MA 108-ODE- D3

Lecture 5

Debanjana Mitra

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

May 08, 2023

Existence - Uniqueness Theorem

Recall: Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0): $R : |x - x_0| < a, \; |y - y_0| < b$.

- $f(x, y)$ be continuous at all points $(x, y) \in R$ in and
- bounded in R, that is, $|f(x, y)| \leq K, \; \forall (x, y) \in R$.

Then, the IVP $y' = f(x, y), \; y(x_0) = y_0$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min \left\{ a, \frac{b}{K} \right\}.$$

In addition to the above conditions, if f satisfies the Lipschitz condition with respect to y in R, that is,

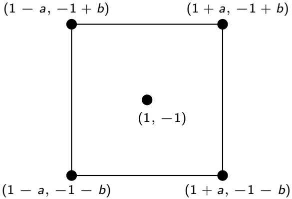
$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \;\; \forall (x, y_1), (x, y_2) \text{ in } R,$$

then,the IVP admits a unique solution on the interval $(x_0 - \alpha, x_0 + \alpha)$. [1].

[1]Existence - Peano, Existence & uniqueness -Picard

Warm up!

Example. Consider $y' = y^2,\ y(1) = -1$. Find α in the existence & uniqueness theorem.



$f(x, y) = y^2,\ f_y = 2y$ are continuous in the closed rectangle
$R : |x - 1| \leq a, |y + 1| \leq b$.

$$|f(x, y)| = |y|^2 \leq |(-b - 1)|^2 \leq (b + 1)^2 \tag{1}$$

Now, $\alpha = \min\left\{a, \frac{b}{(b+1)^2}\right\}$.

Example (contd..)

Consider

$$F(b) = \frac{b}{(b+1)^2}.$$

$F'(b) = \frac{1-b}{(b+1)^3} \Longrightarrow$ the maximum value of $F(b)$ for $b > 0$ occurs at $b = 1$ (Why?); and we find $F(1) = \frac{1}{4}$ and $F(b) \leq 1/4$ for any $b > 0$.

Hence, for any given $a > 0$ and $b > 0$, $\alpha = \min\{a, F(b)\} \leq \frac{1}{4}$.

In particular, for any $a \geq 1/4$ and any $b > 0$, the best possible $\alpha = \frac{1}{4}$ and the theorem gives that the IVP has a unique solution in $|x - 1| < 1/4 \Longrightarrow \boxed{3/4 < x < 5/4}$.

Example - Remarks

1. The theorem guarantees existence and uniqueness only in a very small interval!
2. The theorem DOES NOT give the largest interval where the solution exits.
3. What is the solution in this case by separation of variables and where is it valid? Can you think of extending the solution to a larger interval?
   Ans. $y(x) = \frac{-1}{x}$, solution defined on $(0, \infty)$.

Example. Consider the IVP: $y' = f(x, y), \quad y(0) = 0$, where $f(x, y) = y^2 + \cos(x^2), \quad \forall\, (x, y)\, \text{with} \quad |x| < 1,\, |y| < 1$. Does the IVP has a solution on the interval $(-\frac{1}{2}, \frac{1}{2})$? If yes, is the solution unique on the interval?

Ans. Yes. For the rectangle $R := \{(x, y) \in \mathbb{R}^2 \mid |x| < 1, \quad |y| < 1\}$, all the hypothesis of existence and uniqueness theorem holds. i.e., f is continuous and bounded on R, with $|f(x, y)| \leq K$ for all $(x, y) \in R$, where $K = 2$. The function f satisfies Lipschitz condition with respect to y on R:

$$|f(x, y_1) - f(x, y_2)| = |y_1^2 - y_2^2| \leq M|y_1 - y_2|, \quad \forall\, (x, y_1) \in R, \quad (x, y_2) \in R,$$

for some positive condition M, independent of x, y.

So, the theorem is applicable and the IVP admits a unique solution on $(-\alpha, \alpha)$, where recall $\alpha = \min\{a, b/k\}$, and here $a = 1, b = 1, K = 2$. Hence, IVP has a solution on $(-\frac{1}{2}, \frac{1}{2})$ and the solution is unique on the interval.

Existence and Uniqueness

Example: Consider the IVP

$$y' = \frac{10}{3} x y^{2/5}, y(x_0) = y_0. \tag{2}$$

(i) For what points (x_0, y_0), does the Theorem imply that (2) has a solution?

(ii) For what points (x_0, y_0), does the Theorem imply that (2) has a unique solution on some open interval that contains x_0?

Ans.(i) Since $f(x,y) = \frac{10}{3} x y^{2/5}$ is continuous for all (x, y), it follows that the above IVP has a solution for every (x_0, y_0).

(ii) f is not Lipschitz with respect to y on any rectangle R containing the points $(x, 0)$ for any $x \in \mathbb{R}^2$. Otherwise f satisfies the Lipschitz condition on any rectangle R not containing $(x, 0)$.
Therefore, if $y_0 \neq 0$, there is an open rectangle on which f satisfies the Lipschitz condition with respect to y, and hence and hence the above IVP has a unique solution on some interval that contains x_0.
If $y_0 = 0$, then on any rectangle containing $(x_0, 0)$, f does not satisfy the Lipschitz condition with respect to y, and thus Theorem for the uniqueness is not applicable to this IVP if $y_0 = 0$.

Linear first order ODEs

Consider the linear equation

$$y' + p(t)y = q(t), \tag{3}$$

where $p(\cdot)$ and $q(\cdot)$ are continuous functions defined on an interval I.

(i) There is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation, i.e.,

$$y(t) = e^{-\int p(t)dt} \left(\int e^{\int p(t)dt} \cdot q(t)dt + c \right).$$

A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.

(ii) Let $t_0 \in I$. The solution of the linear equation (3) subject to the initial condition $y(t_0) = y_0$ exists for all $t \in I$ and the solution can be obtained from the above expression determining c by $y(t_0) = y_0$. The solution is unique. (Check! why?)

Linear ODE contd...

Let ϕ and ψ be two solutions of the above IVP. Set

$$w(t) = \phi(t) - \psi(t), \quad \forall\, t \in I.$$

Then $w(\cdot)$ satisfies

$$w'(t) + p(t)w(t) = 0, \quad \forall\, t \in I, \quad w(t_0) = 0.$$

Using Integrating factor denoting $\mu(t) = e^{\int p(t)\,dt}$, we deduce

$$\frac{d}{dt}\Big(\mu(t)w(t)\Big) = 0, \quad \forall\, t \in I, \quad w(t_0) = 0,$$

and hence $\mu(t)w(t) = \mu(t_0)w(t_0) = 0$ for all $t \in I$.

Thus, from above it follows $w(t) = 0$ for all $t \in I$, as $\mu(t) \neq 0$ for all t and

$$\phi(t) = \psi(t), \quad \forall\, t \in I.$$

Picard's iteration method

[2] AIM : To solve

$$y' = f(x, y),\ y(x_0) = y_0 \tag{4}$$

METHOD

1. Integrate both sides of (4) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{5}$$

Note that any solution of (4) is a solution of (5) and vice-versa.

[2] Picard used this in his existence-uniqueness proof

Picard's method

2. Solve (5) by iteration:

$$
\begin{aligned}
\phi_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt \\
\phi_2(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\, dt \\
&\vdots \\
\phi_n(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\, dt
\end{aligned}
$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $\phi(x)$ of (4). That is,

$$\phi(x) = \lim_{n\to\infty} \phi_n(x).$$