Lecture 13

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Lorentz Equations

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

The inverse transformation

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Lorentz Transformation

$$x' = \frac{x - \mathrm{v}t}{\sqrt{1 - \mathrm{v}^2 / c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \mathrm{v}x / c^2}{\sqrt{1 - \mathrm{v}^2 / c^2}}$$

Length contraction

Simultaneity problems

Time Dilation

$$x = \frac{x' + \mathrm{v}t'}{\sqrt{1 - \mathrm{v}^2 / c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \mathrm{v}x' / c^2}{\sqrt{1 - \mathrm{v}^2 / c^2}}$$

If $\mathrm{v} << c$, i.e., $\beta \approx 0$ and $\gamma \approx 1$, yielding the familiar Galilean transformation.

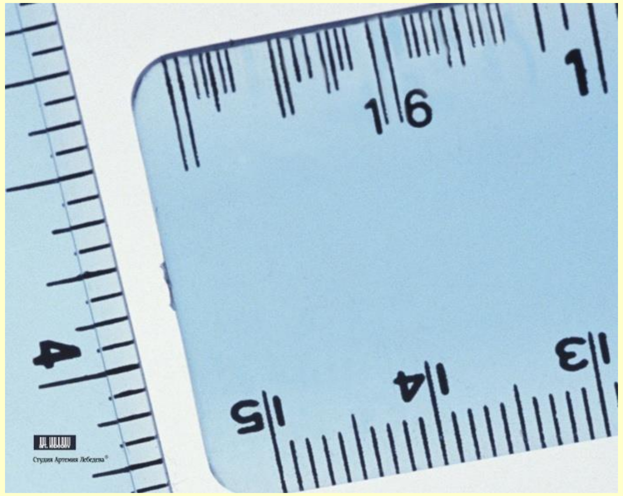
Space and time are now linked, and the frame velocity cannot exceed c.

Consequences of Lorentz transformations (space-time correlation)

1. **Length contraction**
(a moving rod appears shorter)

2. **Time dilation**
(a moving clock runs slower)

Proper Length



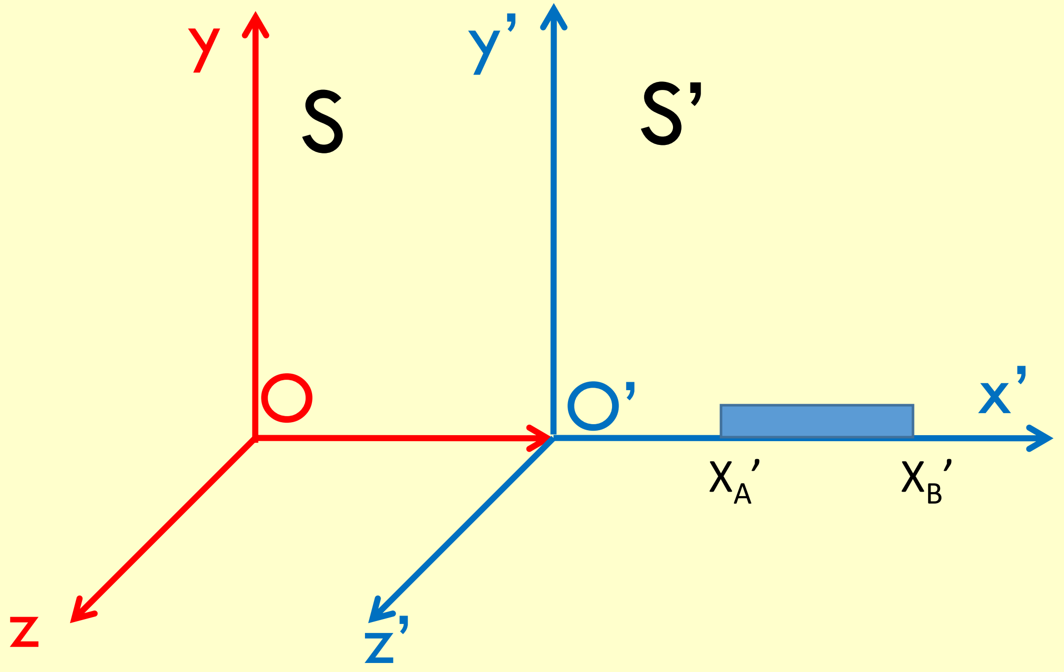
When both endpoints of an object (at rest in a given frame) are measured in that frame, the resulting length is called the **Proper Length**.

The proper length of an object is the length of the object in the frame in which the object is at rest.

We'll find that the proper length is the largest length observed. Observers in motion will see a contracted object.

Length contraction

$L' = x'_B - x'_A$ irrespective of t_B' and t_A'



Let the rod be at rest in *S'* frame. Hence x_B' and x_A' can be measured at any time. But this rod will be observed to be moving is *S* frame. Hence the two ends should be measured at the same time in S; $t_A = t_B$

Length (L) in S

Proper length = $L' = x'_B - x'_A$ since the rod is at rest in this frame.

Event 1: Observer in S measures the co-ordinate of B end of the rod. (x_B, t_B)

Event 2: Observer in S measures the co-ordinate of A start of the rod. (x_A, t_A)

$$x_B - x_A = L \quad \text{only if} \quad t_B = t_A$$

So let us take t_B and t_A to be the same.

Co-ordinates of the events in S'

$$x'_B = \gamma\left(x_B - vt\right),\ t'_B = \gamma\left(t - \frac{vx_B}{c^2}\right)$$

$$t_B = t_A = t$$

$$x'_A = \gamma\left(x_A - vt\right),\ t'_A = \gamma\left(t - \frac{vx_A}{c^2}\right)$$

$$(x_A' - x_B') = \gamma(x_A - x_B)$$

$$\therefore L = (x_A - x_B) = \frac{(x_A' - x_B')}{\gamma} = \frac{L'}{\gamma}$$

Contraction Formula

When there is a relative velocity between the rod and the observer, he/she feels a contracted length.

Observations

1. *In S' the two ends are not measured simultaneously. But as in S' the rod is rest it does not cause any error.*

2. ***L_o is the length in a frame in which the rod is at rest****. It is called the* ***Proper Length****. The frame of reference in which the observed body is rest is called* ***Proper Frame of Reference****.*

3. *The formula ($l=L_o/\gamma$) relates proper length to length in a frame in which the rod is moving.*

4. ***If S was the proper frame (rod at rest in the inertial frame) then t_B' would need to be equal to t_A' and (x_B-x_A) would be the proper length. Using inverse transformation ; $x_B= \gamma(x_B'+vt_B')$ and $x_A=\gamma(x_A'+vt_A')$***

we would get the same result i.e. $(x_B'-x_A')= l=L_o/\gamma$; as expected

Length Contraction: S proper frame

Random guy.

Random Guy, **at rest** in system S, measures the length of his somewhat bulging waist:

$$L_{\mathrm{p}} = x_r - x_\ell \qquad \leftarrow \text{ Proper length}$$

Now, two observers in S', measure it, too, making simultaneous measurements ($t'_{\mathrm{l}} = t'_{\mathrm{r}}$) of the left, x'_l , and the right x'_r endpoints

Random Guy's measurement in terms of the two observers in the *S*':

$$L_p = x_r - x_l = \frac{(x'_r - x'_l) + v\cancel{(t'_r - t'_l)}}{\sqrt{1 - v^2/c^2}} = \frac{L'}{\sqrt{1 - v^2/c^2}} = \gamma L'$$

where the measured length by the two observers in S' is: $L' = x'_r - x'_\ell$

$$L' = \frac{L_p}{\gamma} = L_p \sqrt{1 - v^2/c^2}$$

Moving objects appear thinner!

$x_B = \gamma(x_B' + vt_B')$ **and** $x_A = \gamma(x_A' + vt_A')$

Time Dilation

Proper Time interval is time interval between two events occurring ***at the same place*** in a given frame.

Or

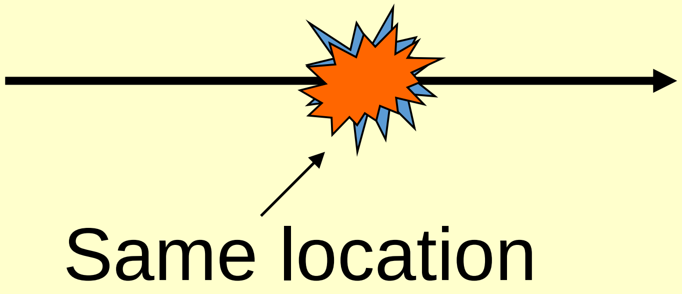
The time interval recorded by a clock which is attached to the observed body.

Implies

The proper time is the time interval between two events measured by a clock which ***travels through both events***

Proper Time

The **Proper Time**, τ, is the time between two events (here two explosions) occurring **at the same position** (i.e., at rest) in a system as measured by a clock at that position.



Proper time measurements are in some sense the most fundamental measurements of a duration. But observers in moving systems, where the explosions' positions differ, will make different measurements.

Time Dilation formula

Let the time interval be proper in S. The coordinates A and B ***should be same*** in S.

$$x' = \gamma\left(x - vt\right)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right)$$

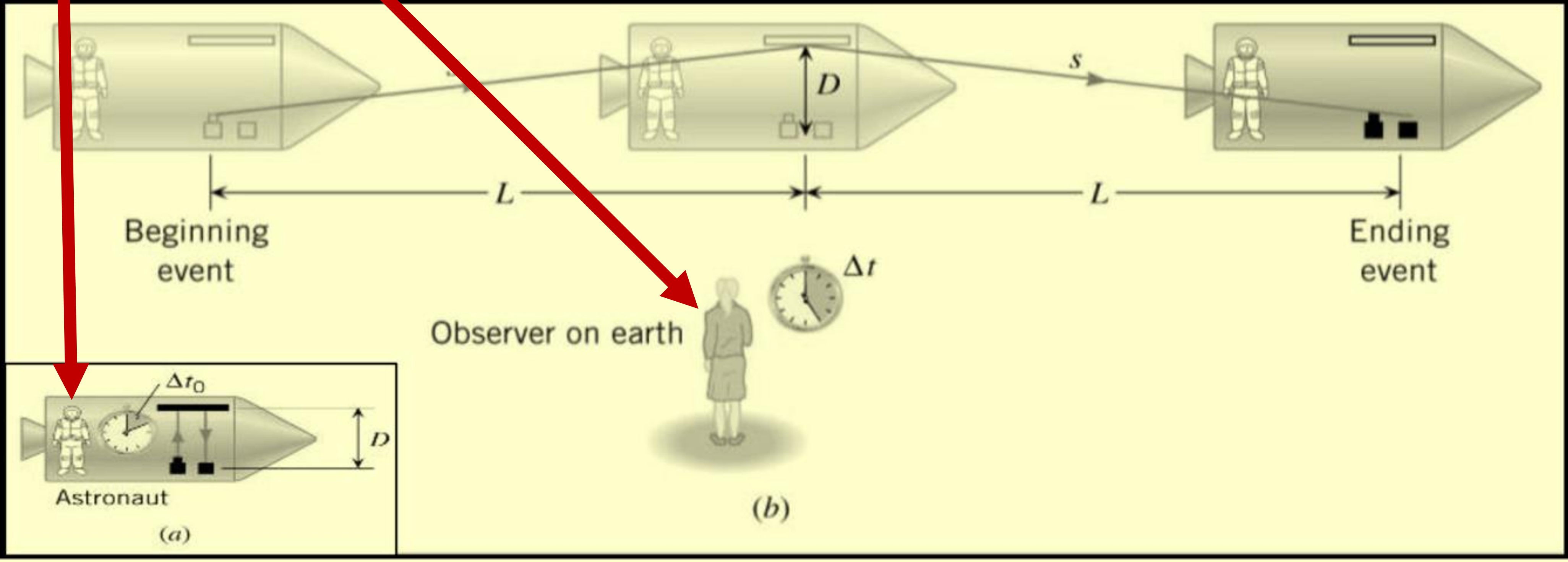
$$(t_A' - t_B') = \gamma(t_A - t_B)$$

i.e., $\Delta t' = \gamma \Delta t$

or $\Delta t' > \Delta t$

Time Dilation (Intuitive)

- Astronaut measures time by aiming a laser at a mirror. The light reflects from the mirror and hits a detector.
- The person on earth says that the time of the event must be longer because she sees the laser beam go farther.



Muons - An Example of Time Dilation and Length Contraction

Muons are unstable particles created when cosmic rays interact with the upper atmosphere. They move at very high velocities ($\beta \sim 0.9999$) and have very short lifetimes, $\tau = 2 \times 10^{-6}$ s, as measured in the lab.

Do muons reach the ground, given an atmospheric "thickness" of about 10 km?

"Classical" answer: distance = velocity $\times$ time = $0.9999c \times 2 \times 10^{-6}$ s $\cong 0.6$ km

$\therefore$ conclude that muons will never reach the ground.

However, they do! What is wrong? Because muons move so quickly, relativistic effects are important. The classical answer mixes up reference frames.

- τ - refers to lifetime in the muon reference frame where τ is measured)
- atmospheric thickness - refers to length in the Earth's reference frame

(1) Time dilation approach – from Earth frame

$\rightarrow$ treat the muons as a clock in A'

$t' = 0$ muons are created

$t' = \tau$ muons are destroyed

$\Delta t' = \tau$

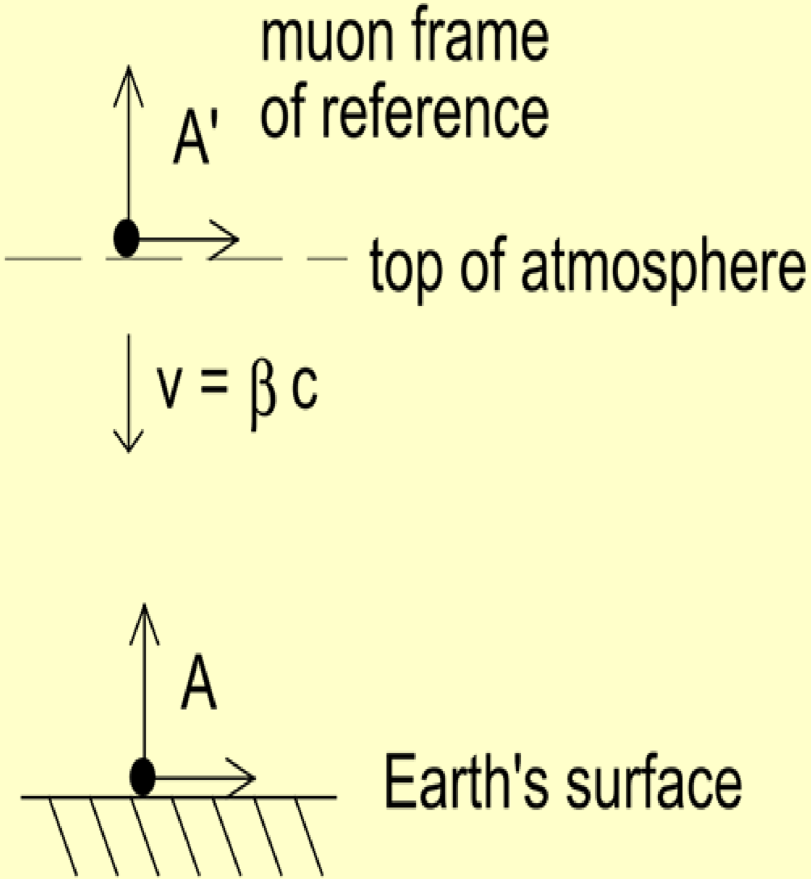
The creation and destruction of muons occurs at the same place in A'. **Proper Time**

What is the time interval as measured in A?

$\Delta t = \gamma \Delta t' = \gamma \tau = 1.4 \times 10^{-4}\,\mathrm{s}$ i.e., it takes 70 times longer

$\therefore$ distance $= 0.9999c \times 1.4 \times 10^{-4}\,\mathrm{s} \cong 42\,\mathrm{km}$

Since 42 km > 10 km, the muons will reach the ground.

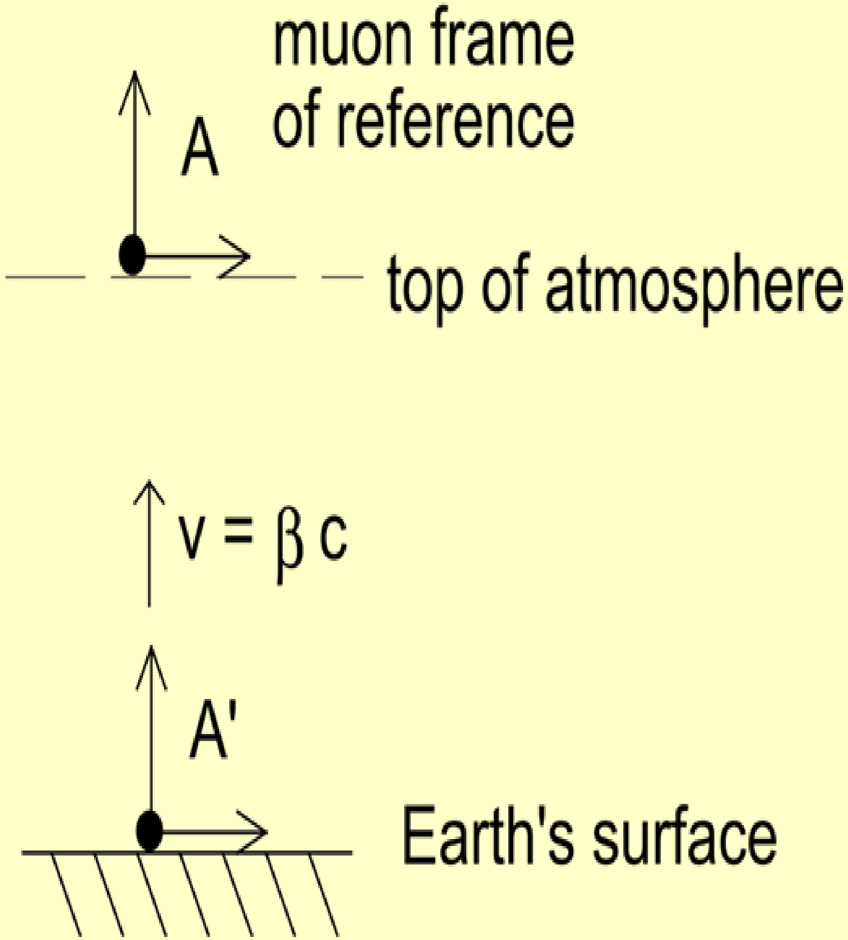


(2) Length contraction approach – from muon frame

In reference frame A (now with the muons)

lifetime = $\tau = 2 \times 10^{-6}$ s

velocity of ground = 0.9999c



So the distance that the ground travels before the muon decays is 0.6 km. But what is the thickness of the atmosphere that the muon sees?

proper length of atmosphere = 10 km ← **Proper length**

length of atmosphere in muon frame is

$$\Delta x = \Delta x' \sqrt{1 - v^2/c^2} = 10\sqrt{1 - 0.9999^2} = 0.14 \text{ km}$$

i.e., the atmosphere that the muon sees is 70 times thinner

0.6 km > 0.14 km and so the ground will reach the muon.

Therefore, the lab frame uses time dilation to get the result, while the muon frame uses length contraction! That is, these two are correlated.

So not only time dilation and length contraction are correlated they are real

The Twin Paradox

The Set-up

Mary and Frank are twins. Mary, an astronaut, leaves on a trip many lightyears (ly) from the Earth at great speed and returns; Frank decides to remain safely on Earth.

The Problem

Frank knows that Mary's clocks measuring her age must run slow, so she will return younger than he. However, Mary (who also knows about time dilation) claims that Frank is also moving relative to her, and so his clocks must run slow.

The Paradox

Who, in fact, is younger upon Mary's return?

The Twin-Paradox Resolution

Frank's clock is in an **inertial system** during the entire trip. But Mary's clock is not. As long as Mary is traveling at constant speed away from Frank, both of them can argue that the other twin is aging less rapidly.

But when Mary slows down (**decelerates**) to turn around, she leaves her original inertial system and eventually returns in a completely different inertial system.

Mary's claim is no longer valid, because she doesn't remain in the same inertial system. Frank does, however, and **Mary ages less than Frank**.

Go see Interstellar !

A spaceship is measured to be 100 m long while it is at rest with respect to an observer. If this spaceship now flies by the observer with a speed of 0.99c, what length will the observer find for the spaceship?

$$L' = \frac{L_p}{\gamma} = L_p\sqrt{1 - v^2/c^2}$$

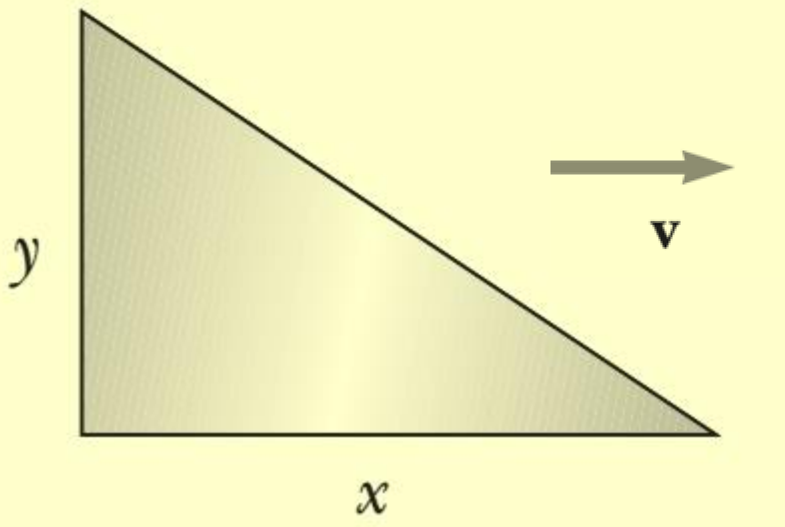
$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} = (100\text{ m})\sqrt{1 - \frac{(0.99c)^2}{c^2}} = 14\text{ m}$$

0.01000c, what length will the observer measure?

99.99 m.

NEWTON !!!

A spaceship in the form of a triangle flies by an observer at $0.950c$. When the ship is measured by an observer at rest with respect to the ship the distances x and y are found to be 50.0 m and 25.0 m, respectively. What is the shape of the ship as seen by an observer who sees the ship in motion along the direction shown in Fig-

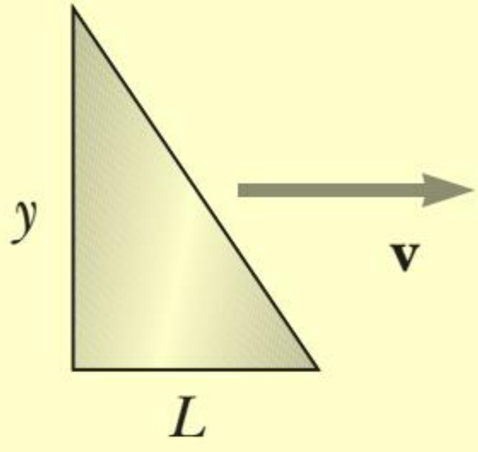


$$L' = \frac{L_p}{\gamma} = L_p\sqrt{1 - v^2/c^2}$$

Solution The observer sees the horizontal length of the ship to be contracted to a length of

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}}$$

$$= (50.0 \text{ m})\sqrt{1 - \frac{(0.950c)^2}{c^2}} = 15.6 \text{ m}$$



The 25-m vertical height is unchanged because it is perpendicular to the direction of relative motion between the observer and the spaceship. **Figure** represents the shape of the spaceship as seen by the observer who sees the ship in motion.

A meter stick, at an angle of 30 degrees with the x-axis, is traveling at 0.6c in the direction of the positive y-axis. To a stationary observer, how long does the meter stick appear to be?

$$L' = \frac{L_p}{\gamma} = L_p\sqrt{1 - v^2/c^2}$$

Length contraction only occurs in the direction of motion. This means that the x component of the length, which is cos(30), does not change; length contraction only occurs the the y component, which is sin(30).

First, we find the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{9}{25}}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}$$

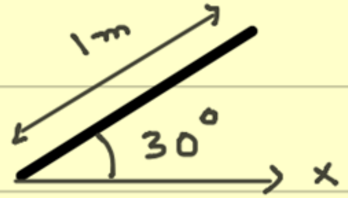
$$L'_y = L_y/\gamma = \frac{4}{5}\sin(30) = \frac{2}{5}$$

Finally, we find the total length by combining the length-contracted y component and the unchanged x component:

$$L = \sqrt{L_x^2 + {L'_y}^2} = \sqrt{\frac{3}{4} + \frac{4}{25}} = \frac{\sqrt{91}}{10}$$

$$\sqrt{\frac{3}{4} + \frac{4}{25}} = \frac{\sqrt{91}}{10}$$

Tutorial # 5) A meter stick is positioned so that it makes an angle 30º with the x-axis in its rest frame. Determine its length and its orientation as seen by an observer who is moving along the x-axis with a speed of 0.8c



The x-component of length would contract in S′ frame, while the y-component would remain unchanged.

Hence $l'_x = \frac{l_x}{\gamma} = \frac{1 \cos 30^\circ}{(5/3)} = \frac{\sqrt{3}}{2} \times \frac{3}{5} = 0.3\sqrt{3}$

$l'_y = l_y = \frac{1}{2}$ $\quad \therefore\ l' = \sqrt{0.09 \times 3 + .25} \simeq 0.72$ m

$\tan\theta' = \frac{1}{2 \times 0.3\sqrt{3}} \Rightarrow \theta' \simeq 43.9^\circ$

Tutorial # 4) Observer B is at rest in frame S' moving horizontally past an inertial frame S at a speed of 0.6c. A boy in the frame S drops a ball, which according to the clock of observer B falls for 1.5s before reaching ground. How long will the ball fall for an observer A at rest in S frame?

$$(t_A' - t_B') = \gamma(t_A - t_B)$$

The two events are

E1 : Ball being dropped E2 : Ball reaching ground

$$\gamma = 1.25 \Rightarrow \Delta t = \frac{1.5}{1.25} = 1.2\,s$$

Let's say the USS Enterprise's 1/3 impulse speed is one-quarter the speed of light. If Spock, in the ship, says the planet will blow up in 10 minutes, how long does the away team have to beam up?

$$\Delta t = 10\,minutes, v = 0.25c$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \qquad (t_A' - t_B') = \gamma(t_A - t_B)$$

$$10\,minutes = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{(0.25c)^2}{c^2}\right)}} \rightarrow \Delta t_0 = 10\,\mathrm{min}\sqrt{1 - (0.0625)} = 9.682458\,min$$

$$= 580.9\,s$$

Picard is on Rigel 7 and needs to go to Earth 776.6 light-years away, but the Enterprise's warp drive is broken. If full impulse is ¾ the speed of light, how long will a Rigelian think it will take the Enterprise to get to Earth?

How long will the Enterprise's crew think it will take?

1 light–year is the distance light will travel in a year.

a) $\Delta t = \frac{776.6\ ly}{0.75} = 1035.47\ years$ (the Rigelian measures dilated time because the events of leaving and arriving are at rest to the starship – they both happen outside the starship's windows)

b) $\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}} \rightarrow 1035.47\ yrs = \frac{\Delta t_0}{\sqrt{1-\left(\frac{(0.75c)^2}{c^2}\right)}} \rightarrow \Delta t_0 = 1035.47\ yrs\sqrt{1-0.5625} =$ $684.90\ yrs$

According to Bob, an observer on Earth, a rocket carrying Martha from Earth directly to the planet Zorg travels at a speed of 0.80 c and takes 30 years to reach Zorg. Zorg is at rest relative to the Earth.

How long does Martha, the observer on the rocket, think the trip takes?

Martha measures the proper time, in this case.

equation: $\Delta t_{proper} = \Delta t \times \sqrt{1 - \frac{v^2}{c^2}} = 30 \text{ years} \times \sqrt{1 - \frac{0.8c^2}{c^2}} = 30 \text{ years} \times 0.6 = 18 \text{ years}$

1. The time interval between two ticks of two identical clocks is 2.0 sec. One of the two clocks is set in motion, so that its speed relative to the observer, who holds the other clock is $0.6c$. What is the time interval between the ticks of the moving clock as measured by the observer with the stationary clock?

2. The incoming primary cosmic rays create μ-mesons in the upper atmosphere. The lifetime of μ-mesons at rest is 2 μs. If the mean speed of μ-mesons is $0.998c$, what fraction of the mu-mesons created at a height of 20 km reach the sea level?

3. Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be inertial frame of reference.

4. Observer A is at rest in frame S' moving horizontally past an inertial frame [illegible]ed of $0.6c$. A boy in the frame S, drops a ball, which according to the clock of observer [illegible] 1.5sec. How long will the ball fall for an observer at rest in S frame ?

6. A rod flies with constant velocity past a mark, which is stationary in reference frame S. In reference frame S, it takes 20 ns for the rod to fly past the mark. In the reference frame fixed to the rod, S', the mark moves past the rod for 25 ns. Find the length of the rod in S and S' and the speed of S' with respect to S.

7. A rod of length 60 cm in its rest frame is traveling along its length with a speed of $0.6c$ in the frame S. A particle moving in the opposite direction to the rod, with a speed $0.6c$ in S, passes the rod. How much time will the particle take to cross the rod

   (a) in the frame S.

   (b) in the rest frame of the particle.

8. Two spacehips pass each other, travelling in opposite directions. The speed of ship B, measured by a passenger in ship A is $0.96c$. This passenger has measured the length of the ship A as 100 mt and determines that the ship B is 30 mt long. What are the lengths of the two ships as measured by a passenger in ship B ?

9. An observer O is at the origin of an inertial frame. He notices a vehicle A to pass by him in $+x$ direction with constant speed. At this instant, the watch of the observer O and the watch of the driver of A show time equal to zero. 50 μs after A passed by, O sees another vehicle B pass by him, also in $+x$ direction and again with constant speed. After sometime B catches A and sends a light signal to O, which O receives at 200 μs according to his watch. The driver of B notices that, in his frame, the time between passing O and catching A is 90 μs Assume that drivers A and B are at the origins of their respective frames. Find

(a) the speeds of B and A, in the frame of O.

(b) position of A in O's frame when B passes O.

(c) the position of O in the frame of A, when B passes O.

10. An intertial frame S' moves relative to another frame S with a velocity $v_1\hat{\imath} + v_2\hat{\jmath}$ in such a way that the x and x' axes, y and y' axes and z and z' axes are always parallel. Let the time $t = t' = 0$ when the origins of the two frames are co-incident. Find the Lorenz transformation relating the co-ordinates and time of S' to those in S.