Lecture 6

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RECAP Rotating Coordinate System

Taking $\mathbf{A} = \mathbf{r}$, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{r}$$

$$\implies \boxed{\mathbf{v}_{in} = \mathbf{v}_{rot} + \mathbf{\Omega} \times \mathbf{r}}$$

Note: no subscript in **r**

On taking $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{v}_{in}$$

$$\implies \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} (\mathbf{v}_{rot} + \mathbf{\Omega} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{v}_{rot} + \mathbf{\Omega} \times \mathbf{r})$$

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$$\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + \left[\frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}\right]_{rot}$$

$$+\vec{\Omega}\times(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})$$

Let us assume a constant angular velocity. Then the above equation can be written as

$$\boxed{\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

Having calculated the accelerations, we can calculate the pseudo force as before. In *S* frame:

$$\vec{F} = m\vec{a}_{in}$$

where $\vec{F}$ is sum of all real forces

In S', on the other hand, the observed acceleration would be

$$\boxed{\vec{a}_{rot} = \vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

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Thus the force found by an observer in S' would be

$$\boxed{\vec{F}_{rot} = m\vec{a}_{rot} = m(\vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}))}$$

This force can be written as: $$\vec{F}_{rot} = \vec{F} + \vec{F}_{fict}$$

where

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$$\boxed{\vec{F}_{fict} = \boxed{-2m\vec{\Omega}\times\vec{v}_{rot}} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

The first term in above expression is called **Coriolis force**. This force is present only when a particle is observed to move in the rotating frame of reference.

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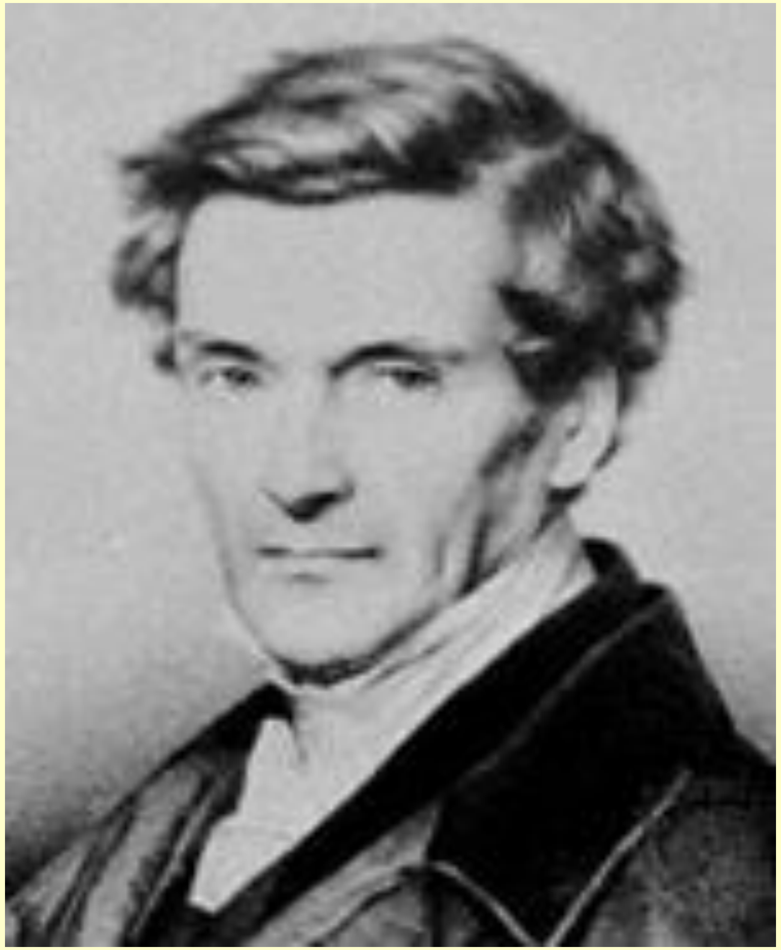
The second term on the other hand is called **Centrifugal force**. This force is present whenever the particle is at a non zero distance from origin.

Note: ***In order to apply these forces we must know the angular speed of S′ relative to S like we have to know the acceleration in the case of uniformly accelerating frames of reference.***

Gaspard-Gustave Coriolis

- May 21, 1792- September 19, 1843
- French mathematician, mechanical engineer, and scientist
- kinetic energy and work to rotating systems like waterwheels

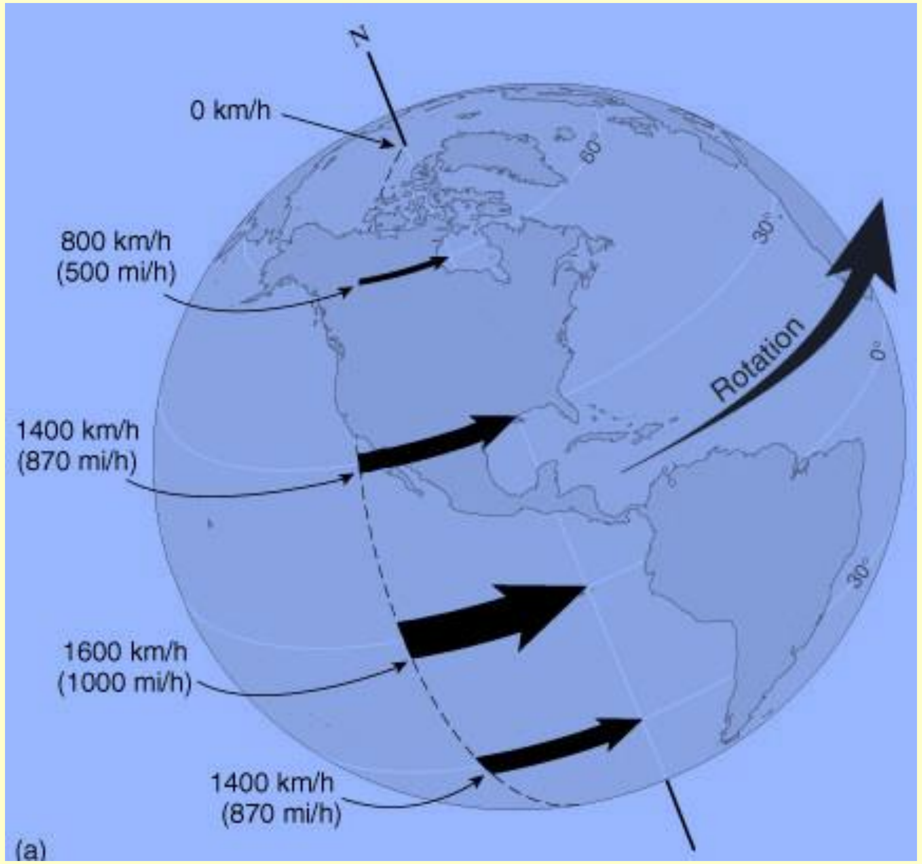


The Coriolis effect

- The Coriolis effect
  - Is a result of Earth's rotation
  - Causes moving objects to follow curved paths:
    - In Northern Hemisphere, curvature is to right
    - In Southern Hemisphere, curvature is to left
  - Changes with latitude:
    - No Coriolis effect at Equator
    - Maximum Coriolis effect at poles

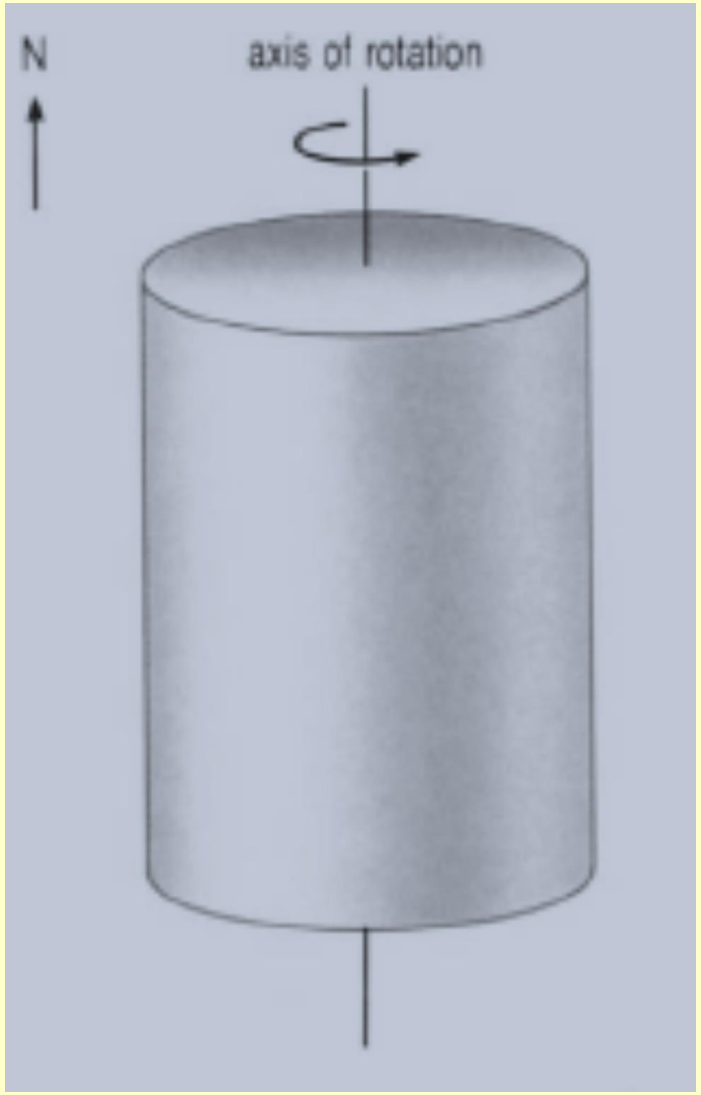
The Coriolis effect

- As Earth rotates, different latitudes travel at different speeds
- The change in speed with latitude causes the Coriolis effect



Coriolis Force

- If the earth is a cylinder shape and rotating about its axis, will there still any Coriolis effect exists?



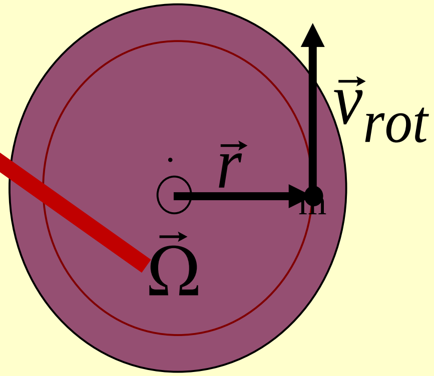
Coriolis Force

- The magnitude of the Coriolis force increases from zero at the Equator to a maximum at the poles.
- The Coriolis force acts at right angles to the direction of motion, so as to cause deflection to the *right* in the Northern Hemisphere and to the *left* in the Southern Hemisphere.

Example: A particle of mass *m* is rotating in *S*′ with angular velocity ω about the same axis of rotation as *S*′ and in the same direction.

According to *S*: The particle rotates with angular velocity (Ω+ω). Therefore, some real force must provide a centripetal force.

$$\therefore \vec{F}_{real} = -m(\Omega + \omega)^2 r\hat{\mathbf{r}}$$



According to *S*′: The particle rotates with angular velocity ω.

$$\therefore \vec{F}_{real} + \vec{F}_{fict} = -m\omega^2 r\hat{\mathbf{r}}$$

But

$$\vec{F}_{centrifugal} = m\Omega^2 r\hat{\mathbf{r}}$$

$$\vec{F}_{coriolis} = 2m\Omega\, v_{rot}\hat{\mathbf{r}} = 2m\Omega\omega\, r\hat{\mathbf{r}}$$

Please take care where ω is to be used and where Ω is to be used.

According to observer in S'

$$\vec{F}_{net} = \vec{F}_{real} + \vec{F}_{centrifuga\,l} + \vec{F}_{coriolis}$$

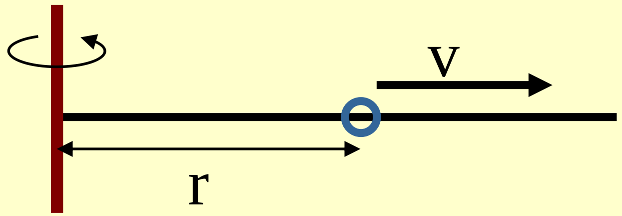
$$= -m(\Omega + \omega)^2 r\hat{\mathbf{r}} + m\Omega^2 r\hat{\mathbf{r}} + 2m\Omega\omega\, r\hat{\mathbf{r}}$$

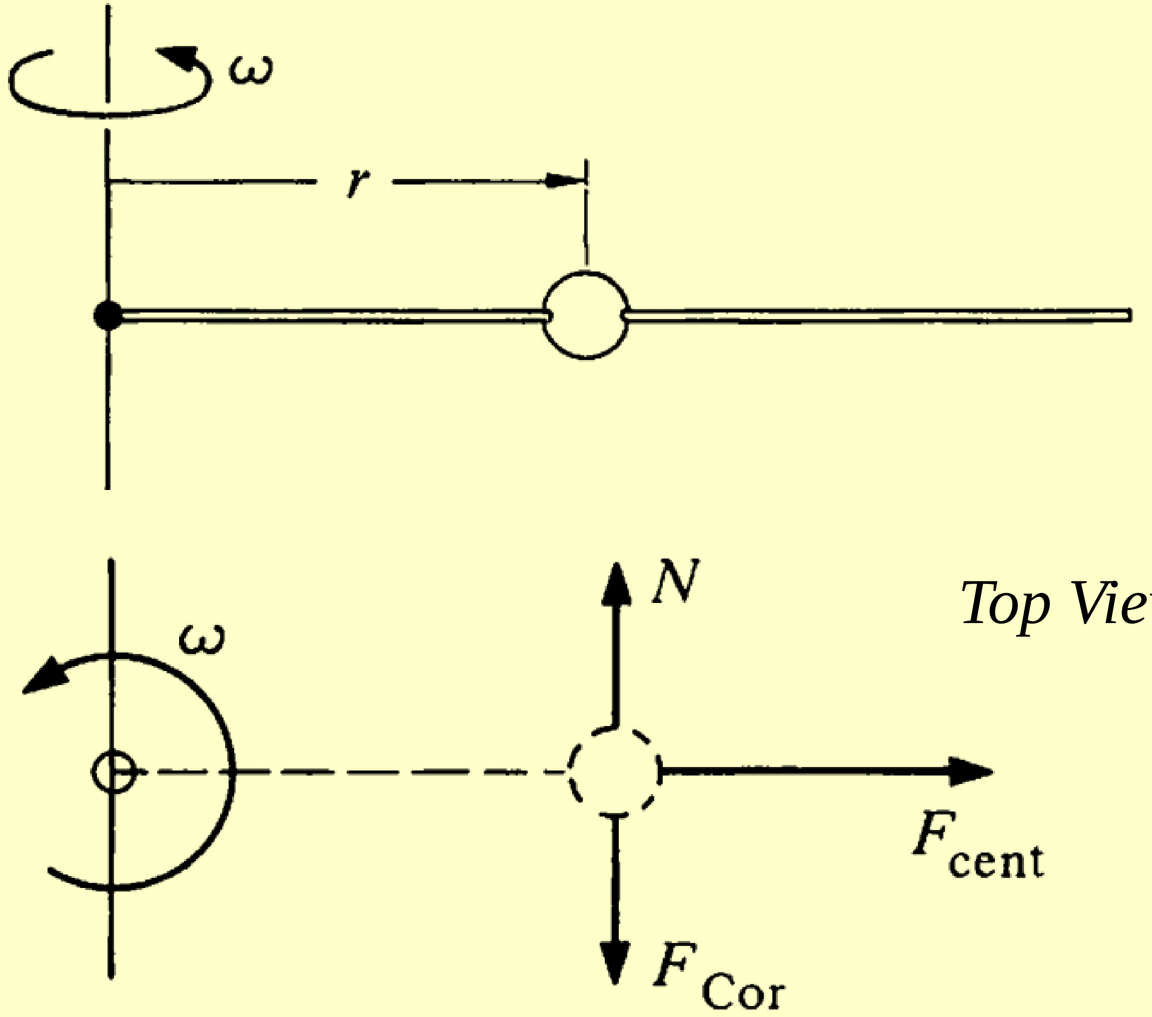
$$= -m\omega^2 r\hat{\mathbf{r}}$$

as expected

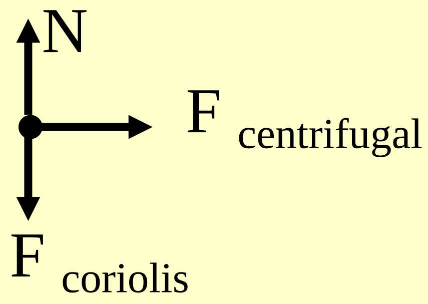
EXAMPLE

A bead slides without $\boxed{\text{friction}}$ on a rigid wire rotating at constant angular speed ω. The problem is to find the force exerted by the wire on the bead. **Neglect Gravity**





Top View



$m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

In a coordinate system rotating with the wire the motion is purely radial. F_{cent} is the centrifugal force and F_{Cor} is the Coriolis force. Since the wire is frictionless, the contact force N is normal to the wire. In the rotating system the equations of motion are

$$F_{\text{cent}} = m\ddot{r} \quad \text{and} \quad N - F_{\text{Cor}} = 0.$$

Since, $F_{\text{cent}} = m\omega^2 r$, $\quad m\ddot{r} - m\omega^2 r = 0,$

Hence, $\quad r = Ae^{\omega t} + Be^{-\omega t},$

where A and B are constants depending on the initial conditions.

The other equation gives:

$$N = F_{\mathrm{Cor}} = 2m\dot{r}\omega$$

$$\boxed{-2m\vec{\Omega}\times\vec{v}_{rot}}$$

$$= 2m\omega^2(Ae^{\omega t} - Be^{-\omega t}).$$

To complete the problem, apply the initial conditions which specify *A* and *B*.

$$\text{at } t = 0, r = a, \dot{r} = 0.$$

$$r = a\cosh\omega t$$

Questions:

1. In inertial frame is radial acceleration zero?

2. In inertial frame is tangential acceleration zero?

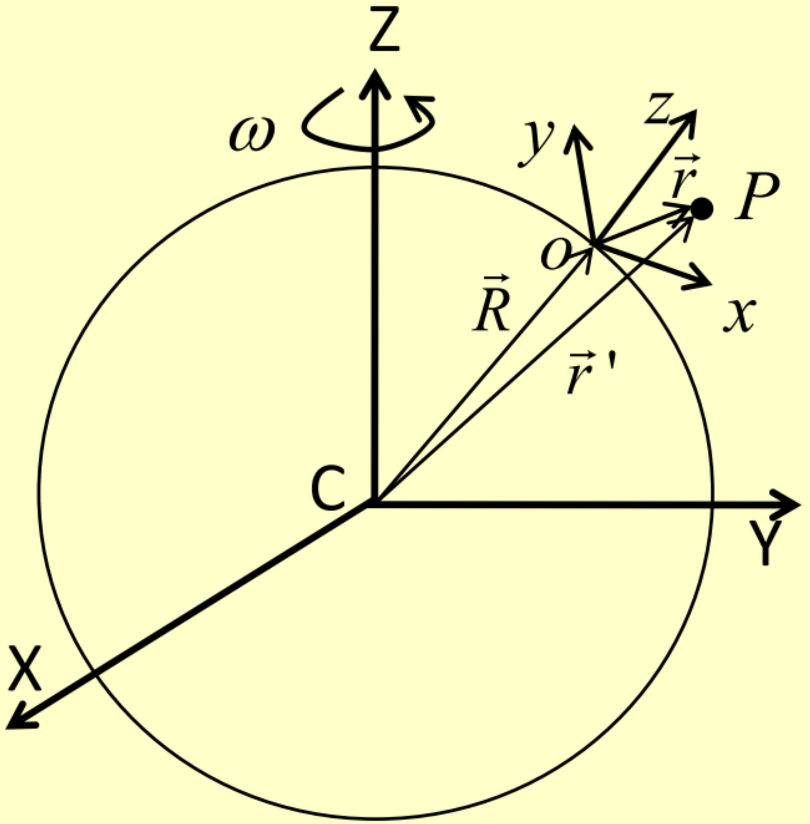
Answers:

1. Yes, there is no radial force

2. No, there is a real force normal reaction acting. It is interesting because ω is constant

Motion on rotating earth

Fix an inertial frame at the center of the earth. Fix another coordinate system at some point on the surface of the earth but rotating along with the earth with angular velocity ω. A particle *P* of mass *m* is moving above the surface of the earth subject to a force ***F*** and acted upon by the gravitational force m***g***.



$$\vec{r}\,' = \vec{R} + \vec{r},$$

$$\vec{r}\,' = \vec{R} + \vec{r},$$

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{in} = \left(\frac{d\vec{R}}{dt}\right)_{in} + \left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{R}}{dt}\right)_{rot} + \vec{\omega}\times\vec{R} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}$$

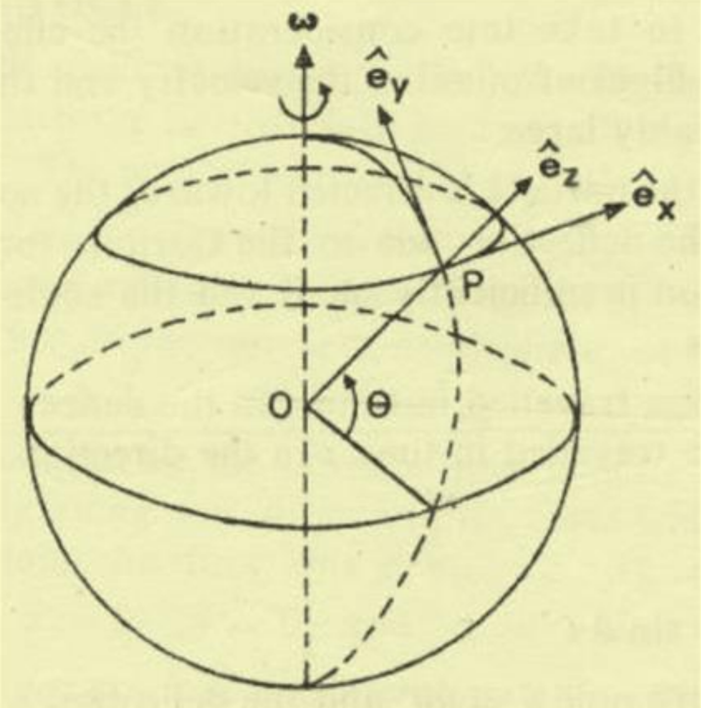
$$= \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\left(\vec{R}+\vec{r}\right)$$

$$\left(\frac{d^2\vec{r}\,'}{dt^2}\right)_{in} = \frac{d}{dt}\left[\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\left(\vec{R}+\vec{r}\right)\right]_{rot} + \vec{\omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\left(\vec{R}+\vec{r}\right)\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\left[\vec{\omega}\times\left(\vec{R}+\vec{r}\right)\right]$$

Freely falling particle **(Effective Gravity)**

$$\vec{a}_{rot} = \vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$



Acceleration: $\mathbf{a} = \mathbf{g} - 2\boldsymbol{\omega}\times\mathbf{v}$

X

Components of **ω**:

$$\omega_x = 0,\ \omega_y = \omega\cos\theta \text{ and } \omega_z = \omega\sin\theta$$

Since the Coriolis force produces small velocity components along x and y directions, they can be neglected in comparison to the vertical component.

$$\dot{x} \simeq 0, \quad \dot{y} \simeq 0 \quad \text{and} \quad \dot{z} = -gt$$

The components of acceleration are:

$$a_x = \ddot{x} = 2\omega g t\cos\theta$$
$$a_y = \ddot{y} = 0$$
$$a_z = \ddot{z} = -g$$

After integration:

$$x = \tfrac{1}{3}\omega g t^3\cos\theta$$
$$z = z_0 - \tfrac{1}{2}gt^2$$

Since $z_0=h$ $\quad t = \sqrt{\dfrac{2h}{g}}$ $\quad$ and $\quad d= \tfrac{1}{3}\omega\cos\theta\left(\dfrac{8h^3}{g}\right)^{1/2}$

Toward east

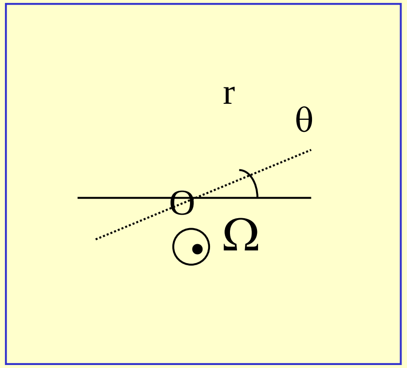
If h=100 m and θ=45° then the deflection would be 1.55 cm

Foucault Pendulum

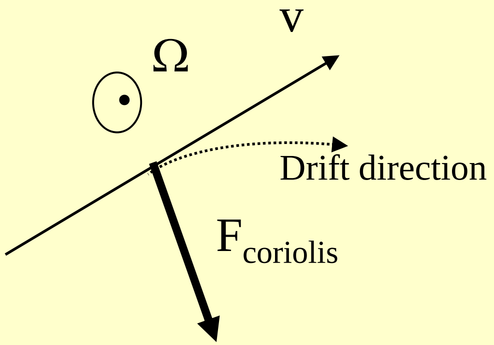
The Foucault Pendulum was conceived by Léon Foucault in the middle of the 19th century, with the goal of proving Earth's rotation through the effect of the *Coriolis Force*. In essence, the *Foucault Pendulum* is a Pendulum with a long enough damping rate such that the precession of its plane of oscillations can be observed after typically an hour or more. A whole revolution of the plane of oscillation takes anywhere between a day if it is at the pole, or longer at lower latitudes. At the equator the plane of oscillation does not rotate at all. (Note that if a precession period is defined as a rotation of 180° of the plane of oscillation, then the period at the pole is 12 hrs).

Foucault's Pendulum, *(the Panthéon, Paris, Photograph taken by Michael Reeve, 30/1/04)*

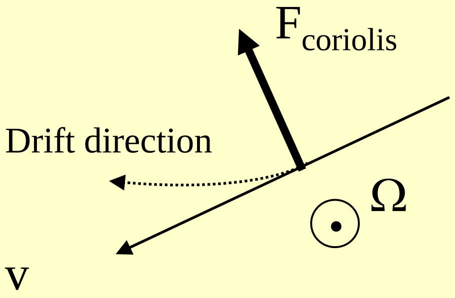
Foucault's Pendulum



Demonstrates that the earth is rotating. The plane of oscillation of a simple pendulum changes at a rate determined by the latitude.



Let a Foucault's pendulum be set at north pole. Let the position of the bob of the pendulum be described by r, θ co-ordinates in the horizontal plane



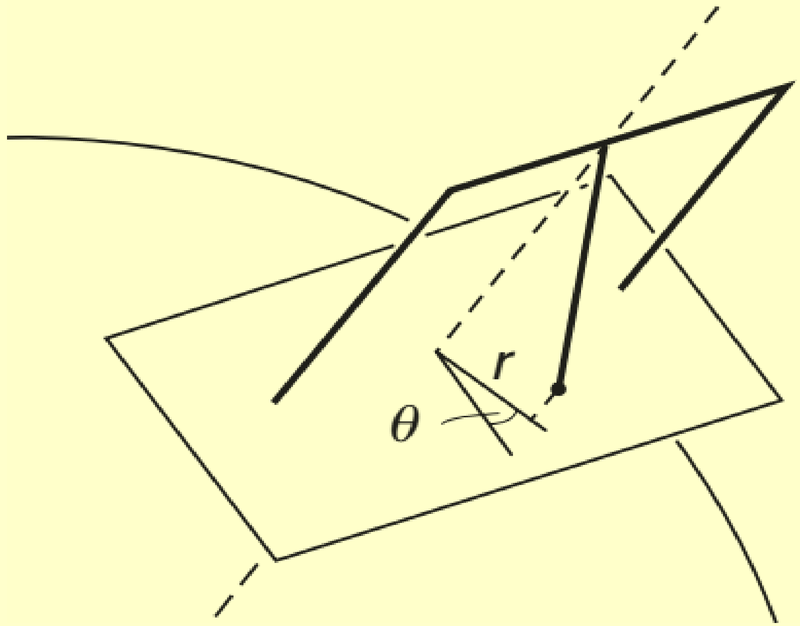
Consider a pendulum of mass m that is swinging with frequency $\omega = \sqrt{g/l}$, where l is the length of the pendulum. If we describe the position of the pendulum's bob in the horizontal plane by coordinates r, θ, then

$$r = r_0 \sin \omega t,$$

where r_0 is the amplitude of the motion. In the absence of the Coriolis force, there are no tangential forces and θ is constant.

$$r = r_o \sin \omega t; \omega = \sqrt{\frac{g}{l}}$$

θ (Precession angle along the horizontal plane)



As the pendulum oscillates, a tangential Coriolis force is felt by the bob, being zero at the end and maximum at *O*. This causes the pendulum to continuously change the plane of oscillation.

The tangential equation of motion $ma_\theta =$

$$\boxed{= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

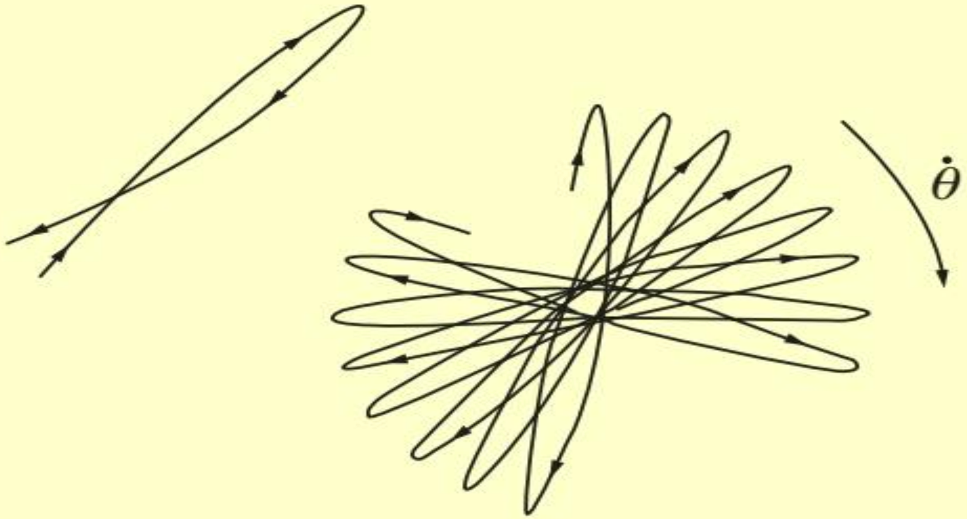
$$\vec{F}_{coriolis} = -2m\Omega \frac{dr}{dt}\hat{\theta}$$

We thus get

$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = -2m\Omega\frac{dr}{dt}$$

The simplest solution correspond s to $\frac{d^2\theta}{dt^2} = 0$;

giving $\frac{d\theta}{dt} = -\Omega$



The pendulum $\boxed{\text{precesses}}$ uniformly in a clockwise direction. The time for the plane of oscillation to rotate once is

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\Omega}$$

If T is the time period to complete one revolution, this would be $\boxed{\text{24 hrs.}}$ This result is obvious from the point of view of an inertial observer, with respect to whom the plane is fixed and the earth is rotating below it.

For a latitude λ, only the vertical component of angular velocity would cause tangential coriolis force. Hence, $\boxed{T = 24\text{hr} / \sin \lambda.}$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