2.9. Tutorial Sheet No. 8: Multiple Integrals

(1) For the following, write an equivalent iterated integral with the order of integration reversed:

(i) $\displaystyle\int_0^1 \left[\int_1^{e^x} dy\right] dx.$ (ii) $\displaystyle\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy.$

(2) Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi} \left[\int_x^{\pi} \frac{\sin y}{y} dy\right] dx.$ (ii) $\displaystyle\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy.$

(iii) $\displaystyle\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)dx.$

(3) Find $\displaystyle\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0,\ x = 1$ and $y = 2x$.

(4) Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)d(x,y),$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

(5) Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find $\displaystyle\iint_D d(x,y)$ by transforming it to $\displaystyle\iint_E d(u,v)$, where $x = \dfrac{u}{v},\ y = uv,\ v > 0$.

(6) Find

$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$

where $D(r)$ equals:

(i) $\{(x,y) : x^2 + y^2 \le r^2\}$.
(ii) $\{(x,y) : x^2 + y^2 \le r^2,\ x \ge 0,\ y \ge 0\}$.
(iii) $\{(x,y) : |x| \le r,\ |y| \le r\}$.
(iv) $\{(x,y) : 0 \le x \le r,\ 0 \le y \le r\}$.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

(8) Express the solid $D = \{(x,y,z) | \sqrt{x^2 + y^2} \le z \le 1\}$ as

$$\{(x,y,z) | a \le x \le b, \phi_1(x) \le y \le \phi_2(x), \xi_1(x,y) \le z \le \xi_2(x,y)\}.$$

(9) Evaluate
$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$
Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

(10) Using suitable change of variables, evaluate the following:

(i)
$$I = \iiint_D (z^2x^2 + z^2y^2) dxdydz$$
where D is the cylindrical region $x^2+y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(ii)
$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$
over the region enclosed by the unit sphere in $\mathbb{R}^3$.

2.10. Tutorial Sheet No.9: Vector fields, Curves, parameterization

(1) Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:

(i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.

(ii) $\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.

(iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

(2) For any two scalar functions f, g on $\mathbb{R}^m$ establish the relations:

(i) $\nabla(fg) = f\nabla g + g\nabla f$.

(ii) $\nabla f^n = nf^{n-1}\nabla f$.

(iii) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

(3) Prove the following:

(i) $\nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + (\nabla f) \cdot \mathbf{v}$.

(ii) $\nabla \times (f\mathbf{v}) = f(\nabla \times \mathbf{v}) + \nabla f \times \mathbf{v}$.

(iii) $\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$,
where $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

(iv) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

(v) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$

(vi) $\nabla \times (\nabla f) = 0$.

(vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0$.

(4) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Show that

(i) $\nabla^2 f = \text{div }(\nabla f(r)) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$.
(ii) div $(r^n\mathbf{r}) = (n+3)r^n$.
(iii) $\text{curl}(r^n\mathbf{r}) = 0$
(iv) div $(\nabla\frac{1}{r}) = 0$ for $r \neq 0$.

(5) Prove that
(i) $\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v})$
Hence, if $\mathbf{u}, \mathbf{v}$ are irrotational, $\mathbf{u}\times\mathbf{v}$ is solenoidal.
(ii) $\nabla\times(\mathbf{u}\times\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} + (\nabla\cdot\mathbf{v})\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{v}$.
(iii) $\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{u}) + \mathbf{u}\times(\nabla\times\mathbf{v})$.
Hint: Write $\nabla = \sum \mathbf{i}\dfrac{\partial}{\partial x}$, $\nabla\times\mathbf{v} = \sum \mathbf{i}\dfrac{\partial}{\partial x}\times\mathbf{v}$ and $\nabla\cdot\mathbf{v} = \sum \mathbf{i}\dfrac{\partial}{\partial x}\cdot\mathbf{v}$.

(6) (i) If $\mathbf{w}$ is a vector field of constant direction and $\nabla\times\mathbf{w} \neq 0$, prove that $\nabla\times\mathbf{w}$ is always orthogonal to $\mathbf{w}$.
(ii) If $\mathbf{v} = \mathbf{w}\times\mathbf{r}$ for a constant vector $\mathbf{w}$, prove that $\nabla\times\mathbf{v} = 2\mathbf{w}$.
(iii) If $\rho\mathbf{v} = \nabla p$ where $\rho(\neq 0)$ and p are continuously differentiable scalar functions, prove that

$$\mathbf{v}\cdot(\nabla\times\mathbf{v}) = 0.$$

(7) Calculate the line integral of the vector field

$$f(x,y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1,1)$ to $(1,1)$ along $y = x^2$.

(8) Calculate the line integral of

$$f(x,y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

(9) Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

(10) Calculate

$$\oint_C y\,dx + z\,dy + x\,dz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

2.11. Tutorial Sheet No.10: Line integrals and applications

(1) Consider the helix

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k} \text{ lying on } x^2 + y^2 = a^2.$$

Parameterize this in terms of arc length.

(2) Evaluate the line integral

$$\oint_C \frac{x^2ydx - x^3dy}{(x^2+y^2)^2}$$

where C is the square with vertices $(\pm 1, \pm 1)$ oriented in the counterclockwise direction.

(3) Let $\mathbf{n}$ denote the outward unit normal to $C : x^2 + y^2 = 1$. Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot d\mathbf{n}.$$

(4) Evaluate

$$\int_{(0,0)}^{(2,8)} \text{grad } (x^2 - y^2) \cdot d\mathbf{r}$$

where C is $y = x^3$.

(5)

Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices $(1, 0), (0, 1)$, $(-1, 0)$ and $(0, -1)$ traversed once in the counter clockwise direction.

(6) A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0, 0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b, find the value of a.

(7) Calculate the work done by the force field $F(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ where $z \geq 0, a > 0$ (specify the orientation of C that you use.)

(8) Determine whether or not the vector field $f(x, y) = 3xy\mathbf{i} + x^3y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

(9) Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} := f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

Show that $\dfrac{\partial}{\partial y} f_1(x,y) = \dfrac{\partial}{\partial x} f_2(x,y)$ on S while $\mathbf{F}$ is not the gradient of a scalar field on S.

(10) For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\oint_C \mathbf{v} \cdot d\mathbf{r}$ where C is any arbitrary smooth closed curve.

(11) A radial force field is one which can be expressed as $\mathbf{F} = f(r)\mathbf{r}$ where $\mathbf{r}$ is the position vector and $r = \|\mathbf{r}\|$. Show that $\mathbf{F}$ is conservative if f is continuous.

2.12. Tutorial Sheet No.11: Green's theorem and applications

(1) Verify Green's theorem in each of the following cases:
   (i) $f(x,y) = -xy^2$; $g(x,y) = x^2y$; $R:\ x \geq 0,\ 0 \leq y \leq 1 - x^2$;
   (ii) $f(x,y) = 2xy$; $g(x,y) = e^x + x^2$; where R is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

(2) Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2\,dx + x\,dy$ where:
   (i) R is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.
   (ii) R is the square with vertices $(\pm 1, \pm 1)$.
   (iii) R is the disc of radius 2 and center $(0,0)$ (specify the orientation you use for the curve.)

(3) For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2}\oint_C r^2 d\theta.$$

Use this to compute the area enclosed by the following curves:
   (i) The cardioid: $r = a(1 - \cos\theta),\ 0 \leq \theta \leq 2\pi$.
   (ii) The lemniscate: $r^2 = a^2\cos 2\theta,\ -\pi/4 \leq \theta \leq \pi/4$.

(4) Find the area of the following regions:
   (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.
   (ii) The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \leq t \leq 2\pi.$$

   (iii) The region bounded by the limaçon

$$r = 1 - 2\cos\theta,\ 0 \leq \theta \leq \pi/2$$

and the two axes.

(5) Evaluate

$$\oint_C xe^{-y^2}dx + [-x^2ye^{-y^2} + 1/(x^2+y^2)]dy$$

around the square determined by $|x| \le a$, $|y| \le a$ traced in the counter clockwise direction.

(6) Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3dy - y^3dx,$$

where I_0 is the polar moment of inertia of the region R enclosed by C.

(7) Consider $a = a(x,y)$, $b = b(x,y)$ having continuous partial derivatives on the unit disc D. If

$$a(x,y) \equiv 1,\ b(x,y) \equiv y$$

on the boundary circle C, and

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j};\ \mathbf{v} = (a_x - a_y)\mathbf{i} + (b_x - b_y)\mathbf{j}, \mathbf{w} = (b_x - b_y)\mathbf{i} + (a_x - a_y)\mathbf{j},$$

find

$$\iint_D \mathbf{u}\cdot\mathbf{v}\, dxdy \ \text{ and } \ \iint_D \mathbf{u}\cdot\mathbf{w}\, dxdy.$$

(8) Let C be any closed curve in the plane. Compute $\oint_C \nabla(x^2-y^2)\cdot\mathbf{n}ds$.

(9) Recall the Green's Identities:

(i) $\displaystyle\iint_R \nabla^2 w\, dxdy = \oint_{\partial R} \frac{\partial w}{\partial \mathbf{n}}\, ds.$

(ii) $\displaystyle\iint_R [w\nabla^2 w + \nabla w\cdot\nabla w]\, dxdy = \oint_{\partial R} w\frac{\partial w}{\partial \mathbf{n}}\, ds.$

(iii) $\displaystyle\oint_{\partial R}\left(v\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial v}{\partial \mathbf{n}}\right)\, ds = \iint_R (v\nabla^2 w - w\nabla^2 v)\, dxdy.$

(a) Use (i) to compute

$$\oint_C \frac{\partial w}{\partial \mathbf{n}}\, ds$$

for $w = e^x \sin y$, and R the triangle with vertices $(0,0), (4,2), (0,2)$.

(b) Let D be a plane region bounded by a simple closed curve C and let $\mathbf{F}, \mathbf{G} : U \longrightarrow \mathbb{R}^2$ be smooth functions where U is a region containing $D \cup C$ such that

$$\operatorname{curl}\mathbf{F} = \operatorname{curl}\mathbf{G},\ \operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G} \ \text{ on } \ D\cup C$$

and

$$\mathbf{F}\cdot\mathbf{N} = \mathbf{G}\cdot\mathbf{N} \ \text{ on } \ C,$$

where $\mathbf{N}$ is the unit normal to the curve. Show that $\mathbf{F} = \mathbf{G}$ on D.

(10) Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)
$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}$$
where C is any simple closed curve not passing through the origin.

(ii)
$$\oint_C \frac{x^2y dx - x^3 dy}{(x^2+y^2)^2},$$
where C is the square with vertices $(\pm 1, \pm 1)$.

(iii) Let C be a smooth simple closed curve lying in the annulus $1 < x^2+y^2 < 2$. Find
$$\oint_C \frac{\partial(\ln r)}{\partial y} dx - \frac{\partial(\ln r)}{\partial x} dy.$$

2.13. Tutorial Sheet No.12: Surface area and surface integrals

(1) Find a suitable parameterization $\mathbf{r}(u,v)$ and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ for the following surface:

(i) The plane $x - y + 2z + 4 = 0$.

(ii) The right circular cylinder $y^2 + z^2 = a^2$.

(iii) The right circular cylinder of radius 1 whose axis is along the line $x = y = z$.

(2) (a) For a surface S let the unit normal $\mathbf{n}$ at every point make the same acute angle α with z-axis. Let SA_{xy} denote the area of the projection of S onto the xy plane. Show that SA, the area of the surface S satisfies the relation: $SA_{xy} = SA\,\cos\alpha$.

(b) Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the ares of S is $\sqrt{S_1^2+S_2^2+S_3^2}$.

(3) Compute the surface area of that portion of the sphere $x^2+y^2+z^2 = a^2$ which lies within the cylinder $x^2+y^2 = ay$, where $a > 0$.

(4) A parametric surface S is described by the vector equation
$$\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\mathbf{k},$$
where $0 \le u \le 4$ and $0 \le v \le 2\pi$.

(i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

(ii) Compute the vector $\mathbf{r}_u \times \mathbf{r}_v$ in terms of u and v.

(iii) The area of S is $\frac{\pi}{n}(65\sqrt{65} - 1)$ where n is an integer. Compute the value of n.

(5) Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

(6) A sphere is inscribed in a right circular cylinder. The sphere is sliced by two parallel planes perpendicular the axis of the cylinder. Show that the portions of the sphere and the cylinder lying between these planes have equal surface areas.

(7) Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, using

(i) The vector representation $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + (1 - 2u)\mathbf{k}$.

(ii) An explicit representation of the form $z = f(x, y)$.

(8) If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, compute the value of the surface integral (with the choice of outward unit normal)

$$\iint_S xz dy \wedge dz + yz dz \wedge dx + x^2 dx \wedge dy.$$

Choose a representation in which the fundamental vector product points in the direction of the outward normal.

(9) A fluid flow has flux density vector

$$\mathbf{F}(x, y, z) = x\mathbf{i} - (2x + y)\mathbf{j} + z\mathbf{k}.$$

Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and let $\mathbf{n}$ denote the unit normal that points out of the sphere. Calculate the mass of the fluid flowing through S in unit time in the direction of $\mathbf{n}$.

(10) Solve the previous exercise when S includes the planar base of the hemisphere also with the outward unit normal on the base being $-\mathbf{k}$.

2.14. Tutorial Sheet No.13: Divergence theorem and applications

(1) Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$$

for the region

$$R : y^2 + z^2 \leq x^2; 0 \leq x \leq 4.$$

(2) Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(3) Let R be a region bounded by a piecewise smooth closed surface S with outward unit normal

$$\mathbf{n} = n_x\,\mathbf{i} + n_y\,\mathbf{j} + n_z\,\mathbf{k}.$$

Let $u, v : R \to \mathbb{R}$ be continuously differentiable. Show that

$$\iiint_R u\,\frac{\partial v}{\partial x}dV = -\iiint_R v\,\frac{\partial u}{\partial x}dV + \int_{\partial R} u\,v\,n_x\,dS.$$

[Hint: Consider $\mathbf{F} = u\,v\,\mathbf{i}$.]

(4) Suppose a scalar field ϕ, which is never zero has the properties

$$\|\nabla\phi\|^2 = 4\phi \text{ and } \nabla\cdot(\phi\nabla\phi) = 10\phi.$$

Evaluate $\displaystyle\iint_S \frac{\partial\phi}{\partial\mathbf{n}}\,dS$, where S is the surface of the unit sphere.

(5) Let V be the volume of a region bounded by a closed surface S and $\mathbf{n} = (n_x, n_y, n_z)$ be its outer unit normal. Prove that

$$V = \iint_S x\,n_x\,dS = \iint_S y\,n_y\,dS = \iint_S z\,n_z\,dS$$

(6) Compute $\displaystyle\iint_S (x^2 dy\wedge dz + y^2\,dz\wedge dx + z^2 dx\wedge dy)$, where S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

(7) Compute $\displaystyle\iint_S yzdy\wedge dz + zxdz\wedge dx + xydx\wedge dy$, where S is the unit sphere.

(8) Let $\mathbf{u} = -x^3\mathbf{i} + (y^3 + 3z^2\sin z)\mathbf{j} + (e^y\sin z + x^4)\mathbf{k}$ and S be the portion of the sphere $x^2+y^2+z^2 = 1$ with $z \ge \frac{1}{2}$ and $\mathbf{n}$ is the unit normal with positive z-component. Use Divergence theorem to compute $\displaystyle\iint_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\,dS$.

(9) Let p denote the distance from the origin to the tangent plane at the point (x,y,z) to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Prove that

(a) $\displaystyle\iint_S p\,dS = 4\pi abc.$ (b) $\displaystyle\iint_S \frac{1}{p}\,dS = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2).$

(10) Interpret Green's theorem as a divergence theorem in the plane.

2.15. Tutorial Sheet No.14: Stoke's theorem and applications

(1) Consider the vector field $\mathbf{F} = (x-y)\mathbf{i} + (x+z)\mathbf{j} + (y+z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by
(a) $x^2 + (y-a)^2 + z^2 = a^2 : z \geq 0$ (b) $x^2 + (y-a)^2 = a^2$

(2) Evaluate using Stokes Theorem, the line integral
$$\oint_C yz\,dx + xz\,dy + xy\,dz,$$
where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

(3) Compute
$$\iint_S (\operatorname{curl} \mathbf{v}) \cdot \mathbf{n} dS,$$
where $\mathbf{v} = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$ and $\mathbf{n}$ is the outward unit normal to S, the surface of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = -3$.

(4) Compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for
$$\mathbf{v} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2},$$
where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes Theorem?

(5) Compute
$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$
where C is the curve cut out of the boundary of the cube
$$0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a$$
by the plane $x + y + z = \frac{3}{2}a$ (specify the orientation of C.)

(6) Calculate
$$\oint_C ydx + zdy + xdz,$$
where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

(7) Consider a plane with unit normal $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. For a closed curve C lying in this plane, show that the area enclosed by C is given by
$$A(C) = \frac{1}{2}\oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz,$$

where C is given the anti-clockwise orientation. Compute $A(C)$ for the curve C given by

$$\mathbf{u}\cos t + \mathbf{v}\sin t,\ 0 \leq t \leq 2\pi.$$

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