MA 108-ODE- D3

Lecture 3

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Exact ODE

Integrating Factors

Linear first order ODEs

Warm up!

1. $3x(xy-2)dx + (x^3+2y)dy = 0$ Is the ODE separable? Homogeneous? Closed form? Exact?

   Ans. Closed form for all $(x,y) \in \mathbb{R}^2$ and hence Exact since the domain $D = \mathbb{R}^2$ is convex.

2. $\frac{-y}{x^2+y^2} + \frac{x}{x^2+y^2} y' = 0, \quad \forall\,(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$. Is the differential form closed? Is the ODE exact?

   Ans. It is closed but not exact. Note $D = \mathbb{R}^2 \backslash \{(0,0)\}$ is not convex. (Why?)

3. ODE: Solve the IVP: $y'(t) = y(1-y)$, $y(0) = 0$ by finding the solution $y(t;\epsilon)$ of the problem $y' = y(1-y)$, $y(0) = \epsilon$ and then computing $\lim_{\epsilon \to 0} y(t;\epsilon)$.

   Ans. $y(t;\epsilon) = \frac{\epsilon e^x}{1-\epsilon+\epsilon e^x}$ and $\lim_{\epsilon \to 0} y(t;\epsilon) = 0$ for all t.

Recall...

Proposition

Let M, N and their first order partial derivatives exist and be continuous in a region $D \subseteq \mathbb{R}^2$.

(i) *If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then it is closed.*

(ii) *If D is convex, then any closed form is exact.*

The above result gives us that under certain hypothesis, the ODE is exact i.e., there exist a function u with continuous first order derivatives satisfying

$$\frac{\partial u}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y}(x, y) = N(x, y), \quad \forall\,(x, y) \in D.$$

Then the solution (implicit) of ODE is given $u(x, y) = C$ for all $(x, y) \in D$.

How to find u?

Solving Exact ODE's

Given an exact ODE as above, the function $u(x, y)$ can be found either by inspection or by the following method:
Step I: Integrate $\frac{\partial u}{\partial x}(x, y) = M(x, y)$ with respect to x, treating y as fixed parameter, to get

$$u(x, y) = \int M(x, y)dx + k(y),$$

where $k(y)$ is a constant of integration.
Step II: To determine $k(y)$ differentiate the above equation in Step I with respect to y, to get:

$$\frac{\partial u}{\partial y}(x, y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

As the given ODE is exact, we get

$$N(x, y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

We use this to determine $k(y)$ and hence u.

Example

Example: Solve the ODE:

$$(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$$

Let $M(x, y) = y \cos x + 2xe^y$ and $N(x, y) = \sin x + x^2 e^y - 1$, for all $(x, y) \in D$. Check: $M_y(x, y) = N_x(x, y)$ for all $(x, y) \in D$. Thus,

$$Mdx + Ndy$$

is closed. Can we conclude that it is exact? Yes (why?). Thus, we have an exact ODE. Need to find $u(x, y)$ such that $u_x = M$ and $u_y = N$ on $\mathbb{R}^2$.

Example continued

Step I:

$$u(x,y) = \int (y\cos x + 2xe^y)dx + k(y) = y\sin x + x^2e^y + k(y).$$

Step II:

$$u_y(x,y) = \sin x + x^2e^y + k'(y) = \sin x + x^2e^y - 1.$$

Thus, $k'(y) = -1$. Choosing $k(y) = -y$, we get:

$$u(x,y) = y\sin x + x^2e^y - y = c$$

as an implicit solution to the given DE.

Integrating Factors

Suppose the first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{1}$$

is not exact; i.e., $M_y \neq N_x$. In this situation, we sometimes find a function $\mu(x,y)$ such that

$$\mu(x,y) \cdot M(x,y) + \mu(x,y) \cdot N(x,y)\frac{dy}{dx} = 0 \tag{2}$$

is exact; i.e.,

$$(\mu(x,y) \cdot M(x,y))_y = (\mu(x,y) \cdot N(x,y))_x.$$

Thus,

$$\mu_y(x,y)M(x,y) + \mu(x,y)M_y(x,y) = \mu_x(x,y)N(x,y) + \mu(x,y)N_x(x,y).$$

Such a function $\mu(x,y)$ is called an integrating factor of the given ODE.

Integrating Factors

Note: In practice, we start by looking for an IF which depends only on one variable x or y. Suppose μ is a function of x. Then, $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ becomes

$$\mu M_y = \mu_x N + \mu N_x;$$

i.e.,

$$-N\mu_x + (M_y - N_x)\mu = 0.$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

If further, $\frac{M_y - N_x}{N}$ is a function of x (alone) then the above is a separable ODE which we try to solve to find μ.

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}.}$$

Integrating Factors

If we assume μ to be a function of y alone in

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0,$$

then we get an analogous equation:

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

If further, $\frac{N_x - M_y}{M}$ is a function of y (alone) then the above DE is separable & we try to solve it to find $\mu(y)$.

$$\boxed{\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}.}$$

Examples

Example: Solve the ODE: for $x \neq 0$:

$$(8xy - 9y^2) + (2x^2 - 6xy)\frac{dy}{dx} = 0.$$

Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$. Thus, $M_y = 8x - 18y$ and $N_x = 4x - 6y$. As $M_y \neq N_x$, the given ODE is not exact.
We first try to find an IF depending only upon one variable. Note that

$$\frac{M_y - N_x}{N} = \frac{4x - 12y}{2x(x - 3y)} = \frac{2}{x}.$$

Hence by the earlier discussion, we have:

$$\frac{d\mu}{dx} = \frac{2}{x}\mu.$$

Solving this separable ODE, we get $\ln|\mu| = \ln x^2$. Hence,

$$\mu(x) = x^2$$

can be chosen as an IF for the given ODE.

Example continued

Multiplying the given ODE by $\mu(x) = x^2$, we get:

$$(8x^3y - 9x^2y^2) + (2x^4 - 6x^3y)\frac{dy}{dx} = 0.$$

Check that this is an exact ODE. To solve this exact ODE, we need to find $u(x, y)$ such that

$$8x^3y - 9x^2y^2 = u_x \;\&\; 2x^4 - 6x^3y = u_y.$$

To find $u(x, y)$:
Step I: $u(x, y) = 2x^4y - 3x^3y^2 + k(y)$.
Step II: $u_y = 2x^4 - 6x^3y + k'(y) = 2x^4 - 6x^3y$.
Thus, $k'(y) = 0$. Hence,

$$u(x, y) = 2x^4y - 3x^3y^2 = c$$

is an implicit solution of the given ODE.

Examples

Example: Solve the ODE: $-y + x\frac{dy}{dx} = 0, \quad x \neq 0.$
$y(x) = 0$ is a solution.
Is there any non-zero solution? Check that this is not an exact DE.
Let $M(x, y) = -y$ and $N(x, y) = x$.
To find an IF μ: note that $\frac{N_x - M_y}{M} = -\frac{2}{y}$. By the earlier discussion, we get:

$$\frac{d\mu}{dy} = -\frac{2}{y}\mu.$$

Thus, $\ln|\mu| = -2\ln|y|$. So we choose

$$\mu(y) = \frac{1}{y^2}$$

as an IF. Then, $-\frac{1}{y} + \frac{x}{y^2}\frac{dy}{dx} = 0$ is exact. Thus,

$$d\left(-\frac{x}{y}\right) = 0.$$

Therefore, a solution is given by $\frac{x}{y} = c$.

Linear DE's

Definition

A first order linear DE of the type

$$\frac{dy}{dx} + p(x)y = g(x)$$

is called a linear DE in standard form.

Here we assume $p(x)$ and $g(x)$ are continuous on an interval I.

Is there an integrating factor? Check: $e^{\int p dx}$ is an integrating factor for the given DE.

Linear DE's

$e^{\int p dx}$ is an integrating factor for $\dfrac{dy}{dx} + p(x)y = g(x)$, i.e.,

$$e^{\int p dx}(py - g) + e^{\int p dx}\frac{dy}{dx} = 0$$

or equivalently,

$$\frac{d}{dx}\left(e^{\int p dx} \cdot y\right) = e^{\int p dx} \cdot g$$

is exact. Solving this, we get:

$$y = e^{-\int p dx}\left(\int e^{\int p dx} \cdot g \; dx + c\right).$$

Example

Example: Solve the IVP: $t \neq 0$,

$$ty' + 2y = 4t^2;\ y(1) = 2.$$

The standard form of the given DE is:

$$y' + \frac{2y}{t} = 4t.$$

An integrating factor is

$$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2.$$

Multiplying the given DE by t^2, we get:

$$t^2 y' + 2yt = 4t^3;$$

i.e.,

$$\frac{d}{dt}(t^2 y) = 4t^3.$$

Example continued

Integrating

$$\frac{d}{dt}(t^2 y) = 4t^3,$$

we get:

$$y(t) = t^2 + \frac{c}{t^2}.$$

The initial condition $y(1) = 2$ gives $c = 1$. Hence $y(t) = t^2 + \frac{1}{t^2}$ is a solution of the IVP.

Bernoulli DE

Definition

The first order DE

$$y' + p(x)y = q(x)y^n$$

is called Bernoulli's DE.

If $n = 0$ or 1, Bernoulli's DE is a linear differential equation.

Bernoulli DE $y' + p(x)y = q(x)y^n$

Let $n \geq 2$. Divide the above equation by y^n to get:

$$\frac{y'}{y^n} + \frac{p(x)}{y^{n-1}} = q(x).$$

Put

$$u(x) = \frac{1}{y^{n-1}}.$$

Then,

$$\frac{du}{dx} = \frac{1-n}{y^n}\frac{dy}{dx}.$$

Substituting this in the given DE, we get:

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u(x) = q(x).$$

This is a first order linear DE. We solve it by finding an integrating factor.

Example

Example: Solve the DE

$$6x^2\frac{dy}{dx} - yx = 2y^4.$$

The equation in standard form is

$$\frac{dy}{dx} - \frac{y}{6x} = \frac{y^4}{3x^2},$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. This is a Bernoulli DE for the unknown function $y = y(x)$. If y is a non-trivial solution of the above ODE, there must be some open interval I on which y has no zeroes. On I,

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{6y^3x} = \frac{1}{3x^2}.$$

Put $u = \frac{1}{y^3}$. Then,

$$\frac{du}{dx} = -\frac{3}{y^4}\frac{dy}{dx}.$$

Example continued

Substituting in the given DE, we get:

$$-\frac{1}{3}\frac{du}{dx} - \frac{u}{6x} = \frac{1}{3x^2};$$

i.e.,

$$\frac{du}{dx} + \frac{u}{2x} = -\frac{1}{x^2}.$$

An integrating factor is:

$$\mu(y) = e^{\int \frac{1}{2x}dy} = \sqrt{|x|}.$$

Thus,

$$\sqrt{|x|}\frac{du}{dx} + \frac{\sqrt{|x|}u}{2x} = -\frac{\sqrt{|x|}}{x^2}.$$

If $x > 0$, then

$$\frac{d}{dx}(u\sqrt{x}) = -x^{-\frac{3}{2}},$$

Example continued

which implies that

$$u\sqrt{x} = \frac{2}{\sqrt{x}} + c,$$

i.e.,

$$\frac{\sqrt{x}}{y^3} = \frac{2}{\sqrt{x}} + c.$$

Thus,

$$y^3 = \frac{x}{2 + c\sqrt{x}} \text{ for } x > 0.$$

Similarily, it can be checked that

$$y^3 = \frac{-x}{2 + c'\sqrt{-x}} \text{ for } x < 0.$$

Existence and Uniqueness

The IVP's that we have considered usually have unique solutions. This need not always be the case.

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\ y(0) = y_0.$$

This is a separable first order DE. First, suppose that $y_0 \neq 0$. Hence, separating the variables, we get:

$$y^{-\frac{1}{3}} dy = dx;$$

i.e.,

$$\frac{3}{2} y^{\frac{2}{3}} = x + c,$$

or

$$y = \left[\frac{2}{3}(x + c)\right]^{\frac{3}{2}}, \quad x \geq -c.$$

The initial condition $y(0) = y_0$ gives $c = \frac{3}{2} y_0^{\frac{2}{3}}$. Hence,

$$y = \left[\frac{2}{3}\left(x + \frac{3}{2} y_0^{\frac{2}{3}}\right)\right]^{\frac{3}{2}}, \quad x \geq -\frac{3}{2} y_0^{\frac{2}{3}}$$

is a solution of the given IVP. **This is also a solution if $y_0 = 0$.**

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\ y(0) = 0.$$

$$y = \phi(x) = \left[\frac{2x}{3}\right]^{\frac{3}{2}};\ x \geq 0$$

is a solution.

$$y = -\phi(x) = -\left[\frac{2x}{3}\right]^{\frac{3}{2}};\ x \geq 0$$

is also a solution of the IVP.

$$y = \psi(x) \equiv 0$$

is also a solution of the IVP.
For any $a > 0$,

$$y = \phi_a(x) = \begin{cases} 0 & \text{if } x \in [0, a) \\ \pm\left[\frac{2}{3}(x-a)\right]^{\frac{3}{2}} & \text{if } x \geq a \end{cases}$$

is continuous, differentiable, and gives a solution of the given IVP.

Existence and Uniqueness

That is, we get infinitely many solutions of the given IVP.

No solution of IVP

It may happen there exists no differentiable function satisfying the ODE and initial value!

Ex. Solve: $y(t)y'(t) = \frac{1}{2}, y(0) = 0$.

Ans. no solution, because if there exists any solution ϕ, then $\phi(t)\phi'(t) = \frac{1}{2}$ and putting $t = 0$, $\phi(0) = 0$ contradicts the equation.

How to determine if an IVP has a solution? In case a solution exists, when it has to be unique? Existence and uniqueness theorem.