Lecture 16

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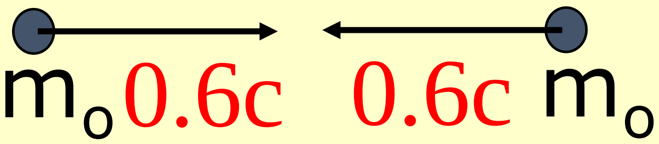
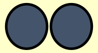
Before collision



After collision



S' Frame

(First Particles frame)

Recall: The velocity of the first particle before collision:

RECAP

$$u'_{1x} = \frac{0.6c - 0.6c}{1 - 0.36} = 0$$

$$u'_{1y} = 0$$

$$u'_{1z} = 0$$

$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

The momentum of the first particle before collision is, therefore, zero.

$$u'_{1x} = \frac{0.6c - 0.6c}{1 - 0.36} = 0,\ \ u'_{1y} = 0,\ \ u'_{1z} = 0$$

S' Frame (**First Particles frame**)

RECAP

Recall: The velocity of the second particle before collision.

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

$$u'_{2y} = 0$$

$$u'_{2z} = 0$$

$$u'_{2x} = \frac{-0.6c - 0.6c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

$$u'_{2y} = 0$$

$$u'_{2z} = 0$$

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

S' Frame (First Particles frame) RECAP

The momentum of the second particle before collision is thus.

$$p'_{2x} = -\frac{\dfrac{1.2}{1.36}}{\sqrt{1-\left(\dfrac{1.2}{1.36}\right)^2}}m_0c$$

$$= -\frac{2.125\times 1.2}{1.36}m_0c$$

$$= -1.875m_0c$$

$$\sum_k p'_{xkI} = -1.875m_0c$$

$$\gamma_u \equiv \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$$

$$p_x \equiv m_o\gamma_u u_x$$

$$p_y \equiv m_o\gamma_u u_y$$

$$p_z \equiv m_o\gamma_u u_z$$

$$u'_{2x} = \frac{-0.6c-0.6c}{1+0.36} = -\frac{1.2}{1.36}c$$

S' Frame (First Particles frame)

Recall: The velocity of the combined particle after collision.

$$u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c$$

$$u'_{fy} = 0$$

$$u'_{fz} = 0$$

$$u'_x = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

$$u'_{fx} = \frac{0 - 0.6c}{1 - \dfrac{0 \times 0.6c}{c^2}} = -0.6c$$

$$u'_{fy} = 0$$

$$u'_{fz} = 0$$

RECAP

To show that momentum is conserved in S$'$, we use the fact that M is not simply equal to $2m$ in relativity. As shown, the combined mass, M, formed from the collision of two particles, each of mass m moving toward each other with speed v, is greater than $2m$. This occurs because of the equivalence of mass and energy, that is, the kinetic energy of the incident particles shows up in relativity theory as a tiny increase in mass, which can actually be measured as thermal energy. Thus, from Equation 2.13, which results from imposing the conservation of mass–energy, we have

$$M = \frac{2m}{\sqrt{1 - (v^2/c^2)}}$$

Consider the combined rest mass of the two objects each of mass *m* after the collision. When $\gamma_u = 1$ this mass is 2*m*, as we would expect from the classical law of conservation of mass. But this is not true for relativistic situations.

Law: The sum of relativistic masses before a collision is equal to the sum of relativistic masses after the collision.

S' Frame

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0$$

$$M_o = 2.5 m_o$$

The momentum of the combined particle after collision is thus.

$$p'_{xF} = -1.25 \times 0.6 M_0 c$$

$$u'_{fx} = \frac{0 - 0.6c}{1 - \frac{0 \times 0.6c}{c^2}} = -0.6c$$

$$= -1.25 \times 0.6 \times 2.5 m_0 c$$

$$= -1.875 m_0 c$$

since $\quad p = \gamma_u m_0 u)$

RECAP

since $\quad \gamma_u = 1.25)$

Energy Conservation along similar lines

Summary of formulae

RECAP

$$\mathbf{p} = m\mathbf{u} = m_0\mathbf{u}\gamma$$

$$K = mc^2 - m_0c^2 = m_0c^2(\gamma - 1)$$

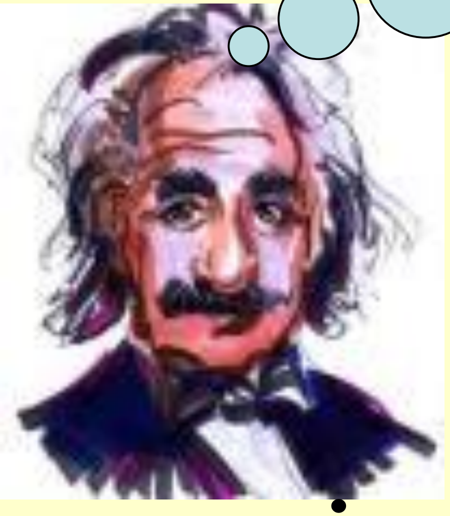
$$E = mc^2 = m_0c^2\gamma$$

$$E^2 = (pc)^2 + (m_0c^2)^2.$$

Will Revisit this mass energy Equivalence in the next lecture

Summer
1905

What is the significance of the increase mass of a moving object??



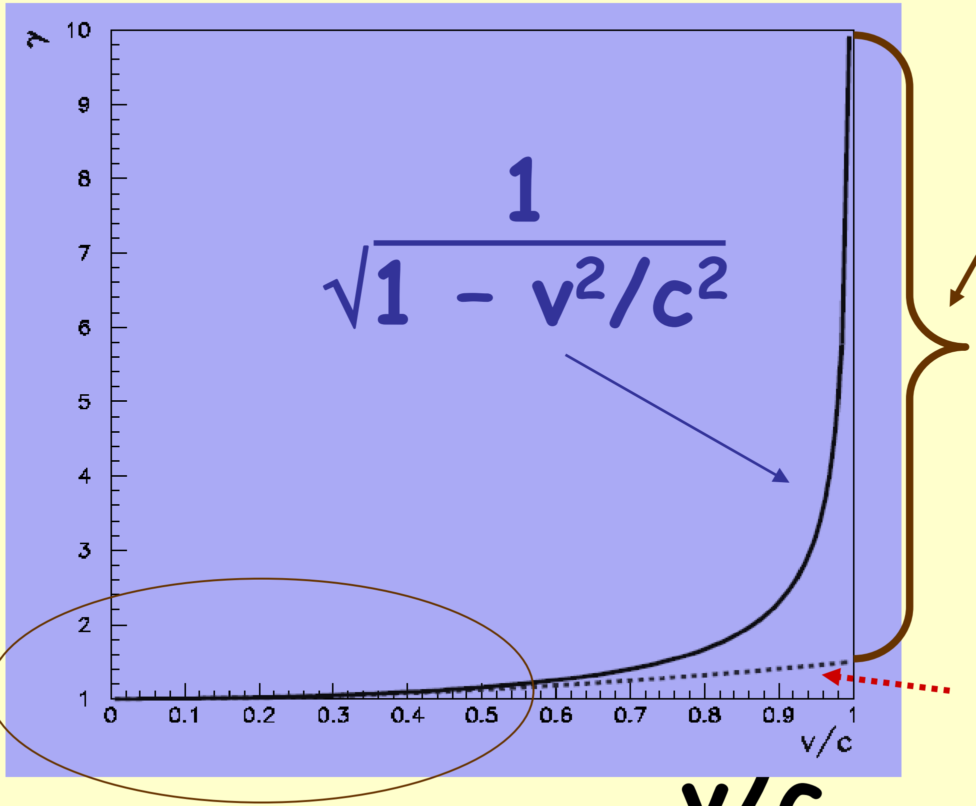
$$m = \gamma m_0$$

$$\text{increase in mass} = m - m_0$$

$$\text{increase in } m = \gamma m_0 - m_0 = (\gamma - 1) m_0$$

Expansion yields:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \text{other stuff}$$

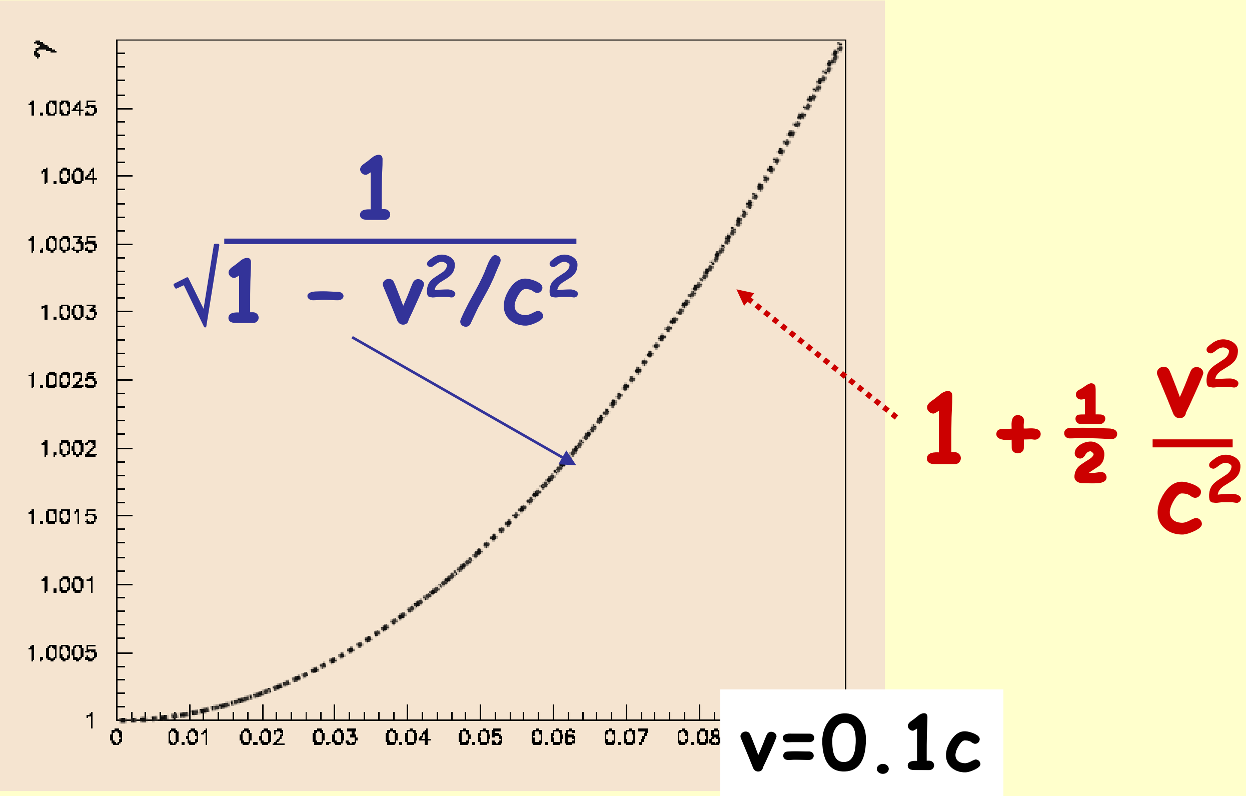


What about here?

v/c

Below v/c = 0.1, "other stuff" is negligible

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \tfrac{1}{2}\frac{v^2}{c^2} + \text{other stuff}$$



Increase in mass = $(\gamma-1)m_0$

for $v<0.1c = \gamma - 1 = \frac{1}{2}\frac{v^2}{c^2}$

$=(\gamma-1)m_0$ =increase in mass = $\frac{1}{2} m_0 \frac{v^2}{c^2}$

Multiply both sides by c^2

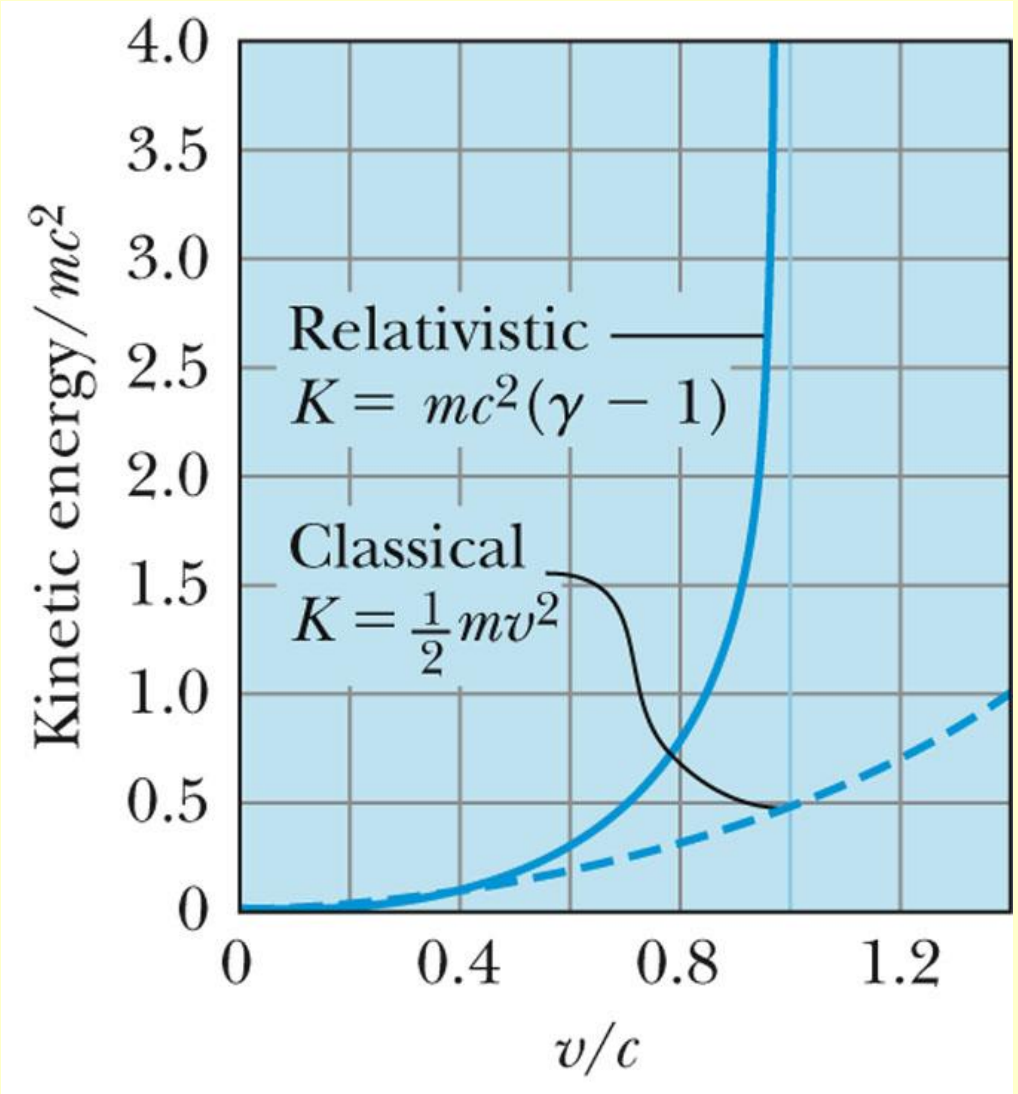
(increase in mass)c^2 = $\frac{1}{2} m_0 v^2$

Classical Kinetic energy

$$K = mc^2 - m_0c^2 = m_0c^2(\gamma - 1)$$

Relativistic K.E

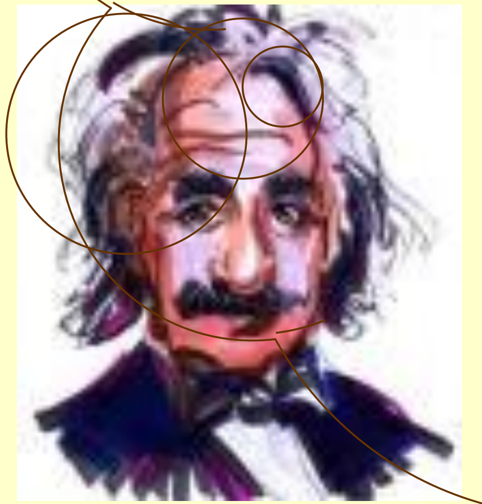
Relativistic and Classical Kinetic Energies



$$E = mc^2$$

This must be the total energy

$$mc^2 = m_0c^2 + \text{kinetic energy}$$



This must also be a form of energy:

➔"rest mass energy"

Mass is a form of energy

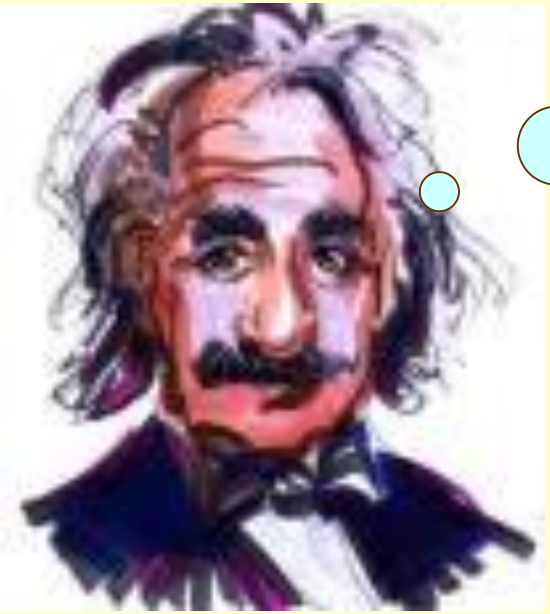
$$E = mc^2$$

The conversion factor from mass to energy is c^2, a huge number!

$$\text{Energy (Joules)} = \text{Mass (kg)} \times (3\times10^8)^2$$

$$=9\times10^{16}$$

$$E_{\text{(Joules)}} = 90{,}000{,}000{,}000{,}000{,}000 \times m_{\text{(kg)}}$$



Mass contains
an *enormous*
amount of
energy

Mass energy equivalence: Binding energy

Consider the following reversible reaction/process

- 1 proton+1 neutron $\leftrightarrow$ 1 deutron
- m_0(proton)=1.00731 a.m.u.
- m_0(neutron)=1.00867 a.m.u.
- m_0(deutron)=2.01360 a.m.u.
- m_0(product)<m_0(reactants), Δm_0=0.00238 a.m.u.
- Equivalent energy=$\Delta m_0 c^2$=2.2 Mev

(1 a.m.u.=1.66×10^{-27} kg)

Mass energy equivalence: Binding energy

- **When the forward reaction happens, this energy is given out as a photon (gamma)**
- ***For the reverse reaction (dissociation) to occur, this much (2.2 Mev) must be supplied from outside.***
- ***Therefore, this energy is called the binding energy of the deutron.***

Mass energy equivalence: Implications

- Mass is a property of energy
- Energy is a property of mass (m_0 of electron = 0.51MeV/c^2 and that of a proton is 0.94 GeV/c^2)
- Classical distinction between mechanical and other forms of energy disappears in relativity
- Rest mass corresponds to internal energy
- ***Massless (zero rest mass) particles exist***?

ZERO REST MASS PARTICLE

The rest mass $\boxed{m_o}$ is zero,

$$m = \frac{m_o}{\sqrt{1 - u^2 / c^2}}$$

if and only if $u=c$.

$$E^2 = p^2c^2 + (mc^2)^2$$

When the particle is at rest, $p = 0$, and so we see that $E = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons (massless, chargeless particles of light), we set $m = 0$ in Equation above and find

$$E = pc$$

This equation is an *exact* expression relating energy and momentum for photons, which always travel at the speed of light.

PHOTON

- **A particle with zero rest mass.**
- **Light consists of photons.**
- ***We started treating light as a wave and used its speed as an argument to come to STR.***
- **Dual nature??**
- ***The photon associated with the e.m. wave of frequency ν has energy $E=h\nu$ and a momentum $p=E/c=h\nu/c$ (particle like)***

Black Body Radiation

- Energy transfer is possible only as a multiple of a minimum called quantum (quanta)
- This quantum (quanta) was postulated to be proportional to the frequency.

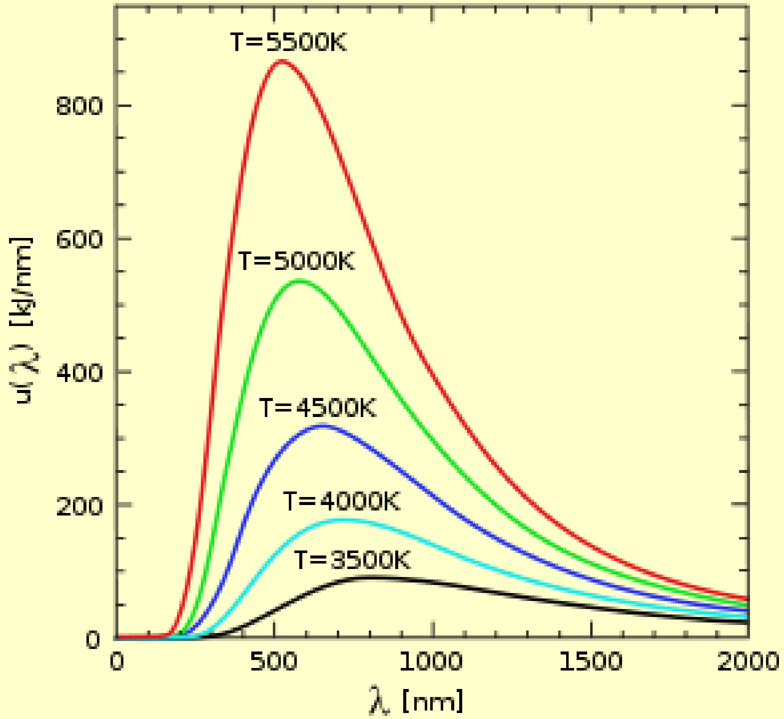
$$\varepsilon_n = nh\nu = n\hbar\omega$$

h is the Planck's constant, 6.56×10^{-34} J.s and ($\hbar = h/2\pi$)

Indication of duality in radiation

X-ray diffraction

BBR



X-RAYS AS WAVES

EM waves as particles

Bragg and Laue

Planck

Matter from radiation and vice versa

$$\gamma \longrightarrow e^- + e^+$$

Pair production

$$\gamma \longleftarrow e^- + e^+$$

Annihilation of matter

Photoelectric Effect

- Consolidated the particle (photon) picture of light.

- Demonstrated that photon has an energy $h\nu$, combining the wave and particle aspects.

Properties of Photon

- Undergoes particle-like collisions with fundamental particle such as electrons.
- Moves with speed c.
- Zero rest mass and rest mass energy
- Carries energy ($h\upsilon$) and linear momentum ($h\upsilon/c=h/\lambda$).
- Number not conserved. Can be created (during emission of radiation) and destroyed (during absorption of radiation).

Electromagnetism and Relativity

- Einstein was convinced that magnetic fields appeared as electric fields observed in another inertial frame. That conclusion is the key to electromagnetism and relativity.
- Einstein's belief that ***Maxwell's equations describe electromagnetism in any inertial frame*** was the key that led Einstein to the Lorentz transformations.
- Maxwell's assertion that all electromagnetic waves travel at the speed of light and Einstein's postulate that the speed of light is invariant in all inertial frames seem intimately connected.

Finally, Einstein about his discovery….

"If the relativity theory is proven true,
the Germans will say I am a German,
the Swiss I am a Swiss and
the French that I am a great man."

If not,

"Germans will call me Swiss,
Swiss will call me German, and
the French will say I am a Jew."

Space-time geometry

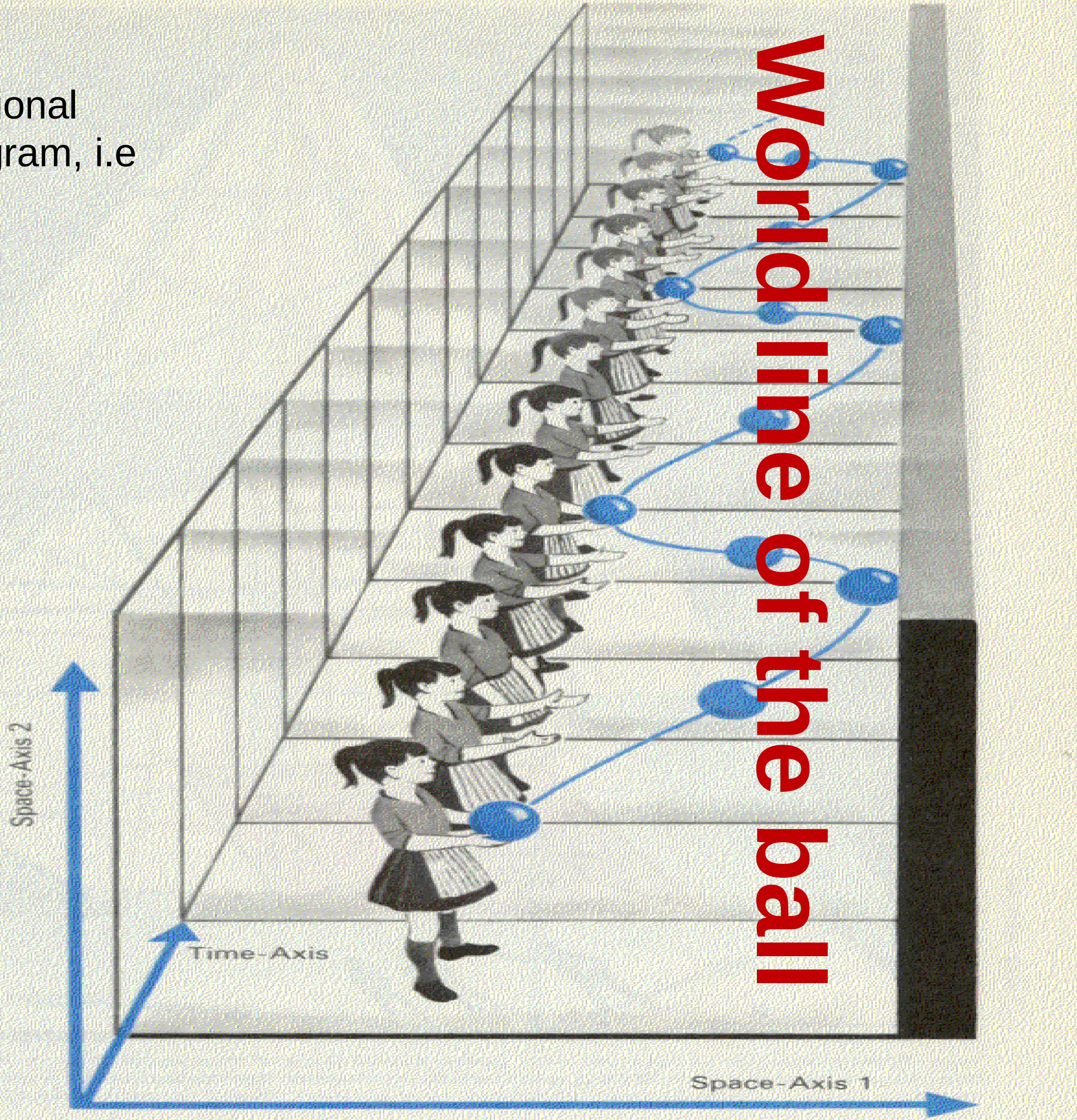
Hermann Minkowski’s contribution

- “***Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality***” (1908)

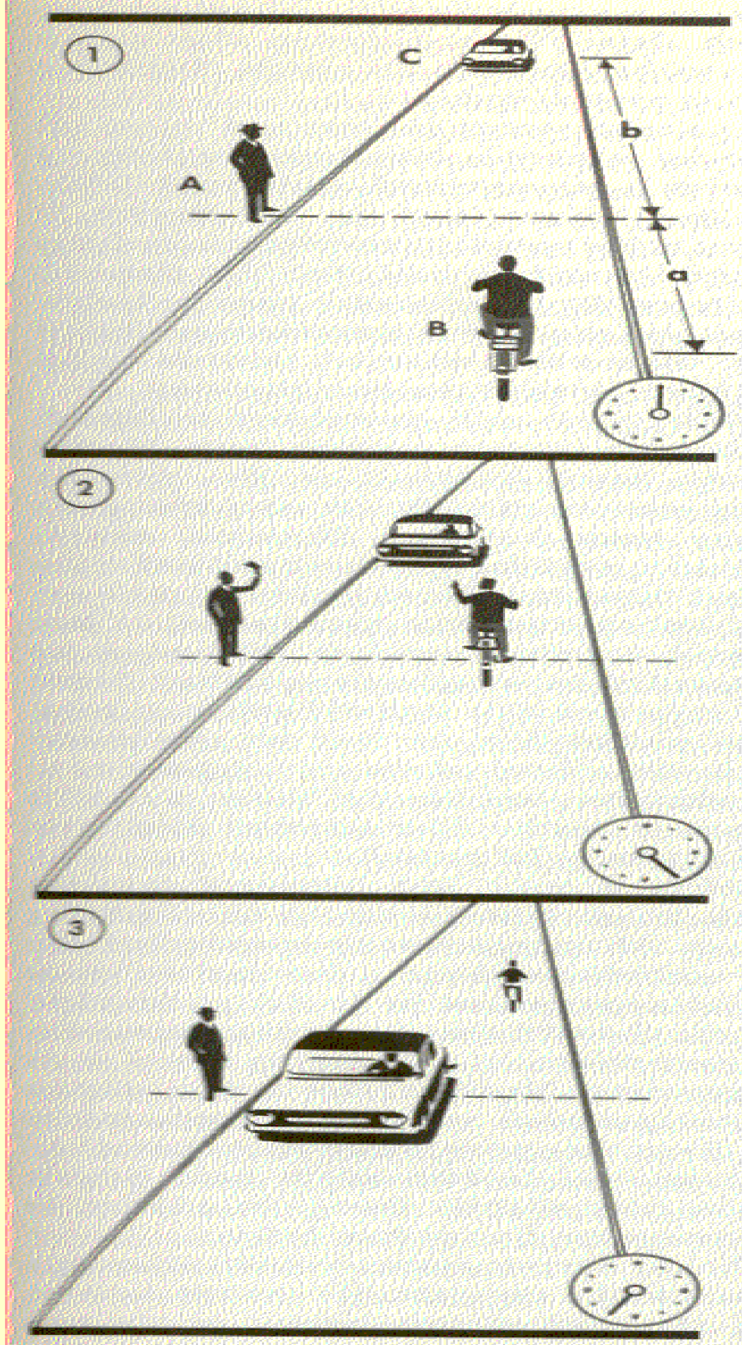
- Geometrized Physics!

- Hyperbolic nature of space time frame (4D) is richer than the circular geometry of 3D space.

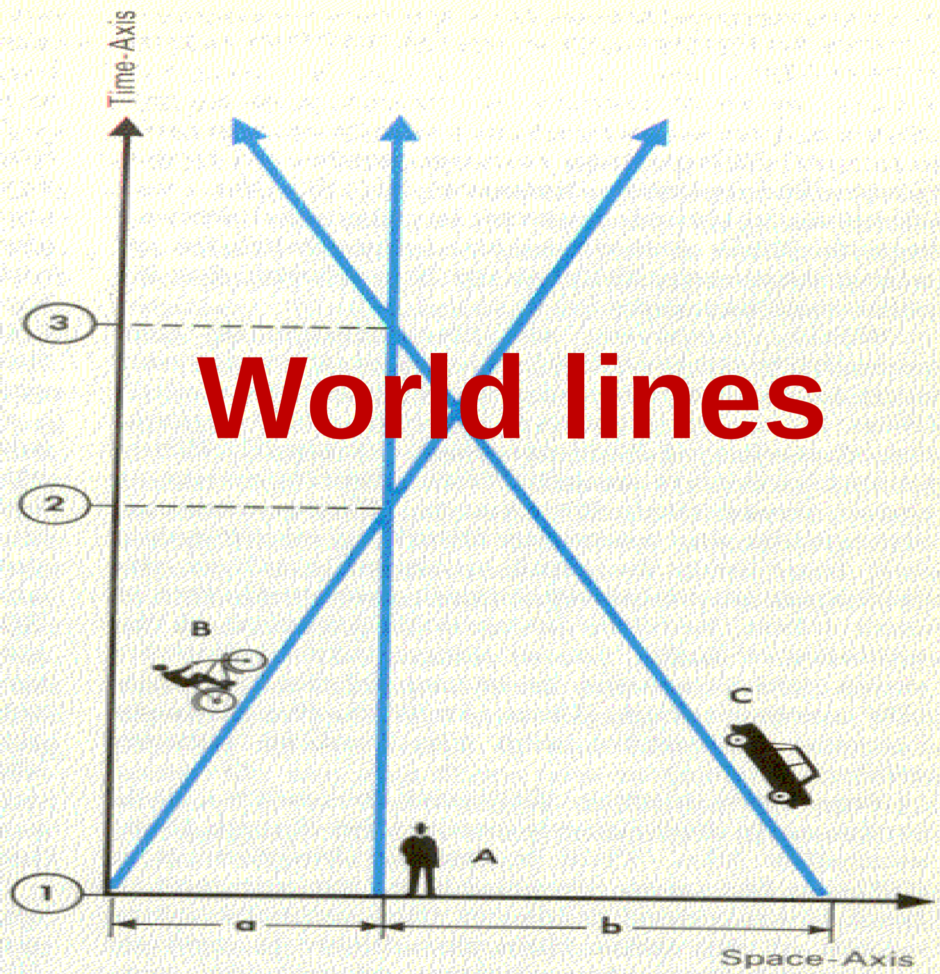
Two dimensional space-time diagram, i.e 3-D



A series of instantaneous snapshots of a girl playing ball – arranged one after the other in a stack – forms a three-dimensional picture of a four-dimensional space–time event. The world line of the ball is shown.



World lines



Space time diagram

An everyday occurrence is represented in terms of a space–time diagram. An observer A stands at the edge of a street, a cyclist B passes him, then from the opposite direction a motorist C passes by. *Left:* the three situations during this process.
Above: the space–time diagram.

Four Vector

That is space-time is a 4D space, and any vector in that space is called a "four vector"

Can we call (x, y, z, t) a four vector?

This is problematic because t doesn't have the dimensions of space coordinates (x, y, z)

This problem is solved by taking the fourth coordinate as ct instead of t

Thus, we can define a position vector R in the 4D space as

$$\mathrm{R} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

A general four vector A in this 4D space will be given by

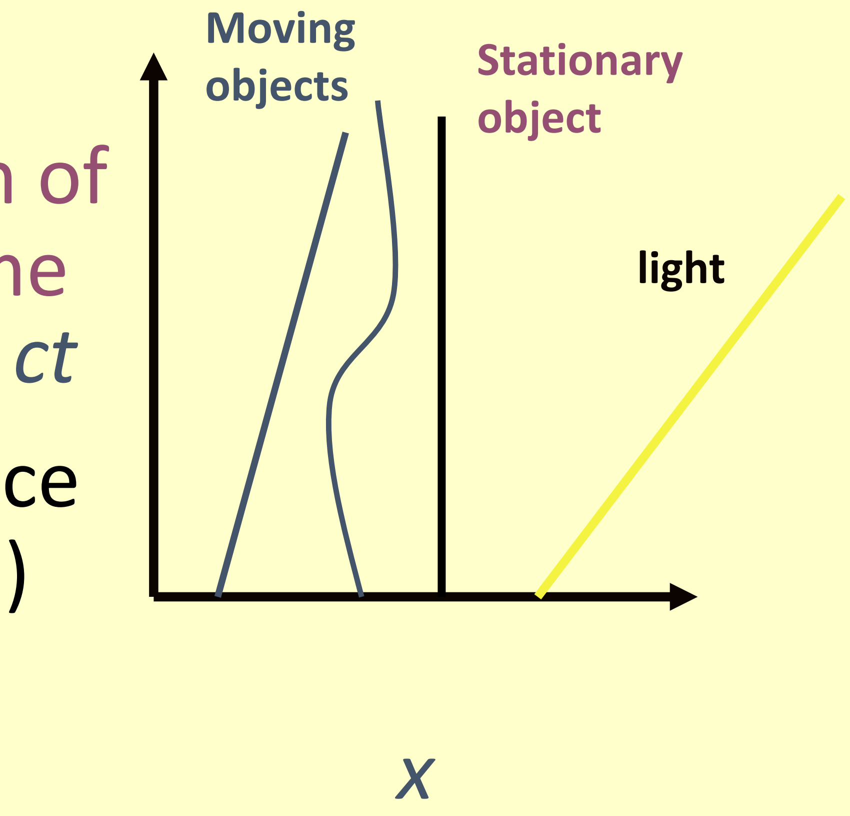
$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

In physics this 4D space, or four-space, is called the Minkowski space

The Lorentz transformation equations
can be seen as “rotations” in the Minkowski space

Space-time diagrams

- Because space and time are "mixed up" in relativity, it is often useful to make a diagram of events that includes both their space and time coordinates.
- This is simplest to do for events that take place along a line in space (one-dimensional space)
  - Plot as a 2D graph
  - use two coordinates: x and ct



World lines of events

Care should be taken of units if light at 45 degrees

Space-time diagrams

- Can be generalized to events taking place in a plane (two-dimensional space) using a 3D graph (volume rendered image): *x, y* and *ct*

- Can also be generalized to events taking place in 3D space using a 4D graph, but this is difficult to visualize

- **Space time diagrams** were first used by H. Minkowski in 1908 and are often called **Minkowski diagrams**. Paths in Minkowski space time are called **world lines**.

Minkowski's space-time diagrams

Consider S' frame moving with a speed v with respect to S. Let their origins coincide at t=t'=0.

Let a burst of light be emitted at this time. After a time t, the light reaches a point on the sphere given by

$$x^2 + y^2 + z^2 = c^2 t^2$$

Using inverse Lorentz transformation, one can show that

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Space-time Interval

Since all observers "see" the same speed of light, then all observers, regardless of their velocities, must see spherical wave fronts.

$$s^2 = x^2 - c^2t^2 = (x')^2 - c^2(t')^2 = (s')^2$$

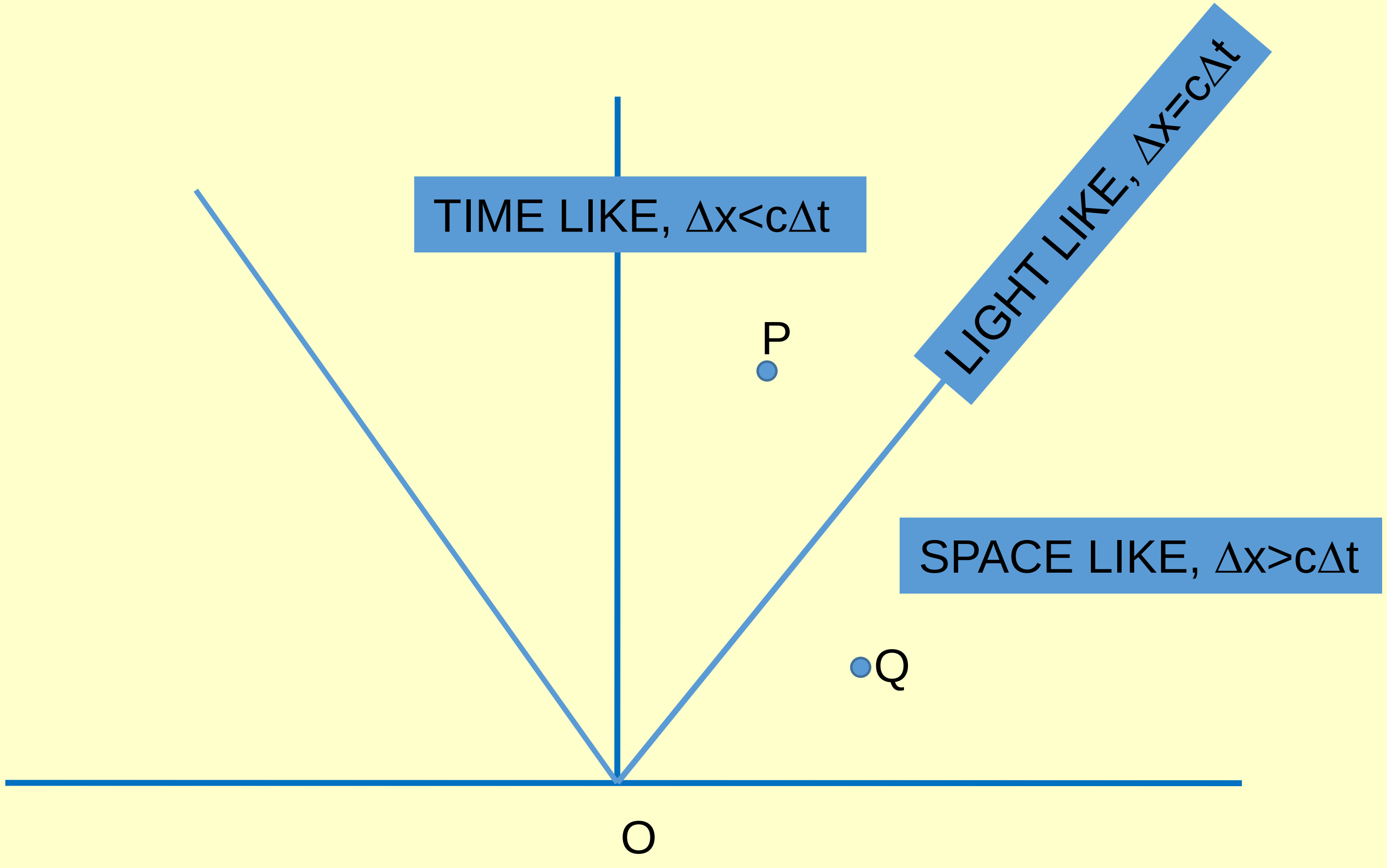
Space-time Invariants

If we consider two events, we can determine the quantity **Δs^2** between the two events, and we find that it is **invariant (Lorentz Invariant)** in any inertial frame. The quantity **Δs^2** is known as **the *space-time interval*** between two events.

Space-time Invariants

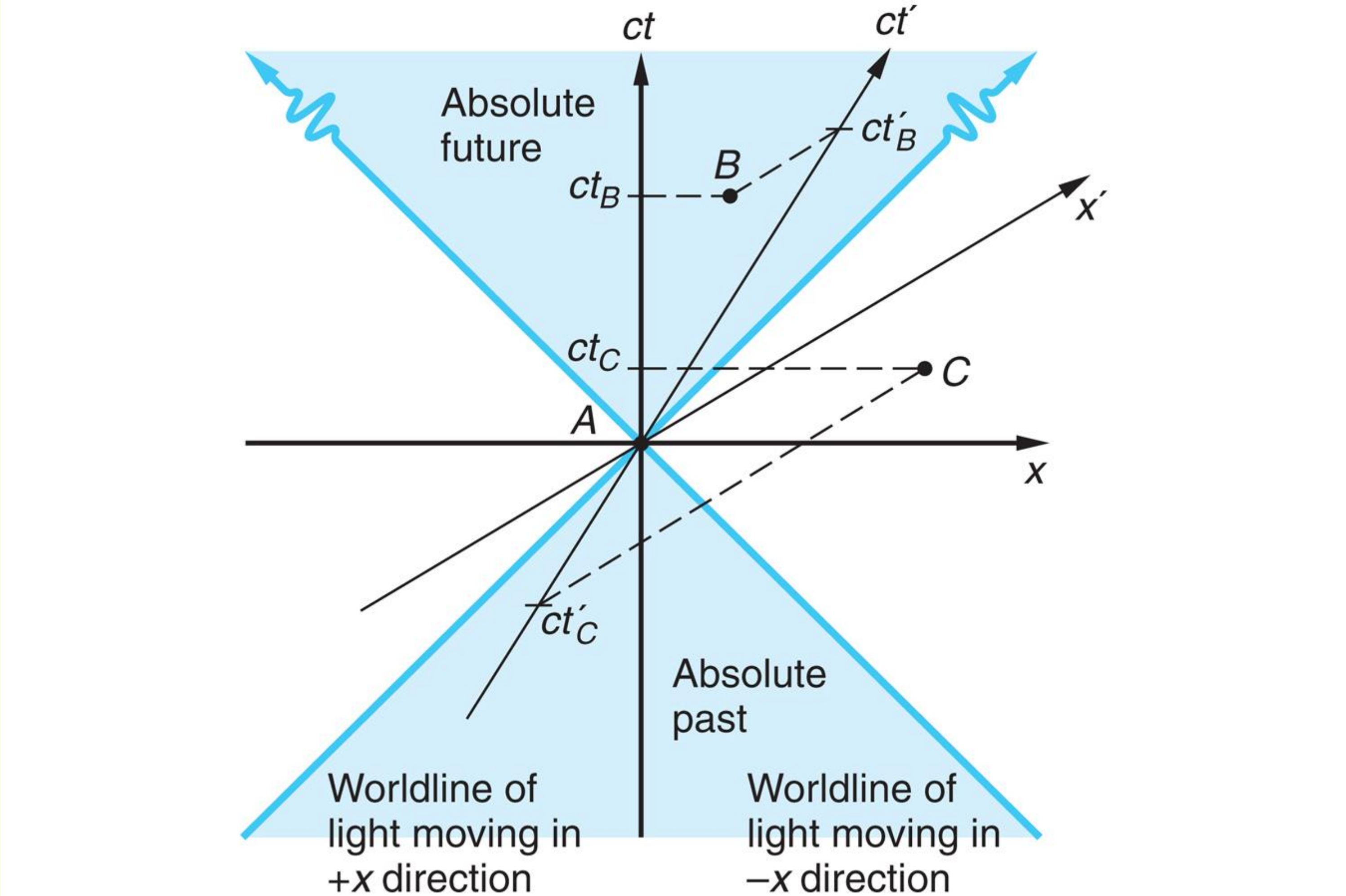
There are three possibilities for the invariant quantity Δs^2:

1) $\Delta s^2 = 0$: $\Delta x^2 = c^2 \Delta t^2$, and the two events can be connected only by a light signal. The events are said to have a **light like** separation.
2) $\Delta s^2 > 0$: $\Delta x^2 > c^2 \Delta t^2$, and no signal can travel fast enough to connect the two events. The events are not causally connected and are said to have a **space like** separation.
3) $\Delta s^2 < 0$: $\Delta x^2 < c^2 \Delta t^2$, and the two events can be causally connected. The interval is said to be **time like**.

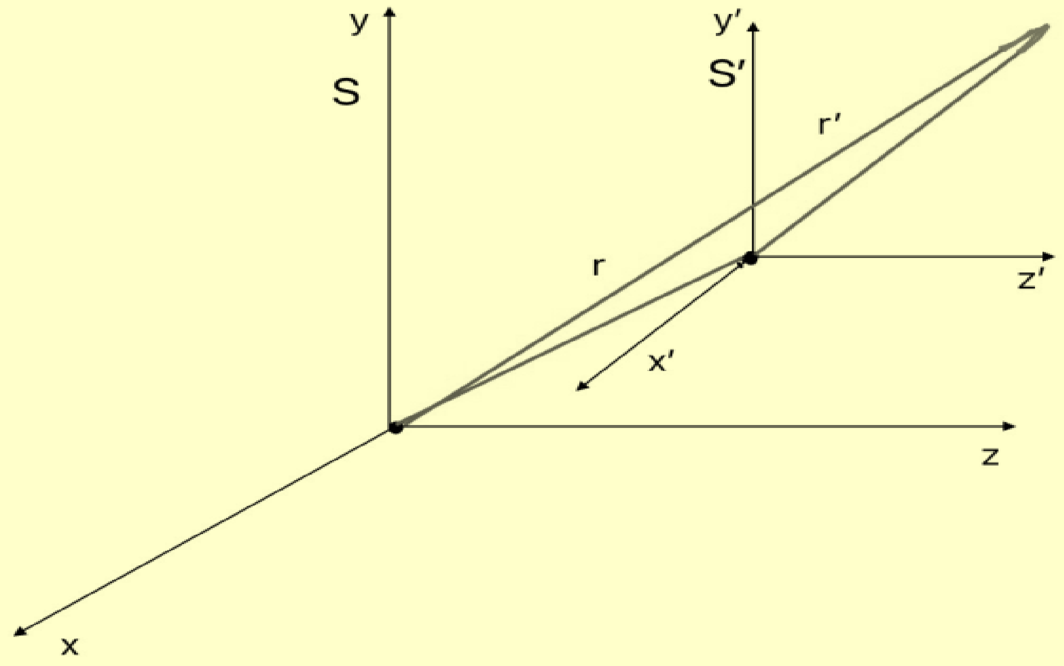

TIME LIKE, Δx<cΔt
LIGHT LIKE, Δx=cΔt
P
SPACE LIKE, Δx>cΔt
Q
O

The Light Cone

The past, present, and future are easily identified in space-time diagrams. And if we add another spatial dimension, these regions become cones.



- **Prob 10:** An intertial frame S' moves relative to another frame S with a velocity $v_1\hat{i} + v_2\hat{j}$ in such a way that the x and x' axes, y and y' axes and z and z' axes are always parallel. Let the time $t = t' = 0$ when the origins of the two frames are co-incident. Find the Lorenz transformation relating the co-ordinates and time of S' to those in S.
- **Soln:**



Solve the general problem as follows.

Write a position vector $\vec{r}$ in the S frames as a sum of two vectors, one projected along the direction of the velocity $\vec{v}$, and the other one perpendicular to it.

$$\vec{r}_{\parallel} = (\vec{r}.\hat{n})\hat{n} = \frac{(r \cdot \vec{v})\vec{v}}{v^2}$$
$$\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}$$

The Lorenz transformation only affects $\vec{r}_{\parallel}$, therefore, we obtain

$$\vec{r}' = \vec{r}_{\perp} + \gamma\left(\vec{r}_{\parallel} - \vec{v}\,t\right)$$
$$\vec{r}' = \vec{r} + (\gamma - 1)\vec{r}_{\parallel} - \gamma\vec{v}\,t$$

above $\gamma = 1/\sqrt{1-\beta^2}$. And the analogue of the time coordinate transformation

$$t' = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

is obtained by replacing vx by $\vec{v}\cdot\vec{r}$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right)$$

Taking $\vec{v} = v_1\hat{i} + v_2\hat{j} = v\cos\theta\hat{i} + v\sin\theta\hat{j}$, with $v = \sqrt{v_1^2 + v_2^2}$, we have

$$\vec{r}_{\|} = \frac{(\vec{r}\cdot\vec{v})\vec{v}}{v^2} = \frac{(xv\cos\theta + yv\sin\theta)\vec{v}}{v^2}$$

we obtain from $\overrightarrow{r}' = \vec{r} + (\gamma - 1)\vec{r}_{\|} - \gamma\vec{v}t$

$$\begin{aligned}
x' &= x + (\gamma - 1)\frac{(xv\cos\theta + yv\sin\theta)\psi\cos\theta}{v^2} - \gamma v\cos\theta t \\
&= x\left[1 + (\gamma - 1)\cos^2\theta\right] + y(\gamma - 1)\sin\theta\cos\theta - \gamma v\cos\theta t \\
y' &= y + (\gamma - 1)(x\cos\theta + y\sin\theta)\sin\theta - \gamma v\sin\theta t \\
&= x(r - 1)\sin\theta\cos\theta + y\left[\cos^2\theta + r\sin^2\theta\right] - rv\sin\theta t. \\
z' &= z \\
t' &= r\left[t - (x\cos\theta + y\sin\theta)\frac{v}{c^2}\right].
\end{aligned}$$

- **Prob 12:** Two identical spaceships, each 200 m long, pass one another traveling in opposite directions. If the relative velocity of the two space ships is $0.58c$: (a) how long does it take for the other ship to pass by as measured by a passenger in one of the ships, and (b) if these spaceships are moving along the x-direction with velocities $\pm u$ with respect to a frame S, what are their lengths as measured by an observer in S.
- **Soln:** (a) One spaceship will see the Lorentz contracted length of the other given by

$$L = L_0\sqrt{1 - v^2/c^2} = 200\sqrt{1 - 0.58^2} = 162.92329482 \text{ m}$$

Time taken will be L divided by the relative speed $0.58c$

$$\Delta t = 162.92329482/(0.58 \times 3 \times 10^8) = 9.363\,\mu\text{s}$$

(b) Clearly, from the rule of addition of velocities, we have

$$\frac{u - (-u)}{1 - u(-u)/c^2} = 0.58c$$

$$\implies 0.58u^2/c - 2u + 0.58c = 0$$

$$\Longrightarrow\ u=\frac{2\pm\sqrt{4-4(0.58)^2}}{2\times(0.58/c)}=c\left\{\frac{1\pm\sqrt{1-0.58^2}}{0.58}\right\}=c\frac{1\pm0.814616}{0.58}$$

$$\Longrightarrow\ u=3.128648c \text{ or } 0.319628$$

The only acceptable value is $u=0.319628$. Thus the equal contracted lengths of the two space ships with respect to S

$$L'=L_0\sqrt{1-u^2/c^2}=200\sqrt{1-0.319628^2}=189.509\ \text{m}$$