APPLICATIONS - STOKES' THEOREM

COMPUTING LINE INTEGRALS

COMPUTE $\oint_C F \cdot dr$, $F = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$

AND $C := \{(x,y,z) \mid x^2+y^2 = 4,\ z = -3\}$,

ORIENTED ANTI-CLOCKWISE.

CHECK THAT STOKES IS APPLICABLE :

$$\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot n \, dS \quad (\text{NOTE THAT } n = \vec{k})$$

$$\text{curl}(F) = \text{curl}(y, xz^3, -zy^3) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & xz^3 & -zy^3 \end{vmatrix}$$

$$= (\qquad\qquad)i + (\qquad\qquad)j$$

$$+ (z^3 - 1)\vec{k}$$

$$\iint_S (\text{curl } F) \cdot n \, dS = \iint_S (z^3-1) \, dS = -28 \iint_S dS$$

$$= -28(4\pi)$$

EVALUATE

$$\oint_C F \cdot dr \;, \quad F = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

$C := \{(x,y,z) \mid z=0,\ x^2+y^2=1\}$, ORIENTED (ANTI) CLOCKWISE.

RECALL THAT

- F SATISFIES $\mathrm{curl}(F) = 0$, BUT F IS NOT CONSERVATIVE.
- IN FACT, WE HAVE ALREADY CALCULATED

$$\oint_C F \cdot dr = -2\pi \quad (C \text{ ORIENTED CLOCKWISE})$$

STOKES' THEOREM IS NOT APPLICABLE PER SE, BUT, THERE IS A WAY TO WORK AROUND THIS DIFFICULTY.

CONSIDER $\tilde{F}(x,y,z) = \frac{-y}{x^2+y^2+z^2}\vec{i} + \frac{x}{x^2+y^2+z^2}\vec{j}$

ON $\mathbb{R}^3 \setminus \{(0,0,0)\}$. ON THE CIRCLE C,

$\tilde{F} = F$. NOW, LET S DENOTE THE UPPER

HEMISPHERE $S = \{x^2+y^2+z^2 = 1,\ z \geq 0\}$.

NOW, STOKES' THEOREM IS APPLICABLE FOR $\tilde{F}$

ON S (CHECK!), SO

$$\oint_C F \cdot dr = \int_C \tilde{F} \cdot dr = \iint_S (\operatorname{curl} \tilde{F}) \cdot n \, dS$$

CONSIDER PARAMETRIZING S BY

$$r(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\right). \text{ (OUTWARD NORMAL)}$$

ALSO, $\operatorname{curl}(\tilde{F}) = (2xz, 2yz, 2z^2)$

HENCE $\iint_S (\operatorname{curl} \tilde{F}) \cdot n \, dS = -\iint_R 4\sqrt{1-(x^2+y^2)} \, dx\, dy$

$$= -4\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r \, dr\, d\theta = 4\pi \int_0^1 \sqrt{1-r^2}\,(2r)\, dr$$

$$= 4\pi\left(\frac{-1}{2}\right)(1-r^2)^{3/2}\Big|_0^1$$

$$= 2\pi.$$

CALCULATION OF SURFACE INTEGRALS

CALCULATE $\iint_S (\nabla u \times \nabla v)\cdot n \, dS$

$S \equiv x^2+y^2+z^2=1, \quad z \geq 0,$

$u \equiv x^3-y^3+z^2, \quad v = x+y+z.$

FIRST, OBSERVE THAT $\nabla u \times \nabla v = \text{curl}(u\nabla v)$

HENCE WE NEED TO EVALUATE $\iint_S \text{curl}(u\nabla v)\cdot n \, dS$

BY STOKES',

$$\iint_S \text{curl}(u\nabla v)\cdot n \, dS = \oint_C (u\nabla v)\cdot dr$$

WHERE $C = \left\{ (x,y) \;\middle|\; \begin{array}{l} x^2+y^2=1 \\ \& \; z=0 \end{array} \right\}$ (ANTICLOCKWISE)

$u\nabla v = (u,u,u)$

HENCE $\oint_C u\,dx + u\,dy = 2\oint_C u\,dx$

$$= 2\int_0^{2\pi} (\cos^3 t - \sin^3 t)(-\sin t)\,dt = 2\int_0^{2\pi} \sin^4 t \, dt$$

$$= 2\cdot 4 \cdot \int_0^{\pi/2} \sin^4 t \, dt$$

$$= 2\cdot 4 \cdot \left(\frac{3}{4}\right)\cdot\left(\frac{1}{2}\right)\cdot\frac{\pi}{2} = \frac{3\pi}{2}.$$

GREEN'S THEOREM

(VIA STOKES' THEOREM)

$F = M\vec{i} + N\vec{j}$. $D \subseteq \mathbb{R}^2$ AND $\partial D = C$, A SIMPLE CLOSED CURVE. CLEARLY $\mathbf{n} = \vec{k}$

BY STOKES',

$$\iint_D (\text{curl } F) \cdot \vec{k} \, dS = \int_{\partial C} F \cdot dr \qquad — (*)$$

BUT SINCE THE PARAMETRIZATION FOR THE SURFACE IS $\mathbf{r}(x,y) = (x, y, 0)$, FOR $(x,y) \in D$,

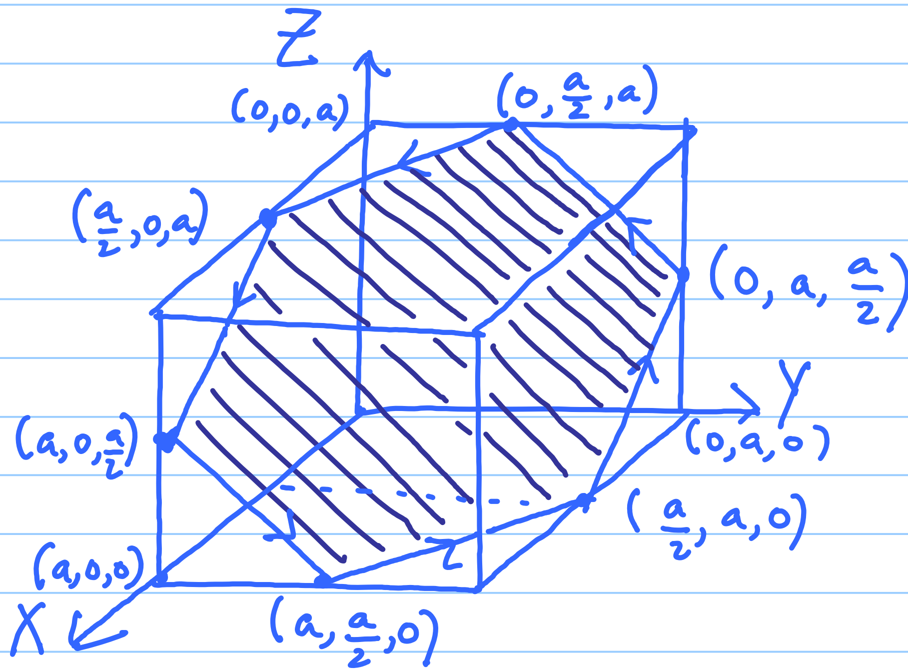
$$\iint_D (\text{curl } F) \cdot \vec{k} \, dS = \iint_D (\text{curl } F) \cdot \vec{k} \, dx\, dy$$

THIS $(*)$ IS PRECISELY GREEN'S THEOREM IN FLUX FORM.

FURTHER EXAMPLES

🚩 EVALUATE $\oint_C (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz$,

C IS THE CURVE CUT FROM THE BOUNDARY OF THE CUBE $[0,a] \times [0,a] \times [0,a]$ BY THE PLANE $x + y + z = \frac{3}{2}a$. C IS ANTI CLOCKWISE.



STOKES' THEOREM $\Rightarrow$

$$\oint_C F \cdot dr = \iint_S (\text{curl}(F) \cdot n)\,dS.$$

THE SURFACE IS PARAMETRIZED BY

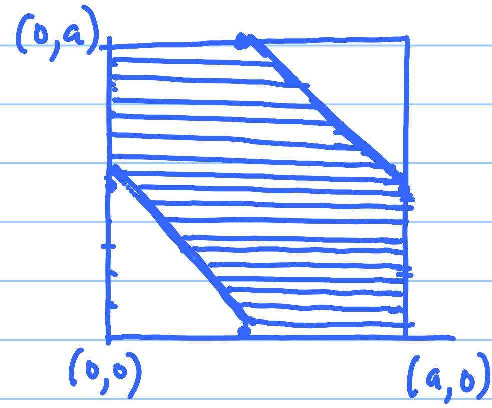
$$z = \frac{3}{2}a - x - y, \quad \text{so} \quad n = \frac{(1,1,1)}{\sqrt{3}}.$$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix} = -2(y+z)\vec{i} - 2(x+z)\vec{j} - 2(x+y)\vec{k}.$$

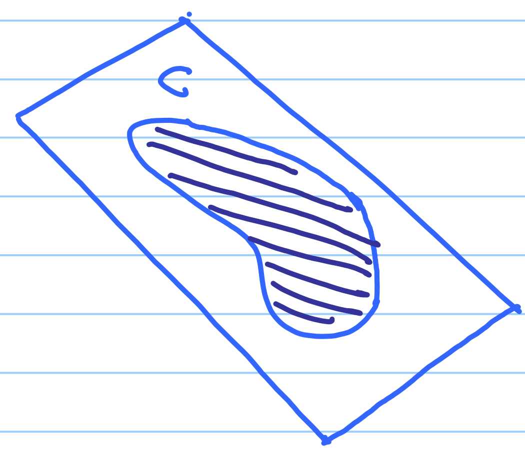
HENCE

$$\oint_C F\cdot dr = -4\iint_R \frac{3}{2}a\,dx\,dy = -6a\left(a^2 - \frac{a^2}{8}\cdot 2\right) = -\frac{9}{2}a^3.$$

WHERE R IS THE REGION:



CONSIDER A CLOSED CURVE C ON THE PLANE WITH UNIT NORMAL $a\vec{i} + b\vec{j} + c\vec{k}$. GIVE A FORMULA FOR THE AREA OF THE REGION ENCLOSED BY C. (DENOTE A(C))



CONSIDER $F(x,y,z) = (bz-cy,\ cx-az,\ ay-bx)$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ bz-cy & cx-az & ay-bx \end{vmatrix}$$

$$= 2a\vec{i} + 2b\vec{j} + 2c\vec{k}$$

HENCE $\frac{1}{2}\iint_S \text{curl}\, F \cdot n \, dS = \iint_S dS = A(C)$.

HENCE STOKES' $\Rightarrow$

$$A(C) = \frac{1}{2}\oint_C (bz-cy)\,dx + (cx-az)\,dy + (ay-bx)\,dz$$