MA108 ENDSEM 12-06-2023 **10:00-12:00** **Maximum Marks: 30**

Name: Pink **Division:**
Roll No: **Tutorial Batch:**

1. Write your Name, Roll No., Division, Tutorial Batch.
2. This is a question paper cum answer booklet. At the end of the exam, ***only*** *this booklet will be collected for evaluation. Write the answers in the space provided against each question. Separate sheets will be provided for rough work.*
3. There are ***sixteen*** *questions.*
4. No books, notes, calculators, mobile phones, electronic devices are permitted.
5. There is ***no*** *negative marking.*

1. The curve in the x-y plane through the point $(1, 0)$ and having the slope $2x$ at each point (x, y) is given by [1]

$$y(x) = \boxed{x^2 - 1}.$$

2. Consider the IVP: $y' = \frac{y}{x} + 2x^2 e^{-\frac{y}{x}}$, $y(1) = 0$. The solution of the above IVP is [2]

$$y(x) = \boxed{x \ln x^2} \quad \forall\, x > 0.$$

3. Let x^{-2} and $x^{-2}\ln x$ be two solutions of $x^2y'' + axy' + by = 0$, for $x > 0$, and $a, b \in \mathbb{R}$. Then [2]

$$a = \boxed{5} \text{ and } b = \boxed{4}.$$

4. Let $L(y)(x) = (1 + x^2)y''(x) - 2xy'(x) + 2y(x)$ for all $x > 0$. [3]

   a. Let ϕ_1, ϕ_2 be two linearly independent solutions of $L(y)(x) = 0$ for $x > 0$. Given that $\phi_1(x) = x$, for $x > 0$, find

$$\phi_2(x) = \boxed{x^2 - 1}.$$

   Or $\boxed{\phi_2(x) = C_1x + C_2(x^2 - 1), \text{ for any } C_1, C_2 \in \mathbb{R} \text{ with } C_2 \neq 0.}$

   b. If $y_p(x) = v_1(x)\phi_1(x) + v_2(x)\phi_2(x)$ is a particular solution of $L(y)(x) = x^3 + x$, then

$$v_1(x) = \boxed{-\frac{x^2}{2} + \ln(x^2 + 1)} \text{ and } v_2(x) = \boxed{x - \tan^{-1} x}.$$

   Or $\boxed{v_1(x) = -\frac{C_1}{C_2}\left(x - \tan^{-1} x\right) - \frac{x^2}{2} + \ln(x^2 + 1) + d_1}$, and $\boxed{v_2(x) = \frac{1}{C_2}\left(x - \tan^{-1} x\right) + d_2}$, for any constants $d_1, d_2 \in \mathbb{R}$ and C_1, C_2 same as in [a].

5. The inverse Laplace transform $\mathcal{L}^{-1}$ of the function $F(s) = \dfrac{s^2 + 9}{(s^2 - 9)^2}$ for $s > 3$ is given by [2]

$$\mathcal{L}^{-1}(F)(t) = \boxed{t \cosh 3t}.$$

(i.e., $\mathcal{L}^{-1}(F)(t) = \boxed{t\dfrac{(e^{3t} + e^{-3t})}{2}}$.)

6. *Possibly multiple correct answers.* Let $\phi_1(x) = \begin{cases} 1+x^3, & x<0 \\ 1, & x \geq 0 \end{cases}$, $\phi_2(x) = \begin{cases} 1, & x<0 \\ 1+x^3, & x \geq 0 \end{cases}$, and $\phi_3(x) = 3 + x^3, \quad x \in \mathbb{R}$. [1]
Write the correct option(s) here: Ans. $\boxed{\text{b}}$

   a. The Wronskian $W(\phi_1, \phi_2, \phi_3)(x) \neq 0$ for all $x \in [-1, 1]$.

   b. The functions ϕ_1, ϕ_2, ϕ_3 are linearly independent on $[-1, 1]$.

   c. There exist functions p_1, p_2, p_3 defined and continuous on $[-1, 1]$ such that ϕ_1, ϕ_2, ϕ_3 are solutions of

$$y'''(x) + p_1(x)y''(x) + p_2(x)y'(x) + p_3(x)y(x) = 0,$$

   for all $x \in [-1, 1]$.

7. *Possibly multiple correct answers.* Every solution of the DE $y''(x) + \alpha y'(x) + \beta y(x) = 0$, where $\alpha, \beta \in \mathbb{R}$, converges to 0 as $x \to \infty$, if [1]

   a. $\alpha > 0$, $\beta > 0$ and $\alpha^2 - 4\beta > 0$.

   b. $\alpha < 0, \beta < 0$, $\alpha^2 - 4\beta > 0$.

   c. $\alpha > 0, \alpha^2 - 4\beta < 0$.

Write the correct option(s) here: Ans. $\boxed{\text{(a), (c)}}$.

8. *Possibly multiple correct answers.* The function $r(x) = xe^x + xe^{-x}$ is annihilated by [1]

   a. $D^2 - 2D + 1$

   b. $D^5 + D^4 - 2D^3 - 2D^2 + D + 1$.

   c. $D^4 - 2D^2 + 1$

Write the correct option(s) here: Ans. $\boxed{\text{(b), (c)}}$.

9. Let p, q, r be continuous functions on $\mathbb{R}$ and $L(y)(x) = y''(x) + p(x)y'(x) + q(x)y(x)$. If

$$\phi_1(x) = 1 + e^{x^2}, \ \phi_2(x) = 1 + xe^{x^2}, \ \phi_3(x) = (1+x)e^{x^2} + 1$$

are solutions of $L(y)(x) = r(x), x \in \mathbb{R}$, then [4]

   a. Two linearly independent solutions of $L(y)(x) = 0$ on $\mathbb{R}$ are given by

$$y_1(x) = \boxed{e^{x^2}} \text{ and } y_2(x) = \boxed{xe^{x^2}}.$$

   (or, $\boxed{y_1(x) = c_1 e^{x^2} + c_2}$, for some constants c_1, c_2, with $c_1 \neq 0$ and $\boxed{y_2(x) = d_3 x e^{x^2} + d_2 e^{x^2} + d_3}$, for some constants d_1, d_2, d_3, with $d_3 \neq 0$.)

   b. The functions p and r are given by

$$p(x) = \boxed{-4x} \text{ and } r(x) = \boxed{4x^2 - 2}.$$

10. The solution set of

$$(\sin x)y'''(x) + xy''(x) + x^2y'(x) + x^3y(x) = 0$$

for $\frac{\pi}{4} < x < \frac{\pi}{2}$ is a vector space of dimension d, where [1]

$$d = \boxed{3}$$

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11. The least possible n for which $y(x) = \sin^2 x$ is a solution of some n^{th}-order linear differential equation [2]

$$y^{(n)}(x) + a_1y^{(n-1)}(x) + \ldots + a_ny(x) = 0$$

for $a_1, \ldots, a_n \in \mathbb{R}$ is

Ans: $\boxed{3}$.

12. Let $L(y)(x) = y'''(x) - 5y''(x) + 6y'(x)$. [3+1]

   a. A basis of solutions of $L(y)(x) = 0$ is given by

$$y_1(x) = \boxed{1},$$
$$y_2(x) = \boxed{e^{2x}},$$
$$\text{and } y_3(x) = \boxed{e^{3x}}.$$

   b. The solution of the IVP: $L(y)(x) = 12x,\ y(0) = 0,\ y'(0) = \frac{5}{3},\ y''(0) = 2$ is given by

$$y(x) = \boxed{x^2 + \frac{5}{3}x}.$$

13. *Possibly multiple correct answers.* Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function of exponential order. Let $F(s) = \mathcal{L}(f)(s)$, for $s > 0$, denote the Laplace transform of f. [1]

   a. If f is differentiable on $[0, \infty)$, then f' is also of exponential order.
   b. If $F(s) = 0$ for all $s > 0$, then $f(t) = 0$, for all $t \geq 0$.
   c. $\lim\limits_{s\to\infty} F(s)e^{\frac{-s^2}{2}} = 1$.

Write the correct option(s) here: Ans. $\boxed{\text{(b)}}$.

14. Let $g(t) = \int_0^t (t-x)\cos x\, dx$ and $f(t) = \begin{cases} 0 & t < 1 \\ g(t-1) & t \geq 1 \end{cases}$. Then the Laplace transform $\mathcal{L}(f)$ of f is given by [1]

$$\mathcal{L}(f)(s) = \boxed{\frac{e^{-s}}{s(s^2+1)}}.$$

15. Consider the IVP: $y''(x) + 4y'(x) + 4y(x) = x^2e^{-2x}$, $y(0) = 1$, $y'(0) = 2$. The Laplace transform $\mathcal{L}(y)$ of the solution y of the IVP is given by [2]

$$\mathcal{L}(y)(s) = \boxed{\frac{1}{s+2} + \frac{4}{(s+2)^2} + \frac{2}{(s+2)^5}}.$$

The solution y of the IVP is given by

$$y(x) = \boxed{e^{-2x}\left(1 + 4x + \frac{x^4}{12}\right)}.$$

16. Do there exist functions p, q continuous on $\mathbb{R}$ such that $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$? Justify your answer. [2]

Ans. No, there cannot exist p, q continuous functions on $\mathbb{R}$ such that $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$.

Reason: Let $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ for all $x \in \mathbb{R}$. Note that $y(0) = 0$ and $y'(0) = 0$. If there exist p, q continuous functions on $\mathbb{R}$ such that $y(x)$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$, then $y(x)$ satisfies the IVP

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

But then the uniqueness of solution of the second order linear ODE with continuous coefficients implies that $y(x) = 0$ for all $x \in \mathbb{R}$ which is not the case since $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$.

So, $y(x)$ cannot be a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$, for any continuous functions p, q defined on $\mathbb{R}$.