STOKES' THEOREM

RECALL GREEN'S THEOREM:

$$\oint_C F \cdot dr = \iint_R (\text{curl } F) \cdot \vec{k} \, dx \, dy \quad (\text{WORK FORM})$$

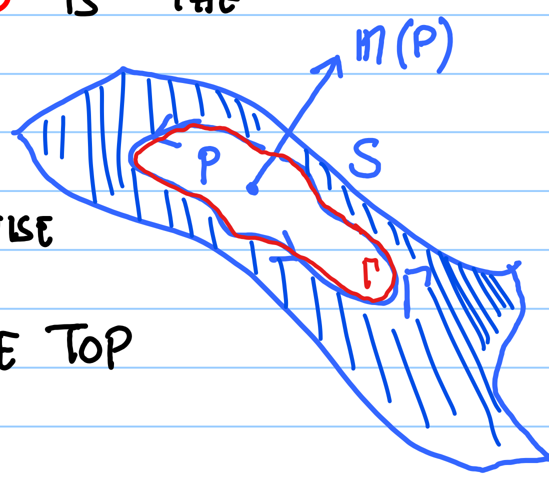
$R \subseteq \mathbb{R}^2$

WHERE $C = \partial R$, ORIENTED COUNTER CLOCKWISE.

LET S BE A PIECEWISE SMOOTH ORIENTABLE SURFACE IN $\mathbb{R}^3$, WHOSE BOUNDARY IS A SIMPLE CLOSED CURVE C. LET US FIX n, A CONTINUOUS UNIT NORMAL VECTOR FIELD ON S.

LET Γ BE A SIMPLE CLOSED CURVE ON S AND LET P BE A POINT 'INSIDE' Γ.

THE POSITIVE ORIENTATION ON Γ RELATIVE TO THE ORIENTATION ON S IS THE ORIENTATION ON Γ THAT IS COUNTER CLOCKWISE WHEN VIEWED FROM THE TOP OF $n(P)$

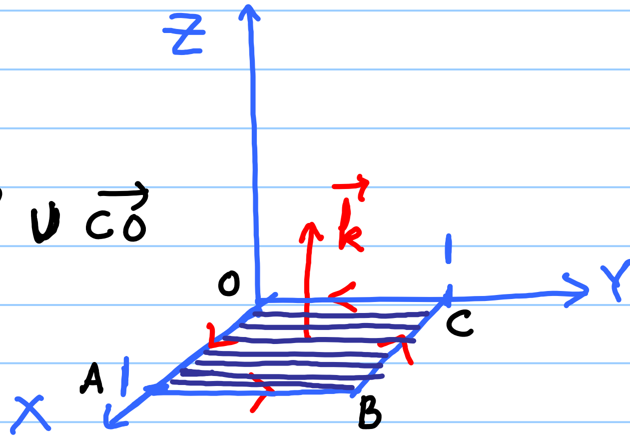


EXAMPLES

$S = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, z = 0\}$

$C = \partial S;$

$C = \overrightarrow{OA} \cup \overrightarrow{AB} \cup \overrightarrow{BC} \cup \overrightarrow{CO}$



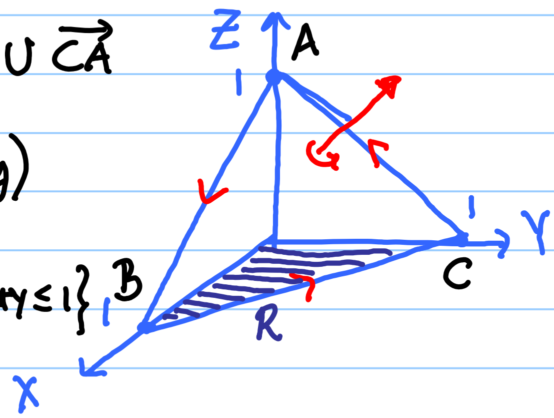
$S = \{(x,y,z) \mid x, y, z \ge 0, \ x+y+z = 1\}$

$C = \partial S = \overrightarrow{AB} \cup \overrightarrow{BC} \cup \overrightarrow{CA}$

$r(x,y) = (x, y, 1-x-y)$

FOR $(x,y) \in R = \{(x,y) \mid 0 \le x+y \le 1\}$,

$n = \dfrac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$.



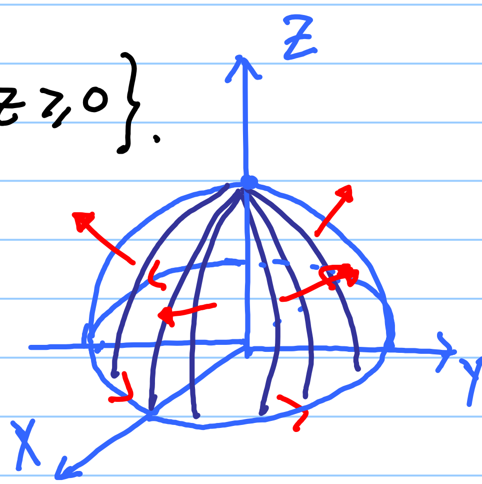
$S = \{(x,y,z) \mid z = 4 - x^2 - y^2, \ z \ge 0\}$.

$C = \partial S$

C IS THE CIRCLE

$x^2 + y^2 = 4$ ORIENTED

ANTICLOCKWISE



STOKES' THEOREM

🚩 LET S BE A PIECEWISE SMOOTH ORIENTED (BNDED SURFACE, AND SUPPOSE $C = \partial S$ BE A PIECEWISE CLOSED, SIMPLE CLOSED CURVE. IF F IS A CONTINUOUSLY DIFFERENTIABLE VECTOR FIELD ON D, A REGION IN $\mathbb{R}^3$ WHICH CONTAINS S (AND C).

THEN

$$\iint_S (\text{curl}\, F) \cdot \mathbf{n}\, dS = \oint_C F \cdot dr$$

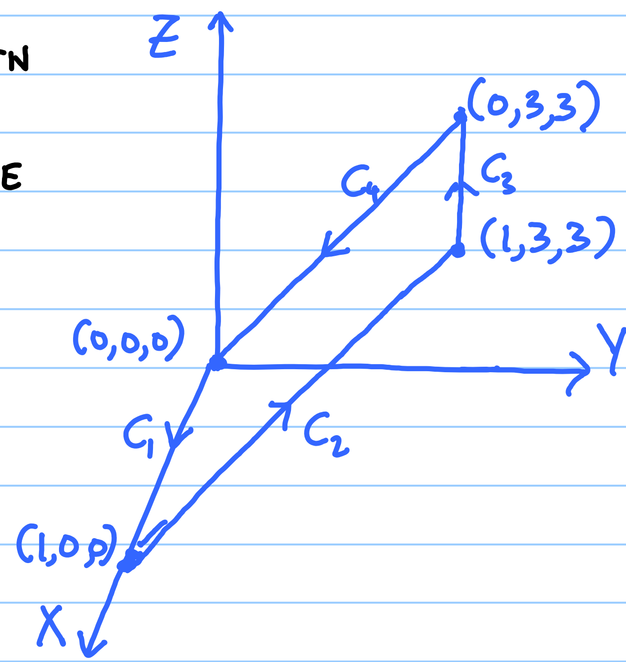
WHERE $\mathbf{n}$ IS A CONTINUOUS NORMAL ON S, AND C IS GIVEN THE POSITIVE ORIENTATION RELATIVE TO THE ORIENTATION OF S.

VERIFICATION OF STOKES'

SUPPOSE $F = x^2\vec{i} + 4xy^3\vec{j} + xy^2\vec{k}$, AND

$S \equiv$ THE RECTANGULAR REGION IN THE PLANE

$z = y$ WITH VERTICES $(0,0,0), (1,0,0), (1,3,3), (0,3,3)$.



THE SURFACE IS GIVEN

BY $z = g(x,y)$, WHERE

$g(x,y) = y$.

SET $G(x,y,z) = z - y$

THE PARAMETRIZATION

FOR S IS

$\mathbf{r}(x,y) = (x,y,y)$ FOR $x \in [0,1]$, $y \in [0,3]$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & 4xy^3 & xy^2 \end{vmatrix} = 2xy\vec{i} - y^2\vec{j} + 4y^3\vec{k}.$$

HENCE

$$\iint_S (\operatorname{curl} F)\cdot \mathbf{n}\, dS = \int_0^3\int_0^1 (2xy, -y^2, 4y^3)\cdot(0,-1,1)\, dx\, dy$$

$$= \int_0^3\int_0^1 (y^2 + 4y^3)\, dx\, dy = \int_0^3 y^2 + 4y^3\, dy = 90.$$

ON THE OTHER HAND,

$$\oint_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr + \int_{C_4} F\cdot dr$$

WHERE

$$C_1 : (x,0,0) \quad (0 \le x \le 1)$$

$$C_2 : (1, y, y) \quad (0 \le y \le 3)$$

$$C_3 : (1-x, 3, 3) \quad (0 \le x \le 1)$$

$$C_4 : (0, 3-y, 3-y) \quad (0 \le y \le 3)$$

$$\int_C F\cdot dr = \int_C P\,dx + Q\,dy + R\,dz \quad (F = (P,Q,R)).$$

HENCE

$$\int_{C_1} x^2 dx + 4xy^3 dy + ay^2 dz$$

$$= \int_0^1 x^2 dx = \frac{1}{3}$$

SIMILARLY ONE CALCULATES $\int_{C_2}$, $\int_{C_3}$ $\int_{C_4}$

VERIFY THAT STOKES' THEOREM HOLDS

$F = -y\vec{i} + 2yz\vec{j} + y^2\vec{k}$

$S \equiv$ UPPER HALF OF $x^2+y^2+z^2 = 1.$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & 2yz & y^2 \end{vmatrix} = \vec{k}.$$

$S \equiv z = \sqrt{1-x^2-y^2} \quad \left(g(x,y) = \sqrt{1-x^2-y^2}\right)$

HENCE
$$\iint_S (\text{curl } F)\cdot n \, dS = \iint_R 1 \, dx\, dy = \pi$$

LET $r(\theta) = (\cos\theta, \sin\theta, 0)$, $\theta \in [0, 2\pi]$

$$\oint_C F.dr = \int_0^{2\pi} (-\sin\theta)(-\sin\theta)\, d\theta$$
$$= 4\int_0^{\pi/2} \sin^2\theta \, d\theta = 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= \pi.$$

THIS VERIFIES STOKES' THEOREM.

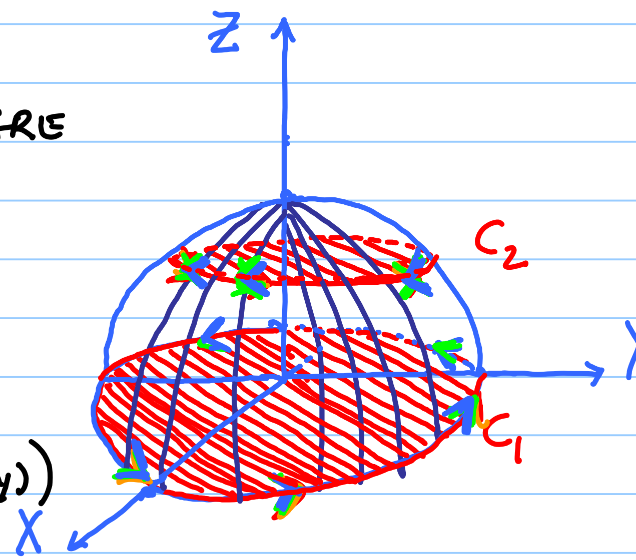
EXTENDING STOKES' THEOREM

WE CAN EXTEND STOKES' THEOREM TO 'SURFACES WITH HOLES' AS IN THE FOLLOWING EXAMPLE. THIS IS ANOTHER VERIFICATION.

$$S = \{(x,y,z) \mid z = 9-x^2-y^2, \ 0 \le z \le 4\}$$

$\partial S = C_1 \cup C_2$;

THE ORIENTATIONS ARE AS INDICATED.



SUPPOSE

$$F = (z-y, \ z+x, \ -(x+y))$$

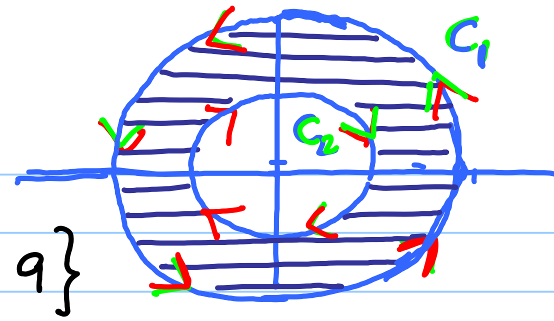
SUPPOSE WE WANT TO CALCULATE $\iint_S (\nabla \times F)\cdot \mathbf{n} \, dS$.

$$\nabla \times F = -2\vec{i} + 2\vec{j} + 2\vec{k}.$$

A PARAMETRIZATION FOR S:

$$\mathbf{r}(x,y) = (x, \ y, \ 9-x^2-y^2), \ (x,y) \in D.$$

WHERE



$$D = \{(x,y) \mid 5 \le x^2+y^2 \le 9\}$$

$$\mathbf{r}_x \times \mathbf{r}_y = (2x, 2y, 1) \quad \Big(\underset{\text{curl}(F)}{\underset{\shortparallel}{\nabla \times F}} = (-2,2,2) \Big)$$

HENCE

$$\iint_S (\nabla \times F)\cdot \mathbf{n}\, dS = \iint_D (-4x + 4y + 2)\, dx\, dy$$

NOTE THAT $\displaystyle\iint_D x \, dx\, dy = \iint_D y \, dy\, dx$, SO

$$\iint_S (\nabla \times F)\cdot \mathbf{n}\, dS = 2\iint_D dx\, dy = 2(9\pi - 5\pi) = 8\pi.$$

OTOH, $\displaystyle\oint_C F\cdot dr = \oint_{C_1} F\cdot dr + \oint_{C_2} F\cdot dr$

$$\oint_{C_1} F\cdot dr = \int_{C_1} (z-y)\, dx + (z+x)\, dy = \oint_{C_1} x\, dy - y\, dx$$

$$= 2 \text{ Area}\{x^2+y^2 \le 9\} = 18\pi$$

AND SIMILARLY

$$\oint_{C_2} (z-y)\, dx + (z+x)\, dy = \oint_{C_2} x\, dy - y\, dx + 4\oint_{C_2} dx + dy$$

$$= -10\pi. \text{ (CHECK)}$$