FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Curso MIEM / MIEGI Data / 12/ 20

Disciplina Álgebra Linear e Geometria Analítica Ano 1º Semestre 1º

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Espaço reservado para o avaliador

Notas de apoio ao Capítulo 4 do manual:
"Noções sobre Álgebra Linear".

Matriz Mudança de Base

Base $E = \{ \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n \} \subset V$

$$\vec{x}_E = (\alpha_1, \alpha_2, \dots, \alpha_n)_E = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n$$

Base $U = \{ \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \} \subset V$

$$\vec{x}_U = (\beta_1, \beta_2, \dots, \beta_n)_U = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2 + \dots + \beta_n \vec{u}_n$$

$$\alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2 + \dots + \beta_n \vec{u}_n \Leftrightarrow$$

$$\Leftrightarrow \alpha_1 \begin{bmatrix} e_{11} \\ e_{21} \\ \vdots \\ e_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} e_{12} \\ e_{22} \\ \vdots \\ e_{n2} \end{bmatrix} + \dots + \alpha_n \begin{bmatrix} e_{1n} \\ e_{2n} \\ \vdots \\ e_{nn} \end{bmatrix} =$$

$$= \beta_1 \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} + \beta_2 \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{n2} \end{bmatrix} + \dots + \beta_n \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix} \Leftrightarrow$$

pág 2

$$\Leftrightarrow \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_E = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}_U \Leftrightarrow$$

$$\vec{e}_1 \quad \vec{e}_2 \cdots \vec{e}_n \qquad \vec{X}_E \qquad \vec{u}_1 \quad \vec{u}_2 \cdots \vec{u}_n \qquad \vec{X}_U$$

$$\Leftrightarrow \boxed{E X_E = U X_U}$$

Expressão que regula a mudança de coordenadas entre duas bases

- $X_E = \underbrace{(E^{-1} U)} X_U \implies \boxed{M_{U \to E} = E^{-1} U} \leftarrow X_E = M_{U \to E} X_U$

- $\underbrace{(U^{-1} E)} X_E = X_U \implies \boxed{M_{E \to U} = U^{-1} E} \leftarrow X_U = M_{E \to U} X_E$

- $M_{U \to E} = M_{E \to U}^{-1}$ já que

$$(U^{-1} E)^{-1} = E^{-1} (U^{-1})^{-1} = E^{-1} U$$

$$|M_{E \to U}| = |U^{-1} E| = |U^{-1}|\,|E| = \frac{|E|}{|U|} = \frac{1}{|M_{U \to E}|} \neq 0$$

- Se as base E e U forem <u>bases ortonormadas</u>, então:

  i) As matrizes E e U são <u>matrizes ortogonais</u>, ou seja

$$E^{-1} = E^{T} \quad e \quad U^{-1} = U^{T}$$

ii) $$M_{U \to E} = E^T U$$

$$M_{E \to U} = U^T E$$

iii) $$M_{E \to U} = M^T_{U \to E}$$ já que

$$(U^T E)^T = E^T (U^T)^T = E^T U$$

pág. 3

Aplicação a transformações lineares

Seja a representação matricial

$$m(T)_{E,E^*}$$

tal que

$$Y_{E^*} = m(T)_{E,E^*} X_E \qquad (1)$$

Pretende-se definir a representação matricial

$$m(T)_{U,U^*}$$

tal que

$$Y_{U^*} = m(T)_{U,U^*} X_U \qquad (2)$$

pág. 8

Sabendo que

$$X_E = M_{U \to E} \; X_U$$

$$Y_{E^*} = M_{U^* \to E^*} \; Y_{U^*}$$

recorrendo a (1) obtém-se

$$M_{U^* \to E^*} \; Y_{U^*} = m(T)_{E,E^*} \; M_{U \to E} \; X_U.$$

Multiplicando ambos os membros por $M^{-1}_{U^* \to E^*}$ (multiplicação à esquerda), resulta

$$\underbrace{\left[M^{-1}_{U^* \to E^*} \; M_{U^* \to E^*} \right]}_{= I} Y_{U^*} = M^{-1}_{U^* \to E^*} \; m(T)_{E,E^*} \; M_{U \to E} \; X_U \iff$$

$$\iff \quad Y_{U^*} = \left[M^{-1}_{U^* \to E^*} \; m(T)_{E,E^*} \; M_{U \to E} \right] X_U$$

Tendo em atenção a expressão (2), conclui-se que

$$\boxed{m(T)_{U,U^*} = M^{-1}_{U^* \to E^*} \; m(T)_{E,E^*} \; M_{U \to E}}$$

pág. 8

Wrz

• $$m(T)_{E,E^*} = M_{U^*\to E^*}\; m(T)_{U,U^*}\; M^{-1}_{U\to E}$$

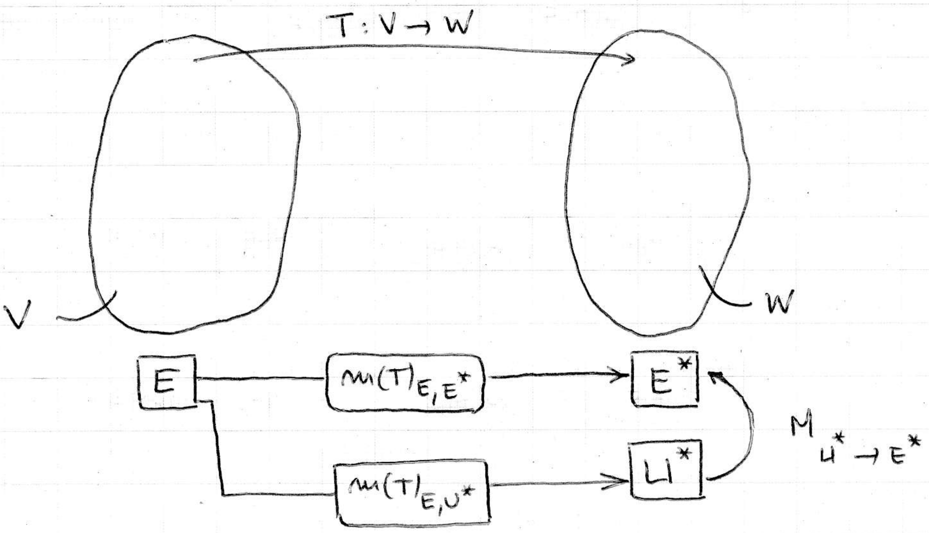
ou

$$m(T)_{E,E^*} = M^{-1}_{E^*\to U^*}\; m(T)_{U,U^*}\; M_{E\to U}$$

• Se $\underline{U = E}$ (no espaço V), então

$$M_{U\to E} = I$$

$$\boxed{m(T)_{E,U^*} = M^{-1}_{U^*\to E^*}\; m(T)_{E,E^*}}$$

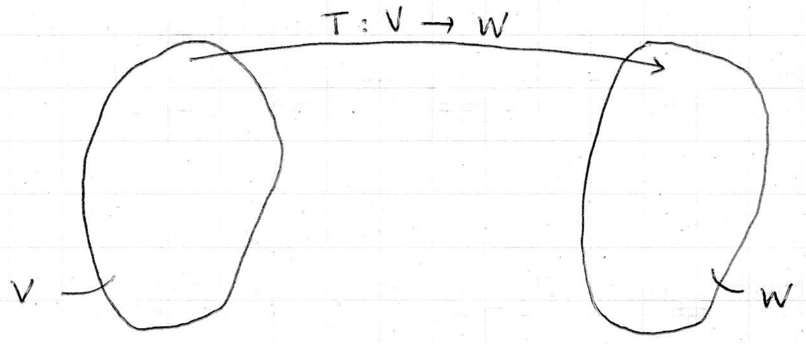


• Se $\underline{U^* = E^*}$ (no espaço W), então

$$M_{U^*\to E^*} = I$$

$$\boxed{m(T)_{U,E^*} = m(T)_{E,E^*}\; M_{U\to E}}$$

pág. 9



$$\begin{tikzcd}
\boxed{E} \arrow[r, "\boxed{m(T)_{E,E^*}}"] & \boxed{E^*} \\
\boxed{U} \arrow[u, bend left, "M_{U\to E}"] \arrow[ur, "\boxed{m(T)_{U,E^*}}"'] &
\end{tikzcd}$$

- Se E, U, E^*, U^* são <u>bases ortonormadas</u>, então as matrizes $M_{U\to E}$ e $M_{U^*\to E^*}$ são <u>matrizes ortogonais</u> pelo que

$$m(T)_{U,U^*} = M^T_{U^*\to E^*}\, m(T)_{E,E^*}\, M_{U\to E}$$

$$m(T)_{E,E^*} = M_{U^*\to E^*}\, m(T)_{U,U^*}\, M^T_{U\to E} =$$

$$= M^T_{E^*\to U^*}\, m(T)_{U,U^*}\, M_{E\to U}$$

<u>pág. 9/10</u>

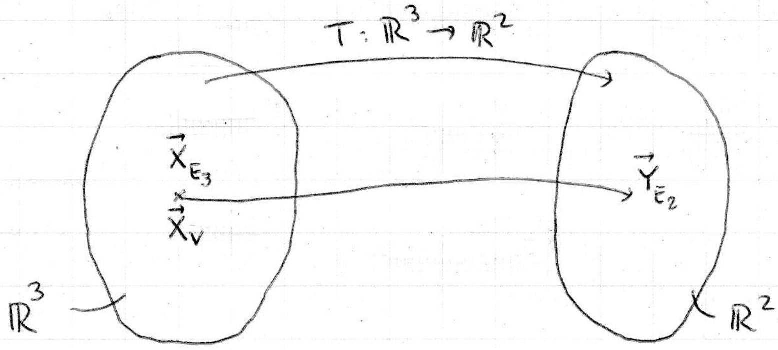
Exemplo 2 [4.4]

a) Sejam as bases

$$E_3 = \{\vec{\imath}, \vec{\jmath}, \vec{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\} \quad : \mathbb{R}^3 \quad \text{(canónica)}$$

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1,-1,0), (0,1,1), (1,0,-1)\} \quad : \mathbb{R}^3$$

$$E_2 = \{\vec{\imath}_1, \vec{\jmath}_1\} = \{(1,0), (0,1)\} \quad : \mathbb{R}^2 \quad \text{(canónica)}$$



$$\begin{tikzcd}
\boxed{E_3} \arrow[rr, "\boxed{m(T)}"] & & \boxed{E_2} \\
\boxed{V} \arrow[u, "M_{V \to E_3}"] \arrow[rru, "\boxed{m(T)_{V,E_2}}"'] & &
\end{tikzcd}$$

Designando $\quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad , \quad V = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

obtém-se

$$\boxed{E_3 X_{E_3} = V X_V} \quad \Rightarrow \quad M_{V \to E_3} = E_3^{-1} V = I_3 V = V$$

Resulta, então,

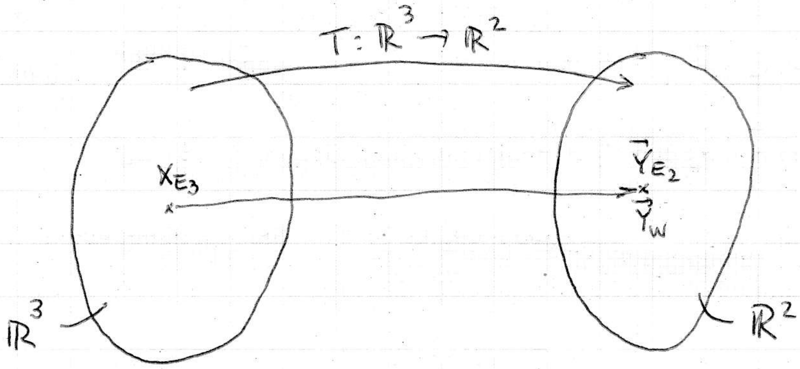
$$\boxed{m(T)_{V,E_2} = m(T)\, M_{V \to E_3}} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}_{V,E_2}$$

Wmj

b) Seja a base

$$W = \{\vec{w}_1, \vec{w}_2\} = \{(1,1),(1,-1)\} \; : \; \mathbb{R}^2$$



$$\begin{tikzcd}
E_3 \arrow[rr, "m(T)"] \arrow[d] & & E_2 \\
{} \arrow[rr, "m(T)_{E_3,W}"] & & W \arrow[u, bend right, "M_{W \rightarrow E_2}"']
\end{tikzcd}$$

Designando

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \, , \quad W = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

obtém-se

$$\boxed{E_2 \, Y_{E_2} = W \, Y_W} \Rightarrow M_{W \rightarrow E_2} = E_2^{-1} W = I_2 W = W$$

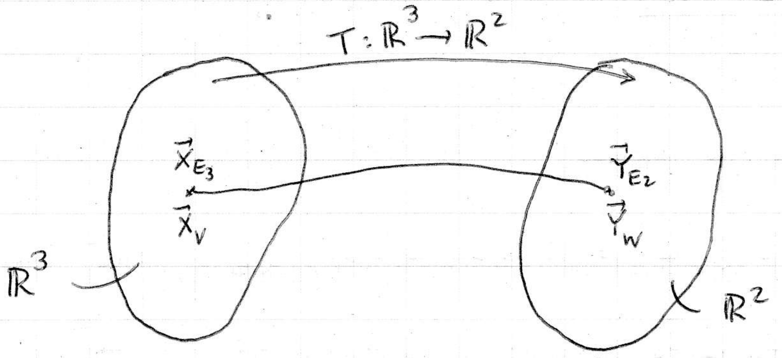
Resulta, então,

$$\boxed{m(T)_{E_3,W} = M^{-1}_{W \rightarrow E_2} \, m(T)}$$

$$M^{-1}_{W \rightarrow E_2} = W^{-1} = \frac{1}{|W|}\left[\mathrm{Cof}\, W\right]^T = \frac{1}{(-2)}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$m(T)_{E_3,W} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -1 \\ 3 & -1 & -3 \end{bmatrix}_{E_3,W}$$

c)



$$
\begin{tikzcd}
\boxed{E_3} \arrow[rr, "m(T)"] & & \boxed{E_2} \\
\boxed{V} \arrow[u, bend left, "M_{V\to E_3}"] \arrow[rr, "m(T)_{V,W}"] & & \boxed{W} \arrow[u, bend right, "M_{W\to E_2}"']
\end{tikzcd}
$$

$$\boxed{m(T)_{V,W} = M^{-1}_{W\to E_2}\left[m(T)\ M_{V\to E_3}\right]} =$$

$$= M^{-1}_{W\to E_2}\ m(T)_{V,E_2} =$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}_{V,E_2} =$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 4 \\ 4 & -4 & 6 \end{bmatrix}_{V,W} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 3 \end{bmatrix}_{V,W}$$

páj. 12

Wáj

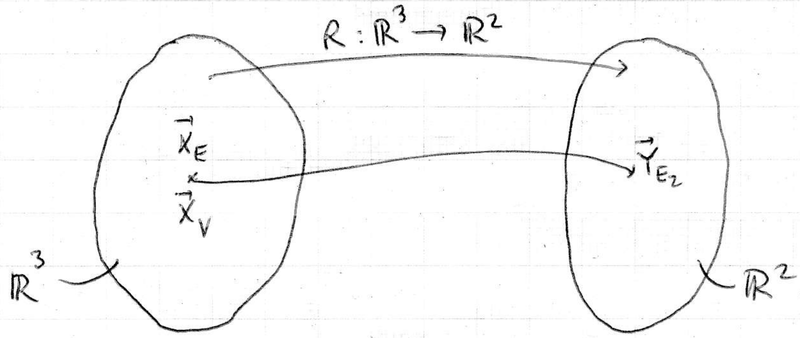
Exemplo 3 [4.5]

a) Sejam as bases

$$E_3 = \{\vec{\imath}, \vec{\jmath}, \vec{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\} : \mathbb{R}^3 \text{ (canónica)}$$

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1,0,1), (1,1,0), (1,0,2)\} : \mathbb{R}^3$$

$$E_2 = \{\vec{\imath}_1, \vec{\jmath}_1\} = \{(1,0), (0,1)\} : \mathbb{R}^2 \text{ (canónica)}$$



$$\begin{tikzcd}
\boxed{E_3} \arrow[rr, "\boxed{m(R)}"] & & \boxed{E_2} \\
\boxed{V} \arrow[u, "M_{V\to E_3}"] \arrow[rr, "\boxed{m(R)_{V,E_2}}"'] & & \arrow[u]
\end{tikzcd}$$

Designando

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad , \quad V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

obtém-se

$$E_3 X_{E_3} = V X_V \Rightarrow M_{V\to E_3} = E_3^{-1} V = I_3 V = V$$

Resulta, então,

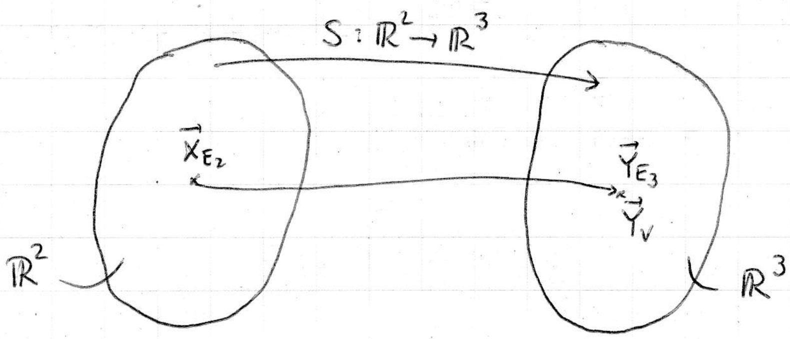
$$m(R)_{V,E_2} = m(R)\, M_{V\to E_3} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}_{V,E_2}$$

pág. 13/14

Wij

b)



$$\begin{tikzcd} \boxed{E_2} \arrow[rr, "m(S)"] \arrow[drr, "m(S)_{E_2,V}"'] & & \boxed{E_3} \\ & & \boxed{V} \arrow[u, bend right, "M_{V\to E_3}"'] \end{tikzcd}$$

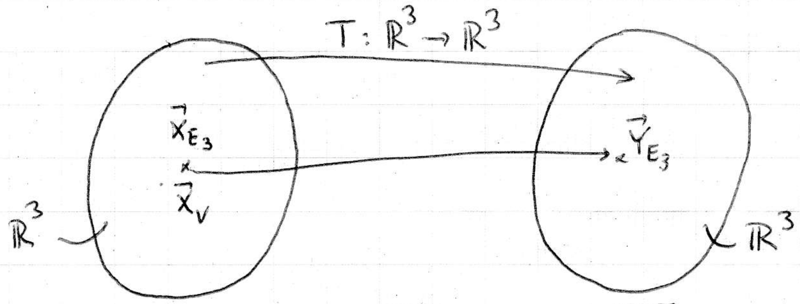
$$m(S)_{E_2,V} = M^{-1}_{V\to E_3}\ m(S)$$

$$M^{-1}_{V\to E_3} = V^{-1} = \frac{1}{|V|}\left[\operatorname{Cof} V\right]^T = \frac{1}{1}\begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$|V| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = (1)(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$m(S)_{E_2,V} = \begin{bmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 1 & 2 \\ 4 & 2 \end{bmatrix}_{E_2,V}$$

c)



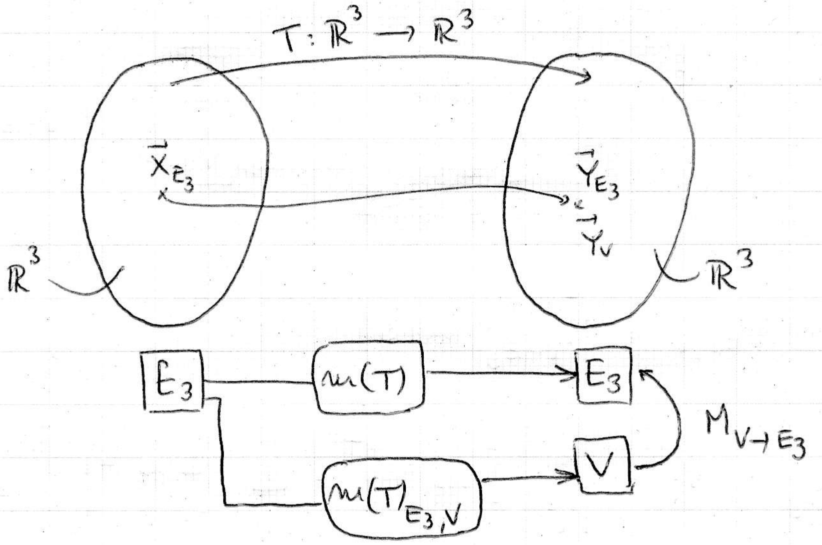
$$\begin{tikzcd} \boxed{E_3} \arrow[rr, "m(T)"] & & \boxed{E_3} \\ \boxed{V} \arrow[u, bend left, "M_{V\to E_3}"] \arrow[urr, "m(T)_{V,E_3}"'] & & \end{tikzcd}$$

pág. 14

$$m(T)_{V,E_3} = m(T)\, M_{V\to E_3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}_{V,E_3}$$

d)

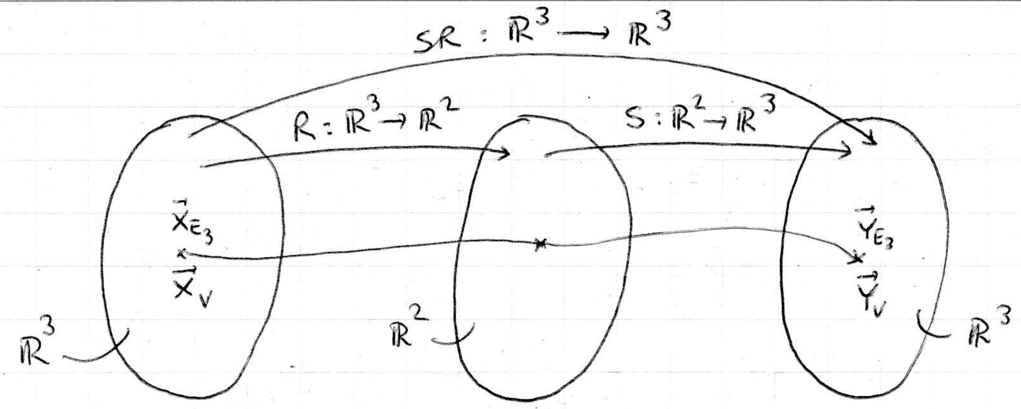


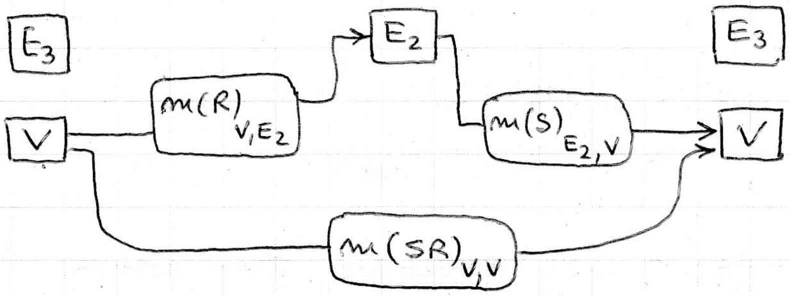
$$m(T)_{E_3,V} = M^{-1}_{V\to E_3}\, m(T) = \begin{bmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{E_3,V}$$

e)

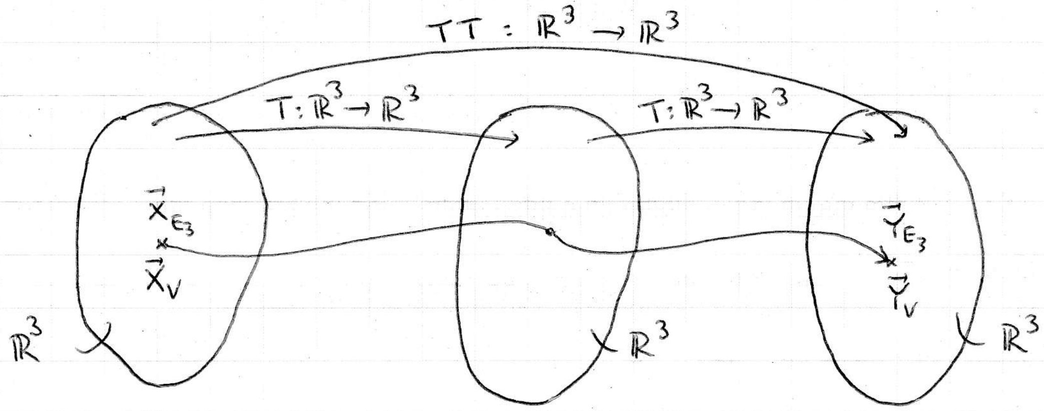
$$m(SR + T^2)_{V,V} = m(SR)_{V,V} + m(TT)_{V,V} \qquad (1)$$

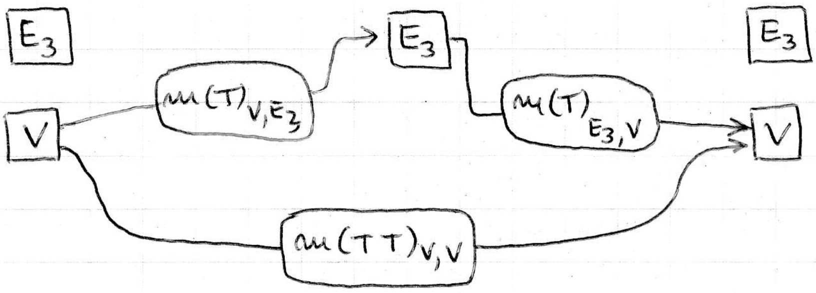




$$m(SR)_{V,V} = m(S)_{E_2,V} \; m(R)_{V,E_2} =$$

$$= \begin{bmatrix} -6 & -3 \\ 1 & 2 \\ 4 & 2 \end{bmatrix}_{E_2,V} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}_{V,E_2} = \begin{bmatrix} 0 & -6 & 3 \\ 0 & 1 & 1 \\ 0 & 4 & -2 \end{bmatrix}_{V,V}$$





pág. 15

$$m(TT)_{V,V} = m(T)_{E_3,V} \; m(T)_{V,E_3} =$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{E_3,V} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}_{V,E_3} =$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}_{V,V}$$

Recorrendo a (1) obtém-se finalmente

$$m(SR+T^2)_{V,V} = \begin{bmatrix} 0 & -6 & 3 \\ 0 & 1 & 1 \\ 0 & 4 & -2 \end{bmatrix}_{V,V} + \begin{bmatrix} 0 & -1 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}_{V,V} =$$

$$= \begin{bmatrix} 0 & -7 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix}_{V,V}$$

pág. 15

f) A representação matricial da transformação linear $SR + T^2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ em relação à base E_3 (canónica) para o espaço $\mathbb{R}^3$ é

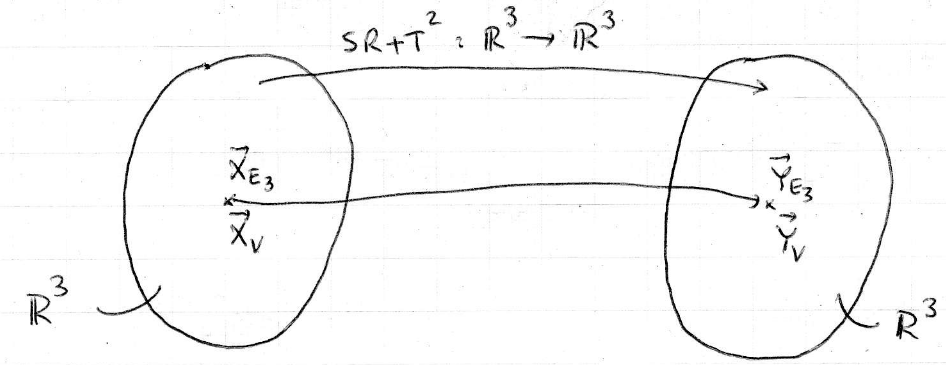
$$m(SR+T^2) = m(S)\,m(R) + m(T)\,m(T) =$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Leftrightarrow$$

pág. 15/16

(e) $$m(SR+T^2) = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 3 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$



$$\begin{tikzcd} \boxed{E_3} \arrow[rr, "m(SR+T^2)"] & & \boxed{E_3} \\ \boxed{V} \arrow[u, bend left, "M_{V\to E_3}"] \arrow[rr, "m(SR+T^2)_{V,V}"] & & \boxed{V} \arrow[u, bend right, "M_{V\to E_3}"'] \end{tikzcd}$$

$$m(SR+T^2)_{V,V} = \left[M^{-1}_{V\to E_3}\, m(SR+T^2) \right] M_{V\to E_3} =$$

$$= \begin{bmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} M_{V\to E_3} =$$

$$= \begin{bmatrix} -2 & -5 & 2 \\ 0 & 3 & 2 \\ 2 & 3 & -1 \end{bmatrix}_{E_3,V} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -7 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix}_{V,V}$$

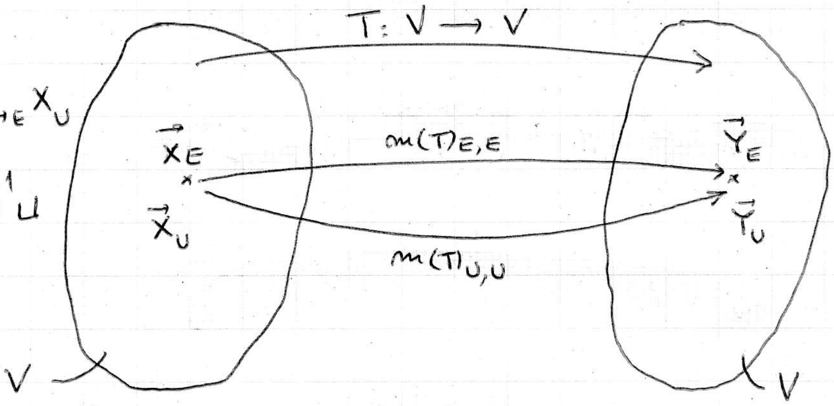
Wwij

Matrizes semelhantes

Seja a transformação linear $T : V \to V$, em que $\dim V = n$.

$X_E = M_{U \to E} X_U$

$M_{U \to E} = E^{-1} U$



$Y_E = M_{U \to E} Y_U$

$M_{U \to E} = E^{-1} U$

$$\begin{tikzcd}
\boxed{E} \arrow[rr, "\boxed{m(T)_{E,E}}"] & & \boxed{E} \\
\boxed{U} \arrow[u, bend left, "M_{U \to E}"] \arrow[rr, "\boxed{m(T)_{U,U}}"] & & \boxed{U} \arrow[u, bend right, "M_{U \to E}"']
\end{tikzcd}$$

$Y_E = m(T)_{E,E} X_E$ e $Y_U = m(T)_{U,U} X_U$ em que

$$m(T)_{U,U} = M^{-1}_{U \to E}\, m(T)_{E,E}\, M_{U \to E}$$

$m(T)_{U,U} \approx m(T)_{E,E} \to$ as matrizes dizem-se matrizes semelhantes, já que existe uma matriz $P = M_{U \to E}$ não singular, tal que

$$m(T)_{U,U} = P^{-1}\, m(T)_{E,E}\, P \quad (\text{definição } [4.2])$$

pág. 17/18

Teorema [4.6]

Considere-se as matrizes semelhantes $B \simeq A$, ou seja,

$$B = P^{-1} A P$$

$$|B| = |P^{-1}(AP)| = |P^{-1}|\,|AP| =$$

$$= \frac{1}{|P|}\,|A|\,|P| = |A|$$

NOTAS:

$|AB| = |A|\,|B|$

$|P^{-1}| = \frac{1}{|P|}$

$|P| \neq 0$

Teorema [4.7]

Considere-se as matrizes semelhantes $B \simeq A$, ou seja,

$$B = P^{-1} A P$$

que representam a mesma transformação linear $T : V \longrightarrow V$.
Então

$$\dim T(V) = r(B) = r(A) \Rightarrow r(A) = r(B)$$

Teorema [4.8]

Considere-se as matrizes semelhantes $B \simeq A$, ou seja,

$$B = P^{-1} A P$$

e $k \in \mathbb{Z}^+$.

Demonstração pelo método de indução.

i) $k = 1$; $B^1 = P^{-1} A^1 P$

$$B^1 = B^0 B = I B = B = P^{-1} A P = P^{-1} (A I) P =$$

$$= P^{-1} (A A^0) P = P^{-1} A^1 P$$

pág. 18

ii) Admitindo que $B^k = P^{-1} A^k P$ é verdadeiro, pretende-se mostrar que

$$B^{k+1} = P^{-1} A^{k+1} P$$

$$B^{k+1} = B B^k = (P^{-1} A P)(P^{-1} A^k P) =$$

$$= P^{-1} A (P P^{-1}) A^k P = P^{-1} (A I) A^k P =$$

$$= P^{-1} A A^k P = P^{-1} A^{k+1} P$$

Teorema [4.9]

Sejam as matrizes semelhantes $B \simeq A$, ou seja,

$$B = P^{-1} A P$$

Admite-se que A é uma matriz não singular. Neste caso, B é também uma matriz não singular, já que

$$|B| = |A| \neq 0$$

$$B^{-1} = (P^{-1}(AP))^{-1} = (AP)^{-1}(P^{-1})^{-1} =$$

NOTAS

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= P^{-1} A^{-1} P$$

ou seja, as matrizes B^{-1} e A^{-1} são matrizes semelhantes, $B^{-1} \simeq A^{-1}$.

Teorema [4.10]

Sejam as matrizes semelhantes $B \simeq A$, ou seja,

$$B = P^{-1} A P$$

Admite-se que A é uma matriz não singular; neste caso, B é também uma matriz não singular, já que

$$|B| = |A| \neq 0$$

Papel 100% Reciclado

Sendo $k \in \mathbb{Z}^+$ então

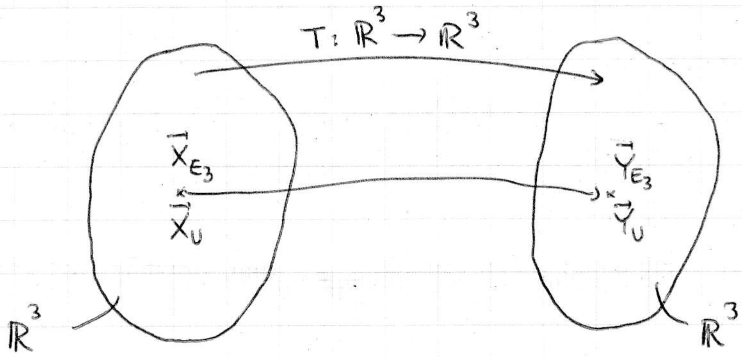
$$\bar{B}^{k} = (B^k)^{-1} = [P^{-1}(A^k P)]^{-1} =$$

$$= [A^k P]^{-1} (P^{-1})^{-1} = P^{-1} [A^k]^{-1} P =$$

$$= P^{-1} A^{-k} P$$

ou seja, as matrizes $\bar{B}^{k}$ e $\bar{A}^{k}$ são matrizes semelhantes, $\bar{B}^{k} \approx A^{-k}$.

<u>Exemplo 4 [4.6]</u>



$$\begin{tikzcd}
\boxed{E_3} \arrow[r, "\boxed{m(T)}"] & \boxed{E_3} \\
\boxed{U} \arrow[u, bend left, "M_{U\to E_3}"] \arrow[r, "\boxed{m(T)_{U,U}}"] & \boxed{U} \arrow[u, bend right, "M_{U\to E_3}"']
\end{tikzcd}$$

a) Seja a base para o espaço $\mathbb{R}^3$

$$E_3 = \{\vec{i}, \vec{j}, \vec{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\} \quad \text{(canónica)}$$

Sabendo que
$$T(1,0,0) = (3,-2,3)$$
$$T(0,1,0) = (2,-2,6)$$
$$T(0,0,1) = (-1,2,-1)$$

a representação matricial de T em relação à base E_3 é

$$m(T) = \begin{bmatrix} 3 & 2 & -1 \\ -2 & -2 & 2 \\ 3 & 6 & -1 \end{bmatrix}$$

b) Designando

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad e \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

obtém-se

$$E_3 X_{E_3} = U X_U \Rightarrow M_{U \to E_3} = E_3^{-1} U = I_3 U = U$$

Além disso

$$M^{-1}_{U \to E_3} = U^{-1} = \frac{1}{|U|} \left[\mathrm{Cof}\, U \right]^T = \frac{1}{(-1)} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$|U| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (1)(-1)^2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

As matrizes $m(T)_{U,U}$ e $m(T)$ são matrizes semelhantes, isto é,

$$m(T)_{U,U} = M^{-1}_{U \to E_3} \left[m(T)\ M_{U \to E_3} \right] = M^{-1}_{U \to E_3} \begin{bmatrix} 3 & 2 & -1 \\ -2 & -2 & 2 \\ 3 & 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 7 & 1 & 2 \\ -6 & 0 & -2 \\ 15 & 5 & 6 \end{bmatrix}_{U,E_3} = \begin{bmatrix} 7 & 1 & 2 \\ 15 & 5 & 6 \\ -35 & -7 & -12 \end{bmatrix}_{U,U}$$

Uma vez que $m(T)_{U,U} \approx m(T)$ então

$$|m(T)_{U,U}| = |m(T)| = -16$$

$$|m(T)| = \begin{vmatrix} 3 & 2 & -1 \\ -2 & -2 & 2 \\ 3 & 6 & -1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & -1 \\ -1 & -1 & 1 \\ 3 & 6 & -1 \end{vmatrix} \begin{matrix} \\ \leftarrow L_2 + L_1 \\ \leftarrow L_3 - L_1 \end{matrix} = 2 \begin{vmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 0 & 4 & 0 \end{vmatrix} =$$

$$= 2\,(-1)\,(-1)^4 \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = -16$$

Designando $\vec{X}_U = (x_1, y_1, z_1)_U = x_1\vec{U}_1 + y_1\vec{U}_2 + z_1\vec{U}_3$ (coordenadas do vector $\vec{X}$ em relação à base U), então

$$Y_U = m(T)_{U,U}\, X_U = \begin{bmatrix} 7 & 1 & 2 \\ 15 & 5 & 6 \\ -35 & -7 & -12 \end{bmatrix}_{U,U} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}_U =$$

$$= \begin{bmatrix} 7x_1 + y_1 + 2z_1 \\ 15x_1 + 5y_1 + 6z_1 \\ -35x_1 - 7y_1 - 12z_1 \end{bmatrix}_U$$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x_1, y_1, z_1)_U \longrightarrow (7x_1 + y_1 + 2z_1,\ 15x_1 + 5y_1 + 6z_1,\ -35x_1 - 7y_1 - 12z_1)_U$$

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