MAXIMA / MINIMA

SUPPOSE $f : U \to \mathbb{R}$ AND $(x_0, y_0) \in U$ IS INTERIOR.

LET $\vec{u} = (u_1, u_2)$ BE A UNIT VECTOR. IF

f HAS A LOCAL MAX/MIN. AT (x_0, y_0)

AND $(D_u f)(x_0, y_0)$ EXISTS, THEN $(D_u f)(x_0, y_0) = 0$

IN PARTICULAR, IF f_x, f_y BOTH EXIST AT (x_0, y_0)

THEN $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

<u>PROOF</u>: IF (x_0, y_0) IS A LOCAL MAX. THEN

$\exists\ \delta > 0$ S.T. $\forall\ (x, y) \in B_\delta(x_0, y_0)$

$$f(x, y) \leq f(x_0, y_0) \Rightarrow f(x_0 + tu_1, y_0 + tu_2) \leq f(x_0, y_0)$$

SINCE $D_u f(x_0, y_0)$ EXISTS,

$$\lim_{t \to 0^+} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \leq 0$$

$$\lim_{t \to 0^-} \quad " \quad \geq 0$$

$$\Rightarrow \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} = 0$$

(x_0, y_0) IS CALLED A CRITICAL POINT OF f

IF

(i) EITHER f_x, f_y EXIST AND

$$f_x(x_0, y_0) = f_x(x_0, y_0) = 0$$

(ii) ONE OF $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ DOES NOT EXIST. (AT LEAST ONE)

IF $f : K \to \mathbb{R}$ IS CONTINUOUS, (K IS CLOSED AND BOUNDED) THEN $\max\limits_{(x,y)\in K} f(x,y)$ $\left(\text{resp. } \min\limits_{(x,y)\in K} f(x,y)\right)$ IS ATTAINED AT A CRITICAL POINT OF K, OR AT A BOUNDARY POINT OF K.

$f : U \to \mathbb{R}$ AND $P \in U$ IS A CRITICAL POINT.

WE SAY THAT P IS A **SADDLE POINT** IF IN EACH $B_\delta(P) \cap U$, THERE EXIST POINTS $Q_1, Q_2 \in B_\delta(P) \cap U$ S.T

$$f(Q_1) < f(P) < f(Q_2).$$

DETERMINING MAX/MIN

GIVEN $f: U \to \mathbb{R}$, AND A CRITICAL POINT (x_0, y_0) FOR f.

HOW DO WE DETERMINE IF THIS IS A POINT OF LOCAL MAX. OR LOCAL MINIMA?

SECOND DERIVATIVE TEST.

SUPPOSE ALL THE SECOND ORDER PARTIAL DERIVATIVES OF f EXIST AT (x_0, y_0) (viz. $f_{xx}, f_{xy}, f_{yx}, f_{yy}$)

WE DENOTE BY $\Delta f(x_0, y_0)$, THE HESSIAN OF f AT (x_0, y_0):

$$\Delta f(x_0, y_0) := \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

$$= (f_{xx} f_{yy} - f_{xy} f_{yx})(x_0, y_0).$$ (2x2 DETERMINANT)

SUPPOSE $f: U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^2$) AND (x_0, y_0) IS AN INTERIOR POINT. SUPPOSE f_{xx}, f_{xy}, f_{yy} EXIST AND ARE CONTINUOUS ON $B_\delta(x_0, y_0)$ FOR SOME $\delta > 0$. SUPPOSE FURTHER THAT

(GRAD.) $\leftarrow \nabla f(x_0, y_0) = (0, 0)$. THEN

(i) f HAS A LOCAL MAXIMUM AT (x_0, y_0) IF $f_{xx}(x_0, y_0) < 0$ AND $\Delta f(x_0, y_0) > 0$. (HESSIAN)

(ii) f HAS A LOCAL MINIMUM AT (x_0, y_0) IF $f_{xx}(x_0, y_0) > 0$ AND $\Delta f(x_0, y_0) > 0$

(iii) f HAS A SADDLE POINT AT (x_0, y_0) IF $\Delta f(x_0, y_0) < 0$.

IF $\Delta f(x_0, y_0) = 0$, THEN NO DEFINITE CONCLUSION CAN BE DRAWN. IN THIS CASE ONE HAS TO TRY SOMETHING ELSE.

EXAMPLE

$f(x,y) = 4xy - x^4 - y^4$. FIND MAX/MIN

$f_x = 4y - 4x^3 \quad f_y = 4x - 4y^3$

f IS DIFF. EVERYWHERE. $(U = \mathbb{R}^2)$

CRITICAL POINTS: $\quad y = x^3, \; x = y^3$

$\Rightarrow y = y^9 = y(y^8 - 1) = 0$

$\Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$f_{xx} = -12x^2, \; f_{yy} = -12y^2 \quad f_{xy} = 4 = f_{yx}$

$$\Delta f = \begin{vmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{vmatrix}$$

AT $(0,0)$: $\quad \Delta f = -16 \Rightarrow (0,0)$ IS SADDLE

$(1,1)$: $\quad \Delta f = 128 > 0; \; -12x^2\Big]_{x=1} = -12 < 0$

$\Rightarrow (1,1)$ IS LOCAL MAX.

$(-1,-1)$, $\Delta f = 128 > 0$, $(-1,-1)$ IS LOCAL MAX

MAX/MIN WITH CONSTRAINTS

THE METHOD OF LAGRANGE MULTIPLIERS

CONSIDER THE FOLLOWING PROBLEMS:

🚩 FIND THE CLOSEST POINT ON THE PLANE $2x + y - 3z = 5$, TO THE ORIGIN.

🚩 A SATELLITE IN THE SHAPE OF THE ELLIPSOID $4x^2 + y^2 + 4z^2 = 16$ ENTERS THE EARTH'S SURFACE AND IT'S SURFACE BEGINS TO HEAT. THE TEMPERATURE ON THE SURFACE AT (x, y, z) IS GIVEN BY $T(x, y, z) = 8x^2 + xyz - 16z + 600$ CELSIUS. WHICH IS THE HOTTEST POINT ON THE SATELLITE?

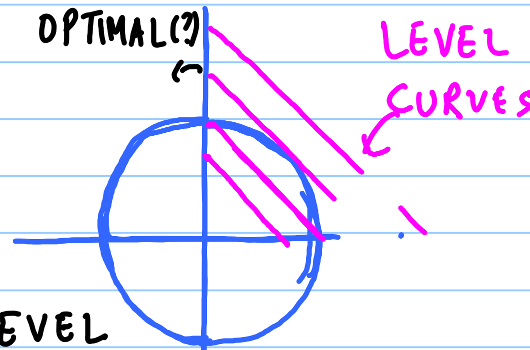
THESE PROBLEMS ARE INSTANCES OF CONSTRAINED MAX/MIN PROBLEMS.

FIND MAX $f(x, y, z)$, SUBJECT TO $g(x, y, z) = 0$.

EXAMPLE:

FIND MAX $X+Y$ SUBJECT TO $X^2+Y^2=1$.

NOTE THAT AT THE MAXIMAL VALUE OF $X+Y$, THE 'CONSTRAINT' AND THE LEVEL CURVE '$X+Y$' ARE TANGENTIAL.



LET $(x_0, y_0) \in \mathbb{R}^2$, AND

$f, g : B_r(x_0, y_0) \to \mathbb{R}$ SATISFYING

- f_x, f_y, g_x, g_y ARE CONTINUOUS AT (x_0, y_0).
- $g(x_0, y_0) = 0$, $\nabla g(x_0, y_0) \neq (0,0)$
- f HAS A LOCAL EXTREMUM AT (x_0, y_0) WHEN RESTRICTED TO THE LEVEL CURVE

$$\mathcal{C} = \{(x,y) \mid g(x,y) = 0\}.$$

THEN $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ FOR SOME $\lambda \in \mathbb{R}$.

FOR THE PROBLEM

MAX/MIN $f(x,y)$ SUBJECT TO

$g(x,y)=0$, CONSIDER

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$

$$S_1 = \{(x_0,y_0) \mid \nabla F(x_0,y_0) = (0,0)\}$$

$$S_2 = \{(x_0,y_0) \mid g(x_0,y_0)=0 \text{ AND } f_x(x_0,y_0) \text{ OR } f_y(x_0,y_0) \text{ DNE OR } \nabla g(x_0,y_0) = (0,0)\}.$$

CHECK ALL POINTS IN $S_1 \cup S_2$ FOR MAX/MIN VALUES.

EXAMPLES

MIN : $x^2+y^2+z^2$ SUBJECT TO $2x+y-3z=5$.

$f(x,y,z) = x^2+y^2+z^2$; $g(x,y,z) = 2x+y-3z-5$

$F = f - \lambda g \quad (F \equiv F(x,y,z;\lambda))$

$\frac{\partial F}{\partial x}=0 \Rightarrow 2x-2\lambda=0$; $\frac{\partial F}{\partial y}=0 \Rightarrow 2y-\lambda=0$

$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+3\lambda=0$ $\frac{\partial F}{\partial \lambda}=0 \Rightarrow 2x+y-3z=5$.

$x=\lambda,\ y=\frac{\lambda}{2},\ z=\frac{-3\lambda}{2} \Rightarrow \frac{5\lambda}{2}+\frac{9\lambda}{2}=5 \Rightarrow \lambda=\frac{5}{7}$.

MAX. $T(x,y,z) = 8x^2 + xyz - 16z + 600$ SUBJECT

TO $4x^2 + y^2 + 4z^2 = 16$.

$f = 8x^2 + xyz - 16z + 600$, $g = 4x^2 + y^2 + 4z^2 - 16$

$F = f - \lambda g \quad (F(x,y,z,\lambda))$

$\nabla f = (16x + yz,\ xz,\ xy - 16)$

$\nabla g = (8x,\ 2y,\ 8z) = (0,0,0)$ IFF $x = y = z = 0$

FIND $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}, \dfrac{\partial F}{\partial \lambda} = 0$, AND SOLVE.

MULTIPLE CONSTRAINTS

MAX $f(x,y,z)$ SUBJECT TO

$g_1(x,y,z)=0, \ g_2(x,y,z)=0, \ \cdots, \ g_p(x,y,z)=0.$

IN THIS CASE WE CONSIDER

$$F(x,y,z;\lambda_1,\lambda_2,\cdots,\lambda_p) := f - \lambda_1 g_1 - \lambda_2 g_2 - \cdots - \lambda_p g_p$$

THEN WE PROCEED AS IN THE SINGLE CONSTRAINT CASE.

MIN. $x^2+y^2+z^2$ SUBJECT TO $x+y+z=1$ AND $3x+2y+z=6$.

$f = x^2+y^2+z^2, \quad g_1 = x+y+z-1, \quad g_2 = 3x+2y+z-6$

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \qquad \nabla g_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \nabla g_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$F = f - \lambda g_1 - \mu g_2.$

SET $\nabla F = (0,0,0,0)$ AND SOLVE FOR (x,y,z,λ,μ). (EXERCISE)

MAX/MIN ON BOUNDED REGIONS

EXAMPLE: FIND ALL TRIANGLES IN WHICH THE PRODUCT OF THE SINES OF THE ANGLES IS MAXIMUM.

IF THE ANGLES ARE $x, y, \pi-(x+y)$, WE WISH TO MAXIMIZE

$$f(x,y) = \sin x \sin y \sin(x+y) \quad \text{IN THE REGION}$$

$$R = \{(x,y) \mid 0 < x, y < \pi,\ 0 < x+y < \pi\}.$$

CONSIDER $\bar{R} = \{(x,y) \mid 0 \leq x, y \leq \pi,\ 0 \leq x+y \leq \pi\}$

$\bar{R}$ IS CLOSED (WHY? PROVE!), AND f IS DIFFERENTIABLE IN R AND CONTINUOUS ON $\bar{R}$ (WHY? CHECK)

$$\frac{\partial f}{\partial x} = \sin y \left[\cos x \sin(x+y) + \sin x \cos(x+y)\right]$$

$$= \sin y \sin(2x+y) = 0.$$

SIMILARLY $\frac{\partial f}{\partial y} = 0 \Rightarrow \sin x \sin(x+2y) = 0.$

CHECK THAT $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ IS THE <u>ONLY</u> POINT OF MAXIMUM IN $\bar{R}$. (EXERCISE)