$$f(x) = \sin\frac{1}{x} \qquad x \neq 0$$
$$= 0 \qquad x = 0.$$

f is cont. at $c \neq 0$.

- What about continuity at $x = 0$. ??

Consider a seq.

$$x_n = \frac{1}{(2n+1)\pi/2} \longrightarrow 0.$$

$$|f(x_n)| = 1.$$

Let f be cont. at $x = 0$.

For any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x| < \delta \implies |f(x)| < \varepsilon. \quad \text{——} (1)$$

$x_n \to 0 \implies \exists\ N \in \mathbb{N}$ s.t.

$$|x_n| < \delta \quad \forall n \geq N.$$

$(1) \Rightarrow \quad |f(x_n)| < \epsilon, \ \forall n \geq N.$

but $|f(x_n)| = 1$.

$\longrightarrow$ a contradiction.

• **Thm:** $f : [a, b] \longrightarrow \mathbb{R}$ is cont. at $c \in [a, b] \iff$ for every seq. $\{x_n\} \longrightarrow c$, $\{f(x_n)\}_n \longrightarrow f(c)$.

Proof: Let f be cont. at $x = c$.

Let $\{x_n\} \longrightarrow c$. **Claim:** $\{f(x_n)\}_n \to f(c)$.

For $\epsilon > 0$, $\exists \ \delta > 0$ s.t.

$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$. $\longrightarrow (1)$

but $\{x_n\} \longrightarrow c$ so, $\exists N \in \mathbb{N}$

s.t. $|x_n - c| < \delta \quad \forall n \geq N$

1 $\Rightarrow \quad |f(x_n) - f(c)| < \epsilon, \ \forall n \geq N.$

Conversely, $\boxed{\text{For any seq. } x_n \longrightarrow c \Rightarrow f(x_n) \to f(c).}$

Aim: f is cont. at $x = c$.

Suppose f is not cont. at c.

Then, $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$,

$\exists x_\delta$ with $|x_\delta - c| < \delta$

such that $|f(x_\delta) - f(c)| \geq \epsilon_0$. $\quad (*)$

$\delta = 1$; $\exists x_1 \in (c-1, c+1)$

& $|f(x_1) - f(c)| \geq \epsilon_0$.

$\delta = \frac{1}{2}$, $\exists\, x_2 \in \left(c - \frac{1}{2}, c + \frac{1}{2}\right)$

& $|f(x_2) - f(c)| \geq \epsilon_0$.

. . . .

$\delta = \frac{1}{n}$, $\exists\, x_n \in \left(c - \frac{1}{n}, c + \frac{1}{n}\right)$

& $|f(x_n) - f(c)| \geq \epsilon_0$.

. . . .

got a seq. $\{x_n\}_n \longrightarrow c$.

but $\{f(x_n)\}_n \not\longrightarrow f(c)$.

$\therefore$ $\circledR \longrightarrow$ a contradiction.

Ex. $f(x) = \begin{cases} 1 & x \in \mathbb{Q}. \\ 0 & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$

f is nowhere cont.

Let <u>f be cont. at $x = c \in \mathbb{R} - \mathbb{Q}$.</u>

get a seq. of rational no. $\{r_n\}$

s.t. $r_n \longrightarrow c$.

$$\Longrightarrow \underset{1}{\underline{\underline{f(r_n)}}} \longrightarrow \underset{0}{\overset{}{f(c)}}$$

$\longrightarrow$ not possible.

- Let f be cont. at $x = d \in \mathbb{Q}$.

get a seq. of irrational no. $\{s_n\}$

s.t. $s_n \longrightarrow s$.

$$f(s_n) \longrightarrow f(s)$$

$$\begin{array}{ccc} f(s_n) & \longrightarrow & f(s) \\ \shortparallel & & \shortparallel \\ 0 & & 1 \end{array}$$

contradiction.

- $g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

g is cont. only at 0.

1. Prove that g is cont. at 0.

Fix $\epsilon > 0$.

$$|g(x) - g(0)|$$

$$= \begin{cases} |x| & x \in \mathbb{Q} \end{cases}$$

$0 \quad x \notin \mathbb{Q}.$

$\therefore |g(x) - g(0)| < \epsilon$

if $|x| < \epsilon := \delta.$

2. Prove that f is discont. at $x = c\,(\neq 0) \in \mathbb{Q}.$

Let f be cont. at $c \in \mathbb{Q}\,(c \neq 0).$

let $r_n\,(\in \mathbb{R}-\mathbb{Q}) \longrightarrow c.$

$$\begin{array}{ccc} g(r_n) & \longrightarrow & g(c) \\ \| & & \| \\ 0 & & c \end{array}$$

— not possible.

- ex. of a func. cont. only at 2 - pts.

$$f(x) = \begin{cases} x(x-1) & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- Function cont. only on $\mathbb{Z}$:

$$f(x) = \begin{cases} \sin \pi x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- Funct. which is cont. only at $\mathbb{Q}$:

— do not exist.