MA 108-ODE- D3

Lecture 17

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Laplace transform

Laplace Transforms: Recall

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform $\mathcal{L}(f)$ of f is the function defined by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t) dt,$$

for all values of s for which the integral exists.

Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$ and is of exponential order. Moreover, if the piecewise continuous function f is of exponential order a, for some $a \in \mathbb{R}$, then the $\mathcal{L}(f)(s)$ exists for all $s > a$.

Denote by $F(s) = \mathcal{L}(f)(s)$.
The inverse Laplace transform (if defined) f of F is denoted by
$f = \mathcal{L}^{-1}(F)$.

Examples

1. $\mathcal{L}(1)(s) = \frac{1}{s}, s > 0.$
2. $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}, s > a.$
3. $\mathcal{L}(\sin at)(s) = \frac{a}{s^2+a^2}, s > 0.$
4. $\mathcal{L}(\cos at)(s) = \frac{s}{s^2+a^2}, s > 0.$
5. $\mathcal{L}(\sinh at)(s) = \frac{a}{s^2-a^2}, s > a \geq 0.$
6. $\mathcal{L}(\cosh at)(s) = \frac{s}{s^2-a^2}, s > a \geq 0.$
7. For $p > -1, \mathcal{L}(t^p)(s) = \frac{\Gamma(p+1)}{s^{p+1}}, s > 0.$

Properties

For large enough s, for which the Laplace transform of functions given below exist:

1.	Linearity	$\mathcal{L}(af+bg)(s) = a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s)$
2.	Scaling	$\mathcal{L}\big(f(ct)\big)(s) = \frac{1}{c}\mathcal{L}(f)\left(\frac{s}{c}\right),\ c>0$
3.	Shifting	$\mathcal{L}\big(e^{ct}f(t)\big)(s) = \mathcal{L}(f)(s-c)$
4.	Laplace transform of derivative	$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ $\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$
5.	L.T. of integral	$\mathcal{L}\left(\int_0^t f(x)\,dx\right)(s) = \frac{\mathcal{L}(f)(s)}{s}$

Derivative of Laplace Transforms

Theorem

Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and of exponential order. Let $F(s) = \mathcal{L}(f)(s)$. Then,

$$\frac{dF(s)}{ds} = -\mathcal{L}(t \cdot f(t))(s),$$

at those s, where both the terms $\frac{dF(s)}{ds}$ and $\mathcal{L}(t \cdot f(t))(s)$ exist.

Laplace Transforms

Example: Find $\mathcal{L}^{-1}$ of

$$\text{(i)}\ \frac{1}{(s^2+\beta^2)^2} \qquad \text{(ii)}\ \frac{s}{(s^2+\beta^2)^2} \qquad \text{(iii)}\ \frac{s^2}{(s^2+\beta^2)^2}$$

Recall

$$\mathcal{L}(\cos\beta t)(s) = \frac{s}{s^2+\beta^2},\ \mathcal{L}(\sin\beta t)(s) = \frac{\beta}{s^2+\beta^2}.$$

Therefore,

$$\mathcal{L}(t\cdot\cos\beta t)(s) = -\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2},$$

and

$$\mathcal{L}(t\cdot\sin\beta t)(s) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}.$$

I.e.,

$$\mathcal{L}\left(\frac{t\cdot\sin\beta t}{2\beta}\right)(s) = \frac{s}{(s^2+\beta^2)^2}.$$

Laplace Transforms

$$\mathcal{L}\left(\frac{t \cdot \sin \beta t}{2\beta}\right)(s) = \frac{s}{(s^2+\beta^2)^2},$$

or equivalently

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+\beta^2)^2}\right) = \frac{t \sin \beta t}{2\beta}.$$

Thus,

$$\frac{1}{(s^2+\beta^2)^2} = \frac{1}{s}\mathcal{L}\left(\frac{t \cdot \sin \beta t}{2\beta}\right) = \mathcal{L}\left(\int_0^t \frac{x \sin \beta x}{2\beta}\,dx\right),$$

which implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+\beta^2)^2}\right) = \int_0^t \frac{x \sin \beta x}{2\beta}\,dx.$$

Laplace Transforms

Recall

$$\mathcal{L}\left(\frac{t \cdot \sin \beta t}{2\beta}\right) = \frac{s}{(s^2+\beta^2)^2},$$

so that

$$\frac{s^2}{(s^2+\beta^2)^2} = s\mathcal{L}\left(\frac{t \cdot \sin \beta t}{2\beta}\right) = \mathcal{L}\left(\frac{d}{dt}\left(\frac{t \sin \beta t}{2\beta}\right)\right).$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+\beta^2)^2}\right) = \frac{d}{dt}\left(\frac{t \sin \beta t}{2\beta}\right).$$

Laplace Transforms

Theorem

Suppose $f : [0, \infty) \to \mathbb{R}$ *is 'piecewise continuous' of exponential order. Let* $F(s) = \mathcal{L}(f)(s)$. *Suppose further that* $\lim_{t \to 0^+} \frac{f(t)}{t}$ *exists. Then,*

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(x)dx,$$

at those s, *where both the terms* $\mathcal{L}\left(\frac{f(t)}{t}\right)$ *and* $\int_s^\infty F(x)dx$ *exist.*

Outline:

$$\begin{aligned}
\int_s^\infty F(x)dx &= \int_s^\infty \left(\int_0^\infty e^{-xt} f(t)dt\right) dx \\
&= \int_0^\infty \left(\int_s^\infty e^{-xt} f(t)dx\right) dt \\
&= \int_0^\infty f(t) \left[\frac{e^{-xt}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = \mathcal{L}\left(\frac{f(t)}{t}\right)(s).
\end{aligned}$$

Laplace Transforms

Example: Find $\mathcal{L}^{-1}$ of $\ln(1+\frac{w^2}{s^2})$.
We have:

$$\begin{aligned}
\frac{d}{ds}\left(\ln(1+\frac{w^2}{s^2})\right) &= -\frac{2w^2}{s(s^2+w^2)} \\
&= -\frac{2}{s}+\frac{2s}{s^2+w^2} \\
&= \mathcal{L}(-2+2\cos wt).
\end{aligned}$$

Hence,

$$\mathcal{L}\left(\frac{-2+2\cos wt}{t}\right)=\int_s^{\infty}\mathcal{L}(-2+2\cos wt)(x)dx=-\ln\left(1+\frac{w^2}{s^2}\right),$$

so that

$$\mathcal{L}\left(\frac{2-2\cos wt}{t}\right)=\ln\left(1+\frac{w^2}{s^2}\right).$$

Laplace Transforms

We have now seen $\mathcal{L}(f'), \mathcal{L}(\int f), (\mathcal{L}(f))', \int \mathcal{L}(f)$. All these were useful in finding $\mathcal{L}^{-1}$ of a given function. It will be useful to know

$$\mathcal{L}^{-1}(\mathcal{L}(f) \cdot \mathcal{L}(g)).$$

This turns out to be

$$f * g$$

where $*$ is the convolution operation.

Laplace Transforms

The convolution of $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

Check:

1. $f * g = g * f$ (use change of variable $y = t - x$.)
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$.

Example. Find $f * g$, where $f(t) = \sin t, \quad g(t) = 1$.
Ans. $(f * g)(t) = 1 - \cos t$.

Laplace Transforms

Theorem

Suppose $\mathcal{L}(f)$ and $\mathcal{L}(g)$ exist for all $s > a \geq 0$. Then,

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s),$$

for $s > a$.

Proof:

$$\begin{aligned}
\mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s) &= \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-sy} g(y)dy \\
&= \int_0^\infty e^{-sy} g(y) \left(\int_0^\infty e^{-sx} f(x)dx \right) dy \\
&= \int_0^\infty g(y) \left(\int_0^\infty e^{-s(x+y)} f(x)dx \right) dy.
\end{aligned}$$

Laplace Transforms

$$\mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s) = \int_0^\infty g(y) \left(\int_0^\infty e^{-s(x+y)} f(x) dx \right) dy.$$

Put $x + y = t$, for fixed y, so that $dx = dt$, and

$$\begin{aligned}
\mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s) &= \int_0^\infty g(y) \left(\int_y^\infty e^{-st} f(t-y) dt \right) dy \\
&= \int_0^\infty e^{-st} \left(\int_0^t f(t-y) g(y) dy \right) dt \\
&= \int_0^\infty e^{-st} (f * g)(t) dt \\
&= \mathcal{L}(f * g)(s).
\end{aligned}$$

Laplace Transforms

Example: Find $\mathcal{L}^{-1}$ of

$$H(s) = \frac{a}{s^2(s^2 + a^2)}.$$

Recall for $f(t) = t$,

$$\mathcal{L}(f)(s) = \frac{1}{s^2},$$

and for $g(t) = \sin at$,

$$\mathcal{L}(g)(s) = \frac{a}{s^2 + a^2}.$$

Thus,

$$\mathcal{L}(f * g)(s) = H(s).$$

Now,

$$(f * g)(t) = \int_0^t (t - x) \sin ax \; dx = \frac{at - \sin at}{a^2}.$$

Laplace Transforms

Example: Solve the IVP:

$$y'' + 4y = g(t),\ \ y(0) = 3, y'(0) = -1.$$

Taking Laplace transforms:

$$\mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) = \mathcal{L}(g)(s) = G(s).$$

Thus,

$$s^2\mathcal{L}(y)(s) - sy(0) - y'(0) + 4\mathcal{L}(y)(s) = G(s),$$

i.e.,

$$(s^2 + 4)\mathcal{L}(y)(s) = G(s) + 3s - 1.$$

Laplace Transforms

Therefore,

$$\begin{aligned}
\mathcal{L}(y)(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\
&= 3 \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{1}{2} \cdot \frac{2}{s^2+4} \cdot G(s) \\
&= 3\mathcal{L}(\cos 2t)(s) - \frac{1}{2}\mathcal{L}(\sin 2t)(s) + \frac{1}{2}\mathcal{L}(\sin 2t)(s) \cdot \mathcal{L}(g)(s) \\
&= 3\mathcal{L}(\cos 2t)(s) - \frac{1}{2}\mathcal{L}(\sin 2t)(s) + \frac{1}{2}\mathcal{L}(\sin 2t * g)(s).
\end{aligned}$$

Hence,

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-x)g(x)dx.$$

Variable coefficients - an example

Tutorial sheet 5, Q12: Compute the Laplace transform of a solution of

$$ty'' + y' + ty = 0,\ t > 0,\ \ y(0) = k,\ \mathcal{L}(y)(1) = 1/\sqrt{2}.$$

Taking Laplace transform,

$$\mathcal{L}(ty'' + y' + ty)(s) = 0.$$

Hence

$$-\frac{d}{ds}\mathcal{L}(y'')(s) + (s\mathcal{L}(y)(s) - y(0)) - \frac{d}{ds}(\mathcal{L}(y)(s)) = 0.$$

I.e.,

$$-\frac{d}{ds}\left(s^2\mathcal{L}(y)(s) - sy(0) - y'(0)\right) + (s\mathcal{L}(y)(s) - y(0)) - \frac{d}{ds}(\mathcal{L}(y)(s)) = 0.$$

Writing $\mathcal{L}(y)(s) = Y(s)$, it follows that
$(s^2 + 1)Y'(s) + sY(s) = 0 \Longrightarrow Y(s) = \frac{C}{\sqrt{s^2+1}}$.
Since $Y(1) = 1/\sqrt{2}, \Longrightarrow Y(s) = \frac{1}{\sqrt{s^2+1}}$.