TRIPLE INTEGRALS

SUPPOSE $F : K \to \mathbb{R}$ WHERE

$K = [a,b] \times [c,d] \times [e,f]$

A PARTITION OF K IS AN ORDERED TRIPLE

OF PARTITIONS

$$\mathcal{P}_x \equiv a = x_0 < x_1 < \cdots < x_m = b$$

$$\mathcal{P}_y \equiv c = y_0 < y_1 < \cdots < y_n = d$$

$$\mathcal{P}_z \equiv e = z_0 < z_1 < \cdots < z_p = f.$$

LET $V_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$

AND LET $(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}) \in A_{ijk}$

THEN IF $\lim\limits_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, F)$ EXISTS, WHERE

$$S(\mathcal{P}, f) = \sum_{(i,j,k)}' F(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk})\, |V_{ijk}|$$

$$\Big(|V_{ijk}| = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \Big)$$

THEN WE SAY F IS INTEGRABLE AND WE

WRITE $\displaystyle\iiint_K F(x,y,z)\,dx\,dy\,dz = \lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, F)$

- IF F IS CONTINUOUS THEN F IS INTEGRABLE.
  (SET OF DISCONTINUITIES HAS 3-MEASURE ZERO $\Rightarrow$ F IS INTEGRABLE)
- F, G ARE INTEGRABLE $\Rightarrow$

$$\iiint_K F \pm G = \iiint_K F \pm \iiint_K G$$

$$\iiint_K \alpha F = \alpha \iiint_K F \qquad (\alpha \in \mathbb{R})$$

$$\left| \iiint_K F \right| \leq \iiint_K |F|$$

- LIKE IN THE 2-VARIABLE CASE, WE CAN EXTEND THE TRIPLE INTEGRAL OVER CLOSED BOUNDED REGIONS.

$$dV = dx\,dy\,dz$$

FUBINI'S THEOREM

SUPPOSE D IS CLOSED AND BOUNDED AND

$$D = \{(x,y,z) \mid (x,y) \in R,\ g(x,y) \leq z \leq h(x,y)\} \quad (R \subseteq \mathbb{R}^2)$$

WHERE $g, h : R \to \mathbb{R}$ ARE CONTINUOUS. IF

$f : D \to \mathbb{R}$ IS CONTINUOUS, THEN

$$\iiint_D f(x,y,z)\, dV = \iint_R \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right] dx\, dy$$

IF R IS A TYPE-I ELEMENTARY FUNCTION,

$R = \{(x,y) \mid a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$ THEN

$$\iiint_K f(x,y,z)\, dV = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right) dy \right) dx$$

WE CAN WRITE A SIMILAR FORMULA IF R IS ELEMENTARY TYPE-II.

$$\mathrm{Vol}(K) := \iiint_K dV$$ FOR A BOUNDED (CLOSED) REGION K

EXAMPLES

FIND THE VOLUME OF THE SOLID K BOUNDED BY $4x^2+4y^2+z^2=16$

$$R=\{(x,y)\mid x^2+y^2\le 4\}$$

$$g(x,y)\ (\text{TOP}) = \sqrt{16-4(x^2+y^2)} = 2\sqrt{4-(x^2+y^2)}$$

$$h(x,y)\ (\text{BOTTOM}) = -\sqrt{16-4(x^2+y^2)} = -2\sqrt{4-(x^2+y^2)}$$

HENCE THE VOLUME EQUALS

$$\iiint_K dV = \iint_R \left[\int_{-2\sqrt{4-(x^2+y^2)}}^{2\sqrt{4-(x^2+y^2)}} 1\, dz\right] dx\, dy \quad (\text{FUBINI})$$

R IS A TYPE I ELEMENTARY REGION, SO THE INTEGRAL

$$\iint_R = \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\right] dx$$

HENCE, VOLUME OF K EQUALS

$$\int_{-2}^{2}\left\{\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[\int_{-2\sqrt{4-(x^2+y^2)}}^{2\sqrt{4-(x^2+y^2)}} dz\right] dy\right\} dx$$

EXAMPLES (CONTINUED)

FIND VOLUME OF SOLID ENCLOSED BETWEEN

$Z = X^2 + 3Y^2$ AND $z = 8 - X^2 - Y^2$.

$$\text{MIN}\left\{x^2+3y^2, 8-(..)\right\} \le Z \le \text{MAX}\left\{x^2+3y^2, 8-(x^2+y^2)\right\}$$

SET $X^2 + 3Y^2 = 8 - X^2 - Y^2$, SO,

$$2x^2 + 4y^2 = 8, \text{ i.e. } \frac{x^2}{4} + \frac{y^2}{2} = 1$$

$\therefore$ VOLUME $=$

$$\iint_{\frac{x^2}{4}+\frac{y^2}{2}\le 1} \left[\int_{x^2+3y^2}^{8-(x^2+y^2)} dz\right] dx\,dy$$

$=$

$\vdots$

CHANGE OF VARIABLE FORMULA

SUPPOSE D IS AN ELEMENTARY REGION, $f: D \to \mathbb{R}$ IS CONTINUOUS, $\Omega \subseteq \mathbb{R}^3$ IS OPEN, AND

$g: \Omega \to \mathbb{R}^3$; $g(u,v,w) = (g_1, g_2, g_3)$ IS 1-1

WITH ALL PARTIAL DERIVATIVES CONTINUOUS.

FURTHER, SUPPOSE THE JACOBIAN $J(u,v,w) \neq 0$ $\forall\, (u,v,w) \in \Omega$, AND $g(E) = D$ FOR SOME ELEMENTARY REGION E. THEN

$$\iiint_D f\, dV = \iiint_E f(g_1, g_2, g_3)(u,v,w)\, |J(u,v,w)|\, dW$$

WHERE $dW = du\,dv\,dw$, AND

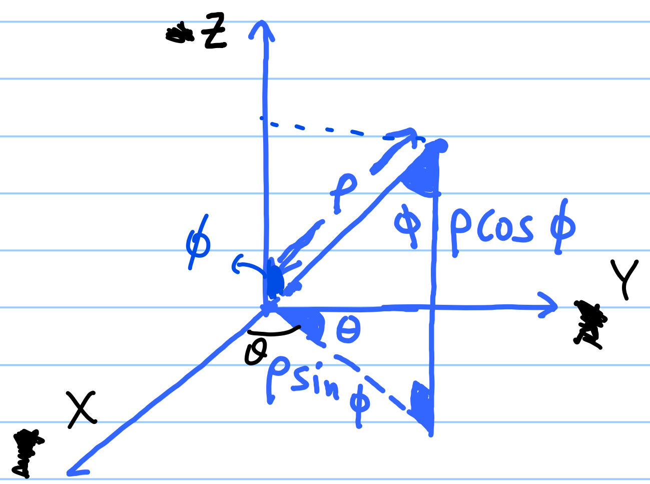
$$J(u,v,w) = \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} & \frac{\partial g_3}{\partial w} \end{vmatrix}$$

SPHERICAL COORDINATES

$x = \rho \sin\phi \cos\theta \qquad \rho \geq 0$

$Y = \rho \sin\phi \sin\theta \qquad 0 \leq \theta < 2\pi$

$z = \rho \cos\phi \qquad 0 \leq \phi \leq \pi$



$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

THE JACOBIAN EQUALS

$$J(\rho, \theta, \phi) = \begin{array}{c} \begin{array}{ccc} \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \end{array} \\ \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix} \end{array}$$

$$= \rho^2 \sin\phi$$

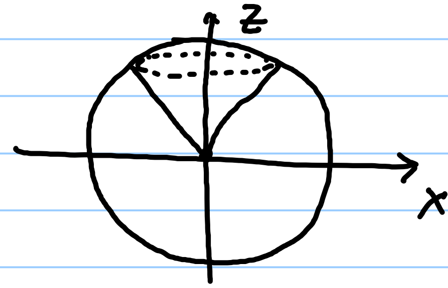
HENCE

$$\iiint_D f \, dV =$$

$$\iiint f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \, \rho^2 \sin\phi \, dr \, d\theta \, d\phi$$

EXAMPLE

FIND VOLUME OF THE SOLID CUT FROM $x^2+y^2+z^2=9$ BY THE CONE $z=\sqrt{x^2+y^2}$



LET US DESCRIBE THESE IN SPHERICAL COORDINATES:

$$x^2+y^2+z^2=9 = \{(\rho,\theta,\phi) \mid \rho\le 3,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi\} = \mathcal{R}_1$$

THE CONE $z^2=x^2+y^2$, $(z\ge 0)$ IN SPHERICAL COORDINATES:

$z^2=\rho^2\cos^2\phi$

$\begin{cases} x^2=\rho^2\sin^2\phi\cos^2\theta \\ y^2=\rho^2\sin^2\phi\sin^2\theta \end{cases}$

$$2z^2=\rho^2 \iff 2\rho^2\cos^2\phi=\rho^2$$
$$\iff \underline{\cos^2\phi=\frac{1}{2}}$$

SO CONE IN SPHERICAL COORDINATES IS:

$$\mathcal{R}_2=\{(\rho,\theta,\phi) \mid \rho\ge 0,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{4}\}$$

$$\mathcal{R}_1\cap\mathcal{R}_2 = \{0\le\rho\le 3,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{4}\}$$

SO VOLUME$(\mathcal{R}_1\cap\mathcal{R}_2)$ =

$$\int_0^3\int_0^{2\pi}\int_0^{\pi/4} \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

THIS IS NOW A ROUTINE CALCULATION

(COMPLETE IT: EXERCISE)

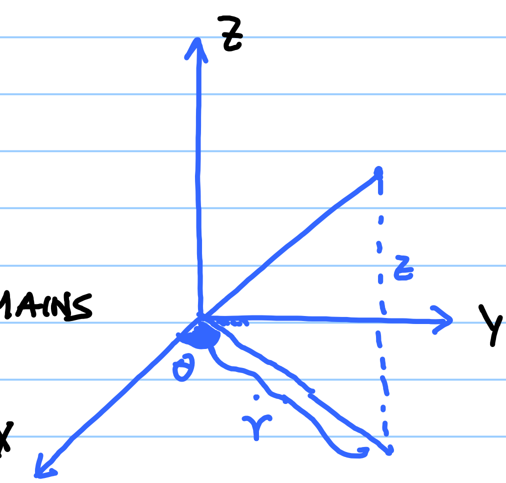
CYLINDRICAL COORDINATES

A POINT (x, y, z) IN CYLINDRICAL COORDINATES IS GIVEN BY THE TRIPLE (r, θ, z)

$$x = r\cos\theta$$

$$y = r\sin\theta$$

AND THE z-COORDINATE REMAINS THE SAME.



CHANGE OF VARIABLE FORMULA FOR CHANGING INTO CYLINDRICAL COORDINATES:

IF $g(r, \theta, z) = (x, y, z)$, $g(E) = D$, THEN

$$\iiint_D f(x, y, z)\, dV = \iiint_E f \circ g(r, \theta, z)\, |J(r, \theta, z)|\, d(r, \theta, z)$$

$$= \iiint f \circ g(r, \theta, z)\, r\, dr\, d\theta\, dz.$$

SINCE

$$J(r, \theta, z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

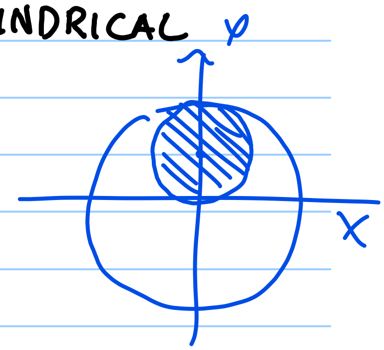
EXAMPLE

FIND THE VOLUME OF THE SOLID CUT FROM

$x^2+y^2+z^2=1$ BY THE CYLINDER $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$.

LET US EXPRESS THIS SOLID IN CYLINDRICAL

COORDINATES:



$x^2+y^2-y \le 0$

$x^2+y^2 \le y$

$r^2 \le r\sin\theta$

$r \le \sin\theta$

$$\mathcal{R}_1 = \{(r,\theta,z) \mid r^2+z^2 \le 1\}$$

$$\mathcal{R}_2 = \{(r,\theta,z) \mid \underline{0 \le r \le \sin\theta}\}$$

$$\mathcal{R}_1 \cap \mathcal{R}_2 = \{(r,\theta,z) \mid 0 \le r \le \sin\theta,\ z^2 \le 1-r^2\}$$

So, $\mathrm{Vol}(\mathcal{R}_1 \cap \mathcal{R}_2) =$

$$\int_0^{\pi}\left(\int_0^{\sin\theta}\left(\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dz\right) dr\right) d\theta .$$

THE LIMITS FOR θ ARE $[0,\pi]$ AND

NOT $[0,2\pi]$ (WHY?!)