Lecture 8

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(3) The motion is confined to a plane: As the angular momentum is constant and its direction is always perpendicular to the velocity direction, the motion is always confined to a plane.

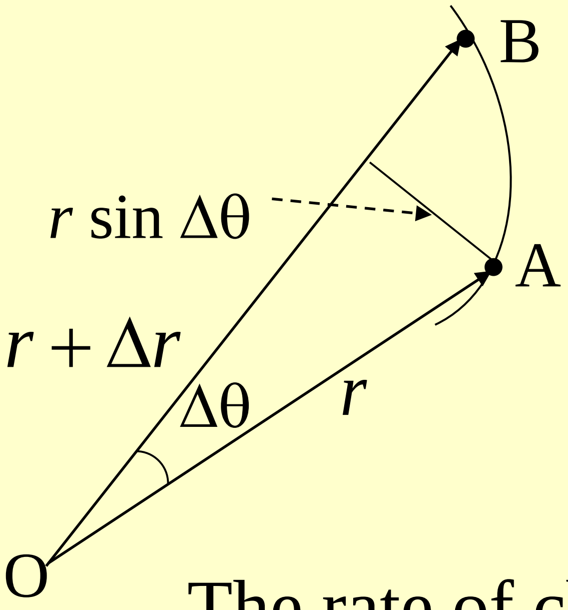
(4) The Kepler's second law (Law off equal areas) is obeyed: The expression for the tangential expression can be written as follows.

RECAP

$$a_t = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

If there is no force in the tangential direction, the tangential acceleration would be zero. This would imply that $(r^2\dot{\theta})$ would be constant.



If a particle moves between positions A and B in time Δt, then area ΔA subtended by it at the origin is given by

$$\Delta A = \frac{1}{2}(r+\Delta r)r\sin\Delta\theta$$

$$\approx \frac{1}{2}r^2\Delta\theta$$

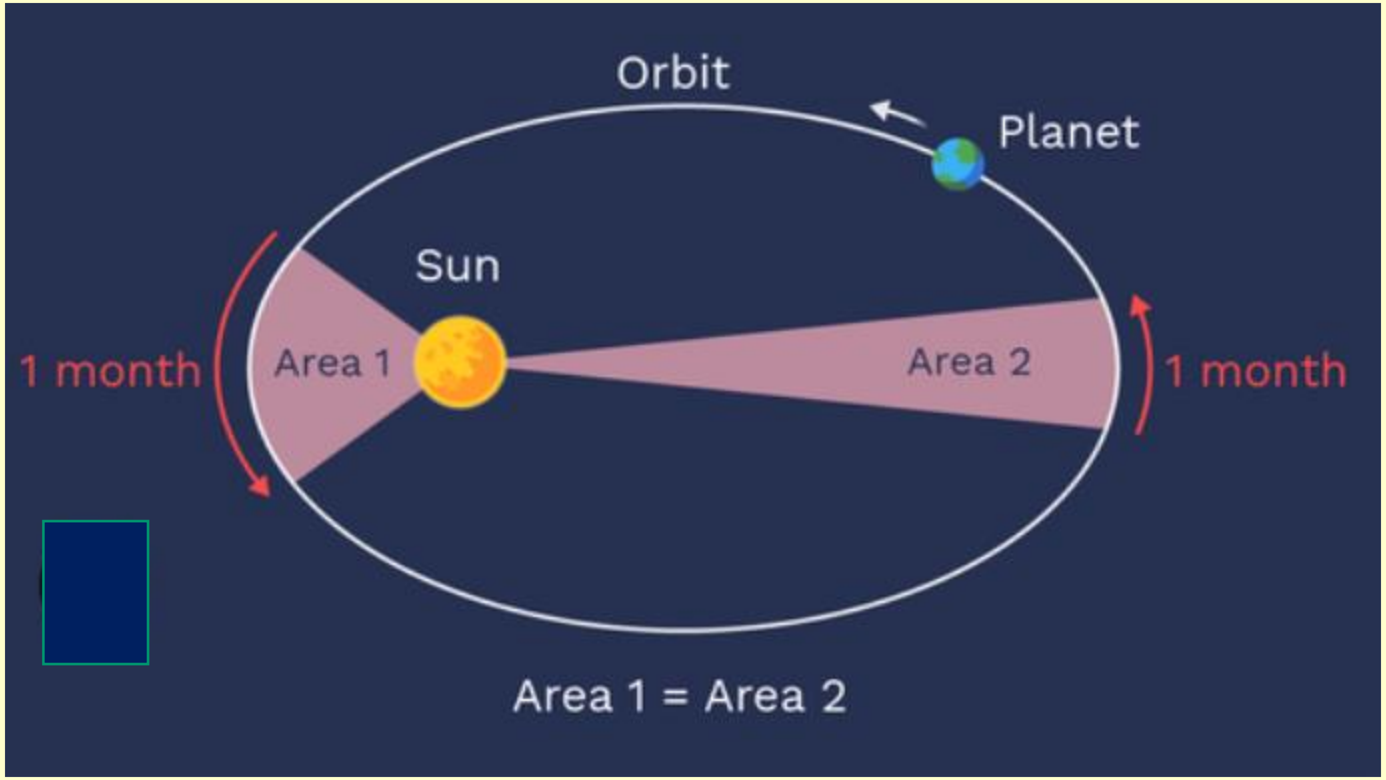
RECAP

The rate of change of area is, therefore, given by the following equation

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$$

We, thus, get that if the tangential acceleration is zero, which would happen if the tangential force is zero, ***then the rate of change of area at the center would be constant. This is known as Kepler's second law,*** observed for the motion of the planets. Clearly the gravitational force is always radial in the case of planets, hence the tangential acceleration is zero.

RECAP



Kepler's Second Law says that a line running from the sun to the planet sweeps out equal areas of the ellipse in equal times. This would imply that the planet speeds up as it is closer to the sun and slows down if is far from it

For a central force motion the torque is zero.

$$\vec{\tau} = r\hat{r} \times f(r)\hat{r}$$

RECAP

$$= 0$$

Hence, the total energy and the angular momentum are constants of motion One can, then solve the equation for $\dot{r}$ and $\dot{\theta}$, to evaluate r and θ as a function of time. The equation of the orbit can be obtained by dividing the two expressions.

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

$$\frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\frac{dr}{d\theta} = \frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

Reduction of a two-body central force problem to a one-body problem

An ideal central force is of the form

RECAP

$$\mathbf{F}(r) = f(r)\hat{\mathbf{r}},$$

i.e., it is a one-body force depending on the coordinates of only the particle on which it acts

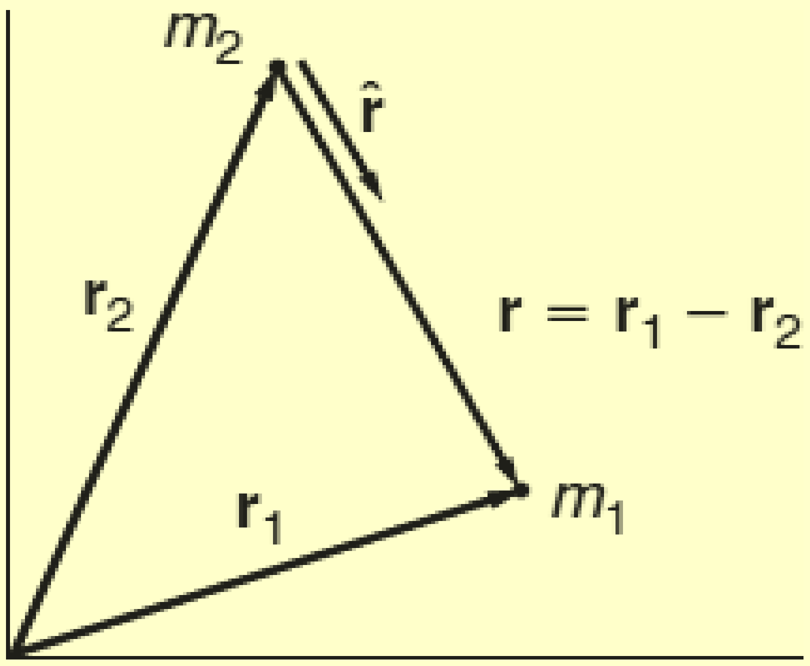
But gravity and Coulomb forces are two-body forces, of the form

$$\mathbf{F}(r_{12}) = f(r_{12})\hat{\mathbf{r}}_{12}$$

But, fortunately, they can be reduced to a pure one-body form

Reduction of a two-body central force problem to a one-body problem

Relevant coordinates are shown in the figure



Define:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\Rightarrow \; r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$$

Given $\mathbf{F}_{12} = f(r)\hat{\mathbf{r}}$, we have

$$m_1 \ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$

$$m_2 \ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

Decoupling equations of motion

Both the equations above are coupled, because both depend upon $\mathbf{r}_1$ and $\mathbf{r}_2$.

In order to decouple them, we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

(called relative coordinate), and center of mass coordinate **R**

$$\mathsf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$\ddot{\mathsf{R}} = \frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2} = \frac{f\hat{\mathbf{r}} - f\hat{\mathbf{r}}}{m_1 + m_2} = 0$$

$$\Longrightarrow \mathsf{R} = \mathsf{R}_0 + \mathbf{V}t,$$

above $\mathbf{R}_0$ is the initial location of center of mass, and **V** is the center of mass velocity.

Decoupling equations of motion

This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.

$$m_1 \ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$
$$m_2 \ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

Moreover, $\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = f(r)(\frac{1}{m_1} + \frac{1}{m_2})\hat{\mathbf{r}}$

$$\Longrightarrow \ddot{\mathbf{r}} = \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(r)\hat{\mathbf{r}}$$

RECAP

$$\mu \ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}},$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, is the reduced mass

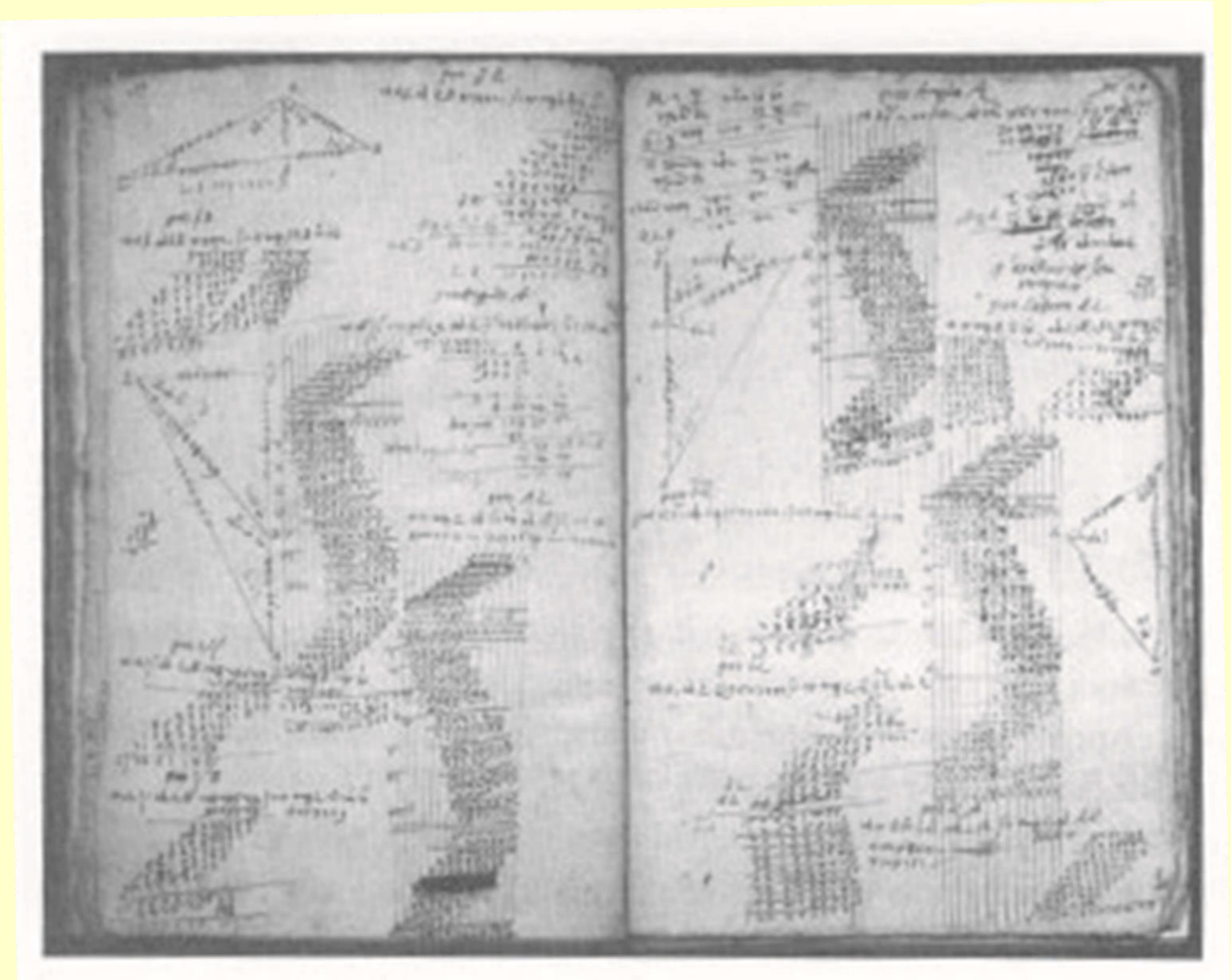
Two-Body-Central-Force-Problem

Historical Background: Kepler's Laws on celestial bodies (~1605)

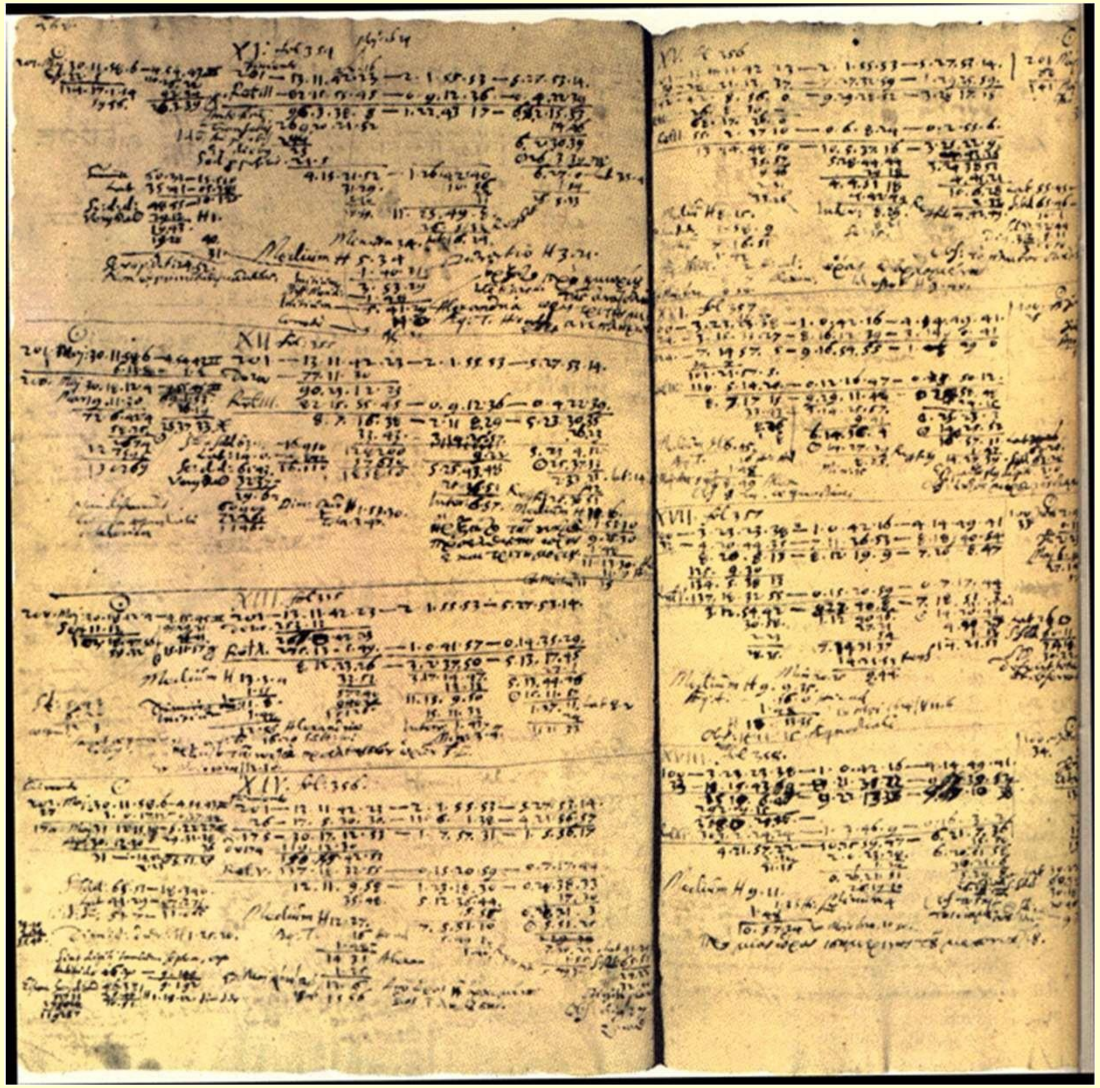
- Johannes Kepler based his 3 laws on observational data from Tycho Brahe
- Formulate his famous 3 laws:
    1) Law of Orbits: Orbit of each planet is an ellipse with sun at one of its foci
    2) Law of Areas: Equal areas swept out in equal time by an orbit
    3) Law of Periods: The ratio t^2/R^3 is the same for all planets, where "t" is the period and "R" is the semi-major axis

All these results were obtained through sheer mathematical efforts: AMAZING

NOTES



NOTES(From Tyco Brahe’s data)



Keplerian Orbits

- We want to use the theory developed to calculate the orbits of different planets around sun

- Planets are bound to sun because of gravitational force Therefore

$$f(r) = -\frac{GMm}{r^2}$$

So that

$$\boxed{V(r) = -\frac{GMm}{r} = -\frac{C}{r},}$$

above $C = GMm$, where G is gravitational constant, M is mass of the Sun, and m is mass of the planet in question.

Inverse of Kepler's 1st Law

A particle under the influence of a central force, executes a motion, the equation of which is given by

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

Show that the force is of inverse square law type.

We first differentiate to find time derivative of *r*.

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

$$\dot{r} = -\frac{r_o \varepsilon \sin\theta}{(1 - \varepsilon \cos\theta)^2}\dot{\theta}$$

Using Kepler's second law we get the following equation, where *k* is a constant.

$$r^2\dot{\theta} = k \Rightarrow \dot{\theta} = \frac{k}{r^2}$$

$$\dot{\theta} = \frac{k}{r^2} \qquad \dot{r} = -\frac{r_o \varepsilon \sin\theta}{(1-\varepsilon\cos\theta)^2}\dot{\theta} \qquad r = \frac{r_0}{1-\varepsilon\cos\theta}$$

Substituting this and eliminating the denominator in the first equation we get

$$\dot{r} = -\frac{k\varepsilon\sin\theta}{r_o}$$

Differentiating again w.r.t time.

$$\ddot{r} = -\frac{k\varepsilon}{r_o}\cos\theta\,\dot{\theta}$$

Substituting $\cos\theta$ in terms of r from equation of orbit, we get

$$\ddot{r} = -\frac{k\varepsilon}{r_o\varepsilon}\left[1 - \frac{ro}{r}\right]\frac{k}{r^2}$$

$$r = \frac{r_0}{1 - \varepsilon\cos\theta}$$

The radial acceleration, therefore, would be

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$= -\frac{k^2}{r_o r^2} + \frac{k^2}{r^3} - r\frac{k^2}{r^4}$$

$$\dot{\theta} = \frac{k}{r^2}$$

DONE

$$= -\frac{k^2}{r_o r^2}$$

Verify that the tangential acceleration is zero

Kepler's 1st law

Recall:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

$$\frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\frac{dr}{d\theta} = \frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

$$\theta - \theta_0 = L\int_{r_0}^{r} \frac{dr}{r^2\sqrt{2\mu(E - \frac{L^2}{2\mu r^2} + \frac{C}{r})}}$$

$$= L\int \frac{dr}{r\sqrt{2\mu E r^2 + 2\mu C r - L^2}}$$

Converting the definite integral on the RHS to an indefinite one, because θ_0 is a constant of integration in which the constant contribution of the lower limit $r = r_0$ can be absorbed. This orbital integral can be done by the following substitution

$$r = \frac{1}{s - \alpha}$$

$$\Longrightarrow \quad dr = -\frac{ds}{(s-\alpha)^2}$$

$$\Longrightarrow \quad \frac{dr}{r} = -\frac{ds}{(s-\alpha)}$$

$$\theta - \theta_0 = -L \int \frac{ds}{(s-\alpha)\sqrt{\frac{2\mu E}{(s-\alpha)^2} + \frac{2\mu C}{s-\alpha} - L^2}}$$

$$= -L \int \frac{ds}{\sqrt{2\mu E + 2\mu C(s-\alpha) - L^2(s-\alpha)^2}}$$

$$= -L \int \frac{ds}{\sqrt{2\mu E + 2\mu Cs - 2\mu C\alpha - L^2 s^2 + 2L^2\alpha s - L^2\alpha^2}}$$

$$\boxed{r = \frac{1}{s-\alpha}}$$

The integrand is simplified if we choose $\alpha = -\frac{\mu C}{L^2}$, leading to

$$\theta - \theta_0 = -L \int \frac{ds}{\sqrt{2\mu E + 2\frac{(\mu C)^2}{L^2} - L^2 s^2 - \frac{(\mu C)^2}{L^2}}}$$

$$= -L \int \frac{ds}{\sqrt{2\mu E + \frac{(\mu C)^2}{L^2} - L^2 s^2}}$$

Finally, the integral is

$$\theta - \theta_0 = -L^2 \int \frac{ds}{\sqrt{2\mu E L^2 + (\mu C)^2 - L^4 s^2}}$$
$$= -\int \frac{ds}{\sqrt{\frac{2\mu E L^2 + (\mu C)^2}{L^4} - s^2}}$$

On substituting $s = a\sin\phi$, where $a = \sqrt{\frac{2\mu E L^2+(\mu C)^2}{L^4}}$, the integral transforms to

$$\theta - \theta_0 = -\phi = -\sin^{-1}\left(\frac{s}{a}\right)$$
$$s = -a\sin(\theta - \theta_0)$$
$$\implies \frac{1}{r} + \alpha = -a\sin(\theta - \theta_0)$$
$$\implies r = \frac{1}{-\alpha - a\sin(\theta - \theta_0)}$$

$$\boxed{r = \frac{1}{s - \alpha}}$$

$\boxed{\text{We define } r_0 = -\frac{1}{\alpha} = \frac{L^2}{\mu C}, \text{ to obtain}}$

$$r = \frac{r_0}{1 - \sqrt{1 + \frac{2EL^2}{\mu C^2}} \sin(\theta - \theta_0)}$$

Conventionally, one takes $\theta_0 = -\pi/2$, and we define

C = GMm

$$\boxed{\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}}$$

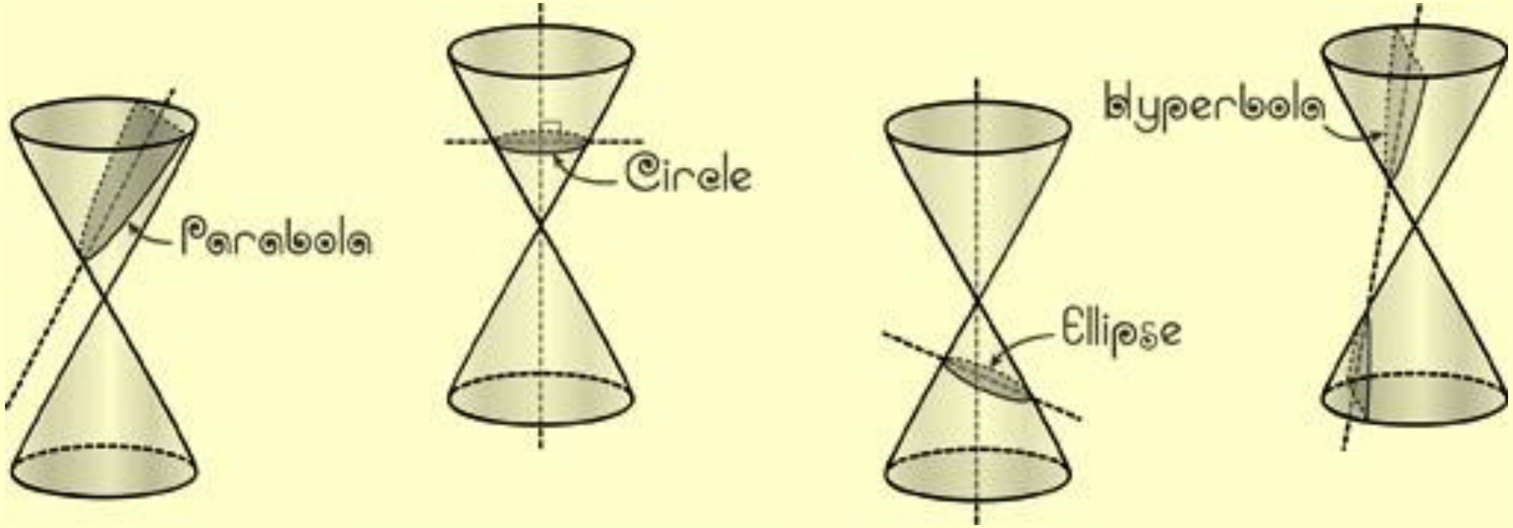
Need these later

To obtain the final result

$$\boxed{r = \frac{r_0}{1 - \varepsilon \cos\theta}}$$

Digression: Conic Sections

Curves such as circle, parabola, ellipse, and hyperbola are called conic sections



$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

in plane polar coordinates, denotes different conic sections for various values of ε, which is nothing but the eccentricity. Eccentricity is a measure of how much a conic section deviates from being circular.

Using the fact that $r = \sqrt{x^2 + y^2}$, and $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$, we obtain

$$\sqrt{x^2 + y^2} = \frac{r_0}{1 - \frac{\varepsilon x}{\sqrt{x^2+y^2}}}$$

$$\boxed{r = \frac{r_0}{1 - \varepsilon\cos\theta}}$$

$$\implies \sqrt{x^2 + y^2} = r_0 + \varepsilon x$$

$$\implies \boxed{x^2(1 - \varepsilon^2) - 2r_0\varepsilon x + y^2 = r_0^2}$$

$$\boxed{\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}}$$

Case I: $\varepsilon = 1$, which means $E = 0$, we obtain

$$y^2 = 2r_0 x + r_0^2$$

which is nothing but a parabola. This is clearly an open or unbound orbit. This is typically the case with comets.

Case II: $\varepsilon > 1 \implies E > 0$, let us define $A = \varepsilon^2 - 1 > 0$ With this, the equation of the orbit is

$$\boxed{x^2(1-\varepsilon^2) - 2r_0\varepsilon x + y^2 = r_0^2}$$

$$y^2 - Ax^2 - 2r_0\sqrt{1+A}x = r_0^2$$

Here, the coefficients of x^2 and y^2 are opposite in sign, therefore, the curve is unbounded, i.e., open. It is actually the equation of a hyperbola. Therefore, whenever $E > 0$, the particles execute unbound motion, and some comets and asteroids belong to this class.

Case III: $\varepsilon = 0$, we have

$$x^2 + y^2 = r_0^2$$

which denotes a circle of radius r_0, with center at the origin. This is clearly a closed orbit, for which the system is bound. $\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}} = 0 \implies E = -\frac{\mu C^2}{2L^2} < 0$. Satellites launched by humans are put in circular orbits many times, particularly the geosynchronous ones.

Elliptical Orbits

$$\boxed{x^2(1-\varepsilon^2) - 2r_0\varepsilon x + y^2 = r_0^2}$$

Case IV: $0 < \varepsilon < 1 \Rightarrow E < 0$, here we define

$$\boxed{\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}}$$

$A = (1 - \varepsilon^2) > 0$, to obtain

$$Ax^2 - 2r_0\sqrt{1-A}\,x + y^2 = r_0^2$$

Because coefficients of x^2 and y^2 are both positive, orbit will be closed (i.e. bound), and will be an ellipse.

To summarize, when $E \geq 0$, orbits are unbound, i.e., hyperbola or parabola

When $E < 0$, orbits are bound, i.e., circle or ellipse.

Kepler's 3 law

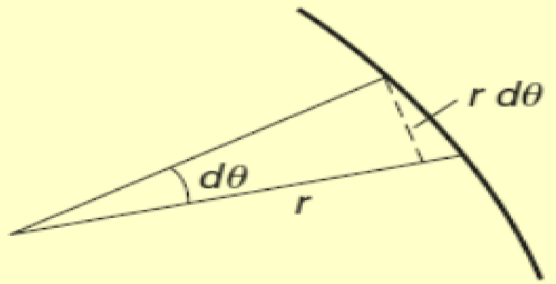
Alternate approach to calculate the time period

- We use the constancy of angular momentum

$$L = \mu r^2 \frac{d\theta}{dt}$$

$$\Longrightarrow \frac{L}{2\mu} dt = \frac{1}{2} r^2 d\theta$$

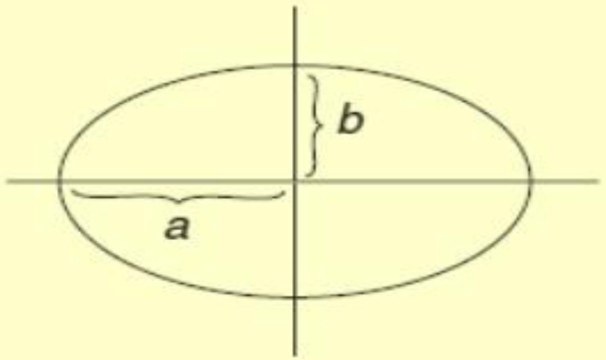
- R.H.S. of the previous equation is nothing but the area element swept as the particle changes its position by $d\theta$



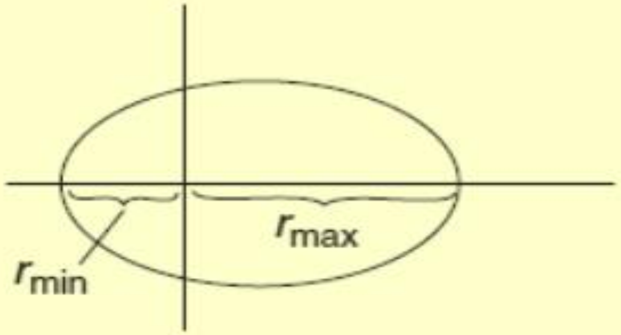
- Now, the integrals on both sides can be carried out to yield

$$\boxed{\frac{LT}{2\mu} = \text{area of ellipse} = \pi ab.}$$

- a and b in the equation are semi-major and semi-minor axes of the ellipse as shown



- Now, we have



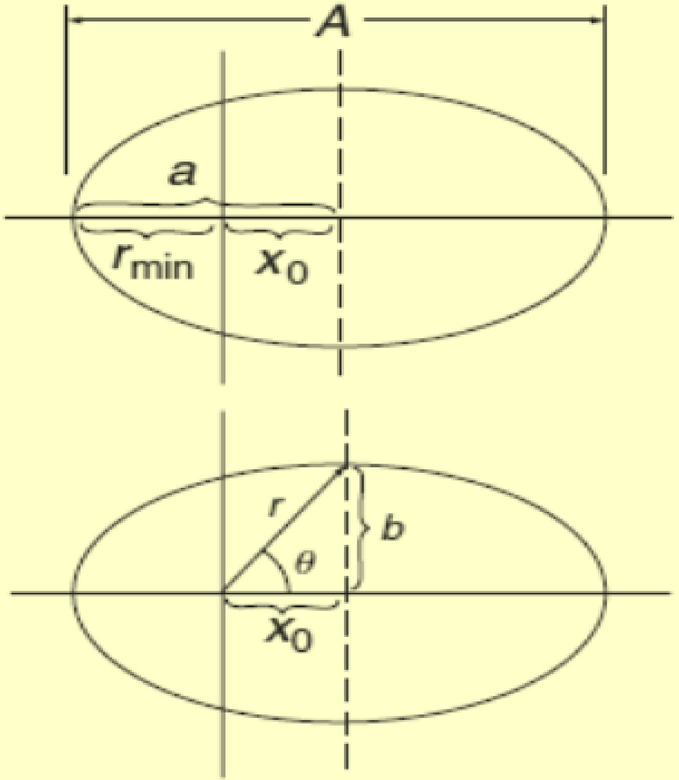
- Therefore

$$a = \frac{A}{2} = \frac{(r_{min} + r_{max})}{2}$$

- Using the orbital equation $r = \frac{r_0}{1-\varepsilon\cos\theta}$, we have

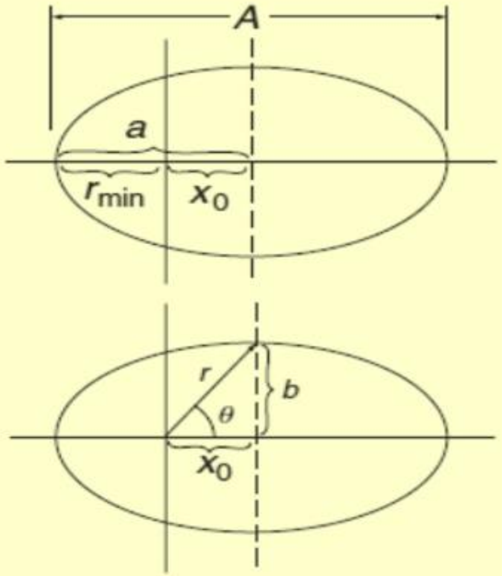
$$\boxed{a = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon\cos\pi} + \frac{r_0}{1-\varepsilon\cos 0}\right) = \frac{r_0}{2}\left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2}}$$

- Calculation of b is slightly involved. Following diagram is helpful



- x_0 is the distance between the focus and the center of the ellipse, thus

$$x_0 = a - r_{min} = \frac{r_0}{1-\varepsilon^2} - \frac{r_0}{1+\varepsilon} = \frac{r_0\varepsilon}{1-\varepsilon^2}$$



- In the diagram $b = \sqrt{r^2 - x_0^2}$, and for θ, we have $\cos\theta = \frac{x_0}{r}$, which on substitution in orbital equation yields

$$r = \frac{r_0}{1-\varepsilon\cos\theta} = \frac{r_0}{1-\frac{\varepsilon x_0}{r}}$$

$$\Longrightarrow r = r_0 + \varepsilon x_0 = r_0 + \frac{r_0\varepsilon^2}{1-\varepsilon^2} = \frac{r_0}{1-\varepsilon^2}$$

- So that

$$b = \sqrt{r^2 - x_0^2} = \sqrt{\frac{r_0^2}{(1-\varepsilon^2)^2} - \frac{r_0^2\varepsilon^2}{(1-\varepsilon^2)^2}} = \frac{r_0}{\sqrt{1-\varepsilon^2}}$$

- Now

$$1-\varepsilon^2 = 1-\left(1+\frac{2EL^2}{\mu C^2}\right) = -\frac{2EL^2}{\mu C^2}$$

- Using $r_0 = \frac{L^2}{\mu C}$, we have

$$\varepsilon = \sqrt{1+\frac{2EL^2}{\mu C^2}}$$

$\boxed{\text{We define } r_0 = -\frac{1}{\alpha} = \frac{L^2}{\mu C}\text{, to obtain}}$

$$A = 2a = \frac{2r_0}{1-\varepsilon^2} = \frac{2L^2}{\mu C}\times\left(-\frac{\mu C^2}{2EL^2}\right) = -\frac{C}{E}$$

$$a = \frac{r_0}{1-\varepsilon^2}$$

$$b = \frac{r_0}{\sqrt{1-\varepsilon^2}} = \frac{L^2}{\mu C}\times\sqrt{-\frac{\mu C^2}{2EL^2}} = L\sqrt{-\frac{1}{2\mu E}}$$

- Using this, we have $\boxed{\frac{LT}{2\mu} = \text{area of ellipse } = \pi ab.}$

$$T = \frac{2\pi\mu}{L}ab = \frac{2\pi\mu}{L}\times\left(-\frac{C}{2E}\right)\times L\sqrt{-\frac{1}{2\mu E}} = \pi\sqrt{\frac{\mu}{2C}}\left(-\frac{C}{E}\right)^{3/2}$$

- Which can be written as $\boxed{T = \pi\sqrt{\frac{\mu}{2C}}\left(-\frac{C}{E}\right)^{3/2}}$

$$A = 2a = \frac{2r_0}{1-\varepsilon^2} = \frac{2L^2}{\mu C} \times \left(-\frac{\mu C^2}{2EL^2}\right) = -\frac{C}{E}$$

$$T = \pi\sqrt{\frac{\mu}{2C}}A^{3/2}$$

$$\implies T^2 = \frac{\pi^2\mu}{2C}A^3,$$

which is nothing but Kepler's third law.

The square of the period of a planet's orbit is proportional to the cube of its semi-major axis.

How Newton, perhaps, derived his law of gravitation from Kepler's laws

Newton had at hand Kepler's third law, stating that the squares of the periods of the planets are proportional to the cubes of the semi-major axes of their elliptical orbits. Introducing a constant of proportionality K, we can write this law in the form

$$T_p^{\,2} = K a_p^{\,3}.$$

Assume that the planets all move precisely in circles of radius r_p

Kepler's third law for these circles may be written

$$T_p^{\,2} = K r_p^{\,3}.$$

Centripetal acceleration for a circular orbit:

$$a = \frac{v_P^{\,2}}{r_P}.$$

In order to take advantage of Kepler's third law, we replace the speed v_p in this expression by the orbital circumference divided by the period:

$$v_P = \frac{2\pi r_P}{T_P}.$$

(Speed equals distance divided by time.)

The formula for the planet's acceleration ***(centripetal)*** then becomes

$$a = \frac{1}{r_P}\left(\frac{2\pi r_P}{T_P}\right)^2 = \frac{4\pi^2 r_P}{T_P^2}.$$

Substituting the expression for Kepler's third law, $T_p^2 = K r_p^3$, into the denominator of the equation

The following formula for planetary acceleration is obtained

$$a = \frac{4\pi^2}{K}\frac{1}{r_P^2}.$$

Recall that the derivation is based on the approximation of circular motion

The combination $4\pi^2/K$ is a constant; the quantities a and r_p are variables. This equation states that the acceleration of a planet toward the Sun is inversely proportional to the square of its distance from the Sun. Multiplication of the acceleration of a planet by its mass gives the force acting on it:

$$F_P = \frac{4\pi^2}{K}\frac{m_P}{r_P{}^2}\,.$$

Compare this with the law of gravitational force,

$$F = G\,\frac{m_1 m_2}{d^2}\,;$$

From Kepler's third law one gets a major part of the final form of the law of force: The gravitational force experienced by a planet is proportional to its mass and inversely proportional to the square of its distance from the sun. All that is missing is the proportionality of this force to the mass of the Sun.