Lecture 12

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Einstein’s Postulates RECAP

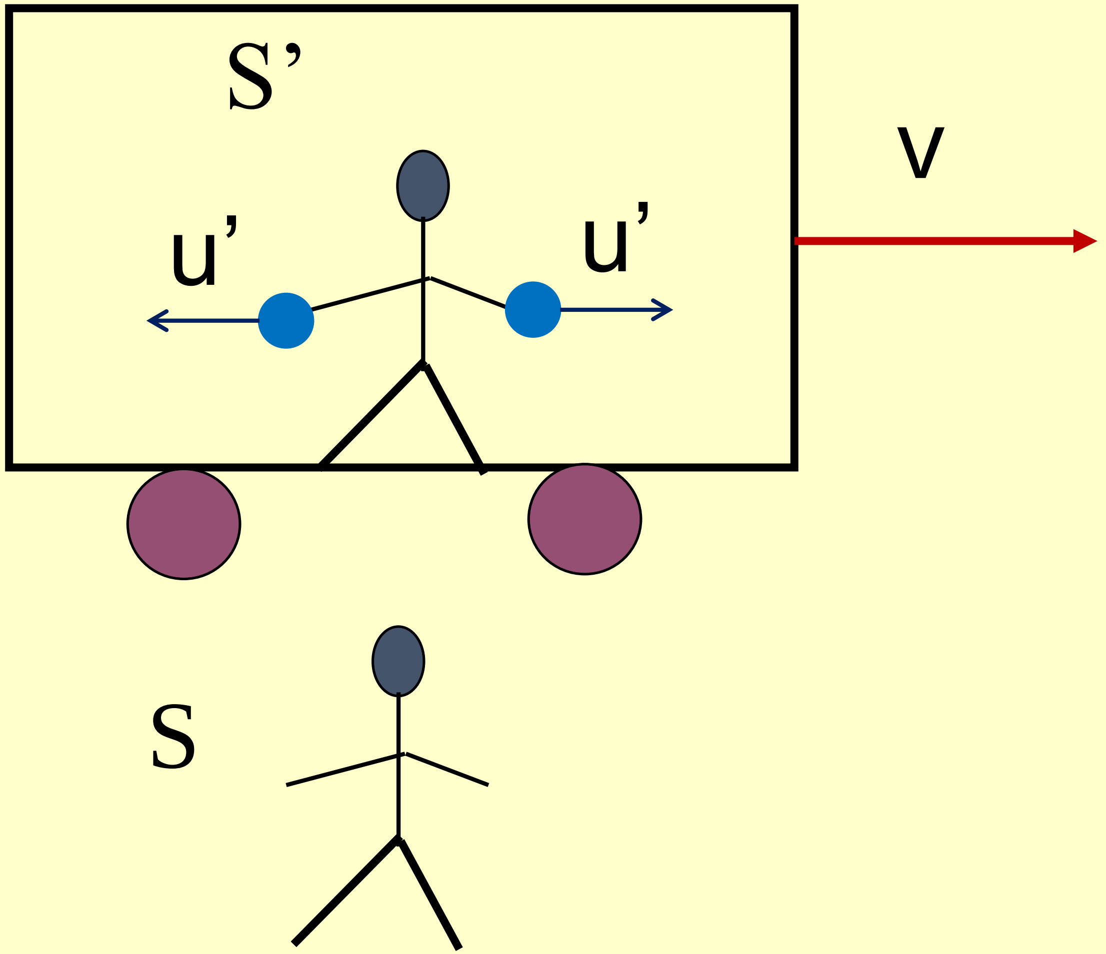
- Laws of Physics are same in all inertial frames of references. No preferred inertial frame exists.

- The speed of light ‘*c*’ is same in all inertial frames.

Time is suspect RECAP

- Time is related to the simultaneity of two events.
  - Train leaves at 10:00 implies: ***train leaving*** and ***clock showing 10:00*** are simultaneous events.
- We shall show that simultaneity is relative under the second postulate.

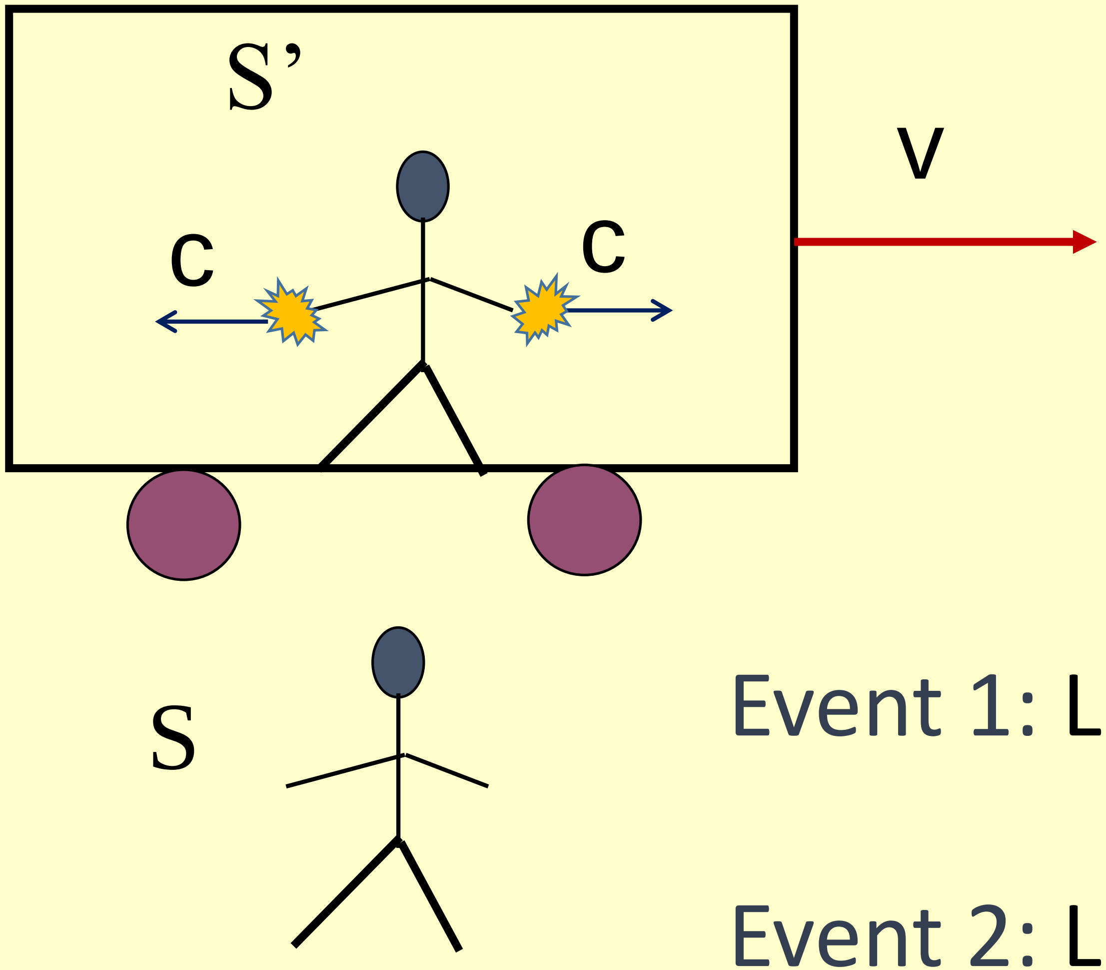
An observer is exactly half way in a running compartment of length *L*. He throws two balls at the same time (t'=0) with same speed *u'* as measured by him, one towards the front wall and other toward back wall.



RECAP

Use Light instead of Balls

RECAP



Event 1: Light reaches the front wall

Event 2: Light reaches the back wall

In *S'* frame

RECAP

Time (t'_1) for event 1: $$t'_1 = \frac{L}{2c}$$

Time (t'_2) for event 2: $$t'_2 = \frac{L}{2c}$$

The two events are simultaneous because in S' the light covers the same distance in the front direction as in the back, and with the same speed.

Hence the two events are simultaneous in this frame as before. ($\Delta t'=0$)

In S Frame RECAP

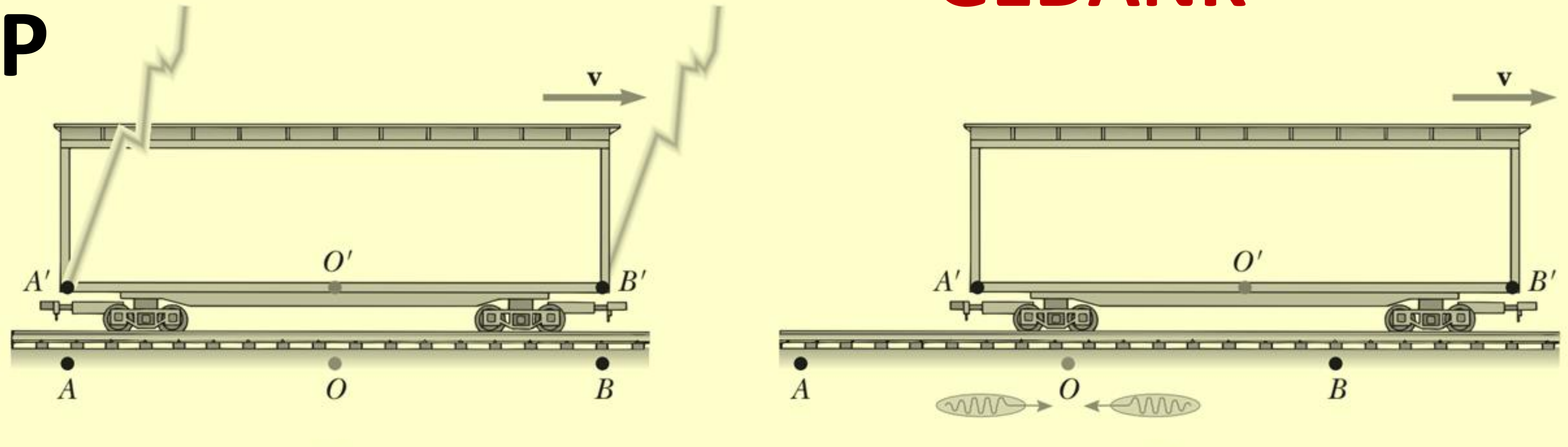
- The speed of light is still *c* in both the directions, according to the second postulate. But it has to travel a larger distance to reach the front wall than the back wall.
- $t_1 > t_2$ or $(t_2 - t_1)$ is negative.

Conclusion RECAP

- There is simultaneity of two events in S' but not in S.
- Simultaneity of two events thus depends on the frame chosen.
- In other words, two events that are simultaneous in a given frame will not be seen to be simultaneous in another frame. That means, ***time is a frame dependent quantity!*** ***Or time is no longer absolute!***
- ***This is a big deviation from the Galilean transformation.***

GEDANK

RECAP



(a) (b)

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike the ends of the boxcar, as in Figure above leaving marks on the boxcar and ground. The marks left on the boxcar are labeled A' and B'; those on the ground are labeled A and B. An observer at O' moving with the boxcar is midway between A' and B', and a ground observer at O is midway between A and B. The events recorded by the observers are the light signals from the lightning bolts.

Lorentz Transformation

Derived under numerous, albeit reasonable, assumptions

Hendrik Antoon Lorentz (/ˈlɒrənts/; 18 July 1853 – 4 February 1928) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect. He derived the Lorentz transformation which Albert Einstein subsequently used to make claims to special theory of relativity, as well as the Lorentz force which describes the combined electric and magnetic forces acting on a charged particle in an electromagnetic field. Lorentz was also responsible for the Lorentz oscillator model, a classical model used to describe the anomalous dispersion observed in dielectric materials when the driving frequency of the electric field was near the resonant frequency, resulting in abnormal refractive indices.

Hendrik Lorentz

Assumption #1: Space is homogeneous

For example, the result of measurement of a length or a time interval should not depend on where and when the interval happens to be in our frame. ***Space is isotropic.***

This implies that transformation equations must be linear.

If it is not linear, a *1 meter* long rod in *S* between $x=2$ and $x= 1\ m$ may measure differently in *S'* if it is moved to between $x=2$ and $x=3$ meters [If quadratic; $x_2^2-x_1^2=3$ *and* 5 *depending upon the location*].

Most General Transformation: 20 Unknowns

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t + C_1$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t + C_2$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t + C_3$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t + C_4$$

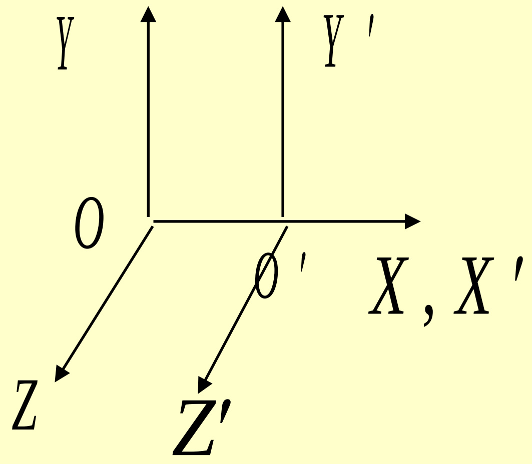
Assumption #2: Demand coincidence of Origins

Special Choice of axes and time reference

Assume when $x=y=z=0$; $x'=y'=z'=0$ and when $t=0$; $t'=0$.

This renders $C_1=C_2=C_3=C_4=0$

$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\
y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\
z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\
t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t
\end{aligned}
$$



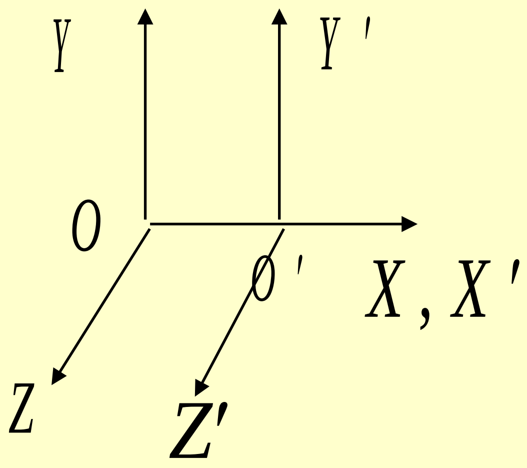
$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = \cancel{a_{21}}x + a_{22}y + a_{23}z + \cancel{a_{24}}t$$
$$z' = \cancel{a_{31}}x + a_{32}y + a_{33}z + \cancel{a_{34}}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

The x- axis coincides continuously with x' axis.

This gives for $y=0$, $z=0$; $y'=0$, $z'=0$,

Hence; $a_{21}=a_{24}=a_{31}=a_{34}=0$

Similarly a point in x-z plane (characterized by y=0) should transform to x’-z’ plane (characterized by y’=0)



Analogously, a point in x-y plane (i.e. z=0) should transform to x’-y’ plane (i.e. z’=0)

This gives $a_{23}=0$; and analogously $a_{32}=0$

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t +$$
$$y' = \cancel{a_{21}x} + a_{22}y + \cancel{a_{23}z} + \cancel{a_{24}t}$$
$$z' = \cancel{a_{31}x} + \cancel{a_{32}y} + a_{33}z + \cancel{a_{34}t}$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Hence the current equations have the following form:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{22}y$$

$$z' = a_{33}z$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Assumption #3: Principle of relativity: First Postulate

This ensures all the inertial frames of reference (ifor) are equivalent.

$$y' = a_{22}\, y$$

If we put 1 meter long stick parallel to *y-axis* between $y=0$ and $y=1\ m$ in S; that would appear to be of length a_{22} meter long in S'. If, however, the same rod was put in S' between $y'=0$ and $y'=1\ m$, this would appear to be of $1/a_{22}$ m in S.

The first postulates mandates that since these measurements are reciprocal they must be identical then only the frames will be equivalent.

Hence, we must have $a_{22} = 1/a_{22}$ or $a_{22} = 1$.

Identical argument yields $a_{33} = 1.$

And then there were 8

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t.$$

$$y' = y \qquad \text{and} \qquad z' = z.$$

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$t' = a_{41}x + \cancel{a_{42}y} + \cancel{a_{43}z} + a_{44}t.$$

Let us look first at the t'-equation. For reasons of symmetry, we assume that t' does not depend on y and z. Otherwise, clocks placed symmetrically in the y-z plane (such as at $+y$, $-y$ or $+z$, $-z$) about the x-axis would appear to disagree as observed from S', which would contradict the isotropy of space. Hence, $a_{42} = a_{43} = 0$. As for the x'-equation, we know that a point having $x' = 0$ appears to move in the direction of the

$$x' = a_{11}x + \cancel{a_{12}y} + \cancel{a_{13}z} + a_{14}t$$

positive x-axis with speed v, so that the statement $x' = 0$ must be identical to the statement $x = vt$. Therefore, we expect $\boxed{x' = a_{11}(x - vt)}$ to be the correct transformation equation. (That is, $x = vt$ always gives $x' = 0$ in this equation.) Hence, $\boxed{x' = a_{11}\,x - a_{11}\,vt = a_{11}\,x + a_{14}\,t.}$ This gives us $\boxed{a_{14} = -va_{11},}$ and our four equations have now been reduced to

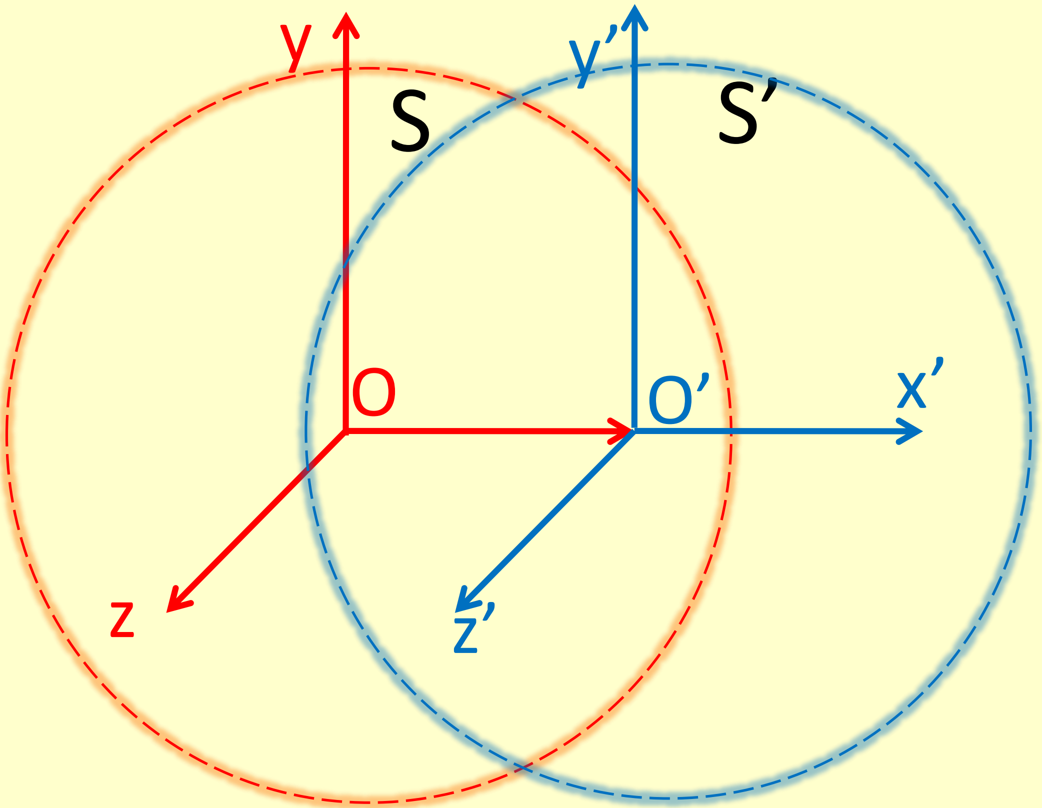
And then there were 3; Nearly done

Now the transformation equations become of the following form:

$$t' = a_{41}x + \cancel{a_{42}y} + \cancel{a_{43}z} + a_{44}t.$$

Assumption #4: Principle of relativity: Second Postulate

At $t = t' = 0$, a spherical light wave is emitted form the origin. The observers in both S and S′ will find that the spherical wave-front is emerging from their respective centers ***with the same speed c.***



Hence each observer would write the equation of the wave front in their own frame of reference as follows.

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Constancy of *c*

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Substituting the transformation equations in the above equation, we get

$$a_{11}^2 (x-vt)^2 + y^2 + z^2 = c^2 (a_{41} x + a_{44} t)^2$$

$$x' = a_{11}(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = a_{41}x + a_{44}t.$$

$$a_{11}{}^2(x-vt)^2+y^2+z^2=c^2(a_{41}x+a_{44}t)^2$$

$$(a_{11}{}^2-a_{41}{}^2c^2)x^2+y^2+z^2 - 2(a_{11}{}^2v+a_{41}a_{44}c^2)xt = (a_{44}{}^2c^2-a_{11}{}^2v^2)t^2$$

Comparing with the other equation

$$x^2 + y^2 + z^2 = c^2t^2$$

we get

$$(a_{11}{}^2 - a_{41}{}^2 c^2) = 1$$

$$(a_{11}{}^2 v + a_{41} a_{44} c^2) = 0$$

Since no "<u>xt</u>" term

$$(a_{44}{}^2 c^2 - a_{11}{}^2 v^2) = c^2$$

Three Equations three unknowns; Solving yields

$$a_{11} = 1 / \sqrt{1 - (v^2 / c^2)}$$

$$a_{41} = -(v / c^2) / \sqrt{1 - (v^2 / c^2)}$$

$$a_{44} = 1 / \sqrt{1 - (v^2 / c^2)}$$

$$x' = a_{11}(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = a_{41}x + a_{44}t.$$

Phew !

$$x' = \frac{x - vt}{\sqrt{1 - (v^2 / c^2)}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - (v / c^2)x}{\sqrt{1 - (v^2 / c^2)}}$$

Common abbreviations

$$\beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1 - (v^2 / c^2)}}$$

Using abbreviations

Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - (v^2/c^2)}}$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Lorentz Equations

$$x' = \gamma\left(x - vt\right)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The inverse transformation

$$x = \gamma\left(x' + vt'\right)$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

In the limit v<<<<c the Lorentz transformation equations reduce to Galilean transformation as expected/hoped.

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

$$\gamma = \frac{1}{\sqrt{1 - (v^2 / c^2)}}$$

Lorentz factor

HAPPY NEWTON/GALILEO

The Lorentz Transformations can be written in the following matrix form

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

Sometimes we use a notation in which $\omega = ict$ and $\omega' = ict'$

$$\text{For example, when v} = 0.9\text{c},$$

$$\beta = 0.9$$

$$\gamma = \frac{1}{\sqrt{1-0.81}} = 2.29$$

$$For \text{ v} = 0.6c,\ \gamma = 1.25 = 5/4$$

$$For \text{ v} = 0.8c,\ \ \gamma = 1.67 = 5/3$$

Properties of γ

Recall that $\beta = \mathrm{v}/c < 1$ for all observers.

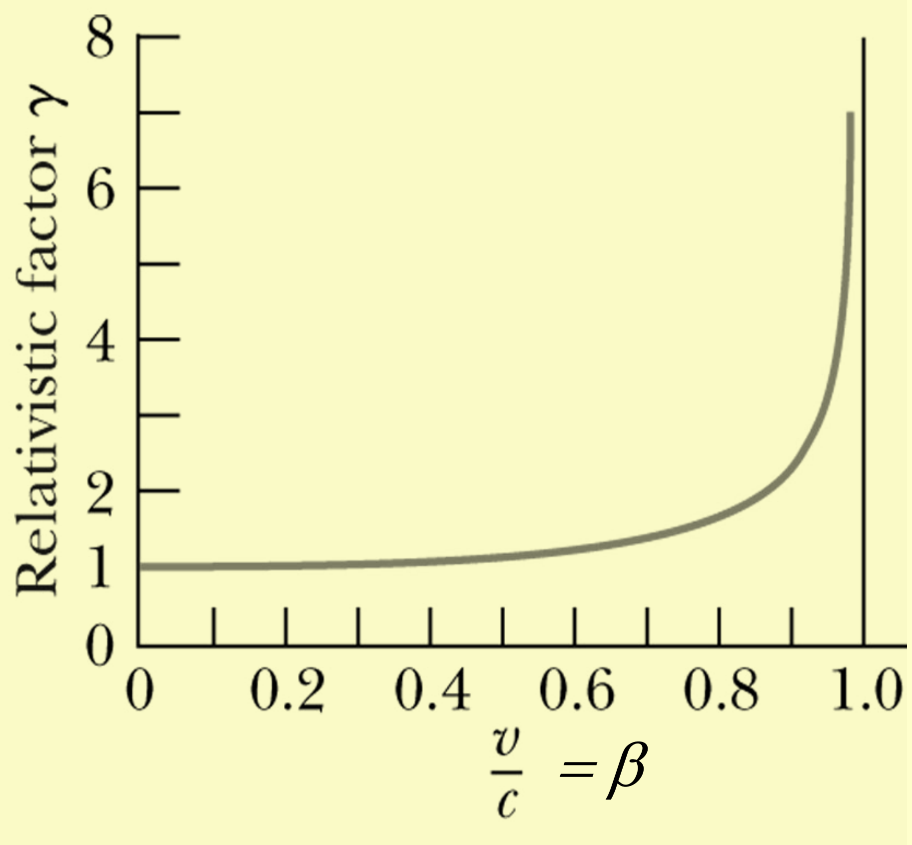
γ equals 1 only when $\mathrm{v} = 0$.

In general:

$$\gamma \geq 1$$

Graph of γ vs. β:

(note $\mathrm{v} < c$)



Final Comment

If an event occurs at $x=0$ in S, it would appear to occur at $x'=0$ in S' only when the event occurred at $t=0$. At a later time the origins are no longer coincident (True even classically)

If an event occurs at $t=0$ in S, it would appear to occur at $t'=0$ in S' only when the event took place at origin ($x=0$). (***Relativistic Effect;*** **(coincident space))**

Final Final Comment

Assume that the frames are inertial. Further, relative velocity of S’ frame is along the x-axis of frame S, which is also coincident with x’ axis of S’ frame, y and y’ axes and similarly z and z’ axes are assumed to be parallel to each other. Time is assumed to be zero both in S and S’ frame at the incident when the origins of the frames are coincident.

Lorentz Transformation

$$x' = \frac{x - \mathrm{v}t}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \mathrm{v}x/c^2}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

Length contraction

Simultaneity problems

Time Dilation

$$x = \frac{x' + \mathrm{v}t'}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \mathrm{v}x'/c^2}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

If $\mathrm{v} << c$, i.e., $\beta \approx 0$ and $\gamma \approx 1$, yielding the familiar Galilean transformation.

Space and time are now linked, and the frame velocity cannot exceed c.

Show that $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz Transformation

$$x^2+y^2+z^2-c^2t^2=\gamma^2(x'+vt')^2+y'^2+z'^2-c^2\gamma^2\left[t'+\left(vx'/c^2\right)\right]^2$$

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}}$$
$$y'=y$$
$$z'=z$$
$$t'=\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}$$

$$=x'^2\gamma^2\left(1-\frac{v^2}{c^2}\right)+y'^2+z'^2-t'^2\gamma^2c^2\left(1-\frac{v^2}{c^2}\right)$$

$$=x'^2+y'^2+z'^2-c^2t'^2$$

$$x=\frac{x'+vt'}{\sqrt{1-v^2/c^2}}$$
$$y=y'$$
$$z=z'$$
$$t=\frac{t'+vx'/c^2}{\sqrt{1-v^2/c^2}}$$

That is, $x^2 + y^2 + z^2 - c^2 t^2$ is invariant under Lorentz transformation.

Lorentz Invariant

Spacecraft S' is on its way to Alpha Centauri when Spacecraft **S** passes it at relative speed **c**/2. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship **S**.

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Because the position of the clock in S' is fixed, $\Delta x' = 0$, and the time interval Δt becomes:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}.$$

$$\Delta t = \frac{1.2\,s}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$$

$$= 1.6\,s.$$

$$x = \frac{x' + \mathrm{v}t'}{\sqrt{1-\mathrm{v}^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \mathrm{v}x'/c^2}{\sqrt{1-\mathrm{v}^2/c^2}}$$

A surveyor measures a street to be $L = 100\,m$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed $0.20c$, assuming the x coordinates of the two frames coincide at time $t = 0$.

$$\begin{aligned} x_2' - x_1' &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{L}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

$$\begin{aligned} L' &= (100\,m)\sqrt{1 - v^2/c^2} \\ &= (100\,m)\sqrt{1 - (0.20)^2} = 98.0\,m. \end{aligned}$$

1. According to O' a flash of light strikes at x' = 60 m, y' = z' = 0, t' = 8 x 10^{-8} O' has a velocity of 0.6 c along x from O. Find the spacetime coordinates of the lightning strike according to O ?

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$x - vt = x'\sqrt{1 - v^2/c^2}$$

$$x = \frac{x' + vt}{\sqrt{1 - v^2/c^2}} = \frac{60\,m + \left(0{,}6 \times 3 \times 10^8 \frac{m}{s}\right)(8 \times 10^{-8}\,s)}{\sqrt{1 - (0{,}6\,c)^2/c^2}}$$

$$x = \frac{60\,m + 14{,}4\,m}{\sqrt{1 - 0{,}36\,c^2/c^2}}$$

$$x = \frac{74{,}4\,m}{\sqrt{0{,}64}} = \frac{74{,}4\,m}{0{,}8} = 93\ \text{m}$$

$$t = \frac{t' - \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}} = \frac{(8 \times 10^{-8}\,s) - 0{,}6\,c/c^2\,(60\,m)}{\sqrt{1 - (0{,}6\,c)^2/c^2}}$$

$$t = \frac{(8 \times 10^{-8}\,s) - 0{,}6\,(60\,m)/3 \times 10^8\,m/s}{\sqrt{1 - 0{,}36\,c^2/c^2}}$$

$$t = \frac{(8 \times 10^{-8}\,s) - 36\,s/3 \times 10^8}{\sqrt{0{,}64}}$$

$$t = \frac{(8 \times 10^{-8}\,s) - (12 \times 10^{-8}\,s)}{0{,}8}$$

$$t = \frac{-4 \times 10^{-8}\,s}{0{,}8}$$

$$t = -5 \times 10^{-8}\ \text{s}$$

$$y' = y = 0$$

$$z' = z = 0$$

$$\boxed{\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

Observer O’ has a speed of 0.6 c relative to O and his time is adjusted so that t = t’ = 0 when x = x’ = 0. If O observes that the flash fires at point x = 100 m and at t = 24 x 10⁻⁶ s, what is the time for this occurrence when viewed by observer O’ ?

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{(24 \times 10^{-6}\ s) - \frac{0{,}6\ c}{c^2}(100\ m)}{\sqrt{1-\frac{(0{,}6\ c)^2}{c^2}}}$$

$$t' = \frac{(24 \times 10^{-6}\ s) - \frac{0{,}6\ (160\ m)}{3 \times 10^8\ m/s}}{\sqrt{1-\frac{0{,}36\ c^2}{c^2}}}$$

$$t' = \frac{(24 \times 10^{-6}\ s) - (20 \times 10^{-6}\ s)}{0{,}8}$$

$$t' = \frac{(4 \times 10^{-6}\ s)}{0{,}8}$$

$$t' = 5 \times 10^{-6}\ s$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Two guns, A and B, located at $x_A = 0$ and $x_B = 1.5$ km in S frame, fire at an enemy plane. The gun A fires at t = 0 s, while the gun B fires at t =1. 0 μs. An observer in S’, which is moving in x-direction relative to S finds that both the guns are fired at the same time. Determine the speed of S’ in S frame.

$E1: (x_1 = 0, t_1 = 0)$ $E2: (x_2 = 1.5 \times 10^3, t_2 = 1 \times 10^{-6})$

In S' frame $t_1' = t_2'$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\Rightarrow \gamma\left(1 \times 10^{-6} - \frac{v \times 1.5 \times 10^3}{9 \times 10^{16}}\right) = 0$$

$$\Rightarrow v = \frac{1 \times 10^{-6} \times 9 \times 10^{16}}{1.5 \times 10^3} = 6 \times 10^7 \text{ m/s} = 0.2c$$