THE DIVERGENCE THEOREM

WE HAVE SEEN THE FLUX FORM OF GREEN'S THEOREM:

$$\oint_C (F \cdot n)\, ds = \iint_R (\text{div } F)\, dx\, dy.$$

THE DIVERGENCE THEOREM IS AN EXTENSION OF THE SAME TO SURFACE INTEGRALS:

DIVERGENCE THEOREM:

SUPPOSE D IS A CLOSED BOUNDED REGION IN $\mathbb{R}^3$ WHOSE BOUNDARY $S = \partial D$ IS A SMOOTH ORIENTABLE SURFACE. LET $F = P\vec{i} + Q\vec{j} + R\vec{k}$ BE A CONTINUOUSLY DIFFERENTIABLE VECTOR FIELD IN AN OPEN SET CONTAINING D. THEN

$$\iint_S (F \cdot n)\, dS = \iiint_D (\text{div } F)\, dx\, dy\, dz$$

WHERE n IS THE OUTWARD NORMAL TO S.

PROOF: (FOR SIMPLE REGIONS)

SUPPOSE ANY STRAIGHT LINE PARALLEL TO ANY OF X, Y, Z AXES INTERSECTS D IN A LINE SEGMENT, A POINT, OR THE EMPTY SET.

SUPPOSE $\mathbf{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$

NOW

$$\iint_S (F\cdot \mathbf{n})\,dS = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS$$

$$\left(F = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}\right)$$

AND

$$\iiint_D \operatorname{div}(F)\,dx\,dy\,dz = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

WE WILL SHOW:

$$\iint_S R\cos\gamma\,dS = \iiint_D \frac{\partial R}{\partial z}\,dx\,dy\,dz$$

AND TWO OTHER CORRESPONDING EQUALITIES.

SINCE D IS SIMPLE, WE SUPPOSE

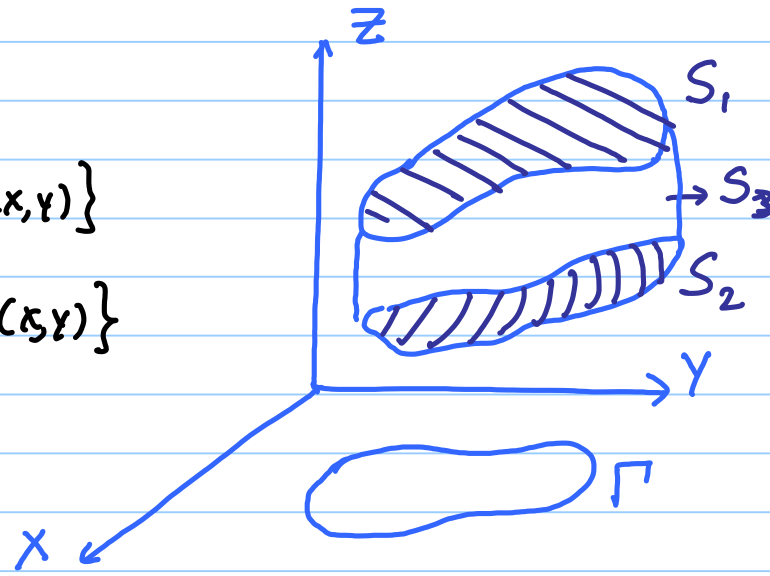
$$D = \{(x,y,z) \mid (x,y) \in \Gamma \subseteq \mathbb{R}^2,\ g(x,y) \le z \le h(x,y)\}.$$

$\Gamma \equiv$ PROJECTION OF S ONTO XY PLANE.

$S = S_1 \cup S_2 \cup S_3$

WHERE $S_1 = \{z = h(x,y)\}$

$S_2 = \{z = g(x,y)\}$



$S_3 \equiv$ A (POSSIBLE)

CYLINDER WITH

BASE Γ, AND AXIS PARALLEL TO Z-AXIS.

HENCE

$$\iint_S R\cos\gamma\, dS = \iint_{S_1} R\cos\gamma\, dS + \iint_{S_2} R\cos\gamma\, dS + \iint_{S_3} R\cos\gamma\, dS$$

$$= \iint_D R(x,y,h(x,y))\,dx\,dy - \iint_R R(x,y,g(x,y))\,dx\,dy$$

$$= \iint_D R(x,y,h(x,y)) - R(x,y,g(x,y))\ dx\,dy$$

$$= \iint_D \left(\int_{g(x,y)}^{h(x,y)} \frac{\partial R}{\partial z}\, dz \right) dx\,dy = \iiint_D \frac{\partial R}{\partial z}\, dz\, dx\, dy$$

EXAMPLE

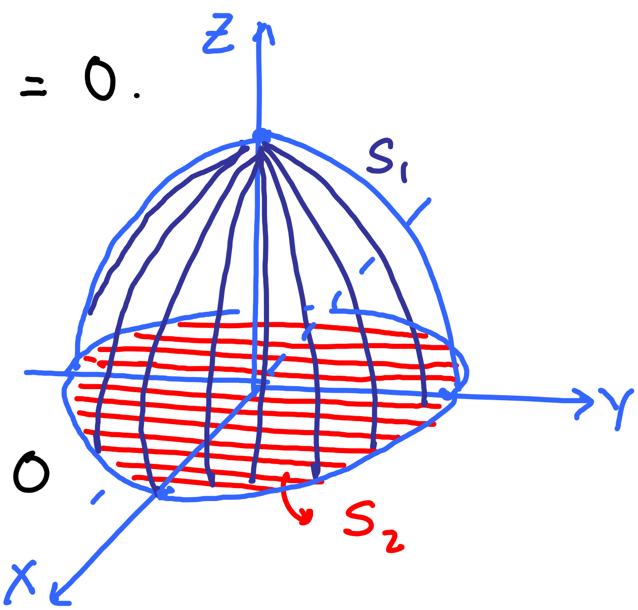
LET US VERIFY THE DIVERGENCE THEOREM:

$$F = 2z\vec{i} + x\vec{j} + y^2\vec{k}$$

D ≡ REGION BOUNDED BY THE PARABOLOID

$$z = 4 - x^2 - y^2.$$

NOTE THAT $\operatorname{div}(F) = 0$.



HENCE

$$\iiint_D (\operatorname{div} F)\, dx\, dy\, dz = 0$$

TO CALCULATE $\iint_{\partial D} (F \cdot n)\, dS$, NOTE THAT

$\partial D = S_1 \cup S_2$, WHERE D ≡ REGION IN QUESTION

S_1 ≡ PARABOLOID SURFACE

S_2 ≡ FLAT (BOTTOM) DISK.

HENCE

$$\iint_{S_1} (F \cdot n)\, dS = \iint_{\Gamma} (2z, x, y^2) \cdot (2x, 2y, 1)\, dx\, dy$$

$$= \iint_{\Gamma} \underline{(4xz + 2xy + y^2)}\, dx\, dy.$$

$\Gamma \equiv \{ x^2 + y^2 \leq 4 \}$

FOR S_2, $$\iint_{S_2} (F \cdot n)\, dS = \iint_{\Gamma} (2z, x, y^2) \cdot (0, 0, -1)\, dx\, dy$$

$$= -\iint_{\Gamma} y^2\, dx\, dy.$$

HENCE $$\iint_{S} (F \cdot n)\, dS = \iint_{\Gamma} (4xz + 2xy)\, dx\, dy$$

$$= \iint_{\Gamma} \left[4x(4 - x^2 - y^2) + 2xy \right] dx\, dy$$

CALCULATE AND BE CONVINCED!

CONSIDER $G = G_1 \cup G_2$

$$G_1 = \{ 4 \leq x^2+y^2+z^2 \leq 9,\ z \geq 0 \}$$

$$G_2 = \{ 4 \leq x^2+y^2+z^2 \leq 9,\ z \leq 0 \}$$

$$S_1 : \quad x^2+y^2+z^2 = 4$$

$$\mathcal{A} = \{ (x,y) \mid 4 \leq x^2+y^2 \leq 9 \}$$

$$S_2 : \quad x^2+y^2+z^2 = 9$$

$$\partial G_1 = S_1^+ \cup S_2^+ \cup \mathcal{A},$$

$$\partial G_2 = S_1^- \cup S_2^- \cup \mathcal{A}$$

APPLICATIONS: DIVERGENCE THEOREM

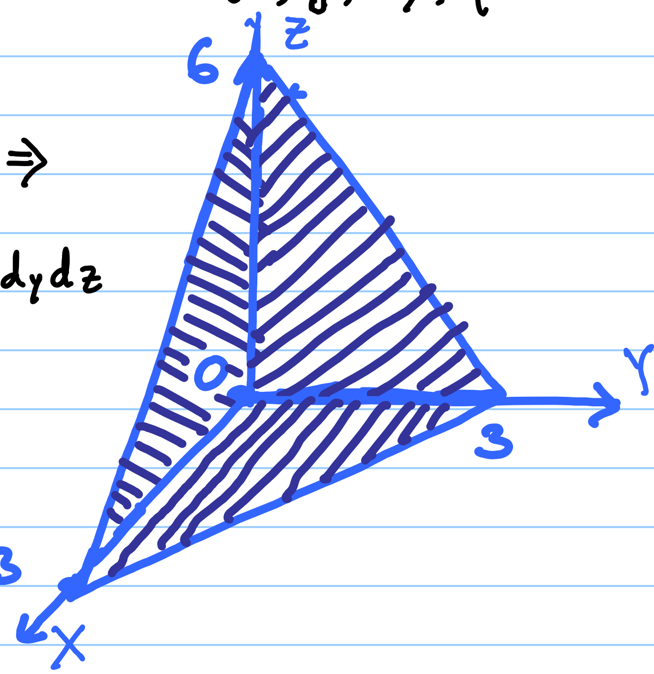
CALCULATION OF SURFACE INTEGRALS:

EXAMPLE

COMPUTE $\iint_S F \cdot dS$ WHERE $S = \partial G$,

AND G IS BOUNDED BY THE COORDINATE PLANES

AND $2x + 2y + z = 6$, AND $F = (x, y^2, 1)$. (FIRST OCTANT)



DIVERGENCE THEOREM $\Rightarrow$

$$\iint_S F \cdot dS = \iiint_G (\operatorname{div} F)\, dx\, dy\, dz$$

NOW,

$$\operatorname{div}(F) = 1 + 2y$$

HENCE,

$$\iint_G \operatorname{div} F = \int_0^3 \int_0^{3-x} \int_0^{6-2(x+y)} (1+2y)\, dz\, dy\, dx$$

AND THIS IS A ROUTINE CALCULATION.

SUPPOSE G IS THE REGION (IN $\mathbb{R}^3$) ENCLOSED BETWEEN TWO SURFACES (ORIENTABLE). SUPPOSE S_1 IS THE 'INNER' SURFACE BOUNDARY OF G, AND S_2 IS THE 'OUTER' BOUNDARY

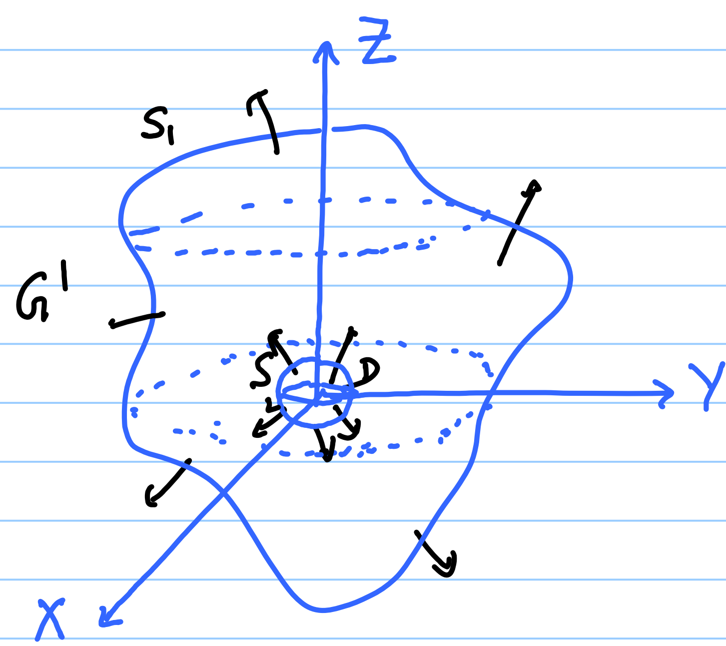
IF F IS A VECTOR FIELD s.t $\text{div}(F) = 0$,

THEN DIVERGENCE THEOREM $\Rightarrow$

$$\iint_{S_1} (F \cdot n)\, dS + \iint_{S_2} (F \cdot n)\, dS = \iiint_G \text{div}\, F = 0$$

EXAMPLE: SUPPOSE $F = -\dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$

AND SUPPOSE S IS SOME SURFACE:



$$\iint_{S_1} F \cdot n + \iint_{S_2} F \cdot n = \iiint_{G'} \text{div}\, F = 0$$

FIRST, NOTE THAT

$$\mathrm{div}(F) = 0.$$

LET D DENOTE A SMALL SPHERE OF RADIUS a (FOR SMALL ENOUGH $a > 0$) WHICH IS CONTAINED INSIDE G.

LET $G' = G \setminus D$. APPLY DIVERGENCE THEOREM TO F ON G':

$$\iint_{\partial G'} (F \cdot n)\, dS = \iiint_{G'} \mathrm{div}(F) = 0$$

NOTE THAT $\displaystyle\iint_{\partial G'} (F \cdot n)\, dS = \iint_{\partial G} (F \cdot n)\, dS - \iint_{\partial D} (F \cdot n)\, dS$

HENCE
$$\iint_{\partial G} (F \cdot n)\, dS = \iint_{\partial D} (F \cdot n)\, dS$$

THE RHS ABOVE CAN BE CALCULATED TO BE 4π.