MA 108-ODE- D3

Lecture 14

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nth order linear DE: Constant coefficient

Laplace transform

Constant Differential Operators

Aim. Give a formulation for the basis of $ker\ L$, where

$$L = D^n + a_1 D^{n-1} + \ldots + a_n I,$$

with constant coefficients $a_1, a_2, \cdots a_n$.

Constant Differential Operators

Constant differential operator: $L = D^n + a_1 D^{n-1} + \ldots + a_n I$, with constant coefficients $a_1, a_2, \cdots a_n$.

Case I: P_L has distinct real roots:

Theorem

Let L be a constant coefficient linear differential operator of order n such that

$$P_L(x) = (x - r_1)\ldots(x - r_n)$$

where $r_1, r_2, \ldots, r_n$ are distinct real numbers. Then the general solution of $L(y) = 0$ is given by

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Proof: We have

$$L = (D - r_1)\ldots(D - r_n)$$

and the Null-space

$$N(D - r_i) = \{ce^{r_i x} : c \in \mathbb{R}\}.$$

Constant Differential Operators

It follows that

$$e^{r_1 x}, \ldots, e^{r_n x} \in N(L).$$

Check! $\{e^{r_1 x}, \ldots, e^{r_n x}\}$ are linearly independent over $\mathbb{R}$ as $r_1, r_2, \ldots, r_n$ are distinct real numbers.

As dimension of $N(L)$ is n by the Dimension Theorem, we get

$$N(L) = \{c_1 e^{r_1 x} + \ldots + c_n e^{r_n x} : c_1, \ldots, c_n \in \mathbb{R}\}.$$

Constant Differential Operators

Case II: $P_L(x)$ has some repeated real roots: What happened in the $n = 2$ case? $m_1 = m_2 = m$ gave us only one solution - $f(x) = e^{mx}$. The other solution was obtained using the method of looking for a solution of the form vf. This method yielded xe^{mx} as the other solution.

Proposition

For any real number r, the functions,

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \ldots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \ldots, u_m(x) \in \mathrm{Ker}((D - r)^m).$$

Constant Differential Operators

Proof. Since $\{1, x, x^2, \ldots, x^m\}$ is linearly independent, it follows that $\{e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}\}$ is also linearly independent. We need to show that these functions are in Ker $(D-r)^m$. Let's first verify this when $m = 1$. We need to show that

$$u_1(x) = e^{rx} \in \mathrm{Ker}((D-r)),$$

which is true, since

$$(D-r)(e^{rx}) = re^{rx} - re^{rx} = 0.$$

Suppose $m = 2$. Since u_1 is in Ker of $(D-r)$, it's in Ker of $(D-r)^2$ (why?). What about u_2?

$$\begin{aligned}(D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\ &= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\ &= (D-r)(e^{rx}) = 0.\end{aligned}$$

Constant Differential Operators

So how do we prove this in general? Induction. Assume

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in \mathrm{Ker}((D-r)^m).$$

That

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^m)$$

is easy since

$$\mathrm{Ker}((D-r)^{m-1}) \subseteq \mathrm{Ker}((D-r)^m).$$

To show that u_m is also in $\mathrm{Ker}((D-r)^m)$, consider

$$\begin{aligned}
(D-r)^m\left(u_m(x)\right) &= (D-r)^m(x^{m-1}e^{rx}) \\
&= (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
= (D-r)^{m-1}\big(x^{m-1}re^{rx} &+ (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}\big) \\
&= (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) \\
&= 0.
\end{aligned}$$

Constant Differential Operators

Thus, if

$$P_L(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \dots (x - r_\ell)^{m_\ell},$$

where $\displaystyle\sum_{i=1}^{\ell} m_i = n$, a basis of Ker L is given by

$$e^{r_1 x}, \dots, x^{m_1 - 1} e^{r_1 x}, e^{r_2 x}, \dots, x^{m_2 - 1} e^{r_2 x}, \dots, e^{r_\ell x}, \dots, x^{m_\ell - 1} e^{r_\ell x}.$$

Note that that the above functions are linearly independent and since dim Ker $L = n$, these form a basis.

Exercise: Check that the above functions are linearly independent.

Constant Differential Operators

Example: Find the general solution of the DE:

$$L(y) = (D^3 - D^2 - 8D - 12)(y) = 0.$$

We have

$$P_L(x) = x^3 - x^2 - 8x - 12 = (x-2)^2(x+3),$$

and therefore,

$$L = (D-2)^2(D+3).$$

Hence, $e^{2x}, xe^{2x} \in N\big((D-2)^2\big)$ and $e^{-3x} \in N\big((D+3)\big)$. As dimension of $N(L) = 3$ and e^{2x}, xe^{2x} and e^{-3x} are linearly independent, they form a basis of the solution space of L. Thus the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Constant Differential Operators

Example: Find the general solution of the DE:

$$L(y) = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

Now,

$$P_L(x) = x^6 + 2x^5 - 2x^3 - x^2 = x^2(x-1)(x+1)^3.$$

Hence,

$$L = D^2(D-1)(D+1)^3.$$

Ker D^2 is $\langle 1, x \rangle$. Ker $D-1$ is $\langle e^x \rangle$. Ker $(D+1)^3$ is $\langle e^{-x}, xe^{-x}, x^2e^{-x} \rangle$.
Thus, the general solution is

$$c_1 + c_2x + c_3e^x + c_4e^{-x} + c_5xe^{-x} + c_6x^2e^{-x},$$

with $c_i \in \mathbb{R}$.

Proposition (Linear independency of a set of products of polynomial and exponential functions)

Let $Q_1, Q_2, \ldots, Q_n$ be non-zero polynomials and let $r_1, r_2, \ldots, r_n$ be distinct real numbers. Then $Q_1(x)e^{r_1 x}, Q_2(x)e^{r_2 x}, \ldots, Q_n(x)e^{r_n x}$ are linearly independent functions.

Proof: Induction on n.

Proposition

Consider the functions

$$u_{pq}(x) = x^{q-1}e^{r_p x},$$

where $p = 1, 2, \ldots, k$*,* $q = 1, 2, \ldots, m_p$*,* $r_1, \ldots, r_k$ *are distinct real numbers and* $m_1, m_2, \ldots, m_k$ *are positive integers such that*

$$m_1 + \ldots + m_k = n.$$

Then the n *functions* u_{pq} *are linearly independent.*

Proof: Suppose that

$$\sum_{p=1}^{k} \sum_{q=1}^{m_p} c_{pq} u_{pq}(x) = 0$$

Proof: Suppose that

$$\sum_{p=1}^{k}\sum_{q=1}^{m_p} c_{pq}u_{pq}(x) = 0$$

i.e.,

$$\sum_{p=1}^{k}\left(\sum_{q=1}^{m_p} c_{pq}x^{q-1}\right)e^{r_p x} = 0$$

Denoting $Q_p(x) = \sum_{q=1}^{m_p} c_{pq}x^{q-1}$ for each $p = 1, \cdots k$, we have

$$\sum_{p=1}^{k} Q_{pq}(x)e^{r_p x} = 0.$$

From the above Proposition [Linear independency of a set of products of polynomial and exponential functions], it follows that

$$\sum_{q=1}^{m_p} c_{pq}x^{q-1} = 0.$$

This implies that $c_{pq} = 0$ for every p, q as $1, x, \ldots, x^{m_p-1}$ are linearly independent.

Constant Differential Operators

III. P_L has some complex roots: In the second order case, if $m_1 = a + \imath b, m_2 = a - \imath b$, we got a basis:

$$\frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx, \frac{e^{m_1 x} - e^{m_2 x}}{2i} = e^{ax} \sin bx.$$

If

$$P_L(x) = x^n + p_1 x^{n-1} + \ldots + p_n$$

has a complex root $a + \imath b$, it also has $a - \imath b$ as a root. Thus,

$$(x - (a + \imath b))(x - (a - \imath b)) = x^2 - 2ax + (a^2 + b^2)$$

is a factor of $P_L(x)$, i.e., $D^2 - 2aD + (a^2 + b^2)$ is a factor of L. Ker $(D^2 - 2aD + (a^2 + b^2))$ has a basis

$$\{e^{ax} \cos bx, e^{ax} \sin bx\}.$$

Constant Differential Operators

If $a \pm \imath b$ is a root of multiplicity m, we need to consider

$$\operatorname{Ker}\left(\left(D^2 - 2aD + (a^2 + b^2)\right)^m\right).$$

Can you guess a basis for this space?

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx.$$

Exercise: Check this.

Constant Differential Operators

Example: Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(x-1)(x^2 - 4x + 5)^2.$$

The roots are

$$1, 2 \pm \imath, 2 \pm \imath.$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} \cos x + c_3 x e^{2x} \cos x + c_4 e^{2x} \sin x + c_5 x e^{2x} \sin x,$$

where $c_i \in \mathbb{R}$.

Summary: n-th order linear ODE with constant coefficients

$L = D^n + a_1 D^{n-1} + \ldots + a_n I$, with constant coefficients $a_1, a_2, \cdots a_n$.
Characteristic polynomial: $P_L(x) = x^n + a_1 x^{n-1} + \cdots + a_n x$.

Case I If $P_L(x) = (x - r_1) \cdots (x - r_n)$, where $r_1, r_2, \ldots, r_n$ are distinct real numbers, then the general solution of $L(y) = 0$ is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Case II If $P_L(x) = (x - r_1)^{m_1} (x - r_2)^{m_2} \ldots (x - r_\ell)^{m_\ell}$, where $\displaystyle\sum_{i=1}^{\ell} m_i = n$,

then the general solution of $L(y) = 0$ is given by

$$y(x) = \sum_{p=1}^{\ell} \sum_{q=1}^{m_p} c_{pq} x^{q-1} e^{r_p x},$$

where c_{pq}, for $p = 1, \cdots, \ell$ and $q = 1, \cdots m_p$ are real constants.

Summary: Contd....

Case 3 If $P_L(x)$ has a factor $\left(D^2 - 2aD + (a^2 + b^2)\right)^m$, then the general solution of $L(y) = 0$ will have terms involving

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m-1}e^{ax}\sin bx.$$

Non-homogeneous 2nd order Linear ODE's - RECALL

Consider

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

where p, q, r are continuous functions on an interval I.

Let y_p be any solution of (1) and y_1, y_2 be a basis of the solution space of the corresponding homogeneous DE.

Then the set of solutions of (1) is

$$\{c_1y_1 + c_2y_2 + y_p \mid c_1, c_2 \in \mathbb{R}\}.$$

Summary: In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE.

Annihilator Operator

If A is a linear differential operator with constant coefficients and $r(\cdot)$ is a sufficiently smooth differentiable function such that

$$Ar(x) = 0,$$

then A is said to be the annihilator of the function $r(x)$.

Examples :

1. $D + 3$ annihilates e^{-3x}.
2. $(D-2)^2$ annihilates $4e^{2x} - 10xe^{2x}$.
3. $D^2 + 16$ annihilates $\cos 4x,\ \sin 4x$ or any of their linear combinations.
4. $D^2 + 2D + 5$ annihilates $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$.
5. D^4 annihilates $1 - 5x^2 + 8x^3$.
6. D^n annihilates $1, x, x^2, \cdots, x^{n-1}$.
7. $(D-\alpha)^n$ annihilates $e^{\alpha x},\ xe^{\alpha x}, \cdots, x^{n-1}e^{\alpha x}$.
8. $(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$ annihilates $e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x, \cdots$ $x^{n-1}e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x, \cdots\ x^{n-1}e^{\alpha x}\sin\beta x$.

Annihilator Method - Formalising the method of undetermined coefficients

- Let $L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r(\cdot)$ be a non-homogeneous DE with constant coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $r(\cdot)$ is a sum or product of exponential, polynomial, or sinusoidal terms.
- In the method of annihilators, we assume that the function r is in Null-space of A, where A is a constant coefficient linear differential operator, i.e., $Ar(x) = 0$.
- Since $L(y) = r(\cdot)$, we get $(AL)(y) = A(r(\cdot)) = 0$.
- If y is a solution of the non-homogeneous DE $L(y) = r(\cdot)$, then y is a solution of the homogeneous DE with constant coefficients:

$$AL(y) = 0.$$

- To find solutions of the above homogeneous DE, we look for roots of its charactersitic polynomial:

$$P_{AL}(x) = P_A(x) P_L(x).$$

Example

Example: Find a particular solution of DE:

$$L(y) = (D^2 - 5D + 6)(y) = xe^x.$$

Note that for $r(x) = xe^x$,

$$(D-1)^2(xe^x) = 0.$$

Thus, for $r(x) = xe^x$, an annihilator operator is $A = (D-1)^2$.

If $Ly = xe^x$,

$$AL(y) = A(xe^x) = 0,$$

i.e.,

$$(D-1)^2(D^2 - 5D + 6)y = 0.$$

We know that the characteristic polynomial of the above DE is

$P_{AL}(x) = P_A(x)P_L(x) = (x-1)^2(x^2 - 5x + 6) = (x-1)^2(x-2)(x-3).$

The roots of this polynomial are

$$1, 1, 2, 3.$$

Hence, the linearly independent solutions of $AL(y) = 0$ are

$$e^x, xe^x, e^{2x}, e^{3x},$$

Annihilator Method

The general solution of $AL(y) = 0$ is given by

$$ae^x + bxe^x + ce^{2x} + de^{3x},$$

for $a, b, c, d \in \mathbb{R}$.

We need to find $a, b, c, d \in \mathbb{R}$ such that

$$L(ae^x + bxe^x + ce^{2x} + de^{3x}) = xe^x.$$

Since for $Ly = (D^2 - 5D + 6)y = 0$, the characteristic polynomial is

$$m^2 - 5m + 6 = (m-2)(m-3).$$

Thus, $N(L) = \langle e^{2x}, e^{3x} \rangle$.

As $L(ce^{2x} + de^{3x}) = 0$, we may choose

$$c = d = 0.$$

Annihilator Method

So we need to find a, b such that

$$L(ae^x + bxe^x) = xe^x.$$

i.e.,

$$(D^2 - 5D + 6)(ae^x + bxe^x) = xe^x.$$

To find a, b satisfying the above equation, set

$$y_1 = ae^x + bxe^x,$$

$$Dy_1 = ae^x + be^x + bxe^x,$$

$$D^2y_1 = ae^x + 2be^x + bxe^x.$$

Annihilator Method

Hence,

$$\begin{aligned} L(y_1) =& (D^2 - 5D + 6)(y_1), \\ =& D^2(y_1) - 5D(y_1) + 6y_1, \\ =& (a + 2b)e^x + bxe^x - 5(a + b)e^x - 5bxe^x + 6ae^x + 6bxe^x, \\ =& xe^x. \end{aligned}$$

This gives $(2a - 3b)e^x + 2bxe^x = xe^x$. Thus $a = 3/4$ and $b = 1/2$ and hence,

$$y = \frac{3}{4}e^x + \frac{1}{2}xe^x$$

is a particular solution of the non-homgeneous DE.

Example

Find a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1.$$

Here, the linear differential operator associated to the above DE is

$$L = D^4 - 16,$$

and note that $r(x) = x^4 + x + 1$ is in Null-space of the operator D^5. Let us take

$$A = D^5.$$

Then $Ar(x) = D^5(x^4 + x + 1) = 0$. Hence a solution y of $L(y) = r(x)$ is also a solution of

$$AL(y) = D^5(D^4 - 16)y = 0.$$

To find solutions of $AL(y) = 0$, consider the characteristic equation of AL

$$P_{AL}(x) = m^5(m^4 - 16) = m^5(m-2)(m+2)(m^2+4).$$

Roots are $0, \pm 2, \pm 2\iota$.

Annihilator Method

Thus, a general solution of $(AL)(y) = 0$ is of the form

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^{2x} + c_7 e^{-2x} + c_8 \cos 2x + c_9 \sin 2x.$$

So in order to solve

$$L(y) = x^4 + x + 1,$$

we should look for a solution of the form

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4.$$

Why? First note that L annihilates the last four terms. Hence, we may choose c_6, c_7, c_8, c_9 arbitrarily. We choose

$$c_6 = c_7 = c_8 = c_9 = 0.$$

We need to find c_1, c_2, c_3, c_4, c_5 so that

$$L\left(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4\right) = r(x).$$

i.e.,

$$(D^4 - 16)(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) = x^4 + x + 1.$$

Annihilator Method

$$(D^4 - 16)(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1$$

gives

$$24c_5 - 16(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1.$$

Comparing coefficients in the above equation, we get

$$24c_5 - 16c_1 = 1, -16c_2 = 1, -16c_3 = 0, -16c_4 = 0 \text{ and } -16c_5 = 1,$$

and hence

$$c_5 = -\frac{1}{16}, c_4 = 0, c_3 = 0, c_2 = -\frac{1}{16}, c_1 = -\frac{5}{32}.$$

Hence, a particular solution of the given non-homogeneous equation is

$$y_p = -\frac{1}{16}x^4 - \frac{1}{16}x - \frac{5}{32}.$$

Annihilator Method

$r(x)$	Annihilator A such that $A\big(r(x)\big) = 0$.
$r(x) = x^{m-1}$	$A = D^m$
e^{ax}	$(D - a)$
$x^{m-1}e^{ax}$	$(D - a)^m$
$\cos bx$ or $\sin bx$	$(D^2 + b^2)$
$e^{ax}\cos bx$ or $e^{ax}\sin bx$	$(D^2 - 2aD + a^2 + b^2)$
$x^{m-1}e^{ax}\cos bx$	$(D^2 - 2aD + a^2 + b^2)^m$
$x^{m-1}e^{ax}\sin bx$	$(D^2 - 2aD + a^2 + b^2)^m$.

Annihilator Method

Exercise: Get candidate solutions by the annihilator method:

1. $y^{(4)} + 2y'' + y = 3\sin x - 5\cos x$.
2. $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x\sin x$.
3. $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$.

Undetermined coefficient method and variation of parameters

Remark As described for the 2nd order non-homogeneous linear differential equation, the undetermined coefficient method and variation of parameters method are as well applicable to the case of n-order non-homogeneous linear differential equation with a suitable modifications.

Variation of Parameters- finding y_p

Let

$$L(y) = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

Here, $p_1, p_2, \ldots, p_n, r$ are continuous on an open interval I.

Recall the case $n = 2$: We replace c_1, c_2 in the general solution of the associated homogeneous equation by functions $v_1(x), v_2(x)$, so that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

We also demanded

$$v_1' y_1 + v_2' y_2 = 0.$$

Variation of Parameters

Then we substituted y, y', y'' in the given non-homogeneous ODE, and rearranged terms, to obtain:

$$v_1' y_1' + v_2' y_2' = r(x).$$

Together with,

$$v_1' y_1 + v_2' y_2 = 0,$$

this yielded :

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \; v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)},$$

and hence

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \; v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Thus,

$$y = v_1 y_1 + v_2 y_2 = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx.$$

Variation of Parameters

For general n, we proceed exactly the same way.

The method of variation of parameters assumes that a basis of solutions $y_1, \ldots, y_n$ of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

is known, i.e., the general solution of the above homogeneous equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where $c_1, \cdots, c_n$ are arbitrary constants. The idea is to replace c_i in the above equation by functions v_i, which are continuous on I and such that

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$$

is a solution of the non-homogeneous equation.

Variation of Parameters

$$
\begin{aligned}
c_1 y_1 + c_2 y_2 \quad &\to \quad c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \\
y = v_1 y_1 + v_2 y_2 \quad &\to \quad y = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n
\end{aligned}
$$

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

$$\to$$

$$y' = v_1 y_1' + \ldots + v_n y_n' + v_1' y_1 + \ldots + v_n' y_n$$

Variation of Parameters

Demand

$$v_1' y_1 + v_2' y_2 = 0$$

$$\rightarrow$$

Demand

$$v_1' y_1 + \ldots + v_n' y_n = 0.$$

Take

$$y'' = v_1 y_1'' + \ldots + v_n y_n'' + v_1' y_1' + \ldots + v_n' y_n'$$

and demand

$$v_1' y_1' + \ldots + v_n' y_n' = 0.$$

$$\ldots\ldots$$

Variation of Parameters

Take

$$y^{(n-1)} = v_1 y_1^{(n-1)} + \ldots + v_n y_n^{(n-1)} + v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)}$$

and demand

$$v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)} = 0.$$

Variation of Parameters

$$\begin{array}{rcl}
\text{Substituting } y, y', y'', & \rightarrow & \text{Substituting } y, y', \ldots, y^{(n)}, \\
\text{rearranging} & \rightarrow & \text{rearranging} \\
\text{and using } L(y_1) = L(y_2) = 0 & \rightarrow & \text{and using } L(y_1) = \ldots = L(y_n) = 0 \\
\text{get } v_1' y_1' + v_2' y_2' = r(x) & \rightarrow & \text{get} \\
 & & v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} = r(x).
\end{array}$$

Variation of Parameters

Thus,

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

takes the shape

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for

$$v_1', v_2', \ldots, v_n',$$

and thus get

$$v_1, v_2, \ldots, v_n,$$

and form

$$y = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n.$$

Variation of Parameters

Let $w(x) = \det W(y_1, y_2, \ldots, y_n)(x)$ be the determinant of the Wronskian matrix of the functions $y_1, y_2, \ldots, y_n$. By Cramer's rule,

$$v_k'(x) = \frac{\det \begin{vmatrix} y_1(x) & \cdots & 0 & \cdots & y_n(x) \\ y_1'(x) & \cdots & 0 & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & \cdots & r(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}}{w(x)} =: \frac{w_k(x)}{w(x)}$$

for each $k = 1, 2, \ldots, n$. Hence,

$$v_k(x) = \int_{x_0}^{x} \frac{w_k(t)}{w(t)}\,dt$$

and therefore,

$$y(x) = y_1(x) \int_{x_0}^{x} \frac{w_k(t)}{w(t)}\,dt + \ldots + y_n(x) \int_{x_0}^{x} \frac{w_k(t)}{w(t)}\,dt.$$

Variation of Parameters

Example: Solve

$$y^{(3)} - y^{(2)} - y^{(1)} + y = r(x).$$

Characteristic polynomial for the homogeneous equation is

$$x^3 - x^2 - x + 1 = (x-1)^2(x+1).$$

Hence, a basis of solutions is

$$\{e^x, xe^x, e^{-x}\}.$$

We need to calculate $W(t)$. Use Abel's formula:

$$W(x) = W(0)e^{-\int_0^x p_1(t)dt} = W(0) \cdot e^x.$$

Variation of Parameters

Now,

$$W(x) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}.$$

Thus,

$$W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

Hence,

$$W(x) = 4e^x.$$

Variation of Parameters

Let

$$W_1(x) = \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} = -r(x)(2x+1).$$

Similarly,

$$W_2(x) = 2r(x), \;\; W_3(x) = r(x)e^{2x}.$$

Therefore,

$$y(x) = e^x \int_0^x \frac{-r(t)(2t+1)}{4e^t} dt + xe^x \int_0^x \frac{2r(t)}{4e^t} dt + e^{-x} \int_0^x \frac{r(t)e^{2t}}{4e^t} dt.$$

Laplace Transforms

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform $\mathcal{L}(f)$ of f is the function defined by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{a \to \infty} \int_0^a e^{-st} f(t) dt,$$

for all values of s for which the integral exists. Sometimes we denote $F(s) = \mathcal{L}(f)(s)$.

The integral above may not converge for every s.

We may impose suitable restrictions on f later under which the intergral exists. What is the meaning of the improper integral?

The Improper Integral of the first kind

Definition

Let a, b be two real numbers such that $0 < a < b < \infty$. A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous on $[a, b]$ if there is a partition

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

such that

(i) f is continuous on (t_{i-1}, t_i) for $i = 1, 2, \ldots, n$.

(ii) $\lim_{t \to t_i^+} f(t)$ and $\lim_{t \to t_i^-} f(t)$ both exist for $i = 1, 2, \ldots, n-1$ and $\lim_{t \to t_0^+} f(t)$ and $\lim_{t \to t_n^-} f(t)$ both exist.

A piecewise continuous function on an interval $[a, b]$ is continuous except possibly for finitely many jump discontinuties.