MA 108-ODE- D3

Lecture 18

Debanjana Mitra

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Laplace transform

Appendix: Improper integrals

Laplace Transforms: Recall

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform $\mathcal{L}(f)$ of f is the function defined by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t)dt,$$

for all values of s for which the integral exists.

Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$ and is of exponential order. Moreover, if the piecewise continuous function f is of exponential order a, for some $a \in \mathbb{R}$, then the $\mathcal{L}(f)(s)$ exists for all $s > a$.

Denote by $F(s) = \mathcal{L}(f)(s)$.
The inverse Laplace transform (if defined) f of F is denoted by $f = \mathcal{L}^{-1}(F)$.

Examples

1. $\mathcal{L}(1)(s) = \frac{1}{s}, s > 0.$
2. $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}, s > a.$
3. $\mathcal{L}(\sin at)(s) = \frac{a}{s^2+a^2}, s > 0.$
4. $\mathcal{L}(\cos at)(s) = \frac{s}{s^2+a^2}, s > 0.$
5. $\mathcal{L}(\sinh at)(s) = \frac{a}{s^2-a^2}, s > a \geq 0.$
6. $\mathcal{L}(\cosh at)(s) = \frac{s}{s^2-a^2}, s > a \geq 0.$
7. For $p > -1, \mathcal{L}(t^p)(s) = \frac{\Gamma(p+1)}{s^{p+1}}, s > 0.$

Laplace Transforms

For $c \geq 0$, the function

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

is called the unit step function or the Heaviside function.

Laplace Transforms

Example: Write the following piecewise continuous function in terms of Heaviside functions:

$$f(t) = \begin{cases} 2 & t \in [0,4) \\ 5 & t \in [4,7) \\ -1 & t \in [7,9) \\ 1 & t \geq 9. \end{cases}$$

Note that $u_c - u_d$ takes 1 on $[c, d)$ and 0 everywhere else. Thus,

$$\begin{aligned} f(t) &= 2(u_0 - u_4) + 5(u_4 - u_7) - (u_7 - u_9) + u_9 \\ &= 2u_0 + 3u_4 - 6u_7 + 2u_9. \end{aligned}$$

Laplace Transforms

Does the Heaviside has a Laplace?

$$\begin{aligned}\mathcal{L}(u_c(t))(s) &= \int_0^\infty e^{-st} u_c(t)dt \\ &= \int_c^\infty e^{-st} dt \\ &= \frac{e^{-cs}}{s},\end{aligned}$$

for $s > 0$.

Laplace Transforms

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Consider the new function

$$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c. \end{cases}$$

Note that

$$g(t) = u_c(t) f(t-c).$$

Laplace Transforms

Theorem

Suppose $\mathcal{L}(f(t))(s) = F(s)$ *for* $s > a \geq 0$. *If* $c > 0$, *then for* $s > a$,

$$\mathcal{L}\left(u_c(t)f(t-c)\right)(s) = e^{-cs}F(s).$$

Proof:

$$\begin{aligned}
\mathcal{L}\left(u_c(t)f(t-c)\right)(s) &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&= \int_0^\infty e^{-s(u+c)}f(u)du \\
&= e^{-cs}\int_0^\infty e^{-su}f(u)du \\
&= e^{-cs}F(s).
\end{aligned}$$

Laplace Transforms

Example: Find the Laplace transform of

$$f(t) = \begin{cases} \sin t & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4}. \end{cases}$$

Write

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

Hence,

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(\sin t) + \mathcal{L}(u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})) \\ &= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1} \\ &= \frac{1 + e^{-\frac{\pi}{4}s}s}{s^2+1}. \end{aligned}$$

Laplace Transforms

Example: Solve the IVP:

$$2y'' + y' + 2y = u_5(t) - u_{20}(t),\ \ y(0) = 0, y'(0) = 0.$$

Take Laplace transforms:

$$2(s^2\mathcal{L}(y) - sy(0) - y'(0)) + (s\mathcal{L}(y) - y(0)) + 2\mathcal{L}(y) = \mathcal{L}(u_5(t) - u_{20}(t));$$

i.e.,

$$(2s^2 + s + 2)\mathcal{L}(y)(s) = \frac{e^{-5s} - e^{-20s}}{s}.$$

Put

$$H(s) = \frac{1}{s(2s^2 + s + 2)},$$

and

$$\mathcal{L}(h)(s) = H(s).$$

Laplace Transforms

Then,

$$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20).$$

To find h, write the partial fraction expansion of H:

$$\frac{1}{s(2s^2+s+2)} = \frac{a}{s} + \frac{bs+c}{2s^2+s+2}.$$

Check:

$$a = \frac{1}{2}, b = -1, c = -\frac{1}{2}.$$

Thus,

$$\begin{aligned} H(s) &= \frac{1/2}{s} + \frac{(-s-\frac{1}{2})}{2s^2+s+2} \\ &= \frac{1/2}{s} - \frac{1}{2} \cdot \frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}. \end{aligned}$$

Laplace Transforms

Thus,

$$H(s) \quad = \quad \frac{1/2}{s} - \frac{1}{2} \cdot \frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{8}\frac{4}{\sqrt{15}} \cdot \frac{\sqrt{15}/4}{(s+\frac{1}{4})^2 + \frac{15}{16}}.$$

Recall: $\mathcal{L}(1) = \dfrac{1}{s}$, $\mathcal{L}(\cos at) = \dfrac{s}{a^2+s^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$.
Therefore,

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{t}{4}} \cos \frac{\sqrt{15}t}{4} - \frac{1}{2\sqrt{15}}e^{-\frac{t}{4}} \sin \frac{\sqrt{15}t}{4}.$$

$$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20).$$

Note that the function $y(t)$ is defined everywhere and $y'(t)$ exists everywhere, but $y''(t)$ does not exist at $t = 5, 20$. You could have found the solutions on the intervals $(0,5), (5,20), (20,\infty)$ directly by earlier methods as well.

Laplace Transforms

Observation: Suppose f is piecewise continuous on all $[0, \infty)$ and of exponential order a. (i.e., there exist positive constants K, t_0 such that

$$|f(t)| \leq Ke^{at}, \quad \forall\, t > t_0 > 0.$$

Note that piecewise continuity of f implies that

$$|f(t)| \leq K_*,$$

for all $t \in [0, t_0]$ for some positive constant K_*. Thus, on $[0, t_0]$,

$$|f(t)| \leq K_{**}e^{at},$$

for some postive constant K_{**}. (Why?) Choose $M = \max(K, K_{**})$ to conclude.

Laplace Transforms

Thus,

$$|\mathcal{L}(f)(s)| \leq \int_0^\infty |e^{-st} f(t)| dt \leq M \int_0^\infty e^{-(s-a)t} dt = \frac{M}{s-a},$$

for $s > a$.
In particular, it follows that

$$\mathcal{L}(f)(s) \to 0,$$

as $s \to \infty$.
Remark: This limiting behaviour is true for any f for which $\mathcal{L}(f)$ exists; i.e., even without assuming exponential order etc. In particular, $\frac{s-1}{s+1}, \frac{e^s}{s}, s^2, \frac{s}{\ln s}$ etc cannot be the Laplace transform of any function!

Example

Example: Solve $y'' + ty' - 2y = 4,\ y(0) = -1,\ y'(0) = 0.$

Take Laplace transform

$$\mathcal{L}(y'')(s) + \mathcal{L}(ty')(s) - 2\mathcal{L}(y)(s) = \mathcal{L}(4)(s),$$

so that

$$\left(s^2\mathcal{L}(y)(s) - sy(0) - y'(0)\right) - \frac{d}{ds}\mathcal{L}(y')(s) - 2\mathcal{L}(y)(s) = \frac{4}{s}.$$

I.e. $s^2\mathcal{L}(y)(s) - sy(0) - y'(0)) - (s\mathcal{L}(y)(s) - y(0))' - 2\mathcal{L}(y)(s) = \dfrac{4}{s}.$

Writing $Y(s) = \mathcal{L}(y)(s)$, we obtain

$$s^2 Y(s) + s - (Y(s) + sY'(s)) - 2Y(s) = \frac{4}{s}.$$

$$Y'(s) + (\frac{3}{s} - s)Y(s) = 1 - \frac{4}{s^2}.$$

Solving this DE, we obtain :

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{c}{s^3}e^{s^2/2},$$

where c is a constant.

Example Continued

Solving this DE, we obtain :

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{c}{s^3} e^{s^2/2},$$

where c is a constant.
By using the remark in the previous slide, we have $c = 0$!
($\lim_{s \to \infty} Y(s) = 0$.) Hence,

$$Y(s) = \frac{2}{s^3} - \frac{1}{s},$$

which implies that

$$y(t) = t^2 - 1.$$

Laplace Transforms: Initial value at $t_0 \neq 0$

Example: Solve the IVP:

$$y'' + y = 2t,\ y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\ y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$$

Set $t = \tilde{t} + \frac{\pi}{4}$, so that the problem is given by

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{\pi}{4}\right),\ \tilde{y}\,(0) = \frac{\pi}{2},\ \tilde{y}'\,(0) = 2 - \sqrt{2},$$

where $\tilde{y}(\tilde{t}) = y(t)$. Taking the Laplace transform,

$$s^2\mathcal{L}(\tilde{y})(s) - s\tilde{y}(0) - \tilde{y}'(0) + \mathcal{L}(\tilde{y})(s) = 2\frac{\Gamma(2)}{s^2} + \frac{\pi}{2s}.$$

Thus,

$$(s^2+1)\mathcal{L}(\tilde{y})(s) = \frac{2}{s^2} + \frac{\pi}{2s} + \frac{\pi s}{2} + 2 - \sqrt{2}.$$

Example Continued

$$\mathcal{L}(\tilde{y})(s) = \frac{2}{s^2(s^2+1)} + \frac{\pi}{2s(s^2+1)} + \frac{\pi s}{2(s^2+1)} + \frac{2-\sqrt{2}}{(s^2+1)}.$$

Taking the inverse Laplace transform, we get

$$\begin{aligned}\tilde{y}(\tilde{t}) &= 2\left(\tilde{t} - \sin\tilde{t}\right) + \frac{\pi}{2}\left(1 - \cos\tilde{t}\right) + \frac{\pi}{2}\cos\tilde{t} + \left(2 - \sqrt{2}\right)\sin\tilde{t} \\ &= 2\tilde{t} + \frac{\pi}{2} - \sqrt{2}\sin\tilde{t}.\end{aligned}$$

Since $\tilde{t} = t - \frac{\pi}{4}$, it follows that

$$\sin\tilde{t} = \sin\left(t - \frac{\pi}{4}\right) = \frac{\sin t - \cos t}{\sqrt{2}},$$

so that

$$y(t) = 2t - \sin t + \cos t.$$

Solve a system of DEs using Laplace transform

Example: (Q4, Tutorial sheet 5) Solve

$$\begin{aligned} 2y_1' - y_2' - y_3' &= 0, \\ y_1' + y_2' &= 4t + 2, \\ y_2' + y_3 &= t^2 + 2; \\ y_1(0) = 0, y_2(0) = 0, y_3(0) &= 0. \end{aligned}$$

Taking Laplace transforms, we have

$$\begin{aligned} 2s\mathcal{L}(y_1)(s) - s\mathcal{L}(y_2)(s) - s\mathcal{L}(y_3)(s) &= 0 \\ s\mathcal{L}(y_1)(s) + s\mathcal{L}(y_2)(s) &= \frac{4}{s^2} + \frac{2}{s} \\ s\mathcal{L}(y_2)(s) + \mathcal{L}(y_3)(s) &= \frac{2}{s^3} + \frac{2}{s}. \end{aligned}$$

Solving:

$$\mathcal{L}(y_1)(s) = \frac{2}{s^3}, \quad \mathcal{L}(y_2)(s) = \frac{2}{s^3} + \frac{2}{s^2}, \quad \mathcal{L}(y_3)(s) = \frac{2}{s^3} - \frac{2}{s^2}.$$

Thus,

$$y_1(t) = t^2, \quad y_2(t) = t^2 + 2t, \quad y_3(t) = t^2 - 2t.$$

Laplace transform: Solving integro-differential equation

Tutorial sheet 5, Qn 16 (iii): Solve:
$y'(t) = 1 - \int_0^t y(t-\tau)\, d\tau, \quad y(0) = 1.$
Ans. Note that the integral in the RHS

$$\int_0^t y(t-\tau)\, d\tau = (y * h)(t), \quad \text{where} h(t) = 1.$$

Applying Laplace transform to the integro-differential equation,

$$\begin{aligned}
&\mathcal{L}(y')(s) = \mathcal{L}(1)(s) - \mathcal{L}(y * h)(s) = \tfrac{1}{s} - \mathcal{L}(y)(s)\mathcal{L}(h)(s) \\
&s\mathcal{L}(y)(s) - 1 = \tfrac{1}{s} - \tfrac{1}{s}\mathcal{L}(y)(s).
\end{aligned}$$

Thus, $\left(s + \frac{1}{s}\right)\mathcal{L}(y)(s) = 1 + \frac{1}{s}$, and then

$$\mathcal{L}(y)(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1} = \mathcal{L}(\cos t) + \mathcal{L}(\sin t).$$

Therefore, $y(t) = \cos t + \sin t$.

Properties

For large enough s, at which value Laplace transform of functions given below exist:

1.	Linearity	$\mathcal{L}(af+bg)(s) = a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s)$
2.	Scaling	$\mathcal{L}(f(ct))(s) = \frac{1}{c}F\left(\frac{s}{c}\right),\ c>0$
3.	Laplace transform of derivative	$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ $\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$
4.	L.T. of integral	$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)(s) = \frac{F(s)}{s}$.
5.	Dervative of L.T.	$F'(s) = -\mathcal{L}(tf(t))(s)$ $\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)$
6.	Integral of L.T.	$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty F(\tilde{s})\,d\tilde{s}$.
7.	shifting theorem	$\mathcal{L}(u_c(t)f(t-c))(s) = e^{-cs}F(s)$
8.	Convolution & L.T.	$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f*g)(s) = \mathcal{L}(f)(s)\cdot\mathcal{L}(g)(s)$

Here $F(s) = \mathcal{L}(f)(s)$.

Appendix

Additional notes for those interested

The Improper Integral of the first kind : Recall

- Let $f : [a, \infty) \to \mathbb{R}$ be a function. If f is such that, for any $b \geq a$, $f : [a, b] \to \mathbb{R}$ is piecewise continuous, then we say that f is piecewise continuous on $[a, \infty)$.
- Note that such an f is bounded on $[a, b]$ for every $b \geq a$.
- Note that, for f as above, the usual Riemann integral

  $$I(b) = \int_a^b f(x)\, dx$$

  exists for any $b \geq a$.

Definition

An improper integral of first kind of the function f with the property mentioned above is defined to be

$$\int_a^\infty f(x)\, dx := \lim_{b \to \infty} \int_a^b f(x)\, dx,$$

if this limit exists.

If the above limit exists, we say that $\int_a^\infty f(x)\, dx$ converges, otherwise it is said to diverge.

The Improper Integral of the first kind

Note that we can define similarily

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx,$$

provided the limit exists.

Definition

The integral $\int_{-\infty}^{\infty} f(x)\, dx$ is said to be convergent if there is a $c \in \mathbb{R}$ such that $\int_{-\infty}^{c} f(x)\, dx$ is convergent and $\int_{c}^{\infty} f(x)\, dx$ is convergent. We then define

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx.$$

Exercise: Show that the above definition is independent of the choice of c.

The Improper Integral of the first kind

Let f be piecewise continuous on $[a, \infty)$.

Theorem (Convergence Tests for Improper Integral)

Suppose there is a real number $M > 0$ such that

$$\int_a^b |f(x)|\, dx \leq M$$

for every $b \geq a$. Then $\int_a^\infty f(x)\, dx$ and $\int_a^\infty |f(x)|\, dx$ are convergent.

The Improper Integral of the first kind

Assume that f is piecewise continuous on $[a, \infty)$.

Theorem (Comparison Test)

Suppose $0 \le f(x) \le g(x)$ *for every* $x \ge a$*. If* $\int_a^\infty g(x)\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *also converges and*

$$\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx.$$

Example: As

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$$

on $[1, \infty)$, and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, it follows that $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ also converges.

Theorem

If $\int_a^\infty |f(x)|\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *converges.*

The Improper Integral of the first kind

Theorem (Limit Comparison Test)

Suppose $f(x) \geq 0$ and $g(x) > 0$ on $[a, \infty)$. Suppose that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist for every $b \geq a$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

(i) *If $c \neq 0$, then either both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge.*

(ii) *If $c = 0$, and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*

Example: Consider $\int_1^\infty e^{-x} x^s\,dx$ for $s \in \mathbb{R}$.
Note that

$$\lim_{x \to \infty} \frac{e^{-x} x^s}{x^{-2}} = 0$$

and $\int_1^\infty \frac{dx}{x^2}$ converges. Hence, by the above theorem $\int_1^\infty e^{-x} x^s dx$ converges for every $s \in \mathbb{R}$.

Gamma Function

Define the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(y) = \int_0^\infty e^{-x} x^{y-1} dx.$$

How do we know that the right hand side integral converges? Can write it as

$$\int_0^1 e^{-x} x^{y-1} dx + \int_1^\infty e^{-x} x^{y-1} dx,$$

and we need to check that both these integrals do converge. Why do these integrals converge?
Hint: For the first, $e^{-x} x^{y-1} \leq x^{y-1}$. Thus, the first integral $\leq \frac{1}{y}$, and thus converges.

Hence, $\Gamma(y)$ is well-defined for $y > 0$.

Gamma Function

The gamma function satisfies a nice functional equation:

$$\Gamma(y+1) = y\Gamma(y) \text{ for } y > 0.$$

Proof: Let $0 < a < b$. Use integration by parts to see:

$$\begin{aligned}\int_a^b e^{-x}x^y dx &= [-x^y e^{-x}]_a^b + y\int_a^b e^{-x}x^{y-1}dx \\ &= a^y e^{-a} - b^y e^{-b} + y\int_a^b e^{-x}x^{y-1}dx.\end{aligned}$$

Take limit as $b \to \infty$ and $a \to 0^+$ to get

$$\int_0^\infty e^{-x}x^y dx = y\Gamma(y),$$

i.e., $\Gamma(y+1) = y\Gamma(y)$. In particular, for $n = 1, 2, \ldots$

$$\Gamma(n+1) = n!.$$

Check! Use Induction to verify the above equation.
Thus, the gamma function interpolates the factorial function.

Example

Exercise: Prove that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Note that e^{-x^2} is continuous for every x and $\int_0^1 e^{-x^2} dx$ is a proper integral. We need to check that the improper integral $\int_1^\infty e^{-x^2} dx$ converges. To see this, note that

$$\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}.$$

Hence, $\int_1^\infty e^{-x^2} dx$ converges. To find it's value, note that

$$I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$$

so that

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$$

Hint: Compute I^2 as a double integral by changing to polar coordinates.

Contd...

By definition,

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx.$$

Put $x = t^2$. Thus,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$