VECTOR FIELDS

SUPPOSE $U \subseteq \mathbb{R}^m$ A FUNCTION

$f : U \to \mathbb{R}^n$ IS CALLED A VECTOR FIELD.

($f : U \to \mathbb{R}^2$ IS ALSO A VECTOR FIELD) $(m, n \in \{2,3\})$

WE WRITE $f = (f_1, f_2, f_3)$, WHERE

$f_i : U \to \mathbb{R}$ $(i = 1,2,3)$ ARE SCALAR FIELDS.

LIMITS

SUPPOSE $a = (a_1, a_2, a_3) \in U$ AND $f : U \to \mathbb{R}^3$, AND

SUPPOSE U IS OPEN. WE SAY f HAS A LIMIT

AT a IF GIVEN $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$0 < \| x - a \| < \delta \Rightarrow \| \underset{\sim}{f(x)} - \underset{\sim}{L} \| < \varepsilon$$

WHERE FOR $\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\| \underset{\sim}{x} \| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

AND $\underset{\sim}{L} = (l_1, l_2, l_3) \in \mathbb{R}^3$.

WE WRITE

$$\lim_{x \to a} f(\underset{\sim}{x}) = \underset{\sim}{L}.$$

EQUIVALENTLY, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x})$ EXISTS IFF $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f_i$ $(i = 1,2,3)$

<u>ALL</u> EXIST

We say f is continuous at $\underset{\sim}{a}$ if

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = f(\underset{\sim}{a}).$$

Suppose $f: U \to \mathbb{R}^3$, $f = (f_1, f_2, f_3)$. We say f is <u>differentiable</u> at $\underset{\sim}{a}$, if each f_i $(i = 1, 2, 3)$ is a scalar field differentiable at $\underset{\sim}{a}$.

We say f has i,j^{th} partial derivatives at $\underset{\sim}{a}$ iff $\dfrac{\partial f_i}{\partial x_j}$ exists at $\underset{\sim}{a}$. ($\dfrac{\partial f}{\partial x_2} = \dfrac{\partial f}{\partial y}$ and so on)

f is continuously differentiable at $\underset{\sim}{a}$ if each f_i is continuously differentiable at $\underset{\sim}{a}$.

$$Df = (\nabla f_1, \nabla f_2\ \nabla f_3)$$

LET $U \subseteq \mathbb{R}$ AND $f : U \to \mathbb{R}^3$. WRITE $f(x) = (f_1(x), f_2(x), f_3(x))$. f IS DIFFERENTIABLE IFF f_1, f_2, f_3 ARE DIFFERENTIABLE.

SUPPOSE $g : f(U) \to \mathbb{R}$. IS ALSO DIFFERENTIABLE.

THEN $g \circ f : U \to \mathbb{R}$ IS DIFFERENTIABLE AND

$$(g \circ f')(t) = \frac{\partial g}{\partial x}(f(t)) \cdot f_1'(t) + \frac{\partial g}{\partial y}(f(t)) \cdot f_2'(t) + \frac{\partial g}{\partial z}(f(t)) \cdot f_3'(t).$$

SUPPOSE $U \to \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ AND $g : \mathbb{R}^m \to \mathbb{R}^k$

$g \circ f : U \to \mathbb{R}^k$, AND SUPPOSE

$g \circ f = (F_1, F_2, \ldots, F_k)$ WHERE

$F_i = g_i \circ f$, $i = 1, 2, \ldots, k$.

IF f, g ARE BOTH DIFFERENTIABLE, THEN SO IS $g \circ f$ AND

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial (g_i \circ f)}{\partial x_j} \quad \forall\ 1 \leq j \leq n, \quad 1 \leq i \leq k.$$

GRAD, DIV, CURL

SUPPOSE $U \subseteq \mathbb{R}^3$, AND $\phi : U \to \mathbb{R}$ AND ALL PARTIAL DERIVATIVES EXIST. WE HAVE SEEN GRAD ϕ, DENOTED $\nabla\phi$ DEFINED AS

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

LET $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ BE A DIFFERENTIABLE VECTOR FIELD, AND SUPPOSE $F = (F_1, F_2, F_3)$.

THE DIVERGENCE OF F $(\mathrm{div}(F))$ IS DEFINED AS

$$\mathrm{div}(F)(P) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P)$$

THIS IS SOMETIMES DENOTED $\mathrm{div}(F) = \nabla \cdot F$

(THIS IS ONLY NOTATION!) $\left(\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right)$

SUPPOSE $F = (F_1, F_2, F_3)$ IS A VECTOR FIELD. DEFINE

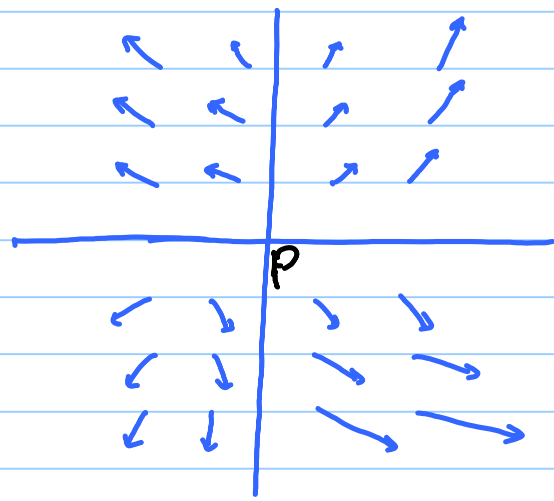
$$\mathrm{curl}(F) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

$$\mathrm{curl}(F) = \nabla \times F$$

THIS IS SOMETIMES WRITTEN

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$, OR SOMETIMES AS $\nabla \times F$.

SO, WHAT ARE div, curl ?

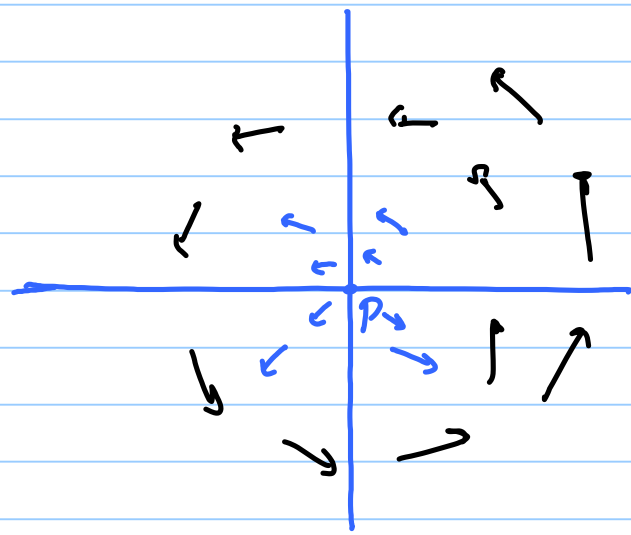


FLOW IS 'OUTWARDS' AT P.

VECTOR FIELD OF FLUID VELOCITY

DIVERGENCE ($\text{div}(F)$) AT A POINT P MEASURES WHETHER THE FLUID HAS A TENDENCY TO 'EXPAND' (FLOW OUTWARDS) OR 'SHRINK' (FLOW INWARDS) AT THAT POINT.

IF $\text{div}(P) = 0$, THEN P IS NEITHER 'SOURCE' NOR 'SINK'.



curl(F)(P) MEASURES 'MICROSCOPIC' ROTATIONAL TENDENCY OF THE FLUID AT THE POINT P.

THIS IS DONE SO BY INDICATING THE AXIS OF ROTATION, THE CORRESPONDING 'DIRECTION' i.e. CLOCKWISE/ANTI CLOCKWISE, AND THE MAGNITUDE.

A VECTOR FIELD F IS CALLED IRROTATIONAL IF $\text{curl}(F) = \underset{\sim}{0}$

SOME FACTS ON div

SUPPOSE ϕ, ψ ARE DIFFERENTIABLE SCALAR FIELDS AND F, G ARE DIFFERENTIABLE VECTOR FIELDS. THEN

- $\operatorname{div}(\alpha F) = \alpha \operatorname{div}(F) \quad \forall\ \alpha \in \mathbb{R}$
- $\operatorname{div}(F+G) = \operatorname{div}(F) + \operatorname{div}(G)$
- $\operatorname{div}(\phi F) = \phi \operatorname{div}(F) + F \cdot \nabla\phi.$
- $\operatorname{div}(\phi \nabla\psi) = \phi \nabla^2 \psi + (\nabla\phi)\cdot(\nabla\psi).$

  WHERE $\nabla^2\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial^2\psi}{\partial z^2}$

  (∇^2 IS CALLED THE LAPLACIAN)
- $\operatorname{div}(F \times G) = \operatorname{curl}(F)\cdot G - \operatorname{curl}(G)\cdot F.$

- IF F IS A VECTOR FIELD, THE VECTOR FIELD

$$i \times \frac{\partial F}{\partial x} + j \times \frac{\partial F}{\partial y} + k \times \frac{\partial F}{\partial z}$$

IS THE SAME AS $\operatorname{curl}(F)$. THIS EXPRESSION IS PARTICULARLY USEFUL FOR CALCULATIONS.

SOME FACTS ON curl

LET F, G BE CONTINUOUSLY DIFFERENTIABLE VECTOR FIELDS AND ϕ A CONTINUOUSLY DIFFERENTIABLE SCALAR FIELD. THEN

🚩 $\operatorname{curl}(F+G) = \operatorname{curl}(F) + \operatorname{curl}(G)$

🚩 $\operatorname{curl}(\phi F) = \phi \operatorname{curl}(F) + \nabla\phi \times F$

🚩 $\operatorname{curl}(F \times G) = (G \cdot \nabla) F - (F \cdot \nabla) G + (\operatorname{div} G) F - (\operatorname{div} F) G$

WHERE

- $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} := (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

- $$F \cdot \nabla := F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}$$

(THIS IS AN OPERATOR), SO

$$(F \cdot \nabla) G = F_1 \frac{\partial G}{\partial x} + F_2 \frac{\partial G}{\partial y} + F_3 \frac{\partial G}{\partial z}$$

CONSERVATIVE VECTOR FIELDS

$F: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ IS SAID TO HAVE A POTENTIAL FUNCTION ϕ IF THERE EXISTS A DIFFERENTIABLE $\phi: U \to \mathbb{R}$ s.t.

$$F(P) = (\nabla \phi)(P).$$

WE THEN SAY F IS CONSERVATIVE WITH POTENTIAL ϕ

LET $F = (F_1, F_2, F_3)$ BE CONSERVATIVE WITH POTENTIAL ϕ. IF THE SECOND ORDER PARTIAL DERIVATIVES OF ϕ EXIST AND ARE CONTINUOUS, THEN

$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

IN THE DOMAIN OF F.