MA 108-ODE- D3

Lecture 15

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Laplace transform

Laplace Transforms

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform $\mathcal{L}(f)$ of f is the function defined by

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{a \to \infty} \int_0^{a} e^{-st} f(t) dt,$$

for all values of s for which the integral exists. Sometimes we denote $F(s) = \mathcal{L}(f)(s)$.

The integral above may not converge for every s.

We may impose suitable restrictions on f later under which the intergral exists. What is the meaning of the improper integral?

The Improper Integral of the first kind

Definition

Let a, b be two real numbers such that $0 < a < b < \infty$. A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous on $[a, b]$ if there is a partition

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

such that

(i) f is continuous on (t_{i-1}, t_i) for $i = 1, 2, \ldots, n$.

(ii) $\lim\limits_{t \to t_i^+} f(t)$ and $\lim\limits_{t \to t_i^-} f(t)$ both exist for $i = 1, 2, \ldots, n-1$ and $\lim\limits_{t \to t_0^+} f(t)$ and $\lim\limits_{t \to t_n^-} f(t)$ both exist.

A piecewise continuous function on an interval $[a, b]$ is continuous except possibly for finitely many jump discontinuties.

The Improper Integral of the first kind

- Let $f : [a, \infty) \to \mathbb{R}$ be a function. If f is such that, for any $b \geq a$, $f : [a, b] \to \mathbb{R}$ is piecewise continuous, then we say that f is piecewise continuous on $[a, \infty)$.
- Note that such an f is bounded on $[a, b]$ for every $b \geq a$.
- Note that, for f as above, the usual Riemann integral

  $$I(b) = \int_a^b f(x)\, dx$$

  exists for any $b \geq a$.

Definition

An improper integral of first kind of the function f with the property mentioned above is defined to be

$$\int_a^\infty f(x)\, dx := \lim_{b \to \infty} \int_a^b f(x)\, dx,$$

if this limit exists.

If the above limit exists, we say that $\int_a^\infty f(x)\, dx$ converges, otherwise it is said to diverge.

The Improper Integral of the first kind

Example: Consider the improper integral $\int_1^\infty \frac{dx}{x^s}$ for $s \in \mathbb{R}$.

For $s \neq 1$, consider $I(b) = \int_1^b \frac{dx}{x^s} = \frac{b^{1-s}-1}{1-s}$.

$$I(b) = \begin{cases} \frac{b^{1-s}-1}{1-s} & \text{if} \quad s \neq 1, \\ \ln b & \text{if} \quad s = 1. \end{cases}$$

So that

$$\lim_{b\to\infty} I(b) = \begin{cases} \frac{1}{s-1} & \text{if} \quad s > 1, \\ \infty & \text{if} \quad s \leq 1. \end{cases}$$

The Improper Integral of the first kind

Example: Consider the improper integral $\int_0^\infty \sin x \; dx$. Consider

$I(b) = \int_0^b \sin x \; dx = 1 - \cos b$.

Since $\lim_{b \to \infty} I(b)$ does not exist, the integral $\int_0^\infty \sin x \; dx$ diverges.

Laplace Transforms

Example: $\boxed{\text{Let } f(t) = 1 \text{ for all } t \geq 0. \text{ Then, } \mathcal{L}(f)(s) = \frac{1}{s}, \quad \forall\, s > 0.}$

Proof.

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} dt = \left[-\frac{e^{-st}}{s} \right]_0^\infty = \frac{1}{s}$$

for $s > 0$.

Example:

$\boxed{\text{Let } f(t) = e^{at}, t \geq 0 \text{ where } a \text{ is a constant. Then } \mathcal{L}(f)(s) = \frac{1}{s-a}, \quad \forall\, s > a.}$

Proof.

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \frac{1}{s-a},$$

for $s > a$.

Laplace Transforms

Example: $\boxed{\text{Let } f(t) = \sin at,\, t \geq 0, \text{ where } a \text{ is a constant.}}$

$\boxed{\text{Then } \mathcal{L}(f)(s) = \frac{a}{s^2+a^2}, \quad \forall\, s > 0.}$

Proof. If $s > 0$, then

$$\begin{aligned}
\mathcal{L}(f)(s) &= \int_0^\infty e^{-st} \sin at \; dt \\
&= \lim_{b \to \infty} \int_0^b e^{-st} \sin at \; dt \\
&= \lim_{b \to \infty} \left[-\frac{e^{-st} \cos at}{a} \right]_0^b - \frac{s}{a} \int_0^\infty e^{-st} \cos at \; dt \\
&= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \; dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \; dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} \mathcal{L}(f)(s)
\end{aligned}$$

Therefore, for $s > 0$ $\mathcal{L}(f)(s) = \frac{a}{s^2+a^2}$.

Gamma function

Definition

For any given $a > 0$, the gamma function is given by

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1}\, dt.$$

Properties of Gamma function

1. $\Gamma(a+1) = a\Gamma(a), \quad a > 0.$
2. $\Gamma(1) = 1, \quad \Gamma(k+1) = k!, k \in \mathbb{N}.$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

The justification of the convergence of the Gamma function and its properties are given in the Appendix, at the end of the slides.

Example: Let $f(t) = t^p$, for any given $p > -1$. Determine $\mathcal{L}(t^p), p > -1$.

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} t^p dt.$$

Put $x = st$.Thus, $dt = \frac{dx}{s}$.Thus,

$$\begin{aligned}
\mathcal{L}(f)(s) &= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \cdot \frac{dx}{s} \\
&= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx \\
&= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^{(1+p)-1} \, dx \\
&= \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0.
\end{aligned}$$

For $p = n$, where $n = 1, \ldots$, i.e, $f(t) = t^n$, $\mathcal{L}(f)(s) = \frac{n!}{s^{n+1}}, \quad s > 0$.
For $p = \frac{-1}{2}$, i.e., $f(t) = t^{-\frac{1}{2}}$, for all $s > 0$,

$$\mathcal{L}(f)(s) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}},$$

Laplace Transforms

Exercise: Show that

1. For $f(t) = \cos at$, $\mathcal{L}(f)(s) = \frac{s}{s^2+a^2}, s > 0$.
2. For $f(t) = \cosh at$, $\mathcal{L}(f)(s) = \frac{s}{s^2-a^2}, s > a \geq 0$.
3. For $f(t) = \sinh at$, $\mathcal{L}(f)(s) = \frac{a}{s^2-a^2}, s > a \geq 0$.

Existence of Laplace transforms

- For a given f, $\mathcal{L}(f)$ may or may not exist.
- Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$ and is of exponential order on $(0, \infty)$.
- A function $f : (0, \infty) \to \mathbb{R}$ is said to be of exponential order if there exists $a \in \mathbb{R}$ and positive constants t_0 and K such that

  $$|f(t)| \leq Ke^{at},$$

  for all $t \geq t_0 > 0$.

Examples

1. Every bounded function is of exponential order with the constant $a = 0$. Thus, $\sin bt$ and $\cos bt$ are of exponential order. Also, if $|f(t)| \leq K$ for $t \geq t_0 > 0$, then f is of exponential order.
2. $e^{\alpha t} \sin bt$ is of exponential order, with constant $a = \alpha$.
3. t^n for $n > 0$ is of exponential order, since for $a > 0$, $\lim_{t\to\infty} e^{-at} t^n = 0$ and thus, there exists $K > 0$ and $t_0 > 0$ such that
$$e^{-at}|f(t)| = e^{-at} t^n < K, \quad \text{for } t > t_0.$$
4. e^{t^2} is not of exponential order, for in this case,
$$e^{-at}|f(t)| = e^{t^2 - at}$$
and this becomes unbounded as $t \to \infty$, no matter what is value of a.
5. Sum of functions of exponential order is also of exponential order.

Existence theorem

Theorem

Suppose $f(\cdot)$ is piecewise continuous on $[0,\alpha]$ for all $\alpha > 0$. Further suppose

$$|f(t)| \leq Ke^{at},$$

for $t \geq t_0 > 0$, where $K > 0,\ a,\ t_0 \in \mathbb{R}$. Then

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)dt$$

exists for $s > a$.

Proof: We have:

$$\mathcal{L}(f)(s) = \int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^\infty e^{-st} f(t)dt.$$

As f is piecewise continuous on $[0, t_0]$, $\int_0^{t_0} e^{-st} f(t)dt$ exists.

Proof contd...

We need to show that $\int_{t_0}^{\infty} e^{-st} f(t) dt$ converges.
For $t \geq t_0$, we have:

$$|e^{-st} f(t)| \leq e^{-st} K e^{at} = K e^{-(s-a)t}.$$

For $s > a$, $\int_{t_0}^{\infty} e^{-(s-a)t} dt$ converges. Hence,

$$\int_{t_0}^{\infty} |e^{-st} f(t)| dt,$$

and thus

$$\int_{t_0}^{\infty} e^{-st} f(t) dt$$

converges (Why?) Use comparison test.

The Improper Integral of the first kind

Assume that f is piecewise continuous on $[a, \infty)$.

Theorem (Comparison Test)

Suppose $0 \leq f(x) \leq g(x)$ *for every* $x \geq a$. *If* $\int_a^\infty g(x)\, dx$ *converges, then* $\int_a^\infty f(x)\, dx$ *also converges and*

$$\int_a^\infty f(x)\, dx \leq \int_a^\infty g(x)\, dx.$$

Example: As

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

on $[1, \infty)$, and $\int_1^\infty \frac{1}{x^2}\, dx$ converges, it follows that $\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ also converges.

Theorem

If $\int_a^\infty |f(x)|\, dx$ *converges, then* $\int_a^\infty f(x)\, dx$ *converges.*

List of Laplace tranforms

Often, in practice to simplify the notation, the notation $\mathcal{L}(e^t)(s)$, $\mathcal{L}(\sin at)(s)$... are used instead of $\mathcal{L}(f)(s)$, where $f(t) = e^t$, $f(t) = \sin at$ respectively...

1. $L(1)(s) = \frac{1}{s}, s > 0$.
2. $L(e^{at})(s) = \frac{1}{s-a}, s > a$.
3. $L(\sin at)(s) = \frac{a}{s^2+a^2}, s > 0$.
4. $L(\cos at)(s) = \frac{s}{s^2+a^2}, s > 0$.
5. $L(\sinh at)(s) = \frac{a}{s^2-a^2}, s > a \geq 0$.
6. $L(\cosh at)(s) = \frac{s}{s^2-a^2}, s > a \geq 0$.
7. For $p > -1, L(t^p)(s) = \frac{\Gamma(p+1)}{s^{p+1}}, s > 0$.

Laplace Transforms: Linearity

Theorem

Let f, g be two functions such that $\mathcal{L}(f)(s)$ and $\mathcal{L}(g)(s)$ exist for $s > a_1$ and $s > a_2$ respectively. Then for $s > \max\{a_1, a_2\}$,

$$\mathcal{L}(c_1 f + c_2 g)(s) = c_1 \mathcal{L}(f)(s) + c_2 \mathcal{L}(g)(s),$$

where $c_1, c_2 \in \mathbb{R}$.

Proof: For $s > \max\{a_1, a_2\}$,

$$\begin{aligned}
\mathcal{L}(c_1 f + c_2 g)(s) &= \int_0^\infty e^{-st} \left(c_1 f(t) + c_2 g(t)\right) dt \\
&= \int_0^\infty e^{-st} c_1 f(t) dt + \int_0^\infty e^{-st} c_2 g(t) dt \\
&= c_1 \mathcal{L}(f)(s) + c_2 \mathcal{L}(g)(s).
\end{aligned}$$

Application of the above property

Find the Laplace transform of $f(t) = e^t + t^n + \sin at + c$, where a, c are given constants.

Ans. Note that $\mathcal{L}(e^t)(s)$ exists for all $s > 1$, and $\mathcal{L}(t^2)(s)$, $\mathcal{L}(\sin at)(s)$, $\mathcal{L}(c)(s)$ exist for all $s > 0$.
Using the linearity property of Laplace transform, for all $s > 1$,

$$\begin{aligned}\mathcal{L}(f)(s) &= \mathcal{L}(e^t)(s) + \mathcal{L}(t^2)(s) + \mathcal{L}(\sin at)(s) + \mathcal{L}(c)(s) \\ &= \tfrac{1}{s-1} + \tfrac{2}{s^3} + \tfrac{a}{s^2+a^2} + \tfrac{c}{s}, \quad \forall\, s > 1.\end{aligned}$$

Scaling

Theorem (Scaling)

Let f be a function such that $\mathcal{L}(f)(s)$ exists for $s > a$, for some $a \in \mathbb{R}$, and let $c > 0$ a constant. Let $g(t) = f(ct)$ for all $t > 0$. Then for $s > ca$,

$$\mathcal{L}(g)(s) = \frac{1}{c}\mathcal{L}(f)(\frac{s}{c}).$$

Proof: Let $g(t) = f(ct)$ for all $t > 0$, where $c > 0$ and $\mathcal{L}(f)(s)$ exists for $s > a$. Then for all $s > ca$,

$$\mathcal{L}(g)(s) = \int_0^\infty e^{-st} f(ct)\, dt = \frac{1}{c}\int_0^\infty e^{-\frac{s}{c}\tau} f(\tau)\, d\tau = \frac{1}{c}\mathcal{L}(f)(\frac{s}{c}).$$

Example. Find $\mathcal{L}(f)$, where $f_\pi(t) = \sin(\pi t)$.

Shifting

Theorem (Shifting)

Let f be a function such that $\mathcal{L}(f)(s)$ exists for $s > a$, for some $a \in \mathbb{R}$, and let c be a constant. Let $g(t) = e^{ct} f(t)$ for all $t > 0$. Then for $s > a + c$,

$$\mathcal{L}(g)(s) = \mathcal{L}(f)(s - c).$$

Proof:

$$\begin{aligned} L(e^{ct} f(t))(s) &= \int_0^\infty e^{-st} e^{ct} f(t) dt \\ &= \int_0^\infty e^{-(s-c)t} f(t) dt \\ &= \mathcal{L}(f)(s - c), \quad \forall\, s > a + c. \end{aligned}$$

Example. Let $f(t) = e^{2t} \sin t$. Check $\mathcal{L}(f)(s) = \frac{1}{(s-2)^2+1}, \quad \forall\, s > 2$.

Lerch's Cancellation Law

Theorem

Suppose f, g are continuous functions and

$$\int_0^\infty e^{-st} f(t)dt \text{ and } \int_0^\infty e^{-st} g(t)dt,$$

converge for some s and that $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for all s for which both integrals converge. Then $f(t) = g(t)$ for all $t > 0$.

Application of Lerch's cancellation law

Qn. For a continuous function $\phi : [0, \infty) \to \mathbb{R}$, it is given that $\mathcal{L}(\phi)(s) = \frac{c}{s-a}$, for all $s > a$, where c, a are constants. Find ϕ.

Ans. Recall for $f(t) = ce^{at}$, $\mathcal{L}(f)(s) = \frac{c}{s-a}$, for all $s > a$.
Under the condition that ϕ is continuous, and $\mathcal{L}(\phi)(s) = \mathcal{L}(f)(s)$ for all $s > a$, using Lerch's Cancellation Law we get

$$\phi(t) = ce^{at}, \quad \forall\, t > 0.$$