IF $f: U \to \mathbb{R}$ $(U \subseteq \mathbb{R}^2)$ AND f IS DIFF. AT (x_0, y_0), THEN $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ EXIST

PROOF: f IS DIFF. AT (x_0, y_0), SO,

THERE EXIST $a, b \in \mathbb{R}$ AND $\epsilon: (-\delta, \delta)^2 \to \mathbb{R}$

S.T

$$f(x_0+h, y_0+k) = f(x_0, y_0) + ah + bk + \sqrt{h^2+k^2}\,\epsilon(h,k)$$

WHERE $\lim_{(h,k)\to(0,0)} \epsilon(h,k) = 0.$

LET $k=0$, SO,

$$f(x_0+h, y_0) = f(x_0, y_0) + ah + |h|\,\epsilon(h,0)$$

THIS IS EQUIVALENT TO SAYING $f_x(x_0, y_0) = a.$

THE SAME PROOF WORKS FOR f_y.

$f(x,y) = \sqrt{x^2+y^2}$ IS NOT DIFF. AT $(0,0)$ SINCE $f_x(0,0)$ DNE. (PROVE THIS!)

CHAIN RULE

$(U \subseteq \mathbb{R}^2)$

LET $f: U \to \mathbb{R}$ AND $(x_0, y_0) \in U$. LET $t_0 \in \mathbb{R}$

AND $x, y: (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ FOR SOME $\delta > 0$

SUCH THAT

- $(x(t_0), y(t_0)) = (x_0, y_0)$ AND $(x(t), y(t))$ LIES IN $B_r(x_0, y_0)$ $\forall$ $t \in (t_0 - \delta, t_0 + \delta)$
- IF x, y ARE DIFFERENTIABLE AT t_0, THEN $F(t) = f(x(t), y(t))$ IS DIFFERENTIABLE AT t_0 AND

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$\cdot \left(x'(t), y'(t)\right)$

EXAMPLE: $f(x, y) = x^2 + y^2$, $x(t) = e^t$, $y(t) = t$.

$F = f(x(t), y(t))$.

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t)$$

$$\frac{\partial f}{\partial x} = 2x \; ; \; \frac{\partial f}{\partial y} = 2y \; ; \; x'(t) = e^t, \; y'(t) = 1$$

$$\Rightarrow \frac{dF}{dt} = 2x \cdot e^t + 2y \cdot 1 = 2(e^t)^2 + 2t$$

CHAIN RULE: VERSION II

SUPPOSE $f: U \to \mathbb{R}$, $(x_0, y_0) \in U$. LET

$x, y : B_\delta((s_0, t_0)) \to \mathbb{R}$ S.T

- $x(s_0, t_0) = x_0$, $y(s_0, t_0) = y_0$ AND

$\left(x(B_\delta(s_0,t_0)), y(B_\delta(s_0,t_0))\right) \in B_r(x_0, y_0)$

FOR SOME $B_r(x_0, y_0) \subseteq U$.

- SUPPOSE x_s, x_t, y_s, y_t EXIST AT (s_0, t_0) THEN

$F(s,t)$ HAS PARTIAL DERIVATIVES AT (s_0, t_0)

AND

$$F_s(s_0,t_0) = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s}\right)(s_0,t_0) + \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}\right)(s_0,t_0)$$

$$F_t(s_0,t_0) = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}\right)(s_0,t_0) + \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}\right)(s_0,t_0)$$

$f(x,y) = x^2 + y^2$; $x(t,s) = s^2 - t^2$, $y(t,s) = 2st$

$F_s(s,t)$ $(F = f(x(t,s), y(t,s)))$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$; $\frac{\partial x}{\partial s} = 2s$, $\frac{\partial x}{\partial t} = -2t$

$\frac{\partial y}{\partial s} = 2t$, $\frac{\partial y}{\partial t} = 2s$

IMPLICIT FUNCTION THEOREM

SUPPOSE $(x_0, y_0) \in U$, $g : B_r(x_0, y_0) \to \mathbb{R}$

SATISFIES

- g_x, g_y ARE CONTINUOUS IN $B_r(x_0, y_0)$
- $g(x_0, y_0) = 0$, $g_y(x_0, y_0) \neq 0$

THEN THERE EXISTS $\delta > 0$ AND

$\phi : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$, DIFFERENTIABLE

WITH A CONTINUOUS DERIVATIVE SATISFYING

- $\phi(x_0) = y_0$
- $g(x, \phi(x)) = 0$, $g_y(x, y) \neq 0$

  $\forall \ x \in (x_0 - \delta, x_0 + \delta)$
- $$\phi'(x_0) = -\frac{g_x(x_0, y_0)}{g_y(x_0, y_0)}$$

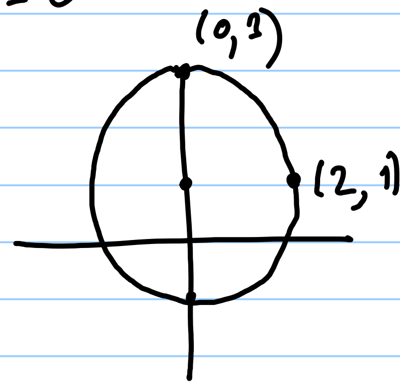
EXAMPLE

$g(x,y) = x^2 + (y-1)^2 - 4$; $g(x,y) = 0$

$(x_0, y_0) = (2,1)$

$g_x = 2x$; $g_y = 2(y-1)$

$g_x(2,1) = 4$; $g_y(2,1) = 0$



$(x_0, y_0) = (0,3)$

$g_x(0,3) = 0$, $g_y(0,3) = 4 \neq 0$, SO IFT IS

APPLICABLE AT (0,3) BUT NOT (2,1)

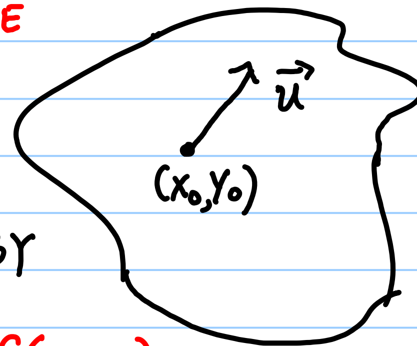
DIRECTIONAL DERIVATIVES

SUPPOSE $f : U \to \mathbb{R}$. $U \subseteq \mathbb{R}^2$ AND LET

$(x_0, y_0) \in U$. (INTERIOR)

LET $\vec{u} = u_1 \vec{i} + u_2 \vec{j} = (u_1, u_2)$, $u_1^2 + u_2^2 = 1$

THE DIRECTIONAL DERIVATIVE OF f ALONG THE DIRECTION $\vec{u}$ IS GIVEN BY



$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

IT IS DENOTED $D_{\vec{u}}(f)(x_0, y_0)$

(DIRECTIONAL DERIVATIVE ALONG $\vec{u}$ OF f, AT (x_0, y_0).

🚩 IF $f : D \to \mathbb{R}$ IS DIFFERENTIABLE AT (x_0, y_0), THEN FOR ANY $\vec{u}$, THE DIRECTIONAL DERIVATIVE OF f ALONG $\vec{u}$ EXISTS.

PROOF: EXERCISE!

🚩 THE CONVERSE OF THE ABOVE IS FALSE!

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{IF } (x,y) \neq (0,0)$$
$$= 0 \quad \text{IF } (x,y) = (0,0).$$

AT $(0,0)$ FOR ANY u_1, u_2, $(u_1^2 + u_2^2 = 1)$

$$\frac{f(u_1 t, u_2 t)}{t} = \frac{u_1 u_2^2}{(u_1^2 + u_2^4 t^2)} \longrightarrow u_2^2 \text{ AS } t \to 0.$$

BUT f IS NOT CONTINUOUS (CHECK THIS!) AT $(0,0)$.