Basic Linear Algebra

Jugal K. Verma and Murali K. Srinivasan

March 4, 2023

Contents

Chapter 1

Vectors, lines and planes

Chapter overview. In this chapter we discuss basic notions related to vectors in the n-space

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

Linear algebra and Euclidean geometry are closely connected. Our discussions about various topics in linear algebra will often be accompanied with related geometric notions. In order to do this precisely, we discuss vectors in $\mathbb{R}^n$ and operations among them such as addition, scalar multiplication, dot product, angle between vectors, projection of a vector onto another vector, equations of lines and planes.

Euclid's postulates. Euclid's book *Elements* written around 300 B. C. contains five postulates:

1. A straight line segment may be drawn from any given point to any other.

2. A straight line segment may be extended to any finite length.

3. A circle may be described with any given point as its center and any distance as its radius.

4. All right angles are congruent.

5. **The Parallel postulate.** If a straight line intersects two other straight lines, and so makes the two interior angles on one side of it together less than two right angles, then the other straight lines will meet at a point if extended far enough on the side on which the angles are less than two right angles.

Euclid's *Elements* consists of 13 books. Euclid defines parallel lines in Book 1 as follows:

Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Many mathematicians tried to deduce the parallel postulate from the first four postulates but failed. However, they proved many equivalent statements such as:

1. The sum of angles in every triangle is two right angles.
2. There exists a pair of straight lines that are at constant distance from each other.
3. Two lines that are parallel to the same line are also parallel to each other.
4. In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagoras' Theorem).
5. The Law of cosines, a generalization of Pythagoras' Theorem.

We introduce the set $\mathbb{R}^n$ of vectors with n components as a model of Euclidean geometry in which the parallel postulate is valid.

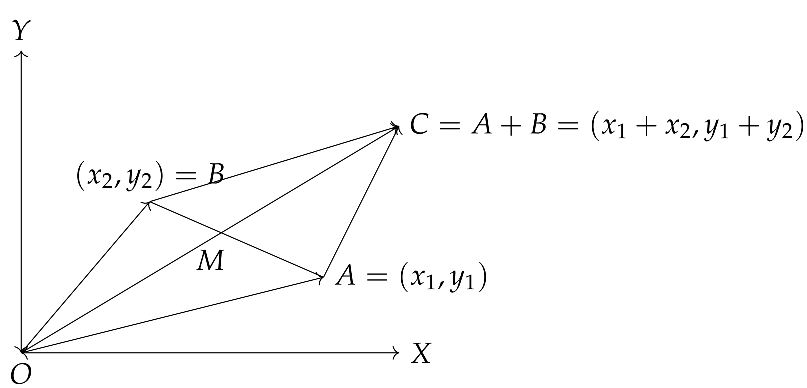
1.1 The n-space $\mathbb{R}^n$

We introduce the n-dimensional Euclidean space $\mathbb{R}^n$. We are familiar with how points in the plane and 3-dimensional space are represented by ordered pairs (x, y) and ordered triplets (x, y, z) where $x, y, z \in \mathbb{R}$. We define a point in n-space to be an n-tuple of real numbers $X = (x_1, x_2, \ldots, x_n)$. The real numbers $x_1, x_2, \ldots, x_n$ are called the **co-ordinates** of the point X. The n-space is the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

The elements of $\mathbb{R}^n$ can be thought of vectors or points. Two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are said to be equal if $x_i = y_i$ for all $i = 1, 2, \ldots, n$. In other words X and Y are equal if they have the same co-ordinates.

Addition of vectors. Vectors in the plane are added using the parallelogram law. This translates into component-wise addition of their co-ordinate vectors.



The diagonals of the parallelogram having sides OA and OB meet at the mid-point M of the line segment AB. Hence the coordinates of M are $((x_1+x_2)/2, (y_1+y_2)/2)$. Let $C = (a, b)$. Then the mid-point of OC has coordinates $(a/2, b/2)$. Therefore $(a, b) = (x_1 + x_2, y_1 + y_2)$. This motivates the definition of sum of vectors in $\mathbb{R}^n$.
Given $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, define their sum

$$X + Y = (x_1 + y_1, x_2 + y_2 + \dots, +x_n + y_n).$$

The following properties follow from the properties of real numbers. Let $X, Y, Z \in \mathbb{R}^n$.

1. Associativity: $(X + Y) + Z = X + (Y + Z)$.
2. Commutativity: $X + Y = Y + X$.
3. The **zero vector** for addition: Let $O = (0, 0, \dots, 0)$. Then $X + O = X$ for all $X \in \mathbb{R}^n$.
4. Existence of additive inverse: Let $X = (x_1, \dots, x_n) \in \mathbb{R}^n$. Define the additive inverse of X to be the vector $-X = (-x_1, -x_2, \dots, -x_n)$. Note that $X + (-X) = O$.

Scalar multiples of vectors. The second important operation on vectors is that of **scalar multiplication** of a vector by a real number. Let $c \in \mathbb{R}$ and $A = (a_1, \dots, a_n) \in \mathbb{R}^n$. We define $cA = (ca_1, ca_2, \dots, ca_n)$. The vector cA is a scaled version of A. The following properties are easy to verify. Let $A, B \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

1. $c(A + B) = cA + cB$.
2. $(c + d)A = cA + dA$.
3. $(cd)A = c(dA)$.
4. $(-1)A = -A$.

Definition 1.1.1. *If $A, B \in \mathbb{R}^n$ and $A = cB$ where $c \neq 0$, then we say that A and B are* **parallel vectors***. If $c > 0$ then we say that A and B have the* **same direction.** *If $c < 0$ then we say that A and B have* **opposite direction.**

Linear combinations of vectors. If $u_1, u_2, \dots, u_m$ are vectors in $\mathbb{R}^n$ and $x_1, x_2, \dots, x_m$ are scalars then the vector

$$x_1 u_1 + x_2 u_2 + \dots + x_m u_m$$

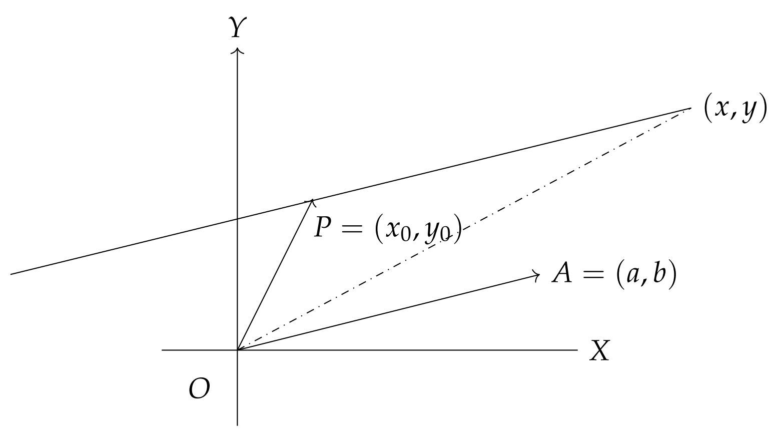
Is called a **linear combination** of the vectors $u_1, u_2, \ldots, u_m$. The scalars $x_1, x_2, \ldots, x_m$ are called the **coefficients** of the linear combination. The vectors $e_i = (0, 0, \ldots, 1, \ldots, 0)$ where $i = 1, 2, \ldots, n$ are called the **standard unit vectors .** Any vector $a = (a_1, a_2, \ldots, a_n)$ can be written as

$$u = a_1e_1 + a_2e_2 + \cdots + a_ne_n.$$

Let a and b are different n-vectors and $x \in \mathbb{R}$. Then $c = xa + (1-x)b$ is a point on the line joining a and b. We say that c is an **affine linear combination** of a and b. If $x \in [0, 1]$ then we say that c is a **convex linear combination** of a and b. In this case c is a point on the line segment joining a and b.

1.2 Parametric equations of lines in $\mathbb{R}^n$

In this section we describe parametric equations of lines in $\mathbb{R}^n$. We shall prove that $\mathbb{R}^2$ is a model of Euclidean geometry in which the parallel postulate is true. We shall also prove that there is a unique line passing through two distinct points. This is one of the axioms of Euclidean geometry.



Let us find the equations of the line $L(P, A)$ in the plane passing through $P = (x_0, y_0)$ that is parallel to a nonzero vector $A = (a, b)$. Let (x, y) be any point on $L(P, A)$. Then $(x, y) - (x_0, y_0) = t(a, b)$ for some real number t. Hence the parametric equations of $L(P, A)$ are

$$x = x_0 + ta \text{ and } y = y_0 + tb.$$

Similarly we can derive the equations of a line in $\mathbb{R}^3$ passing through $P = (x_0, y_0, z_0)$ and parallel to $0 \neq A = (a, b, c)$. These are given by the equations

$$x = x_0 + ta, y = y_0 + tb, z = z_0 + tb \quad t \in \mathbb{R}.$$

Therefore $L(P, A) = \{P + tA \mid t \in \mathbb{R}\}$.

Definition 1.2.1. *Let P be a point in $\mathbb{R}^n$ and $A \in \mathbb{R}^n$ be a nonzero vector. A line through P and parallel to A is the set*

$$L(P, A) = \{P + tA \mid t \in \mathbb{R}\}.$$

The vector A is called the **direction vector** *of the line $L(P, A)$. Two vectors A and B are called* **parallel** *if $A = cB$ for a nonzero real number c. Two lines are called* **parallel** *if their direction vectors are parallel.*

Theorem 1.2.2. *Two lines $L(P, A)$ and $L(P, B)$ are equal if and only if A and B are parallel.*

Proof. Let $L(P, A) = L(P, B)$. Since $A \neq 0$, $P + A = P + tB$ for some $0 \neq t \in \mathbb{R}$. Hence $A = tB$. Therefore A and B are parallel. Conversely let $A = cB$ for a nonzero real number c. Let $Q \in L(P, A)$. Then $Q = P + tA$ for some $t \in \mathbb{R}$. Hence $Q = P + tcB \in L(P, B)$. Thus $L(P, A) \subset L(P, B)$. By symmetry $L(P, B) \subset L(P, A)$. Therefore $L(P, A) = L(P, B)$.

□

Theorem 1.2.3. *Two lines $L(P, A)$ and $L(Q, A)$ are equal if and only if $Q \in L(P, A)$.*

Proof. Let $L(P, A) = L(Q, A)$. Since $Q \in L(Q, A)$, $Q \in L(P, A)$. Conversely let $Q \in L(P, A)$ and $Q = P + cA$ for some $c \in \mathbb{R}$. Let $X \in L(P, A)$. Then $X = P + tA$ for some c. Then $X = Q - cA + tA = Q + (t - c)A \in L(Q, A)$. Hence $L(P, A) \subset L(Q, A)$. We can similarly prove that $L(Q, A) \subset L(P, A)$. Therefore the two lines are equal. □

We now prove that $\mathbb{R}^n$ satisfies Euclid's parallel postulate. The next result is equivalent to Euclid's parallel postulate.

Theorem 1.2.4. *If $Q \notin L(P, A)$ then $L(Q, A)$ unique line containing Q and parallel to $L(P, A)$.*

Proof. Let $L = L(P, A)$. Consider the line $L' = L(Q, A)$. Any other line $L(Q, B)$ that is parallel to L must satisfy $B = cA$ for some nonzero c. Therefore $L' = L(Q, B)$. □

The next result is another axiom of Euclidean Geometry.

Theorem 1.2.5. *Let $P \neq Q \in \mathbb{R}^n$. Then $L(P, Q - P)$ is the unique line containing P and Q.*

Proof. We show that the unique line containing P, Q is $L = L(P, Q - P)$. Note that $L(P, Q - P) = \{P + t(Q - P) \mid t \in \mathbb{R}\}$. Take $t = 0$ and $t = 1$ to see that $P, Q \in L$. Let L' be a line containing P and Q. Then $L' = L(P, A)$ for some A. Then $Q = P + tA$ for some nonzero t. Hence $Q - P$ is parallel to A. Hence $L' = L(P, A) = L(P, Q - P)$. Thus L is uniquely determined as the line $L(P, Q - P)$. □

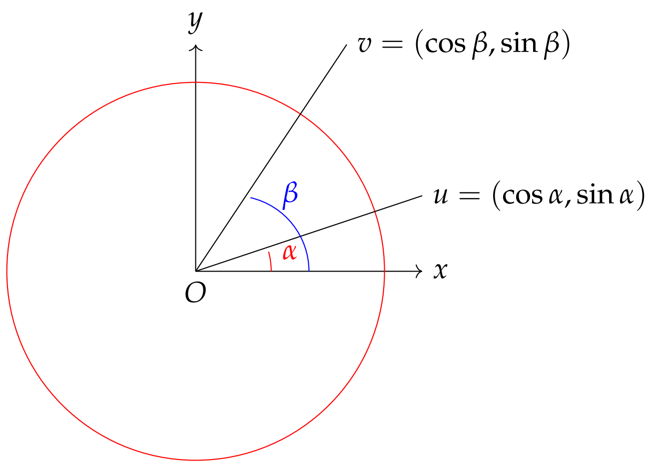
Exercise 1.2.6. Show that two distinct lines in $\mathbb{R}^n$ meet in at most one point.

1.3 The dot product of vectors in $\mathbb{R}^n$

If $u = (a, b) \in \mathbb{R}^2$ then by Pythagoras Theorem, the length of u is $||u|| := \sqrt{a^2 + b^2}$. If $v = (c, d)$ then the distance between the points u and v can be found by the Pythagorean Theorem and it is

$$||u - v|| = \sqrt{(a - c)^2 + (b - d)^2}.$$

These concepts can be captured by the operation of dot product on vectors. Let us see how the idea of dot product arises.



Suppose that we want to find the angle between two nonzero vectors u and $v \in \mathbb{R}^2$. We can assume without loss of generality that they are unit vectors. Hence u, v lie on the unit circle. Therefore $u = (\cos\alpha, \sin\alpha)$ and $v = (\cos\beta, \sin\beta)$. Here α is the angle between the vector u and the x-axis and β is the angle between v and the x-axis. Let us assume that $\alpha \leq \beta$. Then the angle between u and v is $\beta - \alpha$. Recall the trigonometric identity

$$\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

This suggests the following operation on vectors $u = (a, b)$ and $v = (c, d)$: Define the **dot product or scalar product** of u and v as $u \cdot v = ac + bd$. This motivates the definition of dot product in $\mathbb{R}^n$.

Definition 1.3.1. *Let $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then their* **dot product** *is defined as*

$$X \cdot Y = x_1y_1 + \cdots + x_ny_n.$$

Definition 1.3.2. *The* **norm** *or* **length** *of a vector $A \in \mathbb{R}^n$ denoted by $||A||$ is the real number $||A|| = \sqrt{A \cdot A}$. The* **distance** *between A and B in $\mathbb{R}^n$ is defined to be $||A - B||$. Therefore $||A - B|| = \sqrt{(A - B) \cdot (A - B)}$. We say that A is a* **unit vector** *if $||A|| = 1$.*

Let $A, B, C \in \mathbb{R}^n$ and $c \in \mathbb{R}$. The following four properties are easy to verify.

SP 1. *commutativity.* $A \cdot B = B \cdot A$,

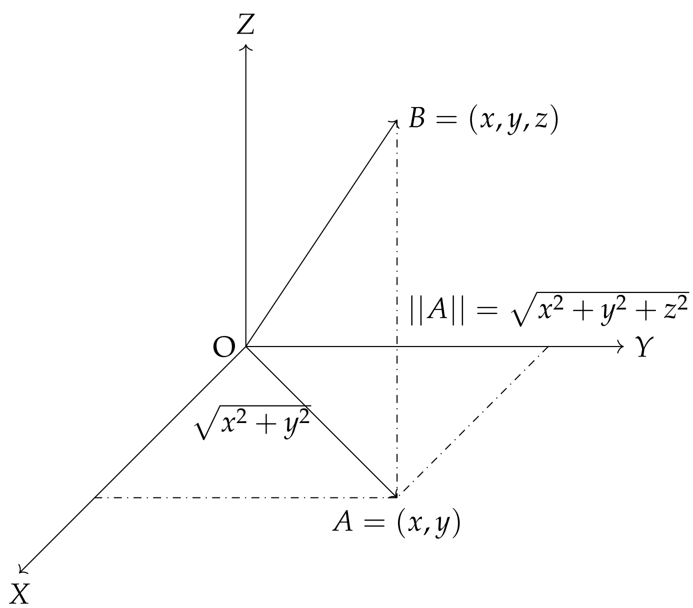
SP 2. *distributivity with vector addition.* $A \cdot (B + C) = A \cdot B + A \cdot C$,

SP 3. *associativity with scalar multiplication.* $(cA) \cdot B = c(A \cdot B) = A \cdot (cB)$,

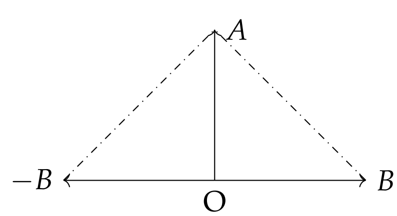
SP 4. *positive-definite.* If $A = O$, then $A \cdot A = 0$ and if $A \neq O$ then $A \cdot A > 0$.

The Pythagoras Theorem in $\mathbb{R}^n$.

If $B = (x, y, z) \in \mathbb{R}^3$ then $||B|| = \sqrt{x^2 + y^2 + z^2}$. Note that that for a nonzero vector A, $||\frac{A}{||A||}|| = \frac{1}{||A||}||A|| = 1$. Hence $\frac{A}{||A||}$ is a unit vector.



We can detect perpendicular vectors using the dot product. Recall that by similarity of triangles, $||A - B|| = ||A + B||$ if and only if $\angle - BOA = \angle AOB = 90°$.



Theorem 1.3.3. *Let $A, B \in \mathbb{R}^n$. Then $||A + B|| = ||A - B|| \iff A \cdot B = 0$.*

Proof.

$$\begin{aligned} ||A+B|| = ||A-B|| &\iff ||A+B||^2 = ||A-B||^2 \\ &\iff A\cdot A + 2A\cdot B + B\cdot B \\ &= A\cdot A - 2A\cdot B + B\cdot B \\ &\iff A\cdot B = 0 \end{aligned}$$

□

Definition 1.3.4. *Two vectors $A, B \in \mathbb{R}^n$ are called* **perpendicular or orthogonal** *if $A \cdot B = 0$.*

Theorem 1.3.5 (General Pythagoras Theorem). *Let $A, B \in \mathbb{R}^n$ be nonzero vectors. Then*

$$A\cdot B = 0 \iff ||A+B||^2 = ||A||^2 + ||B||^2.$$

Proof. If $A \cdot B = 0$ then

$$\begin{aligned} ||A+B||^2 &= (A+B)\cdot(A+B) = A\cdot A + 2A\cdot B + B\cdot B \\ &= ||A||^2 + ||B||^2. \end{aligned}$$

By the above equation, if $||A+B||^2 = ||A||^2 + ||B||^2$ then $A \cdot B = 0$. □

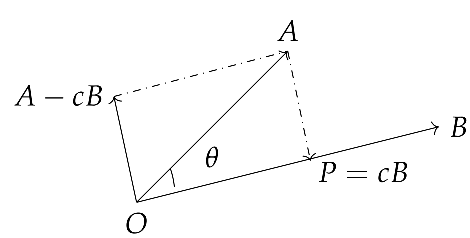
1.4 Projection of a vector along another vector

We have proved a formula for dot product of vectors in the plane which enables us to find the angle between these vectors. We want to prove the same formula for vectors in $\mathbb{R}^n$. For this we shall prove the Cauchy-Schwarz inequality for vectors $A, B \in \mathbb{R}^n$:

$$|A\cdot B| \leq ||A||\ ||B||.$$

This will be obtained as a cosequence of understanding projection of vectors in the direction of a given vector. This is an important construction about vectors. We shall see many applications of this construction later.

Let A and B be two vectors and $B \neq O$. Let P be the point on the line OB so that $(P - A) \perp B$. Then $P = cB$ for a real number c. We wish to find c.



As $(cB - A) \perp B$, we have $(cB - A) \cdot B = cB \cdot B - A \cdot B = 0$. Solve for c to get $c = A \cdot B / B \cdot B$. Conversely, if we take this value of c, then $cB - A \perp B$.

Definition 1.4.1. *The projection of A along B is the vector $P = \frac{A \cdot B}{B \cdot B} B$.*

Note that if we write vectors as column vectors then the projection of A along B is considered as a map:

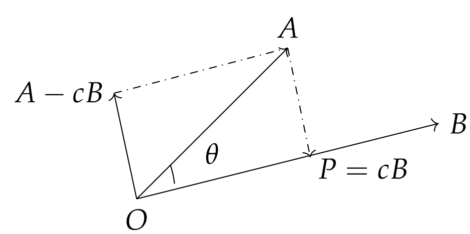
$$P_B : \mathbb{R}^n \to \mathbb{R}^n \quad \text{given by } P_B(A) = \frac{BB^t}{B^t B} A.$$

Here B^t denotes B as a row vector. If B is a unit vector then $B^t B = 1$. Hence $P_B(A) = BB^t A$. Check that for any two vectors $C, D \in \mathbb{R}^n$ and $x, y \in \mathbb{R}$, we have

$$P_B(xC + yD) = xP_B(C) + yP_B(D).$$

Using this property we get

$$P_B(A - cB) = BB^t A - cBB^t(B) = BB^t A - BB^t A = O.$$



We can now provide a geometric interpretation of the dot product. If A and B are two nonzero vectors then from plane geometry we see that if θ is the angle between A and B and cB is the projection of A on B, then

$$\cos\theta = \frac{c||B||}{||A||} = \frac{A \cdot B}{||A||\ ||B||}.$$

This gives the usual definition of the dot product $A \cdot B = ||A||\ ||B|| \cos\theta$ for 2-vectors. In order to extend this for n-vectors, we need to prove the Cauchy-Schwarz inequality.

Theorem 1.4.2 (Cauchy-Schwarz inequality). *Let A, B be vectors in $\mathbb{R}^n$. Then*

$$|A \cdot B| \le ||A||\ \ ||B||.$$

The above inequality is an equality if and only if $A = xB$ for some $x \in \mathbb{R}$.

Proof. If $B = O$ then both the sides are 0. So let $B \neq O$. Note that $A - cB$ and B are perpendicular vectors. Hence by the Pythagoras Theorem,

$$||A||^2 = ||A - cB||^2 + ||cB||^2 = ||A - cB||^2 + c^2||B||^2.$$

Hence $c^2||B||^2 \leq ||A||^2$. But

$$c^2||B||^2 = \frac{(A \cdot B)^2||B||^2}{(B \cdot B)^2} = \frac{|A \cdot B|^2}{||B||^2} \leq ||A||^2.$$

Hence $|A \cdot B| \leq ||A||\ ||B||$. If $A = xB$ for some $x \in \mathbb{R}$ then $|xB \cdot B| = |x|B \cdot B = ||xB||\ ||B||$. Conversely let $|A \cdot B| = ||A||\ ||B||$. If either A or B is the zero vector, then the conclusion is true. So let neither A nor B is O. Then we may divide by $||A||$ and $||B||$ and assume that we have two unit vectors u, v so that $|u \cdot v| = 1$. If $u \cdot v = 1$ then $(u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v = 0$. Hence $u = v$. If $u \cdot v = -1$ then $(u + v) \cdot (u + v) = 0$. Hence $u = -v$. □

Definition 1.4.3. *The angle $\theta \in [0, \pi]$ between nonzero vectors $A, B \in \mathbb{R}^n$ is defined by the equation*

$$A \cdot B = ||A||\ ||B|| \cos\theta.$$

The Cauchy-Schwarz inequality enables us to prove an analogue of the triangle inequality about the lengths of sides of a plane triangle.

Theorem 1.4.4 (Triangle inequality and the law of cosines). *Let $A, B \in \mathbb{R}^n$ be nonzero vectors and let θ be the angle between them. Then*

$$||A + B||^2 = ||A||^2 + 2||A||\ ||B|| \cos\theta + ||B||^2.$$

In particular $||A + B|| \leq ||A|| + ||B||$ for all $A, B \in \mathbb{R}^n$.

Proof. By the properties of dot product we have

$$\begin{aligned} ||A + B||^2 = (A + B) \cdot (A + B) &= A \cdot A + 2A \cdot B + B \cdot B \\ &= ||A||^2 + 2||A||\ ||B|| \cos\theta + ||B||^2 \\ &\leq ||A|^2 + 2||A||\ ||B|| + ||B||^2 \\ &= (||A|| + ||B||)^2 \end{aligned}$$

Take square roots on both the sides to get the inequality. □

1.5 The cross and the scalar triple product of vectors

Let e_1, e_2, e_3 be standard unit vectors in $\mathbb{R}^3$. Let $u = (a, b, c), v = (d, e, f) \in \mathbb{R}^3$. We wish to find a vector that is perpendicular to u and v. The cross product of u and v is such a vector. It is defined to be the vector

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ec)e_1 + (cd - af)e_2 + (ae - bd)e_3.$$

It is clear that if either u or v is zero then $u \times v = 0$. Check that $u \cdot (u \times v)$ is the determinant of a 3×3 matrix in which the first two rows are identical. Similarly $v \cdot (u \times v) = 0$. Hence $u \times v$ is perpendicular to both u and v.

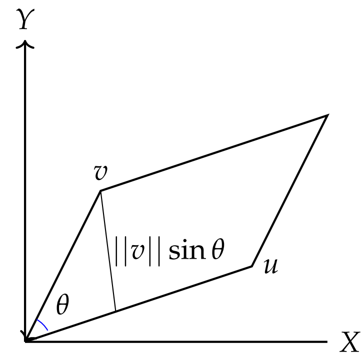
Theorem 1.5.1. *For all $u, v, w \in \mathbb{R}^3$ and $x \in \mathbb{R}$ we have*
(1) $u \times v = -(v \times u)$.
(2) $u \times (v + w) = u \times v + u \times w$.
(3) $x(u \times v) = (xu) \times v$.
(4) $||u \times v||^2 + (u \cdot v)^2 = ||u||^2 ||v||^2$, *(Lagrange's identity).*
(5) *If $\theta \in [0, \pi]$ is the angle between u and v then $u \times v = ||u|| ||v|| \sin\theta$.*
(6) *If u and v are nonzero then $u \times v = 0$ if and only if u and v are parallel.*
(7) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$.

Proof. (1) The vector $v \times u$ is obtained by interchanging the second and the third rows of the matrix whose determinant is $u \times v$. Hence $u \times v = -(v \times u)$.
(2) Let $w = (x, y, z)$. Using the fact that the determinant function is linear, we obtain

$$u \times (v + w) = \begin{vmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ d + x & e + y & f + z \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ d & e & f \end{vmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ x & y & z \end{vmatrix} = u \times v + u \times w.$$

(3) This also follows from the properties of the determinants.
(4) Substitute the co-ordinates of u, v, expand and check.
(5) Use the fact that $u \cdot v = ||u|| ||v|| \cos\theta$.
(6) Use (5) to see that if $u \times v = 0$ and $u, v \neq 0$ then $\sin\theta = 0$. Hence $\theta = 0$ or $\theta = \pi$. Thus u and v are parallel. Conversely, if u and v are parallel then $\theta = 0$ or $\theta = \pi$. In both cases $\sin\theta = 0$. Hence $u \times v = 0$.
(7) Prove it for $w = e_i$ for $i = 1, 2, 3$ and then use (2) to prove it for w.

□



Geometric interpretation of $u \times v$. The length of $u \times v$ has an interesting geometric interpretation. Let $\theta \in [0, \pi]$ be the angle between $u, v \in \mathbb{R}^n$. Then $u \times v = ||u||||v|| \sin\theta$. The height of the parallelogram with side lengths $||u||$ and $||v||$ is $||v|| \sin\theta$. Therefore its area is $||u \times v|| = ||u||||v|| \sin\theta$.

Heron's formula for the area of a triangle. Let a, b, c be the lengths of the sides of a triangle and $s = (a + b + c)/2$. Then the area of the triangle is

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. Let the triangle have vertices at O, A, B with $||A|| = a, ||B|| = b, ||B - A|| = c$. Use the two identities

$$||A \times B||^2 = ||A||^2||B||^2 - (A \cdot B)^2, \quad -2A \cdot B = ||A - B||^2 - ||A||^2 - ||B||^2.$$

Since $2S = ||A \times B||$, we get

$$\begin{aligned} 4S^2 = a^2b^2 - \frac{1}{4}(c^2 - a^2 - b^2)^2 &= \frac{1}{4}(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2) \\ \implies S^2 &= \frac{1}{16}(a+b+c)(a+b-c)(c-a+b)(c+a-b) \\ \implies S &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

□

Definition 1.5.2. *If u, v and w are vectors in 3-space, then $u \cdot (v \times w)$ Is called the* **scalar triple product** *of u, v and w.*

Theorem 1.5.3. (1) *Let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. Then*

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(2) *The volume of the parallelepiped in $\mathbb{R}^3$ determined by the vectors u, v and w is given by $|u \cdot (v \times w)|$.*

Proof. (1) follows from the determinantal formula for $v \times w$.
(2) Consider he parallelogram P determined by the vectors v and w. The area of P is $||v \times w||$. The height h of the parallelepiped determined by u, v, w is the length of the orthogonal projection of u on the vector $v \times w$. Hence

$$h = ||P_{v\times w} u|| = \frac{|u \cdot (v \times w)|}{||v \times w||}.$$

Thus the volume V of the parallelepiped is

$$V = \text{ (area of the base) } \cdot \text{ height } = ||v \times w|| \frac{|u \cdot (v \times w)|}{||v \times w||} = |u \cdot (v \times w)|.$$

□

1.6 Planes in $\mathbb{R}^n$

We describe planes in the 3-space by equations similar to those of lines in $\mathbb{R}^2$. Let P be a point in 3-space and N be a nonzero vector. The plane passing through P and perpendicular to N consists of $X \in \mathbb{R}^3$ so that $(X - P) \cdot N = 0$. Therefore the equation of the plane is

$$X \cdot N = P \cdot N.$$

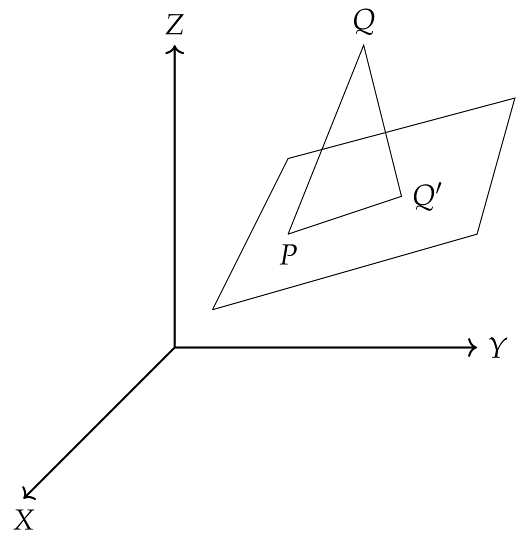
Let $N = (a, b, c)$ and $P = (x_0, y_0, z_0)$. Then the equation of the plane is $ax + by + cz = ax_0 + by_0 + cz_0$. In other words

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example 1.6.1. Let $P = (2, 1, -1)$ and $N = (-1, 1, 3)$. Let $X = (x, y, z)$. Then $P \cdot N = -4$. $N \cdot X = -x + y + 3z$. Therefore the equation of the plane passing through P and perpendicular to N is $-x + y + 3z = -4$.

Definition 1.6.2. *Two planes are* **parallel** *if their normal vectors are parallel. The* **angle between two planes** *is defined to be the angle between their normals. Two planes are said to be* **perpendicular** *if their normals are so.*

Distance between a point and a plane.



Consider the plane Π defined by the equation $(X - P) \cdot N = 0$. Let $Q \in \mathbb{R}^n$. We wish to find a formula for the distance between Q and Π. This is defined to be the distance between Q and Q' where Q' is the point of intersection of the plane with a line passing through Q so that $Q - Q' \| N$. The distance between Q and the plane is the length of the projection of $Q - P$ on $Q - Q'$. Therefore the distance of Q from the plane is given by the formula

$$\frac{|(Q - P) \cdot N|}{||N||}.$$

Let $Q = (x, y, z), P = (x_0, y_0, z_0)$ and $N = (a, b, c)$. Then the distance of Q from Π is given by

$$\frac{|(x - x_0, y - y_0, z - z_0) \cdot (a, b, c)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 1.6.3. Let $Q = (1, 3, 5), \quad P = (-1, 1, 7), \quad N = (-1, 1, -1)$. The equation of the plane passing through P and perpendicular to N is given by

$$(x, y, z) \cdot N = -x + y - z = P \cdot N = -5.$$

The unit vector that is perpendicular to Π is $N/||N|| = \frac{1}{\sqrt{3}}(-1, 1, -1)$. Therefore the distance of Q from the plane is given: $|(Q - P) \cdot \frac{1}{\sqrt{3}}(-1, 1, -1)| = 2/\sqrt{3}$.