MA 108-ODE- D3

Lecture 12

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Linear ODE's of higher order

Linear DE's of Higher Order: Recall

- $$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0.$$

  can be written as

  $$Ly = 0$$

  in terms of a differential operator

  $$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I,$$

  where $D^k = \frac{d^k}{dx^k}$, $k \geq 0$, and I is the identity operator.

- We have seen the IVP for n-th order linear differential equation and the existence and uniqueness theorem regarding the solution of an IVP.

Linear DE's of Higher Order: Recall

Theorem (Dimension Theorem)

Let I be an interval in $\mathbb{R}$, $p_1, p_2, \ldots, p_n$ be continuous on I and

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I.$$

Then null space $N(L)$ of L is of dimension n.

Proof of Dimension Theorem

We prove the Dimension theorem using the Existence and Uniqueness theorem for IVPs. We need to show that dimension of $N(L) = n$.

Fix a point x_0 in the interior of the interval I. Define

$$T : N(L) \to \mathbb{R}^n$$

by

$$T(f) = \big(f(x_0), f^{(1)}(x_0), \ldots, f^{(n-1)}(x_0)\big).$$

Then T is a linear transformation (Check).
T is one-one (by the uniqueness of solution to an IVP).
T is onto (using the existence of solution to an IVP).
Hence by the rank-nullity theorem applying to T^{-1}, we get

$$\text{Dimension of } N(L) = \text{ Dimension of } \mathbb{R}^n = n.$$

Linearly independent & dependent functions: Recall

Definition

The functions $f_1, f_2, \cdots, f_n$ are said to be linearly independent on an interval I if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \;\; \forall x \in I \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

The functions are said to be linearly dependent on an interval I if they are not linearly independent on I.

Examples : 1. The functions $f_1(x) = x, f_2(x) = x^2, f_3(x) = \sin x$ are linearly independent on $(-\infty, \infty)$.

2. The functions $f_1(x) = \sin 2x, f_2(x) = \sin x \cos x, f_3(x) = x$ are linearly dependent on $(-\infty, \infty)$.

Linear DE's of Higher Order: Wronskian

The Wronskian of differentiable functions $y_1(x)$ and $y_2(x)$ is defined by

$$W(y_1, y_2; x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}.$$

The Wronskian of n functions $y_1(x), y_2(x), \cdots, y_n(x)$ such that each of them is $(n-1)$-times differentiable is defined by

$$W(y_1, \cdots, y_n; x) := \det \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}.$$

Linear dependency and Wronskian

Proposition

Suppose $f_1, \cdots, f_n$ are linearly dependent and $(n-1)$ times differentiable on an interval I. Then, $W(f_1, \cdots, f_n; x) = 0$ on I.
In other words, if the differentiable functions $f_1, \cdots, f_n$ have $W(f, \cdots, f_n; x_0) \neq 0$, for some $x_0 \in I$, then the functions $f_1, \cdots, f_n$ are linearly independent on I.

Note: The converse of the Proposition is not true.

Test for Linear Independence

Consider the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where p *and* q *are continuous on interval* I*. Then two solutions* y_1 *and* y_2 *of the DE on* I *are linearly dependent iff their Wronskian is* 0 *at some* $x_0 \in I$*.*

Theorem

Consider the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

where $p_1, p_2, \ldots, p_n$ *are continuous on interval* I*. Then* n *solutions* y_1*,* $y_2, \cdots, y_n$ *of the DE on* I *are linearly dependent iff their Wronskian determinant is* 0 *at some* $x_0 \in I$*.*

Proof for n th order - $\Longrightarrow$

Let $y_1, \ldots, y_n$ be linearly dependent in I. That is, $\exists$ non-trivial $k_1, \ldots, k_n$ such that

$$
\begin{aligned}
& k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \\
& k_1 y_1'(x) + \cdots + k_n y_n'(x) = 0 \\
& \vdots \\
& k_1 y_1^{(n-1)}(x) + \cdots + k_n y_n^{(n-1)}(x) = 0
\end{aligned}
$$

For $x_0 \in I$, in particular,

$$
\begin{pmatrix}
y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\
y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\
\vdots & \vdots & \vdots & \vdots \\
y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0)
\end{pmatrix}
\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

has a non-trivial solution $\Longrightarrow \det W(y_1, \cdots, y_n; x_0) = 0$.

$\Longleftarrow$

Conversely, let $\det W(y_1, \cdots, y_n; x_0) = 0$ for some $x_0 \in I$.
Consider the linear system of equations :

$$\begin{aligned}
&k_1 y_1(x_0) + \cdots + k_n y_n(x_0) = 0 \\
&k_1 y_1'(x_0) + \cdots + k_n y_n'(x_0) = 0 \\
&\vdots \\
&k_1 y_1^{(n-1)}(x_0) + \cdots + k_n y_n^{(n-1)}(x_0) = 0
\end{aligned}$$

$\det W(y_1, \cdots, y_n)(x_0) = 0 \Longrightarrow \exists$ non-trivial $k_1, \cdots, k_n$ solving the above linear system, i.e., not all the $k_i, 1 \leq i \leq n$ are zero.

Let

$$y(x) = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x).$$

Now, $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$.
By the existence-uniqueness theorem, $y(x) \equiv 0$ is the unique solution of the IVP

$$\begin{aligned} y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0, \\ y(x_0) = \cdots = y^{(n-1)}(x_0) = 0. \end{aligned}$$

$$\implies k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = 0$$

with $k_1, k_2, \cdots, k_n$ not all identically zero.
Hence, $y_1, y_2, \cdots, y_n$ are linearly dependent, proving the theorem.

Abel's Formula

We have seen that the Wronskian of any two solutions y_1, y_2 of $y'' + p(x)y' + q(x)y = 0$ is given by

$$W(y_1, y_2; x) = W(y_1, y_2; x_0)e^{-\int_{x_0}^{x} p(t)dt},$$

for any $x_0, x \in I$. Note that the formula depends only on p and not on q. What should be the n-th order analogue?

Theorem (Abel's Formula)

Let $p_1, p_2, \ldots, p_n$ be continuous on an interval I and let $y_1, y_2, \ldots, y_n$ be solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

on I. Then,

$$W(y_1, \ldots, y_n; x) = W(y_1, \ldots, y_n; x_0)e^{-\int_{x_0}^{x} p_1(t)dt}.$$

for any $x_0 \in I$.

Proof of Abel's Formula

Proof: Let $y = (y_1, y_2, \ldots, y_n)$ be the first row vector in $\mathcal{M}(y_1, \ldots, y_n)(x)$. Then $y', y^{(2)}, \ldots, y^{(n-1)}$ are the other row vectors in $\mathcal{M}(y_1, \ldots, y_n)(x)$. Hence

$$
\begin{aligned}
w(x) = W(y_1, \ldots, y_n; x) &= \det \mathcal{M}(y_1, \ldots, y_n)(x) \\
&= \det(y(x), y'(x), y^{(2)}(x), \ldots, y^{(n-1)(x)}).
\end{aligned}
$$

<u>Exercise:</u> Show that

$$
w'(x) = \det(y(x), y'(x), y^{(2)}(x), \ldots, y^{(n-2)}(x), y^{(n)}(x)).
$$

Now, multiply the last row of $\mathcal{M}(y_1, \ldots, y_n)(x)$ by $p_1(x)$ to get

$$
\begin{aligned}
&p_1(x)w(x) \\
&= \det(y(x), y'(x), y^{(2)}(x), \ldots, y^{(n-2)}(x), p_1(x)y^{(n-1)}(x)).
\end{aligned}
$$

Proof of Abel's Formula

Adding the above two equations, we get

$$\begin{aligned}&w'(x) + p_1(x)w(x)\\ &= \det(y(x), y'(x), y^{(2)}(x), \ldots, y^{(n-2)}(x), y^{(n)}(x) + p_1(x)y^{(n-1)}(x)).\end{aligned}$$

For each fixed x, as the rows of this last determinant are linearly dependent, it follows that

$$w'(x) + p_1(x)w(x) = 0.$$

$$w'(x) + p_1(x)w(x) = 0.$$

It follows that

$$w(x) = ce^{-\int_{x_0}^{x} p_1(t)dt}$$

for some constant c. Put $x = x_0$ to get $c = w(x_0)$ and hence

$$w(x) = w(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}.$$

n th Order Linear Homogeneous ODE: Summary

Theorem

Suppose $p_1, p_2, \ldots, p_n$ are continuous on an interval $I \subset \mathbb{R}$ and let $y_1, y_2, \ldots, y_n$ be solutions of

$$y^n + p_1(x)y^{n-1} + \ldots + p_n(x)y = 0 \tag{1}$$

on I. Then the following statements are equivalent:

(i) $\{y_1, y_2, \ldots, y_n\}$ *is linearly independent on I.*

(ii) *Every solution of (1) on I can be written as a linear combination of $y_1, y_2, \ldots, y_n$.*

(iii) *The Wronskian of $\{y_1, y_2, \ldots, y_n\}$ is non-zero at some point in I.*

(iv) *The Wronskian of $\{y_1, y_2, \ldots, y_n\}$ is non-zero at all points in I.*

Constant Differential Operators

Set

$$D^k = \frac{d^k}{dx^k}, k = 0, 1, 2, \ldots$$

and let

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n I,$$

and

$$M = b_0 D^m + b_1 D^{m-1} + \ldots + b_{m-1} D + b_m I$$

be constant coefficient linear differential operators, i.e., $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m \in \mathbb{R}$. Since,

$$D^r \cdot D^s = D^s \cdot D^r,$$

for $r, s \geq 0$, it follows that

$$L(M(\cdot)) = M(L(\cdot)).$$

Homogeneous n th order ODE with constant coefficients

If

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n I,$$

the characteristic polynomial of the differential operator L is defined by:

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$

Constant Differential Operators

Theorem

Let L and M be two constant coefficient linear differential operators. Then,

1. $L = M$ *if and only if* $P_L = P_M$.
2. $P_{L+M} = P_L + P_M$.
3. $P_{LM} = P_L \cdot P_M$.
4. $P_{\lambda L} = \lambda \cdot P_L$, *for every* $\lambda \in \mathbb{R}$.

Proof: (2),(3) and (4) are straightforward from the definition of the characteristic polynomial.

Constant Differential Operators

Proof of (1): Suppose

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \;\; M = \sum_{i=0}^{m} b_{m-i} D^i.$$

Then,

$$P_L(x) = \sum_{i=0}^{n} a_{n-i} x^i, \;\; P_M(x) = \sum_{i=0}^{m} b_{m-i} x^i.$$

Thus, $P_L = P_M$ iff $n = m$ and $a_i = b_i$ for $0 \leq i \leq n$ and hence $L = M$.
Conversely, suppose $L = M$. In particular,

$$L(e^{rx}) = M(e^{rx}) \text{ for every } r \in \mathbb{R},$$

It follows that

$$\sum_{i=0}^{n} a_{n-i} r^i e^{rx} = \sum_{i=0}^{m} b_{m-i} r^i e^{rx}.$$

Conclude that $P_L(r) = P_M(r)$ for every $r \in \mathbb{R}$ and hence $P_L = P_M$.