PH-112 (2023 Spring): Tutorial Sheet 5

Notes:

1. * marked problems will be solved in the Wednesday tutorial class.
2. Please make sure that you do the assignment by yourself. You can consult your classmates and seniors and ensure you understand the concept. However, do not copy assignments from others.

Scattering problems:

1. * A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy E ($< V_0$) approaches the barrier from left.
   (a) Find the value of $x = x_0$ ($x_0 > 0$), for which the probability density is $1/e$ times the probability density at $x = 0$.
   (b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.

2. Consider a potential

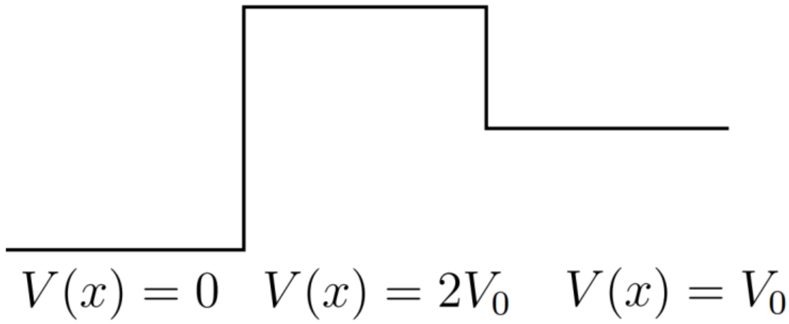
$$\begin{aligned} V(x) &= 0 \ \text{ for } x < 0, \\ &= -V_0 \text{ for } x > 0 \end{aligned}$$

   Consider a beam of non-relativistic particles of energy $E > 0$ coming from $x \to -\infty$ and being incident on the potential. Calculate the reflection and transmission coefficients.

3. A potential barrier is defined by $V = 0$ eV for $x < 0$ and $V = 7$ eV for $x > 0$. A beam of electrons with energy 3 eV collides with this barrier from left. Find the value of x for which the probability of detecting the electron will be half the probability of detecting it at $x = 0$.

4. * A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.
   (a) Obtain an expression for the transmission coefficient.
   (b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.

5. A beam of particles of energy $E < V_0$ is incident on a barrier (see figure below) of height $V = 2V_0$. It is claimed that the solution is $\psi_I = A\exp(-k_1 x)$ for region I ($0 < x < L$) and $\psi_{II} = B\exp(-k_2 x)$ for region II ($x > L$), where $k_1 = \sqrt{\frac{2m(2V_0 - E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$. Is this claim correct? Justify your answer.

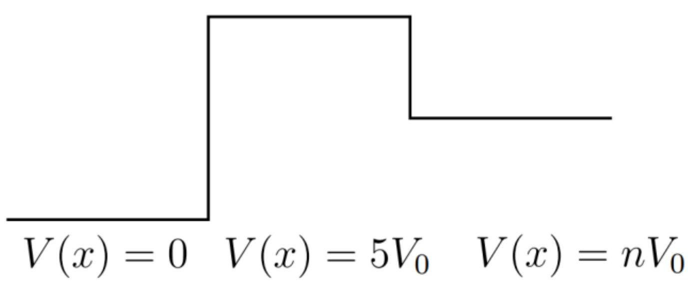


6. * A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0$, $V = 5V_0$ for $x \leq d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \pi h/\sqrt{8mV_0}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.

   (a) Find n. Are there more than one possible values for n?

   (b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

   (c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used



7. A scanning tunneling microscope (STM) can be approximated as an electron tunneling into a step potential [$V(x) = 0$ for $x \leq 0$, $V(x) = V_0$ for $x > 0$]. The tunneling current (or probability) in an STM reduces exponentially as a function of the distance from the sample. Considering only a single electron-electron interaction, an applied voltage of 5 V

and the sample work function of 7 eV, calculate the amplification in the tunneling current if the separation is reduced from 2 atoms to 1 atom thickness (take approximate size of an atom to be 3 Å).

Simple Harmonic Oscillator and 2D/3D Systems

1. Using the uncertainty principle, show that the lowest energy of an oscillator is $\hbar\omega/2$.

2. Determine the expectation value of the potential energy for a quantum harmonic oscillator (with mass m and frequency ω) in the ground state. Use this to calculate the expectation value of the kinetic energy. The ground state wavefunction of quantum harmonic oscillator is:
$$\psi_0(x) = C_0 \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad C_0 \text{ is constant;} \tag{1}$$

3. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant $k = 12Nm^{-1}$ and mass $m = 5.6 * 10^{-26}kg$
   (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state ?
   (b) Find the ground state energy of vibrations for this diatomic molecule.

4. Vibrations of the hydrogen molecule can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 * 10^3 Nm^{-2}$ and mass $m = 1.67 * 10^{27}$ kg.
   (a) What is the vibrational frequency of this molecule ?
   (b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states ?

5. * A two-dimensional isotropic harmonic oscillator has the Hamiltonian
$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}k(x^2 + y^2)$$
   (a) Show that the energy levels are given by
$$E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1) \quad \text{where} \quad n_x, n_y \in (0, 1, 2...) \quad \omega = \sqrt{\frac{k}{m}}$$
   (b) What is the degeneracy of each level?

6. Consider the Hamiltonian of a two-dimensional anisotropic harmonic oscillator $(\omega_1 \neq \omega_2)$
$$H = \frac{{p_1}^2}{2m} + \frac{{p_2}^2}{2m} + \frac{1}{2}m{\omega_1}^2{q_1}^2 + \frac{1}{2}m{\omega_2}^2{q_2}^2$$

(a) Exploit the fact that the Schrödinger eigenvalue equation can be solved by separating the variables and find a complete set of eigenfunctions of H and the corresponding eigenvalues.

(b) Assume that $\frac{\omega_1}{\omega_2} = \frac{3}{4}$ .Find the first two degenerate energy levels. What can one say about the degeneracy of energy levels when the ratio between ω_1 and ω_2 is not a rational number.

7. A particle of mass m is confined to move in the potential $(m\omega^2 x^2)/2$. Its normalized wave function is

$$\psi(x) = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-(\beta x^2/2)}$$

where β is a constant of appropriate dimension.

(a) Obtain a dimensional expression for β in terms of m, ω and $\hbar$.

(b) It can be shown that the above wave function is the linear combination

$$\psi(x) = a\psi_0(x) + b\psi_2(x)$$

where $\psi_0(x)$ is the normalized ground state wave function and $\psi_2(x)$ is the normalized second excited state wave function of the potential. Evaluate b and hence calculate the expectation value of the energy of the particle in this state $\psi(x)$.

Given: $I_0(\beta) = \int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$, $\quad I_n(\beta) = \int_{-\infty}^{+\infty} (x^2)^n e^{-\beta x^2} dx = (-1)^n \frac{\partial^n}{\partial \beta^n}(I_0(\beta))$,

$\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{\frac{-\beta x^2}{2}}$

8. Consider an 3D isotropic harmonic oscillator show that the degeneracy g_n of the nth excited state, which is equal the number of ways the non negative integers n_x, n_y, n_z may be chosen to total to n, is given by

$$g_n = \frac{1}{2}(n+1)(n+2)$$

9. * A charged particle of mass ' m ' and charge ' q ' is bound in a 1-dimensional simple harmonic oscillator potential of angular frequency ' ω '. An electric field E_0 is turned on.

(a) What is the total potential V(x) experienced by the charge ?

(b) Express the total potential in the form of an effective harmonic oscillator potential.

(c) Sketch V (x) versus x.

(d) What is the ground state energy of the particle in this potential?

(e) What is the expectation value of the position (x) if the charge is in its ground state ?