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Properties of real numbers:

Theorem 2: (Archimedean property):-

Let $x, y \in \mathbb{R}$ and $x > 0$. Then $\exists n \in \mathbb{N}$ such that $nx > y$.

Proof: If the statement is not true then $\forall n \in \mathbb{N}$

$$nx \leq y. \quad \text{--- (1)}$$

Let $A = \{nx : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

From (1) above, it follows that A is bounded above. Hence by least upper bound property, A has a supremum. Let $\alpha = \sup A$.

Therefore, $\alpha - x$ is not an upper bound of A ($\because x > 0$). So, $\exists m \in \mathbb{N}$ such that,

$$\cancel{\alpha - x \leq mx} \quad \alpha - x \leq mx$$

i.e., $\alpha \leq (m+1)x$.

But $(m+1)x \in A$. This contradicts the fact that α is an upper bound of A.

This completes the proof.

Theorem; 3:

Let $x, y \in \mathbb{R}$ and $x < y$. Then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

That is, between any two real number there exists a rational number.

Proof: We have, $y - x > 0$.

Therefore by archemedean property (Theorem 2), $\exists n \in \mathbb{N}$ such that

$$n(y-x) > 1 \quad \text{--- } (1)$$

Also by archemedean property, $\exists m_1 \in \mathbb{N}$, $\exists m_2 \in \mathbb{N}$

such that $m_1 > nx$ (Take two nos. 1 & nx to apply archemedean)

& $m_2 > -nx$. (Two no. 1 & $-nx$)

Hence, $-m_2 < nx < m_1$.

This shows that, there is an integer m, $-m_2 < m < m_1$ such that

$$m - 1 \leq nx < m \quad \text{--- } (2)$$

$\therefore$ From, (1) & (2)

$$nx < m \leq nx + 1 < ny$$

$\Rightarrow x < \frac{m}{n} < y$. This completes the proof.

Exercise :-

1. prove that $\sqrt{2}$ is not in $\mathbb{Q}$, i.e. $\sqrt{2}$ is irrational.

2. prove that $\sqrt{3}$ is irrational no.

3. For any $n \in \mathbb{N}$, prove that either ~~$\sqrt{n}$~~ n is perfect square or $\sqrt{n}$ is irrational no:

4. Prove that $\sqrt{n+1} + \sqrt{n-1}$ is irrational for any $n \in \mathbb{N}$.

5. Let $e := 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

prove that e is irrational.

Exercises:

1. Prove that between any two rational number, there exists an irrational no. also.

Theorem 4: For every real $x>0$ and $n \in \mathbb{N}$, there is unique $y \in \mathbb{R}$ such that $y^n = x$.

Proof:

- Uniqueness: Obvious, since $y_1 < y_2$
$$\Rightarrow y_1^n < y_2^n.$$

- Existence:
Let $E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}$.

- $E \neq \phi$ as $t_0 := \frac{x}{1+x} \in E$. $\left(\because t_0 < 1 \Rightarrow t_0^n < t_0 < x\right)$.

- E is bounded above, with an upper bound $1+x$: as if $t > 1+x \Rightarrow t^n > t > x \Rightarrow t \notin E$.

- Hence by completeness property,
$$y := \sup E \in \mathbb{R}, \text{ exists.}$$

- Claim: $y^n = x$.

If possible, suppose $y^n < x$. Choose $0 < h < 1$ such that
$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$

Now,

$$(y+h)^n - y^n < h\,n\,(y+h)^{n-1} \qquad \left(\because\ b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \le (b-a)\,n\,b^{n-1} \text{ if } 0<a<b \right).$$

$$< h\cdot n\cdot (y+1)^{n-1}$$

$$< x - y^n$$

$\Rightarrow (y+h)^n < x.$

$\Rightarrow y+h \in E$

— which is a contradiction as $y = \sup E$.

So, $y^n \geq x$.

If possible let $y^n > x$.

Let $k = \dfrac{y^n - x}{n y^{n-1}}$. Then, $0 < k < y$.

If $t \geq y-k$, then

$$y^n - t^n \le y^n - (y-k)^n < k n\, y^{n-1} < y^n - x$$

$$\Rightarrow x < t^n \Rightarrow t \notin E.$$

Therefore, $t \geq y-k \Longrightarrow t \notin E$

Hence, $t \in E \Rightarrow t < y-k$. That is $y-k$ is an upperbound of E — which is a contradiction as $y = \sup E$ & $k > 0$.

$\therefore\ y^n = x$.

- Proof of Theorem 1:

Outline of the proof (see ~~next page~~ attached notes).

- Decimal representations: Every real number has decimal representation. (How?)

Let $x > 0$ be a real number. Let n_0 be the largest +ve integer such that $n_0 \le x$. (why it exists?).

Then let n_1 be the largest +ve integer such that

$$n_0 + \frac{n_1}{10} \le x.$$

Let n_2 be the largest +ve integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} \le x$$

... continue ...

Let $E = \left\{ n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} : k = 0, 1, 2, \dots \right\}$

Then $x = \sup E$ and x has decimal repn.

$$x = n_0 . n_1 n_2 n_3 \dots$$