Department of Physics, Indian Institute of Technology Bombay

13-06-2023 PH 112: End-Semester Model solution (40 marks) 9:30 - 12:30 hrs

1. Non-programmable calculators are permitted.
2. Useful constants and integrals:

Speed of light in vacuum : $c = 3 \times 10^8$ m.s^{-1}
Planck's constant : $h = 6.63 \times 10^{-34}$ J.s
1 electron Volt $= 1.6 \times 10^{-19}$ J
Rest mass of electron : 5.1×10^5 eV$/c^2$
Or 9.1×10^{-31} kg

$\int x \sin x dx = \sin x - x \cos x$

$\int x^2 \sin x dx = 2 \cos x + 2x \sin x - x^2 \cos x$

$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}$; $\sinh(x) = \frac{e^x - e^{-x}}{2}$

1. A muon can be considered to be a heavy electron with a mass $m_\mu = 200 m_e$. In a Compton scattering experiment, replace the electron with muon. Calculate the maximum change in photon's wavelength in femtometer ($1 \text{ femtometer} = 10^{-15} m$). **[3 marks]**

Answer The maximum change in the photon's energy is obtained in backscattering, i. e., $(\theta = 180°) \Longrightarrow 1 - \cos\theta = 2$. We then have:

$$\Delta\lambda = \frac{2h}{m_\mu c} \sim 2.42 \times 10^{-14}\ m.$$

Thus, $\alpha \sim 24.2$. [acceptable range: 23.0–25.4]

2. Most of the particles produced in Large Hadron Collider (LHC) experiment in Geneva are unstable. For example, the lifetime of the neutral pion (represented by π^0) is about 8.4×10^{-17} s. Its rest mass is $135.0\ \text{MeV}/c^2$. **[2 marks]**

(a) Taking the uncertainty product as $\hbar$, calculate the uncertainty in the mass of π^0. [1 mark]

(b) What is the relative uncertainty $\Delta m/m$ of the pion's mass? [1 mark]

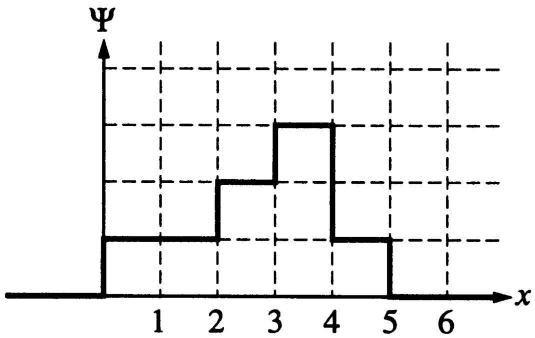
Answer (a) The uncertainty principle provides the relationship between uncertainly in energy and time as, $\Delta E \Delta t \geq \hbar$ or

$$\Delta E \geq \hbar/(\Delta t) = \left(6.5821 \times 10^{-16} \text{eV} \cdot \text{s}\right) / \left(8.4 \times 10^{-17}\ \text{s}\right) = 7.84\text{eV}$$

[acceptable range 7.45–8.23 eV]
(b) The uncertainty in mass is given by the energy-mass relation as $\Delta m = \Delta E/\text{c}^2$. With a mass of $135\text{MeV}/c^2$, the relative uncertainty is

$$\frac{\Delta m}{m} = \frac{\Delta E}{mc^2} = \frac{7.84\text{eV}}{135 \times 10^6\text{eV}} = 5.8 \times 10^{-8}$$

[acceptable range: 5.5×10^{-8}—6.1×10^{-8}]

3. 

The wave function for a particle constrained to move in one dimension is shown in the graph below ($\Psi = 0$ for $x \le 0$ and $x \ge 5$). The probability that the particle would be found between $x = 2$ and $x = 4$ is P_0. Find P_0. **[3 marks]**

Answer The probability of finding a particle in any position by taking the integral of the squared wave function,

$$P_{ab} = \int_a^b |\Psi(x)|^2 dx$$

The integral of a curve is just the area underneath it. Since we are concerned with the probability of the particle being located between $x = 2$ to $x = 4$, we need to compare that with the total area of the squared wave function. Doing so, from left to right, we have

$$\text{Area}_{\ 2\to 4} = (2)^2 + (3)^2 = 13$$
$$\text{Area}_{\ 0\to 6} = (1)^2 + (1)^2 + (2)^2 + (3)^2 + (1)^2 + (0)^2 = 16$$

thus the probability is $13/16 \sim 0.8125$

4. Consider 1-dimensional wave function: $\psi(x) = A\,(x/x_0)^n\, e^{-x/x_0}$ for $x \ge 0$

where A, n and x_0 are real constants. $\psi(x)$ is an energy eigenfunction of a potential $V(x)$ with energy eigenvalue E. Obtain $V(x)$ and E if $V(x) \to 0$ as $x \to \infty$. **[5 marks]**

Answer Differentiating the given wave function,

$$\begin{aligned}
\frac{d}{dx}\psi(x) =& A\frac{n}{x_0}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} + A\left(\frac{x}{x_0}\right)^{n}\left(-\frac{1}{x_0}\right) e^{-x/x_0} \\
\frac{d^2}{dx^2}\psi(x) =& A\frac{n(n-1)}{x_0^2}\left(\frac{x}{x_0}\right)^{n-2} e^{-x/x_0} \\
& - 2A\frac{n}{x_0^2}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} + A\frac{1}{x_0^2}\left(\frac{x}{x_0}\right)^{n} e^{-x/x_0} \\
=& \left[\frac{n(n-1)}{x^2} - 2\frac{n}{x_0 x} + \frac{1}{x_0^2}\right]\psi(x)
\end{aligned}$$

and substituting it in the time-independent Schroedinger equation

$$\left(-\frac{\hbar^2}{2m} d^2/dx^2 + V(x)\right)\psi(x) = E\psi(x)$$

we have

$$E - V(x) = -\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right].$$

As $V(x) \to 0$ when $x \to \infty$, we have

$$E = -\frac{\hbar^2}{2m x_0^2} \quad V(x) = \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x}\right]$$

5. A particle of mass m is represented by the wave function

$$\Psi(x,t) = \mathrm{e}^{i\omega t}[\alpha\cos(kx) + \beta\sin(kx)]$$

where α and β are complex constants, and ω and k are real constants. Consider the following quantity:

$$J(x,t) = i\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

Calculate $J(x,t)$ for the above wave-function. **[3 marks]**

Answer Substituting the above form of Ψ in

$$J(x,t) = i\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right),$$

we have

$$\begin{aligned}
J(x,t) &= i\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)\\
&= ik\big(-|\alpha|^2\cos(kx)\sin(kx) + |\beta|^2\cos(kx)\sin(kx) + \alpha\beta^*\cos^2(kx) - \beta\alpha^*\sin^2(kx)\\
&+|\alpha|^2\cos(kx)\sin(kx) - |\beta|^2\cos(kx)\sin(kx) - \alpha^*\beta\cos^2(kx) + \beta^*\alpha\sin^2(kx)\big)\\
&= \frac{k}{i}\left(\alpha^*\beta - \beta^*\alpha\right)
\end{aligned}$$

6. Consider a wave packet described by **[4 marks]**

$$\psi(x) = Ce^{-a|x|}, \qquad a > 0$$

(a) Obtain the value of C by imposing an appropriate condition on the wave function. [1 mark]

(b) Obtain the expression for $|g(k)|^2$, where $g(k)$ is the momentum distribution function for this wave packet. [3 marks]

Soln (a): Normalization condition requires

$$\begin{aligned}
\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = |C|^2\left\{\int_{-\infty}^{0}e^{2ax} + \int_{0}^{\infty}e^{-2ax}dx\right\} = \frac{|C|^2}{a} = 1\\
\implies |C| = C = \sqrt{a}
\end{aligned}$$

above we chose C to be real.

(b): We know that

$$\begin{aligned}
g(k) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\psi(x,0)e^{-ikx}dx\\
&= \sqrt{\frac{a}{2\pi}}\left\{\int_{-\infty}^{0}e^{ax-ikx} + \int_{0}^{\infty}e^{-ax-ikx}dx\right\}\\
&= \sqrt{\frac{a}{2\pi}}\left\{\frac{1}{(a-ik} + \frac{1}{a+ik}\right\}\\
&= \sqrt{\frac{a}{2\pi}}\frac{2a}{(a^2+k^2)}\\
\implies |g(k)|^2 &= \frac{2a^3}{\pi(a^2+k^2)^2}
\end{aligned}$$

7. Consider a particle in a box with the potential $V(x) = 0$ for $0 \le x \le a$ and $V(x) = \infty$ everywhere else. At time $t = 0$ particle is described by the normalized wave function $\psi(x) = \sqrt{\frac{30}{a^5}}\, x(a-x)$. **[5 marks]**

   (a) Calculate the energy expectation value of this particle [2 marks]

   (b) What is the probability that this particle be found in the ground state of the system? [3 marks]

Soln (a): $\langle E\rangle = \int \psi^*(x) H \psi(x) dx$. Here, because $V(x) = 0$, we have

$$\begin{aligned}
\langle E\rangle &= \int \psi^*(x) \frac{p^2}{2m} \psi(x) dx = \frac{30\hbar^2}{2ma^5} \int_0^a x(x-a) \frac{d^2}{dx^2} \{x(x-a)\}\, dx \\
&= \frac{15\hbar^2}{ma^5} \int_0^a 2x(x-a) dx = \frac{30\hbar^2}{ma^5} \int_0^a \left(x^2 - ax\right) dx = \frac{30\hbar^2}{ma^5} \left(\frac{a^3}{3} - \frac{a^3}{2}\right) \\
&= \frac{5\hbar^2}{ma^2}
\end{aligned}$$

(b): If $\phi_1(x)$ is the lowest eigenfunction of the 1D particle in a box, the required probability P is

$$P = \left| \int_0^a \psi(x)\phi_1(x) dx \right|^2 = \left| \sqrt{\frac{60}{a^6}} \int_0^a x(x-a) \sin\frac{\pi x}{a} dx \right|^2$$

Calculating the two integrals

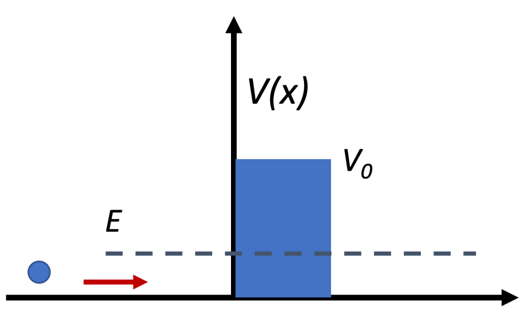
$$\begin{aligned}
& \int_0^a x^2 \sin\frac{\pi x}{a} dx - a \int_0^a x \sin\frac{\pi x}{a} dx \\
&= \left\{ -\frac{a}{\pi} x^2 \cos\frac{\pi x}{a} + \frac{2a^2}{\pi^2} x \sin\frac{\pi x}{a} + \frac{2a^3}{\pi^3} \cos\frac{\pi x}{a} \right\}_0^a \\
&\qquad - a \left\{ -\frac{a}{\pi} x \cos\frac{\pi x}{a} + \frac{a^2}{\pi^2} \sin\frac{\pi x}{a} \right\}_0^a = \left\{ \frac{a^3}{\pi} - \frac{4a^3}{\pi^3} \right\} - \left\{ \frac{a^3}{\pi} \right\} \\
&= -\frac{4a^3}{\pi^3}
\end{aligned}$$

Therefore

$$P = \left| \left(-\frac{4a^3}{\pi^3} \right) \sqrt{\frac{60}{a^6}} \right|^2 = \frac{960}{\pi^6} \approx 0.9986$$

Either the expression $(\frac{960}{\pi^6})$ or the numerical value ≈ 0.9986 will get full marks.

8. **[5 marks]**



An electron with energy $E = 1$ eV is incident upon a rectangular barrier of potential energy $V_0 = 2$ eV (see adjacent figure). About how wide must the barrier be so that the transmission probability is 10^{-3}?

Answer Using the approximate formula, the transmission probability is

$$\begin{aligned}
T &\simeq \frac{16E\,(V_0 - E)}{V_0^2} \cdot \exp\left[-\frac{2d}{\hbar} \sqrt{2m\,(V_0 - E)} \right] \\
&= 4 \exp\left[-\frac{2d}{\hbar} \sqrt{2m\,(V_0 - E)} \right]
\end{aligned}$$

whence

$$d = -\frac{\ln\left(\frac{T}{4}\right)}{2}\frac{\hbar c}{\sqrt{2mc^2\left(V_0 - E\right)}}$$
$$= -\frac{\ln\left(\frac{10^{-3}}{4}\right)}{2} \times \frac{6.58 \times 10^{-16} \times 3 \times 10^{10}}{\sqrt{2 \times 0.51 \times 10^6}} = 8.1 \times 10^{-8}\ \mathrm{cm} = 8.10\ \text{Å}.$$

Alternatively, using the exact formula:

Using $\quad \alpha = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$ we can write

$$T(E) = \frac{1}{1 + \dfrac{V_0^2}{4E(V_0 - E)}\sinh^2 \alpha d} = 10^{-3} \Rightarrow \sinh \alpha d = \sqrt{999} \Rightarrow \alpha d \approx 4.147$$

$$d = \frac{\hbar \times \sinh^{-1}\sqrt{999}}{\sqrt{2m(V_0 - E)}} = \frac{1.05 \times 10^{-34} \times 4.147}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}}} \approx 8.07 \times 10^{-10}\ \text{meter} = 8.07\ \text{Å}$$

[acceptable range: 7.7–8.5 Å]

9. Consider a one-dimensional simple harmonic oscillator of frequency ω. At the time $t = 0$, the harmonic oscillator is in the state **[5 marks]**

$$\psi(x, 0) = \sum_{n=0}^{N} C_n \phi_n(x),$$

where $\phi_n(x)$ are the eigenfunctions of the harmonic oscillator, C_ns are real constants, and $N > 0$ is an integer. Assume that C_ns have been chosen in such a way that $\psi(x, 0)$ is normalized.

(a) What is the wave function $\psi(x, t)$ of this system at a later time $t > 0$? [2 marks]

(b) Calculate the integral I, defined as [3 marks]

$$I = i\hbar \int_{-\infty}^{\infty} \psi^*(x, t)\frac{\partial \psi(x, t)}{\partial t} dx$$

Soln (a): Clearly

$$\psi(x, t) = \sum_{n=0}^{N} C_n \phi_n(x) e^{-iE_n t/\hbar} = \sum_{n=0}^{N} C_n \phi_n(x) e^{-iE_n t/\hbar}$$

using the fact that $E_n = (n + \frac{1}{2})\hbar\omega$, we have

$$\psi(x, t) = \sum_{n=0}^{N} C_n \phi_n(x) e^{-i(n+1/2)\omega t}$$

(b): Using the expression for $\frac{\partial\psi(x,t)}{\partial t}$, we obtain

$$
\begin{aligned}
I &= i\hbar \sum_{m=0}^{N}\sum_{n=0}^{N} C_m C_n \int_{-\infty}^{\infty} \phi_m(x) e^{i(m+1/2)\omega t}\left\{-i\omega\left(n+\frac{1}{2}\right)\right\} e^{-i(n+1/2)\omega t}\phi_n(x)dx \\
&= \hbar\omega \sum_{m=0}^{N}\sum_{n=0}^{N} C_m C_n \left(n+\frac{1}{2}\right) e^{i(m-n)\omega t}\int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)dx \\
&= \hbar\omega \sum_{m=0}^{N}\sum_{n=0}^{N} C_m C_n \left(n+\frac{1}{2}\right) e^{i(m-n)\omega t}\delta_{m,n} \\
&= \sum_{n=0}^{N} C_n^2 \left(n+\frac{1}{2}\right)\hbar\omega
\end{aligned}
$$

above we used the orthonormality condition of the SHO eigenfunctions

$$\int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)dx = \delta_{m,n}$$

10. A 3-dimensional simple harmonic oscillator potential is given by **[5 marks]**

$$V(x,y,z) = \frac{1}{2}m\omega_0^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2 \quad \text{where} \quad \omega_z = 10\omega_0$$

   (a) What are the energy eigenvalues of this system in terms of $\hbar\omega_0$? [2 marks]

   (b) What are the degeneracies of the (i) ground state, (ii) first excited state, and (iii) the second excited state of the system? [3 marks]

Soln (a): Clearly, the energy eigenvalues will be

$$E_{n_x,n_y,n_z} = (n_x+n_y+1)\hbar\omega_0 + 10(n_z+1/2)\hbar\omega_0$$

(b): From above it is obvious that the lowest three eigenstates will correspond to the quantum numbers (n_x, n_y, n_z): (a) (0,0,0), (b) (1,0,0) and (0,1,0), and (c) (2,0,0), (1,1,0), and (0,2,0). Thus, their degeneracies will be 1, 2, and 3, respectively.