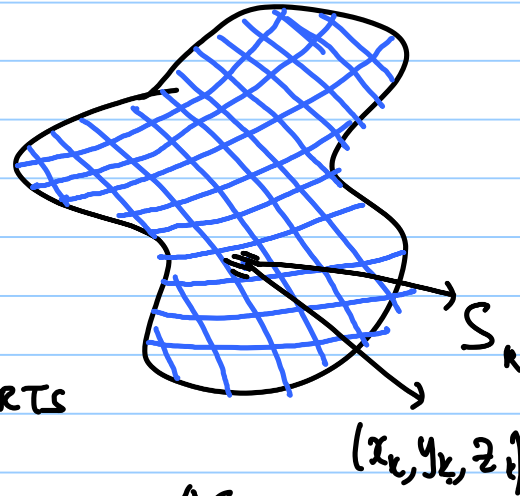
SURFACE INTEGRALS



SUPPOSE f IS A CONTINUOUS SCALAR FIELD, ON S, A SURFACE OF FINITE AREA.

PARTITION S INTO SMALLER PARTS $S_1, S_2, \ldots, S_N$ WITH AREAS $\Delta S_1, \ldots, \Delta S_N$.

LET $(x_k, y_k, z_k) \in S_k$. THEN DEFINE THE RIEMANN SUMS

$$\sigma_N = \sum_{k=1}^{N} f(x_k, y_k, z_k)\, \Delta S_k .$$

IF $\sigma_N \to A$ AS $\max(\Delta S_i) \to 0$ THEN WE DEFINE THE SURFACE INTEGRAL:

$$\iint_S f(x,y,z)\, dS = \lim_{\Delta S_i \to 0} \sigma_N .$$

SUPPOSE S IS SMOOTH, REGULAR, AND HAS PARAMETRIZATION $\mathbf{r} : R \to \mathbb{R}^3$ $(R \subseteq \mathbb{R}^2)$.

IF $\mathbf{r}(u,v)$ IS CONTINUOUS ON R, AND R IS CLOSED AND BOUNDED, AND $f : R \to \mathbb{R}$ IS CONTINUOUS, THEN

$$\iint_S f(x,y,z)\, dS := \iint_R f(x(u,v), y(u,v), z(u,v)) \|\mathbf{r}_u \times \mathbf{r}_v\| \, du\, dv$$

IS THE SURFACE INTEGRAL OF f OVER S

EXAMPLE

SUPPOSE $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

EVALUATE $\iint_S y^2 \, dS$

CONSIDER THE PARAMETRIZATION:

$$\mathbf{r}(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

$$R = \{(\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

THEN

$$\mathbf{r}_\theta = (-\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0)$$

$$\mathbf{r}_\phi = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi)$$

$$\Rightarrow \|\mathbf{r}_\theta \times \mathbf{r}_\phi\|^2 = \sin^2\phi$$

HENCE,

$$\iint_S y^2 \, dS = \iint_R (\sin\theta\sin\phi)^2 \sin\phi \, d\phi \, d\theta$$

WHERE $R = [0,2\pi] \times [0,\pi]$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \sin^3\phi \, d\phi \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$

$$=$$

SURFACE INTEGRALS ON

EXPLICITLY DESCRIBED SURFACES

SUPPOSE S IS GIVEN BY

$$z = h(x,y), \quad (x,y) \in R$$

WE CAN USE THE PARAMETRIZATION

$$\mathbf{r}(x,y) = x\vec{i} + y\vec{j} + h(x,y)\vec{k}.$$

AND THE SURFACE INTEGRAL $\iint_S f\,dS$ IS GIVEN BY

$$\iint_S f\,dS = \iint_R f(x,y,h(x,y)) \underbrace{\sqrt{1+h_x^2+h_y^2}}_{\|\mathbf{r}_x \times \mathbf{r}_y\|}\, dx\,dy$$

IF THE SURFACE IS GIVEN BY

$x = g(y,z)$, $(y,z) \in D$ WE HAVE (SIMILARLY)

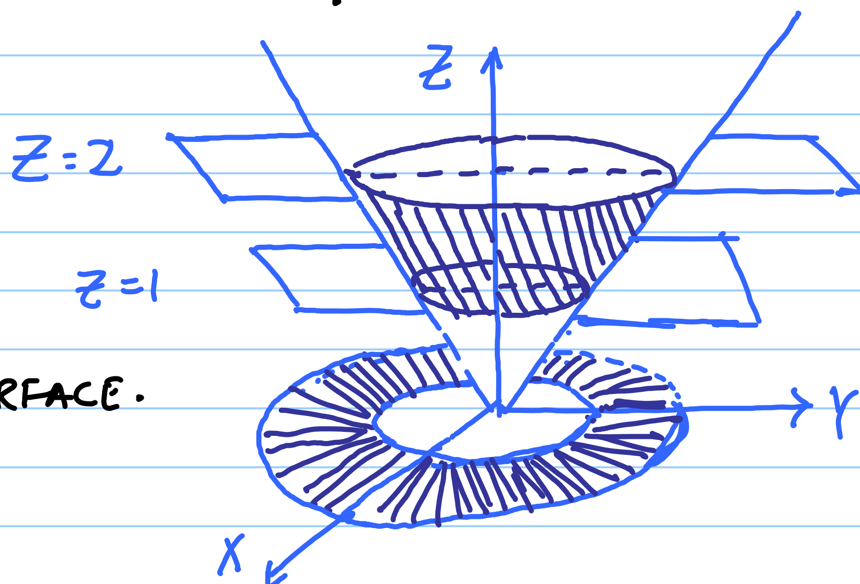
$$\iint_S f\,dS = \iint_D f(g(y,z),y,z)\sqrt{1+g_y^2+g_z^2}\, dy\,dz$$

AND SO ON.

EXAMPLE

🚩 $I = \iint_S z^2\, dS$ WHERE S IS THE SURFACE OF

$z^2 = x^2 + y^2$ BETWEEN $z=1$ & $z=2$



WRITE

$z = \sqrt{x^2+y^2}$ TO

DESCRIBE THE SURFACE.

FURTHERMORE,

$$S_{xy} = \{(x,y) \mid 1 \le \sqrt{x^2+y^2} \le 2\} =: R$$

SO, $$\iint_S z^2\, dS = \iint_R (x^2+y^2)\sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

WHERE $f(x,y) = \sqrt{x^2+y^2}$, SO $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$; $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

HENCE,

$$I = \iint_R (x^2+y^2)\sqrt{2}\, dx\, dy$$
$$= \sqrt{2} \iint_R (x^2+y^2)\, dx\, dy$$

THIS CAN BE CALCULATED BY MAKING A POLAR TRANSFORMATION (EXERCISE)

RECALL THAT WE HAD

$$\text{SURFACE AREA} = \iint_R \sec\gamma \, dx\, dy$$

WHERE γ IS THE ACUTE ANGLE BETWEEN

$\mathbf{r}_x \times \mathbf{r}_y$ AND $\vec{k}$. THIS YIELDS ANOTHER FORMULA.

SUPPOSE $z = h(x,y)$ DESCRIBES THE SURFACE

S:

$$\iint_S f\, dS = \iint_R f(x,y,h(x,y))\sqrt{1+h_x^2+h_y^2}\, dx\, dy$$

$$\Rightarrow \iint_S f(x,y,z)\left(\frac{1}{\sqrt{1+h_x^2+h_y^2}}\right) dS = \iint_R f(x,y,h(x,y))\, dx\, dy$$

So,

$$\iint_S f \cos\gamma\, dS = \iint_R f(x,y,h(x,y))\, dx\, dy$$

SINCE

$$\cos\gamma = \frac{1}{\sqrt{1+h_x^2+h_y^2}}.$$