MA108 ENDSEM 12-06-2023 **10:00-12:00 Maximum Marks: 30**

Name: Green **Division:**
Roll No: **Tutorial Batch:**

1. Write your Name, Roll No., Division, Tutorial Batch.
*2. This is a question paper cum answer booklet. At the end of the exam, **only** this booklet will be collected for evaluation. Write the answers in the space provided against each question. Separate sheets will be provided for rough work.*
*3. There are **sixteen** questions.*
4. No books, notes, calculators, mobile phones, electronic devices are permitted.
*5. There is **no** negative marking.*

1. The curve in the x-y plane through the point $(0,1)$ and having the slope $4x^3$ at each point (x,y) is given by [1]

$$y(x) = \boxed{x^4 + 1}$$

2. Consider the IVP: $y' = \frac{y}{x} + 4x^4 e^{-\frac{y}{x}}$, $y(1) = 0$. The solution of the above IVP is [2]

$$y(x) = \boxed{x \ln x^4} \quad \forall\, x > 0.$$

3. The solution set of

$$(\sin x)y'''(x) + xy''(x) + x^2 y'(x) + x^3 y(x) = 0$$

for $\frac{\pi}{4} < x < \frac{\pi}{2}$ is a vector space of dimension d, where [1]

$$d = \boxed{3}.$$

4. *Possibly multiple correct answers.* Every solution of the DE $y''(x) + \alpha y'(x) + \beta y(x) = 0$, where $\alpha, \beta \in \mathbb{R}$, tends to 0 as $x \to \infty$, if [1]

   a. $\alpha < 0, \beta < 0$, $\alpha^2 - 4\beta > 0$.
   b. $\alpha > 0$, $\beta > 0$ and $\alpha^2 - 4\beta > 0$.
   c. $\alpha > 0, \alpha^2 - 4\beta < 0$.

   Write the correct option(s) here: Ans. $\boxed{\text{(b), (c)}}$.

5. Let x^{-1} and $x^{-1}\ln x$ be two solutions of $x^2y'' + axy' + by = 0$, for $x > 0$, and $a, b \in \mathbb{R}$. Then [1+1]

$$a = \boxed{3} \text{ and } b = \boxed{1}.$$

6. *Possibly multiple correct answers.* Let $\phi_1(x) = \begin{cases} 1 + x^4, & x < 0 \\ 1, & x \geq 0 \end{cases}$, $\phi_2(x) = \begin{cases} 1, & x < 0 \\ 1 + x^4, & x \geq 0 \end{cases}$,
and $\phi_3(x) = 4 + x^4, \quad x \in \mathbb{R}$. [1]
Write the correct option(s) here: Ans. $\boxed{\text{(a)}}$.

a. The functions ϕ_1, ϕ_2, ϕ_3 are linearly independent on $[-1,1]$.

b. The Wronskian $W(\phi_1, \phi_2, \phi_3)(x) \neq 0$ for all $x \in [-1,1]$.

c. There exist functions p_1, p_2, p_3 defined and continuous on $[-1,1]$ such that ϕ_1, ϕ_2, ϕ_3 are solutions of

$$y'''(x) + p_1(x)y''(x) + p_2(x)y'(x) + p_3(x)y(x) = 0,$$

for all $x \in [-1,1]$.

7. Let $L(y)(x) = (1+x^2)y''(x) - 2xy'(x) + 2y(x)$ for all $x > 0$. [3]

a. Let ϕ_1, ϕ_2 be two linearly independent solutions of $L(y)(x) = 0$ for $x > 0$. Given that $\phi_1(x) = x$, for $x > 0$, find

$$\phi_2(x) = \boxed{x^2 - 1}.$$

Or $\boxed{\phi_2(x) = C_1 x + C_2(x^2-1), \text{ for any } C_1, C_2 \in \mathbb{R} \text{ with } C_2 \neq 0.}$

b. If $y_p(x) = v_1(x)\phi_1(x) + v_2(x)\phi_2(x)$ is a particular solution of $L(y)(x) = x^3 + x$, then

$$v_1(x) = \boxed{-\frac{x^2}{2} + \ln(x^2+1)} \text{ and } v_2(x) = \boxed{x - \tan^{-1} x}.$$

Or $\boxed{v_1(x) = -\frac{C_1}{C_2}\left(x - \tan^{-1} x\right) - \frac{x^2}{2} + \ln(x^2+1) + d_1}$, and $\boxed{v_2(x) = \frac{1}{C_2}\left(x - \tan^{-1} x\right) + d_2}$, for any constants $d_1, d_2 \in \mathbb{R}$ and C_1, C_2 same as in [a].

8. *Possibly multiple correct answers.* The function $r(x) = xe^x + xe^{-x}$ is annihilated by [1]

a. $D^4 - 2D^2 + 1$

b. $D^2 - 2D + 1$

c. $D^5 + D^4 - 2D^3 - 2D^2 + D + 1$.

Write the correct option(s) here: Ans. $\boxed{\text{(a), (c)}}$.

9. The least possible n for which $y(x) = \sin^2 x$ is a solution of some n^{th}-order linear differential equation [2]

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \ldots + a_n y(x) = 0$$

for $a_1, \ldots, a_n \in \mathbb{R}$ is

Ans. $\boxed{3}$.

10. Let p, q, r be continuous functions on $\mathbb{R}$ and $L(y)(x) = y''(x) + p(x)y'(x) + q(x)y(x)$. If

$$\phi_1(x) = 1 + e^{x^2},\ \phi_2(x) = 1 + xe^{x^2},\ \phi_3(x) = (1+x)e^{x^2} + 1$$

are solutions of $L(y)(x) = r(x), x \in \mathbb{R}$, then [4]

a. Two linearly independent solutions of $L(y)(x) = 0$ on $\mathbb{R}$ are given by

$$y_1(x) = \boxed{e^{x^2}} \text{ and } y_2(x) = \boxed{xe^{x^2}}.$$

(or, $\boxed{y_1(x) = c_1 e^{x^2} + c_2}$, for some constants c_1, c_2, with $c_1 \neq 0$ and $\boxed{y_2(x) = d_3 x e^{x^2} + d_2 e^{x^2} + d_3}$, for some constants d_1, d_2, d_3, with $d_3 \neq 0$.)

b. The functions p and r are given by

$$p(x) = \boxed{-4x} \text{ and } r(x) = \boxed{4x^2 - 2}.$$

11. Let $L(y)(x) = y'''(x) - 5y''(x) + 6y'(x)$. [3+1]

a. A basis of solutions of $L(y)(x) = 0$ is given by

$$y_1(x) = \boxed{1},$$
$$y_2(x) = \boxed{e^{2x}},$$
$$\text{and } y_3(x) = \boxed{e^{3x}}.$$

b. The solution of the IVP: $L(y)(x) = 12x,\ y(0) = 0,\ y'(0) = \frac{5}{3},\ y''(0) = 2$ is given by

$$y(x) = \boxed{x^2 + \frac{5}{3}x}.$$

12. The inverse Laplace transform $\mathcal{L}^{-1}$ of the function $F(s) = \dfrac{s^2+9}{(s^2-9)^2}$ for $s > 3$ is given by [2]

$$\mathcal{L}^{-1}(F)(t) = \boxed{t\cosh 3t}.$$

(i.e., $\mathcal{L}^{-1}(F)(t) = \boxed{t\dfrac{(e^{3t} + e^{-3t})}{2}}$.)

13. *Possibly multiple correct answers.* Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function of exponential order. Let $F(s) = \mathcal{L}(f)(s)$, for $s > 0$, denote the Laplace transform of f. [1]

a. If $F(s) = 0$ for all $s > 0$, then $f(t) = 0$, for all $t \geq 0$.

b. $\lim_{s\to\infty} F(s)e^{\frac{-s^2}{2}} = 1$.

c. If f is differentiable on $[0, \infty)$, then f' is also of exponential order.

Write the correct option(s) here: Ans. $\boxed{\text{(a)}}$.

14. Let $g(t) = \int_0^t (t-x)\sin x\,dx$ and $f(t) = \begin{cases} 0 & t < 1 \\ g(t-1) & t \geq 1 \end{cases}$. Then the Laplace transform $\mathcal{L}(f)$ of f is given by [1]

$$\mathcal{L}(f)(s) = \boxed{\frac{e^{-s}}{s^2(s^2+1)}}.$$

15. Consider the IVP: $y''(x) + 4y'(x) + 4y(x) = x^3e^{-2x},\ y(0) = 1,\ y'(0) = 3$. The Laplace transform $\mathcal{L}(y)$ of the solution y of the IVP is given by [2]

$$\mathcal{L}(y)(s) = \boxed{\frac{1}{s+2} + \frac{5}{(s+2)^2} + \frac{6}{(s+2)^6}}.$$

The solution y of the IVP is given by

$$y(x) = \boxed{e^{-2x}\left(1 + 5x + \frac{x^5}{20}\right)}.$$

16. Do there exist functions p, q continuous on $\mathbb{R}$ such that $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$? Justify your answer. [2]

Ans. No, there cannot exist p, q continuous functions on $\mathbb{R}$ such that $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$.

Reason: Let $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$ for all $x \in \mathbb{R}$. Note that $y(0) = 0$ and $y'(0) = 0$. If there exist p, q continuous functions on $\mathbb{R}$ such that $y(x)$ is a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$, then $y(x)$ satisfies the IVP

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

But then the uniqueness of solution of the second order linear ODE with continuous coefficients implies that $y(x) = 0$ for all $x \in \mathbb{R}$ which is not the case since $y(x) = e^{x^2}\sin(x^2) + x^3\cos x$.

So, $y(x)$ cannot be a solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ on $\mathbb{R}$, for any continuous functions p, q defined on $\mathbb{R}$.