PH111: Tutorial Sheet 3
Solutions

This tutorial sheet contains problems related to work-energy theorem, conservative force, and potential energy.

1. Mass m rotates on a frictionless table, held to circular path by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of the circle changes from R_0 to R_1. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.
Soln: We know that the radial component of acceleration is given by

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

Here the mass is being pulled very slowly so $\ddot{r} \approx 0$, therefore, the only acceleration experienced by the mass is the centripetal acceleration

$$a_r \approx -r\dot{\theta}^2 = -r\omega^2.$$

If the particle is pulled slowly, the force applied is the tension in the string which should be the centripetal force

$$F = T = -m\omega^2 r,$$

so that the work done will be

$$W = \int_{R_0}^{R_1} F dr = -m \int_{R_0}^{R_1} \omega^2 r dr.$$

r dependence of ω can be calculated using the fact that the angular momentum of the particle about the center of the circle will be conserved because F being radial, does not impart any torque to the particle, with respect to the center. Assuming that the initial angular velocity of the mass was ω_0 corresponding to the radius R_0, conservation of angular momentum ($I\omega = mr^2\omega$), implies

$$mR_0^2\omega_0 = mr^2\omega$$
$$\implies \omega(r) = \frac{R_0^2\omega_0}{r^2},$$

using this we have

$$W = -\int_{R_0}^{R_1} \omega^2 r dr = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{rdr}{r^4} = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{dr}{r^3} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right)$$

Increase in kinetic energy of the particle will be

$$\Delta K = \frac{1}{2} I_1\omega_1^2 - \frac{1}{2} I_0\omega_0^2 = \frac{1}{2}\left\{mR_1^2\left(\frac{R_0^4\omega_0^2}{R_1^4}\right) - mR_0^2\omega_0^2\right\} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right),$$

which is same as the expression for W above.

2. A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed towards the origin with the magnitude B, and and inverse law repulsive force of magnitude A/x^2.

(a) Find the potential energy function $V(x)$
Soln: The force is given by

$$\mathbf{F} = \left(-B + \frac{A}{x^2}\right)\hat{\mathbf{i}}, \text{ for } x \geq 0.$$

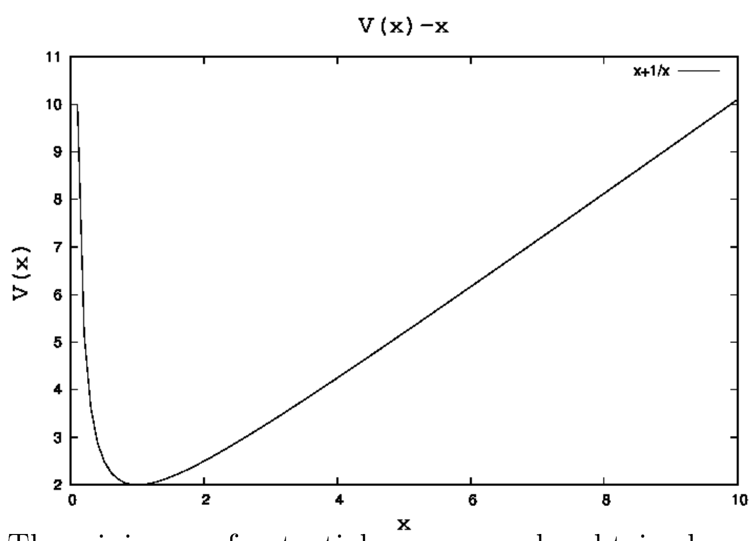
If $U(x)$ is the potential energy, then for $x \geq 0$

$$-\frac{dV}{dx} = -B + \frac{A}{x^2}$$
$$\implies V(x) = Bx + \frac{A}{x} + C,$$

where C is a constant, which we can set to zero so

$$V(x) = Bx + \frac{A}{x}$$

(b) Plot the potential energy as a function of x, and the total energy of the system, assuming that the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.
Soln: The plot of the potential energy for $A = B = 1$ is



The minimum of potential energy can be obtained

$$\frac{dV}{dx} = B - \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

for which $V = B\sqrt{\frac{A}{B}} + A\sqrt{\frac{B}{A}} = 2\sqrt{AB}$. Because total energy is conserved, therefore, $E = K + V =$ constant. Thus, when we have maximum kinetic energy, we will have minimum potential energy, implying that $E = K_{max} + V_{min} = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$. Therefore, total energy of the particle can be represented by a horizontal line in the plot, corresponding to $V(x) = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$.

(c) What is the point of equilibrium, i.e., the point where net force acting on the particle is zero.
Soln: Point of equilibrium is obtained by

$$F = -B + \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

which is the same point where the potential energy is minimum.

3. A particle of mass M is held fixed at the origin. The gravitational potential energy of another particle of m, in the field of the first mass, is given by

$$V(\mathbf{r}) = -\frac{GMm}{r},$$

where G is the gravitational constant, and r is the distance of mass m from the origin.

(a) What is the force acting on the particle of mass m?
Soln: Here

$$V(\mathbf{r}) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$
$$\implies \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$$
$$= GMm\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{k}}\right)$$
$$= -GMm\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{GMm\mathbf{r}}{r^3} = -\frac{GMm\hat{\mathbf{r}}}{r^2}$$

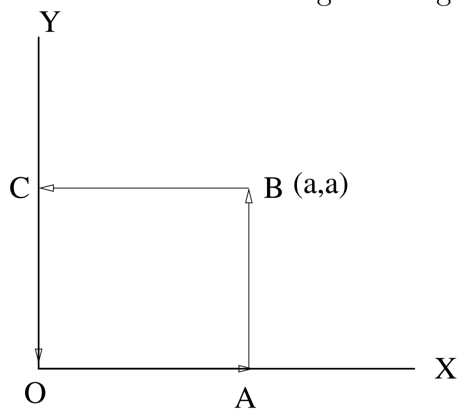
(b) Calculate the curl of this force.
Soln: Because for this force a potential energy function exists, its curl must vanish. We calculate it as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$
$$= -3\left(\frac{yz - yz}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{i}} + \cdots = 0$$

4. Consider a 2D force field $\mathbf{F} = A(y^2\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}})$. Calculate the work done by this force in going around a closed path which is a square made up of sides of length a, lying in

the xy-plane, with two of its vertices located at the origin, and point (a, a). Find the answer by doing the line integral, as well as by using the Stokes' theorem. The path is traversed in a counter clockwise manner.

Soln: Let us first calculate the line integral along the given path



The work done will be

$$W = \oint \mathbf{F}\cdot d\mathbf{r} = \int_{OA} \mathbf{F}\cdot d\mathbf{r} + \int_{AB} \mathbf{F}\cdot d\mathbf{r} + \int_{BC} \mathbf{F}\cdot d\mathbf{r} + \int_{CO} \mathbf{F}\cdot d\mathbf{r}$$

$$= A(0)\int_0^a dx + 2A(a^2)\int_0^a dy + Aa^2\int_a^0 dx + 2(0)\int_a^0 dy = Aa^3$$

To verify Stokes theorem we need to compute

$$\int (\nabla\times\mathbf{F})\cdot d\mathbf{S}.$$

Here

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ay^2 & 2Ax^2 & 0 \end{vmatrix} = (4Ax - 2Ay)\hat{\mathbf{k}}$$

and

$$d\mathbf{S} = dxdy\hat{\mathbf{k}},$$

so that

$$\int (\nabla\times\mathbf{F})\cdot d\mathbf{S} = 4A\int_0^a xdx\int_0^a dy - 2A\int_0^a dx\int_0^a ydy$$

$$= 4A(\frac{a^2}{2})a - 2Aa(\frac{a^2}{2}) = Aa^3.$$

Thus we get the same value of work done by computing the line integral, and by using the Stokes theorem.

5. Find the forces for the following potential energies

   (a) $V(x, y, z) = Ax^2 + By^2 + Cz^2$

   Soln:

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

$$= -2Ax\hat{\mathbf{i}} - 2By\hat{\mathbf{j}} - 2Cz\hat{\mathbf{k}}$$

(b) $V(x,y,z) = A\ln(x^2+y^2+z^2)$
Soln:

$$\mathbf{F} = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}} = \frac{2A}{(x^2+y^2+z^2)}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

(c) $V(r,\theta) = A\cos\theta/r^2$ (r and θ are plane polar coordinates)

Above, A, B, and, C are constants.
Soln:

$$\begin{aligned} V &= \frac{A\cos\theta}{r^2} = \frac{Ax}{r^3} = \frac{Ax}{(x^2+y^2)^{3/2}} \\ \Longrightarrow \mathbf{F} &= -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} = -A\left(\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}\right)\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}} \\ &= \frac{A(2x^2-y^2)}{(x^2+y^2)^{5/2}}\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}} \end{aligned}$$

6. Determine whether each of the following forces is conservative. Find the potential energy function, if it exists. A, α, β are constants.

(a) $\mathbf{F} = A(3\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}})$
Soln: First we compute the curl of $\mathbf{F}$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3A & Az & Ay \end{vmatrix} = 0.$$

Therefore, the force is conservative and it is possible to obtain a potential energy function for it using

$$\begin{aligned} -\frac{\partial V}{\partial x} &= 3A \\ -\frac{\partial V}{\partial y} &= Az \\ -\frac{\partial V}{\partial z} &= Ay. \end{aligned}$$

On integrating the first equation above we have

$$V(x,y,z) = -3Ax + f(y,z),$$

which on substitution in the second equation yields

$$\begin{aligned} -\frac{\partial f}{\partial y} &= Az \\ \Longrightarrow f(y,z) &= -Ayz + C \\ \Longrightarrow V(x,y,z) &= -3Ax - Ayz + C, \end{aligned}$$

where C is a constant. Note that this expression for V satisfies the third equation above, implying that the solution is complete.

(b) $\mathbf{F} = Axyz(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
Soln:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axyz & Axyz & Axyz \end{vmatrix} = A(xz - xy)\hat{\mathbf{i}} + A(xy - yz)\hat{\mathbf{j}} + A(yz - xz)\hat{\mathbf{i}} \neq 0,$$

therefore, a potential energy function does not exist for this force.

(c) $F_x = A \sin(\alpha y) \cos(\beta z)$, $F_y = -Ax\alpha \cos(\alpha y) \cos(\beta z)$, $F_z = Ax \sin(\alpha y) \sin(\beta z)$
Soln:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A \sin(\alpha y) \cos(\beta z) & -Ax\alpha \cos(\alpha y) \cos(\beta z) & Ax \sin(\alpha y) \sin(\beta z) \end{vmatrix} \\ &= A(x\alpha \cos(\alpha y) \sin(\beta z) - x\alpha\beta \cos(\alpha y) \sin(\beta z))\hat{\mathbf{i}} \\ &+ A(-x\beta \sin(\alpha y) \sin(\beta z) - x\alpha \cos(\alpha y) \sin(\beta z))\hat{\mathbf{j}} \\ &+ A(0 - \alpha \cos(\alpha y) \cos(\beta z)\hat{\mathbf{k}} \\ &\neq 0, \end{aligned}$$

hence, for this force also, no potential energy function exists.