MA 108-ODE- D3

Lecture 6

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Existence - Uniqueness

Picard's iteration method

[1] **AIM** : To solve

$$y' = f(x, y),\ y(x_0) = y_0 \tag{1}$$

METHOD

1. Integrate both sides of (1) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$

Note that any solution of (1) is a solution of (2) and vice-versa.

[1] Picard used this in his existence-uniqueness proof

Picard's method

2. Solve (2) by iteration:

$$
\begin{aligned}
\phi_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt \\
\phi_2(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\, dt \\
&\vdots \\
\phi_n(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\, dt
\end{aligned}
$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $\phi(x)$ of (1). That is,

$$\phi(x) = \lim_{n\to\infty} \phi_n(x).$$

Example

Example: Solve the IVP:

$$y' = 2t(1+y);\ \ y(0) = 0$$

by the method of successive approximation.
If $y = \phi(t)$, the corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Example continued

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Hence $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Example continued

Recall that $\phi_n(t)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$. Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n \to \infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Hence, $y(t) = e^{t^2} - 1$ is a solution of the IVP.

Proof of uniqueness of solution: Hints

Let all hypothesis in Existence and uniqueness theorem hold: Let R be a rectangle containing (x_0, y_0) in the domain D,

- $f(x, y)$ be continuous at all points (x, y) in $R : |x - x_0| < a,\ |y - y_0| < b$ and
- bounded in R, that is, $|f(x, y)| \leq K\ \ \forall (x, y) \in R$.
- Lipschitz w.r. to second variable f satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|\ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

Suppose $\phi(x)$ and $\psi(x)$ are solutions of $y' = f(x, y), y(x_0) = y_0$ on an interval $(x_0 - h, x_0 + h)$. Thus, both these satisfy the integral equation as well. Then, for $x_0 < x < x_0 + h$,

$$\phi(x) - \psi(x) = \int_{x_0}^{x} (f(s, \phi(s)) - f(s, \psi(s)))ds.$$

Thus,

$$|\phi(x) - \psi(x)| \leq \int_{x_0}^{x} |f(s, \phi(s) - f(s, \psi(s)|ds.$$

Using Lipschitz condition, we have

$$|\phi(x) - \psi(x)| \leq \int_{x_0}^{x} M\, |\phi(s) - \psi(s)| ds.$$

Let $U(x) = \int_{x_0}^{x} |\phi(s) - \psi(s)| ds$. Clearly, $U(x_0) = 0, U(x) \geq 0$. Also, $U'(x) = |\phi(x) - \psi(x)|$ and from above we get

$$U'(x) - MU(x) \leq 0.$$

It yields $\frac{d}{dx}\left(e^{-Mx}U(x)\right) \leq 0$.
Integrating both side from (x_0, x), we get

$$\int_{x_0}^{x} \frac{d}{ds}\left(e^{-Ms}U(s)\right)\, ds \leq 0,$$

and thus $U(x) \leq U(x_0)e^{M(x-x_0)} = 0, \quad \forall\, x_0 < x < x_0 + h$. Hence $U(x) = 0$ for all $x_0 < x < x_0 + h$.
Similarly, derive $U(x) = 0, \quad \forall\, x_0 - h < x < x_0$
Thus, $\phi(x) = \psi(x)$ for all $x_0 - h < x < x_0 + h$.

Summary - First Order Equations

- Linear Equations - Solution
  - Reducible to linear - Bernoulli
- Non-linear equations
  - Variable separable
  - Reducible to variable separable
  - Exact equations - Integrating factors
  - Reducible to Exact
- Existence & Uniqueness results for IVP : $\boxed{y' = f(x, y),\ y(x_0) = y_0}$
  - Peano's existence theorem
  - Picard's existence-uniqueness theorem
- Picard's iteration method