MA108 QUIZ 19-05-2023 **8:30-9:15AM Maximum Marks: 20**

Name: **Blue** **Division:**
Roll No: **Tutorial Batch:**

1. Write your Name, Roll No., Division, Tutorial Batch.
2. This a question paper cum answer booklet. At the end of the quiz, ***only*** *this booklet will be collected for evaluation. Write the answers in the space provided against each question. Separate sheets will be provided for rough work.*
3. There are ***nine*** *questions.*
4. No books, notes, calculators, mobile phones, electronic devices are permitted.
5. There is ***no*** *negative marking.*

No partial credits for Qn 1- 8.

1. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and $f(0) = -1$. Then, the function f so that the ODE [1+1]

$$y \sin x + f(x)y' = 0$$

is exact, is given by

$$f(x) = -\cos x.$$

(ii) For the function f as in (i), the implicit solution of the ODE is $u(x, y) =$ constant, where

$$u(x, y) = -y \cos x (or \quad y \cos x, or \quad y \cos x + constant, or \quad -y \cos x + constant),$$

(Each of the above mentioned functions is a correct answer).

2. The ODE $(3y + 10x^2) - (2x + 6x^3y^{-1})y' = 0$ has an integrating factor of the form x^ay^b. Find [1+1]

$$a = 2, \quad b = -3.$$

3. Possibly multiple correct answers. Let f and g be two distinct solutions of $y'+p(x)y = q(x)$, where p, q are continuous on $\mathbb{R}$. **Circle** the correct option(s). [2]

a. The solution curves associated to f and g intersect exactly once.

b. The solution curves associated to f and g can never intersect.

c. The solution curves associated to f and g intersect at least twice.

d. None of the above.

Here 2 marks for full set of correct option(s). Otherwise 0.

4. The solution of the IVP $xy' + y = x^4y^3,\ y(1) = 1$, is given by [2]

$$y(x) = \frac{1}{x}\frac{1}{\sqrt{2-x^2}}, \quad 0 < x < \sqrt{2}.$$

Update. Full 2 marks for $y(x) = \frac{1}{x}\frac{1}{\sqrt{2-x^2}}$ or $y(x) = \frac{1}{|x|}\frac{1}{\sqrt{2-x^2}}$ or $y(x) = \frac{1}{x}\frac{1}{\sqrt{|2-x^2|}}$ or $y(x) = \frac{1}{|x|}\frac{1}{\sqrt{|2-x^2|}}$, or $y(x) = \frac{1}{\sqrt{x^2(2-x^2)}}$.

5. Let $f(x) = x^4$ and $g(x) = x^3|x|$ for all $x \in \mathbb{R}$. Are the functions f and g linearly dependent on $\mathbb{R}$? Ans. No [1]

6. Possibly multiple correct answers. Consider the IVP: $xy' - y = 0,\ y(0) = y_0$. **Circle** the correct option(s). [2]

   a. The IVP has no solution for $y_0 \neq 0$.

   b. The IVP has a unique solution for each $y_0 \in \mathbb{R}$.

   c. The IVP has infinitely many solutions for $y_0 = 0$.

   d. The ODE is linear and separable.

Here 2 marks for full set of correct option(s). Otherwise 0.

7. Let $\{\phi_n\}_{n=0}^{\infty}$ be the sequence of functions given by the Picard's iteration method for the IVP $y' = x - y^2 + 1,\ y(0) = 1$, starting with $\phi_0 \equiv 1$. Then the first two Picard iterates are [1+1]

$$\phi_1(x) = 1 + \frac{x^2}{2},$$

$$\phi_2(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^5}{20}.$$

8. Let f and g be two linearly independent solutions of [2]

$$xy'' + y' + y\sin x = 0,$$

on $(0, \infty)$. Let $W(f, g; x)$ be the Wronskian of f and g at a point $x \in (0, \infty)$. Given $W(f, g; 1) = 4$, compute

$$W(f, g; 2) = 2.$$

9. Consider the IVP for $x \neq 0$

$$y' = \sqrt{\frac{|y|}{|x|}},\ y(1) = y_0,$$

(a) Find all y_0 such that the IVP is guaranteed to have a solution in an interval containing the point 1. Justify your answer. [2]

Ans. The IVP has a solution for every $y_0 \in \mathbb{R}$ in an interval containing 1.

Step 1[1 mark] Let $y_0 \in \mathbb{R}$ be any real number. Consider $a > 0, b > 0$ such that $1-a > 0$ and set $R := \{(x,y) \in \mathbb{R}^2 : |x-1| < a, |y-y_0| < b\}$. Now for any $(x,y) \in R$, since $0 < 1-a < x < 1+a$, the function $f(x,y) := \sqrt{\frac{|y|}{|x|}}, \quad \forall\,(x,y) \in R$ is well-defined and f is continuous on R. Moreover, there exists $M > 0$ such that

$$|f(x,y)| \leq \sqrt{\frac{|y_0| + b}{1-a}} := M, \quad \forall\,(x,y) \in R.$$

Step 2[1 mark] Since for any $y_0 \in \mathbb{R}$, there exists a rectangle R containing $(1, y_0)$ such that f is continuous and bounded on R, from the existence theorem, it is guaranteed that the IVP has a solution for every $y_0 \in \mathbb{R}$ in an interval of 1.

Full 2 marks if explicit solution is **computed correctly for all** $y_0 \in \mathbb{R}$**.**

(b) Find all y_0 such that the IVP is guaranteed to have a unique solution in an interval containing the point 1. Justify your answer.

Ans. The IVP admits a unique solution for every $y_0 \in \mathbb{R}\backslash\{0\}$ in an interval containing 1.

Step 1[1 mark] Let $y_0 \neq 0$. Without loss of generality $y_0 > 0$. Then there exists a rectangle

$$R_1 := \{(x,y) \in \mathbb{R}^2 : |x-1| < a, |y-y_0| < \delta\},$$

that does not contain points $\{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$. So that f is Lipschitz with respect to y on R_1.

Step 2[1 mark] because: the partial derivative of f w.r.to y exists on R_1 and

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \frac{1}{2}\sqrt{\frac{1}{xy}} \leq M_2, \forall (x,y) \in R_1$$

for some $M_2 > 0$. OR it can be shown directly from the definition of the Lip cont. Thus the IVP has a unique solution for $y_0 \in \mathbb{R}\backslash\{0\}$.

Step 3[1 Mark] For $y_0 = 0$, the function f is not Lip w.r. to y on any rectangle containing $(1,0)$. Thus, we cannot apply the 'uniqueness Theorem' and the uniqueness of the solution to IVP with $y_0 = 0$ is not guaranteed.

Or, one can give multiple solutions for $y(1) = 0$:

for any $c \geq 1$, $y(x) = \begin{cases} (\sqrt{x} - c)^2, & x \geq c \\ 0, & x < c \end{cases}$