Lecture 15

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Velocity Transformation

RECAP

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}; u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Inverse Velocity Transformation

RECAP

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}}$$

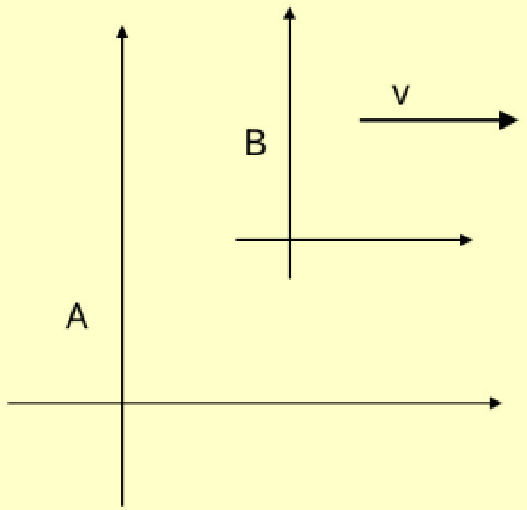
$$u_y = \frac{u'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}; u_z = \frac{u'_z}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

One can show that

- If u<c in S, u<c in S' also irrespective of v.
- If u=c in S, u=c in S' also irrespective of v.

3. Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be inertial frame of reference.

Tutorial **#3**



Assume for the first lighting is at (0,0) for both. For A the second event is at $x_A = d, t_A = t_0$.

t_B and t_A ; time intervals in the two frames

For B

$$t_B = \frac{t_A - vx_A/c^2}{\sqrt{1-\beta^2}} = \frac{t_0 - vd/c^2}{\sqrt{1-\beta^2}} \qquad x_A = d, t_A = t_0$$

But B finds the events to be simultaneous implying

$t_B = 0 \Rightarrow \boxed{v = c^2 t_0/d,}$ so that

$$x' = \gamma(x - vt) \quad x_B = \frac{x_A - vt}{\sqrt{1-\beta^2}} = \frac{d - \frac{c^2 t_o^2}{d}}{\sqrt{1 - \frac{c^4 t_o^2}{c^2 d^2}}} = \frac{d^2 - c^2 t_o^2}{\frac{d\sqrt{d^2 - c^2 t_o^2}}{d}}$$

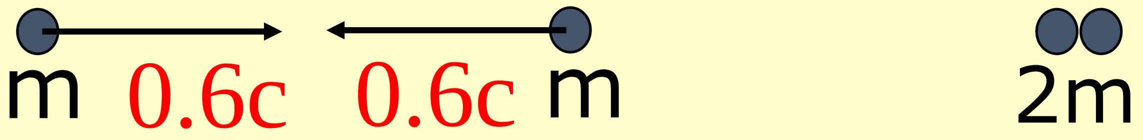
$$x_B = \sqrt{d^2 - c^2 t_o^2}$$

Momentum Conservation

Consider a completely inelastic collision in S frame

Before collision

After collision



In S Frame

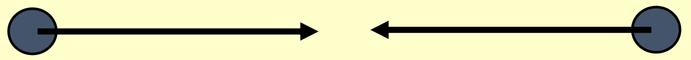
Total initial momentum= 0
Total final momentum= 0

Momentum is conserved.

View this collision from S' frame (belonging to the first particle) with v=0.6c

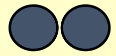
In S’ Frame (First Particles frame)

Before collision



m 0.6c 0.6c m

After collision



2m

The velocity of the first particle before collision ?

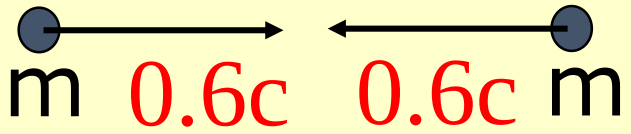
$$u'_{1x} = \frac{0.6c - 0.6c}{1 - 0.36} = 0, \; u'_{1y} = 0, \; u'_{1z} = 0$$

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

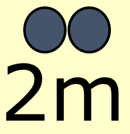
This is as expected

In S' Frame(First Particles frame)

Before collision



After collision



The velocity of the second particle before collision ?

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

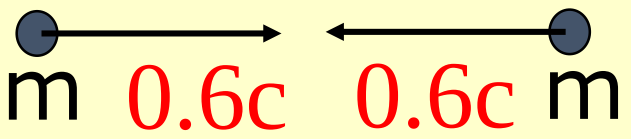
$$u'_{2y} = 0$$

$$u'_{2z} = 0$$

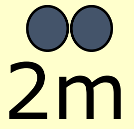
$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

In S’ Frame(First Particles frame)

Before collision



After collision



The velocity of the combined particle after collision ?

$$u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c \qquad u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

$$u'_{fy} = 0$$

$$u'_{fz} = 0$$

In S’ Frame (First Particles frame)

The total initial and the total final momenta in S’ are given as follows.

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c \qquad u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c$$

$$p'_i = -\frac{1.2}{1.36}mc; \; p'_f = -1.2mc$$

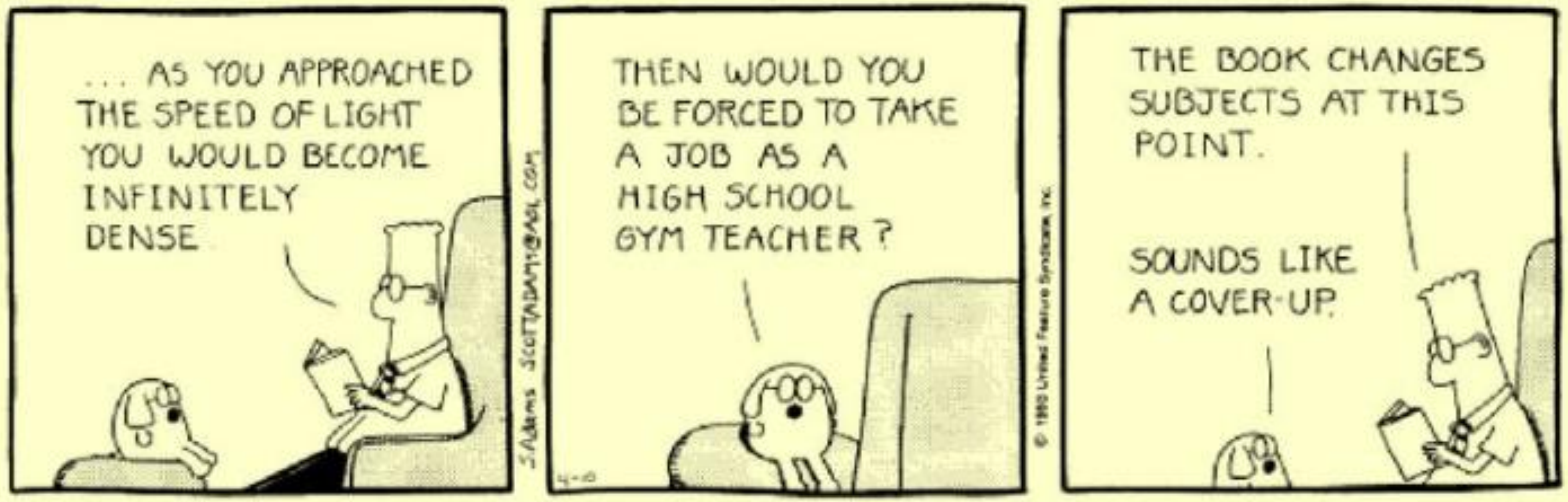
Momentum is NOT conserved in S’!

This result contradicts the basic postulate of special relativity that the laws of physics are the same in all inertial systems. If the conservation of momentum in collisions is to be a law of physics, then the classical definition of momentum cannot be correct in general.

it is possible to preserve the *form* of the classical definition of the momentum of a particle, $\mathbf{p} = m\mathbf{u}$, where $\mathbf{p}$ is the momentum, m the mass, and $\mathbf{u}$ the velocity of a particle, and also to preserve the classical law of the conservation of momentum of a system of interacting particles, providing that we modify the classical concept of mass. We need to let the mass of a particle be a function of its speed u, that is, $m = m_0/\sqrt{1 - u^2/c^2}$, where m_0 is the classical mass and m is the relativistic mass of the particle. Clearly, as u/c tends to zero, m tends to m_0. The relativistic momentum then becomes $\mathbf{p} = m\mathbf{u} = m_0\mathbf{u}/\sqrt{1 - \beta^2}$ and reduces to the classical expression $\mathbf{p} = m_0\mathbf{u}$ as $\beta \to 0$.

The relativistic mass is therefore:

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\, m_0$$



Redefine Momentum in Relativity

We define rest mass m_0, which is the mass of the particle measured in a frame in which it is at rest.

$$p_x \equiv m_o \gamma_u u_x$$

$$p_y \equiv m_o \gamma_u u_y$$

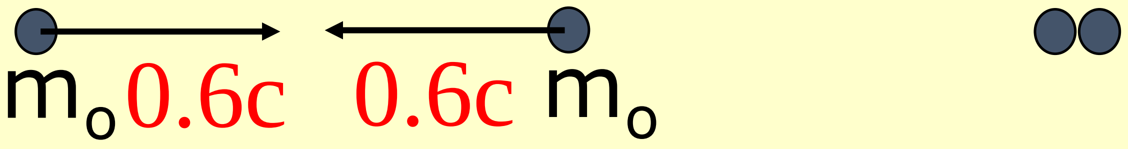
$$p_z \equiv m_o \gamma_u u_z$$

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Revisit the Example – in S frame

Before collision | After collision



$$p_{x1} = m_o \gamma_u u = 1.25 \times 0.6 m_o c$$ since $\gamma_u = 1.25$)

$$= 0.75 m_o c$$

$$p_{x2} = -m_o \gamma_u u = -1.25 \times 0.6 m_o c$$

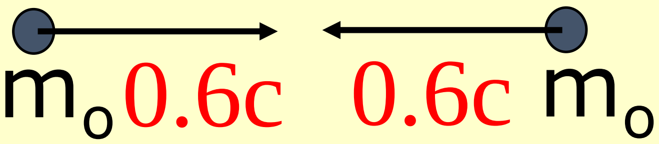
$$= -0.75 m_o c$$

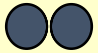
$$\boxed{\sum_k p_{xkI} = 0}$$

Clearly the final momentum is also zero as the speed of the combined particle after collision is zero.

Relativistic Momentum Conserved in S frame

Before collision | After collision





S' Frame

(First Particles frame)

Recall: The velocity of the first particle before collision:

$$u'_{1x} = \frac{0.6c - 0.6c}{1 - 0.36} = 0$$

$$u'_{1y} = 0$$

$$u'_{1z} = 0$$

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

The momentum of the first particle before collision is, therefore, zero.

$$u'_{1x} = \frac{0.6c - 0.6c}{1 - 0.36} = 0, \; u'_{1y} = 0, \; u'_{1z} = 0$$

S' Frame (First Particles frame)

Recall: The velocity of the second particle before collision.

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

$$u'_{2y} = 0$$

$$u'_{2z} = 0$$

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

$$u'_{2y} = 0$$

$$u'_{2z} = 0$$

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

S' Frame (First Particles frame)

The momentum of the second particle before collision is thus.

$$p'_{2x} = -\frac{\dfrac{1.2}{1.36}}{\sqrt{1-\left(\dfrac{1.2}{1.36}\right)^2}} m_0 c$$

$$= -\frac{2.125 \times 1.2}{1.36} m_0 c$$

$$= -1.875 m_0 c$$

$$\sum_k p'_{xkI} = -1.875 m_0 c$$

$$\gamma_u \equiv \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$$

$$p_x \equiv m_o \gamma_u u_x$$

$$p_y \equiv m_o \gamma_u u_y$$

$$p_z \equiv m_o \gamma_u u_z$$

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36} c$$

S' Frame (First Particles frame)

Recall: The velocity of the combined particle after collision.

$$u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c$$

$$u'_{fy} = 0$$

$$u'_{fz} = 0$$

$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

$$u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c$$

$$u'_{fy} = 0$$

$$u'_{fz} = 0$$

S' Frame

Since the rest mass of the combined particle has increased

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0$$

$$M_o = 2.5 m_o$$

The momentum of the combined particle after collision is thus.

$$p'_{xF} = -1.25 \times 0.6 M_0 c$$
$$= -1.25 \times 0.6 \times 2.5 m_0 c$$
$$= -1.875 m_0 c$$

$$u'_{fx} = \frac{0 - 0.6c}{1 - \frac{0 \times 0.6c}{c^2}} = -0.6c$$

since $\quad p = \gamma_u m_0 u)$

since $\quad \gamma_u = 1.25)$

Relativistic Momentum Conserved: Einstein Happy

Energy in Relativity

$$E = m_o \gamma_u c^2 = mc^2$$

$$m \equiv m_o \gamma_u$$

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

What is this energy E?

$$E = m_o \gamma_u c^2 = mc^2$$

$$m \equiv m_o \gamma_u$$

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

When $u = 0, \gamma_u = 1,$

$$\vec{p} = \gamma_u m_0 \vec{u} = 0 \text{ and } E = \gamma_u m_0 c^2 = m_0 c^2$$

When $u << c, \gamma_u \approx 1,$

$$\vec{p} = \gamma_u m_0 \vec{u} \approx m_0 \vec{u} \text{ and } E = \gamma_u m_0 c^2 = m_0 c^2$$

$$p_x \equiv m_o \gamma_u u_x$$

$$p_y \equiv m_o \gamma_u u_y$$

$$p_z \equiv m_o \gamma_u u_z$$

Energy in Relativity

This new form of energy does not resemble any classically known form of energy. It is entirely a new concept of energy.

Energy gain/loss in different form would lead to an increase/decrease in the mass.

"Mass can be expressed in the units of energy, as the two are related through a fundamental constant."

Kinetic Energy

A particle at rest also has an energy called ***Rest Mass Energy***. The increase in its total energy due to motion is called ***Kinetic Energy.***

$$K \equiv mc^2 - m_o c^2$$

Kinetic Energy

$$E = m_o \gamma_u c^2 = mc^2$$

$$m \equiv m_o \gamma_u$$

$$K = \frac{m_o c^2}{\sqrt{1 - u^2 / c^2}} - m_o c^2$$

$$= m_o c^2 (1 - u^2 / c^2)^{-1/2} - m_o c^2$$

$$\approx m_o c^2 (1 + u^2 / 2c^2) - m_o c^2$$

$$= \frac{1}{2} m_o u^2$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Novel Definitions

$$\vec{p} = m\vec{u}$$

$$E = mc^2$$

$$K = mc^2 - m_o c^2$$

$$m = \frac{m_o}{\sqrt{1 - u^2 / c^2}} \Rightarrow m \equiv m_o \gamma_u$$

Energy Momentum Relation

$$p_x \equiv m_o \gamma_u u_x$$

$$E = m_o \gamma_u c^2 = mc^2$$

EVALUATE: $p^2 - \dfrac{E^2}{c^2}$

$$= m_o^2 \gamma_u^2 u^2 - m_o^2 \gamma_u^2 c^2$$

$$= m_o^2 \gamma_u^2 (u^2 - c^2)$$

$$= -m_o^2 c^2$$

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}}$$

$$\boxed{E^2 = p^2 c^2 + m_0^2 c^4}$$

Summary of formulae

$$\mathbf{p} = m\mathbf{u} = m_0\mathbf{u}\gamma$$

$$K = mc^2 - m_0c^2 = m_0c^2(\gamma - 1)$$

$$E = mc^2 = m_0c^2\gamma$$

$$E^2 = (pc)^2 + (m_0c^2)^2.$$

Will Revisit this mass energy Equivalence in the next lecture

6. A rod flies with constant velocity past a mark, which is stationary in reference frame S. In reference frame S, it takes 20 ns for the rod to fly past the mark. In the reference frame fixed to the rod, S', the mark moves past the rod for 25 ns. Find the length of the rod in S and S' and the speed of S' with respect to S.

Let u be the speed of rod in S frame

$\therefore$ length of the rod in S $= u \times 20 \times 10^{-9}$

In rod frame the length of rod $= u \times 25 \times 10^{-9}$

In the frame of rod, rod is stationary

$\therefore$ $u \times 25 \times 10^{-9}$ is proper length

$\therefore$ $u \times 25 \times 10^{-9} = \gamma\, u \times 20 \times 10^{-9}$

$\therefore$ $\gamma = 25/20 \Rightarrow u = 0.6c$

$\Rightarrow l = 0.6 \times 3 \times 10^{8} \times 20 \times 10^{-9} = 3.6\,m$

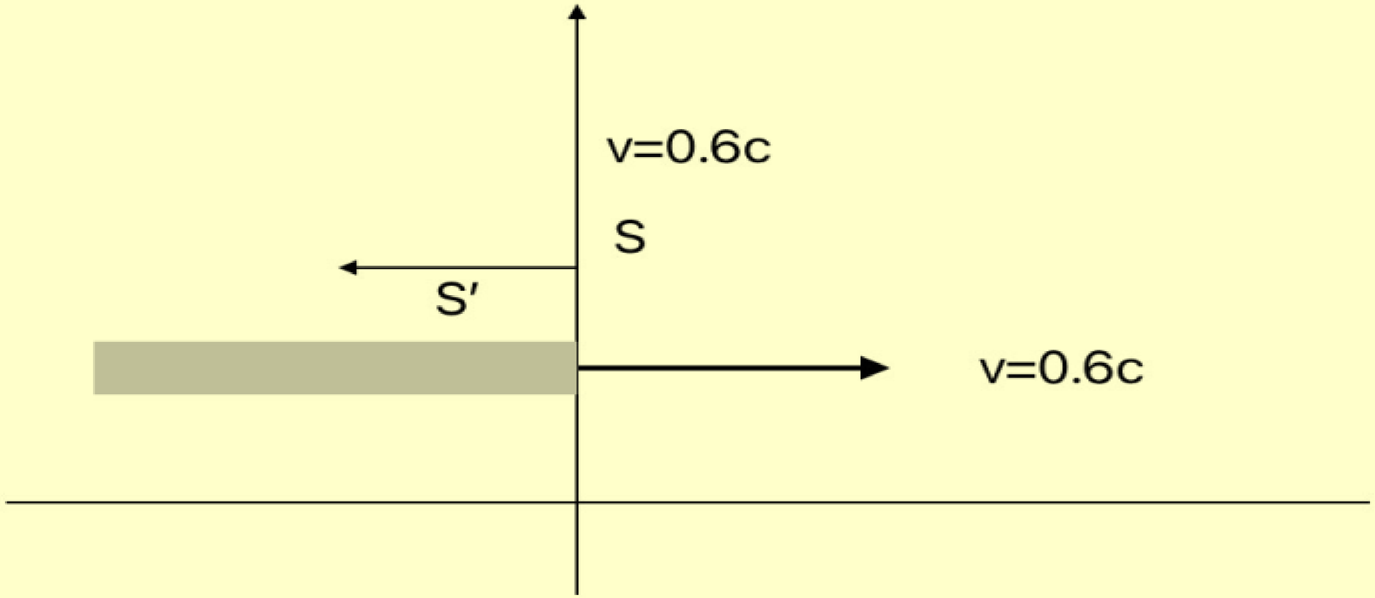
$l' = 1.25 \times 3.6 = 4.5$

7. A rod of length 60 cm in its rest frame is traveling along its length with a speed of $0.6c$ in the frame S. A particle moving in the opposite direction to the rod, with a speed $0.6c$ in S, passes the rod. How much time will the particle take to cross the rod

(a) in the frame S.

(b) in the rest frame of the particle.

Soln:



Let the rod be at rest in S' frame, with its tip at $x' = 0$ and the back end at $x' = -L_0$

In S, the back end position x is computed as

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \Rightarrow x = -L_0\sqrt{1-\beta^2}, \text{ at } t = 0$$

At $t = 0$ the particle is at $x = 0$ in S frame. The time (in S frame) at which the particle will cross the back end of the rod

$$t = \frac{L_0\sqrt{1-\beta^2}}{2v} = \frac{L_0}{2c}\sqrt{\frac{1-\beta^2}{\beta^2}}$$

$$= \frac{60}{100} \times \frac{1}{2 \times 3 \times 10^8}\sqrt{\frac{64}{36}} = \boxed{\frac{4}{3} \times 10^{-9} \text{ seconds}}$$

The crossing happens at $x = -vt$ with $v = 0.6c$
The time in particle's frame

$$t'' = \frac{t + vx/c^2}{\sqrt{1-\beta^2}} = \frac{t\left(1 - v^2/c^2\right)}{\sqrt{1-\beta^2}} = t\sqrt{1-\beta^2}$$

$$= \frac{4}{3} \times 10^{-9}\sqrt{1-(0'6)^2} = \frac{4}{3} \times 10^{-9} \cdot \frac{4}{5} = \boxed{\frac{16}{15} \times 10^{-9}\text{s}}$$

8. Two spacehips pass each other, travelling in opposite directions. The speed of ship B, measured by a passenger in ship A is $0.96c$. This passenger has measured the length of the ship A as 100 mt and determines that the ship B is 30 mt long. What are the lengths of the two ships as measured by a passenger in ship B ?

All the data is given in the frame of ship 'A'. Proper length of ship A is 100m. On the other hand length of ship B is contracted in 'A' and this is 30m.

$$\gamma = \frac{1}{\sqrt{1-0.96)^2}} = \frac{1}{0.28}$$

$\therefore$ length of A in B (contracted) $= 100 \times 0.28 = 28\,m$

length of B in B (Proper) $= \frac{30}{0.28} \simeq 107.1\,m$

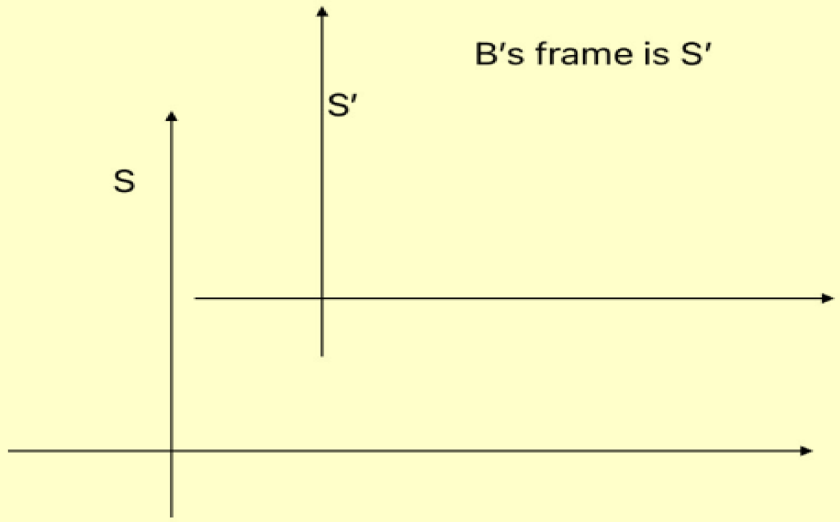
9. An observer O is at the origin of an inertial frame. He notices a vehicle A to pass by him in $+x$ direction with constant speed. At this instant, the watch of the observer O and the watch of the driver of A show time equal to zero. 50 μs after A passed by, O sees another vehicle B pass by him, also in $+x$ direction and again with constant speed. After sometime B catches A and sends a light signal to O, which O receives at 200 μs according to his watch. The driver of B notices that, in his frame, the time between passing O and catching A is 90 μs Assume that drivers A and B are at the origins of their respective frames. Find

(a) the speeds of B and A, in the frame of O.

(b) position of A in O's frame when B passes O.

(c) the position of O in the frame of A, when B passes O.

Soln:



At $t = 0$ $\quad A$ crosses the origin with speed v_A in frame S

$\Rightarrow$ his position $x_A = v_A t$

At $t = t_B$ B crosses the origin with speed v_B

$\Rightarrow$ his position $x_B = v_B(t - t_B) \quad t_B = 50\mu s$

B catches up with A when

$$v_A t = v_B(t - t_B) \Rightarrow t = \frac{v_B t_B}{v_B - v_A}$$

He releases a light signal that reaches origin at $200\mu\mathrm{s}$

$$\therefore t+\frac{1}{c}v_A t=200\mu\mathrm{s}, \qquad \frac{v_B t_B}{v_B-v_A}\times\left(1+\frac{v_A}{c}\right)=200\mu\mathrm{s}$$

$$\Rightarrow \frac{1}{1-v_A/v_B}\left(1+\frac{v_A}{C}\right)=4\ldots\ (1)$$

In the frame of B, he catches up with A 90 μs after crossing the origin of S

In his frame this happens at $x'=0,\ t'=90\,\mu s$

In the S frame the interval was $(t-t_B)$ since B crossed the origin

$$t-t_B=\frac{v_B t_B}{v_B-v_A}-t_B=\frac{t'+v_B x'/c^2}{\sqrt{1-v_B^2/c^2}}$$

$$\left(\frac{v_B}{v_B-v_A}-1\right)t_B=\frac{t'+0}{\sqrt{1-v_B^2/c^2}}$$

$$\frac{1}{\frac{v_B}{v_A}-1}=\frac{90}{50}\times\frac{1}{\sqrt{1-v_B^2/c^2}}$$

$$1+\beta_A = 4\left(1-\frac{\beta_A}{\beta_B}\right)$$
$$\frac{\beta_B}{\beta_A} - 1 = \frac{5}{9}\sqrt{1-\beta_B^2}$$

Solve for β_A and β_B from there two

$$\beta_A\left(1+\frac{4}{\beta_B}\right) = 3 \Rightarrow \beta_A = \frac{3\beta_B}{4+\beta_B}$$
$$\Rightarrow \frac{\beta_B}{\beta_A} - 1 = \frac{4+\beta_B}{3} - 1 = \frac{1+\beta_B}{3}$$

Put this in (2)

$$\frac{(1+\beta_B)^2}{9} = \frac{25}{81}\left(1-\beta_B^2\right)$$
$$9\left(1+\beta_B\right) = 25\left(1-\beta_B\right)$$
$$34\beta_B = 16$$

Or

$$\boxed{\beta_B = \frac{8}{17}}$$

$$\beta_A = \frac{3\beta_B}{4+\beta_B} = \frac{3.8}{17\left(4+\frac{8}{17}\right)}$$
$$= \frac{24}{17} \times \frac{17}{76} = \frac{6}{19} \Rightarrow \boxed{\beta_A = \frac{6}{19}}$$

The position of A in S frame when B crosses origin

$$x_A = v_A t_B = \frac{6}{19} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6}$$
$$= \frac{90}{19} \times 10^3 \mathrm{m} \Rightarrow \boxed{\frac{90}{19}\ \mathrm{km}}$$

The position of the origin of S in $A'S$ frame when B crosses 0.

$$\begin{cases} x & = 0 \\ t & = t_B \end{cases} x'_A = \frac{x - v_A t}{\sqrt{1-\beta_A^2}}$$

$$\begin{aligned} x'_A = \frac{0 - v_A t_B}{1-\beta_A^2} &= -\frac{\beta_A}{\sqrt{1-\beta_A^2}}(ct_B) \\ &= -\frac{6}{19}\frac{1}{\sqrt{1-(6/19)^2}} \cdot 3 \times 10^8 \cdot 50 \times 10^{-6} \\ &= -\frac{6}{19} \cdot \frac{19}{\sqrt{325}} \cdot 15 \times 10^3 \\ &= -\frac{6 \times 15 \times 3}{5\sqrt{13}} \cdot 10^3 \mathrm{m} = \boxed{-4.992\,\mathrm{km}} \end{aligned}$$