Lecture 14

Punit Parmananda

PH111

Email : punit@phy.iitb.ac.in

Office: Room 203, 2nd Floor,
Physics Department

Lorentz Equations

$$x' = \gamma\left(x - vt\right)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

The inverse transformation

$$x = \gamma\left(x' + vt'\right)$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

RECAP

Consequences of Lorentz transformations (space-time correlation)

1. **Length contraction**

(a moving rod appears shorter)

RECAP

2. **Time dilation**

(a moving clock runs slowly)

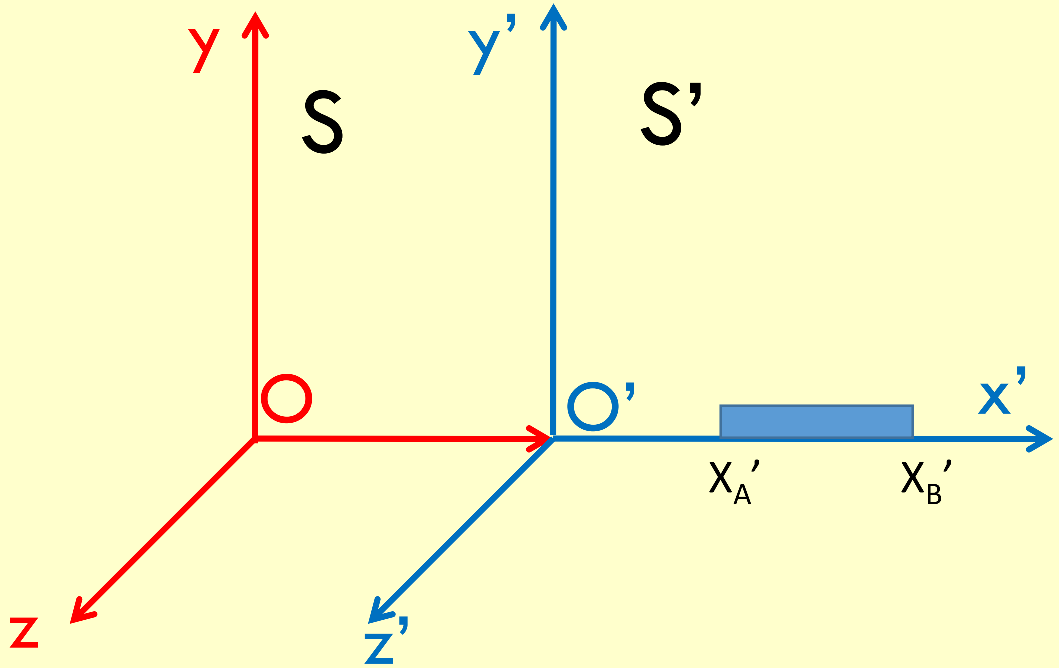
Proper length RECAP

The proper length or rest length of an object is **the length of the object measured by an observer which is at rest relative to it, by applying standard measuring rods on the object**.

The proper length of an object is the length of the object in the frame in which the object is at rest.

Length contraction

$L' = x'_B - x'_A$ irrespective of t_B' and t_A'



Let the rod be at rest in S' frame. Hence x_B' and x_A' can be measured at any time. But this rod will be observed to be moving is S frame. Hence the two ends should be measured at the same time in S; **$t_A = t_B$**

Co-ordinates of the events in S’

RECAP

$$x'_B = \gamma\left(x_B \cancel{- vt}\right), \quad t'_B = \gamma\left(t - \frac{vx_B}{c^2}\right)$$

$$t_B = t_A = t$$

$$x'_A = \gamma\left(x_A \cancel{- vt}\right), \quad t'_A = \gamma\left(t - \frac{vx_A}{c^2}\right)$$

$$(x_A' - x_B') = \gamma(x_A - x_B)$$

$$\therefore L = (x_A - x_B) = \frac{(x_A' - x_B')}{\gamma} = \frac{L'}{\gamma}$$

Contraction Formula

When there is a relative velocity between the rod and the observer, he/she feels a contracted length.

Time Dilation

RECAP

Proper Time interval is time interval between two events occurring ***at the same place*** in a given frame.

Or

The time interval recorded by a clock which is attached to the observed body.

Implies

The proper time is the time interval between two events measured by a clock which ***travels through both events***

Time Dilation formula

RECAP

Let the time interval be proper in S. The coordinates A and B ***should be same*** in S.

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right)$$

$$(t_A' - t_B') = \gamma(t_A - t_B)$$

i.e., $\Delta t' = \gamma \Delta t$

or $\Delta t' > \Delta t$

Velocity Transformation

We know the components of velocity a particle in S and want to find the same in S’ for the same particle.

$\vec{v}$ Relative velocity between frames. Constant as a function of time.

$\vec{u}$ Instantaneous velocity of particle is S. Need not be constant.

$\vec{u}'$ Instantaneous velocity of particle is S’. Need not be constant.

Events related to Displacement

Imagine that a particle is moving in x- direction in a frame S.

Event1: Particle found at x_1 at t_1.

Event2: Particle found at x_2 at t_2.

Even if the velocity of particle is not constant

$$\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

in the limit Δt tending to zero would give the instantaneous velocity of particle in S.

If the motion is in three-dimension, in general

$$u_x = \underset{\Delta t \to 0}{Lt} \frac{\Delta x}{\boxed{\Delta t}}$$

$$u_y = \underset{\Delta t \to 0}{Lt} \frac{\Delta y}{\Delta t}$$

$$u_z = \underset{\Delta t \to 0}{Lt} \frac{\Delta z}{\Delta t}$$

Similarly looking at the same particle in S', we can define:

$$u'_x = \underset{\Delta t' \to 0}{Lt} \frac{\Delta x'}{\boxed{\Delta t'}}$$

$$u'_y = \underset{\Delta t' \to 0}{Lt} \frac{\Delta y'}{\Delta t'}$$

$$u'_z = \underset{\Delta t' \to 0}{Lt} \frac{\Delta z'}{\Delta t'}$$

Note that like displacement, the time difference has also to be measured in one's own frame.

Lorentz Transformation in differential form.

$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right)$$

$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

$$x' = \gamma\left(x - vt\right)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x - v\Delta t}{\Delta t - \frac{v\Delta x}{c^2}} = \frac{\frac{\Delta x}{\Delta t} - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}$$

$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right)$$

$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

?? ← u_x $-$ v → ??

?? ← u'_x

$$\boxed{u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}}$$

$$\frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)} = \frac{\frac{\Delta y}{\Delta t}}{\gamma\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)}$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\boxed{u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}}$$

$$\frac{\Delta z'}{\Delta t'} = \frac{\Delta z}{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)} = \frac{\frac{\Delta z}{\Delta t}}{\gamma\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)}$$

$$\boxed{u'_z = \frac{u_z}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}}$$

Velocity Transformation

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}; u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Inverse Velocity Transformation

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\boxed{u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}} \qquad u_y = \frac{u'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}; u_z = \frac{u'_z}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}}$$

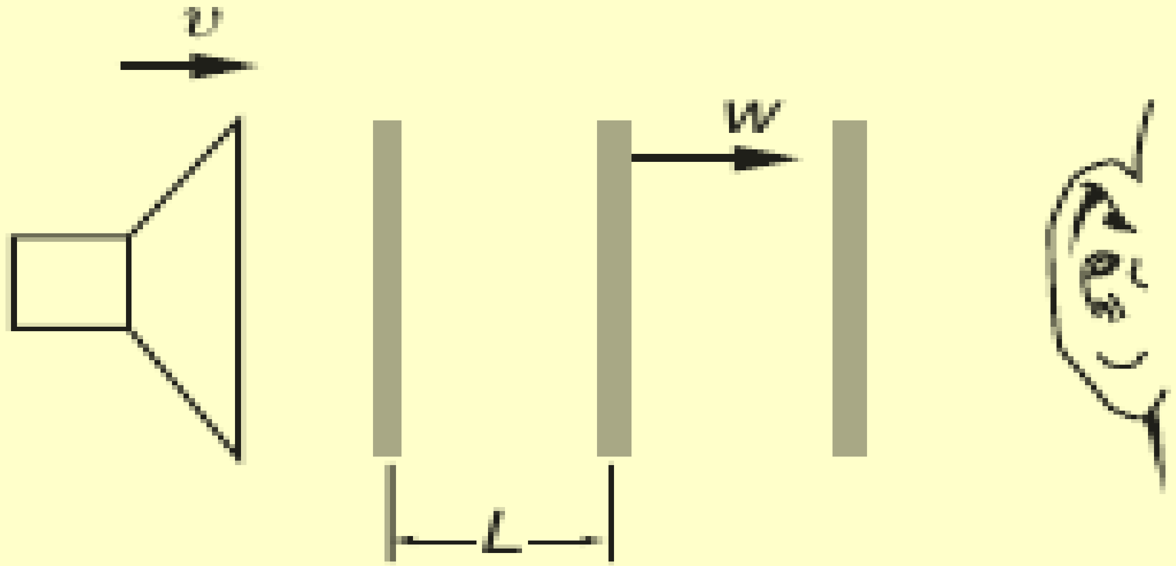
One can show that

- If u<c in S, u<c in S' also irrespective of v.
- If u=c in S, u=c in S' also irrespective of v.

$u_x' = \dfrac{u_x - v}{1 - u_x v/c^2}$	$u_x = \dfrac{u_x' + v}{1 + u_x' v/c^2}$
$u_y' = \dfrac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$	$u_y = \dfrac{u_y'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$
$u_z' = \dfrac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$	$u_z = \dfrac{u_z'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$

Manifestation: Relativistic Doppler Shift of Light

- Before deriving the formula for the Doppler effect of light, let us review the two formulas for the Doppler effect of sound
- When the source of sound moves towards the observer with speed v



- If the frequency of the sound is ν_0, and its speed in that medium is w
- Clearly, the sound pulses arrive with gap of time $\tau_0 = 1/\nu_0$
- The distance between successive crests is the wavelength $\lambda = w/\nu_0$

- For the moving source the distance between successive crests, i.e., $\boxed{\text{Doppler shifted wavelength } \lambda_D,}$ reduces

$$\lambda_D = \lambda - v\tau_0 = \lambda - \frac{v}{\nu_0}$$

- If the $\boxed{\text{Doppler shifted frequency is } \nu_0',}$ we have

$$\lambda = w/\nu_0 \qquad \frac{w}{\nu_0'} = \frac{w}{\nu_0} - \frac{v}{\nu_0}$$

- Leading to the well-known result

$$\boxed{\nu_0' = \nu_0 \left(\frac{1}{1 - v/w} \right)}$$

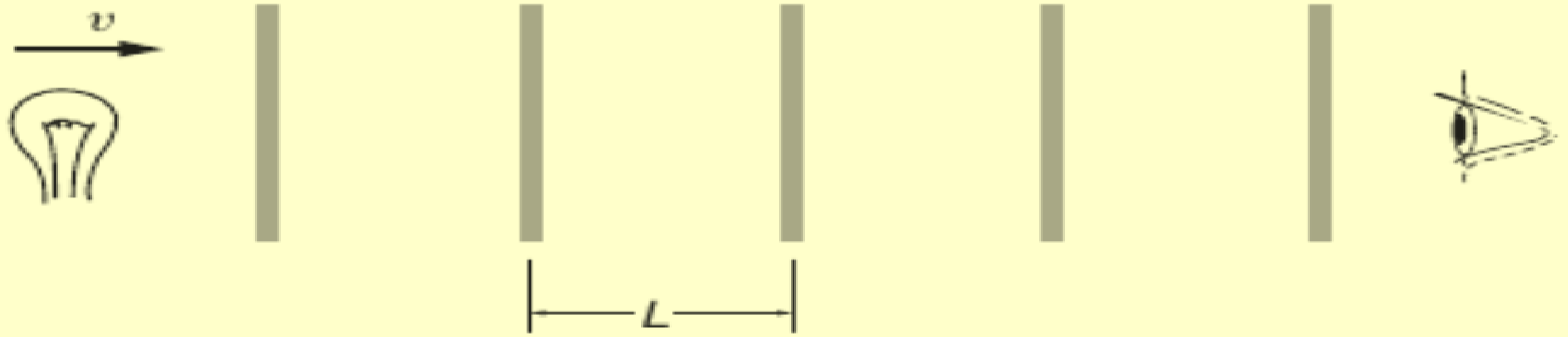
- The change in frequency $\boxed{\Delta\nu = \nu_0' - \nu_0}$ is called the Doppler shift

- Suppose, instead, the observer is moving towards the source with speed v
- Now the wave length of the sound wave doesn't change
- But, the effective sound velocity become $w+v$
- This causes change in frequency

$$\boxed{\nu_0' = \frac{w+v}{\lambda} = \left(\frac{w}{\lambda} + \frac{v}{\lambda}\right) = \left(\nu_0 + \frac{v\nu_0}{w}\right) = \nu_0\left(1 + \frac{v}{w}\right)}$$

- The two results agree to the first order in v/w

- Suppose the light source is moving towards the observer at a speed v



- If the light has frequency ν_0, then the time period between pulses is $\tau_0 = 1/\nu_0$.
- Because the source is moving, this time period will undergo time dilation with respect to the observer, leading to a new time period

$$\boxed{\tau = \gamma\tau_0}$$

- As in case of the sound wave, the distance between the two successive crests in observer's frame will be

$$\boxed{\lambda_D = c\tau - v\tau}$$

- Now Doppler-shifted frequency ν_D will be

$$\boxed{\lambda_D = c\tau - v\tau}$$

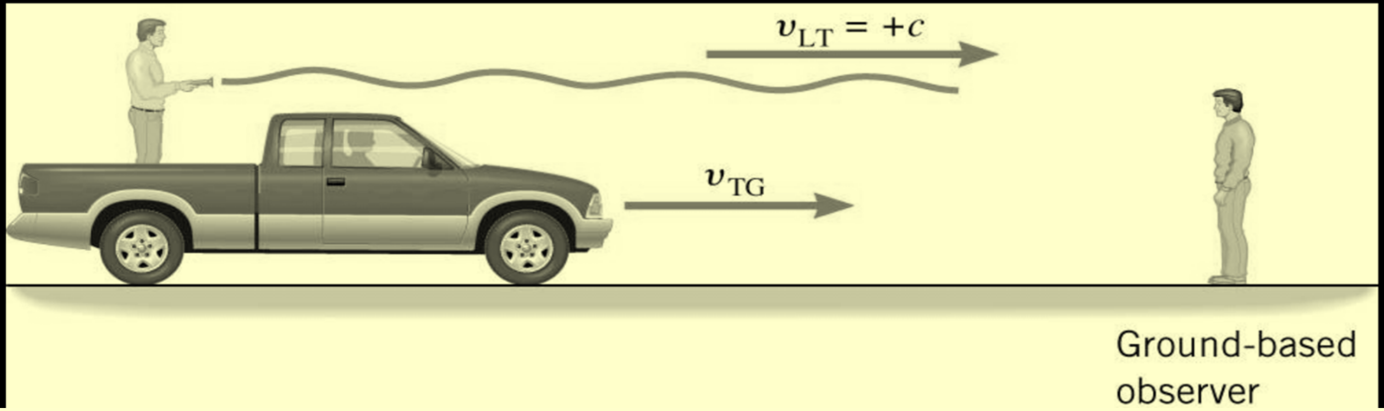
$$\nu_D = \frac{c}{\lambda_D} = \frac{c}{(c-v)\tau} = \frac{c}{\gamma(c-v)\tau_0}$$

$$= \nu_0 \frac{\sqrt{1-v^2/c^2}}{(1-v/c)} = \nu_0 \sqrt{\frac{1+v/c}{1-v/c}} = \nu_D$$

End of Content from Exam's point-of-view

Relativistic Addition of Velocity

$$v_{LG} = \frac{v_{LT} + v_{TG}}{1 + \frac{v_{LT} v_{TG}}{c^2}}$$



$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2}$$

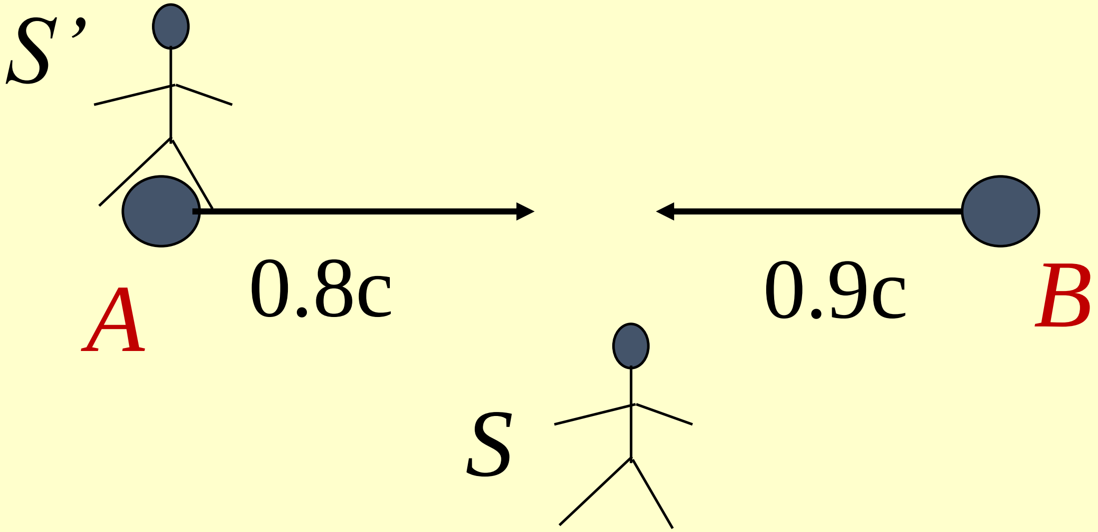
$$u_y = \frac{u_y' \sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$$

$$u_z = \frac{u_z' \sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$$

At what speed does the ground based observer see the light travel?

Second Postulate preserved

$$v_{LG} = \frac{v_{LT} + v_{TG}}{1 + \frac{v_{LT} v_{TG}}{c^2}} = \frac{c + v_{TG}}{1 + \frac{c v_{TG}}{c^2}} = \frac{c + v_{TG}}{1 + \frac{v_{TG}}{c}} = \frac{c + v_{TG}}{\frac{c}{c} + \frac{v_{TG}}{c}} = \frac{c + v_{TG}}{\frac{c + v_{TG}}{c}} = \frac{(c + v_{TG})c}{c + v_{TG}} = c$$

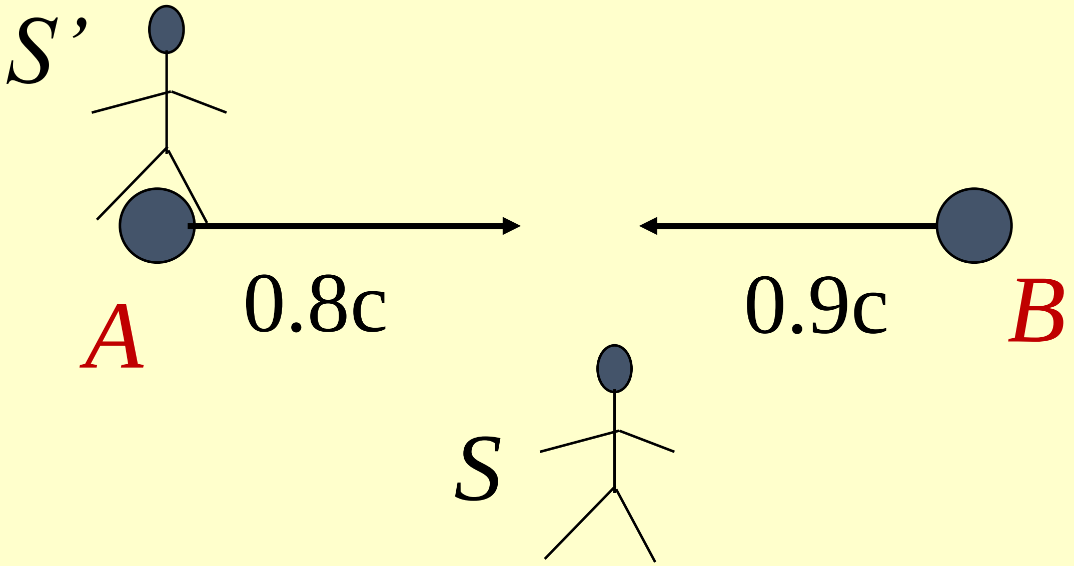


$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}; u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

What is the relative Velocity of B as seen in A ?
One has to use the velocity transformation to obtain that.

$$v = 0.8c; \ u_x = -0.9c,$$
$$u_y = 0, \ u_z = 0$$



$$u'_x = \frac{-(0.9c + 0.8c)}{1 + \frac{0.9c \times 0.8c}{c^2}}$$

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

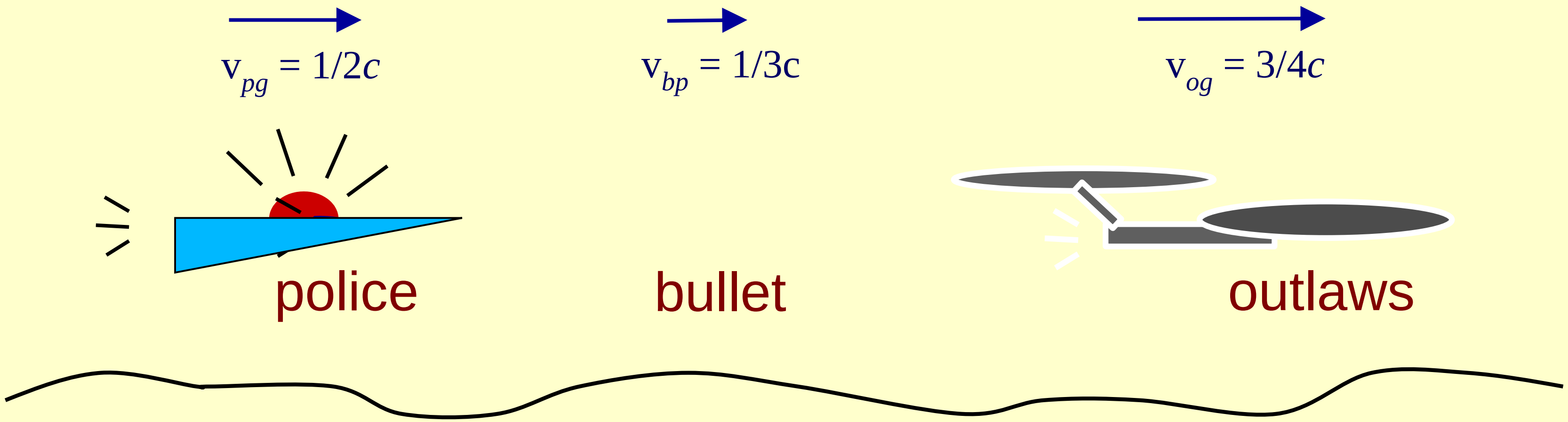
$$= -\frac{1.7}{1.72}c = -0.988c$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}; u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$u'_y = u'_z = 0$$

Relative speed is smaller than c.

As the outlaws escape in their really fast getaway ship at 3/4*c*, the police follow in their pursuit car at a mere 1/2*c*, firing a bullet, whose speed relative to the gun is 1/3*c*. **Question:** does the bullet reach its target *a*) according to Galileo, *b*) according to Einstein?



v_{pg} = velocity of police relative to ground
v_{bp} = velocity of bullet relative to police
v_{og} = velocity of outlaws relative to ground

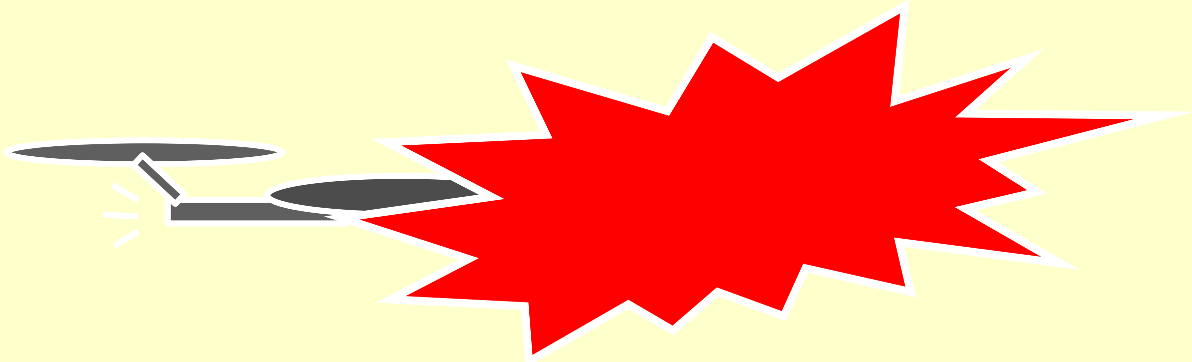
Galileo’s addition of velocities

In order to find out whether justice is met, we need to compute the bullet's velocity relative to the ground and compare that with the outlaw's velocity relative to the ground.

In the Galilean transformation, we simply add the bullet’s velocity to that of the police car:

$$\mathrm{v}_{bg} = \mathrm{v}_{bp} + \mathrm{v}_{pg} \quad \rightarrow \quad \mathrm{v}_{bg} = \tfrac{1}{3}c + \tfrac{1}{2}c = \tfrac{5}{6}c$$

Therefore, $\frac{5}{6}c > \frac{3}{4}c \quad \rightarrow$ justice is served!



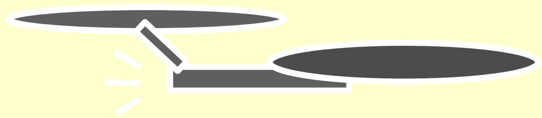
Einstein's addition of velocities

Due to the high speeds involved, we really must **relativistically** add the police ship's and bullet's velocities:

$$u_x = \frac{u'_x + \mathrm{v}}{1 + u'_x \mathrm{v} / c^2} \qquad \mathrm{v}_{bg} = \frac{\mathrm{v}_{bp} + \mathrm{v}_{pg}}{1 + \mathrm{v}_{bp}\mathrm{v}_{pg} / c^2}$$

$$\Rightarrow \quad \mathrm{v}_{bg} = \frac{\frac{1}{3}c + \frac{1}{2}c}{1 + \left(\frac{1}{3}c\right)\left(\frac{1}{2}c\right) / c^2} = \frac{5}{7}c$$

$\frac{5}{7}c < \frac{3}{4}c \quad \rightarrow$ justice is not served!



Tutorial #11 An observer sees two spaceships flying apart with speeds 0.99c. What is the speed of one spaceship as viewed by the other?

Suppose spaceship 1 is moving in the -ve x direction and while spaceship 2 is moving in the +ve x direction. For finding the speed of space ship 2 with respect to 1, we will use the formula

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}.$$

Here $u_x = 0.99c$, $v = -0.99c$ so that

$$u'_x = \frac{0.99c - (-0.99c)}{1 - \frac{0.99c*(-0.99c)}{c^2}} = \frac{1.98c}{1 + 0.99^2} = 0.9999495c < c$$

Tutorial #13: A rod of proper length ℓ is oriented parallel to x-axis in a frame S and is moving with a speed *u* along the same direction. Find its length in a frame S' that is moving with speed *v* along the +x-direction of S.

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

Speed of rod in S'

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

As the length of rod is proper in its own frame, ∴ it would appear to be contracted in S'

$$\therefore \quad \ell' = \frac{\ell}{\gamma} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{c^2 u - c^2 v}{c^2 - uv}\right)^2 / c^2}}$$

$$= \frac{c^2 - uv}{\sqrt{(c^2 - uv)^2 - c^2(u - v)^2}} = \frac{c^2 - uv}{\sqrt{c^4 + u^2 v^2 - 2c^2 \cancel{uv} - c^2 u^2 - c^2 v^2 + 2c^2 \cancel{uv}}}$$

$$= \frac{c^2 - uv}{\sqrt{c^2(c^2 - u^2) - v^2(c^2 - u^2)}}$$

$$\therefore \quad \ell' = \frac{\ell \sqrt{(c^2 - u^2)(c^2 - v^2)}}{c^2 - uv}$$

Tutorial #1

1. The time interval between two ticks of two identical clocks is 2.0 sec. One of the two clocks is set in motion, so that its speed relative to the observer, who holds the other clock is $0.6c$. What is the time interval between the ticks of the moving clock as measured by the observer with the stationary clock?

$$(t_A' - t_B') = \gamma (t_A - t_B)$$

$$\text{i.e., } \Delta t' = \gamma \Delta t$$

$$\text{or } \Delta t' > \Delta t$$

$$t = \frac{2}{\sqrt{1-\beta^2}} = \frac{2}{\sqrt{1-0.36}} = 2.5\,\mathrm{s}$$

2. The incoming primary cosmic rays create μ-mesons in the upper atmosphere. The lifetime of μ-mesons at rest is 2 μs. If the mean speed of μ-mesons is $0.998c$, what fraction of the mu-mesons created at a height of 20 km reach the sea level?

The lifetime of μ-mesons in rest frame = 2 μsTo travel 20 km at $0.998c$ requires

Tutorial #2

$$\triangle t = \frac{20}{0.998 \times 3 \times 10^5} sec = 66.8 \times 10^{-6} sec$$

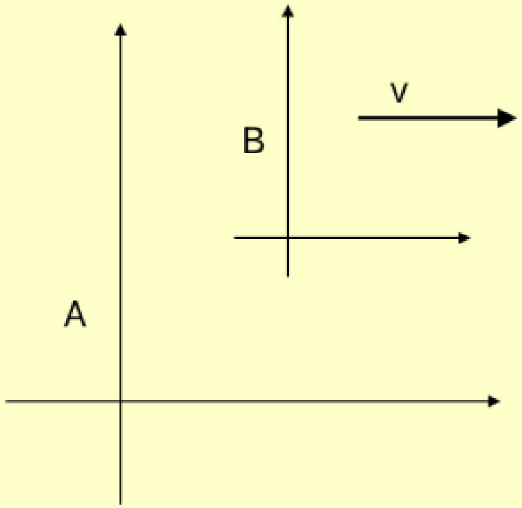
The lifetime τ will appear to be $\frac{\tau}{\sqrt{1-\beta^2}}$

The fraction f that will survive $\boxed{N(t) = N_0 e^{-\frac{t}{\tau}}}$

$$\frac{N(t)}{N_0} = f = \exp\left(-\frac{\Delta t\sqrt{1-\beta^2}}{\tau}\right) = \exp\left(-\frac{66.8\sqrt{1-0.998^2}}{2}\right) = 0.12$$

3. Two observers A and B are close to a point where lightning strikes the earth. According to A, a second lightning strikes t_0 seconds later at a distance d from him. B, on the other hand finds the two events to be simultaneous. Find his velocity with respect to A. Also find the distance between the two lightnings as seen by B. Assume earth to be inertial frame of reference.

Tutorial #3



Assume for the first lighting is at $(0,0)$ for both. For A the second event is at $x_A = d, t_A = t_0$.

t_B and t_B ; time intervals in the two frames

For B

$$t_B = \frac{t_A - vx_A/c^2}{\sqrt{1-\beta^2}} = \frac{t_0 - vd/c^2}{\sqrt{1-\beta^2}} \qquad x_A = d, t_A = t_0$$

But B finds the events to be simultaneous implying

$t_B = 0 \Rightarrow \boxed{v = c^2 t_0/d,}$ so that

$$x_B = \frac{x_A - vt}{\sqrt{1-\beta^2}} = \frac{d - \frac{c^2 t_o^2}{d}}{\sqrt{1 - \frac{c^4 t_o^2}{c^2 d^2}}} = \frac{d^2 - c^2 t_o^2}{\frac{d\sqrt{d^2 - c^2 t_o^2}}{d}}$$

$$x_B = \sqrt{d^2 - c^2 t_o^2}$$

If you read this

Done: ***Tutorial*** #s 1,2,3,4,5,11,13

Tomorrow: 6, 7, 8, 9