SURFACE INTEGRALS - APPLICATIONS

FLUX OF A FLUID ACROSS A SURFACE:

SUPPOSE $V(x,y,z)$ DENOTES THE VELOCITY VECTOR FIELD OF A FLUID, AND LET $\rho(x,y,z)$ DENOTE ITS DENSITY AT (x,y,z). RECALL

$$F(x,y,z) = \rho(x,y,z)\, V(x,y,z)$$

THE FLUX-DENSITY OF THE FLUID.

THE TOTAL MASS OF THE FLUID CROSSING THE SURFACE S IN THE FLOW OF THE FLUID:

$$\iint_S F \cdot n \, dS$$ (THIS IS THE DEFINITION)

WHERE n IS THE UNIT NORMAL VECTOR TO S.

WE ASSUME THAT THE VECTOR FIELDS IN QUESTION ARE CONTINUOUS. TO ENSURE $F \cdot n$ IS CONTINUOUS, WE NEED THE SURFACE S SUCH THAT THE NORMAL VECTOR VARIES CONTINUOUSLY.

ORIENTABLE SURFACES

A SURFACE S IS SAID TO BE **ORIENTABLE** IF $(x,y,z) \mapsto \mathbf{n}(x,y,z)$ IS CONTINUOUS WHERE $\mathbf{n}$ DENOTES THE NORMAL VECTOR TO S AT (x,y,z).

EXAMPLES

- 🚩 PLANES, SPHERES, ELLIPSOIDS (IN FACT, ALL QUADRATIC SURFACES) ARE ORIENTABLE.
- 🚩 NOTE THAT IF $\mathbf{n}(\cdot,\cdot,\cdot)$ IS CONTINUOUS, SO IS $-\mathbf{n}$; THESE GIVE 'INWARD' AND 'OUTWARD' NORMALS.
- 🚩 THE MÖBIUS STRIP IS <u>NOT</u> ORIENTABLE (THIS IS NOT ENTIRELY TRIVIAL!)

FLUX ACROSS SURFACES

SUPPOSE S IS ORIENTABLE, AND F, A CONTINUOUS VECTOR FIELD ON S. THEN

$$\iint_S F \cdot dS := \iint_S (\vec{F} \cdot \mathbf{n})\, dS$$

IS CALLED THE FLUX-INTEGRAL (OR SIMPLY FLUX) OF F ACROSS S.

🚩 IF S HAS PARAMETRIZATION $\mathbf{r}(u,v)$ FOR $(u,v) \in R$, THEN

$$\iint_S F \cdot dS = \pm \iint_R F \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

THE CHOICE OF $\pm$ DEPENDS UPON THE CHOSEN NORMAL.

EXAMPLES

$F = (xz, yz, x^2)$, $S = \{(x,y,z) \mid x^2+y^2+z^2 = a^2\}$

CALCULATE $\iint_S F \cdot dS$.

THE SURFACE IS DESCRIBED BY $G(x,y,z) = 0$,

$G(x,y,z) = x^2+y^2+z^2-a^2 \Rightarrow \nabla G = (2x, 2y, 2z)$

THE UNIT NORMAL IS $\dfrac{\nabla G}{\|\nabla G\|} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$ ON S

HENCE $$\iint_S F \cdot dS = \iint \frac{F \cdot \nabla G}{\|\nabla(G)\|}\, dS$$

$$= \iint_S \frac{(2x^2+y^2)z}{\sqrt{x^2+y^2+z^2}}\, dS = \frac{1}{a}\iint_S (2x^2+y^2)z\, dS$$

TO EVALUATE THIS,

$r(\theta,\phi) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$

WITH $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$

THEN. THE SURFACE INTEGRAL EQUALS 0

EXAMPLE

$$D = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 4\} \,;\; S = \partial D$$

$S = S_1 \cup S_2$; $S_1 \equiv$ OUTER SPHERE, $S_2 \equiv$ INNER SPHERE

$$F = -\frac{x}{r^3}\vec{i} - \frac{y}{r^3}\vec{j} - \frac{z}{r^3}\vec{k}, \quad \text{WHERE}$$

$$r = \sqrt{x^2+y^2+z^2}.$$

$$\iint_S F\cdot dS = \iint_{S_1} F\cdot dS + \iint_{S_2} F\cdot dS$$

$$S_1 = \{(x,y,z) \mid G_1 \equiv x^2+y^2+z^2-4=0\}$$

$$S_2 = \{(x,y,z) \mid G_2 \equiv x^2+y^2+z^2-1=0\}$$

$$n = \frac{\nabla G_1}{\|\nabla G_1\|} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{r}$$

HENCE $\displaystyle\iint_{S_1} F\cdot dS = \iint_{S_1} \frac{(-x^2-y^2-z^2)}{r^4}\, dS = \iint_{S_1} -\frac{dS}{4}$

SIMILARLY, n (ON S_2) $= -\left(\dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{r}\right)$

$$\Rightarrow \iint_{S_2} F\cdot dS = \iint_{S_2} dS$$

$$\Rightarrow \iint_S F\cdot dS = \underbrace{\iint_{S_2} dS}_{4\pi} - \frac{1}{4}\iint dS$$

$$4\pi - \frac{1}{4}(16\pi) = 0$$

SPECIAL FORMS OF THE FLUX INTEGRAL

SUPPOSE S IS EXPLICITLY DESCRIBED BY

$z = g(x,y)$, $(x,y) \in D$, AND A PARAMETRIZATION

OF S IS

$$\mathbf{r}(x,y) = x\vec{i} + y\vec{j} + g(x,y)\vec{k}$$

SO THAT $\mathbf{r}_x \times \mathbf{r}_y = -g_x\vec{i} - g_y\vec{j} + \vec{k}$

HENCE

$$\iint_S (F\cdot\mathbf{n})\,dS = \iint_D (-Pg_x - Qg_y + R)\,dx\,dy$$

WHERE

$$F(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

IF WE WRITE $\mathbf{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$, THEN

$$\iint_S F\cdot dS = \iint_S \left[P\cos\alpha + Q\cos\beta + R\cos\gamma\right] dS$$

IF $\mathbf{n}$ IS THE 'POSITIVE' ORIENTATION OF THE NORMAL

$$\iint_S (R\cos\gamma)\,dS = \iint_D R(x,y,g(x,y))\,dx\,dy \quad \text{IF } \cos\gamma > 0$$

$$= -\iint_D R(x,y,g(x,y))\,dx\,dy \quad \text{IF } \cos\gamma < 0$$

🚩 IF $F = P\vec{i} + Q\vec{j} + R\vec{k}$, AND

$\mathbf{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v)),\ (u,v) \in D$

THEN
$$\mathbf{r}_u \times \mathbf{r}_v = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\vec{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\vec{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\vec{k}$$

THEN,

$$\iint_S P \ dy \wedge dz := \iint_D P(\mathbf{r}(u,v))\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) du\,dv$$

(dy WEDGE dz)

$$\iint_S Q \ dz \wedge dx := \iint_D Q(\mathbf{r}(u,v))\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) du\,dv$$

$$\iint_S R \ dx \wedge dy := \iint_D R(\mathbf{r}(u,v))\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) du\,dv$$

HENCE WE CAN WRITE

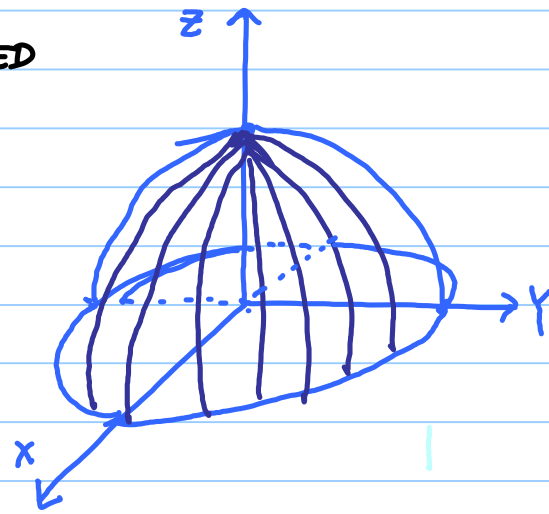
$$\iint_S F \cdot dS = \int P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$

🚩 IF THE SURFACE S IS DESCRIBED AS $z = g(x,y)$, AND $G(x,y,z) = z - g(x,y)$, THEN

$$\iint_S F \cdot dS = \iint_D (F \cdot \nabla G)\, dx\, dy$$

EXAMPLES

$F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$; $S \equiv z = 1 - x^2 - y^2$.



THE SURFACE IS DESCRIBED

AS $G(x,y,z) = 0$, WHERE

$G \equiv x^2 + y^2 + z - 1$

$\nabla G = (2x, 2y, 1)$

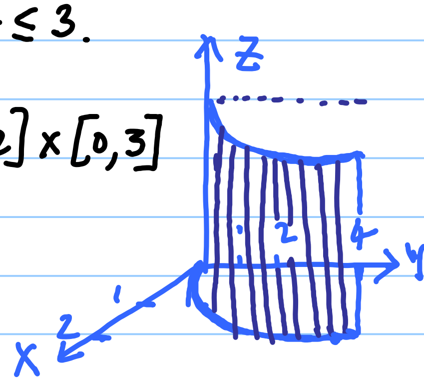
HENCE,

$$\iint_S F \cdot dS = \iint_R (x,y,z)\cdot(2x, 2y, 1)\,dx\,dy$$ WHERE

$R = \{x^2 + y^2 \le 1\}$

$F(x,y,z) = 3z^2\vec{i} + 6\vec{j} + 6xz\vec{k}$

S: $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$.



$r(x,z) = (x, x^2, z)$. ON $D = [0,2]\times[0,3]$

THEN

$r_x \times r_z = 2x\vec{i} - \vec{j}$

So

$$\iint_S F\cdot dS = \iint_S (F\cdot n)\,dS = \iint_D F\cdot(r_x \times r_z)\,dx\,dz$$

$$= \int_0^2\int_0^3 (3z^2\vec{i} + 6\vec{j} + 6xz)\cdot(2x\vec{i} - \vec{j})\,dz\,dx$$

ALTERNATELY,

$$\iint_S (F\cdot n)\,dS = \iint_S P\cos\alpha\,dS + \iint_S Q\cos\beta\,dS + \iint_S R\cos\gamma\,dS$$

WHERE $n = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$

NOW, $n = \dfrac{r_x \times r_z}{\| r_x \times r_z \|} = \dfrac{2x}{\sqrt{1+4x^2}}\,\vec{i} - \dfrac{1}{\sqrt{1+4x^2}}\,\vec{j}$

HENCE

$$\iint_S P\cos\alpha\,dS = \iint_{D'} P\,dy\,dz, \quad \text{WHERE}$$

D' = PROJECTION OF S ON THE YZ PLANE.

$$= \{(y,z) \mid 0 \le y \le 4,\ 0 \le z \le 3\}.$$

AND

$$\iint_S Q\cos\beta\,dS = \iint_D Q\,dx\,dz$$

HENCE $\displaystyle\iint_S F\cdot dS = \iint_{D'} P\,dy\,dz - \iint_D Q\,dx\,dz$

$$= \int_0^4\int_0^3 3z^2\,dy\,dz - \int_0^2\int_0^3 6\,dx\,dz$$