LINE INTEGRALS

LET $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ BE CONTINUOUS AND C A CURVE IN $\mathbb{R}^3$ WITH PARAMETRIZATION $\mathbf{r} : [a,b] \to \mathbb{R}^3$, THE LINE INTEGRAL OF f OVER C IS DEFINED AS

$$\int_C f\, ds := \int_a^b (f \circ r)(s)\, ds. \quad \left(\begin{array}{l} s \text{ IS ARC LENGTH} \\ \text{PARAMETRIZATION} \end{array} \right)$$

(CHECK THAT THIS IS WELL DEFINED)

IF C IS A CLOSED CURVE $(\mathbf{r}(b) = \mathbf{r}(a))$ THEN THE INTEGRAL IS DENOTED $\oint_C f\, ds$

🚩 LET $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ BE CONTINUOUS AND C A SIMPLE REGULAR CURVE WITH PARAMETRIZATION $\mathbf{r}(t)$ $(t \in [c,d])$. THEN

$$\int_C f\, ds = \int_c^d f(x(t), y(t), z(t))\, \|\mathbf{r}'(t)\|\, dt$$

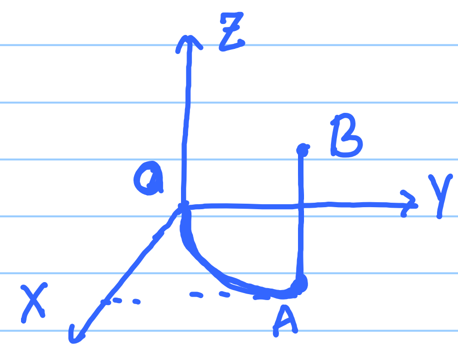
🚩 WORK DONE AGAINST A FORCE FIELD F IN MOVING ALONG C: $W := \int_C F \cdot d\mathbf{r}$ (THIS IS YET NOT DEFINED!)

EXAMPLE

COMPUTE $\int_C f\, ds$ WHERE $f(x,y,z) = x + \sqrt{y} - z^2$

AND C IS GIVEN BY $y = x^2$ FROM $O(0,0,0)$ TO $A(1,1,0)$

AND THEN THE LINE SEGMENT FROM $(1,1,0)$ TO $B(1,1,1)$.



RECALL: IF $\mathbf{r}: [a,b] \to \mathbb{R}^3$

IS A PARAMETRIZATION FOR C,

THEN,

$$\int_C f\, ds = \int_a^b f(x(t), y(t), z(t)) \cdot \|\mathbf{r}'(t)\|\, dt, \quad \text{WHERE}$$

$$\mathbf{r}(t) = (x(t), y(t), z(t)).$$

FIRST PIECE : $\mathbf{r}_1 : [0,1] \to \mathbb{R}^3$, $t \to (t, t^2, 0)$ $\quad$ $\mathbf{r}_2 : [0,1] \to \mathbb{R}^3$, $t \to (0,0,t)$

$$\int_{C_1} f\, ds = \int_0^1 f(t,t^2,0)\sqrt{1+4t^2}\, dt = \int_0^1 2t\sqrt{1+4t^2}\, dt$$

$$\int_{C_2} f\, ds = \int_0^1 f(1,1\ t)\, dt = \int_0^1 (2-t^2)\, dt$$

PROPERTIES OF $\int_C$

SUPPOSE $f, g : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ ARE CONTINUOUS AND C IS A SIMPLE REGULAR CURVE IN D.

THEN

- $$\int_C (f+g)\, ds = \int_C f\, ds + \int_C g\, ds$$
- $$\int_C f = \int_C f\, ds \quad \text{FOR ANY}$$
- SUPPOSE $C = \bigcup_{i=1}^{n} C_i$ WHERE EACH C_i $(1 \leq i \leq n)$ IS REGULAR THEN
$$\int_C f\, ds := \sum_{i=1}^{n} \int_{C_i} f\, ds.$$
- LET C BE A SMOOTH CURVE WITH PARAMETRIZATION $\mathbf{r}(t)$ $(t \in [a,b])$. CONSIDER
$$\bar{\mathbf{r}}(t) = \mathbf{r}(b-(t-a)) \quad t \in [a,b].$$
$\bar{\mathbf{r}}$ IS ALSO SMOOTH, AND IT TRAVERSES C BACKWARDS. (DENOTED $-C$). THEN $$\int_{-C} f\, ds = -\int_C f\, ds$$
- THE LINE INTEGRAL DEPENDS ON STARTING/ENDING POINT(S).

SUPPOSE C IS A SMOOTH CURVE IN $\mathbb{R}^3$ WITH PARAMETRIZATION $\mathbf{r}(t) = (x(t), y(t), z(t))$ $(t \in [a,b])$. LET $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ BE A CONTINUOUS SCALAR FIELD. DEFINE

DEF:
$$\begin{cases} \displaystyle\int_C f\, dx := \int_a^b f(x(t), y(t), z(t)) \left(\frac{dx}{dt}\right) dt \\ \displaystyle\int_C f\, dy := \int_a^b f(x,y,z)(t) \cdot \left(\frac{dy}{dt}\right) dt \\ \displaystyle\int_C f\, dz := \int_a^b f(x,y,z)(t) \left(\frac{dz}{dt}\right) dt \end{cases}$$

IF $F : D \to \mathbb{R}^3$ IS A CONTINUOUS VECTOR FIELD AND $F = (F_1, F_2, F_3)$, DEFINE

$$(\text{WORK}) = \int_C F \cdot d\mathbf{r} = \int_C (F_1, F_2, F_3) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$:= \int_C F_1\, dx + \int_C F_2\, dy + \int_C F_3\, dz$$

THE LINE INTEGRAL OF THE VECTOR FIELD F OVER C.

FUNDAMENTAL THEOREM

LET $D \subseteq \mathbb{R}^3$ BE OPEN, $\phi : D \to \mathbb{R}$ A CONTINUOUSLY DIFFERENTIABLE SCALAR FIELD. LET C BE A SMOOTH CURVE IN C WITH PARAMETRIZATION $\mathbf{r} : [a,b] \to D$. THEN

$$\int_C (\nabla \phi) \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$

LET F BE A CONTINUOUSLY DIFFERENTIABLE VECTOR FIELD, WHICH IS CONSERVATIVE, i.e., $F = \nabla \phi$ FOR SOME CONTINUOUSLY DIFFERENTIABLE SCALAR FIELD ϕ, ON D. THEN FOR ANY SMOOTH CURVE C AS ABOVE,

$$\int_C F \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

APPLICATIONS

- IF F IS A CONSERVATIVE VECTOR FIELD (A FORCE FIELD), THEN THE WORK DONE AGAINST THE FORCE F IN MOVING FROM POINT A TO POINT B DOES NOT DEPEND ON THE PATH TAKEN IN MOVING FROM A TO B.

- SUPPOSE F BE A CONSERVATIVE FORCE FIELD ON A DOMAIN D WITH

$$F(x,y,z) = \nabla\phi(x,y,z)$$

FOR SOME SCALAR FIELD ϕ.

$V(x,y,z) = -\phi(x,y,z)$ IS THE POTENTIAL ENERGY OF THE FIELD AT (x,y,z).

WORK DONE IN MOVING ALONG A CURVE C IN MOVING FROM A TO B IS

$$W = \int_C F \cdot dr = \phi(B) - \phi(A)$$

WHERE A = INITIAL POINT OF C

B = FINAL " " C.

CIRCULATION, FLUX

CIRCULATION OF A FLUID ALONG A CURVE:

SUPPOSE A FLUID FLOWS THROUGH A REGION D. AND LET $V(x,y,z)$ BE THE VELOCITY VECTOR FIELD, AND C A CURVE IN D. IF T DENOTES THE UNIT TANGENT VECTOR ALONG C, THEN

TOTAL FLOW ALONG C $:= \int_C (V \cdot T)\, ds$

$\oint_C (V \cdot T)\, ds$ IS THE CIRCULATION ALONG C. (IF C IS CLOSED)

SUPPOSE $\rho(x,y)$ = DENSITY OF THE FLUID, THEN

$F(x,y) := \rho(x,y)\, V(x,y)$ IS THE RATE OF CHANGE OF MASS PER UNIT TIME PER UNIT AREA.

IF n IS THE UNIT NORMAL TO C, THEN

FLUX ACROSS C $:= \int_C (F \cdot n)\, ds$

CONSERVATIVE VECTOR FIELDS

SUPPOSE $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ IS A SCALAR FIELD.

THE FOLLOWING ARE EQUIVALENT:

🚩 FOR ANY POINTS $P, Q \in D$, THE INTEGRAL

$$\int_{C(P,Q)} f \, ds$$ DOES NOT DEPEND ON C.

($C(P,Q)$ IS A CURVE STARTING AT P TERMINATING AT Q)

🚩 $$\oint_C f \, ds = 0$$ FOR EVERY CLOSED C.

[WE KNOW THAT IF F IS A CONSERVATIVE VECTOR FIELD THEN $\oint_C F \cdot dr = 0$.]

IS THE CONVERSE TRUE ?

SUPPOSE $D \subseteq \mathbb{R}^3$. WE SAY THAT D IS PATH CONNECTED IF FOR ANY TWO POINTS $A, B \in D$ THERE IS A PIECEWISE SMOOTH CURVE $\mathbf{r} : [0,1] \to D$ s.t. $\mathbf{r}(0) = A$, $\mathbf{r}(1) = B$

WE SOMETIMES DROP THE WORD 'PATH' AND SIMPLY WRITE 'CONNECTED'.

LET $F : D \to \mathbb{R}^3$ ($D \subseteq \mathbb{R}^3$) BE A CONTINUOUS VECTOR FIELD, WHERE D IS AN OPEN CONNECTED SET. IF FOR ANY CURVE C, THE LINE INTEGRAL $\int_C F \cdot dr$ DEPENDS ONLY ON THE INITIAL AND TERMINAL POINTS OF C, THEN THERE EXISTS A SCALAR FIELD $\phi : D \to \mathbb{R}$ s.t

$$F(x,y,z) = \nabla\phi(x,y,z) \quad \forall\ (x,y,z) \in D$$

SUMMARIZING:

LET $D \subseteq \mathbb{R}^3$ BE OPEN, CONNECTED, AND $F: D \to \mathbb{R}^3$ A CONTINUOUS VECTOR FIELD. THE FOLLOWING ARE EQUIVALENT:

- THERE EXISTS $\phi: D \to \mathbb{R}$ S.T.

$$F = \nabla \phi$$

- $P, Q \in D \Rightarrow \int_{C(P,Q)} F \cdot dr$ IS INDEPENDENT OF THE PATH JOINING P AND Q.

- $\oint_C F \cdot dr = 0 \quad \forall$ CLOSED CURVES C.

(NECESSARY CONDITION FOR CONSERVATIVENESS)

LET $D \subseteq \mathbb{R}^3$ BE OPEN, CONNECTED AND $F: D \to \mathbb{R}^3$ IS CONTINUOUSLY DIFFERENTIABLE. SUPPOSE

$F = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$, AND F IS ALSO CONSERVATIVE. THEN

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad \begin{matrix} & (f_1\ f_2\ f_3) \\ (x_1, x_2, x_3) \equiv & (x, y, z) \end{matrix}$$

FOR $1 \leq i, j \leq 3$. IN PARTICULAR $\text{curl}(F) = \vec{0}$.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}$$

EXAMPLE

$$F(x,y,z) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2} \quad \forall\ (x,y) \neq (0,0)$$

$$\mathrm{Dom}(F) = \{(x,y,z) \mid (x,y) \neq (0,0)\}$$

$$F_1 = \frac{-y}{x^2+y^2}, \quad F_2 = \frac{x}{x^2+y^2}, \quad F_3 = 0$$

$$\frac{\partial F_1}{\partial y} = \frac{-(x^2+y^2) - (-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

(the two results are marked $\parallel$, i.e. equal)

HENCE $\mathrm{curl}(F) = \underline{0}$.

CONSIDER $C \equiv x^2+y^2=1$.

$F_1(\cos\theta, \sin\theta) = -\sin\theta$

$F_2(\quad) = \cos\theta$

$x = \cos\theta, \quad y = \sin\theta, \quad 0 \leq \theta < 2\pi.$

$$\oint_C F.dr = \int_0^{2\pi} F(\cos\theta, \sin\theta)\cdot(-\sin\theta, \cos\theta)\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta\,d\theta + \int_0^{2\pi} \cos^2\theta\,d\theta = 2\pi \neq 0$$

CONCLUSION: $\mathrm{curl}(F) = 0$ IS NOT SUFFICIENT TO ENSURE THAT F IS CONSERVATIVE.

(BUT IN SOME CIRCUMSTANCES, IT WILL BE...)

EXAMPLE

$$F(x,y,z) = (y^2z^3\cos x - 4x^3z)\vec{i} + (2z^3y\sin x)\vec{j} + (3y^2z^2\sin x - x^4)\vec{k}.$$

IS F CONSERVATIVE (WITH SOME POTENTIAL ϕ)?

WANT ϕ : $\nabla\phi = F$, SO $\frac{\partial\phi}{\partial x} = F_1$, $\frac{\partial\phi}{\partial y} = F_2$, $\frac{\partial\phi}{\partial z} = F_3$.

$$\phi \overset{?}{=} \int F_1\,dx + \alpha(y,z) \quad (\text{ANTIDERIVATIVE FOR } F_1 \text{ w.r.t } x)$$

$$= (\sin x\, y^2z^3 - x^4z) + \alpha(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2z^3y\sin x \Rightarrow 2yz^3\sin x + \alpha'(y,z) = 2z^3y\sin x$$

$$\Rightarrow \alpha'(y,z) = 0 \Rightarrow \alpha(y,z) \equiv \alpha(z)$$

$$\frac{\partial\phi}{\partial z} = 3y^2z^2\sin x - x^4, \text{ SO}$$

$$3z^2y^2\sin x - x^4 + \alpha'(z) = 3y^2z^2\sin x - x^4$$

$$\Rightarrow \alpha(z) = C$$

SO A CANDIDATE POTENTIAL ϕ IS

$$\phi = (\sin x)y^2z^3 - x^4z$$

NOW WE CAN CHECK THAT

$$\nabla\phi = F$$

THIS MAY NOT ALWAYS BE FEASIBLE,
i.e. FINDING A POTENTIAL ϕ EXPLICITLY
MAY BE AN ARDUOUS TASK.

THERE IS ANOTHER WAY TO CALCULATE
LINE INTEGRALS — THIS IS THE CONTENT OF
GREEN'S THEOREM.