MA 108-ODE- D3

Lecture 8

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Second order homogeneous ODEs

Recall

- Definitions of Linear dependency/ independency of functions in an interval, Wroskian of two functions....
- If two differentiable function f, g are linearly dependent in an interval I, then $W(f, g; x) = 0$ for all $x \in I$.
- The converse of the above statement is not true in general.
- Abel's formula: Let p, q be continuous on an interval I and let f, g be solutions of

  $$y'' + p(x)y' + q(x)y = 0$$

  on I. Let a be any point of I. Then

  $$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}, x. \in I$$

Wronskian and Linear Independence

Theorem

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Then,

1. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*
2. *f and g are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*

Thus, f and g are linearly independent on I iff $W(f, g; x) \neq 0$ for all $x \in I$.

Wronskian and Linear Independence

Proof of (1): Suppose $W(a) = 0$ for some $a \in I$. Then, for any $x \in I$,

$$W(x) = W(a)e^{-\int_a^x p(t)dt} = 0.$$

Hence, $W \equiv 0$ on I.

Wronskian and Linear Independence

Proof of (2): $\Rightarrow$ Done earlier. Need to do $\Leftarrow$.
Suppose that $W(f, g; a) = 0$ for some $a \in I$. Then the following linear system

$$\begin{aligned} C_1 f(a) + C_2 g(a) = 0, \\ C_1 f'(a) + C_2 g'(a) = 0. \end{aligned}$$

has non-zero solution $(C_1, C_2) \neq (0, 0)$.
Set $\psi(x) = C_1 f(x) + C_2 g(x)$ for all $x \in I$, for the C_1, C_2 obtained solving the above linear system. Then note that ψ satisfies the ODE with the initial value:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(a) = 0, \quad y'(a) = 0.$$

Also, note that 0 is a solution of the above ODE-IVP. Using the Uniqueness of solution of the second order linear IVP, we get $\psi(x) = 0$ for all $x \in I$. It implies that there exist non-zero C_1 and C_2 such that

$$C_1 f(x) + C_2 g(x) = 0, \quad \forall\, x \in I.$$

Hence f and g are linearly dependent on I.

Existence of set of fundamental solutions

Theorem

There exist two linearly independent solutions of the second order ODE

$$L(y) = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

where $p(\cdot), q(\cdot)$ are continuous on an interval I.

Proof. Let $x_0 \in I$. Recall that for any initial conditions, the 2-nd order linear IVP admits a unique solution.

Let $\phi_1(\cdot)$ be the solution of the ODE-IVP:

$$L(y)(x) = 0, \quad \forall\, x \in I, \quad y(x_0) = 1, \quad y'(x_0) = 0.$$

Let $\phi_2(\cdot)$ be the solution of the ODE-IVP:

$$L(y)(x) = 0, \quad \forall\, x \in I, \quad y(x_0) = 0, \quad y'(x_0) = 1.$$

Claim. The two functions ϕ_1 and ϕ_2 are linearly independent on I.
Check that

$$W(\phi_1, \phi_2; x_0) = \det \begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{pmatrix} = 1.$$

Hence ϕ_1 and ϕ_2 are linearly independent on I.

Proposition

Let f and g be any two solutions of the second order ODE on I with the property that $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ are two linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. The the two solutions f and g are linearly independent on I.

Linear combination

Theorem

If y_1 and y_2 are solutions of homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

on an interval I, then any linear combination

$$y = c_1 y_1 + c_2 y_2$$

of y_1 and y_2 is also a solution of (1) on I.

(Why?).

Basis of Solutions

Theorem

Let f, g be two solutions of the homogeneous second order linear ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an interval I in $\mathbb{R}$. Let $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. Then the solution space is the linear span of f and g.

Proof. Let h be a solution of the given ODE. We want to find c and d such that

$$h(x) = cf(x) + dg(x) \text{ for all } \quad x \in I.$$

Basis of Solutions

To do it, first solve the linear system for c, d

$$\begin{aligned} cf(x_0) + dg(x_0) =& h(x_0) \\ cf'(x_0) + dg'(x_0) =& h'(x_0). \end{aligned}$$

Thus,

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}.$$

As the column vectors

$$\begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} \ \& \ \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix}$$

are linearly independent, the matrix

$$W(x_0) = \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$$

is invertible.

Basis of Solutions

Therefore, there exists a unique solution (c, d) to the linear system given by

$$c = \frac{\begin{vmatrix} h(x_0) & g(x_0) \\ h'(x_0) & g'(x_0) \end{vmatrix}}{\det W(x_0)},$$

and

$$d = \frac{\begin{vmatrix} f(x_0) & h(x_0) \\ f'(x_0) & h'(x_0) \end{vmatrix}}{\det W(x_0)}.$$

(What's this method called?) Let

$$u(x) = h(x) - cf(x) - dg(x) \quad \text{for all} \quad x \in I.$$

Check that u is a solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = 0, y'(x_0) = 0,$$

which implies that $u \equiv 0$ by the uniqueness theorem, i.e.,

$$h(x) = cf(x) + dg(x) \quad \text{for all} \quad x \in I.$$

Note that in the above span of h by f, g, the constants c, d are uniquely determined.

Fundamental Theorem

Given a vector space, what's the most important thing about it? Dimension. The above results give:

Theorem (Dimension Theorem)

Let I be an interval in $\mathbb{R}$, p, q be continuous on I and let

$$L : C^2(I) \to C(I)$$

be defined by

$$L(f) = f'' + p(x)f' + q(x)f$$

and the null space of L

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

The dimension of $N(L) = 2 =$ order of the ODE.

Second Order Linear Homogeneous ODE's: Summary

Theorem

Suppose p and q are continuous on an interval I and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

on I. Then the following statements are equivalent:

(i) $\{y_1, y_2\}$ *is linearly independent on I.*

(ii) *Every solution of (2) on I can be written as a linear combination of y_1 and y_2.*

(iii) *The Wronskian of $\{y_1, y_2\}$ is non-zero at some point in I.*

(iv) *The Wronskian of $\{y_1, y_2\}$ is non-zero at all points in I.*