DIFFERENTIATION

SUPPOSE $f : D \to \mathbb{R}$ WHERE $D \subseteq \mathbb{R}^2$. LET $\vec{x} = (x_0, y_0)$ BE AN INTERIOR POINT IN D, i.e. $B_r(\vec{x}) \subseteq D$ FOR SOME $r > 0$. THE PARTIAL DERIVATIVE OF f w.r.t. x AT (x_0, y_0) IS

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

IF IT EXISTS.

SIMILARLY, THE PARTIAL DERIVATIVE OF f w.r.t y AT (x_0, y_0) IS THE LIMIT

$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

THE PARTIAL DERIVATIVES ARE DENOTED f_x, f_y RESPECTIVELY.

EXAMPLE

$$f(x, y) = \sin(xy)$$

$$\frac{\partial f}{\partial x} = f_x(x, y) = \cos(xy) \cdot y$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \cos(xy) \cdot x$$

PATHOLOGICAL EXAMPLE

CONSIDER $f(x,y) = \dfrac{xy}{x^2+y^2}$ IF $(x,y) \neq (0,0)$

$= 0$ IF $(x,y) = (0,0)$.

$$f_x(x,y) = \frac{y(x^2+y^2) - (xy)(2x)}{(x^2+y^2)^2} = \frac{y\{y^2-x^2\}}{(x^2+y^2)^2}$$

$$f_y(x,y) = (\text{CALCULATE THIS!}) = \frac{x\{x^2-y^2\}}{(x^2+y^2)^2}$$

QUESTION: IS f CONTINUOUS AT $(0,0)$?

WE HAVE SEEN BEFORE THAT IF $y = kx$,

$f(x,y) = \dfrac{k}{1+k^2}$, IS A CONSTANT, SO

f IS <u>NOT</u> CONTINUOUS AT $(0,0)$.

BUT $f_x(x,0) = 0 \ \forall x$

LINEAR MAPS

$f : \mathbb{R}^2 \to \mathbb{R}$ IS A LINEAR MAP IF

$f(x,y) = ax + by$ FOR SOME $a, b \in \mathbb{R}$

$f : \mathbb{R}^3 \to \mathbb{R}$ IS LINEAR IF

$f(x,y,z) = ax + by + cz$ FOR SOME $a, b, c \in \mathbb{R}$.

LET $f : U \to \mathbb{R}$, FOR $U \subseteq \mathbb{R}^2$. LET $(x_0, y_0) \in U$ BE AN INTERIOR POINT

WISH TO APPROXIMATE f IN SOME SMALL NEIGHBORHOOD OF (x_0, y_0), BY SOME LINEAR MAP.

SAY f IS DIFFERENTIABLE AT (x_0, y_0) IF THERE EXISTS A LINEAR MAP $\Lambda : \overset{\mathbb{R}^2}{\cancel{B_{(x_0,y_0)}}}(\delta) \to \mathbb{R}$ (FOR SOME $\delta > 0$) SUCH THAT:

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h, y_0+k) - f(x_0,y_0) - \Lambda(h,k)|}{\sqrt{h^2+k^2}} = 0$$

NOTE THAT $\sqrt{h^2+k^2} = \|(h,k)\|$.

IF Λ, Γ ARE LINEAR AND SATISFY THE PREVIOUS LIMIT, THEN $\Lambda(h,k) = \Gamma(h,k)$ $\forall (h,k)$. (IF f IS DIFFERENTIABLE AT (x_0,y_0) THERE IS A UNIQUE LINEAR MAP APPROXIMATING f).

SKETCH OF PROOF:

Λ SATISFIES:

$$\lim_{(h,k)\to(0,0)} \frac{|\overbrace{f(x_0+h,y_0+k)-f(x_0,y_0)}^{G(h,k)} - \Lambda(h,k)|}{\sqrt{h^2+k^2}} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h,y_0+k)-f(x_0,y_0) - \Gamma(h,k)|}{\sqrt{h^2+k^2}} = 0$$

$$\frac{|\Lambda(h,k) - \Gamma(h,k)|}{\sqrt{h^2+k^2}} \leq \underbrace{\frac{|\Lambda(h,k) - G(h,k)|}{\sqrt{h^2+k^2}} + \frac{|G(h,k) - \Gamma(h,k)|}{\sqrt{h^2+k^2}}}_{\xrightarrow[(h,k)\to(0,0)]{} 0}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{\Lambda(h,k) - \Gamma(h,k)}{\sqrt{h^2+k^2}} = 0$$

$$\Rightarrow \Lambda(h,k) = \Gamma(h,k) \quad \text{(EXERCISE!)}$$

(WRITE $\Lambda(h,k) = ah + bk$, $\Gamma(h,k) = ch + dk$, SIMPLIFY ETC.)

WE SHALL NOW DENOTE THE DERIVATIVE OF f AT (x_0,y_0), BY $Df(x_0,y_0)$.

BASIC OBSERVATIONS

- IF $f : \mathbb{R}^2 \to \mathbb{R}$ IS CONSTANT, THEN IT IS DIFFERENTIABLE AND ITS DERIVATIVE $\equiv 0$.

- IF $f(x,y) = \underline{ax + by}$, THEN f IS DIFFERENTIABLE AND ITS DERIVATIVE IS $\underbrace{(Df(\alpha,\beta))}(h,k) = \underline{ah + bk}$ $\forall (h,k)$

- IF $f : \mathbb{R}^2 \to \mathbb{R}$ IS DIFFERENTIABLE AT (x_0, y_0) & $g : \mathbb{R} \to \mathbb{R}$ IS DIFFERENTIABLE AT $f(x_0,y_0)$, THEN $g \circ f$ IS DIFF. AT (x_0, y_0) AND

$$D(g \circ f(x_0,y_0)) = g'(f(x_0,y_0)) \cdot Df(x_0,y_0)$$

(CHAIN RULE)

- $f, g : \mathbb{R}^2 \to \mathbb{R}$, ARE DIFFERENTIABLE AT (x_0, y_0) THEN

$$D(f \pm g)(x_0,y_0) = Df(x_0,y_0) \pm Dg(x_0,y_0)$$

$$D(\lambda f)(x_0,y_0) = \lambda Df(x_0,y_0) \quad (\lambda \in \mathbb{R})$$

$$D(fg)(x_0,y_0) = Df(x_0,y_0) \cdot g(x_0,y_0) + f(x_0,y_0)\, Dg(x_0,y_0)$$

(PRODUCT RULE).

EXAMPLES

$$f(x,y) = xy.$$

CLAIM: $(Df(a,b))(x,y) = \underbrace{bx + ay}_{\Lambda(x,y)}$.

WANT: $$\lim_{(h,k)\to(0,0)} \frac{|f(a+h, b+k) - f(a,b) - \Lambda(h,k)|}{\sqrt{h^2+k^2}} = 0$$

$$|f(a+h, b+k) - f(a,b) - \Lambda(h,k)| = (a+h)(b+k) - ab - bh - ak$$
$$= |hk|$$

$$\frac{1}{\sqrt{h^2+k^2}} \leq \frac{1}{\sqrt{2|hk|}} \Rightarrow \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} \leq \frac{\sqrt{|hk|}}{\sqrt{2}} \to 0$$

AS $(h,k) \to (0,0)$

THIS COMPLETES THE PROOF. 🏳

ANOTHER WAY TO 'LOOK' AT LINEAR MAPS:

$\Lambda(x,y) = ax + by$. THIS Λ MAY BE IDENTIFIED BY $(a,b) \in \mathbb{R}^2$.

SO, $Df(a,b)$ MAY BE INTERPRETED TO MEAN SOME ELEMENT $(\alpha,\beta) \in \mathbb{R}^2$.

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \quad \text{IF } (x,y) \neq (0,0)$$
$$= 0 \quad \text{IF } (x,y) = (0,0).$$

CLAIM: f IS DIFFERENTIABLE AT $(0,0)$.

$$\frac{f(h,k) - f(0,0) - \Lambda(h,k)}{\sqrt{h^2+k^2}} = \frac{(h^2+k^2)\sin\left(\frac{1}{h^2+h^2}\right) - \Lambda(h,k)}{\sqrt{h^2+h^2}}$$

LET $\Lambda(h,k) = 0$ THEN WE OBSERVE THAT

$$\frac{f(h,k) - f(0,0)}{\sqrt{h^2+k^2}} \to 0 \quad \text{AS} \quad (h,k) \to (0,0)$$

HENCE $Df(0,0) = (0,0)$

EX: $f(x,y) = \tan^{-1}\left(y^3 + \frac{x^2\pi}{\sqrt{2}}\right)$ AT $(1,1)$;

IS THIS DIFF. ?

SUPPOSE $f: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ IS DIFFERENTIABLE AT (x_0, y_0). THEN f IS ALSO CONTINUOUS AT (x_0, y_0).

SUPPOSE $Df(x_0,y_0) = \Lambda$, WHERE $\Lambda(h,k) = \alpha h + \beta k$

WANT: $\lim_{(h,k)\to(0,0)} \left(f(x_0+h, y_0+k) - f(x_0,y_0)\right) = 0$

KNOW: $\lim_{(h,k)\to(0,0)} \dfrac{f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0$

IN OTHER WORDS,

$f(x_0+h, y_0+k) - f(x_0,y_0) = \alpha h + \beta k + \varepsilon(h,k)\sqrt{h^2+k^2}$

FOR SOME $\varepsilon: (-\delta,\delta)\times(-\delta,\delta) \to \mathbb{R}$ S.T. $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$

TAKE $\lim_{(h,k)\to(0,0)}$ ON BOTH SIDES.

$f(x,y) = xy$ $\qquad \dfrac{\partial f}{\partial x} = y$; $\dfrac{\partial f}{\partial y} = x$

$\Lambda = Df(a,b)$, SO $\Lambda(x,y) = bx + ay$

$$\frac{\partial f}{\partial x}(a,b) = b, \quad \frac{\partial f}{\partial y}(a,b) = a$$

$$\Rightarrow \quad \Lambda = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right).$$

DIFFERENTIABILITY CRITERION

SUPPOSE $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^2$, AND $(x_0, y_0) \in U$ (INTERIOR)

SUPPOSE

- f_x AND f_y EXIST AT ALL POINTS IN $B((x_0,y_0), r)$ FOR SOME $r > 0$.
- f_x, f_y ARE CONTINUOUS AT (x_0, y_0)

THEN f IS DIFFERENTIABLE AT (x_0, y_0) AND

$$(Df(a,b))(x,y) = f_x(a,b)\, X + f_y(a,b)\, Y.$$

PROOF: LET $(h,k) \in B((0,0), r)$.

USE MVT ON $x \mapsto f(x, y_0)$ & $y \mapsto f(x_0+h, y)$

$$f(x_0+h, y_0) - f(x_0, y_0) = h f_x(\xi, y_0)$$

$$f(x_0+h, y_0+k) - f(x_0+h, y_0) = k f_y(x_0+h, \eta)$$

WHERE ξ LIES BETWEEN x_0 & x_0+h
η LIES BETWEEN y_0 & y_0+k

NOTE THAT η ALSO DEPENDS ON h!

HENCE,

$$f(x_0+h, y_0+k) - f(x_0, y_0) = h f_x(\xi, y_0) + k f_y(x_0+h, \eta)$$

LET $\epsilon(h,k) = h\left(f_x(\xi, y_0) - f_x(x_0,y_0)\right)$
$+ k\left(f_y(x_0+h, \eta) - f_y(x_0,y_0)\right).$

THEN BY CONTINUITY OF f_x, f_y AT (x_0, y_0),
$\epsilon(h,k) \to 0$ AS $(h,k) \to (0,0)$.

THIS IS A SUFFICIENT CONDITION ONLY.

FOR EXAMPLE,

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \quad (x,y) \neq (0,0)$$
$$f(0,0) = 0.$$

WE HAVE SEEN THAT f IS DIFFERENTIABLE AT $(0,0)$. HOWEVER

$$f_x(x,y) = 2x\sin\left(\frac{1}{x^2+y^2}\right) + \cos\left(\frac{1}{x^2+y^2}\right)\cdot\frac{-2x}{(x^2+y^2)^2}$$

AND $f_x(x,y)$ IS <u>NOT</u> BOUNDED IN A NEIGHBORHOOD OF $(0,0)$.