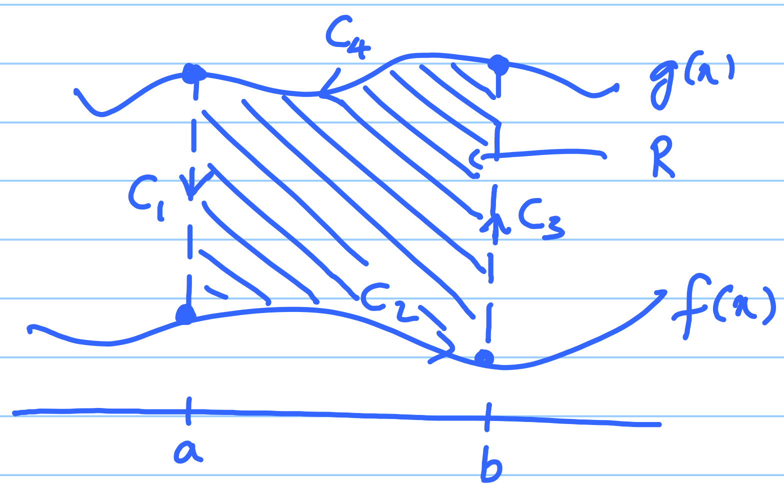
GREEN'S THEOREM

SUPPOSE $R \subseteq \mathbb{R}^2$ IS A TYPE I ELEMENTARY REGION: (VERTICALLY SIMPLE REGION)

$$R = \{(x,y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}.$$



SUPPOSE C IS THE BOUNDARY OF R. IT IS DENOTED BY WRITING $\partial R = C$.

AS INDICATED ABOVE, WRITE

$C = C_1 \cup C_2 \cup C_3 \cup C_4$, WHERE C_i ARE ORIENTED SO THAT C IS COUNTER CLOCKWISE.

A REGION $R \subseteq \mathbb{R}^2$ IS CALLED A SIMPLE REGION IF IT IS A TYPE I AS WELL AS TYPE II ELEMENTARY REGION.

LET $M, N : U \to \mathbb{R}$ BE CONTINUOUSLY $(F = M\vec{i} + N\vec{j})$ DIFFERENTIABLE SCALAR FIELDS. HERE $U \subseteq \mathbb{R}^2$.

SUPPOSE R IS A SIMPLE REGION WITH A SMOOTH BOUNDARY C. SUPPOSE $R, C \subseteq U$. THEN

$$\oint_C F \cdot d\mathbf{r} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

EXAMPLE

$$F(x,y) = M\vec{i} + N\vec{j}, \quad M := -x^2 y, \quad N := xy^2.$$

$$R = \{(x,y) \mid x^2 + y^2 \le a^2\}$$

LET $\mathbf{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}, \quad 0 \le t \le 2\pi$.

THEN

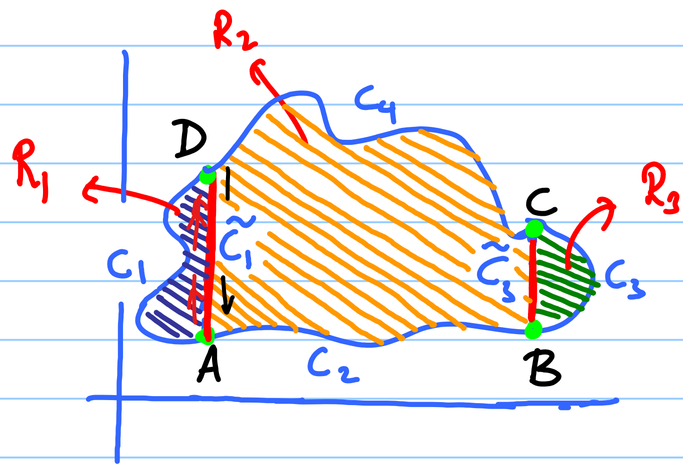
$$\oint_C M\,dx = \int_0^{2\pi} -(a\cos t)^2 (a\sin t)(-a\sin t)\,dt$$

$$\oint_C N\,dy = \int_0^{2\pi} (a\cos t)(a\sin t)^2 (a\cos t)\,dt$$

AND $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R x^2 + y^2\,dx\,dy.$

CHECK THAT THESE ARE EQUAL

BEYOND SIMPLE REGIONS



REGION R_1 ; REGION R_2 ; REGION R_3.

LET US APPLY GREEN'S THEOREM ON R_1, R_2, R_3:

$$\iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_{C_i} M\,dx + N\,dy$$

$i = 1, 2, 3$

SINCE THE INTEGRALS OVER THE 'VERTICAL LINES' CANCEL EACH OTHER,

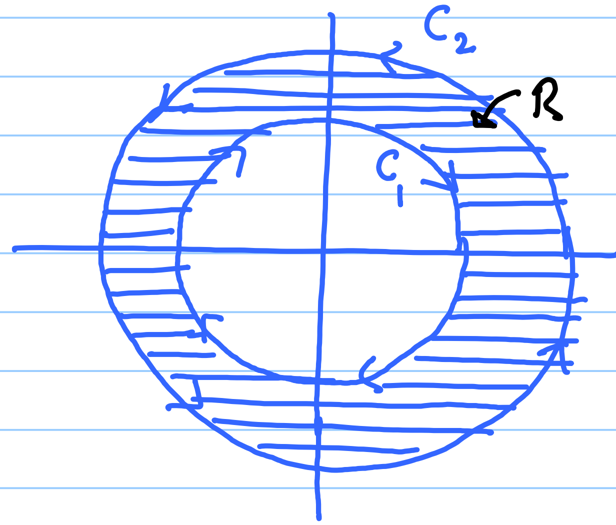
$$\iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_{C} M\,dx + N\,dy$$

GREEN'S THEOREM HOLDS FOR 'ALL' REGIONS BOUNDED BY SIMPLE CLOSED CURVES.

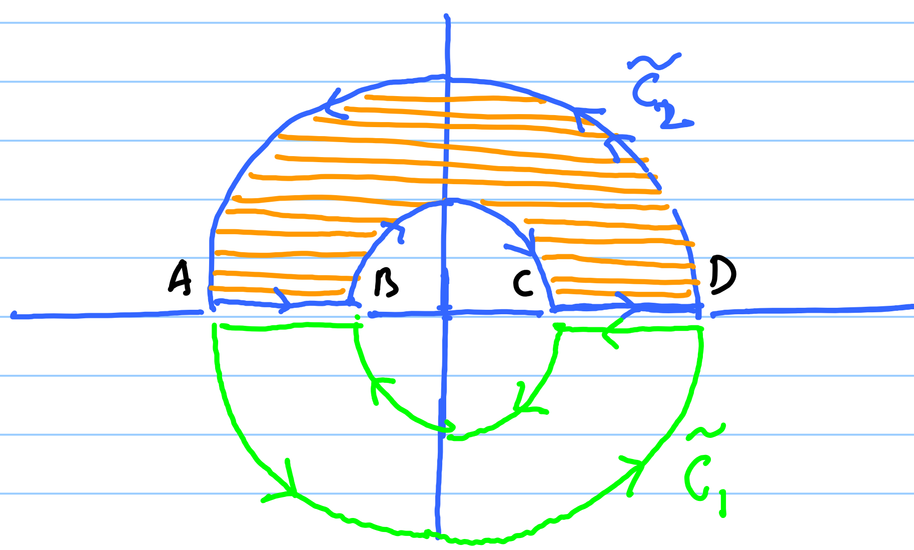
THIS EXTENDS EVEN FURTHER

CONSIDER

$$R = \{(x,y) \mid a^2 < x^2+y^2 < b^2\}$$



THIS MAY BE RENDERED AS FOLLOWS:



SO, WE AGAIN HAVE

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_{C_1} M\,dx + N\,dy + \oint_{C_2} M\,dx + N\,dy$$

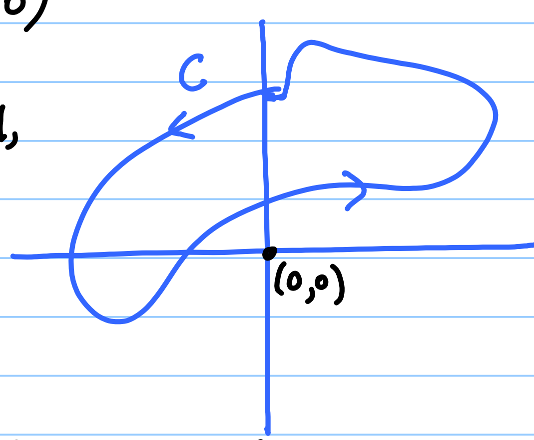
GREEN'S THEOREM CAN BE EXTENDED SIMILARLY TO DOMAINS WITH A FINITE NUMBER OF 'HOLES'.

EXAMPLE

$$F(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} \quad (x,y) \neq (0,0)$$

LET C BE A PIECEWISE SMOOTH CURVE NOT CONTAINING (0,0)

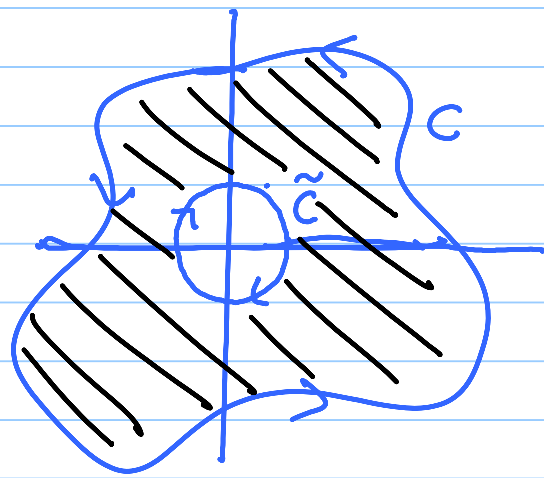
BY GREEN'S THEOREM,



$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$$

ON THE OTHER HAND, IF C CONTAINS $(0,0)$:



LET $\tilde{C}$ BE A CIRCLE $x^2+y^2=r^2$ FOR SOME SUITABLE $r>0$ SO THAT $\tilde{C}$ IS CONTAINED 'INSIDE' C. ORIENT $\tilde{C}$ CLOCKWISE.

GREEN $\Rightarrow$ $$\int_C + \int_{\tilde{C}} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = 0$$

FLUX/WORK FORMS OF GREEN

$C: \mathbf{r}(s) = x(s)\vec{i} + y(s)\vec{j}$ FOR $s \in [0,1]$.

SUPPOSE R IS THE REGION BOUNDED BY C.

SUPPOSE THIS IS THE ARC-LENGTH PARAMETRIZATION FOR C. (UNIT SPEED PARAMETRIZATION FOR C)

$\mathbb{T}(s)$ = UNIT TANGENT VECTOR TO C

AT s $= (x'(s), y'(s))$

$\vec{n}(s)$ = UNIT NORMAL POINTING 'OUTWARD'

$= (y', -x')$

SUPPOSE $F = M\vec{i} + N\vec{j}$ IS CONTINUOUSLY DIFFERENTIABLE, IN D AND $R, C \subseteq D$. WRITE

$G = N\vec{i} - M\vec{j} \Rightarrow \text{div}(G) =$

AND $\vec{G} \cdot \vec{n} =$

HENCE

$$\iint_R \text{div}(G)\, dx\, dy = \oint_C (\vec{G} \cdot \vec{n})\, ds$$

(FLUX FORM OF GREEN'S THEOREM)

SUPPOSE $F = M\vec{i} + N\vec{j}$ AS BEFORE;

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

ALSO, $\oint_C M\,dx + N\,dy$

$$= \oint_C (M\vec{i} + N\vec{j}) \cdot (x'(s)\vec{i} + y'(s)\vec{j})\,ds$$

$$=$$

HENCE GREEN'S THEOREM CAN BE WRITTEN AS

$$\underset{\oint_C F \cdot dr}{\overset{\displaystyle\oint_C (F \cdot T)\,ds}{\parallel}} = \iint_R \left[\text{curl}(F) \cdot \vec{k}\right] dx\,dy$$

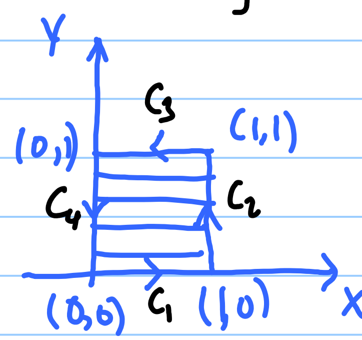
THE FORMER IS THE FLUX FORM OF GREEN'S THEOREM AND THE VERSION ABOVE IS THE WORK FORM OF GREEN'S THEOREM.

APPLICATIONS OF GREEN

EVALUATING LINE INTEGRALS

COMPUTE $$\oint_C \underset{\overset{\shortparallel}{M}}{(5 - xy - y^2)}\, dx - \underset{\overset{\shortparallel}{N}}{(2xy - x^2)}\, dy$$

WHERE $C = \partial R$, $R = \{ (x,y) \mid 0 \le x, y \le 1 \}$



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

C_1: $\mathbf{r}(t) = (x(t), y(t))$

$\mathbf{r}(x) = (x, 0)$, $\quad x \in [0,1]$

C_2: $\mathbf{r}(y) = (1, y) \quad y \in [0,1]$

SIMILARLY, C_3, C_4.

BUT GREEN $\Rightarrow$ $$\oint_C M\,dx + N\,dy = \iint_R (2x - 2y) - (-x - 2y)\, dA$$

$M = 5 - xy - y^2$

$N = x^2 - 2xy$

$$= \int_0^1 \int_0^1 3x\, dx\, dy = \int_0^1 3x\, dx$$

$$= \frac{3}{2}$$

AREA ENCLOSED BY A CURVE

SUPPOSE C IS A SIMPLE CLOSED CURVE IN $\mathbb{R}^2$ ENCLOSING A REGION R. WE WISH TO CALCULATE AREA(R).

CONSIDER $M(x,y) = y, \quad N(x,y) = 0$

$$\oint_C M\,dx + N\,dy \overset{\text{GREEN}}{=} -\iint_R dx\,dy$$

SIMILARLY, SUPPOSE $M(x,y) = 0, \quad N(x,y) = x$

$$\oint_C M\,dx + N\,dy = \iint 1\,dx\,dy$$

HENCE, $\quad \text{AREA}(R) = \frac{1}{2}\int_C -y\,dx + x\,dy$

IF C HAS PARAMETRIZATION $\mathbf{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$a \le t \le b$

THEN THE LINE INTEGRAL ABOVE IS :

$$\frac{1}{2}\int_a^b \left[-y(t)x'(t) + x(t)y'(t)\right] dt$$

IF C HAS POLAR COORDINATES $x(\theta) = r(\theta)\cos\theta$

$y(\theta) = r(\theta)\sin\theta$

$\theta_0 \le \theta \le \theta_1$

THEN THE AREA ENCLOSED BY C EQUALS

$$\frac{1}{2}\int -y\,dx + x\,dy \underset{\text{AFTER A SMALL CALCULATION}}{=} \frac{1}{2}\int_{\theta_0}^{\theta_1} r^2\,d\theta \quad (\text{CHECK?!})$$

SUFFICIENT CONDITION FOR A VECTOR FIELD TO BE CONSERVATIVE

SUPPOSE $D \subseteq \mathbb{R}^2$ S.T. FOR ANY SIMPLE CLOSED CURVE C IN D, THE REGION R ENCLOSED BY C SATISFIES $R \subseteq D$. SUPPOSE $F = M\vec{i} + N\vec{j}$ AND $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. ~~DOES~~ THERE EXISTS $\phi : D \to \mathbb{R}$ S.T $\nabla \phi = F$?!

USE GREEN'S THEOREM

SUCH A DOMAIN IS CALLED SIMPLY CONNECTED.

FORMULAS FOR ∇^2

RECALL THE LAPLACIAN OPERATOR,

$$\nabla^2(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

WHERE f IS A SMOOTH SCALAR FIELD.

DEFINE $\quad F := -\frac{\partial f}{\partial y}\vec{i} + \frac{\partial f}{\partial x}\vec{j}.$

LET C DENOTE A SIMPLE CLOSED CURVE AND

R A REGION S.T. $\quad C = \partial R.$

GREEN $\Rightarrow$

$$\iint_R (\nabla^2 f)\, dx\, dy = \iint_R \left[\frac{\partial}{\partial x}\overbrace{\left(\frac{\partial f}{\partial x}\right)}^{N} - \frac{\partial}{\partial y}\overbrace{\left(-\frac{\partial f}{\partial y}\right)}^{M} \right]$$

$$= \oint_C -\frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial x}\, dy$$

$$= \oint_C \left(-\frac{\partial f}{\partial y} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial x}\frac{dy}{ds} \right) ds$$

$$= \oint_C \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \right) \cdot \left(\frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j} \right) ds$$

$$= \oint_C (\nabla f) \cdot n\, ds$$

WRITE $\frac{\partial f}{\partial n} := (\nabla f)\cdot n$, SO THAT WE HAVE

$$\iint_R (\nabla^2 f)\, dx\, dy = \oint_C \frac{\partial f}{\partial n}\, ds$$

HENCE $\boxed{\frac{\partial f}{\partial n} = 0 \Rightarrow \iint_R \nabla^2 f\, dx\, dy = 0.}$ GREEN'S IDENTITY

SUPPOSE ϕ IS A SCALAR FIELD AS ABOVE.

$$\phi \frac{\partial \phi}{\partial n} = \phi\,(\nabla\phi\cdot n) = \frac{1}{2}(\nabla\phi^2)\cdot n.$$

GREEN $\Rightarrow$ $\oint_C \left(\phi \frac{\partial\phi}{\partial n}\right) ds = \oint_C \frac{1}{2}(\nabla\phi^2)\cdot n\, ds$

$\binom{\text{FLUX}}{\text{FORM}}$ $= \frac{1}{2}\iint_R \operatorname{div}(\nabla\phi^2)\, dx\, dy$

$$\oint_C \phi\frac{\partial\phi}{\partial n} = \iint_R \operatorname{div}(\phi\nabla\phi)\, dx\, dy = \iint_R \left[(\nabla\phi)^2 + \phi\nabla^2\phi\right] dx\, dy$$

IN PARTICULAR, IF $\frac{\partial\phi}{\partial n} = 0$ AND $\nabla^2\phi = 0$ THEN

$$\iint_R |\nabla\phi|^2\, dx\, dy = 0 \Rightarrow \nabla\phi\cdot\nabla\phi = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 0,\ \frac{\partial\phi}{\partial y} = 0.$$

A FUNCTION ϕ THAT SATISFIES

IS CALLED A HARMONIC FUNCTIONS