FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Curso MIEM / MIEGI — Data / 01/ 21

Disciplina Álgebra Linear e Geometria Analítica — Ano 1º — Semestre 1º

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Espaço reservado para o avaliador

Aula Teórico-Prática - Transformações lineares

57)

a) A representação matricial da transformação T em relação à base canónica do espaço $\mathbb{R}^3$

$$E = \{\vec{i}, \vec{j}, \vec{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

é

$$T(\vec{i}) = T(1,0,0) = (1,0,-2)$$
$$T(\vec{j}) = T(0,1,0) = (0,0,0) \Rightarrow \vec{j} \in N(T)$$
$$T(\vec{k}) = T(0,0,1) = (-2,0,4)$$

$$m(T) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

A característica de $m(T)$ é

$$r(m(T)) = r\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow L_3 + 2L_1 \end{matrix} = r\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

Então $\dim T(\mathbb{R}^3) = r[m(T)] = 1$; uma vez que

$$\dim T(\mathbb{R}^3) < \dim \mathbb{R}^3 \Rightarrow T(\mathbb{R}^3) \subset \mathbb{R}^3 \Rightarrow T \text{ não é sobrejectiva}$$

Por outro lado, verifica-se

$$\dim N(T) = \dim \mathbb{R}^3 - \dim T(\mathbb{R}^3) = 3 - 1 = 2 \text{ , pelo que}$$

$$N(T) \neq \{(0,0,0)\} \Rightarrow T \text{ não é injectiva}$$

Cálculo de $N(T)$:

$$N(T) = \{ \vec{x} = (x,y,z) \in \mathbb{R}^3 : T(\vec{x}) = (0,0,0) \} \subset \mathbb{R}^3$$

$$T(x,y,z) = (x-2z, 0, -2x+4z) = (0,0,0) \Leftrightarrow$$

$$\Leftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow L_3 + 2L_1 \end{array} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

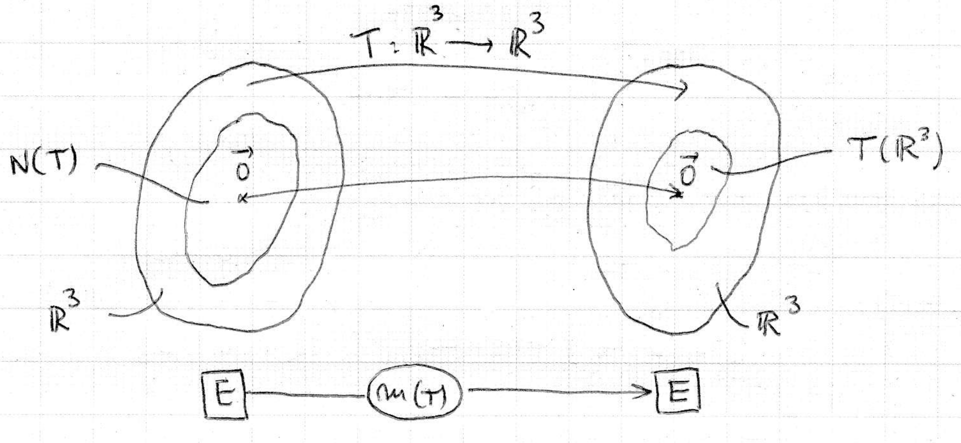
O sistema homogéneo é possível e duplamente indeterminado, tendo como solução

$$x = 2z \quad \wedge \quad y \in \mathbb{R}$$

$$N(T) = \{ \vec{x} = (x,y,z) \in \mathbb{R}^3 : x = 2z \} = \{ \vec{x} = (2z, y, z) \in \mathbb{R}^3 \} =$$

$$= \{ \vec{x} = z(2,0,1) + y(0,1,0) \in \mathbb{R}^3 \}$$

Base $N(T) = \{ (2,0,1), (0,1,0) \}$



Cálculo de $T(\mathbb{R}^3) = Im(T)$

$$T(\mathbb{R}^3) = \{ \vec{y} = (a,b,c) \in \mathbb{R}^3 : \vec{y} = T(\vec{x}), \vec{x} \in \mathbb{R}^3 \} \subset \mathbb{R}^3$$

$$T(x,y,z) = (x-2z, 0, -2x+4z) = (a,b,c) \Leftrightarrow$$

$$\Leftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 0 & 0 & b \\ -2 & 0 & 4 & c \end{array}\right] \leftarrow L_3 + 2L_1 \end{array} \quad \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 2a+c \end{array}\right]$$

O sistema não homogéneo é possível e duplamente indeterminado se e só se

$$b = 0 \quad \wedge \quad 2a + c = 0$$

$$T(\mathbb{R}^3) = \{ \vec{y} = (a,b,c) \in \mathbb{R}^3 : b = 0 \wedge 2a+c=0 \} =$$

$$= \{ \vec{y} = (a, 0, -2a) \in \mathbb{R}^3 \} = \{ \vec{y} = a\,(1,0,-2) \in \mathbb{R}^3 \}$$

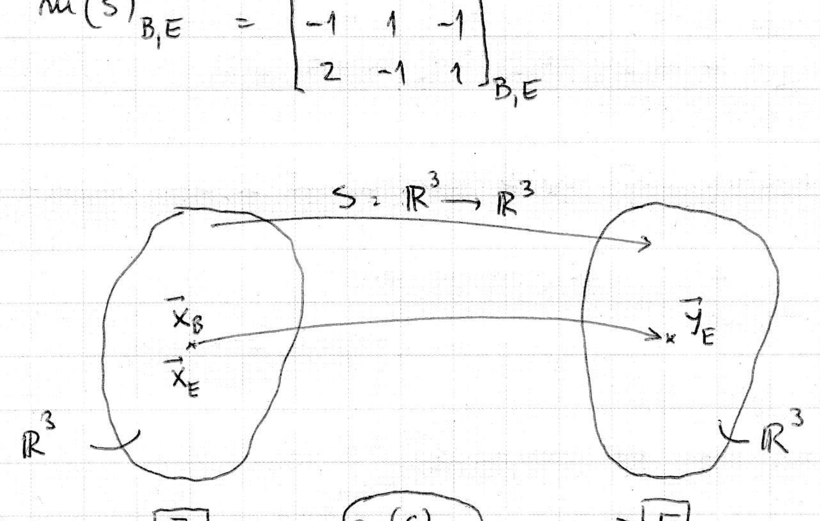
Base $T(\mathbb{R}^3) = \{ (1,0,-2) \}$

b) A transformação linear S está definida através das imagens dos vectores que constituem a base

$$B = \{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \} = \{ (1,0,1), (-1,1,0), (0,0,1) \} \subset \mathbb{R}^3$$

Então a representação matricial de S em relação às bases B (domínio) e E (conjunto de chegada) é

$$m(S)_{B,E} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}_{B,E}$$



$$m(S) = m(S)_{B,E}\, M_{E\to B}$$

Designando

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad e \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

então

$$E X_E = B X_B \Rightarrow X_B = B^{-1} E X_E \Rightarrow M_{E\to B} = B^{-1}E = B^{-1}I = B^{-1}$$

$$M_{E\to B} = B^{-1} = \frac{1}{|B|}\left[\mathrm{Cof}\, B\right]^T = \frac{1}{1}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1)(-1)^4 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$m(S) = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}_{B,E} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$S(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ y-z \\ x+z \end{bmatrix}$$

A lei de transformação de S em relação à base canónica é

$$S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longrightarrow (x+y+z,\ y-z,\ x+z)$$

c) A característica de $m(S)$ é

$$r[m(S)] = r\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow L_3 - L_1 \end{matrix} = r\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow L_3 + L_2 \end{matrix} =$$

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$$= r\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = 3$$

Então $\dim S(\mathbb{R}^3) = r[m(S)] = 3 = \dim \mathbb{R}^3$, pelo que

$$S(\mathbb{R}^3) = \mathbb{R}^3 \Rightarrow S \text{ é sobrejectiva}$$

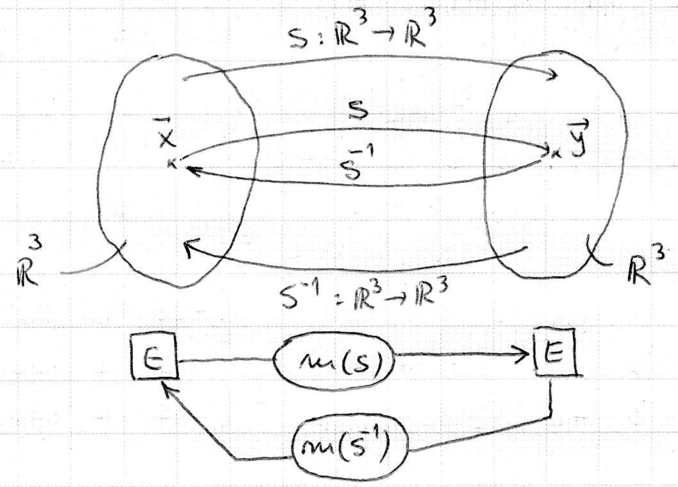
Por outro lado, verifica-se

$$\dim N(S) = \dim \mathbb{R}^3 - \dim S(\mathbb{R}^3) = 3 - 3 = 0$$, pelo que

$$N(S) = \{(0,0,0)\} \Rightarrow S \text{ é injectiva}$$

Como S é injectiva e sobrejectiva, conclui-se que S é bijectiva; neste caso, tem-se

$$S^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$



$$m(S^{-1}) = m^{-1}(S) = \frac{1}{|m(S)|}\left[\text{cof}\, m(S)\right]^T$$

$$|m(S)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow L_3 - L_1 \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{vmatrix} = (1)(-1)^2 \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1$$

$$m(S^{-1}) = \frac{1}{(-1)} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$S^{-1}(a,b,c) = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a+b+2c \\ a-c \\ a-b-c \end{bmatrix}$$

$$S^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(a,b,c) \longrightarrow (-a+b+2c,\ a-c,\ a-b-c)$$

d)

$$m(ST) = m(S)\, m(T) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 0 & -4 \\ -1 & 0 & 2 \end{bmatrix}$$

$$m(S^{-1}T^2) = \left[m(S^{-1})\, m(T)\right] m(T) =$$

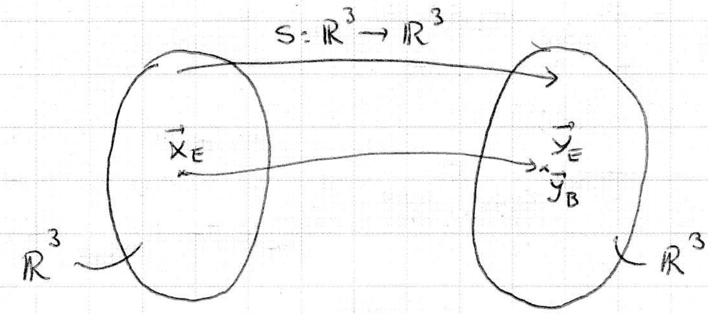
$$= \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} m(T) =$$

$$= \begin{bmatrix} -5 & 0 & 10 \\ 3 & 0 & -6 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -25 & 0 & 50 \\ 15 & 0 & -30 \\ 15 & 0 & -30 \end{bmatrix}$$

g) Conforme se verificou na alínea b),

$$m(S)_{B,E} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}_{B,E}$$

h)



$$\begin{tikzcd} \boxed{E} \arrow[r, "m(S)"] \arrow[dr, "m(S)_{E,B}"'] & \boxed{E} \\ & \boxed{B} \arrow[u, bend right, "M_{B\to E}"'] \end{tikzcd}$$

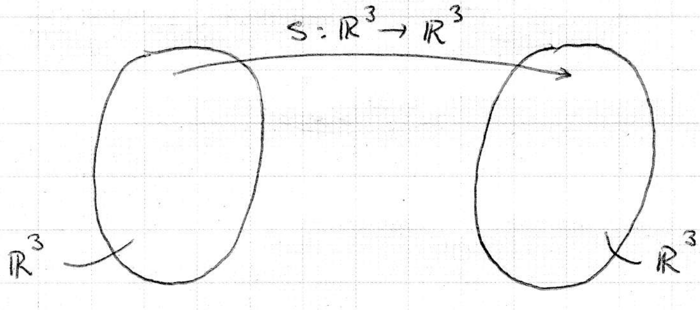
$$m(S)_{E,B} = M^{-1}_{B\to E}\, m(S)$$

Na alínea b) foi definida $M_{E\to B} = B^{-1}$ e, portanto,

$$M^{-1}_{B\to E} = M_{E\to B} = B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$m(S)_{E,B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}_{E,B}$$

i)



$$\begin{tikzcd} \boxed{E} \arrow[r, "m(S)"] & \boxed{E} \\ \boxed{B} \arrow[u, bend left, "M_{B\to E}"] \arrow[r, "m(S)_{B,B}"'] & \boxed{B} \arrow[u, bend right, "M_{B\to E}"'] \end{tikzcd}$$

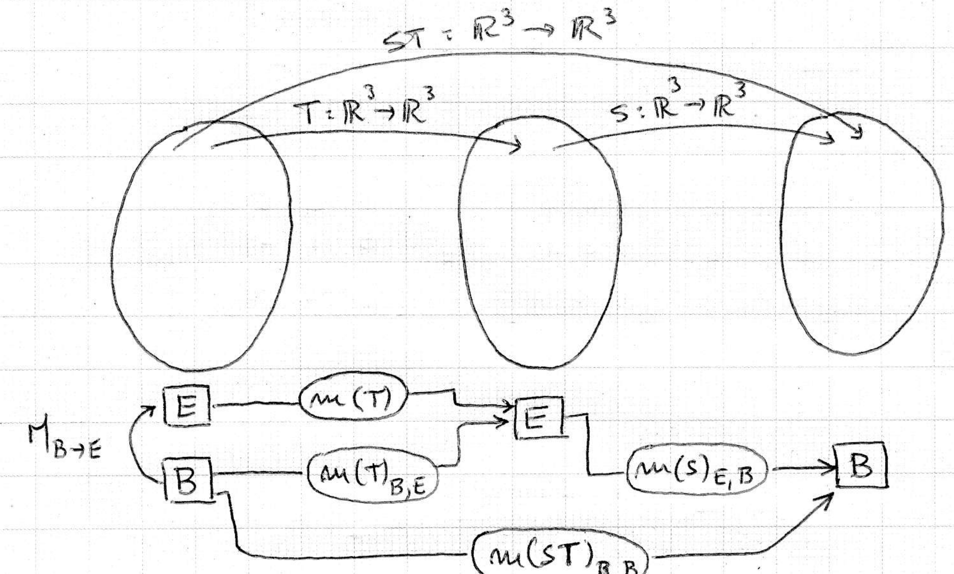
$$m(S)_{B,B} = \left[M^{-1}_{B\to E} \; m(S) \right] M_{B\to E} =$$

$$= m(S)_{E,B} \; M_{B\to E} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}_{E,B} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}_{B,B}$$

$$M_{B\to E} = M^{-1}_{E\to B} = \left[B^{-1} \right]^{-1} = B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

j)



$$m(ST)_{B,B} = m(S)_{E,B} \; m(T)_{B,E}$$

$$m(T)_{B,E} = m(T) \; M_{B\to E} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{bmatrix}_{B,E}$$

$$m(ST)_{B,B} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}_{E,B} \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{bmatrix}_{B,E} =$$

$$= \begin{bmatrix} -1 & -1 & -2 \\ -2 & -2 & -4 \\ 2 & 2 & 4 \end{bmatrix}_{B,B}$$

Designando $\vec{x}_B = (a, b, c)_B$ obtém-se

$$(ST)\,(a,b,c)_B = \begin{bmatrix} -1 & -1 & -2 \\ -2 & -2 & -4 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_B = \begin{bmatrix} -a-b-2c \\ -2a-2b-4c \\ 2a+2b+4c \end{bmatrix}_B$$

$$ST : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(a,b,c)_B \longrightarrow (-a-b-2c, -2a-2b-4c, 2a+2b+4c)_B$$

<u>NOTA</u>: A matriz $m(ST)_{B,B}$ poderia, ainda, ser calculada através dos seguintes processos alternativos

i) $m(ST) = m(S)\, m(T)$

$$m(ST)_{B,B} = M^{-1}_{B\to E}\; m(ST)\; M_{B\to E}$$

ii) $m(ST)_{B,B} = m(S)_{B,B}\; m(T)_{B,B}$

em que

$$m(T)_{B,B} = M^{-1}_{B\to E}\; m(T)\; M_{B\to E}$$

Wiz

46)

a) A característica de $m(S)$ é

$$r[m(S)] = r\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{matrix} \\ \leftarrow L_2+L_1 \\ \leftarrow L_3+2L_1 \end{matrix} = r\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 0 & 3 \end{bmatrix}\begin{matrix} \\ \\ \leftarrow L_3-L_2 \end{matrix} = r\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} = 2$$

Então $\dim S(\mathbb{R}^2) = r[m(S)] = 2$; uma vez que

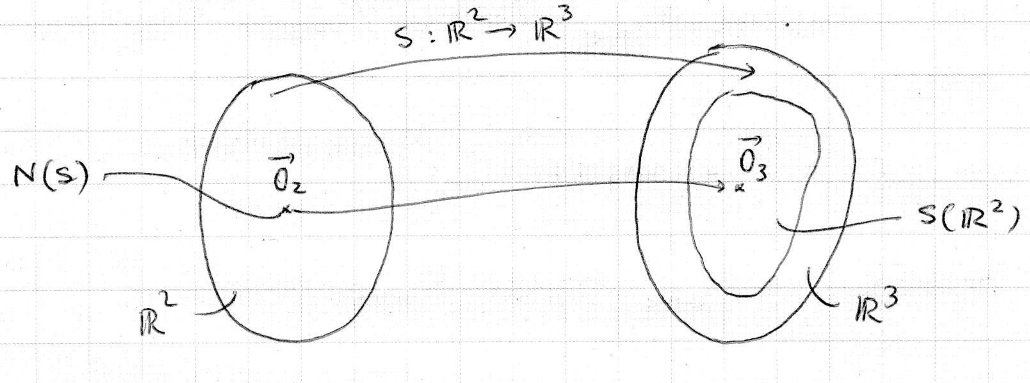
$\dim S(\mathbb{R}^2) < \dim \mathbb{R}^3 \Rightarrow S(\mathbb{R}^2) \subset \mathbb{R}^3 \Rightarrow S$ não é sobrejectiva

Por outro lado, verifica-se

$\dim N(S) = \dim \mathbb{R}^2 - \dim S(\mathbb{R}^2) = 2-2=0$, pelo que

$N(S) = \{(0,0)\} \Rightarrow S$ é injectiva

Base $N(S) = \{\ \}$



Cálculo de $S(\mathbb{R}^2) = Im(S)$

$S(\mathbb{R}^2) = \{\vec{y} = (a,b,c) \in \mathbb{R}^3 : \vec{y} = S(\vec{x}), \vec{x} \in \mathbb{R}^2\} \subset \mathbb{R}^3$

$$S(x,y) = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+y \\ x+2y \\ 2x+y \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} -1 & 1 & a \\ 1 & 2 & b \\ 2 & 1 & c \end{array}\right] \begin{array}{l} \\ \leftarrow L_2+L_1 \\ \leftarrow L_3+2L_1 \end{array} \end{array} \Leftrightarrow \left[\begin{array}{cc|c} -1 & 1 & a \\ 0 & 3 & a+b \\ 0 & 3 & 2a+c \end{array}\right] \begin{array}{l} \\ \\ \leftarrow L_3-L_2 \end{array} \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{cc|c} -1 & 1 & a \\ 0 & 3 & a+b \\ 0 & 0 & a-b+c \end{array}\right]$$

O sistema é possível e determinado, se e só se

$$a-b+c=0 \Leftrightarrow c=b-a$$

$$S(\mathbb{R}^2) = \{ \vec{y} = (a,b,c) \in \mathbb{R}^3 : c = b-a \} =$$

$$= \{ \vec{y} = (a,b,b-a) \in \mathbb{R}^3 \} =$$

$$= \{ \vec{y} = a(1,0,-1) + b(0,1,1) \in \mathbb{R}^3 \}$$

$$\text{Base } S(\mathbb{R}^2) = \{ (1,0,-1), (0,1,1) \}$$

b) Como se verificou na alínea a), S é injectiva.

A característica de $m(R)$ é

$$r[m(R)] = r \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \leftarrow L_2+L_1 \end{array} = r \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 2$$

Então $\dim R(\mathbb{R}^3) = r[m(R)] = 2$ e, portanto,

$\dim N(R) = \dim \mathbb{R}^3 - \dim R(\mathbb{R}^3) = 3 - 2 = 1$, ou seja,

$N(R) \neq \{(0,0,0)\} \Rightarrow R$ não é injectiva

Sabendo que

$$|m(T)| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

conclui-se que $m(T)$ é uma matriz singular, pelo que a transformação linear T não é injectiva.

Recorrendo ao sistema de equações lineares que conduziu ao cálculo de $S(\mathbb{R}^2)$, obtém-se

$$\left[\begin{array}{cc|c} -1 & 1 & a \\ 0 & 3 & a+b \\ 0 & 0 & a-b+c \end{array}\right] \Leftrightarrow \quad y = \frac{a+b}{3}, \quad x = \frac{b-2a}{3} \quad e$$

$$c = b-a$$

Então

$$S^{-1}: S(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$$
$$(a, b, b-a) \longrightarrow \left(\frac{b-2a}{3}, \frac{a+b}{3}\right)$$

c) O operador RTS é possível:

$$RTS: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

O operador TRS não é possível; o conjunto de chegada de R, o espaço $\mathbb{R}^2$, é diferente do domínio de T, o espaço $\mathbb{R}^3$.

O operador SRT é possível:

$$SRT: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$m(RTS) = m(RT)\, m(S) =$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} m(S) =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

Designando $\vec{x} = (x, y) \in \mathbb{R}^2$, então

$$(RTS)(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3y \end{bmatrix}$$

$$\begin{aligned} RTS : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longrightarrow (0, 3y) \end{aligned}$$

$$m(SRT) = m(S)\, m(RT) =$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

Designando $\vec{x} = (x, y, z) \in \mathbb{R}^3$, então

$$(SRT)(x,y,z) = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - z \\ 4y - 2z \\ 2y - z \end{bmatrix}$$

$$\begin{aligned} SRT : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longrightarrow (2y-z, 4y-2z, 2y-z) \end{aligned}$$

Wig

58)

a) Seja a base para o espaço $\mathbb{R}^3$

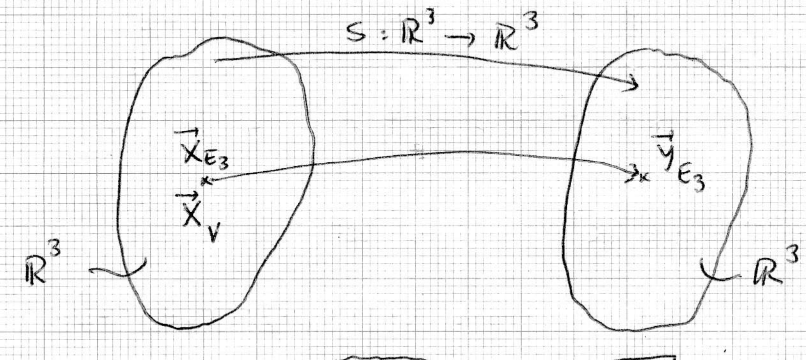
$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1,1,0), (0,-1,1), (1,-1,1)\} \subset \mathbb{R}^3$$

Sabendo que

$$S(\vec{v}_1) = S(1,1,0) = (3,-1,4)$$
$$S(\vec{v}_2) = S(0,-1,1) = (-1,-2,-1)$$
$$S(\vec{v}_3) = S(1,-1,1) = (0,-4,0)$$

a representação matricial de S em relação às base V (domínio) e E_3 (conjunto de chegada) é

$$m(S)_{V,E_3} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & -2 & -4 \\ 4 & -1 & 0 \end{bmatrix}_{V,E_3}$$



$$\begin{tikzcd}
\boxed{V} \arrow[r, "m(S)_{V,E_3}"] & \boxed{E_3} \\
\boxed{E_3} \arrow[u, bend left, "M_{E_3 \to V}"] \arrow[ur, "m(S)"'] &
\end{tikzcd}$$

$$m(S) = m(S)_{V,E_3} \, M_{E_3 \to B}$$

Designando

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{e} \quad V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

então

$$E_3 X_{E_3} = V X_V \Rightarrow X_V = V^{-1} E_3 X_{E_3} \Rightarrow M_{E_3 \to V} = V^{-1} E_3 = V^{-1} I_3 = V^{-1}$$

$$M_{E_3 \to V} = V^{-1} = \frac{1}{|V|} [\text{cof } V]^T = \frac{1}{1} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$|V| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 1(-1)^2 \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = 1$$

$$m(S) = \begin{bmatrix} 3 & -1 & 0 \\ -1 & -2 & -4 \\ 4 & -1 & 0 \end{bmatrix}_{V,E_3} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

Designando $\vec{x} = (x,y,z) \in \mathbb{R}^3$ as coordenadas do vector $\vec{x}$ em relação à base canónica E_3, então

$$S(x,y,z) = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+z \\ -2x+y-z \\ x+3y+2z \end{bmatrix}$$

A lei de transformação de S em relação à base canónica é

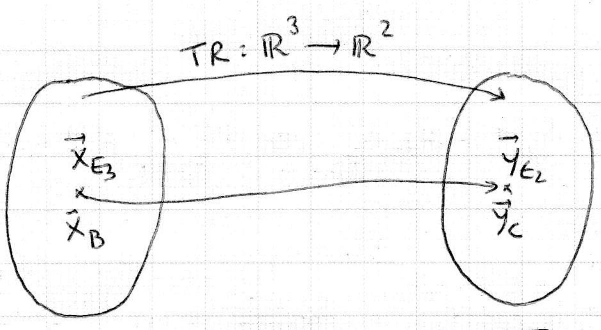
$$S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longrightarrow (x+2y+z, -2x+y-z, x+3y+2z)$$

b)

$$m(TR) = m(T)\, m(R) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & -3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -4 \\ 0 & -2 & -2 \end{bmatrix}$$

$R(1,0,0) = (1,2,0)$
$R(0,1,0) = (-1,-1,1)$
$R(0,0,1) = (-2,-3,1)$

Wij



$$
\begin{tikzcd}
\boxed{E_3} \arrow[rr, "m(TR)"] & & \boxed{E_2} \\
\boxed{B} \arrow[u, bend left, "M_{B\to E_3}"] \arrow[rr, "m(TR)_{B,C}"] & & \boxed{C} \arrow[u, bend right, "M_{C\to E_2}"']
\end{tikzcd}
$$

$$m(TR)_{B,C} = M^{-1}_{C\to E_2}\, m(TR)\, M_{B\to E_3}$$

Designando

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \qquad e \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

então

$$E_3 X_{E_3} = B X_B \Rightarrow X_{E_3} = E_3^{-1} B X_B \Rightarrow M_{B\to E_3} = E_3^{-1} B = I_3 B = B$$

Designando

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad e \qquad C = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

então

$$E_2 Y_{E_2} = C Y_C \Rightarrow Y_{E_2} = E_2^{-1} C Y_C \Rightarrow M_{C\to E_2} = E_2^{-1} C = I_2 C = C$$

$$M^{-1}_{C\to E_2} = C^{-1} = \frac{1}{|C|}\,[\mathrm{cof}\, C]^T = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}^T = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = 4$$

$$m(TR)_{B,C} = \left[M^{-1}_{C\to E_2} \; m(TR) \right] M_{B\to E_3} =$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -4 \\ 0 & -2 & -2 \end{bmatrix} M_{B\to E_3} =$$

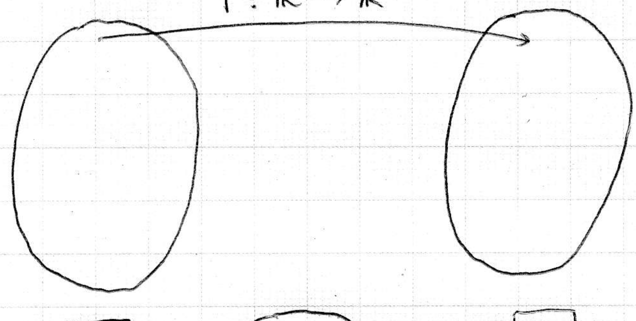
$$= \frac{1}{4}\begin{bmatrix} 6 & 0 & -6 \\ 6 & -4 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 & -6 & 0 \\ 2 & -14 & -4 \end{bmatrix}_{B,C} =$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -3 & 0 \\ 1 & -7 & -2 \end{bmatrix}_{B,C}$$

c)

$$m(TR+T)_{B,C} = m(TR)_{B,C} + m(T)_{B,C}$$

$T : \mathbb{R}^3 \to \mathbb{R}^2$



$$\begin{tikzcd} \boxed{E_3} \arrow[r, "m(T)"] & \boxed{E_2} \\ \boxed{B} \arrow[u, bend left, "M_{B\to E_3}"] \arrow[r, "m(T)_{B,C}"] & \boxed{C} \arrow[u, bend right, "M_{C\to E_2}"'] \end{tikzcd}$$

$$m(T)_{B,C} = \left[M^{-1}_{C\to E_2} \; m(T) \right] M_{B\to E_3} =$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} M_{B\to E_3} \; (=)$$

$$\Leftrightarrow \quad m(T)_{B,C} = \frac{1}{4}\begin{bmatrix} 0 & 3 & 3 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 6 & 3 \\ 5 & 2 & 5 \end{bmatrix}_{B,C}$$

$$m(TR+T)_{B,C} = \frac{1}{4}\begin{bmatrix} 6 & -6 & 0 \\ 2 & -14 & -4 \end{bmatrix}_{B,C} + \frac{1}{4}\begin{bmatrix} 3 & 6 & 3 \\ 5 & 2 & 5 \end{bmatrix}_{B,C} =$$

$$= \frac{1}{4}\begin{bmatrix} 9 & 0 & 3 \\ 7 & -12 & 1 \end{bmatrix}_{B,C}$$