MIEMEC; MIEEICOM

Análise Matemática EE

12 de junho de 2013

Teste 2

1.

a) Pontos críticos:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 27 = 0 \\ 4y^3 + 32 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = 9 \\ y^3 = -8 \end{cases} \Leftrightarrow \begin{cases} x = \pm 3 \\ y = -2 \end{cases}$$

Os pontos $(-3,-2)$ e $(3,-2)$ são os únicos pontos críticos de f.

b) • Discriminante

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$$
$$= 6x \cdot 12y^2 - 0$$
$$= 72xy^2$$

• Classificação dos pontos críticos

- $D(-3,-2) = 72 \times (-3) \times (-2)^2 < 0$

Logo, $(-3,-2)$ é ponto de sela.

- $D(3,-2) = 72 \times 3 \times (-2)^2 > 0$

$$\frac{\partial^2 f}{\partial x^2}(3,-2) = 6 \times 3 > 0$$

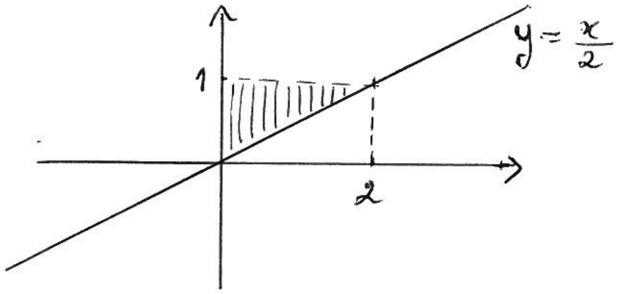
Logo, $(3,-2)$ é um minimizante local de f.

2.

a)

Região de integração:

$$0 \leq x \leq 2 \qquad \text{ou} \qquad 0 \leq y \leq 1$$
$$\frac{x}{2} \leq y \leq 1 \qquad\qquad 0 \leq x \leq 2y$$



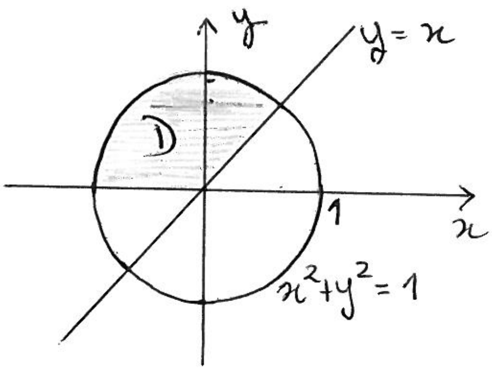
Assim,

$$\int_0^2 \int_{x/2}^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^{2y} \cos(y^2)\,dx\,dy$$

$$= \int_0^1 \Big[x\cos(y^2)\Big]_0^{2y} dy = \int_0^1 2y\cos(y^2)\,dy$$

$$= \Big[\operatorname{sen}(y^2)\Big]_0^1 = \operatorname{sen}(1) - \operatorname{sen}(0) = \operatorname{sen}(1).$$

b) Região de integração:



Usando coordenadas polares,

$$D = \left\{(r,\theta): 0 \leq r \leq 1, \frac{\pi}{4} \leq \theta \leq \pi\right\}$$

e, sendo

$$x = r\cos\theta$$
$$y = r\operatorname{sen}\theta,$$

vem

$$e^{x^2+y^2} = e^{r^2\cos^2\theta + r^2\operatorname{sen}^2\theta} = e^{r^2}$$

Assim,

$$\iint_D e^{x^2+y^2}\,dy\,dx = \int_{\frac{\pi}{4}}^{\pi}\int_0^1 r e^{r^2}\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\pi}\left[\frac{e^{r^2}}{2}\right]_0^1 d\theta = \int_{\frac{\pi}{4}}^{\pi}\frac{e-1}{2}\,d\theta$$

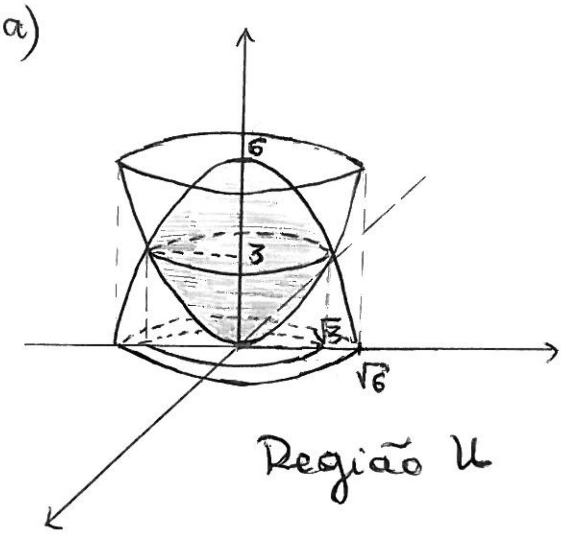
$$= \left[\frac{e-1}{2}\cdot\theta\right]_{\frac{\pi}{4}}^{\pi} = \frac{e-1}{2}\left(\pi-\frac{\pi}{4}\right) = \frac{3}{8}(e-1)$$

c) Usando coordenadas polares,

$$\text{area}(D) = \int_{\frac{\pi}{4}}^{\pi}\int_0^1 r\,dr\,d\theta$$

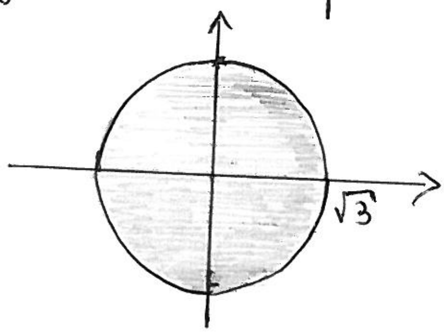
3.

a)



Região U

Projeção de U no plano xOy



- $z = 6 - x^2 - y^2 = 6 - (x^2+y^2)$

  Parabolóide "voltado para baixo"

  – Para $(x,y) = (0,0)$, vem $z = 6$;

  – Para $z=0$, temos a circunferência $x^2+y^2=6$ no plano xOy.

- $z = x^2+y^2$

  Parabolóide "voltado para cima"

  – Para $(x,y) = (0,0)$, vem $z=0$;

  – Para, por exemplo, $z=6$, temos a circunferência $x^2+y^2=6$ (no plano $z=6$).

- Interseção dos dois parabolóides

$$6 - x^2 - y^2 = x^2 + y^2 \iff x^2 + y^2 = 3$$

Circunferência $x^2+y^2=3$ no plano $z=3$.

b) Região U :

$$-\sqrt{3} \leq x \leq \sqrt{3}$$

$$-\sqrt{3-x^2} \leq y \leq \sqrt{3-x^2}$$

$$x^2+y^2 \leq z \leq 6 - x^2 - y^2$$

$$\text{Volume}(U) = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{x^2+y^2}^{6-x^2-y^2} 1 \, dz \, dy \, dx$$

c) Região U, usando coordenadas cilíndricas :

$$0 \leq r \leq \sqrt{3}$$

$$0 \leq \theta \leq 2\pi$$

$$r^2 \leq z \leq 6 - r^2$$

Assim,

$$\text{Volume}(U) = \int_0^{\sqrt{3}} \int_0^{2\pi} \int_{r^2}^{6-r^2} r \, dz \, d\theta \, dr = \int_0^{\sqrt{3}} \int_0^{2\pi} \left[r \cdot z \right]_{z=r^2}^{6-r^2} d\theta \, dr =$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} r\left(6 - r^2 - r^2\right) d\theta \, dr = \int_0^{\sqrt{3}} \int_0^{2\pi} \left(6r - 2r^3\right) d\theta \, dr$$

$$= \int_0^{\sqrt{3}} \left[\left(6r - 2r^3\right)\theta\right]_{\theta=0}^{2\pi} dr = \int_0^{\sqrt{3}} \left(12\pi r - 4\pi r^3\right) dr = \left[6\pi r^2 - \pi r^4\right]_{r=0}^{\sqrt{3}}$$

$$= 18\pi - 9\pi = 9\pi$$

4.

a)

$$r(t) = \left(\int 6t \, dt, \int 2t \, dt, \int t \, dt \right)$$

$$= \left(3t^2 + C_1, \ t^2 + C_2, \ \frac{t^2}{2} + C_3 \right), \quad C_1, C_2, C_3 \text{ constantes}$$

$$r(2) = (14, 5, 2) \Rightarrow \begin{cases} 3 \times 2^2 + C_1 = 14 \\ 2^2 + C_2 = 5 \\ \frac{2^2}{2} + C_3 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = 1 \\ C_3 = 0 \end{cases}$$

Assim,

$$r(t) = \left(3t^2 + 2, \ t^2 + 1, \ \frac{t^2}{2} \right).$$

Posição inicial: $r(0) = (2, 1, 0)$

b)

$$\mathcal{L}(e) = \int_0^2 \| r'(t) \| \, dt = \int_0^2 \| v(t) \| \, dt = \int_0^2 \sqrt{36t^2 + 4t^2 + t^2} \, dt$$

$$= \int_0^2 \sqrt{41 t^2} \, dt = \int_0^2 \sqrt{41} \, t \, dt = \left[\sqrt{41} \cdot \frac{t^2}{2} \right]_0^2 = 2\sqrt{41}$$

e)

Reta tangente em $t=1$:

$$(x,y,z) = r(1) + r'(1)\cdot t, \quad t \in \mathbb{R}$$
$$= \left(5, 2, \tfrac{1}{2}\right) + (6,2,1)\cdot t, \quad t \in \mathbb{R}$$
$$= \left(5+6t,\ 2+2t,\ \tfrac{1}{2}+t\right), \quad t \in \mathbb{R}$$

Plano normal em $t=1$:

$$r'(1) \cdot \left[(x,y,z) - r(1)\right] = 0$$
$$\Leftrightarrow (6,2,1)\cdot\left(x-5,\ y-2,\ z-\tfrac{1}{2}\right) = 0$$
$$\Leftrightarrow 6(x-5) + 2(y-2) + \left(z - \tfrac{1}{2}\right) = 0$$
$$\Leftrightarrow 6x + 2y + z = 30 + 4 + \tfrac{1}{2}$$
$$\Leftrightarrow 6x + 2y + z = \tfrac{69}{2}$$

5.

a) $F(x,y,z) = (2yz,\ 2xz,\ 2xy+2)$

C_1: arco da parábola $y = x^2+1$, $z=0$, de $x=0$ para $x=3$

Parametrização de C_1:

$$\gamma_1(t) = (t,\ t^2+1,\ 0), \quad 0 \le t \le 3$$

C_2: segmento de reta que une o ponto $(3,10,0)$ ao ponto $(3,10,2)$ no sentido ascendente

Parametrização de C_2:

$$\gamma_2(t) = (3,10,0) + t\left[(3,10,2) - (3,10,0)\right] = (3,10,2t), \quad 0 \le t \le 1$$

$$\int_{\mathcal{C}} F \cdot ds = \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds$$

$$= \int_0^3 F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 F(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$$

$$= \int_0^3 F(t, t^2+1, 0) \cdot (1, 2t, 0)\, dt + \int_0^1 F(3, 10, 2t) \cdot (0, 0, 2)\, dt$$

$$= \int_0^3 0\, dt + \int_0^1 (40t, 12t, 62) \cdot (0, 0, 2)\, dt$$

$$= \int_0^1 124\, dt = \left[124t\right]_0^1 = 124$$

b) Uma vez que

$$\vec{rot}\, E = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+1 \end{vmatrix} = (x-x)\vec{i} - (y-y)\vec{j} + (z-z)\vec{k}$$
$$= \vec{0},$$

E é um campo gradiente.

Pretendemos determinar f tal que $E = \vec{\nabla} f$, ou seja,

$$\frac{\partial f}{\partial x} = yz \quad , \quad \frac{\partial f}{\partial y} = xz \quad e \quad \frac{\partial f}{\partial z} = xy + 1$$

a) b) c)

De a), e integrando em ordem a x, obtém-se

$$f(x,y,z) = xyz + h(y,z).$$

Assim, temos

$$\frac{\partial f}{\partial y} = xz + \frac{\partial}{\partial y} h(y,z).$$

Comparando com b), concluímos que $\frac{\partial}{\partial y} h(y,z) = 0$, ou seja, $h(y,z) = g(z)$.

Assim,

$$f(x,y,z) = xyz + g(z)$$

e

$$\frac{\partial f}{\partial z} = xy + \frac{d}{dt} g(z).$$

Comparando com c), devemos ter $\frac{d}{dz} g(z) = 1$, ou seja, $g(z) = z + k$, sendo k uma constante.

Qualquer função

$$f(x,y,z) = xyz + z + k \quad , \; k \text{ constante},$$

é uma função potencial de E.

6. Seja $F = (F_1, F_2, F_3)$.

$$\vec{rot}(fF) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \Big(\frac{\partial}{\partial y} fF_3 - \frac{\partial}{\partial z} fF_2, \frac{\partial}{\partial z} fF_1 - \frac{\partial}{\partial x} fF_3, \frac{\partial}{\partial x} fF_2 - \frac{\partial}{\partial y} fF_1 \Big) =$$

$$= \Big(\left(\frac{\partial}{\partial y} f\right) \cdot F_3 + f \cdot \frac{\partial}{\partial y} F_3 - \left(\frac{\partial}{\partial z} f\right) F_2 - f \frac{\partial}{\partial z} F_2 ,$$

$$\left(\frac{\partial}{\partial z} f\right) \cdot F_1 + f \frac{\partial}{\partial z} F_1 - \left(\frac{\partial}{\partial x} f\right) F_3 - f \frac{\partial}{\partial x} F_3 ,$$

$$\left(\frac{\partial}{\partial x} f\right) F_2 + f \frac{\partial}{\partial x} F_2 - \left(\frac{\partial}{\partial y} f\right) F_1 - f \frac{\partial}{\partial y} F_1 \Big)$$

$$= f \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2 , \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3 , \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right) +$$

$$+ \Big(\left(\frac{\partial}{\partial y} f\right) \cdot F_3 - \left(\frac{\partial}{\partial z} f\right) F_2 , \left(\frac{\partial}{\partial z} f\right) \cdot F_1 - \left(\frac{\partial}{\partial x} f\right) F_3 ,$$

$$\left(\frac{\partial}{\partial x} f\right) \cdot F_2 - \left(\frac{\partial}{\partial y} f\right) F_1 \Big)$$

$$= f \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= f \cdot \vec{rot} F + \vec{\nabla} f \times F$$