xe^{xyz}\right)yz\left[\frac{\partial}{\partial x}\left(x\right)e^{xyz} + x\frac{\partial}{\partial x}\left(e^{xyz}\right)\right] \\
&= yz\left[e^{xyz} + x\frac{\partial}{\partial x}\left(xyz\right)e^{xyz}\right] = yz\left[e^{xyz} + xyze^{xyz}\right] \\
f_y(x,y,z) &= \frac{\partial}{\partial y}\left(xyze^{xyz}\right) = xz\frac{\partial}{\partial y}\left(ye^{xyz}\right) = xz\left[\frac{\partial}{\partial y}\left(y\right)e^{xyz} + y\frac{\partial}{\partial y}\left(e^{xyz}\right)\right] \\
&= xz\left[e^{xyz} + yxze^{xyz}\right] \\
f_z(x,y,z) &= \frac{\partial}{\partial z}\left(xyze^{xyz}\right) = xy\frac{\partial}{\partial y}\left(ze^{xyz}\right) = xy\left[\frac{\partial}{\partial y}\left(z\right)e^{xyz} + z\frac{\partial}{\partial y}\left(e^{xyz}\right)\right] \\
&= xy\left[e^{xyz} + zxye^{xyz}\right]
\end{aligned}
$$

(m) $f(x,y,z) = \mathsf{ln}(1 + x + y^2 + z^3)$;

$$
\begin{aligned}
f_x(x,y,z) &= \frac{\partial}{\partial x}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right) = \frac{\frac{\partial}{\partial x}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right)}{1 + x + y^2 + z^3} = \frac{1}{1 + x + y^2 + z^3} \\
f_y(x,y,z) &= \frac{\partial}{\partial y}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right) = \frac{\frac{\partial}{\partial y}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right)}{1 + x + y^2 + z^3} = \frac{2y}{1 + x + y^2 + z^3} \\
f_z(x,y,z) &= \frac{\partial}{\partial z}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right) = \frac{\frac{\partial}{\partial z}\left(\mathsf{ln}(1 + x + y^2 + z^3)\right)}{1 + x + y^2 + z^3} = \frac{3z^2}{1 + x + y^2 + z^3}
\end{aligned}
$$

(n) $f(r,u,v) = 1 + u + v - \mathsf{sen}(r^2)$;

$$
\begin{aligned}
f_r(r,u,v) &= \frac{\partial}{\partial x}\left(1 + u + v - \mathsf{sen}(r^2)\right) = -\frac{\partial}{\partial x}\left(r^2\right)\mathsf{cos}(r^2) = -2r\,\mathsf{cos}(r^2) \\
f_u(r,u,v) &= \frac{\partial}{\partial x}\left(1 + u + v - \mathsf{sen}(r^2)\right) = 1 \\
f_v(r,u,v) &= \frac{\partial}{\partial v}\left(1 + u + v - \mathsf{sen}(r^2)\right) = 1
\end{aligned}
$$

(o) $f(x,y,z) = e^x\,\mathsf{sen}(x+y) + \mathsf{cos}(z - 3y)$;

$$
\begin{aligned}
f_x(x,y,z) &= \frac{\partial}{\partial x}\left(e^x\,\mathsf{sen}(x+y) + \mathsf{cos}(z - 3y)\right) = \frac{\partial}{\partial x}\left(e^x\right)\mathsf{sen}(x+y) + e^x\frac{\partial}{\partial x}\left(\mathsf{sen}(x+y)\right) \\
&= e^x\,\mathsf{sin}(x+y) + e^x\,\mathsf{cos}(x+y) \\
f_y(x,y,z) &= \frac{\partial}{\partial y}\left(e^x\,\mathsf{sen}(x+y) + \mathsf{cos}(z - 3y)\right) = e^x\frac{\partial}{\partial y}\left(\mathsf{sen}(x+y)\right) - \frac{\partial}{\partial y}\left(z - 3y\right)\mathsf{sin}(z - 3y) \\
&= e^x\,\mathsf{cos}(x+y) + 3\,\mathsf{sin}(z - 3y) \\
f_z(x,y,z) &= \frac{\partial}{\partial z}\left(e^x\,\mathsf{sen}(x+y) + \mathsf{cos}(z - 3y)\right) = \frac{\partial}{\partial z}\left(\mathsf{cos}(z - 3y)\right) = -\,\mathsf{sin}(z - 3y)
\end{aligned}
$$

(p) $f(m,v,r)=\dfrac{mv^2}{r}$;

$$\begin{aligned} f_m(m,v,r) &= \frac{\partial}{\partial m}\left(\frac{mv^2}{r}\right)=\frac{v^2}{r}\\ f_v(m,v,r) &= \frac{\partial}{\partial v}\left(\frac{mv^2}{r}\right)=\frac{2mv}{r}\\ f_r(m,v,r) &= \frac{\partial}{\partial v}\left(\frac{mv^2}{r}\right)=mv^2\frac{\partial}{\partial v}\left(\frac{1}{r}\right)=-\frac{mv^2}{r^2} \end{aligned}$$

(q) $f(x,y,z)=\ln(e^z+x^y)$;

$$\begin{aligned} f_x(x,y,z) &= \frac{\partial}{\partial x}\left(\ln(e^z+x^y)\right)=\frac{\frac{\partial}{\partial x}\left(e^z+x^y\right)}{e^z+x^y}=\frac{0+yx^{y-1}}{e^z+x^y}\\ f_y(x,y,z) &= \frac{\partial}{\partial y}\left(\ln(e^z+x^y)\right)=\frac{\frac{\partial}{\partial y}\left(e^z+x^y\right)}{e^z+x^y}=\frac{0+\frac{\partial}{\partial x}(y)\,x^y\ln x}{e^z+x^y}=\frac{x^y\ln x}{e^z+x^y}\\ f_z(x,y,z) &= \frac{\partial}{\partial z}\left(\ln(e^z+x^y)\right)=\frac{\frac{\partial}{\partial z}\left(e^z+x^y\right)}{e^z+x^y}=\frac{e^z+0}{e^z+x^y} \end{aligned}$$

**3.** Mostre que a função $f$ definida por

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2} & \text{se} \quad (x,y)\neq(0,0)\\ 0 & \text{se} \quad (x,y)=(0,0)\end{cases}$$

possui derivadas parciais em $(0,0)$, embora seja descontínua nesse ponto.

As funções derivadas parciais estão definidas em $\mathbb{R}^2$ por

$$\begin{aligned} f_x(x,y) &= \begin{cases}\frac{\partial}{\partial x}\left(\frac{2xy}{x^2+y^2}\right) & \text{se} \quad (x,y)\neq(0,0)\\ \lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h} & \text{se} \quad (x,y)=(0,0)\end{cases} = \begin{cases}\frac{2y(x^2+y^2)-4x^2y}{(x^2+y^2)^2} & \text{se} \quad (x,y)\neq(0,0)\\ \lim\limits_{h\to 0}\dfrac{\frac{2h\times 0}{h^2+0^2}-0}{h} & \text{se} \quad (x,y)=(0,0)\end{cases}\\ &= \begin{cases}\frac{2y(x^2+y^2)-4x^2y}{(x^2+y^2)^2} & \text{se} \quad (x,y)\neq(0,0)\\ 0 & \text{se} \quad (x,y)=(0,0)\end{cases}\\ f_y(x,y) &= \begin{cases}\frac{\partial}{\partial y}\left(\frac{2xy}{x^2+y^2}\right) & \text{se} \quad (x,y)\neq(0,0)\\ \lim\limits_{h\to 0}\dfrac{f(0,0+h)-f(0,0)}{h} & \text{se} \quad (x,y)=(0,0)\end{cases} = \begin{cases}\frac{2x(x^2+y^2)-4xy^2}{(x^2+y^2)^2} & \text{se} \quad (x,y)\neq(0,0)\\ \lim\limits_{h\to 0}\dfrac{\frac{2\times 0\times h}{0^2+h^2}-f(0,0)}{h} & \text{se} \quad (x,y)=(0,0)\end{cases}\\ &= \begin{cases}\frac{2x(x^2+y^2)-4xy^2}{(x^2+y^2)^2} & \text{se} \quad (x,y)\neq(0,0)\\ 0 & \text{se} \quad (x,y)=(0,0)\end{cases} \end{aligned}$$

No entanto, a função não é contínua em $(0,0)$, porque o limite da função segundo a reta $y=x$ quando $x$ tende para 0 é

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y)=\lim_{x\to 0}\frac{2x^2}{2x^2}=1\neq f(0,0)=0$$

- **Derivadas parciais de ordem superior à primeira**

**4.** Calcule as derivadas parciais de 2ª ordem das funções seguintes:

(a) $f(x,y)=x^4y^3$

$$f_x=4x^3y^3,\;\; f_y=3x^4y^2,\;\; f_{xx}=12x^2y^3,\;\; f_{yy}=6x^4y,\;\; f_{xy}=\frac{\partial}{\partial y}(4x^3y^3)=12x^3y^2,\;\; f_{yx}=\frac{\partial}{\partial x}(3x^4y^2)=12x^3y^2$$

(b) $f(x,y)=\log(x+y)+\log(x-y)$

$$
\begin{aligned}
f_x &= \frac{1}{x+y} + \frac{1}{x-y}, \quad f_y = \frac{1}{x+y} - \frac{1}{x-y},\\
f_{xx} &= -\frac{1}{(x+y)^2} - \frac{1}{(x-y)^2}, \quad f_{yy} = -\frac{1}{(x+y)^2} - \frac{1}{(x-y)^2},\\
f_{xy} &= -\frac{1}{(x+y)^2} + \frac{1}{(x-y)^2}, \quad f_{yx} = -\frac{1}{(x+y)^2} + \frac{1}{(x-y)^2},
\end{aligned}
$$

(c) $f(x, y, z) = \sin(xyz)$

$$
\begin{aligned}
f_x &= yz\cos(xyz), \quad f_y = xz\cos(xyz), \quad f_z = xy\cos(xyz),\\
f_{xx} &= -y^2z^2\sin(xyz), \quad f_{yy} = -x^2z^2\sin(xyz), \quad f_{zz} = -x^2y^2\sin(xyz),\\
f_{xy} &= \frac{\partial}{\partial y}(yz\cos(xyz)) = z\cos(xyz) + yz\frac{\partial}{\partial y}(\cos(xyz)) = z\cos(xyz) + xyz^2\sin(xyz)\\
f_{xz} &= \frac{\partial}{\partial z}(yz\cos(xyz)) = y\cos(xyz) + yz\frac{\partial}{\partial z}(\cos(xyz)) = z\cos(xyz) + xy^2z\sin(xyz)\\
f_{yz} &= \frac{\partial}{\partial z}(xz\cos(xyz)) = x\cos(xyz) + xz\frac{\partial}{\partial z}(\cos(xyz)) = x\cos(xyz) + x^2yz\sin(xyz)
\end{aligned}
$$

(d) $f(x, y, z) = xe^{yz} + y\ln z$

$$
\begin{aligned}
f_x &= e^{yz}, \quad f_y = xze^{yz} + \ln z, \quad f_z = xye^{yz} + \frac{y}{z},\\
f_{xx} &= 0, \quad f_{yy} = xz^2e^{yz}, \quad f_{zz} = xy^2e^{yz} - \frac{y}{z^2},\\
f_{xy} &= \frac{\partial}{\partial y}(yz\cos(xyz)) = z\cos(xyz) + yz\frac{\partial}{\partial y}(\cos(xyz)) = z\cos(xyz) + xyz^2\sin(xyz)\\
f_{xz} &= \frac{\partial}{\partial z}(yz\cos(xyz)) = y\cos(xyz) + yz\frac{\partial}{\partial z}(\cos(xyz)) = z\cos(xyz) + xy^2z\sin(xyz)\\
f_{yz} &= \frac{\partial}{\partial z}(xz\cos(xyz)) = x\cos(xyz) + xz\frac{\partial}{\partial z}(\cos(xyz)) = x\cos(xyz) + x^2yz\sin(xyz)
\end{aligned}
$$

**5.** Seja $f(x,y) = \begin{cases} \dfrac{2yx}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Então $f_x(x,y) = \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{2yx}{x^2+y^2}\right) = \dfrac{2y(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ f_x(0,0), & (x,y) = (0,0) \end{cases}$ , $f_y(x,y) = \begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{2yx}{x^2+y^2}\right) = \dfrac{2x(x^2-y^2)}{(x^2+y^2)^2}, & (x,y) \neq \\ f_y(0,0), & (x,y) = \end{cases}$

onde

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,y) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Calculemos, agora, as segundas derivadas, derivando as funções $f_x(x,y)$ e $f_y(x,y)$ em ordem a $x$ e $y$, respetivamente:

$$f_{xx}(x,y) = \begin{cases} \dfrac{-4xy(3y^2-x^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ f_{xx}(0,0), & (x,y) = (0,0) \end{cases}$$

onde

$$f_{xx}(0,0) = \lim_{h\to 0} \tfrac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \tfrac{0-0}{h} = 0$$

$f_{xy}(x,y)=\dfrac{2(-y^4+6y^2x^2-x^4)}{(x^2+y^2)^3}$ para $(x,y)\neq(0,0)$; não existe $f_{xy}(0,0)$ pq

$$f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{h\to 0}\frac{2k^3/k^4-0}{k}=\infty$$

$$f_{yy}(x,y)=\begin{cases}\dfrac{-4xy(3x^2-y^2)}{(x^2+y^2)^3}, & (x,y)\neq(0,0)\\ f_{yy}(0,0), & (x,y)=(0,0)\end{cases}$$

onde

$$f_{yy}(0,0)=\lim_{k\to 0}\frac{f_y(0,k)-f_y(0,0)}{k}=\lim_{h\to 0}\frac{0-0}{k}=0$$

$f_{yx}(x,y)=\dfrac{2(-x^4+6y^2x^2-y^4)}{(x^2+y^2)^3}$ para $(x,y)\neq(0,0)$; $f_{yx}(0,0)$ não existe porque

$$f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{2h^3/h^4-0}{h}=\infty$$

**6.** Verifique que $w_{xy}=w_{yx}$ para:

(a) $w=xy^4-2x^2y^3+4x^2-3y$

$$w_{xy}=\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial y}\left(y^4-4xy^3+8x\right)=4y^3-12xy^2$$

$$w_{yx}=\frac{\partial^2 w}{\partial y\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=\frac{\partial}{\partial x}\left(4xy^3-6x^2y^2-3\right)=4y^3-12xy^2$$

(b) $w=x^3e^{-2y}+y^{-2}\cos x$;

$$w_{xy}=\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial x^3e^{-2y}+y^{-2}\cos x}{\partial x}\right)=\frac{\partial}{\partial y}\left(4x^2e^{-2y}-y^{-2}\sin x\right)=-8x^2e^{-2y}+4y^{-3}\sin x$$

$$w_{yx}=\frac{\partial^2 w}{\partial y\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=\frac{\partial}{\partial x}\left(4xy^3-6x^2y^2-3\right)=4y^3-12xy^2$$

(c) $w=x^2\cos\dfrac{z}{y}$.

$$w_{xy}=\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial y}\left(2x\cos\frac{z}{y}\right)=\frac{2xz}{y^2}\sin\frac{z}{y}$$

$$w_{yx}=\frac{\partial^2 w}{\partial y\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{x^2z}{y^2}\sin\frac{z}{y}\right)=\frac{2xz}{y^2}\sin\frac{z}{y}$$

**7.** Se $w=r^4s^3t-3s^2e^{rt}$, verifique que $w_{rrs}=w_{rsr}=w_{srr}$.

$$\begin{aligned}
w_{rr}&=12r^2s^3t-3s^2t^2e^{rt}\\
w_{rs}&=12r^3s^2t-6ste^{rt}\\
w_{sr}&=12r^3s^2t-6ste^{rt}\\
w_{rrs}&=\partial_s(12r^2s^3t-3s^2t^2e^{rt})=36r^2s^2t-6st^2e^{rt}\\
w_{rsr}&=\partial_r(12r^3s^2t-6ste^{rt})=36r^2s^2t-6st^2e^{rt}\\
w_{srr}&=\partial_r(12r^3s^2t-6ste^{rt})=36r^2s^2t-6st^2e^{rt}
\end{aligned}$$

**8.** Uma função $f$ de $x$ e $y$ diz-se *harmónica* se $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$. Prove que as funções seguintes são harmónicas.

(a) $f(x,y)=e^{kx}\cos(ky)$, $k\in\mathbb{R}$ Note que

$$f_x=ke^{kx}\cos(ky),\quad f_y=-ke^{kx}\sin(ky),\quad f_{xx}=k^2e^{kx}\cos(ky),\quad f_{yy}=-k^2e^{kx}\cos(ky),$$

donde se concluí que $f_{xx}+f_{yy}=0$.

(b) $f(x,y)=3x^2y-y^3$

Tem-se,

$$f_x=6xy,\quad f_y=3x^2-3y^2,\quad f_{xx}=6y,\quad f_{yy}=-6y^2,$$

donde se concluí que $f_{xx}+f_{yy}=0$.