## 13.3 THE LINEAR TRANSFORMER

We are now ready to apply our knowledge of magnetic coupling to the description of two specific practical devices, each of which may be represented by a model containing mutual inductance. Both of the devices are transformers, a term which we define as a network containing two or more coils which are deliberately coupled magnetically (Fig. 13.15). In this section we consider the linear transformer, which happens to be an excellent model for devices used at radio frequencies, or higher frequencies. In Sec. 13.4 we will consider the ideal transformer, which is an idealized unity-coupled model of a physical transformer that has a core made of some magnetic material, usually an iron alloy.


**FIGURE 13.15** A selection of small transformers for use in electronic applications; the AA battery is shown for scale only.

In Fig. 13.16 a transformer is shown with two mesh currents identified. The first mesh, usually containing the source, is called the ***primary,*** while the second mesh, usually containing the load, is known as the ***secondary.*** The inductors labeled $L_1$ and $L_2$ are also referred to as the primary and secondary, respectively, of the transformer. We will assume that the transformer is *linear*. This implies that no magnetic material (which may cause a *nonlinear* flux-versus-current relationship) is employed. Without such material, however, it is difficult to achieve a coupling coefficient greater than a few tenths. The two resistors serve to account for the resistance of the wire out of which the primary and secondary coils are wound, and any other losses.

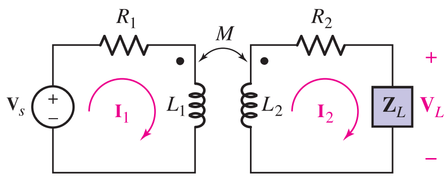


**FIGURE 13.16** A linear transformer containing a source in the primary circuit and a load in the secondary circuit. Resistance is also included in both the primary and the secondary.

### Reflected Impedance

Consider the input impedance offered at the terminals of the primary circuit. The two mesh equations are

$$\mathbf{V}_s = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 \qquad [11]$$

and a phase angle of $-3.819 \times 10^{-8}$ degrees (essentially zero), in agreement with the values calculated by hand in Example 13.5.

PSpice also provides two different transformer models, a linear transformer, **XFRM_LINEAR,** and an ideal transformer **XFRM_NONLINEAR,** a circuit element which is the subject of the following section. The linear transformer requires that values be specified for the coupling coefficient and both coil inductances. The ideal transformer also requires a coupling coefficient, but, as we shall see, an *ideal* transformer has infinite or nearly infinite inductance values. Thus, the remaining parameters required for the part **XFRM_NONLINEAR** are the number of turns of wire that form each coil.

## 13.4 THE IDEAL TRANSFORMER

An ***ideal transformer*** is a useful approximation of a very tightly coupled transformer in which the coupling coefficient is essentially unity and both the primary and secondary inductive reactances are extremely large in comparison with the terminating impedances. These characteristics are closely approached by most well-designed iron-core transformers over a reasonable range of frequencies for a reasonable range of terminal impedances. The approximate analysis of a circuit containing an iron-core transformer may be achieved very simply by replacing that transformer with an ideal transformer; the ideal transformer may be thought of as a first-order model of an iron-core transformer.

### Turns Ratio of an Ideal Transformer

One new concept arises with the ideal transformer: the ***turns ratio*** $a$. The self-inductance of a coil is proportional to the square of the number of turns of wire forming the coil. This relationship is valid only if all the flux established by the current flowing in the coil links all the turns. In order to develop this result quantitatively it is necessary to utilize magnetic field concepts, a subject that is not included in our discussion of circuit analysis. However, a qualitative argument may suffice. If a current $i$ flows through a coil of $N$ turns, then $N$ times the magnetic flux of a single-turn coil will be produced. If we think of the $N$ turns as being coincident, then all the flux certainly links all the turns. As the current and flux change with time, a voltage is then induced *in each turn* which is $N$ times larger than that caused by a single-turn coil. Thus, the voltage induced *in the N-turn coil* must be $N^2$ times the single-turn voltage. From this, the proportionality between inductance and the square of the numbers of turns arises. It follows that

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2 \qquad [20]$$

or

$$\boxed{a = \frac{N_2}{N_1}} \qquad [21]$$

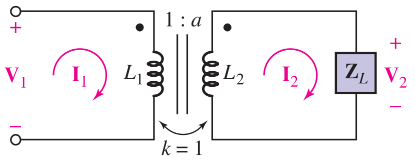


**FIGURE 13.25** An ideal transformer is connected to a general load impedance.

Figure 13.25 shows an ideal transformer to which a secondary load is connected. The ideal nature of the transformer is established by several conventions: the use of the vertical lines between the two coils to indicate the iron laminations present in many iron-core transformers, the unity value of the coupling coefficient, and the presence of the symbol 1:$a$, suggesting a turns ratio of $N_1$ to $N_2$.

Let us analyze this transformer in the sinusoidal steady state. The two mesh equations are

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 \qquad [22]$$

and

$$0 = -j\omega M\mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2 \qquad [23]$$

First, consider the input impedance of an ideal transformer. By solving Eq. [23] for $\mathbf{I}_2$ and substituting in Eq. [22], we obtain

$$\mathbf{V}_1 = \mathbf{I}_1 j\omega L_1 + \mathbf{I}_1 \frac{\omega^2 M^2}{\mathbf{Z}_L + j\omega L_2}$$

and

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = j\omega L_1 + \frac{\omega^2 M^2}{\mathbf{Z}_L + j\omega L_2}$$

Since $k = 1$, $M^2 = L_1 L_2$ so

$$\mathbf{Z}_{\text{in}} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{\mathbf{Z}_L + j\omega L_2}$$

Besides a unity coupling coefficient, another characteristic of an ideal transformer is an extremely large impedance for both the primary and secondary coils, regardless of the operating frequency. This suggests that the ideal case would be for both $L_1$ and $L_2$ to tend to infinity. Their ratio, however, must remain finite, as specified by the turns ratio. Thus,

$$L_2 = a^2 L_1$$

leads to

$$\mathbf{Z}_{\text{in}} = j\omega L_1 + \frac{\omega^2 a^2 L_1^2}{\mathbf{Z}_L + j\omega a^2 L_1}$$

Now if we let $L_1$ become infinite, both of the terms on the right-hand side of the preceding equation become infinite, and the result is indeterminate. Thus, it is necessary to first combine these two terms:

$$\mathbf{Z}_{\text{in}} = \frac{j\omega L_1 \mathbf{Z}_L - \omega^2 a^2 L_1^2 + \omega^2 a^2 L_1^2}{\mathbf{Z}_L + j\omega a^2 L_1} \qquad [24]$$

or

$$\mathbf{Z}_{\text{in}} = \frac{j\omega L_1 \mathbf{Z}_L}{\mathbf{Z}_L + j\omega a^2 L_1} = \frac{\mathbf{Z}_L}{\mathbf{Z}_L / j\omega L_1 + a^2} \qquad [25]$$

Now as $L_1 \to \infty$, we see that $\mathbf{Z}_{\text{in}}$ becomes

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{a^2} \qquad [26]$$

for finite $\mathbf{Z}_L$.

This result has some interesting implications, and at least one of them appears to contradict one of the characteristics of the linear transformer. The input impedance of an ideal transformer is proportional to the load impedance, the proportionality constant being the reciprocal of the square of the turns ratio. In other words, if the *load* impedance is a capacitive impedance, then the *input* impedance is a capacitive impedance. In the linear transformer, however, the reflected impedance suffered a sign change in its reactive part; a capacitive load led to an inductive contribution to the input impedance. The explanation of this occurrence is achieved by first realizing that $\mathbf{Z}_L/a^2$ is *not* the reflected impedance, although it is often loosely called by that name. The true reflected impedance is infinite in the ideal transformer; otherwise it could not "cancel" the infinite impedance of the primary inductance. This cancellation occurs in the numerator of Eq. [24]. The impedance $\mathbf{Z}_L/a^2$ represents a small term which is the amount by which an exact cancellation does not occur. The true reflected impedance in the ideal transformer does change sign in its reactive part; as the primary and secondary inductances become infinite, however, the effect of the infinite primary-coil reactance and the infinite, but negative, reflected reactance of the secondary coil is one of cancellation.

The first important characteristic of the ideal transformer is therefore its ability to change the magnitude of an impedance, or to change impedance level. An ideal transformer having 100 primary turns and 10,000 secondary turns has a turns ratio of 10,000/100, or 100. Any impedance placed across the secondary then appears at the primary terminals reduced in magnitude by a factor of $100^2$, or 10,000. A 20,000 Ω resistor looks like 2 Ω, a 200 mH inductor looks like 20 $\mu$H, and a 100 pF capacitor looks like 1 $\mu$F. If the primary and secondary windings are interchanged, then $a = 0.01$ and the load impedance is apparently increased in magnitude. In practice, this exact change in magnitude does not always occur, for we must remember that as we took the last step in our derivation and allowed $L_1$ to become infinite in Eq. [25], it was necessary to neglect $\mathbf{Z}_L$ in comparison with $j\omega L_2$. Since $L_2$ can never be infinite, it is evident that the ideal transformer model will become invalid if the load impedances are very large.

## Use of Transformers for Impedance Matching

A practical example of the use of an iron-core transformer as a device for changing impedance level is in the coupling of an amplifier to a speaker system. In order to achieve maximum power transfer, we know that the resistance of the load should be equal to the internal resistance of the source; the speaker usually has an impedance magnitude (often assumed to be a resistance) of only a few ohms, while an amplifier may possess an internal resistance of several thousand ohms. Thus, an ideal transformer is required in which $N_2 < N_1$. For example, if the amplifier internal impedance is 4000 Ω and the speaker impedance is 8 Ω, then we desire that

$$\mathbf{Z}_g = 4000 = \frac{\mathbf{Z}_L}{a^2} = \frac{8}{a^2}$$

or

$$a = \frac{1}{22.4}$$

and thus

$$\frac{N_1}{N_2} = 22.4$$

## Use of Transformers for Current Adjustment

There is a simple relationship between the primary and secondary currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in an ideal transformer. From Eq. [23],

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{\mathbf{Z}_L + j\omega L_2}$$

Once again we allow $L_2$ to become infinite, and it follows that

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{j\omega L_2} = \sqrt{\frac{L_1}{L_2}}$$

or

$$\boxed{\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{1}{a}} \qquad [27]$$

Thus, the ratio of the primary and secondary currents is the turns ratio. If we have $N_2 > N_1$, then $a > 1$, and it is apparent that the larger current flows in the winding with the smaller number of turns. In other words,

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

It should also be noted that the current ratio is the *negative* of the turns ratio if either current is reversed or if either dot location is changed.

In our example in which an ideal transformer was used to change the impedance level to efficiently match a speaker to an amplifier, an rms current of 50 mA at 1000 Hz in the primary causes an rms current of 1.12 A at 1000 Hz in the secondary. The power delivered to the speaker is $(1.12)^2(8)$, or 10 W, and the power delivered to the transformer by the power amplifier is $(0.05)^2 4000$, or 10 W. The result is comforting, since the ideal transformer contains neither an active device which can generate power nor any resistor which can absorb power.

## Use of Transformers for Voltage Level Adjustment

Since the power delivered to the ideal transformer is identical with that delivered to the load, whereas the primary and secondary currents are related by the turns ratio, it should seem reasonable that the primary and secondary voltages must also be related to the turns ratio. If we define the secondary voltage, or load voltage, as

$$\mathbf{V}_2 = \mathbf{I}_2\mathbf{Z}_L$$

and the primary voltage as the voltage across $L_1$, then

$$\mathbf{V}_1 = \mathbf{I}_1\mathbf{Z}_{\text{in}} = \mathbf{I}_1\frac{\mathbf{Z}_L}{a^2}$$

The ratio of the two voltages then becomes

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = a^2\frac{\mathbf{I}_2}{\mathbf{I}_1}$$

or

$$\boxed{\frac{\mathbf{V}_2}{\mathbf{V}_1} = a = \frac{N_2}{N_1}} \qquad [28]$$

The ratio of the secondary to the primary voltage is equal to the turns ratio. We should take care to note that this equation is opposite that of Eq. [27], and this is a common source of error for students. This ratio may also be negative if either voltage is reversed or either dot location is changed.

Simply by choosing the turns ratio, therefore, we now have the ability to change any ac voltage to any other ac voltage. If $a > 1$, the secondary voltage will be greater than the primary voltage, and we have what is commonly referred to as a ***step-up transformer.*** If $a < 1$, the secondary voltage will be less than the primary voltage, and we have a ***step-down transformer.*** Utility companies typically generate power at a voltage in the range of 12 to 25 kV. Although this is a rather large voltage, transmission losses over long distances can be reduced by increasing the level to several hundred thousand volts using a step-up transformer (Fig. 13.26*a*). This voltage is then reduced to several tens of kilovolts at substations for local power distribution using step-down transformers (Fig. 13.26*b*). Additional step-down transformers are located outside buildings to reduce the voltage from the transmission voltage to the 110 or 220 V level required to operate machinery (Fig. 13.26*c*).

Combining the voltage and current ratios, Eqs. [27] and [28],

$$\mathbf{V}_2\mathbf{I}_2 = \mathbf{V}_1\mathbf{I}_1$$

and we see that the primary and secondary complex voltamperes are equal. The magnitude of this product is usually specified as a maximum allowable value on power transformers. If the load has a phase angle $\theta$, or

$$\mathbf{Z}_L = |\mathbf{Z}_L|\underline{/\theta}$$

then $\mathbf{V}_2$ leads $\mathbf{I}_2$ by an angle $\theta$. Moreover, the input impedance is $\mathbf{Z}_L/a^2$, and thus $\mathbf{V}_1$ also leads $\mathbf{I}_1$ by the same angle $\theta$. If we let the voltage and current represent rms values, then $|\mathbf{V}_2|\,|\mathbf{I}_2|\cos\theta$ must equal $|\mathbf{V}_1|\,|\mathbf{I}_1|\cos\theta$, and all the power delivered to the primary terminals reaches the load; none is absorbed by or delivered to the ideal transformer.

The characteristics of the ideal transformer that we have obtained have all been determined by phasor analysis. They are certainly true in the sinusoidal steady state, but we have no reason to believe that they are correct for the *complete* response. Actually, they are applicable in general, and the demonstration that this statement is true is much simpler than the phasor-based analysis we have just completed. Our analysis, however, has served to point out the specific approximations that must be made on a more exact model of an actual transformer in order to obtain an ideal transformer. For example, we have seen that the reactance of the secondary winding must be much greater in magnitude than the impedance of any load that is connected to the secondary. Some feeling for those operating conditions under which a transformer ceases to behave as an ideal transformer is thus achieved.

(*a*)

(*b*)

(*c*)

**FIGURE 13.26** (*a*) A step-up transformer used to increase the generator output voltage for transmission. (*b*) Substation transformer used to reduce the voltage from the 220 kV transmission level to several tens of kilovolts for local distribution. (*c*) Step-down transformer used to reduce the distribution voltage level to 240 V for power consumption.
(*Photos courtesy of Dr. Wade Enright, Te Kura Pukaha Vira O Te Whare Wananga O Waitaha, Aotearoa.*)