Thinking Recursively

Part II

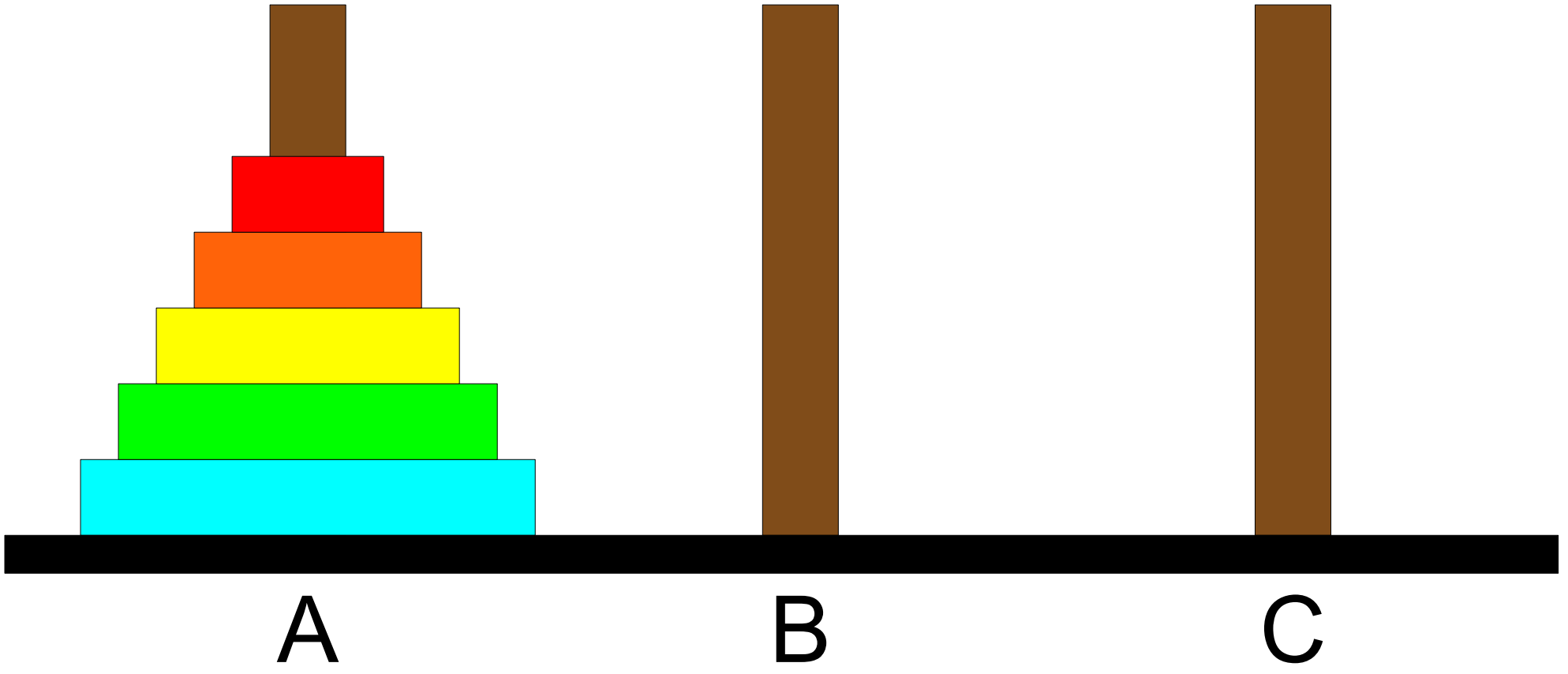
Review from Yesterday...

Recursive Problem-Solving

```
if (problem is sufficiently simple) {
    Directly solve the problem.
    Return the solution.
} else {
    Split the problem up into one or more smaller
        problems with the same structure as the original.
    Solve each of those smaller problems.
    Combine the results to get the overall solution.
    Return the overall solution.
}
```

Solving the Towers of Hanoi

Towers of Hanoi



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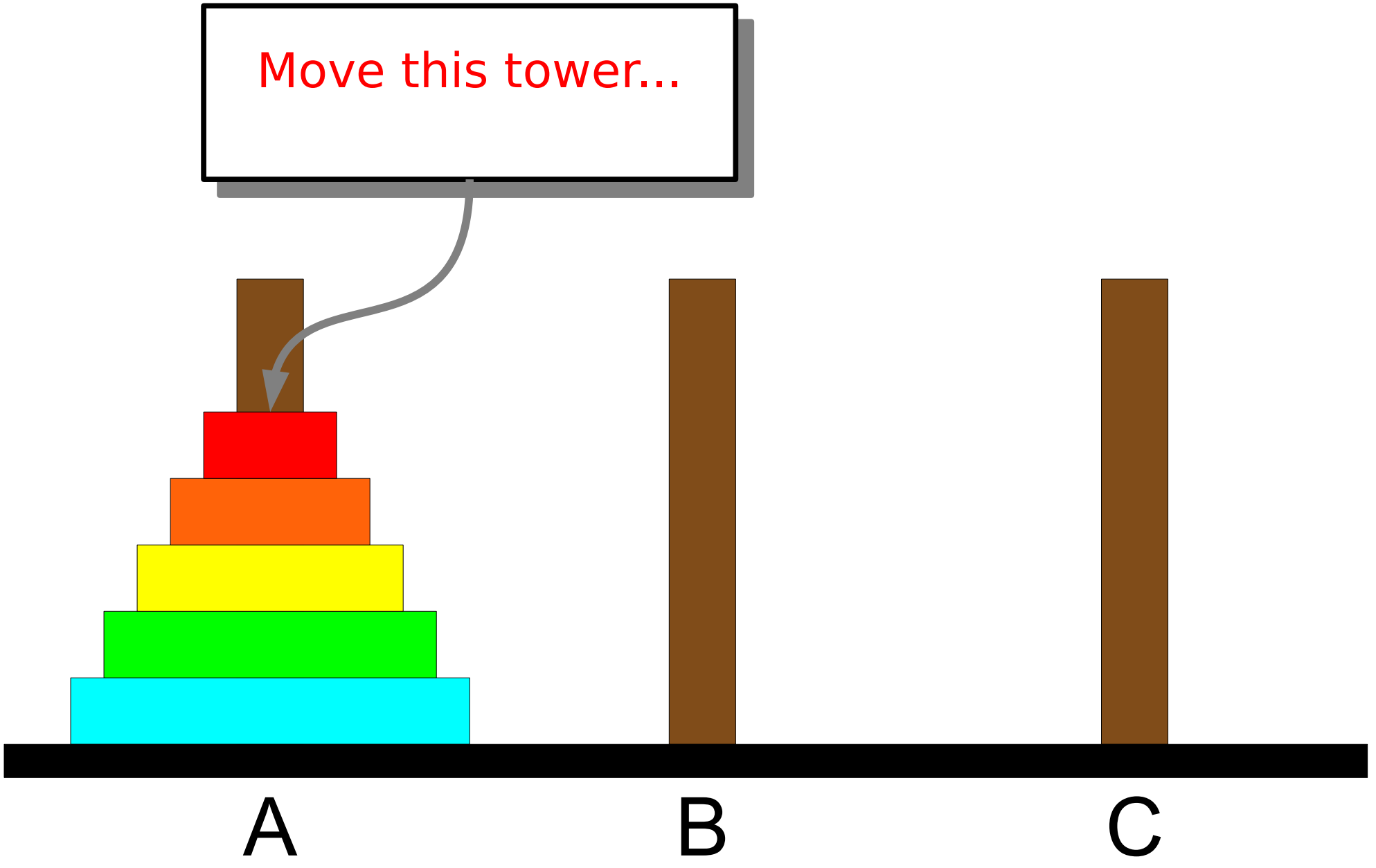

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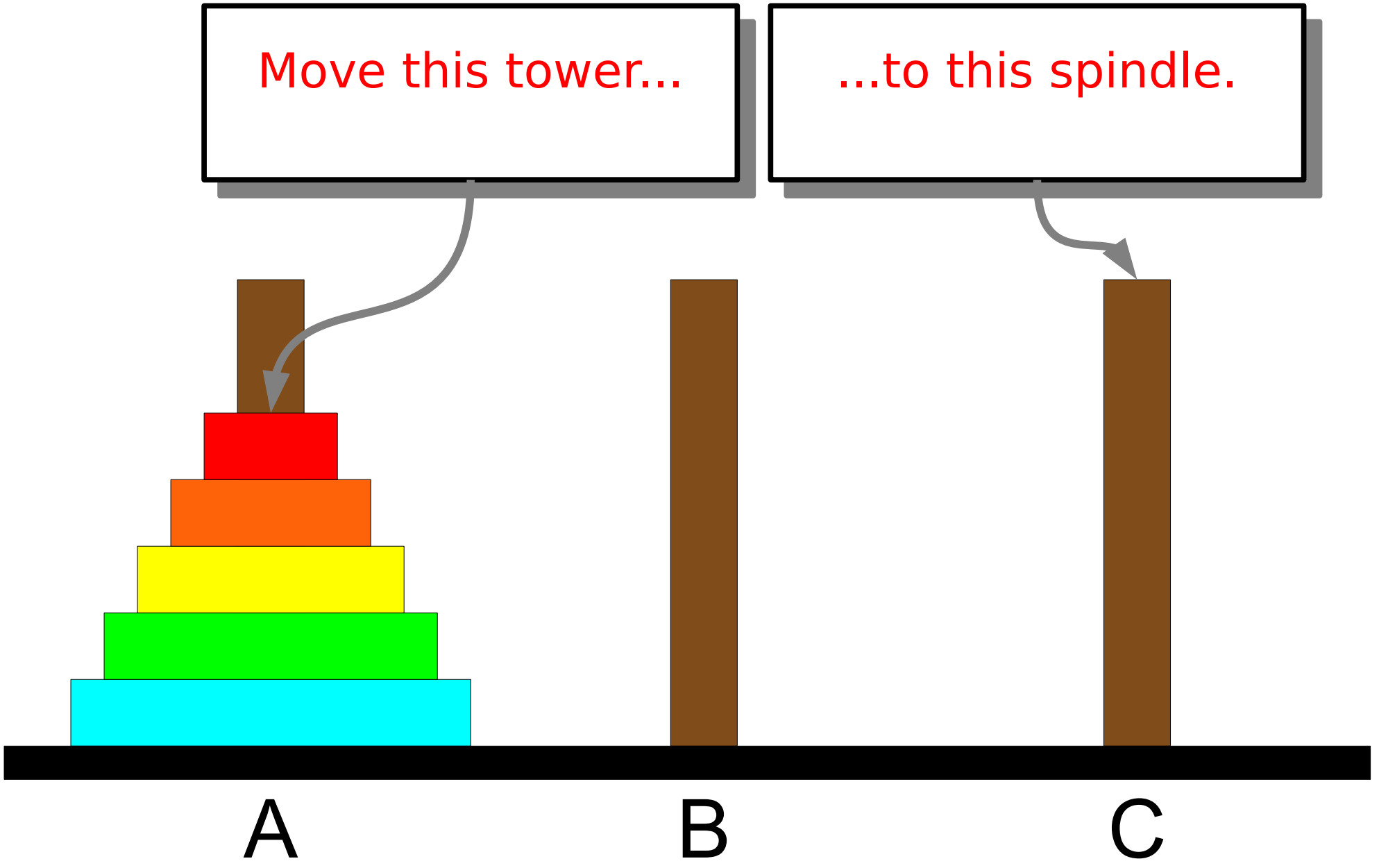

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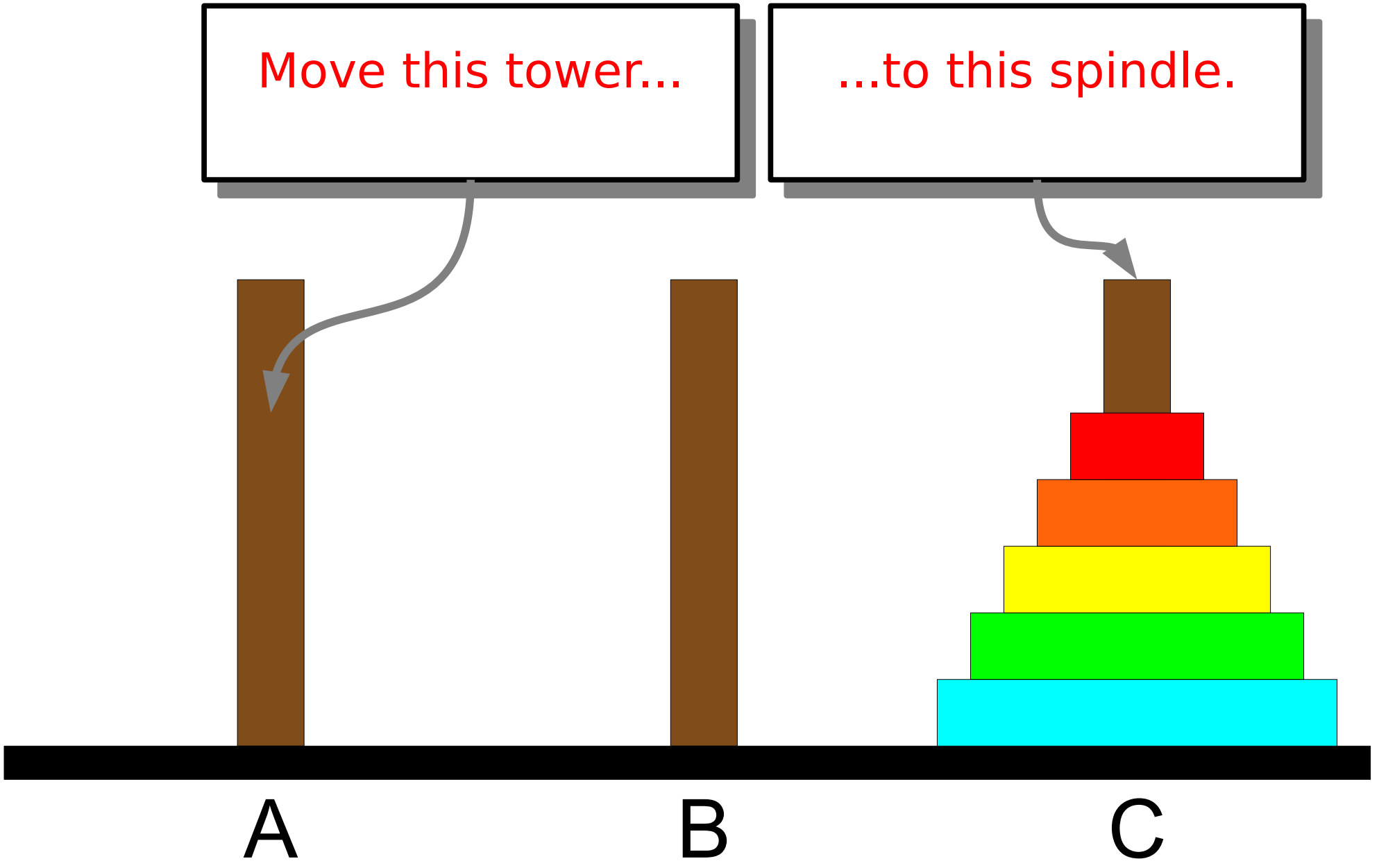

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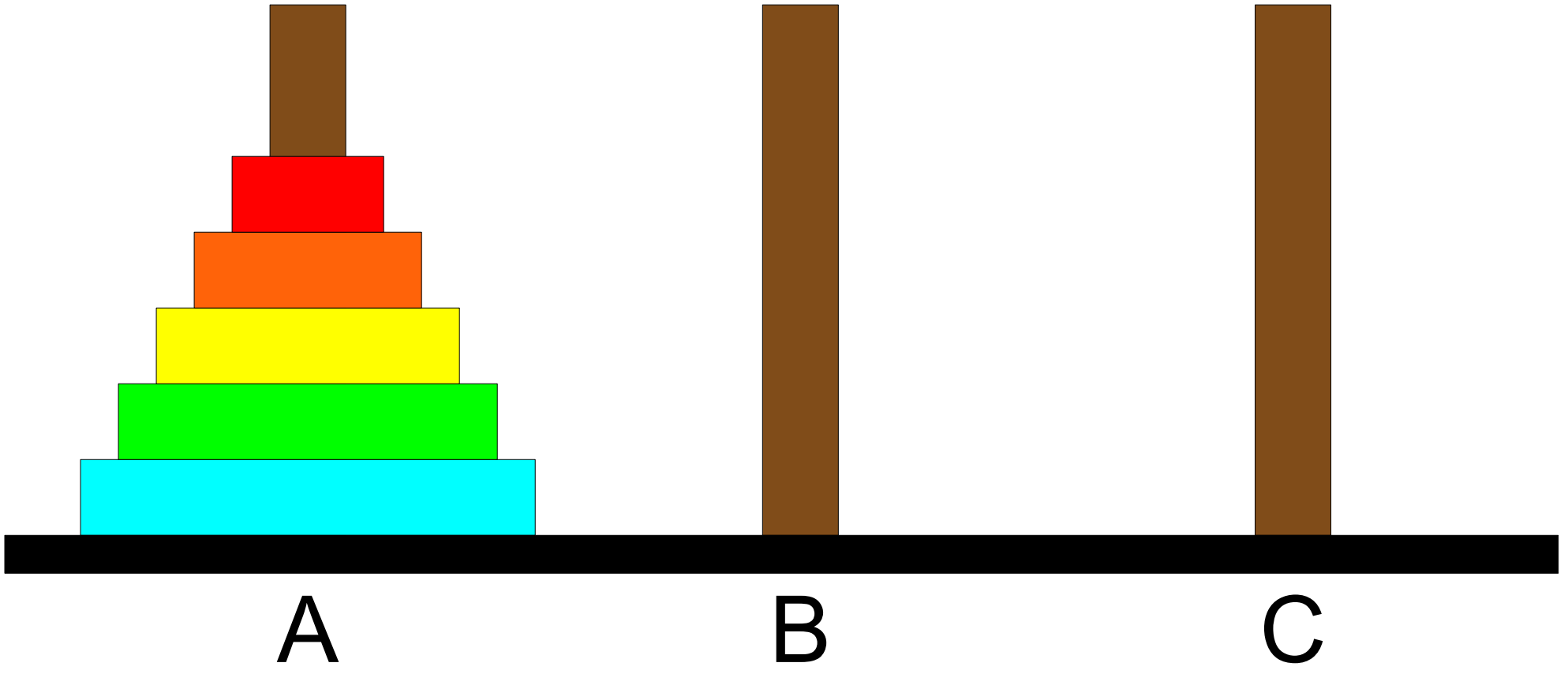

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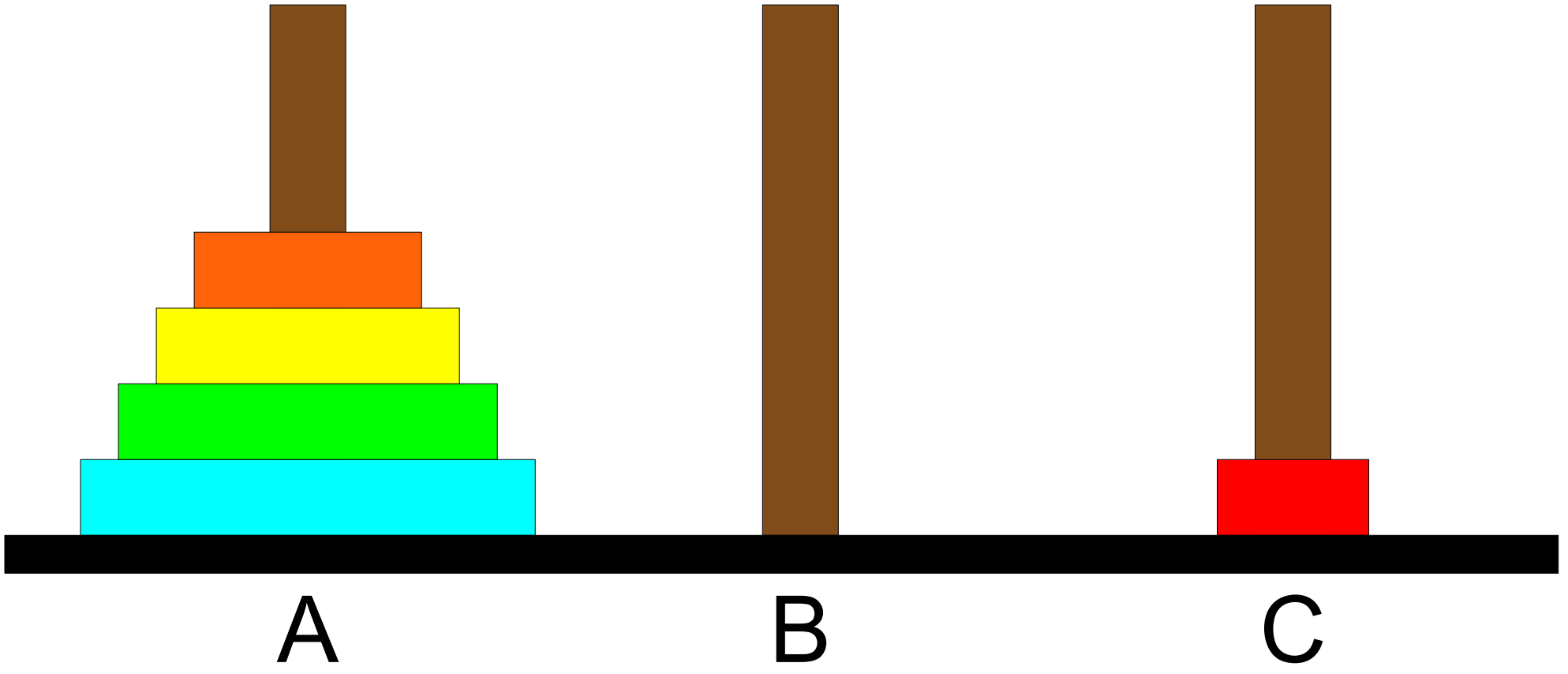

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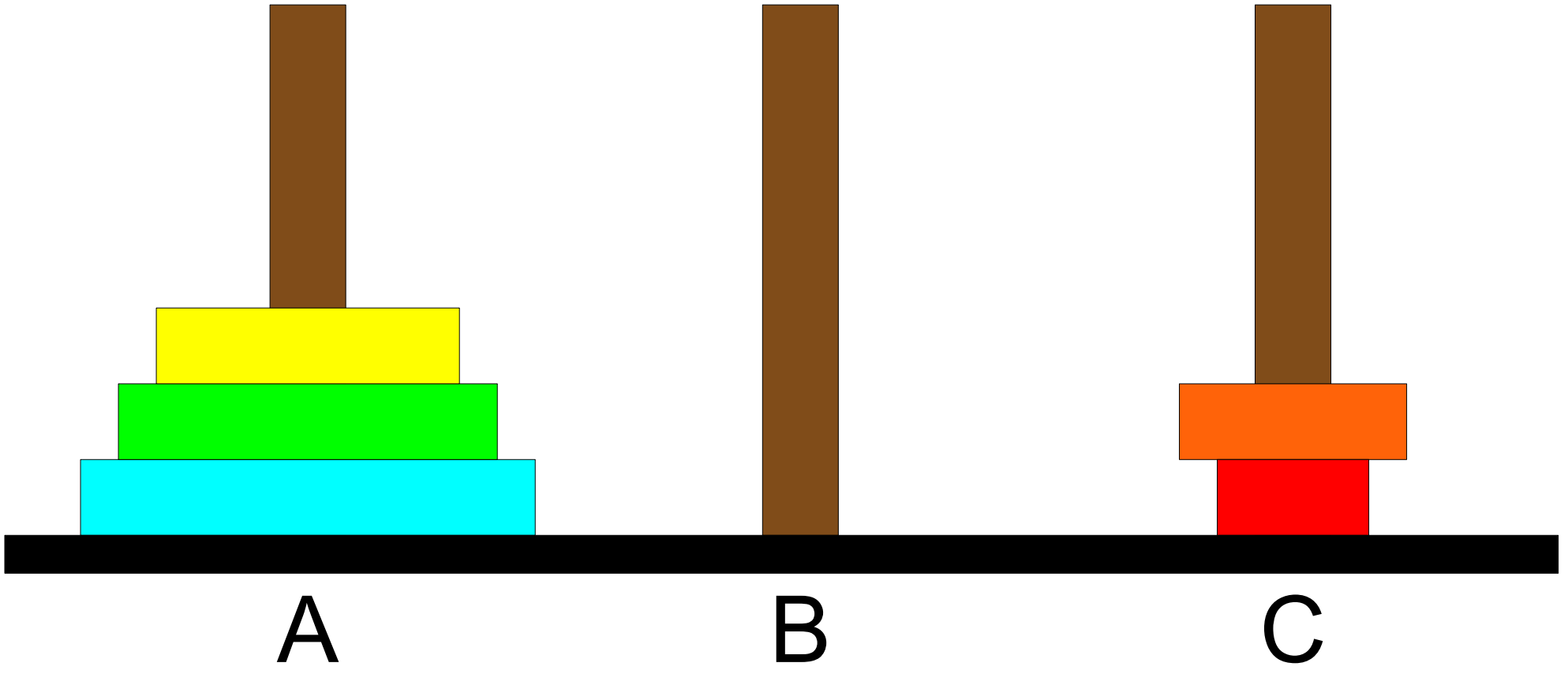

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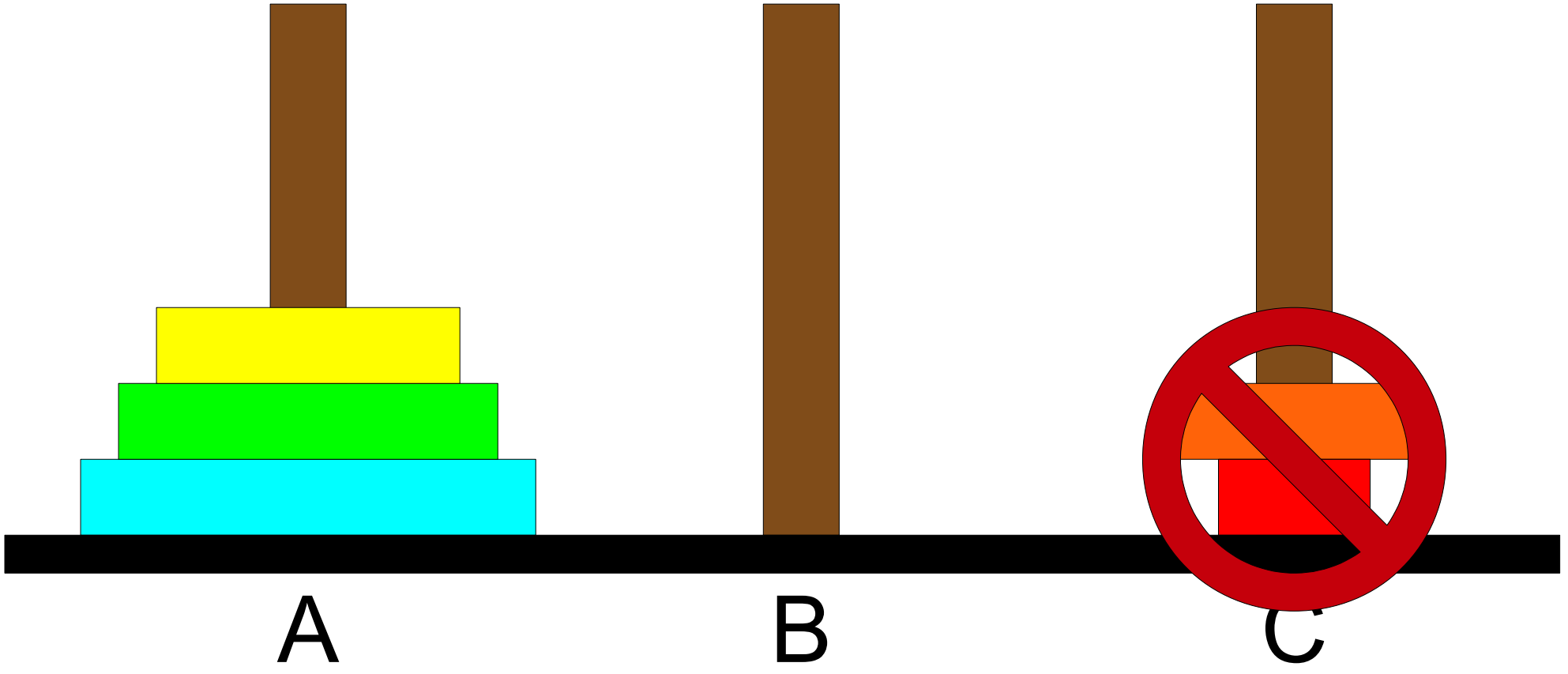

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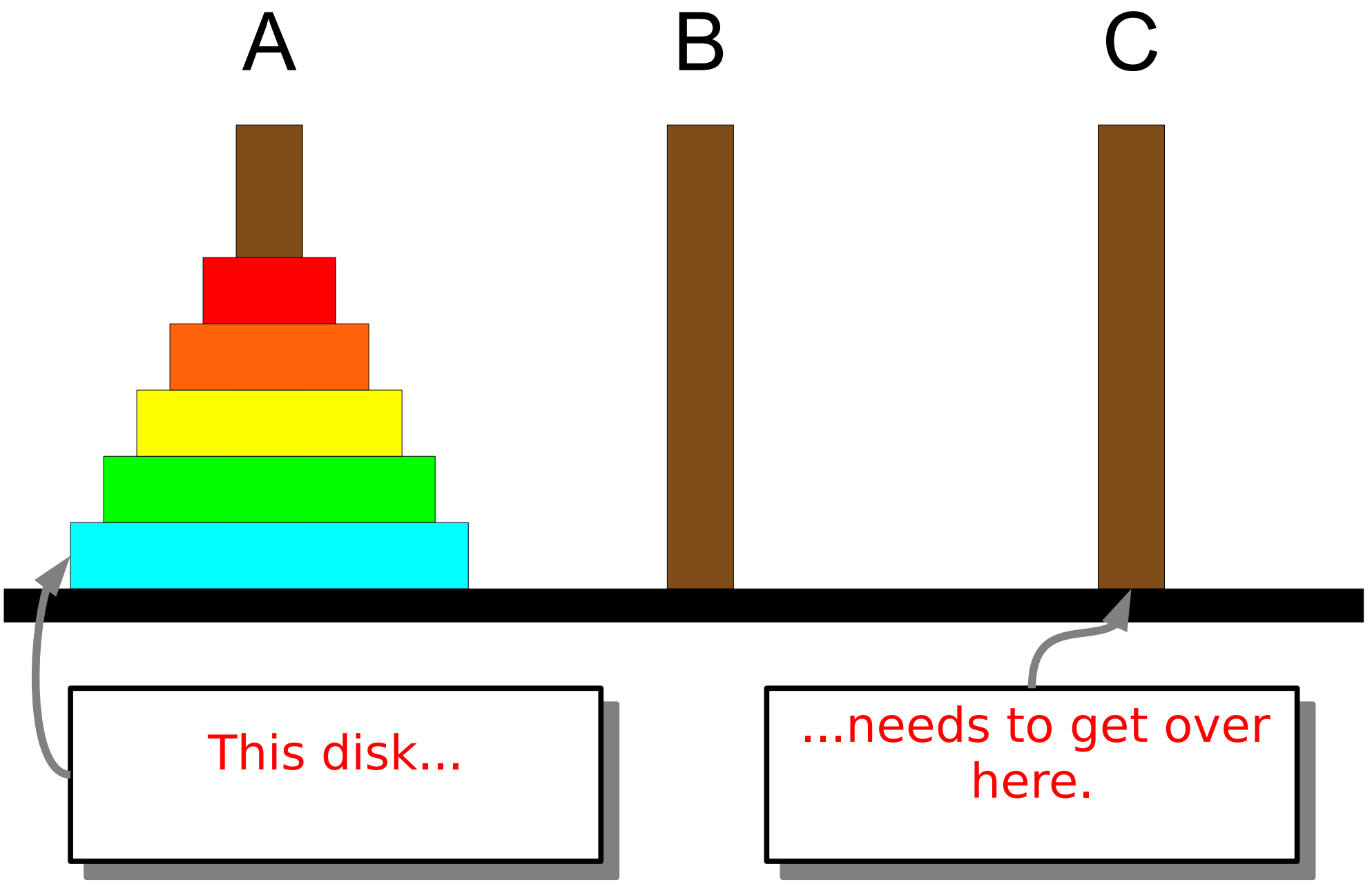

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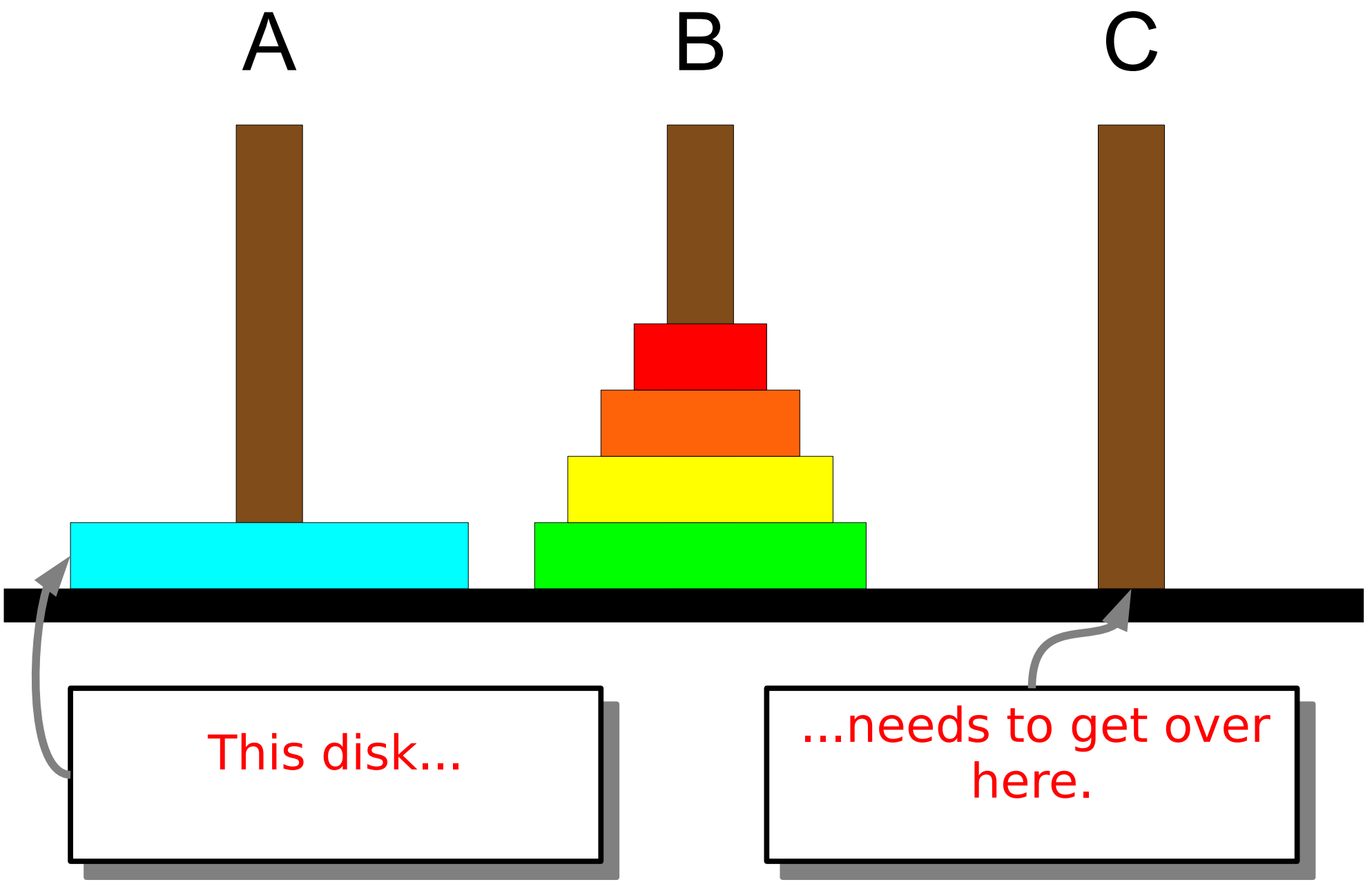

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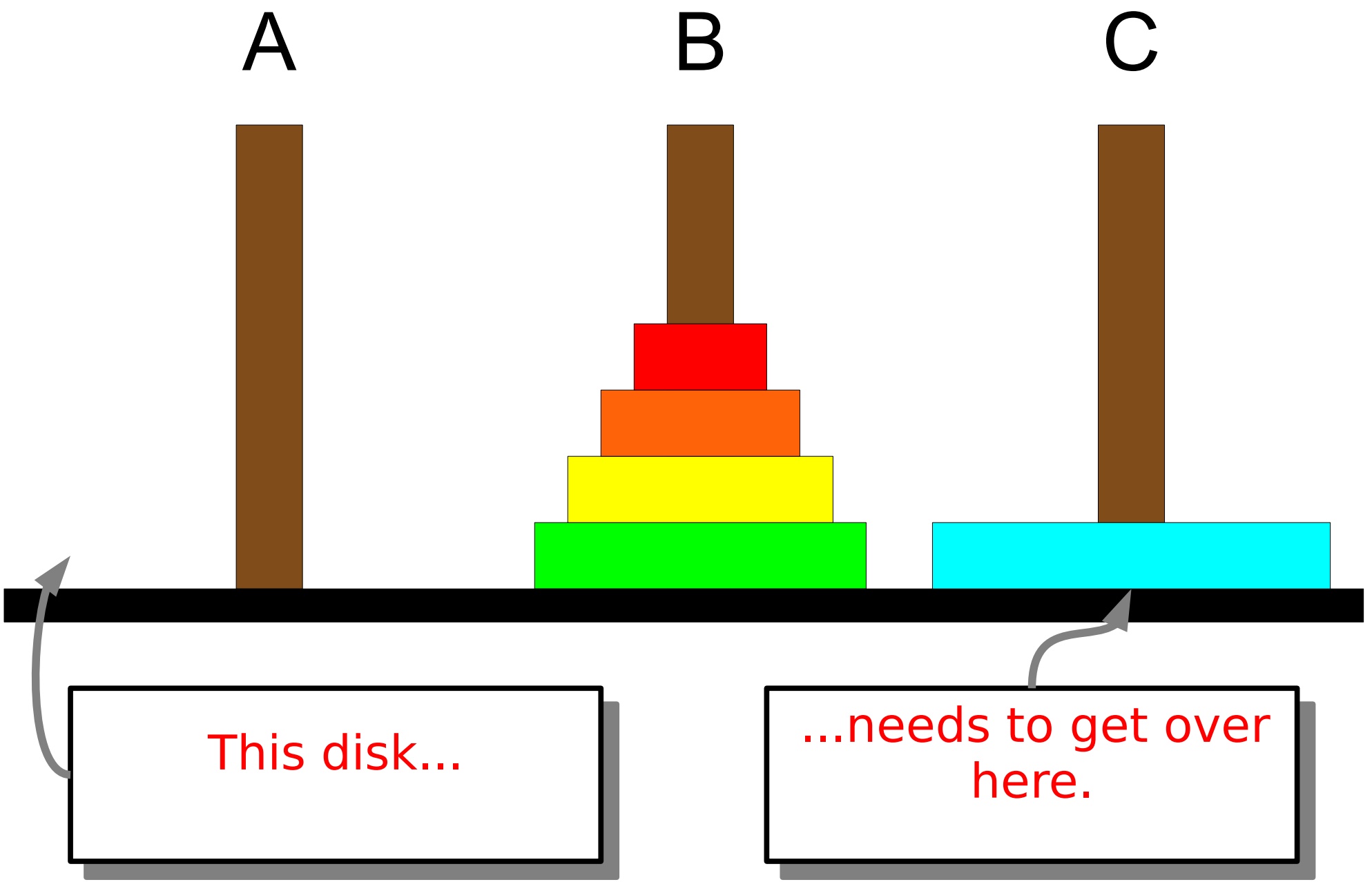

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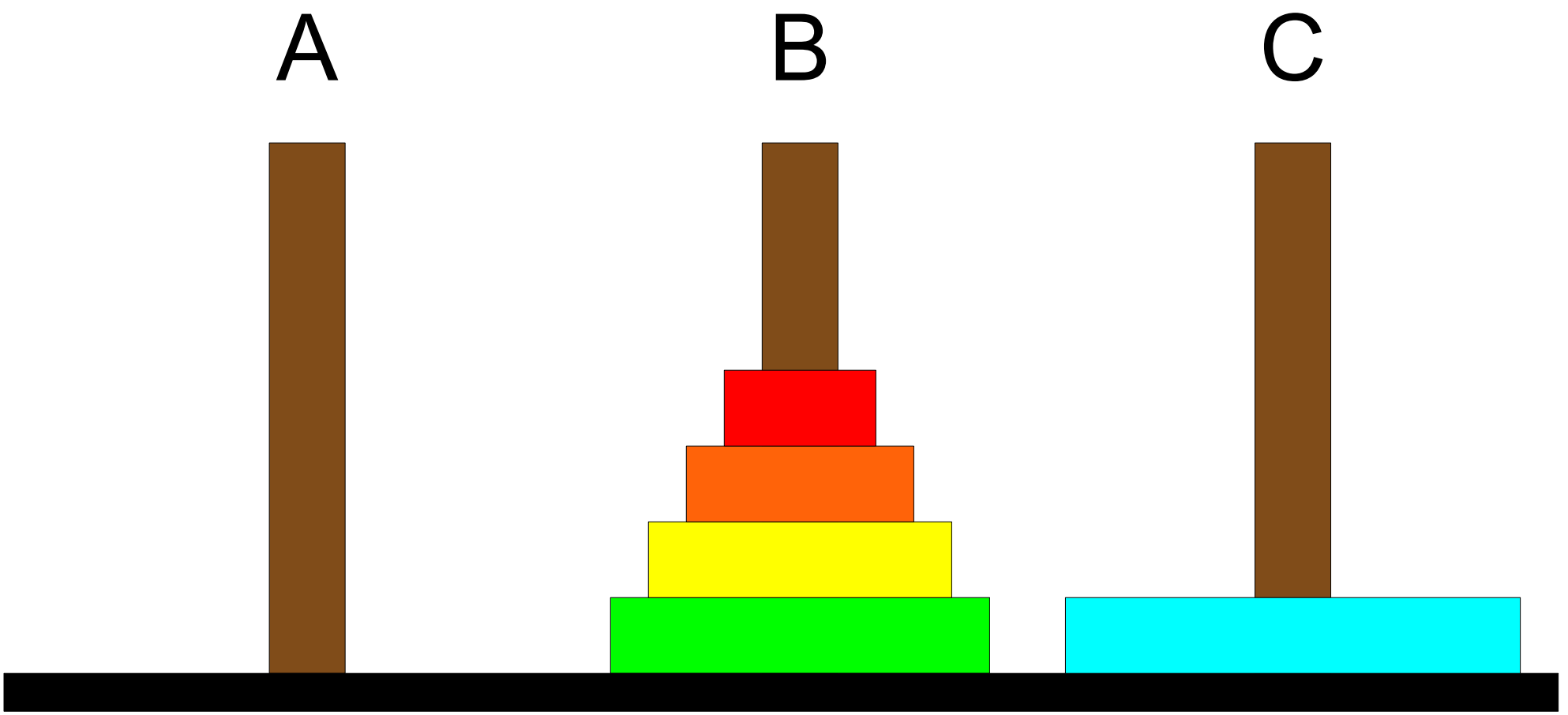

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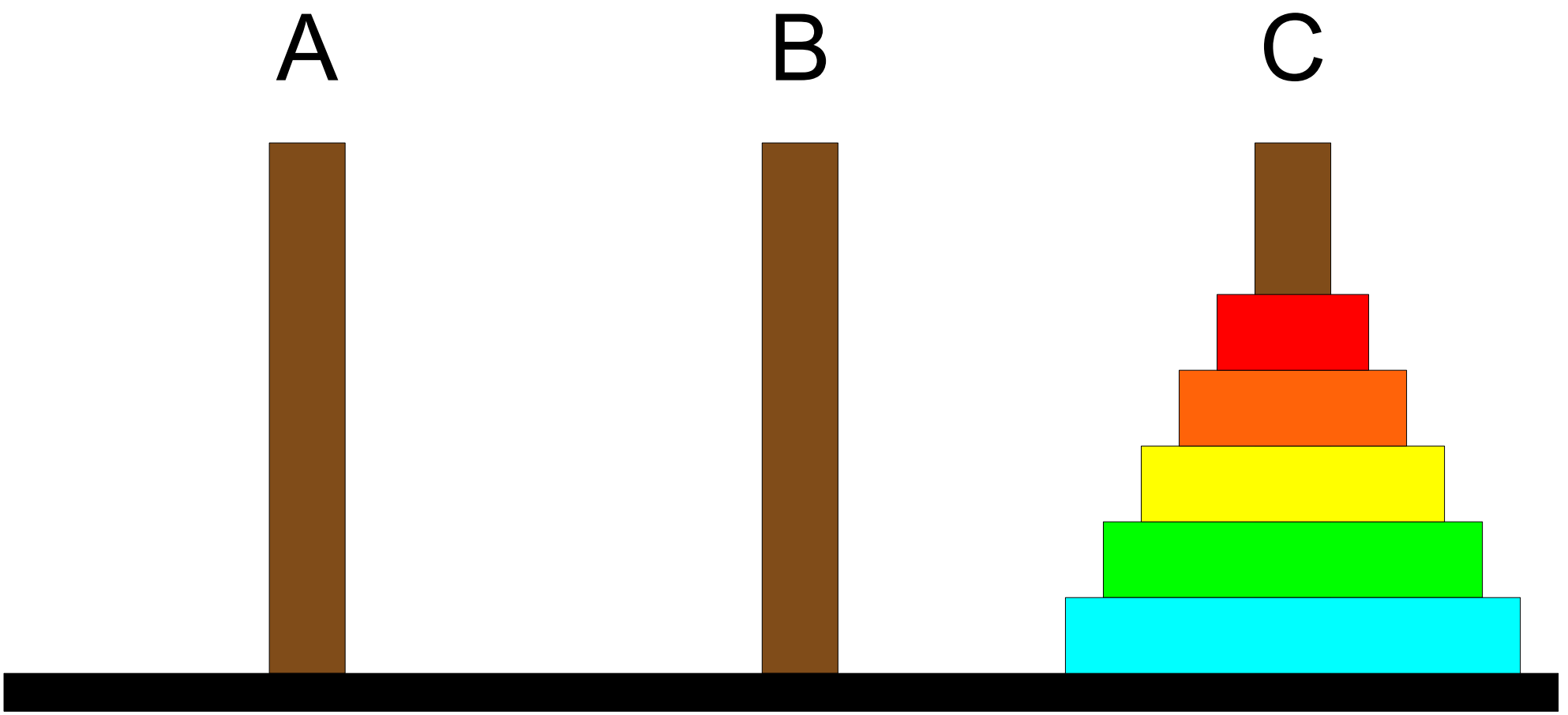
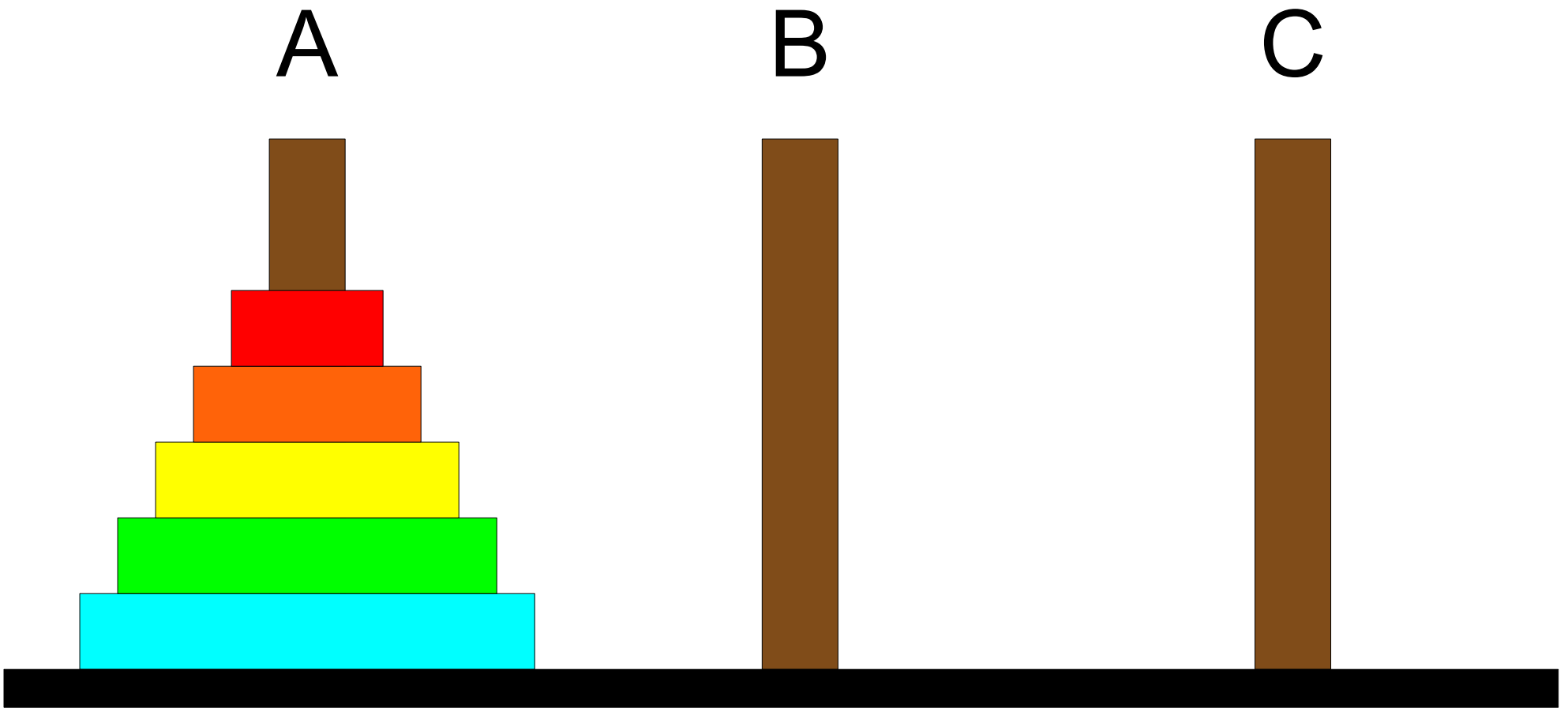


Recursive “Leap of Faith”



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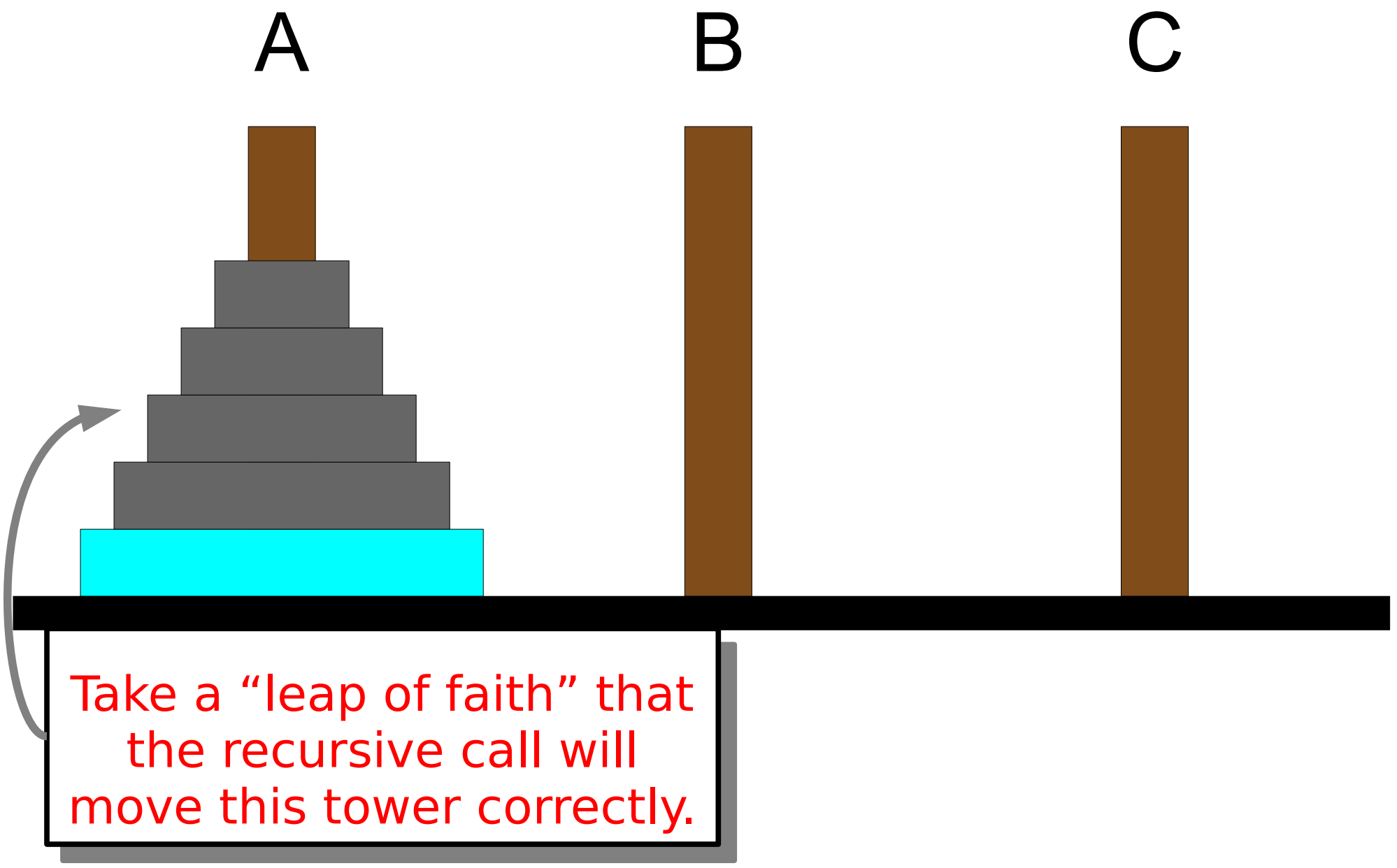

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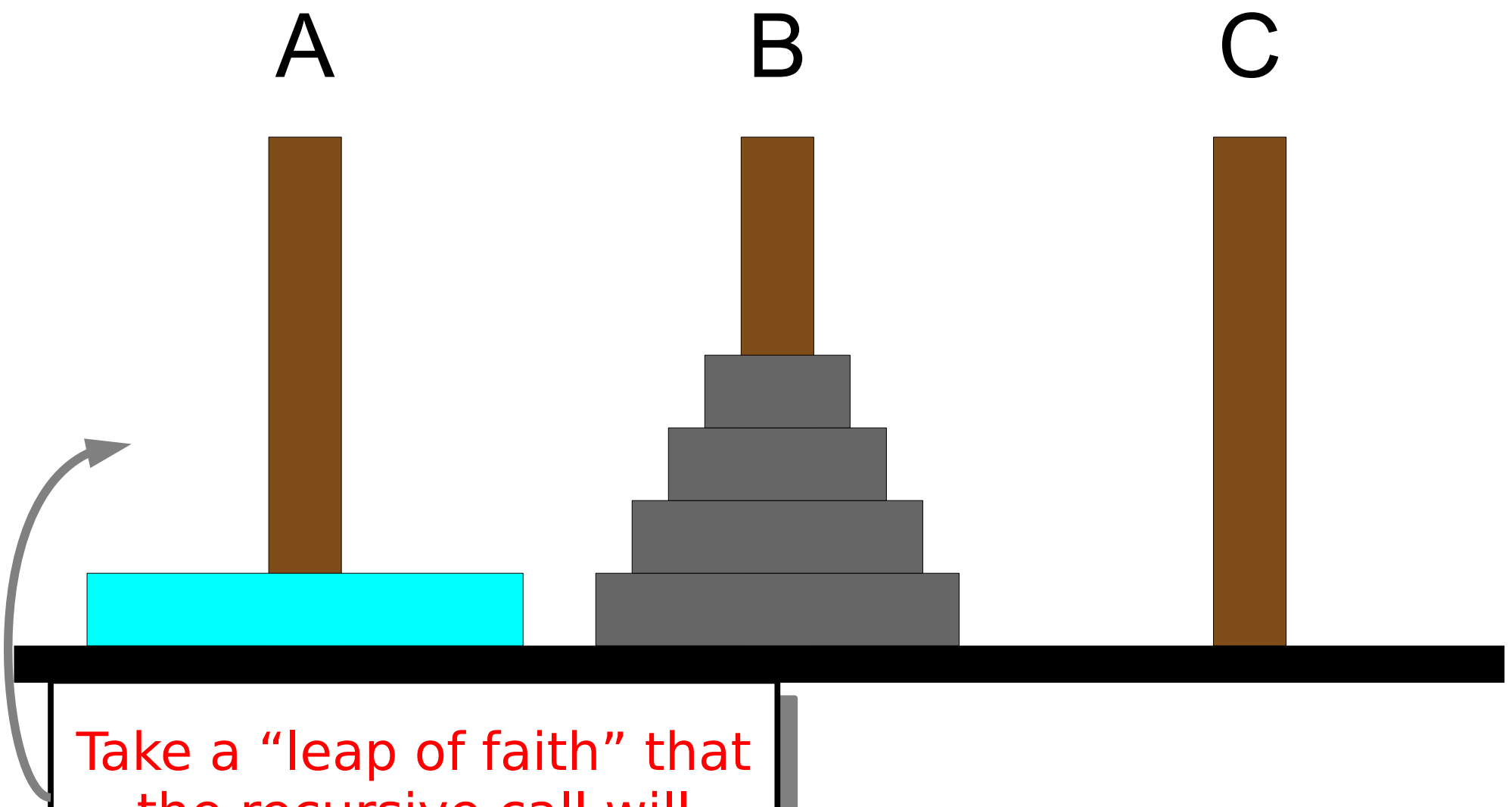

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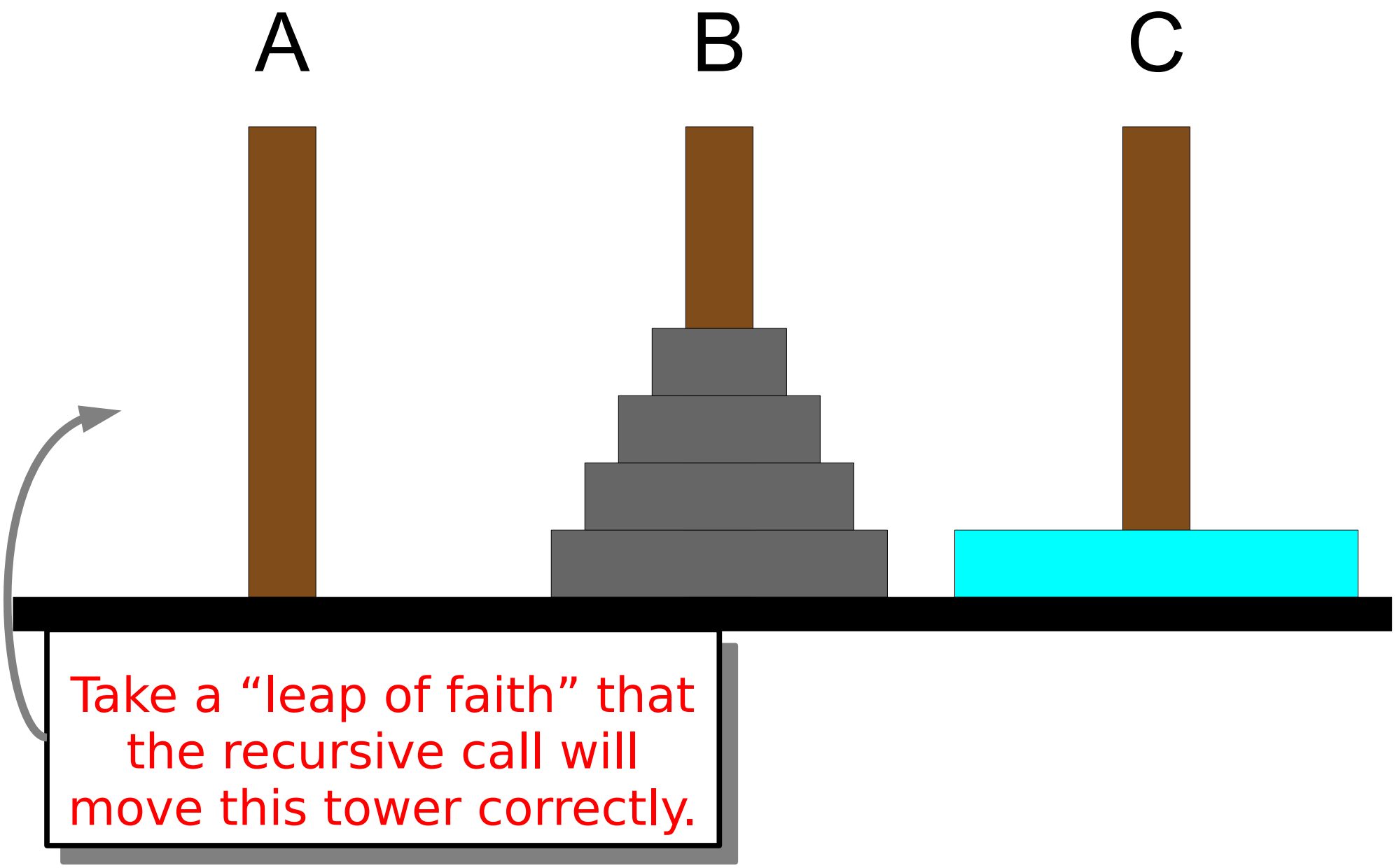

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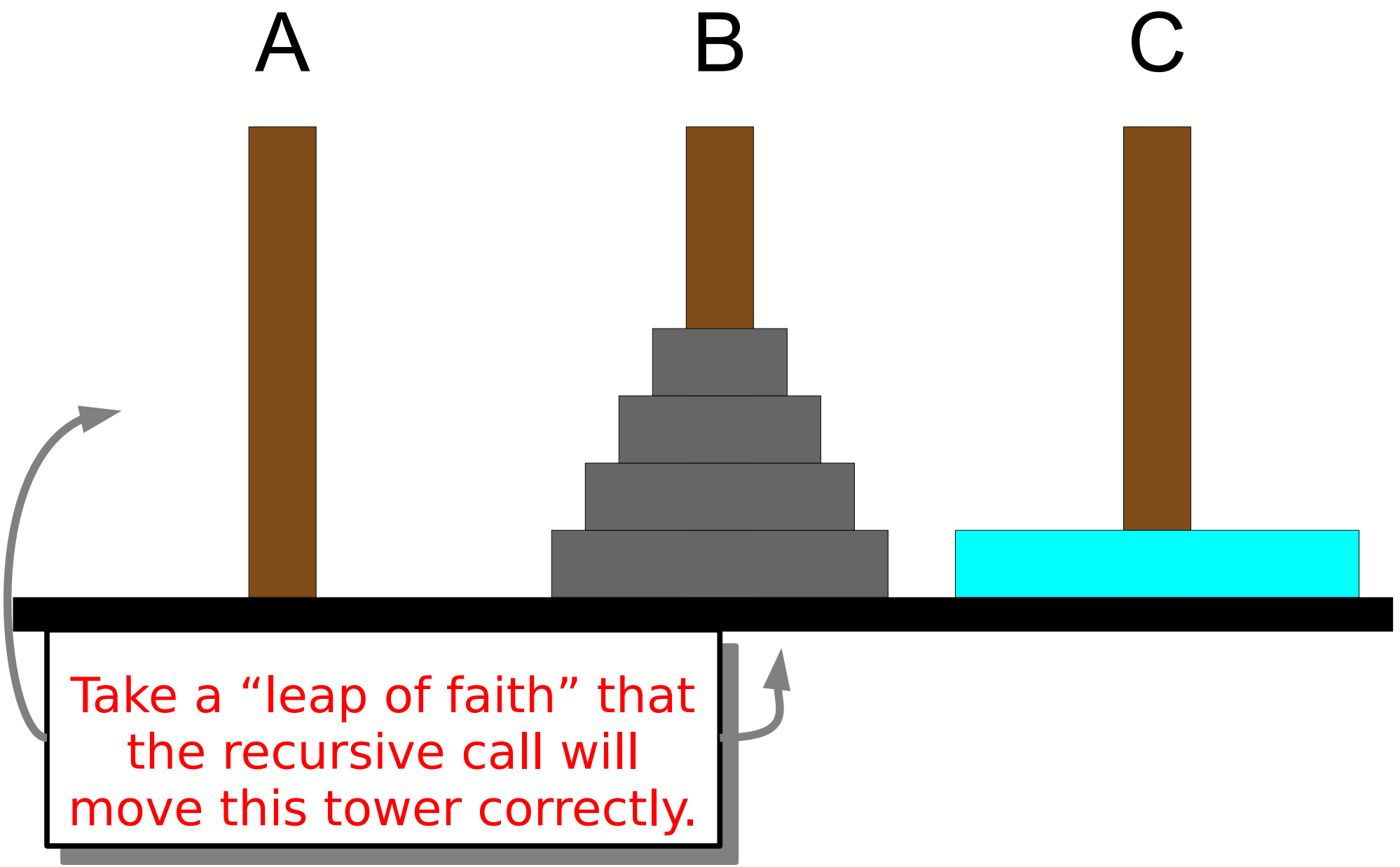

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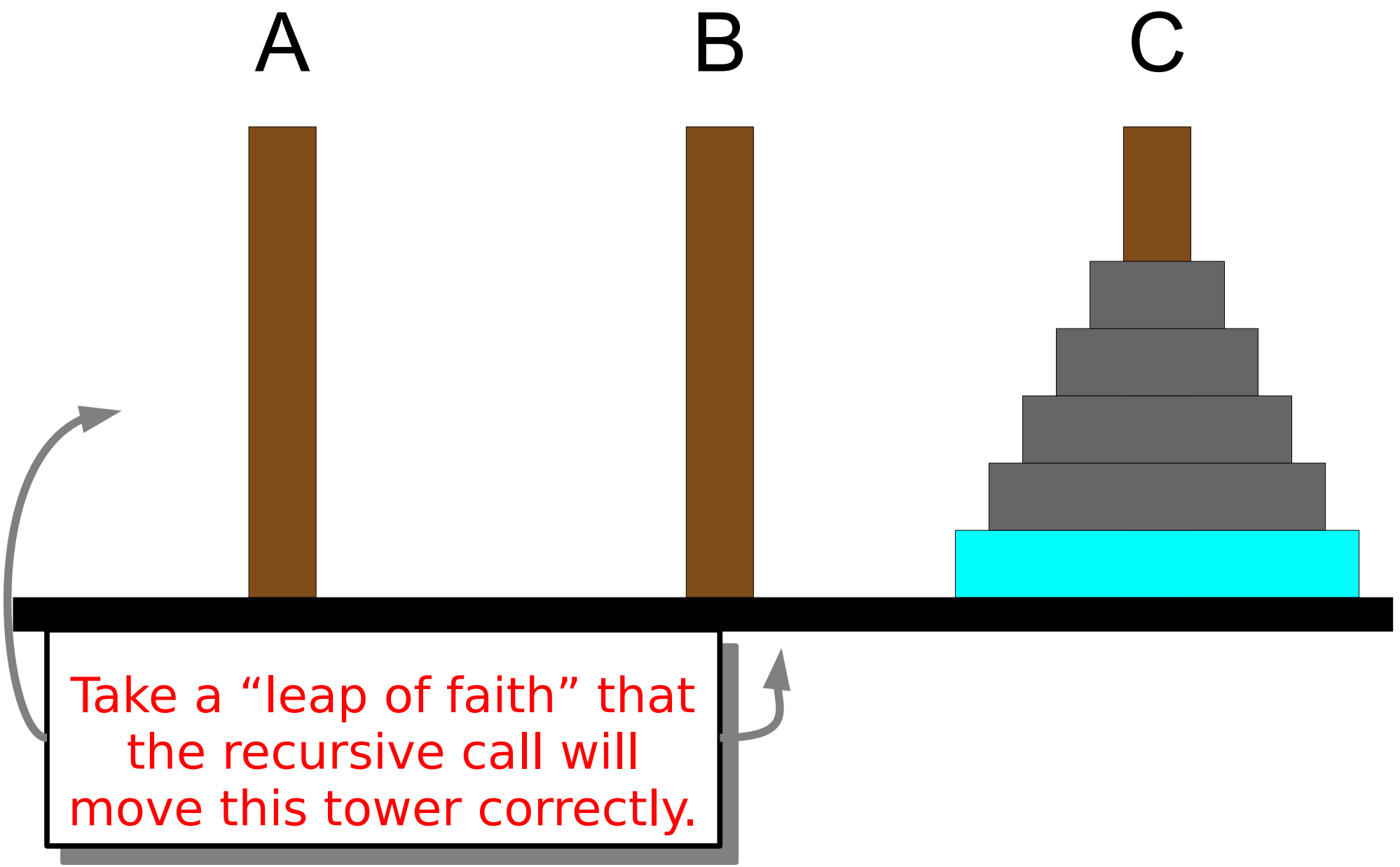
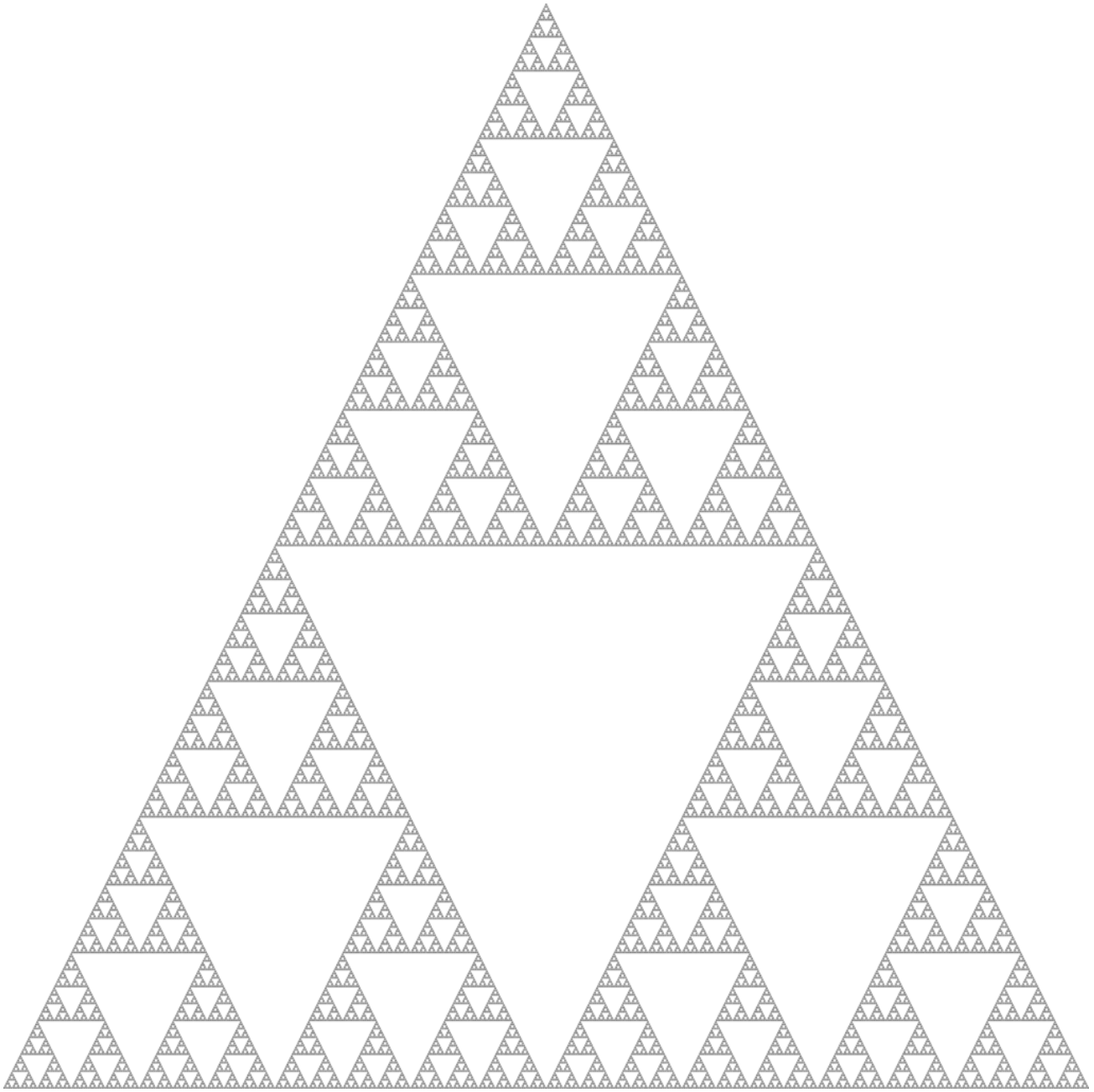


Writing Recursive Functions

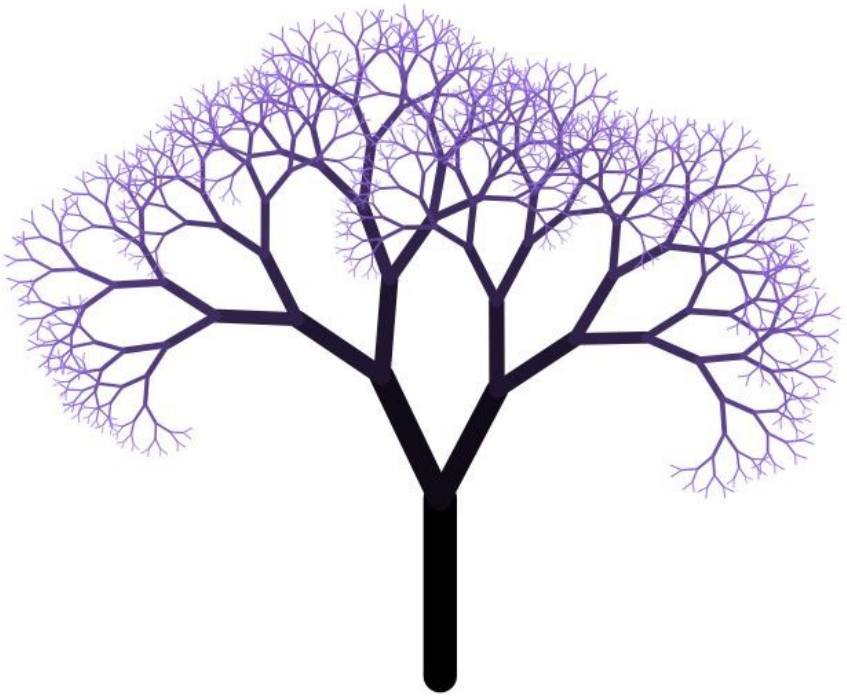
- Another way of putting this: when writing recursive functions it can be very helpful to **assume** that the recursive call will do the right thing.
- This is helpful because it's very hard to think through an **exponential** number of recursive calls.

New Stuff...



A **fractal image** is an image that is defined in terms of smaller versions of itself.

Fractal Trees



- We can generate a fractal tree as follows:
  - Grow in some direction for a period of time.
  - Then, split and grow smaller trees outward at some angle.

Trees
(Pseudocode)

`Trees.cpp`
(Computer)

More Trees

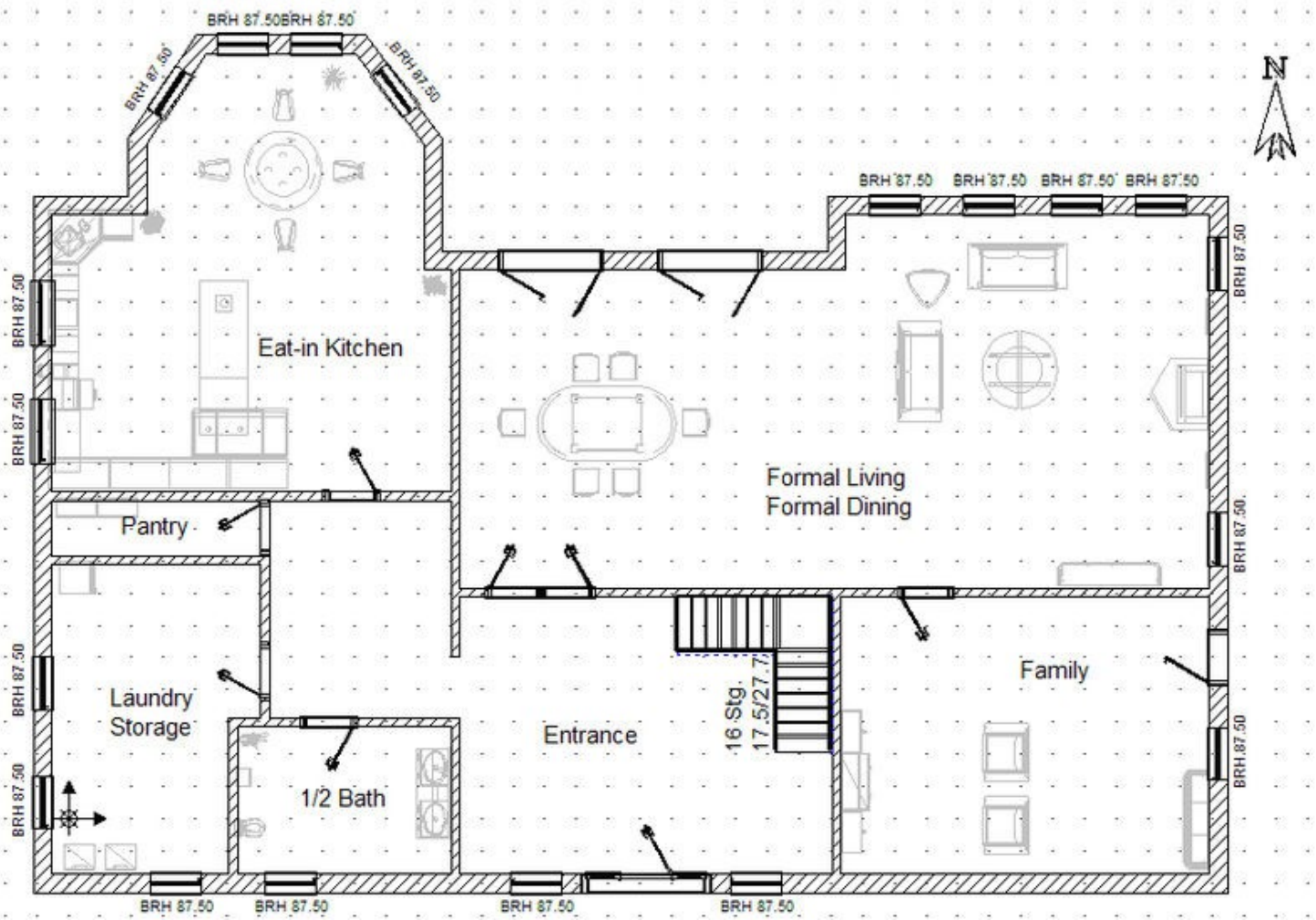
- What if you change the amount of branching?
- What if you make the lines thicker?
- What if you allow the tree to keep growing after it branches?
- What if you color the branches and leaves differently?
- What if you try to space the branches apart more realistically?
- Stanford **Dryad** program uses a combination of recursion, machine learning, and human feedback to design aesthetically pleasing trees.
  - Check it out at **http://dryad.stanford.edu/**

An Amazing Website

http://recursivedrawing.com/

Exhaustive Recursion

Sensor Placement



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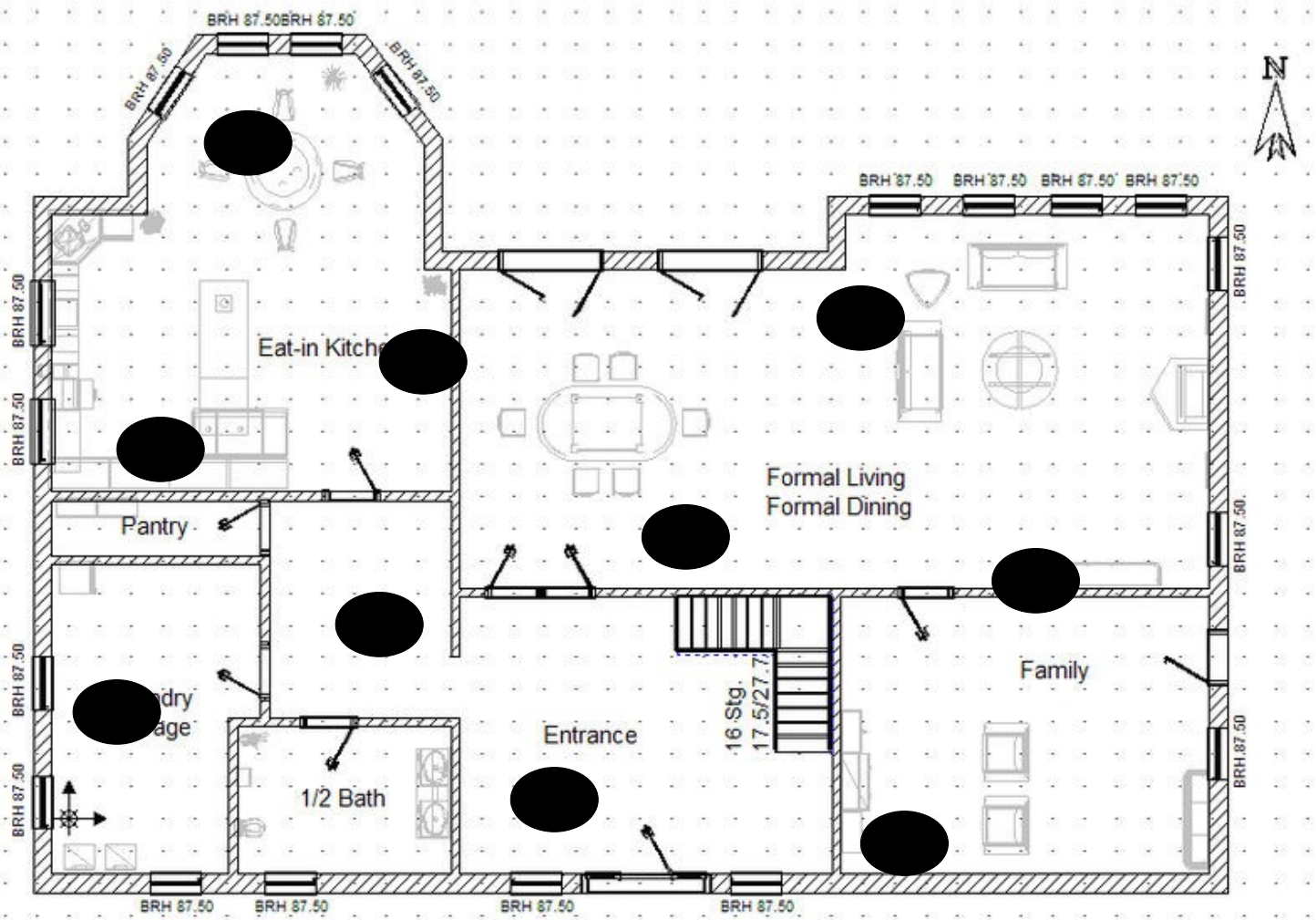

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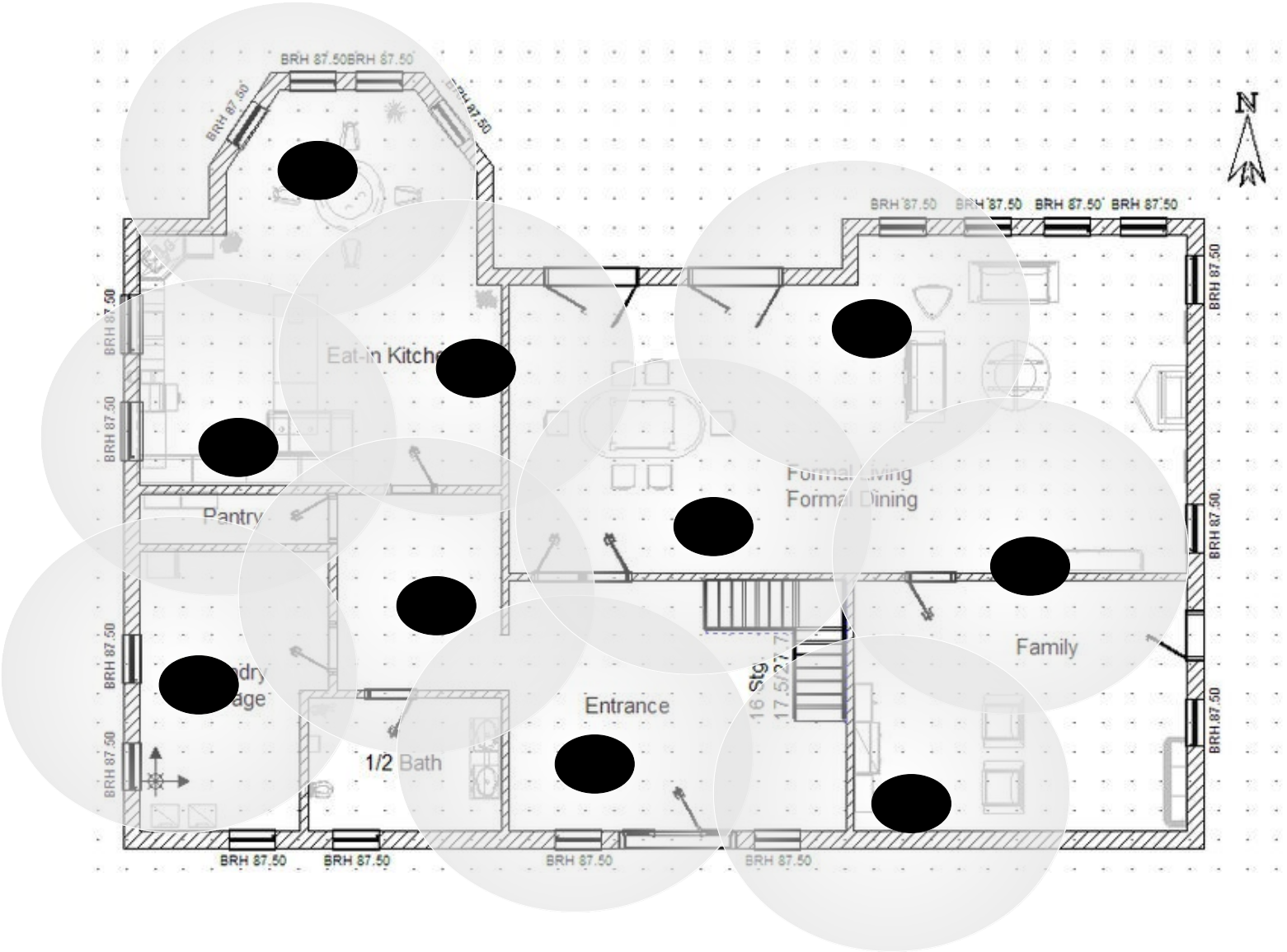

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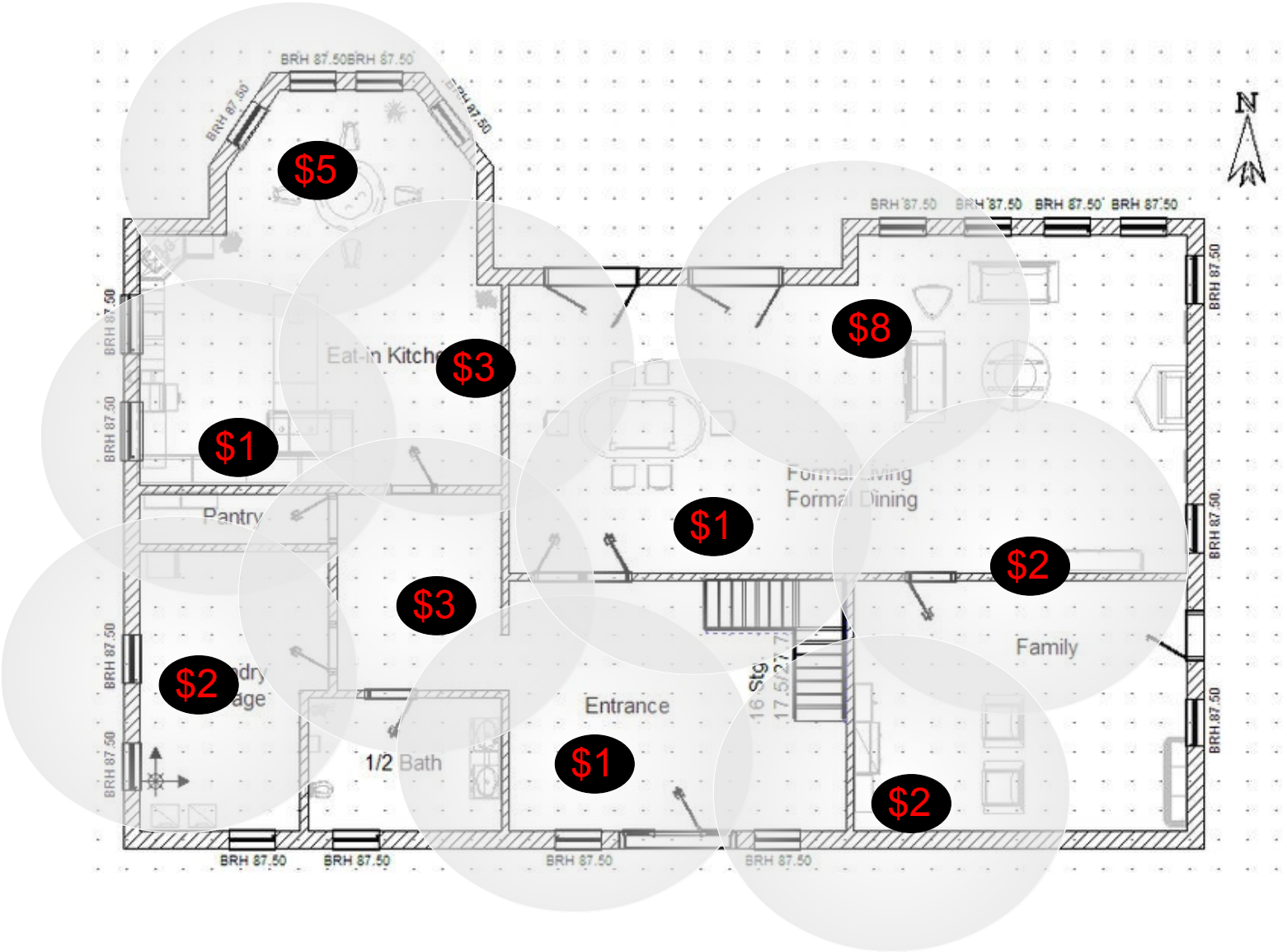


Sensor Placement

- **No known** efficient algorithms for solving this problem *perfectly*
  - Fast algorithms exist for *approximately* solving this
  - Still an active area of research
- If you want the correct answer, the best you can do is **consider all possible choices of sensors** and return the best one.
- How can we generate all possible choices?

Subsets

- Given set S, a **subset** of S is a set formed by choosing some number of elements from S.
  - Note: A set can only hold a single “copy” of an element.
    - e.g The set {0, 1, 1} is identical to the set {0, 1}
- Examples:
  - {0, 1, 4} is a subset of {0, 1, 2, 3, 4, 5}
  - {dikdik, ibex} is a subset of {dikdik, ibex}
  - { } ⊆ {a, b, c}
  - { } ⊆ { }

Generating Subsets

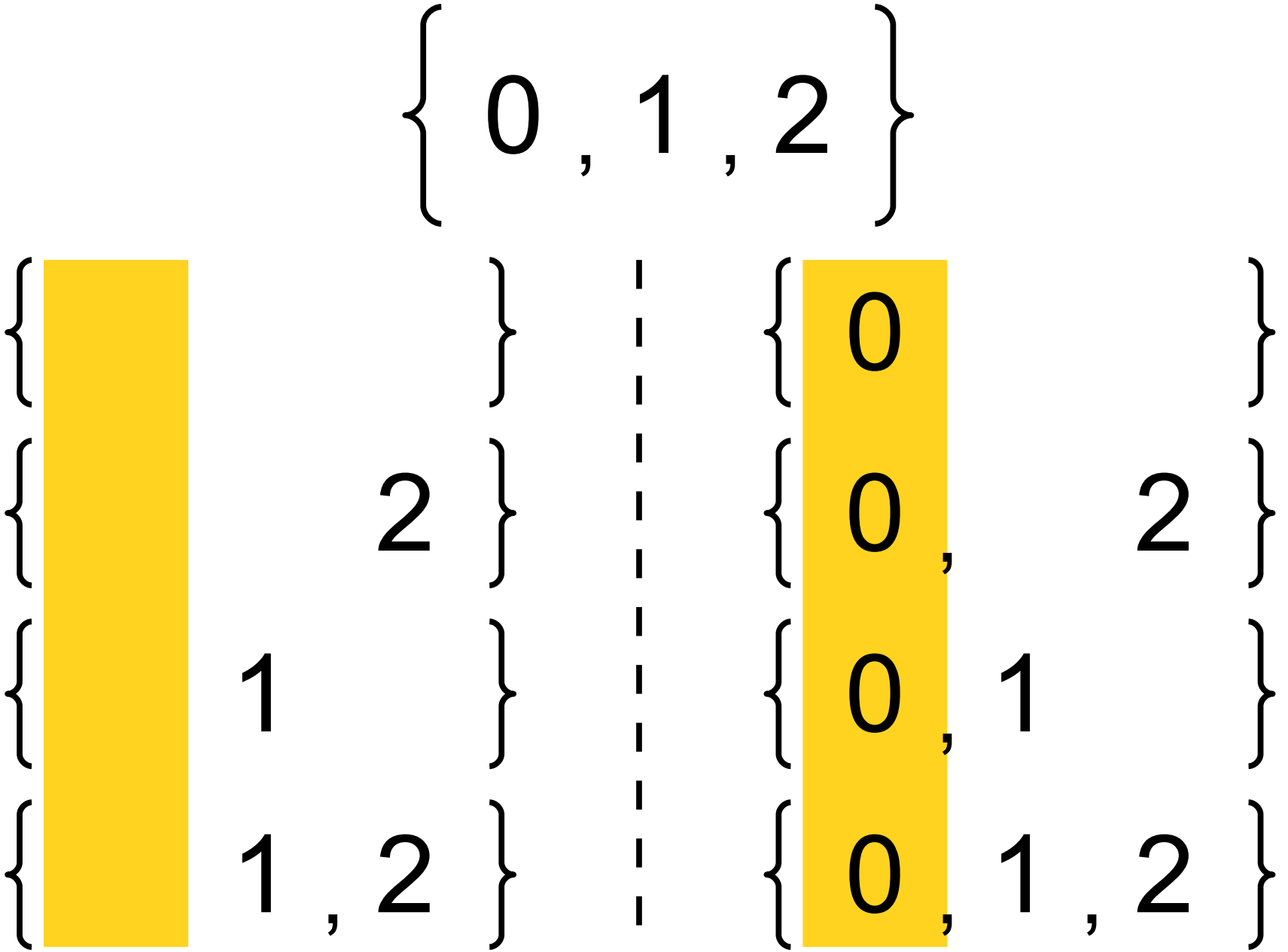
- Many important problems in computer science can be solved by listing all the subsets of a set S and finding the “best” one out of every option.
  - Like optimal sensor placement!

Generating Subsets

$$\{0, 1, 2\}$$

$\{\}$	$\{0\}$
$\{2\}$	$\{0, 2\}$
$\{1\}$	$\{0, 1\}$
$\{1, 2\}$	$\{0, 1, 2\}$

Generating Subsets



Generating Subsets

$$\{0, 1, 2\}$$

$$\{\quad\}$$

$$\{0\}$$

$$\{\quad 2\}$$

$$\{0, 2\}$$

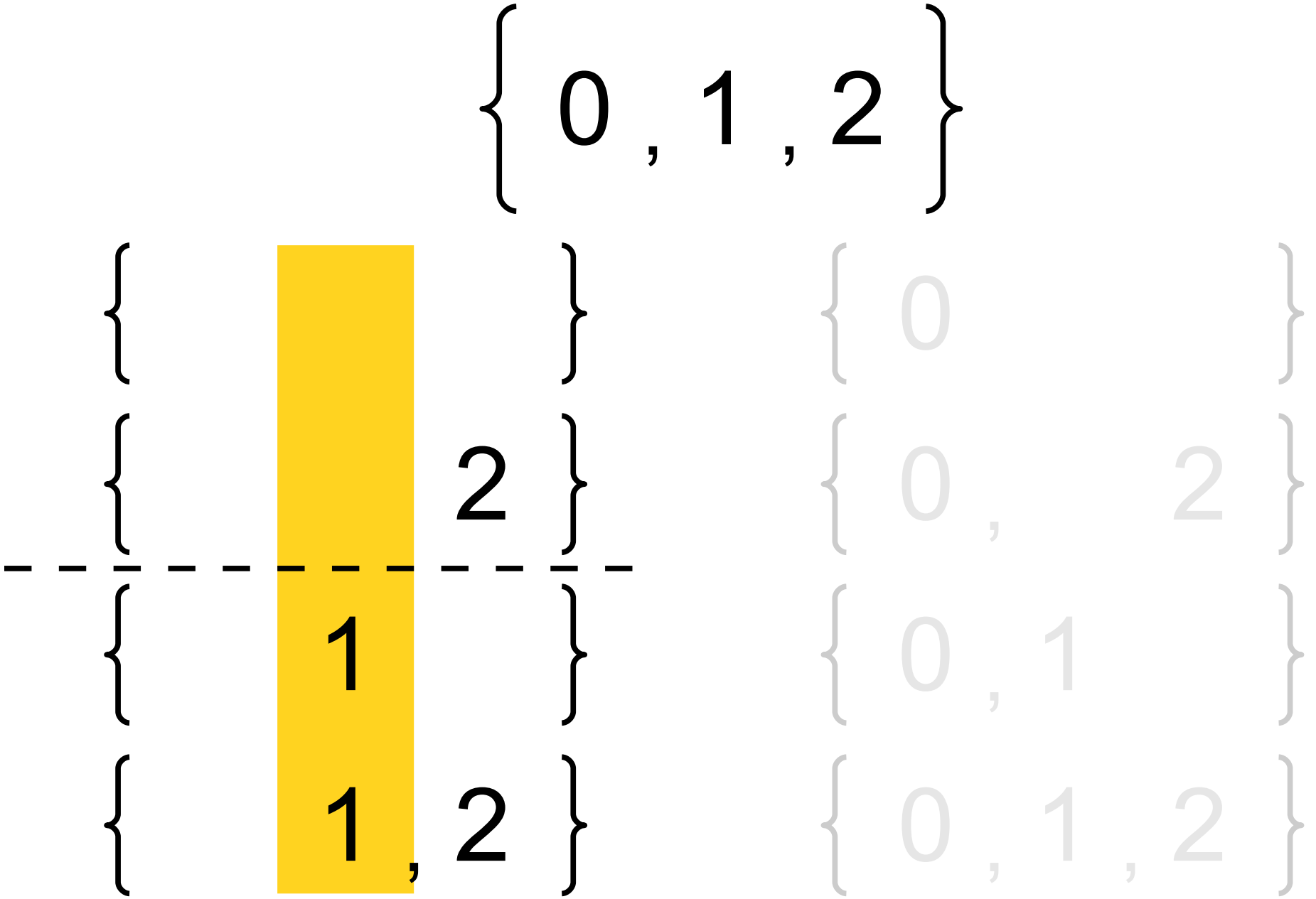
$$\{1\}$$

$$\{0, 1\}$$

$$\{1, 2\}$$

$$\{0, 1, 2\}$$

Generating Subsets



Generating Subsets

$$\{0, 1, 2\}$$

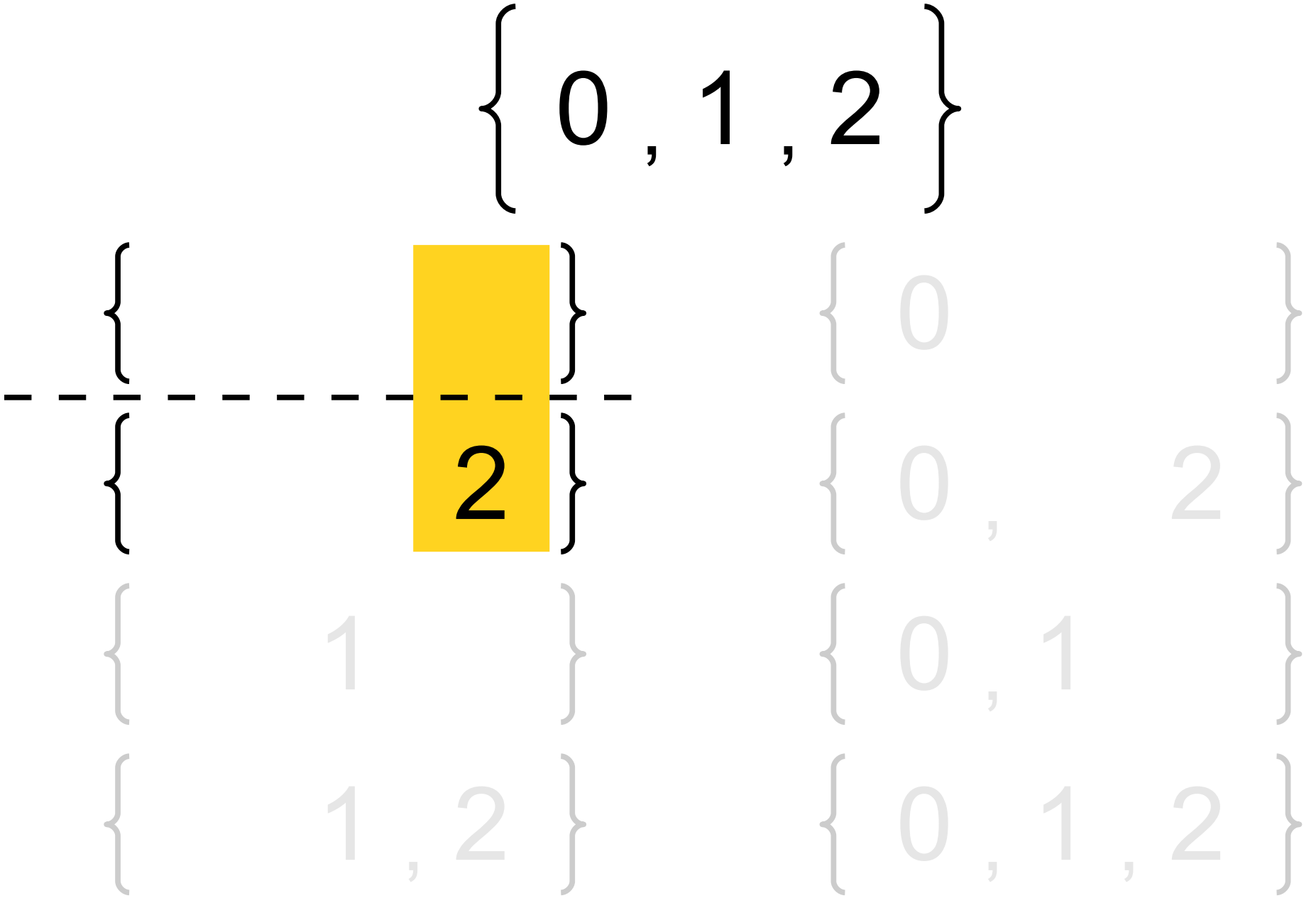
$\{\quad\}$ $\{0\}$

$\{2\}$ $\{0, 2\}$

$\{1\}$ $\{0, 1\}$

$\{1, 2\}$ $\{0, 1, 2\}$

Generating Subsets



`subsetsOf()`
(Pseudocode)

Generating Subsets

- **Base Case:**
  - The only subset of the empty set is the empty set.
- **Recursive Step:**
  - Fix some element x of the set.
  - Generate all subsets of the set formed by removing x from the main set.
  - These subsets are subsets of the original set.
  - All of the sets formed by adding x into those subsets are subsets of the original set.

`subsets.cpp::subsetsOf()`

(Computer)

Tracing the Recursion

Input Set

Subsets Generated

Tracing the Recursion

Input Set

{ A, H, I }

Subsets Generated

Tracing the Recursion

Input Set

{ A, H, I }

{ H, I }

Subsets Generated

Tracing the Recursion

Input Set

Subsets Generated

{ A, H, I }

{ H, I }

{ I }

Tracing the Recursion

Input Set

Subsets Generated

{ A, H, I }

{ H, I }

{ I }

{ }

Tracing the Recursion

Input Set

Subsets Generated

{ A, H, I }

{ H, I }

{ I }

{ }

{ }

Tracing the Recursion

Input Set	Subsets Generated
{ A, H, I }	
{ H, I }	
{ I }	{I}, { }
{ }	{ }

Tracing the Recursion

Input Set	Subsets Generated
{ A, H, I }	
{ H, I }	{H, I}, {H}, {I}, { }
{ I }	{I}, { }
{ }	{ }

Tracing the Recursion

Input Set	Subsets Generated
{ A, H, I }	{A, H, I}, {A, H}, {A, I}, {A} {H, I}, {H}, {I}, { }
{ H, I }	{H, I}, {H}, {I}, { }
{ I }	{I}, { }
{ }	{ }

Analyzing Our Function

- How many subsets are there of a set with n elements?
- We can make a subset by choosing, for each element, whether to include it in the subset or exclude it from the subset.
- We make n choices with 2 options for each choice, so there are $\mathbf{2^n}$ possible subsets.
- The returned collection of sets will use at least 2^n bytes of memory.

A Quick Calculation

- On my computer, an `int` is four bytes ($4 = 2^2$).
- My computer has about 4GB of memory (about 2^{32} bytes).
- If we need 2^n space to hold the return value, what is the largest n we can pick without blowing up my computer (again)?
- **Answer:** $n = 30$.

Reducing Memory Usage

- In many cases, we need to perform some operation on each subset, but don't need to actually store those subsets.
- **Idea:** Generate each subset, process it, and then discard it.
- **Question:** How do we do this?

Recursively Exploring Options

- Our recursive function needs to keep track of
  - What choices we've made so far, and
  - What choices we still need to make.
- **Base Case:**
  - If there are no choices left, output the set we formed from the choices we made.
- **Recursive Step:**
  - Find the next choice to make.
  - For each possible choice, recursively explore all options formed from making that choice.

`visitSubsets`
(Pseudocode)

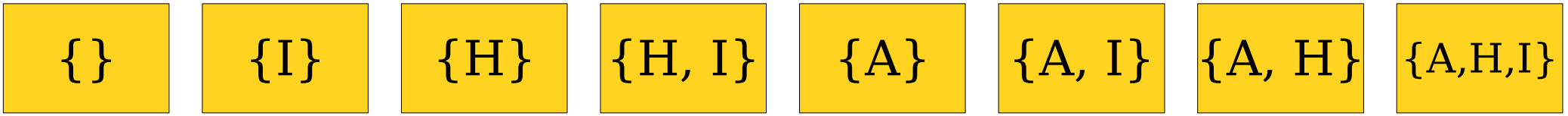
`subsets.cpp::visitSubsets`
(Computer)

Visualizing Subset Generation

- A very useful way of visualizing what happens in a recursive function is with a **decision tree**.
- Idea: Visualize the parameters and "choices" made in the recursion.
- Can provide valuable insights into how solutions are generated via recursive functions.

Subset Decision Tree (Board)

A Decision Tree



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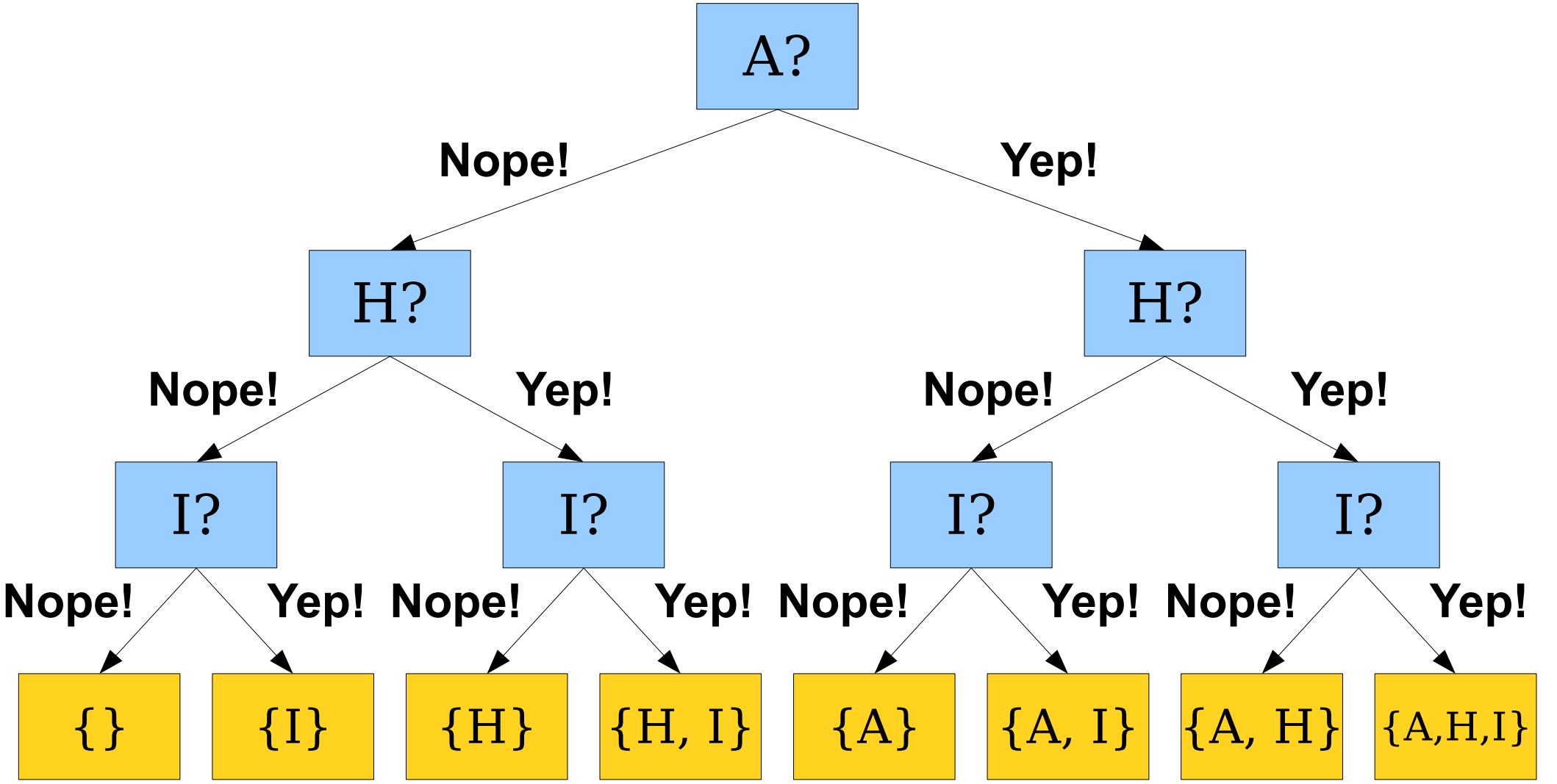

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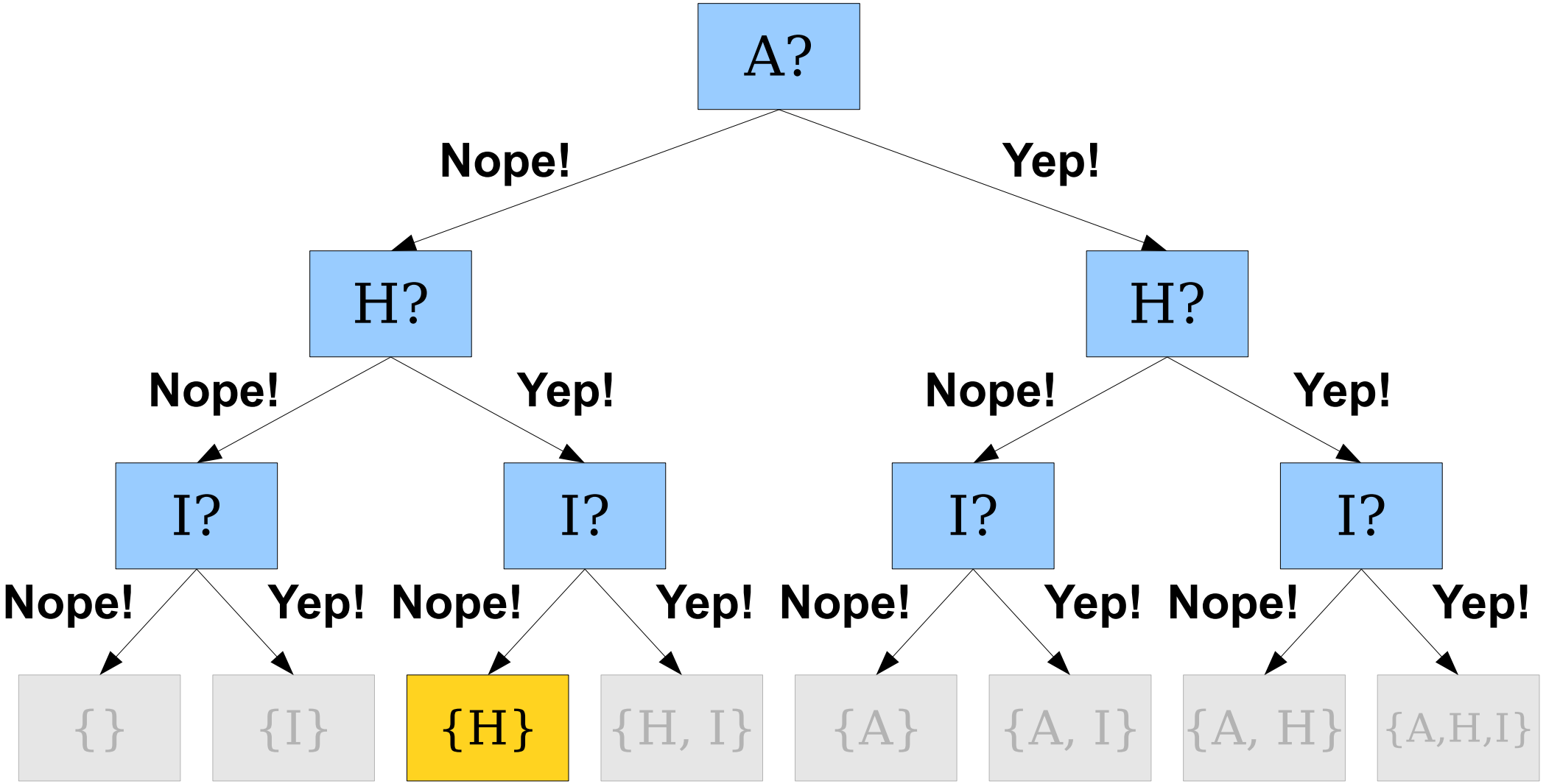

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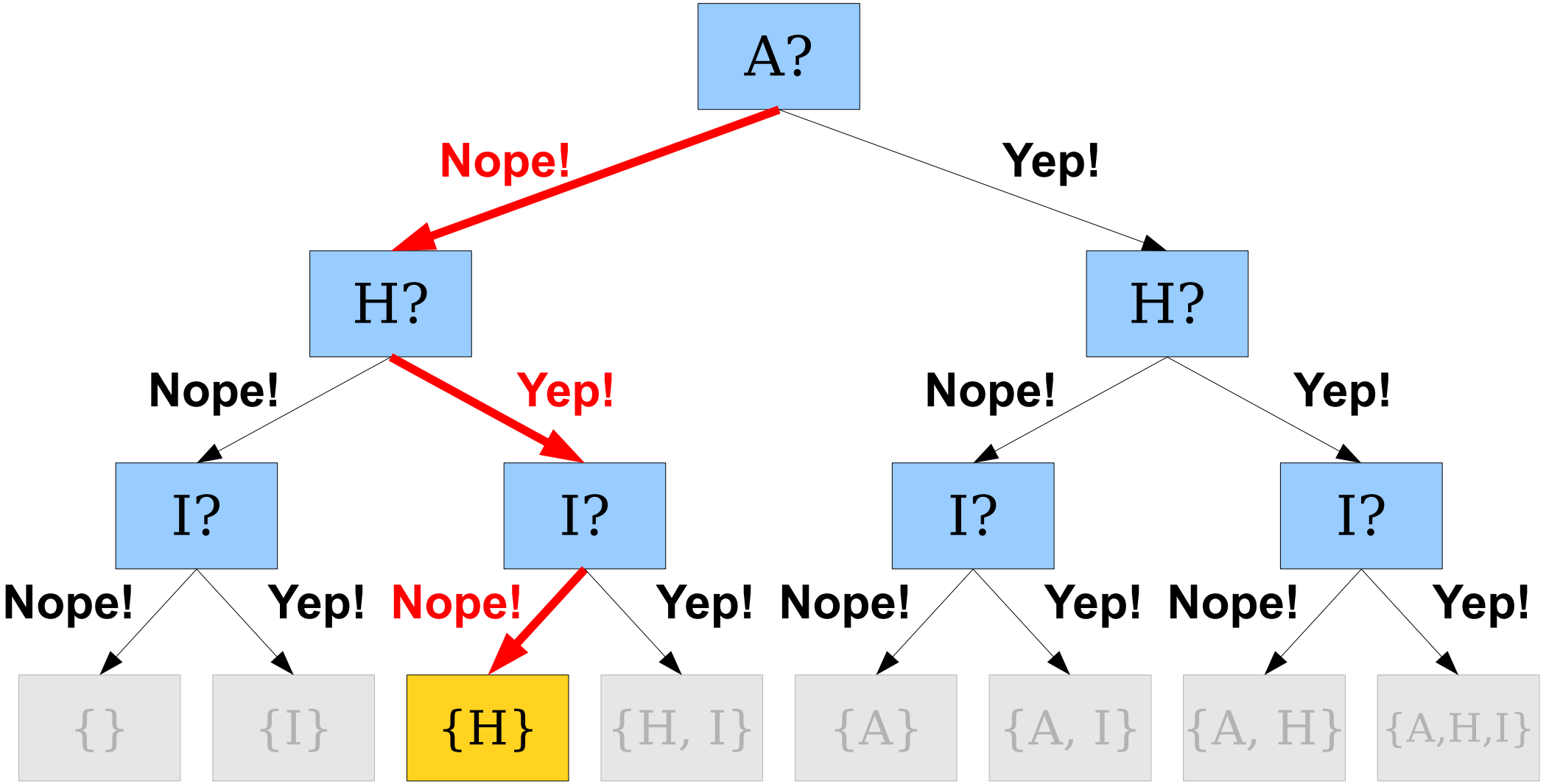

Next Time

- **Exhaustive Recursion II**
  - What other structures can we generate?
  - How do we do so efficiently?
- **Recursive Backtracking**
  - How do you find a needle in a haystack?