- $\mathbb{R}$ — is uncountable.

We will prove: $(0,1)$ — uncountable.

Suppose $(0,1)$ is countable.

$$= \underline{\{x_1, x_2, x_3, \ldots \} \text{ (say)}}.$$

$$x_1 = 0 \cdot y_{11}\, y_{12}\, y_{13} \ldots y_{1n} \ldots \longrightarrow$$

$$x_2 = 0 \cdot y_{21}\, y_{22}\, y_{23} \ldots$$

$$\ldots$$

$$x_n = 0 \cdot y_{n1}\, y_{n2}\, y_{n3} \ldots$$

$$\ldots$$

where $\quad 0 \le y_{ij} \le 9, \quad \forall\, i, j.$

Let.

$$x = 0.y_1 y_2 y_3 \ldots$$

where,

$$y_1 = \begin{cases} 5 & \text{if } y_{11} = 7 \\ 7 & \text{if } y_{11} \neq 7 \end{cases}$$

$$y_2 = \begin{cases} 5 & \text{if } y_{22} = 7 \\ 7 & \text{if } y_{22} \neq 7 \end{cases}$$

$$\vdots$$

$$y_n = \begin{cases} 5 & \text{if } y_{nn} = 7 \\ 7 & \text{if } y_{nn} \neq 7 \end{cases}.$$

$\therefore\ x \in (0,1).$

& $x \neq x_n \ \forall n.$

•

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\leq 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

~~Suppose~~ e is rational.

$e = p/q$, $p, q \in \mathbb{Z}$, $q \neq 0$.

$\Rightarrow q \cdot e = p$.

$\Rightarrow q \cdot \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right)$ – $\hat{\ }$ +ve int.

$\Rightarrow q! \left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) + \cdots\right]$

$\Rightarrow \left(q!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{q!}\right) + \frac{q!}{\ } \right.$ – is a +ve int.

$$\Rightarrow \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots \qquad \frac{q!}{(q+1)!} + \cdots + \frac{q!}{(q+n)!} + \cdots$$

— ve a +ve int.

(— ve a +ve int.

$$\Rightarrow \underbrace{\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots}_{\text{—ve a +ve int.}}$$

$$0 < \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots$$

$$= \frac{1}{q+1}\left(1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots\right)$$

$$\leq \frac{1}{q+1}\left(1 + \frac{1}{q+2} + \frac{1}{(q+2)^2} + \cdots\right)$$

$$= \frac{1}{q+1} \times \frac{1}{1 - \frac{1}{q+2}} = \frac{q+2}{(q+1)^2} < 1.$$

$$\frac{a+2}{(a+1)^2} < 1$$

$$\Leftrightarrow \quad a+2 \le a^2+2a+1$$

$$\Leftrightarrow \quad a^2+a-1 > 0 .$$

- If $\{a_n\}$ is increasing + bounded above $\Longrightarrow$ $\{a_n\}_n$ conv.

- <u>Supremum: of sequence:</u>

$\{a_1, a_2, a_3, \ldots \ldots \}$

$M = \sup\{a_n\}$ if

① $a_n \leq \underline{\underline{M}} \quad \forall n.$

② If M_1 is s.t. $\underline{\underline{a_n \leq M_1, \forall n.}}$

Then $\underline{\underline{M}} \leq M_1$.

<u>Ex:</u> $\left\{0, 1-\frac{1}{2}, 1-\frac{1}{3}, 1-\frac{1}{4}, \ldots\right\}$

<u>Sup:</u> $\underline{1-\frac{1}{n} \leq 1 \quad \forall n.}$

<u>Suppose.</u> Let, $1-\frac{1}{n} \leq M_1$

$\forall n.$

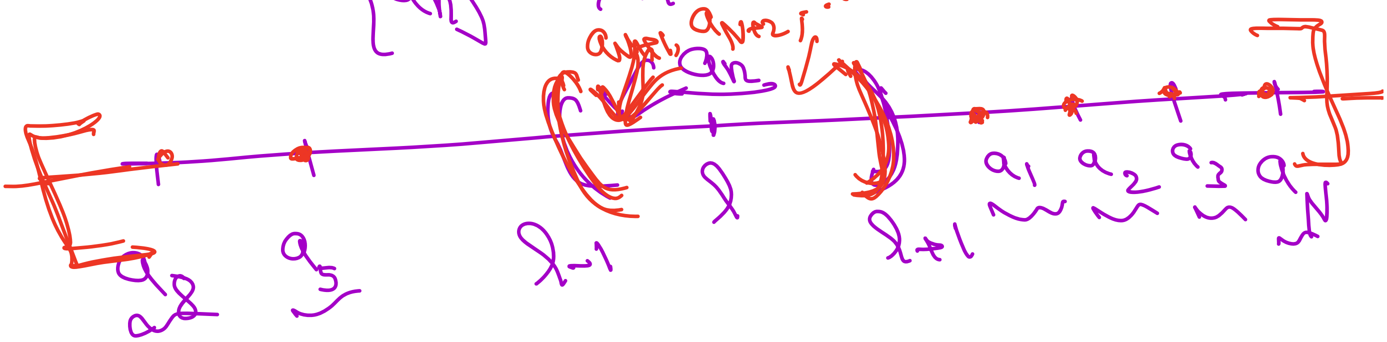
$$\Longrightarrow \lim_{n\to\infty}\left(1-\frac{1}{n}\right) \leq M_1$$

$$\Longrightarrow 1 \leq M_1.$$

$$\sup\left\{1-\frac{1}{n}\right\} = 1.$$

• All convergent sequences are bounded.

$\{a_n\} \to l$. $\quad \underline{\varepsilon = 1.}$



<u>To find</u>: M s.t. $|a_n| \le M \quad \forall n$.

$\underline{a_1 = 3/2}$, $\quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall n.$

$a_n \geq \sqrt{2}$. $\quad \forall n$.

$a_{n+1} - a_n$

$= \frac{a_n}{2} + \frac{1}{a_n} - a_n$

$= \frac{1}{a_n} - \frac{a_n}{2} = \frac{2 - a_n^2}{2a_n} < 0$

$$2-a_n^2$$

- $n^{1/n} = 1+k_n$, $k_n > 0$. $\forall n \geq 2$.

$\therefore$ $n = (1+k_n)^n$

$$= 1 + nk_n + \binom{n}{2}k_n^2 + \cdots + \binom{n}{n}k_n^n$$

$$\geq 1 + \binom{n}{2}k_n^2.$$

$$\Rightarrow n-1 \geq \frac{n(n-1)}{2}k_n^2$$

$$\Rightarrow 0 < k_n \leq \sqrt{\frac{2}{n}} \to 0$$

$\downarrow$

0

$n \geq 1+nk_n$

$\Rightarrow n-1 \geq nk_n$.

$0 < k_n \leq \frac{n-1}{n}$

$$\frac{a_n + \frac{2}{a_n}}{2} \geq \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2}.$$

$\{a_n\}$.

$$\left\{ 1,\ 1+\frac{1}{1!},\ 1+\frac{1}{1!}+\frac{1}{2!},\ \ldots \right\}$$

seq. in Q. $\longrightarrow e$.

$e \notin Q$.

do not converge in Q.

Cauchy seq. in Q.

$$|a_n - a_m| = |a_n - e + e - a_m|$$
$$\leq |a_n - e| + |a_m - e|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon \quad \forall\, n, m \geq N.$$

- $\mathbb{R}$: Every cauchy seq. in $\mathbb{R}$ conv. in $\mathbb{R}$.
- (~~Complete space~~)

$f : \mathbb{R} \longrightarrow \mathbb{R}$ – func.

$a_n : \mathbb{N} \longrightarrow \mathbb{R} \longrightarrow$ func.

seq.

$$e := 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

Rudin's Principle of Math. analysis.

① $x_n > 0 \ \forall n.$

Def. $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda.$

$\lambda < 1.$

Suppose, $0 \le \lambda < 1$.

$\boxed{\text{Choose, } \varepsilon_0 > 0 \text{ s.t. } \lambda + \varepsilon_0 < 1.}$

$\varepsilon > 0$.

$$\left| \frac{x_{n+1}}{x_n} - \lambda \right| < \varepsilon_0 \quad \forall n \ge N.$$

$$\Rightarrow \lambda - \varepsilon_0 < \frac{x_{n+1}}{x_n} < \lambda + \varepsilon_0, \ \forall n \ge N.$$

$$\Rightarrow x_{n+1} < (\lambda + \varepsilon_0) x_n, \quad \forall n \ge N.$$

$$x_{N+1} \le (\lambda + \varepsilon_0) x_N.$$

$$x_{N+2} \le (\lambda + \varepsilon_0) x_{N+1} \le (\lambda + \varepsilon_0)^2 x_N$$

$$x_{N+3} \le (\lambda + \varepsilon_0) x_{N+2} \le (\lambda + \varepsilon_0)^3 x_N.$$

$$0 < x_{N+m} \leq (\lambda+\varepsilon_0)^m x_N$$

$$\downarrow \qquad \qquad \qquad \qquad \downarrow \quad \forall m.$$

$$0 \qquad \qquad 0 \qquad \qquad 0$$

•

$$x_n = \frac{1}{n!}$$

$$\lim \frac{x_{n+1}}{x_n} = \lim \frac{1}{n+1} = 0 = \lambda < 1$$

• $x_n = \dfrac{n!}{n^n}$

$\lambda = 1.$

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$$

$$= \frac{1}{(n+1)^n} \times n^n$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{e} < 1.$$

- $\{x_n\} \rightarrow$ increasing & bounded above.

Let, $M = \sup x_n$.

Fix $\epsilon > 0$

$\therefore \quad x_n \leq M \quad \forall n.$

& $\quad M - \epsilon < x_{n_0} \quad$ for some n_0.

$$\therefore M - \epsilon < \underbrace{x_{n_0} \leq x_{n_0+1} \leq x_{n_0+2} \cdots}_{\leq M < M + \epsilon}$$

$$\Rightarrow \quad |x_n - M| < \epsilon \quad \forall n \geq n_0.$$

$$\Rightarrow \quad x_n \longrightarrow M.$$

$\sqrt{2} = p/q$, $p, q \in \mathbb{Z}$.

$q \neq 0$.

assume $(p, q) = 1$.

$\therefore$ $p^2 = 2q^2$.

p^2 is even.

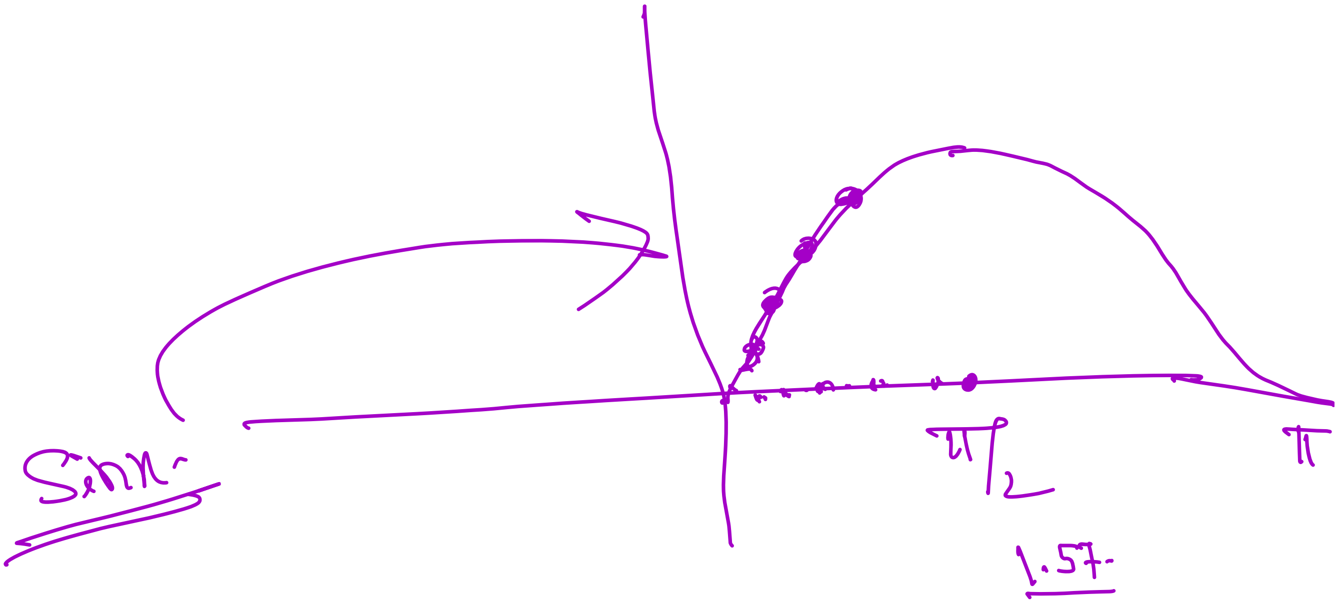
$\Rightarrow$ p is even.

$p = 2m$.

$\therefore$ $4m^2 = 2q^2$.

$\therefore$ q^2 is even.

$\Rightarrow$ q is even.



$\sin\frac{1}{n}$. $n \geq 2$

$\{ f : \mathbb{R} \to \mathbb{R} \text{ cont.} \}$

Set

Space

$f(x) = x$.

$f_1(x) = x^2$.

$f_2(x) = \sin x$.

$$\lim_{n\to\infty}\left(\frac{1}{x}\right)^{1/n} = 1 \qquad \lim y^{1/n}$$

$\Downarrow$

$$\left|\frac{1}{x^{1/n}} - 1\right| < \epsilon \quad \forall n \geq N$$

i.e. $\dfrac{|1 - x^{1/n}|}{|x|^{1/n}} < \epsilon$.

$x < 1$

$\frac{1}{x} > 1$

i.e. $|1 - x^{1/n}| \leq \dfrac{|1-x^{1/n}|}{|x|^{1/n}} < \epsilon$

$\forall n \geq N$.

$\therefore |1 - x^{1/n}| < \epsilon \quad \forall n \geq N.$

$$\Rightarrow \lim_{n\to\infty} x^{1/n} = 1.$$

$$\left.\begin{array}{l} a_n \to l \\ b_n \to m \end{array}\right\}$$

Then $a_n b_n \to lm$.

$$|a_n b_n - lm|$$

$$= |a_n b_n - a_n m + a_n m - lm|$$

$$\leq |a_n| |(b_n - m)| + |a_n - l| |m|$$

$$\leq M |b_n - m| + |a_n - l| \cdot |m|.$$

$$\left\{\begin{array}{l} |b_n - m| < \varepsilon/2M, \ \forall n \geq N_1 \\ |a_n - l| < \varepsilon/2|m| \ \forall n \geq N_2. \end{array}\right.$$

$$N = \max\{N_1, N_2\}.$$

$\forall n \geq N,$

$$|a_n b_n - lm| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$