MA 109, Week-4

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Riemann Integration

Aim: To integrate a bounded function f on a closed and bounded interval $[a, b]$.

Recall: (Least upper bound axiom) If a set of real numbers is bounded above, it has a supremum (or least upper bound). If a set of real numbers is bounded below it has an infimum (or greatest lower bound).

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $[a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$. Indeed $I = \cup_j I_j$ and if two sub-interval intersect, they have at most one point in common. Hence, the notation "partition".

Definition: Let

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

be a partition of $[a, b]$. A partition

$$P' = \{a = x'_0 < x'_1 < \ldots < x'_m = b\}$$

is said to be a refinement of the partition P if $P \subseteq P'$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions have a common refinement.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a bounded function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n.$$

Why M_i, m_i exists?

Defintion: We define the Lower sum as

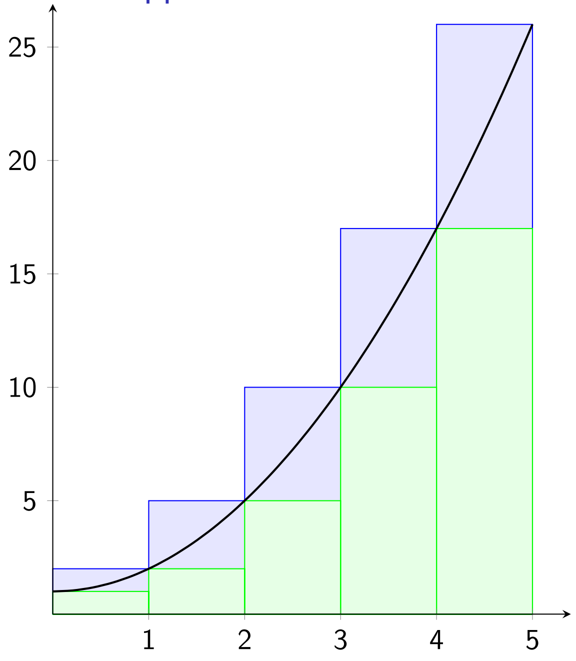
$$L(f, P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

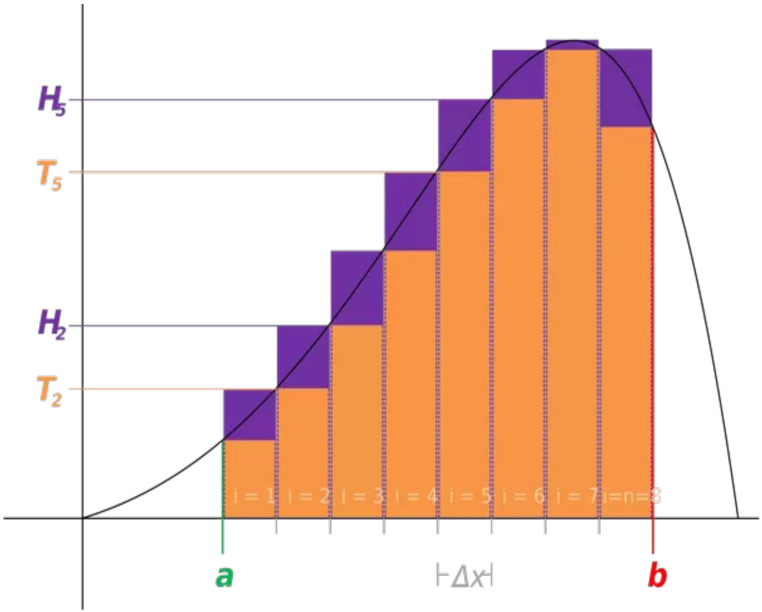
$$U(f, P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}).$$

This is easy to see - the lower sum computes the sum of the areas of rectangles that lie entirely below the curve while the upper sum computes the sum of the areas of rectangles whose "tops" lie above the curve.

Lower and Upper Riemann sums



A picture for a non-monotonic function



One basic example

Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions on an interval is a partition that divides the interval into sub-intervals of equal length. For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \ldots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$ take its infimum?

Clearly, the infimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the supremum $M_j = \frac{j}{n}$ at $\frac{j}{n}$.

Calculate the lower sum $L(P_n, f)$ and the upper sum $U(P_n, f)$.
An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

For any partition P of $[a, b]$, $L(f, P) \leq U(f, P)$.

Let P_1 be partitions of $[a, b]$. Let $P_2 = P_1 \cup \{x_0\}$ where $x_0 \notin P_1$. Then compare $L(f, P_1,), L(f, P_2)$?

$$L(f, P_1) \leq L(f, P_2)$$

and

$$U(f, P_1) \geq U(f, P_2).$$

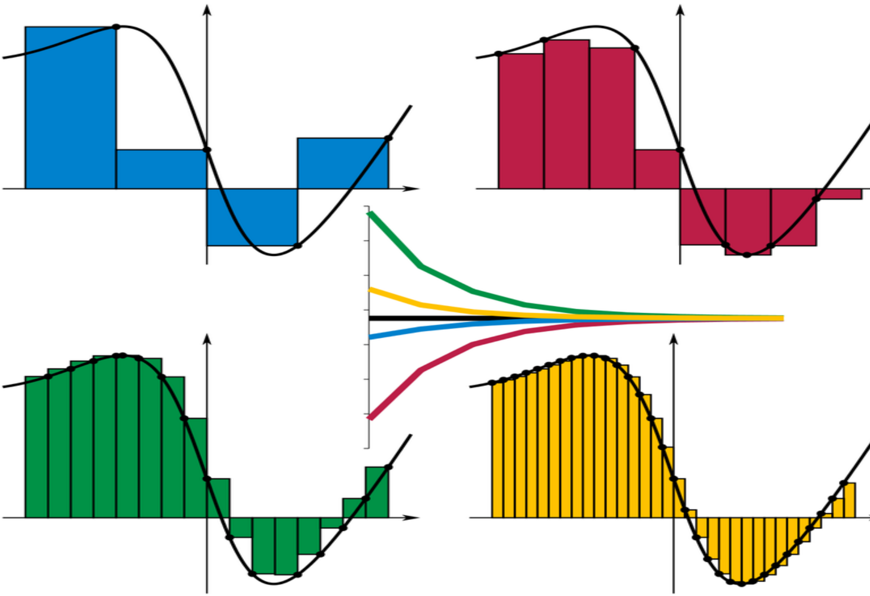
If P_1, P_2 are any two partitions with $P_1 \subseteq P_2$, then

$$L(f, P_1) \leq L(f, P_2)$$

and

$$U(f, P_1) \geq U(f, P_2).$$

Effect of refinement of partition



The Riemann integrals

We now define the lower Riemann integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$ (why $L(f)$ exists?)

and similarly the upper Riemann integral of f by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$. (why $U(f)$ exists?)

If $L(f) = U(f)$, then we say that f is Riemann-integrable and define

$$\int_a^b f(t)\,dt := U(f) = L(f).$$

This common value of the two integrals is called the Riemann integral.

Back to the example

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ in the example we gave (i.e. $f(x) = x, x \in [0, 1]$).

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similary, we can check that

$$U(f, P_n) = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Then

$$\sup_{n \in \mathbb{N}} L(f, P_n) = \frac{1}{2} = \inf_{n \in \mathbb{N}} U(f, P_n).$$

Hence

$$\sup_{n \in \mathbb{N}} L(f, P_n) \le L(f) \le U(f) \le \inf_{n \in \mathbb{N}} U(f, P_n).$$

Therefore

$$L(f) = U(f) = \frac{1}{2}.$$

Riemann Sums

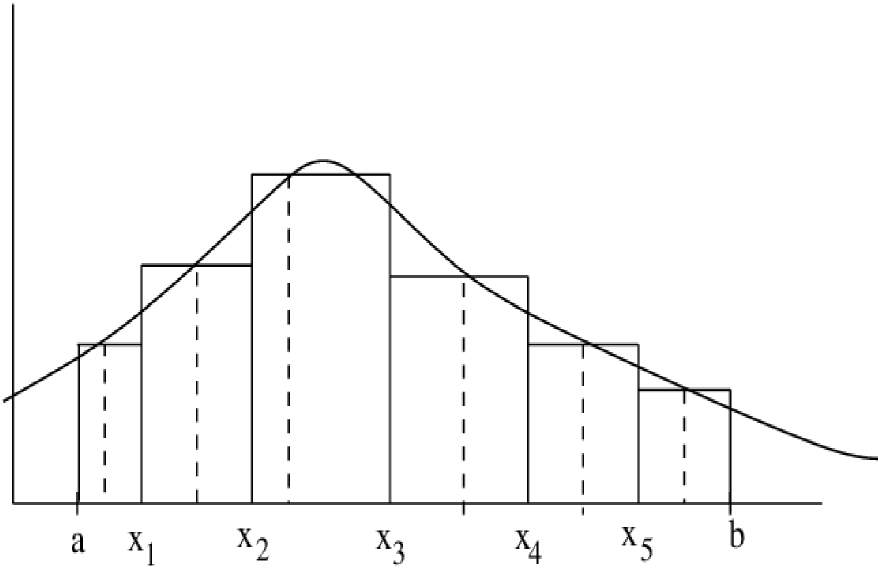
There is another way of getting at the integral which may be a little more intuitive and is better for calculation. This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

Riemann sums



The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise. We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_j \{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.

The Riemann integral

Theorem A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$, for any t. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if

$$\lim_{\|P\| \to 0} R(f, P, t)$$

exists and in that case it is equal to $\int_a^b f(t)\, dt$.

Notice that, as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

The Riemann integral continued

Intuitively, we can see that the smaller or finer the partition, the better the area under the curve is represented by the Riemann sum.

The reason that the Riemann integral is useful is because the definition we have given is actually equivalent to the following apparently weaker definition.

Back to our example

Using Riemann sum it is easy to see that the function $f(x) = x$ is Riemann integrable.
Let $P = P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$.

The Riemann sum is trapped between the upper and lower sums:

$$L(f, P_n) = \frac{1}{2} - \frac{1}{2n} \leq R(f, P_n, t) \leq U(f, P_n) = \frac{1}{2} + \frac{1}{2n},$$

for any choice t of points in the intervals I_j.
Therefore

$$\lim_{n\to\infty} R(f, P_n, t) = \frac{1}{2}.$$

The main theorem for Riemann integration

The main theorem of Riemann integration is the following:
Theorem 21: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In fact, one can allow even countably many discontinuities and the Theorem will remain true.
Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 21 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).

Another example

Let us look at Exercise 3.1 (which was assigned for the tutorial). We have to show that the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \\ 2 \text{ if } x \in (1, 2] \end{cases}$$

is Riemann integrable from definition.

The solution

Let $P = \{0 = x_0 < x_1, \dots x_{n-1}, x_n = 2\}$ be an arbitrary partition. The point 1 lies in only one of the partitions, say $[x_{i-1}, x_i]$. for some i. We assume that $1 \neq x_i$ and treat this case first.

$$L(f, P) = \sum_{j=1}^{i} (x_j - x_{j-1}) + \sum_{j=i+1}^{n} 2(x_j - x_{j-1}) \tag{1}$$

$$= x_i + 2(2 - x_i) = 4 - x_i, \tag{2}$$

where x_i is a point in $(1, 2]$.

If $x_i = 1$ for some i, then

$$L(f, P) = \sum_{j=1}^{i+1} (x_j - x_{j-1}) + \sum_{j=i+2}^{n} 2(x_j - x_{j-1}) \tag{3}$$

$$= (x_{i+1} - x_0) + 2(2 - x_{i+1}) = 4 - x_{i+1}, \tag{4}$$

where x_{i+1} is a point in $(1, 2]$.

In either case $\sup_P L(f, P) = L(f) = 3$.

The Upper sums $U(f, P)$ can be treated in exactly the same way. In either of the cases we have treated above we get $U(f, P) = 4 - x_{i-1}$, for a point $x_{i-1} \in [0, 1)$. It follows that $U(f) = \inf_P U(f, P) = 3$.

We have thus shown that $L(f) = U(f) = 3$ which shows that the function is Riemann integrable.

Properties of the Riemann integral

From the definition of the Riemann integral we can easily prove the following properties. We assume that f and g are Riemann integrable. Then $f \pm g$, $cf, c \in \mathbb{R}$ are Riemann integrable and

$$\int_a^b [f(t) \pm g(t)]dt = \int_a^b f(t)dx \pm \int_a^b g(t)dt,$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt,$$

for any constant $c \in \mathbb{R}$, and finally if $f(t) \leq g(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

Proving the properties of the integral

It is not hard to prove either of the properties. One needs only to use the corresponding properties for inf and sup:

$$\inf_P U(f+g,P) = \inf_P U(f,P) + \inf_P U(g,P),$$
$$\sup_P L(f+g,P) = \sup_P L(f,P) + \sup_P U(g,P),$$

and

$$\inf_P U(cf,P) = c\inf_P U(f,P),$$

$$\sup_P L(cf,P) = c\sup_P L(f,P).$$

Proof of these are left as an exercise.

Products of Riemann Integrable Functions

Theorem E: Let $f : [a, b] \to [m, M]$ be a Riemann integrable function and let $\phi : [m, M] \to \mathbb{R}$ be a continuous function. Then $\phi \circ f$ is Riemann integrable on $[a, b]$.

The above theorem has the following interesting corollaries.

Corollary 1: Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions which are Riemann integrable on $[a, b]$. Then, $f \cdot g$, $|f|$ and f^n (for any positive integer n) are Riemann integrable.

Proof: Exercise (Hint: $f \cdot g = \frac{1}{4}[(f+g)^2 - (f-g)^2]$).

Corollary 2: If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $[c, d] \subseteq [a, b]$. Then the function $g : [c, d] \to \mathbb{R}$ defined as $g(x) = f(x)$ for all $x \in [c, d]$, is Riemann integrable.

Proof: Exercise (Hint: First show that the characteristic function $\chi_{[c,d]}$ is integrable on $[a, b]$ and then show that $f \cdot \chi_{[c,d]}$ is integrable on $[a, b]$ by using the Corollary 1, and $\int_a^b f(t)\chi_{[c,d]}(t)dt = \int_c^d g(t)dt$).

Another property of the Riemann Integral

Theorem 23: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (reps. $L(f)_{[c,b]}$) the Riemann lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a < x_1 < \dots x_{n-1} < b\}$ we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

By the property for refinements for Riemann sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

The same kind of reasoning applies to the Upper sums which allows us to prove the required property.

Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!). In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation. Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems. The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation , especially the product and chain rules. By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable for any $x \in [a, b]$ because of Theorem 21 (every continuous function is Riemann integrable.

The proof of Part I continued

By Theorem 23 we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

We know that if $f(t) \leq g(t)$ on $[a, b]$, then $\int f(t)dt \leq \int g(t)dt$. We apply this to the three functions $m(h)$, f and $M(h)$, where $m(h)$ and $M(h)$ are the constant functions given by the minimum and maximum of the function f on $[x, x+h]$ to get:

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

But f is a continuous function, so $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. By the Sandwich theorem for limits (use version 2), we see that limit in the middle exists and is equal to $f(x)$, that is $F'(x) = f(x)$. This proves the first part of the Fundamental Theorem of Calculus. $\square$

This first form of the Fundamental Theorem allows us to compute definite integrals. Keeping the notation as in the Theorem we obtain

Corollary:

$$\int_c^d f(t)dt = F(d) - F(c),$$

for any two points $c, d \in [a, b]$.

Mean Value Theorem for Integrals

Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof: Since the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$ (by the Fundamental Theorem of Calculus), there is $c \in (a, b)$ such that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)dx$$

(by the Mean Value Theorem).

Thus

$$f(c)(b-a) = \int_a^b f(x)dx. \quad \square$$

The Fundamental Theorem of Calculus Part 2

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and is hence, stronger than the corollary we have just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an arbitrary partition of $[a, b]$. Using the mean value theorem for each of the intervals $I_j = [x_j, x_{j-1}]$, we can write

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

, where $c_i \in (x_{i-1}, x_i)$.

The proof of the Fundamental Theorem part II continued

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

where $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) - f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})].$$

The calculation above is valid for any partition The right hand side obviously represents a Riemann sum. By hypothesis f is Riemann integrable. It follows (using Definition 1, for example) that as $\|P\| \to 0$, the right hand side goes to the Riemann integral. $\square$

Exercise 4.4

Exercise 4.4 Compute

(a) $\frac{d^2y}{dx^2}$, if $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$ where (i) $F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \int_0^{x^2} \cos(t)dt$.

Solution: (a) $\frac{dy}{dx} = \frac{1}{dx/dy} = \sqrt{1+y^2}$. Hence $\frac{d^2y}{dx^2} = y$.

(b)(i) Let $F(x) = \int_1^{2x} \cos(t^2)dt$. Then $F'(x) = 2\cos(4x^2)$.

(b)(ii) Let $F(x) = \int_0^{x^2} \cos(t)dt$. Then $F'(x) = 2x\cos(x^2)$.

Exercise 4.5

Exercise 4.5. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.

Solution: Consider $F(x) = \int_a^x f(t)dt, \; x \in \mathbb{R}$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

Use the Chain rule to see that

$$\begin{aligned} \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x)u'(x) \end{aligned}$$

Using this formula for $F(x) = \int_x^{x+p} f(t)dt$ we see that $F'(x) = 0$.

Exercise 4.2

Exercise 4.2. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is also continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some x.

Solution: (a)
$f(x) \geq 0 \Rightarrow U(P, f) \geq 0, \; L(P, f) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0.$
Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0.$ Assume $f(c) > 0$ for some c in $[a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that $U(P, f) > \delta \times \frac{f(c)}{2}$, for any partition P, and hence, $\int_a^b f(x)dx \geq \delta f(c)/2 > 0$, a contradiction.

(b) On $[0, 1]$ take $f(x) = 0$ for all $x \neq 0$ and $f(0) = 1$.

Exercise 4.6

Exercise 4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Solution: Write $\sin \lambda(x - t) = \sin(\lambda x) \cos(\lambda t) - \cos(\lambda x) \sin(\lambda t)$. Then take the terms in x outside the integral.

Now find $g'(x)$ and $g''(x)$ and simplify.

Fact: A convergent Taylor series (or more generally a convergent "power series") can be differentiated and integrated "term by term". That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.

Exercise 5: Using Taylor series write down a series for the integral

$$\int \frac{e^x}{x} dx.$$

Solution: We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2 \cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

We can obtain Taylor series for the inverse trigonometric functions in this way. Indeed we could define the function $\arcsin x$ in this way:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt.$$

Now we can use the binomial theorem for the integrand. Note that the binomial theorem for arbitrary real exponents is an example of Taylor series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

It is not too hard to prove that the series on the right hand side above converges for $|x| < 1$. Applying the binomial theorem for $\alpha = -1/2$ to the integrand, we get

$$\arcsin x = \int_0^x \left(1 + \frac{1}{2}t^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^4 + \dots\right) dt.$$

Integrating this term by term, you should verify that you get the series for $\arcsin x$ that you can derive directly from Taylor series.

One can treat $\arctan x$ in exactly the same way. I should also add that whether or not one is interested in Taylor series, Riemann integrals can be used to define functions as we have above. For instance we can define

$$\log x = \int_1^x \frac{1}{t} dt \quad \text{for all } x > 0.$$

Note that we know that the integral makes sense because $1/t$ is a continuous function, and hence, Riemann integrable. The exponential function e^x can then be defined as the inverse of this function rather than using the power series definition.

From the definition of log via the integral, one can see immediately that $(\log x)' = 1/x$ and derive some of the other properties of log relatively easily.

The logarithmic function

Definition: The natural logarithmic function is defined for $x > 0$ by

$$\ln x = \int_1^x \frac{1}{t} dt.$$

It is clear that $\ln 1 = 0$, $\ln > 0$ in $(1, \infty)$, and $\ln < 0$ in $(0, 1)$.

Theorem:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln(\frac{x}{y}) = \ln x - \ln y$
3. $\ln(x^r) = r \ln x$, if r is a rational number.

Proof: (1). Let $f(x) = \ln(xy)$. Then, $f'(x) = \frac{1}{x}$. Therefore, $\ln(xy) = \ln x + c$. Put $x = 1$ to see that $c = \ln y$. For (2), put $x = 1/y$ in (1) and get $\ln(1/y) = -\ln y$.

(3) is clear if $r \in \mathbb{N}$. Now, $\ln x = \ln[(x^{1/q})^q] = q \ln(x^{1/q})$. $\square$

The exponential function

Remark: $\ln x$ is increasing and concave. Moreover, by IVT, there exists a number e so that $\ln e = 1$.

$\ln x$ is a strictly increasing function, with range $\mathbb{R}$. Therefore, it has an inverse. We denote this by $\exp(x)$.

That is,

$$\exp(x) = y \iff \ln y = x$$

In particular, $\exp(0) = 1, \exp(1) = e$.

Since, $\ln(e^r) = r \ln e = r$, we get $\exp(r) = e^r$, when r is a rational number. Therefore, we define $e^x = \exp(x)$ for any $x \in \mathbb{R}$.

Laws of Exponents:

$e^{x+y} = e^x e^y, \ e^{x-y} = \frac{e^x}{e^y}, \ (e^x)^r = e^{rx},$ if r is rational.

Proof: Use the laws of exponents for $\ln x$. $\square$

Theorem: $\frac{d}{dx}(e^x) = e^x$.

Proof: If f is differentiable with nonzero derivative, then f^{-1} is also differentiable. In this case,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Thus, $\frac{d}{dx}(e^x) = \frac{1}{(\ln)'(e^x)} = \frac{1}{1/e^x} = e^x$. $\square$

Now, we can define a^x whenever $a > 0$ and $x \in \mathbb{R}$ as

$$a^x = e^{x \ln a}.$$

Exercise: Show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Solution: Let $f(x) = \ln x$. Then $f'(x) = 1/x$. Thus, $f'(1) = 1$. But,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h}.$$

Thus, by using the sequential criterion for limits, if we consider the sequence $\{1/n\}$ converging to 0, then

$$1 = f'(1) = \lim_{n\to\infty} \frac{f(1+(1/n))}{(1/n)} = \lim_{n\to\infty} \log\left(1+\frac{1}{n}\right)^n.$$

Since the log function is continuous, we obtain that

$$\lim_{n\to\infty} \log\left(1+\frac{1}{n}\right)^n = \log\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$$

and hence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. $\square$

Taylor series: Suppose f is a $\mathcal{C}^\infty$ function on $\mathbb{R}$. Then, the Taylor series expansion of f about a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series for e^x

Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. If we choose $N > 2x > 0$, then for all $n > N$,

$$\frac{x^{n+1}}{(n+1)!} = \frac{x^n}{n!} \frac{x}{(n+1)} \leq \frac{x^n}{n!} \frac{x}{N} \leq \frac{x^n}{n!} \frac{1}{2}.$$

Thus, for $m \geq n > N$,

$$s_m - s_n = \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \leq \frac{x^{n+1}}{(n+1)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right)$$

and hence

$$s_m - s_n \leq \frac{2x^{n+1}}{(n+1)!} \leq \frac{x^n}{n!}.$$

This shows that the sequence of partial sums is Cauchy. Hence the series is convergent. Therefore $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Inverse Trigonometric and Trigonometric Functions

Noting that the function $t \in \mathbb{R} \longmapsto 1/(t^2+1) \in \mathbb{R}$ is continuous on $\mathbb{R}$, we proceed as follows.

Definition

The **arctangent function** is defined by

$$\arctan x := \int_0^x \frac{1}{1+t^2} dt \quad \text{for } x \in \mathbb{R}.$$

We can then find properties of the function arctan, and of its inverse function tan in a manner similar to the way we found properties of the functions ln and its inverse function exp.

The theory of inverse trigonometric and trigonometric functions can be developed on these lines. This also allows us to define the polar coordinates (r, θ) of a point $(x, y) \neq (0, 0)$.