MA 108-ODE- D3

Lecture 2

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Homogeneous form

Exact ODE's

Warm up!

1. The DE $e^x y' + 3y = x^2 y$ is linear & separable.
   Qn. TRUE OR FALSE?
   Ans. True

2. The DE $yy' + 3x = 0$ is linear & separable.
   Qn. TRUE OR FALSE?
   Ans. False, non-linear and separable.

3. The DE $2xyy' = y^2 - x^2$ is non-linear & Homogeneous.
   Qn. TRUE OR FALSE?
   Ans. True

4. The DE $\frac{dy}{dx} = \frac{2yx + \cos x}{1 + x^2}$ is linear & separable & Homogeneous.
   Qn. TRUE OR FALSE?
   Ans. False, Linear but neither separable nor homogeneous form.

Initial Value Problem for first order ODE

Definition

Initial value problem (IVP) : A DE along with an initial condition is an IVP.

$$y' = f(x, y),\ y(x_0) = y_0.$$

A solution of the above Initial Value Problem for first order ODE is a real-valued function ϕ defined on an interval (α, β) containing x_0 such that $\phi'(\cdot)$, the derivative of ϕ, exists on the interval (α, β) satisfying

$$\phi'(x) = f(x, \phi(x)), \forall\, \alpha < x < \beta, \quad \phi(x_0) = y_0.$$

Separable ODE's

Example: Escape velocity.
A projectile of mass m moves in a direction perpendicular to the surface of the earth. Suppose v_0 is its initial velocity. We want to calculate the height the projectile reaches.
Its weight at height x (from the surface of the earth) is given by,

$$w(x) = -\frac{mgR^2}{(R+x)^2},$$

where R is the radius of the earth.
Neglect force due to air resistance and other celestial bodies. Therefore, the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{mgR^2}{(R+x)^2};\ v(0) = v_0.$$

Separable ODE's

By chain rule,

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

Thus,

$$v \cdot \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE is separable. Linear or non-linear? (NL)
Separating the variables and integrating, we get:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

For $x = 0$, we get $\frac{v_0^2}{2} = gR + c$, hence, $c = \frac{v_0^2}{2} - gR$, and,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

Separable ODE's

Suppose the body reaches the maximum height H. Then $v = 0$ at this height.

$$v_0^2 - 2gR + \frac{2gR^2}{(R+H)} = 0.$$

Thus,

$$v_0^2 = 2gR - \frac{2gR^2}{R+H} = 2gR\left(\frac{H}{R+H}\right).$$

The escape velocity is found by taking limit as $H \to \infty$. Thus,

$$v_e = \sqrt{2gR} \sim 11 \text{ km/sec.}$$

Homogeneous ODE's

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous if for some $d \in \mathbb{Z}$

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for all $t \neq 0$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The number d is called the degree of $f(x_1, \ldots, x_n)$.

Examples:
$f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.
$f(x, y) = y + x \cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

Definition

The first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of the same degree.

Solving first order homogeneous ODE's

Consider

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of degree d. Put

$$y = xv.$$

Then,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Substituting this in the given ODE, we get:

$$M(x,xv) + N(x,xv)\left(x\frac{dv}{dx} + v\right) = 0.$$

Thus,

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Solving first order homogeneous ODE's Continued

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Let $x \neq 0$. Then,

$$M(1,v) + N(1,v)\cdot v + N(1,v)\cdot x\frac{dv}{dx} = 0.$$

Thus,

$$\frac{dx}{x} + \frac{N(1,v)}{M(1,v) + N(1,v)\cdot v}dv = 0.$$

This is a separable equation.
NOTE: The above method can be applied to any ODE which takes the form

$$y' = f(\frac{y}{x}).$$

Remark

We will later use the term “homogeneous” in a different context to describe DE’s of the form $Dy = 0$ (as opposed to $Dy = b(x)$) for a differential operator D. If you recall, for a linear system, you used the term in this sense, in MA 106.

Example

Example: Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Put $y = vx$. Thus, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)\left(v + x\frac{dv}{dx}\right) + 2x^2v = 0.$$

Thus, for $x \neq 0$,

$$(v^2 - 1)v + x(v^2 - 1)\frac{dv}{dx} + 2v = 0;$$

i.e.,

$$(v^3 + v) + x(v^2 - 1)\frac{dv}{dx} = 0.$$

Thus, we have the separable ODE:

$$\frac{v^2 - 1}{v(v^2 + 1)}dv + \frac{dx}{x} = 0.$$

Homogeneous ODE's

$$\frac{v^2-1}{v(v^2+1)}dv + \frac{dx}{x} = 0.$$

Integrating, we get:

$$\ln|x| + \int \left(\frac{2v}{v^2+1} - \frac{1}{v} \right) dv = 0.$$

Thus,

$$\ln|x| + \ln(v^2+1) - \ln|v| = c.$$

Hence,

$$\frac{x(v^2+1)}{v} = c,$$

or

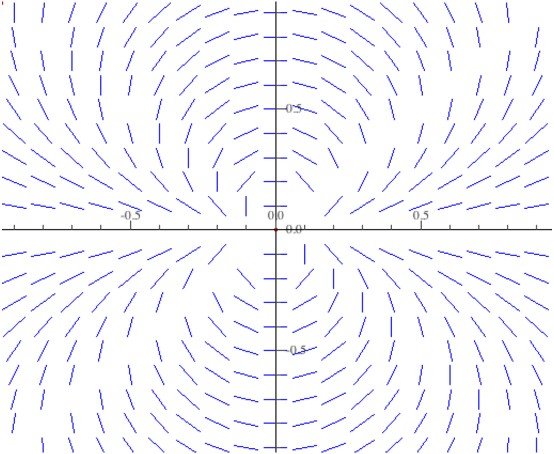
$$y^2 + x^2 = cy,$$

which is

$$x^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}.$$

Homogeneous ODE's

The direction field is $H(x,y) = (1, \frac{2xy}{x^2-y^2})$.



Check that the isoclines are pair of lines.

Recall

So far, we saw how to solve separable ODE's and ODE's which can be put into the form

$$y' = f(\frac{y}{x}).$$

- If the ODE is separable, just integration is enough to solve it.
- The homogeneous ODE's can be converted to separable ODE's by the substitution $y = vx$.

Now, we will look at another class of first order ODE's - exact ODE's.

Exact ODE's

Definition

Let $M(x,y)$ and $N(x,y)$ be defined and continuous for all $(x,y) \in R$, where R is an open rectangle in $\mathbb{R}^2$. A first order ODE $M(x,y) + N(x,y)y' = 0$ is called exact on the open rectangle R if there is a function $u := u(x,y)$ such that

$$\frac{\partial u}{\partial x}(x,y) = M(x,y) \;\&\; \frac{\partial u}{\partial y}(x,y) = N(x,y), \quad \forall\,(x,y) \in R.$$

Suppose that we can identify a function u such that

$$\frac{\partial u}{\partial x}(x,y) = M(x,y) \;\&\; \frac{\partial u}{\partial y}(x,y) = N(x,y)$$

and such that $u(x,y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x.

Exact ODE's

Then

$$M(x,y) + N(x,y)\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{d}{dx}u\left(x, \phi(x)\right)$$

and the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

becomes

$$\frac{d}{dx}u\left(x, \phi(x)\right) = 0.$$

Solutions are given implicitly by

$$u(x,y) = c$$

where c is an arbitrary constant.

Example

Example: Exact ODE:

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0.$$

Note $M(x, y) = 2x + y^2$ and $N(x, y) = 2xy$.

Consider the function

$$u(x, y) = x^2 + xy^2.$$

Note that

$$\frac{\partial u}{\partial x}(x, y) = M(x, y) = 2x + y^2, \quad \frac{\partial u}{\partial y}(x, y) = N(x, y) = 2xy.$$

Hence the ODE is exact and an implicit general solution is given by

$$x^2 + xy^2 = \text{constant}.$$

Qn. How to determine if an ODE is exact? Then how to obtain the function $u(x, y)$?

Closed Forms

Definition

let D be an open region of $\mathbb{R}^2$ and $M(x, y)$ and $N(x, y)$ be defined for all $(x, y) \in D$. The differential form

$$M(x, y)dx + N(x, y)dy$$

is called closed on D if $\frac{\partial M}{\partial y}(x, y)$ and $\frac{\partial N}{\partial x}(x, y)$ both exist for all $(x, y) \in D$ and

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y), \quad \forall\,(x, y) \in D$$

i.e., $M_y(x, y) = N_x(x, y), \quad \forall\,(x, y) \in D.$

Example. The differential form: $(2x + y^2)\,dx + 2xy\,dy$

Note $M(x, y) = 2x + y^2$ and $N(x, y) = 2xy$.
Note $M_y(x, y) = 2y = N_x(x, y)$.
The differential form is exact.

Closed Forms

Proposition

Let M, N and their first order partial derivatives exist and be continuous in a region $D \subseteq \mathbb{R}^2$.

(i) *If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then it is closed.*

(ii) *If D is convex, then any closed form is exact.*

Proof: (i) Let the differential form is exact, i.e., there exists a function $u(x, y)$ with continuous first order derivatives for all $(x, y) \in D$ such that $M(x, y) = \frac{\partial u}{\partial x}(x, y)$ and $N(x, y) = \frac{\partial u}{\partial y}(x, y)$. Then,

$$M_y(x, y) = \frac{\partial^2 u}{\partial y \partial x}(x, y) \ \& \ N_x(x, y) = \frac{\partial^2 u}{\partial x \partial y}(x, y).$$

By the theorem on mixed partials, (what's this?), $M_y = N_x$ and hence $M(x, y)dx + N(x, y)dy$ is closed. Recall from MA 111 that this proof is same as that of "curl of grad is zero".

Closed Forms

(ii) Now, let D be convex, and $Mdx + Ndy$ be a closed form. Consider the vector field

$$H(x, y) = (M(x, y), N(x, y)).$$

By our assumptions, H is continuously differentiable throughout D. What's its curl? The curl of H is given by

$$\nabla \times H = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\mathbf{k} = \mathbf{0}.$$

As D is convex, "curl free is grad"; i.e., there is a function $\phi(x, y)$ such that

$$H = \nabla\phi = (\phi_x, \phi_y).$$

Hence $\phi_x = M$, $\phi_y = N$ and thus $Mdx + Ndy$ is exact.

Question: "curl free is grad" is true on more general regions? What are they called? Examples? How did you prove this in MA 111?