# CHANGE OF VARIABLE

SUPPOSE $\Omega \subseteq \mathbb{R}^2$ IS OPEN AND

$g : \Omega \to \mathbb{R}^2$, $g(u,v) = (g_1(u,v), g_2(u,v))$

WHERE $g_1, g_2 : \Omega \to \mathbb{R}$ ARE SUCH THAT

$(g_i)_x$, $(g_i)_y$ $(i = 1,2)$ EXIST.

THE JACOBIAN OF $g$ IS DEFINED AS

$$J(P) := \begin{vmatrix} \frac{\partial g_1}{\partial u}(P) & \frac{\partial g_1}{\partial v}(P) \\ \frac{\partial g_2}{\partial u}(P) & \frac{\partial g_2}{\partial v}(P) \end{vmatrix}.$$

FOR $P \in \Omega$.

WE SHALL DESCRIBE A FORMULA SIMILAR TO THE CHANGE OF VARIABLE IN THE ONE VARIABLE CASE.

$J : \Omega \to \mathbb{R}$

🚩 SUPPOSE $D$ IS AN ELEMENTARY (TYPE I OR II) REGION IN $\mathbb{R}^2$, AND SUPPOSE $f: D \to \mathbb{R}$ IS CONTINUOUS. LET $g: \Omega \to \mathbb{R}^2$, $g(u,v) = (g_1, g_2)$ $(\Omega \subseteq \mathbb{R}^2)$ (FOR SOME OPEN SET $\Omega$) SATISFY

- 🚩 $g$ IS 1-1
- 🚩 $g_1, g_2 : \Omega \to \mathbb{R}$ HAVE CONT. PARTIAL DERIVATIVES.
- 🚩 $J(u,v) \neq 0 \quad \forall \ (u,v) \in \Omega$
- 🚩 THERE EXISTS $E \subseteq \Omega$ S.T $g(E) = D$.
  ($E$ IS ELEMENTARY)

THEN,

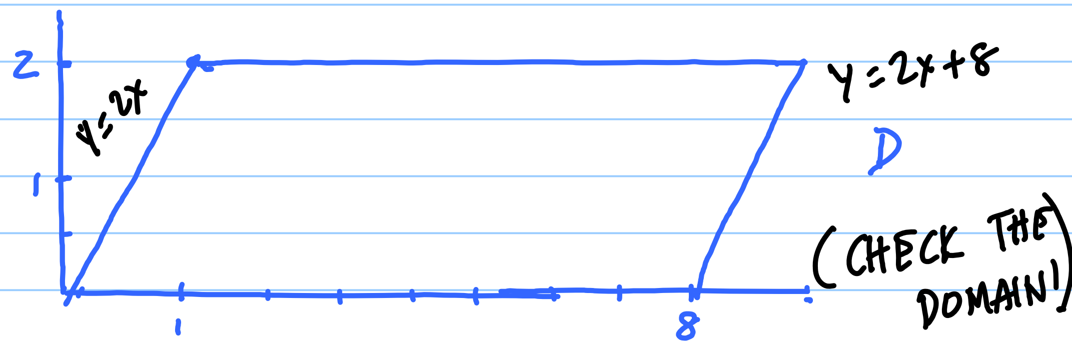
$$\iint_D f(x,y)\,dx\,dy = \iint_E f(g_1(u,v), g_2(u,v))\,|J(u,v)|\,du\,dv$$

🚩 THIS FORMULA EXTENDS TO $D$ WHICH ARE FINITE UNIONS OF NON-OVERLAPPING ELEMENTARY REGIONS.

# EXAMPLE

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, 0 \le y \le 2,\ \frac{y}{2} \le x \le \frac{y+4}{2} \right\}$$

$$f(x,y) = y^3 (2x-y)\, e^{(2x-y)^2} \quad \text{FOR } (x,y) \in D.$$

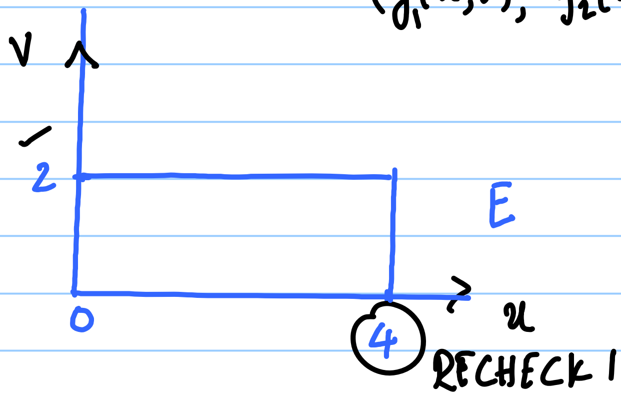



$v = y$

$u = 2x - y$

$f($

$$(g_1(u,v),\ g_2(u,v)) = (x,y)$$




$$x = \frac{u+v}{2} \qquad y = v$$

$g = (g_1, g_2) = (x, y)$, so

$$J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2} \quad \forall\ (u,v)$$

$g(E) = D.$

HENCE, BY CHANGE OF VARIABLE,

$$\iint_D f(x,y)\,dx\,dy = \iint_E v^3 u e^{u^2} \frac{du\,dv}{2} = \frac{1}{2} \iint_E u e^{u^2} v^3 \,du\,dv$$

E IS A RECTANGULAR DOMAIN, i.e,

$$E = [0, \quad ] \times [0,2]$$

COMPLETE THE CALCULATION!

# Polar Coordinates

$$g(r,\theta) = (x,y), \quad g_1(r,\theta) = r\cos\theta, \quad g_2(r,\theta) = r\sin\theta$$

**Change of variable from rectangular to polar:**

$$D \subset \overset{r}{(0,\infty)} \times \overset{\theta}{[0,2\pi)}, \quad E = g(D)$$

If $f$ is integrable over $E$ then $f \circ g$ is integrable over $D$ and

$$\iint_E f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

## Example

$$D = \left\{(x,y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}, \quad f(x,y) = y^2.$$

Consider the change of variable

$$x = ar\cos\theta, \quad y = br\sin\theta$$

$$J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

Consider $E = \{(x,y) \mid x^2 + y^2 \le 1\}$

$$= \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

IF

$$E = [0,1] \times [0,2\pi]$$

THEN $g(D) = E$ SO BY THE CHANGE OF VARIABLE FORMULA,

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_0^{2\pi} (b^2 r^2 \sin^2\theta)\, abr\, dr\, d\theta$$

$$= ab^3 \int_0^1 \int_0^{2\pi} r^3 \sin^2\theta\, dr\, d\theta$$

BY FUBINI'S THEOREM (CHECK!) THIS CAN BE CALCULATED AS

$$\int_0^1 r^3\,dr \int_0^{2\pi} \sin^2\theta\,d\theta$$