Restricted Isometry Property for κ-sparse Vectors in SLR

The main requirement for A.2 Lemma 1 is the absence of distinct $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ that satisfy Eq. (37). With a perfect optimizer, the full rank assumption ensures the presence of a unique inverse $\Sigma^{-1}(\mathbf{w})$ for unique $\mathbf{w}$, leading to no bad local minimas. However, practical optimizers such as SGD might identify distinct $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ as minima for the loss function in Eq. (26) that satisfy $\Sigma^{-1}(\mathbf{w}^{(1)}) \approx \Sigma^{-1}(\mathbf{w}^{(2)})$. We use the RIP bound δ to ensure that the difference in inverse terms large enough to be detectable by SGD.

Step 1: RIP expression for a κ-sparse vector

Let $z = \sum_{i \in S} \alpha_i e_i$, where $S \subseteq \{1, \ldots, n\}$ with $|S| = \kappa$, be a κ-sparse vector and let $\Phi \in R^{d \times n}$ with columns $\phi_1, \ldots, \phi_n$. Then:

$$\Phi z = \sum_{i \in S} \alpha_i \phi_i \tag{1}$$

The squared norm of Φz becomes:

$$\begin{aligned} \|\Phi z\|_2^2 &= \left\| \sum_{i \in S} \alpha_i \phi_i \right\|_2^2 \\ &= \sum_{i \in S} \alpha_i^2 \|\phi_i\|_2^2 + \sum_{\substack{i,j \in S \\ i \neq j}} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle \end{aligned} \tag{2}$$

The squared norm of z is:

$$\|z\|_2^2 = \sum_{i \in S} \alpha_i^2 \tag{3}$$

Step 2: Deviation from isometry and RIP constant

The RIP condition for κ-sparse vectors requires:

$$(1 - \delta_\kappa)\|z\|_2^2 \leq \|\Phi z\|_2^2 \leq (1 + \delta_\kappa)\|z\|_2^2 \tag{4}$$

Subtracting $\|z\|_2^2$, we obtain the deviation:

$$\|\Phi z\|_2^2 - \|z\|_2^2 = \sum_{i \in S} \alpha_i^2 (\|\phi_i\|_2^2 - 1) + \sum_{\substack{i,j \in S \\ i \neq j}} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle \tag{5}$$

Thus, the RIP constant δ_κ is defined as the worst-case deviation over all κ-sparse unit-norm vectors z:

$$\delta_\kappa = \max_{\substack{S \subseteq \{1, \ldots, n\} \\ |S| = \kappa \\ \sum_{i \in S} \alpha_i^2 = 1}} \left| \sum_{i \in S} \alpha_i^2 (\|\phi_i\|_2^2 - 1) + \sum_{\substack{i,j \in S \\ i \neq j}} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle \right| \tag{6}$$