MA 108-ODE- D3

Lecture 10

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Second order non-homogeneous ODEs

- Method of Undetermined Coefficients
- Method of Variation of Parameters

Non-homogeneous Second Order Linear ODE's

Theorem

Let f be any solution of

$$y'' + p(t)y' + q(t)y = r(t), \tag{1}$$

where p, q and r are continuous on an interval I. Let y_1, y_2 be a basis of the solution space of the corresponding homogeneous DE. Then the set of solutions of equation (1) on I is

$$\{c_1y_1(t) + c_2y_2(t) + f(t) \mid c_1, c_2 \in \mathbb{R}\}.$$

Proof: Let ϕ be any solution of

$$L(y) = y'' + p(t)y' + q(t)y = r(t)$$

on I. Then,

$$L(\phi(t) - f(t)) = L(\phi(t)) - L(f(t)) = r(t) - r(t) = 0.$$

Non-homogeneous Second Order Linear ODE's

Hence, $\phi(t) - f(t)$ is a solution of the homogeneous DE. Thus,

$$\phi(t) - f(t) = c_1 y_1(t) + c_2 y_2(t),$$

for $c_1, c_2 \in \mathbb{R}$. Hence,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t) + f(t),$$

for $t \in I$.
Summary: In order to find the general solution of a non-homogeneous DE, we need to

- get one particular solution of the non-homogeneous DE
- get the general solution of the corresponding homogeneous DE.

How to find particular solution

No standard method!

Often useful methods:

- Method of undetermined coefficients,
- Method of variation of parameters.

Method of Undetermined Coefficients

Suppose the non-homogeneous ODE has constant coefficients. In this case, we know how to write down the general solution of the corresponding homogeneous ODE. So we need to find one solution of the non-homogeneous DE. One way to do this is called the method of undetermined coefficients. Thus, we have:

$$y'' + py' + qy = r(t),$$

with $p, q \in \mathbb{R}$, and r is a continuous function on I. The method of undetermined coefficients does not work for any $r(t)$, but only when we know more about $r(t)$. We'll use this method only if $r(t)$ involves $e^{at}, \sin at, \cos at$ or polynomials in t.

Method of Undetermined Coefficients

Example: Find a particular solution of the DE:

$$y'' - 3y' - 4y = 3e^{2t}.$$

We'll search for a solution of the form ae^{2t}, where a is a constant. So put $y = ae^{2t}$. We get:

$$(ae^{2t})'' - 3(ae^{2t})' - 4ae^{2t} = 3e^{2t}.$$

Thus,

$$4ae^{2t} - 6ae^{2t} - 4ae^{2t} = 3e^{2t}.$$

Thus,

$$a = -\frac{1}{2}.$$

Hence $-\frac{1}{2}e^{2t}$ is a particular solution of the DE.
How do you get the general solution? Analyse roots of $m^2 - 3m - 4 = 0$.
So general solution is

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t},$$

where $c_1, c_2 \in \mathbb{R}$.

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

Make a guess as to functions of which form we'll search for as a solution. $a\sin t$? No. $a\sin t + b\cos t$? Yes. So set

$$y(t) = a\sin t + b\cos t.$$

Thus,

$$y' = a\cos t - b\sin t;\ \ y'' = -a\sin t - b\cos t.$$

Substituting, we get:

$$(-5a + 3b - 2)\sin t + (-3a - 5b)\cos t = 0.$$

Thus,

$$-5a + 3b = 2;\ \ 3a + 5b = 0$$

(Why?). Thus, $a = -\frac{5}{17},\ \ b = \frac{3}{17}$, and a particular solution is

$$y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1.$$

Set

$$y(t) = at^2 + bt + c.$$

Substituting, we get:

$$-4at^2 + (-6a - 4b)t + (2a - 3b - 4c) = 4t^2 - 1.$$

Thus,

$$-4a = 4,\ -6a - 4b = 0,\ 2a - 3b - 4c = -1.$$

Thus,

$$a = -1,\ b = \frac{3}{2},\ c = -\frac{11}{8}.$$

Thus, a particular solution is

$$y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

We should search for a solution of the form

$$y(t) = ae^t \cos 2t + be^t \sin 2t.$$

Then,

$$y'(t) = (a + 2b)e^t \cos 2t + (-2a + b)e^t \sin 2t,$$

and

$$y'' = (-3a + 4b)e^t \cos 2t + (-4a - 3b)e^t \sin 2t.$$

Substituting, we get:

$$-10a - 2b = -8, \; 2a - 10b = 0.$$

Thus, a particular solution is

$$y(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Since $r(t) = 3\cos 2t$, you would look for solutions of the form

$$y(t) = a\cos 2t + b\sin 2t.$$

Thus,

$$y'(t) = -2a\sin 2t + 2b\cos 2t,$$

$$y''(t) = -4a\cos 2t - 4b\sin 2t.$$

Substituting in the given DE, we get:

$$(-4a\cos 2t - 4b\sin 2t) + 4(a\cos 2t + b\sin 2t) = 3\cos 2t.$$

But the lhs is 0! So can't solve for a and b.

Method of Undetermined Coefficients

Why this ...? Note that $\sin 2t$ and $\cos 2t$ are also solutions of the associated homogeneous ODE: $y'' + 4y = 0$. So lesson learnt? When we search for solutions of a particular form, we need to make sure that it's not a solution of the associated homogeneous equation.
We now modify the proposed solution as:

$$y(t) = at\cos 2t + bt\sin 2t.$$

Then,

$$y'(t) = (b - 2at)\sin 2t + (a + 2bt)\cos 2t,$$

$$y''(t) = -4at\cos 2t - 4bt\sin 2t - 4a\sin 2t + 4b\cos 2t.$$

Substituting, we get:

$$-4a\sin 2t + 4b\cos 2t = 3\cos 2t.$$

Thus, $a = 0,\ b = \frac{3}{4}$, and a particular solution is $y(t) = \frac{3}{4}t\sin 2t$.

Method of Undetermined Coefficients

If the obvious candidate for a solution, say $y(t) = f(t)$, as well as this one multiplied by t, $y(t) = tf(t)$, turn out to be solutions of the associated homogeneous ODE, then what to do? Modify the proposed solution by multiplying it with t^2; i.e., set

$$y(t) = t^2 f(t).$$

Can this too be a solution of the homogeneous ODE? No, since the solution space is two dimensional.

Method of Undetermined Coefficients

Consider the DE

$$y'' + py' + qy = r(t).$$

where p and q are real numbers. If

$$r(t) = r_1(t) + r_2(t) + \ldots + r_n(t),$$

where $r_i(t)$ are e^{at} or $\sin at$ or $\cos at$ or polynomials in t, consider the n subproblems

$$y'' + py' + qy = r_i(t), 1 \leq i \leq n.$$

If $y_i(t)$ is a particular solution of this problem, then,

$$y(t) = y_1(t) + y_2(t) + \ldots + y_n(t)$$

is a particular solution of

$$y'' + py' + qy = r(t).$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t + 4t^2 - 1 - 8e^t \cos 2t.$$

Here,

$$r(t) = r_1(t) + r_2(t) + r_3(t) + r_4(t).$$

We need to solve

$$y'' - 3y' - 4y = r_i(t),$$

get a particular solution $y_i(t)$, and then

$$y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$$

is a particular solution of the given problem. Thus, a particular solution is

$$y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - t^2 + \frac{3}{2}t - \frac{11}{8} + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' + y = x^3 \sin x.$$

What's your candidate? Presence of $\sin x$ indicates both $\sin x$ and $\cos x$ in the answer. Presence of x^3 indicates a generic cubic polynomial. Thus,

$$(a_1x^3 + b_1x^2 + c_1x + d_1)\cos x + (a_2x^3 + b_2x^2 + c_2x + d_2)\sin x?$$

This wouldn't do since $\sin x$ and $\cos x$ are already solutions of the homogeneous part. So work with

$$x(a_1x^3 + b_1x^2 + c_1x + d_1)\cos x + x(a_2x^3 + b_2x^2 + c_2x + d_2)\sin x.$$

Method of Undetermined Coefficients

Consider the DE
$$y'' + py' + qy = r(t).$$
where p and q are real numbers.
Step I: Find the general solution of the homogeneous equation
$$y'' + py' + qy = 0.$$

Step II: If $r(t) = r_1(t) + r_2(t) + \ldots + r_n(t)$, where $r_i(t)$ are e^{at} or $\sin at$ or $\cos at$ or polynomials in t, consider
$$y'' + py' + qy = r_i(t), 1 \leq i \leq n.$$

Step III: Find a particular solution $y_i(t)$ for each of the above n subproblems. Then
$$y(t) = y_1(t) + y_2(t) + \ldots + y_n(t)$$
is a particular solution of
$$y'' + py' + qy = r(t).$$

Step IV: In case, the trial function is same as the solution of the associated homogeneous equation, then we modify that trial function by multiplying it with t or atmost t^2.

Method of Undetermined Coefficients

Table of particular solutions

$r(t)$	Particular solutions $y(t)$
$P_n(t) = \sum_{i=0}^{n} a_i t^i$	$t^s \left(\sum_{i=0}^{n} A_i t^i\right)$
$P_n(t)e^{\alpha t}$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)$
$P_n(t)e^{\alpha t}\sin\beta t$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)\cos\beta t \quad +$ $e^{\alpha t} t^s \left(\sum_{i=0}^{n} B_i t^i\right)\sin\beta t$
$P_n(t)e^{\alpha t}\cos\beta t$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)\cos\beta t \quad +$ $e^{\alpha t} t^s \left(\sum_{i=0}^{n} B_i t^i\right)\sin\beta t$

Here $s = 0, 1$ or 2 has to be chosen appropriately.