MA 109, Week-2

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Recap

Definition: A sequence $\{a_n\}$ tends to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

A sequence that does not converge is said to diverge, or to be divergent.

Theorem : Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \le b_n \le c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Lemma: Every convergent sequence is bounded.

Theorem: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

The supremum or least upper bound (lub) of sequence $\{a_n\}$ is a real number M such that:

1. $a_n \le M$ for all n and
2. If M_1 is such that $a_n < M_1$ for all n, then $M \le M_1$.

Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Theorem: Every Cauchy sequence in $\mathbb{R}$ converges.

Supremum of a set

Let $A \subseteq \mathbb{R}$. We say M is the supremum or least upper bound of A if

1. $x \leq M$ for all $x \in A$ and
2. If M_1 is such that $x < M_1$ for all $x \in A$, then $M \leq M_1$,

and we write $\sup A = M$.
What is

1. $\sup\{1, 2, 3, \cdots, 2023\}$.
2. $\sup(0, 1]$.
3. $\sup(0, 1)$.
4. $\sup\left\{1 - \frac{1}{2^n} \mid n \in \mathbb{N}\right\}$.
5. $\sup\left\{1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, \cdots, 1 + \frac{1}{1!!} + \cdots + \frac{1}{n!}, \cdots\right\}$.

Supremum of a set

Answers

1. 2023
2. 1
3. 1
4. 1
5. e

Observation: Supremum of every subset of $\mathbb{Q}$ is not in $\mathbb{Q}$.

Theorem:(Least upper bound axiom) If a set of real numbers is bounded above, it has a supremum (or least upper bound). If a set of real numbers is bounded below it has an infimum (or greatest lower bound).

Properties of real numbers

Theorem
(Archimedean Property) For any positive $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$nx > 1$$

That is, for any positive real number, there exists a bigger natural number.

Theorem
Between any two distinct real numbers there is a rational number.

Few exercises

Exercise: Between any two distinct rational numbers there is an irrational number.
Exercise: Let β be an irrational number. There there exist a sequence of rational numbers $\{r_n\}$ such that $\{r_n\}$ converges to β.
Exercise: Let $A \subseteq \mathbb{R}$ be bounded above and let $\alpha = \sup A$. Then for any $\epsilon > 0$, there exists $a_\epsilon \in A$ such that $\alpha - \epsilon < a_\epsilon$.
Consequently there exists a sequence $\{a_n\}$ in A such that $\{a_n\}$ converges to $\alpha = \sup A$.

Series

Given a sequence a_n of real numbers, we can construct a new sequence, namely the sequence of partial sums s_n:

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \dots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

which is also called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

We say that the series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums $\{s_n\}$ converges.
For example, we can define $a_n = r^{n-1}$, for some $r \in \mathbb{R}$ and in this case we obtain a geometric progression $\sum_{k=0}^{\infty} r^k$ for which the n-th partial sum $s_n = \sum_{k=0}^{n-1} r^k$.

Infinite series - a more rigourous treatment

Let us recall what we mean when $|r| < 1$ and we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} ar^{k-1} = \sum_{k=0}^{n-1} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.

So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Theorem

If $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof.

Since $\sum_{n=0}^{\infty} a_n$ is convergent, so is the sequence $\{s_n\}$ of partial sums and hence the sequence $\{s_n\}$ is Cauchy. Therefore $a_n = s_{n+1} - s_n \to 0$ as $n \to \infty$. □

Corollary: If $|x| > 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Theorem (Comparison test)

Let $0 \le a_n \le b_n$ for all $n \ge k$ for some k. Then

1. *The convergence of $\sum b_n$ implies convergence of $\sum a_n$.*
2. *The divergence of $\sum a_n$ implies divergence of $\sum b_n$.*

Exercise:

1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.
2. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \le 1$.

Convergence test

Theorem (Ratio test)

Let $\sum_{n=0}^{\infty} a_n$ be a series of positive real numbers. Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$.

1. *If $\lambda < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.*

Theorem (Root test)

Let $\sum_{n=0}^{\infty} a_n$ be a series of positive real numbers. Let $\lim_{n\to\infty} a_n^{1/n} = \lambda$.

1. *If $\lambda < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.*

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - l_1)^2 + a(n)_2 - l_2)^2} < \epsilon$$

whenver $n > N$. A similar defintion can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$.

The Sandwich Theorem and the theorem about monotonic sequences don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

The completeness of other spaces

Theorem 4 (in Week-1), however, makes perfect sense - one can define Cauchy sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$ exactly as before, using the distance functions - and indeed, remains valid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

So $\mathbb{R}^2$ and $\mathbb{R}^3$ are complete sets too (but $\mathbb{Q}^2$ and $\mathbb{Q}^3$ are not). That is every Cauchy sequence in $\mathbb{R}^2$ and $\mathbb{R}^3$ is convergent but every Cauchy sequence in $\mathbb{Q}^2$ and $\mathbb{Q}^3$ may not converge in $\mathbb{Q}^2$ (or $\mathbb{Q}^3$).

Exercise: Prove that $\{a(n)_1, a(n)_2\}_n$ converges to (l_1, l_2) if and only if $\{a(n)_1\}$ converges to l_1 and $\{a(n)_2\}$ converges to l_2.

Definition of limit of a function

Since we have already defined the limit of a sequence rigourously, it will not be hard to define the limit of a real valued function $f : (a, b) \to \mathbb{R}$.

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0". This is just the rigourous way of saying that the

distance between $f(x)$ and l can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

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$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.

Negation: A function $f : (a, b) \to \mathbb{R}$ do not tend to a limit l at a point $x_0 \in [a, b]$ if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta \in (a, b)$ and $0 < |x_\delta - x_0| < \delta$ such that

$$|f(x_\delta) - l| \geq \epsilon_0.$$

A subtle point and the rules for limits

Notice that in the definition above, the point x_0 can be one of the end points a or b.

Thus the limit of a function may exist even if the function is not defined at that point.

The rules and formulæ for limits of functions are the same as those for sequence and can be proved in almost exactly the same way. If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, then

1. $\lim_{x \to x_0} f(x) \pm g(x) = l_1 \pm l_2$.
2. $\lim_{x \to x_0} f(x)g(x) = l_1 l_2$.
3. $\lim_{x \to x_0} f(x)/g(x) = l_1/l_2$. provided $l_2 \neq 0$

As before, implicit in the formulæ is the fact that the limits on the left hand side exist. We prove the first rule below.

The proof of the addition formula for limits

Proof: We first show that $\lim_{x\to x_0} f(x)+g(x) = l_1 + l_2$. Let $\epsilon > 0$ be arbitrary.
Since $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, there exist δ_1, δ_2 such that

$$|f(x) - l_1| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - l_2| < \frac{\epsilon}{2}$$

whenever $|x - x_0| < \delta_1$ and $|x - x_0| < \delta_2$. If we choose $\delta = \min\{\delta_1, \delta_2\}$ and if $|x - x_0| < \delta$ then both the above inequalities hold. Thus, if $|x - x_0| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - (l_1 + l_2)| &= |f(x) - l_1 + g(x) - l_2| \\ &\leq |f(x) - l_1| + |g(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is what we needed to prove. If we replace $g(x)$ by $-g(x)$ we get the second part of the first rule. □

The Sandwich Theorem(s) for limits of functions

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that $f(x) \le g(x) \le h(x)$ for all $x \in (a, b)$, then

$$l_1 \le l_2 \le l_3.$$

As before, we have a second version.

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \le g(x) \le h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x \to x_0} g(x) = l$$

Once again, note that we do not assume that $g(x)$ converges to a limit in this version of the theorem - we get the convergence of $g(x)$ for free.

Some examples

Let us look at Exercise 1.11. We will use this exercise to explore a few subtle points.

Let $c \in [a, b]$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x \to c} f(x) = 0$. Prove or disprove the following statements.

(i) $\lim_{x \to c}[f(x)g(x)] = 0$.

(ii) $\lim_{x \to c}[f(x)g(x)] = 0$, if g is bounded.($g(x)$ is said to be bounded on (a, b) if there exists $M > 0$ such that $|g(x)| < M$ for all $x \in (a, b)$.

(iii) $\lim_{x \to c}[f(x)g(x)] = 0$, if $\lim_{x \to c} g(x)$ exists.

Before getting into proofs, let us guess whether the statements above are true or false.

(i) false

(ii) true

(iii) true.

(i) Notice that $g(x)$ is not given to be bounded - if this was not obvious before, you should suspect that such a condition is needed after looking at part (ii). So the most natural thing to do is to look for a counter-example to this statement by taking $g(x)$ to be an unbounded function. What is the simplest example of an unbounded function $g(x)$ on an open interval?

How about $g(x) = \frac{1}{x}$ on $(0, 1)$?

What would a candidate for $f(x)$ be - what is the simplest example of a function $f(x)$ which tends to 0 for some value of c in $[0, 1]$.

$f(x) = x$, and $c = 0$ is a pretty simple candidate.

Clearly $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1 \neq 0$, which shows that (i) is not true in general.

Exercise 1: Can you find a counter-example to (i) with c in (a, b) (that is, c should not be one of the end points)? (Hint: Can you find an unbounded function on a closed interval $[a, b]$?)

Let us move to part (ii).

Suppose $g(x)$ is bounded on (a, b). This means that there is some real number $M > 0$ such that $|g(x)| < M$. Let $\epsilon > 0$. We would like to show that

$$|f(x)g(x) - 0| = |f(x)g(x)| < \epsilon,$$

if $|x - c| < \delta$ for some $\delta > 0$.

Since $\lim_{x\to c} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon/M$ for all $|x - c| < \delta$. It follows that

$$|f(x)g(x)| = |f(x)||g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $|x - c| < \delta$, and this is what we had to show.

Part (iii) follows immediately from the product rule, but can one deduce part (iii) from (ii) instead?
Hint: Think back to the lemma on convergent sequences that we proved: Every convergent sequence is bounded. What is the analogue for functions which converge to a limit at some point? Indeed, you can easily show the following

Lemma 7: Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x \to c} f(x)$ exists for some $c \in [a, b]$. If $c \in (a, b)$, there exists an (open) interval $I = (c - \eta, c + \eta) \subset (a, b)$ such that $f(x)$ is bounded on I. If $c = a$, then there is a half-open interval $I_1 = (a, a + \eta)$ such that $f(x)$ is bounded on I_1. Similarly if $c = b$, there exists a half-open interval $I_2 = (b - \eta, b)$ such that $f(x)$ is bounded on I_2.

The proof of the lemma above is almost the same as the the lemma for convergent sequences. Basically, replace "N" by "δ" in the proof.

If one applies the Lemma above to $g(x)$, we see that $g(x)$ is bounded in some (possibly) smaller interval $(0, \eta)$. Now apply part (ii) to this interval to deduce that (iii) is true.

Limits at infinity

There is one further case of limits that we need to consider. This occurs when we consider functions defined on open intervals of the form $(-\infty, b)$, (a, ∞) or $(-\infty, \infty) = \mathbb{R}$ and we wish to define limits as the variable goes to plus or minus infinity. The definition here is very similar to the definition we gave for sequences. Let us consider the last case.

Definition: We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that

$$|f(x) - l| < \epsilon,$$

whenever $x > X$ (resp. $x < X$), and we write

$$\lim_{x \to \infty} f(x) = l \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l.$$

or, alternatively, $f(x) \to l$ as $x \to \infty$ or as $x \to -\infty$, depending on which case we are considering.

Limits from the left and right

If $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$, then it is possible to approach c from either the left or the right on the real line.

We can define the limit of the function $f(x)$ as x approaches c from the left (if it exists) as a number l^- such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l^-| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (a, c)$.

Our notation for this is $\lim_{x \to c-} f(x) = l^-$, and it is also called the left hand (side) limit.

Exercise 2: Write down a definition for the limit of a function from the right. We usually denote the right hand (side) limit by $lim_{x \to c+} f(x)$. Show, using the definitions, that $\lim_{x \to c} f(x)$ exists if and only if the left hand and right hand limits both exist and are equal.

We can also think of the left hand limit as follows. We restrict our attention to the interval (a, c), that is we think of f as a function only on this interval. Call this restricted function f_a. Then, another way of defining the left hand limit is

$$\lim_{x \to c-} f(x) = \lim_{x \to c} f_a(x).$$

It should be easy to see that it is the same as the definition before One can make a similar definition for the right hand limit.

The notions of left and right hand limits are useful because sometimes a function is defined in different ways to the left and right of a particular point. For instance, $|x|$ has different definitions to the left and right of 0.

Calculating limits explicitly

As with sequences, using the rules for limits of functions together with the Sandwich theorem allows one to treat the limits of a large number of expressions once one knows a few basic ones:

(i) $\lim_{x\to 0} x^\alpha = 0$ if $\alpha > 0$, (ii) $\lim_{x\to\infty} x^\alpha = 0$ if $\alpha < 0$,
(iii) $\lim_{x\to 0} \sin x = 0$, (iv) $\lim_{x\to 0} \sin x/x = 1$
(v) $\lim_{x\to 0}(e^x - 1)/x = 1$, (vi) $\lim_{x\to 0} \ln(1+x)/x = 1$
Exercise 3: Find

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

I will give the solution next time, together with the history of the problem (if I mention the history right away you will be able to get the solution by googling!), but feel free to use any method you like.

Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is dicontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

Continuity of familiar functions

Example: Polynomials are continuous functions. More generally rational functions, that is functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, with $Q(x) \neq 0$ are continuous functions.

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is non-zero.

Continuity of other familiar functions

What are the other (continuous) functions we know?
Exercise: Prove that $\sin x$ is a continuous function on $\mathbb{R}$. (Use $\epsilon - \delta$ definition).
How about the other trignometric functions? What we can say about $\cos x$?

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$!).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trignometric functions, that is functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.

The continuity of the square root function

Thus in order to prove the continuity of $\cos x$ (assuming the continuity of $\sin x$) we need only prove the continuity of the square root function.

The main observation is that continuity is a local property, that is, only the behaviour of the function near the point being investigated is important.

Let $x_0 \in [0, \infty)$. To show that the square root function is continuous at x_0 we need to show that $\lim_{y \to x_0} \sqrt{y} = \sqrt{x_0}$, that is we need to show that $|\sqrt{y} - \sqrt{x_0}| < \epsilon$ whenever $|y - x_0| < \delta$ for some δ. First assume that $x_0 \neq 0$. Then

$$|\sqrt{y} - \sqrt{x_0}| = \left| \frac{y - x_0}{\sqrt{y} + \sqrt{x_0}} \right| \leq \frac{|y - x_0|}{\sqrt{x_0}}.$$

If we choose $\delta = \epsilon\sqrt{x_0}$, we see that

$$|\sqrt{y} - \sqrt{x_0}| < \epsilon,$$

which is what we needed to prove. When $x_0 = 0$, I leave the proof as an exercise. $\square$

The intermediate value theorem

One of the most important properties of continuous functions is the Intermediate Value Property (IVP). We will use this property repeatedly to prove other results.

Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property. Theorem 9 can thus be restated as saying that continuous functions have the IVP.

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut every line $y = u$ with u between $f(a)$ and $f(b)$.

The IVT in a picture

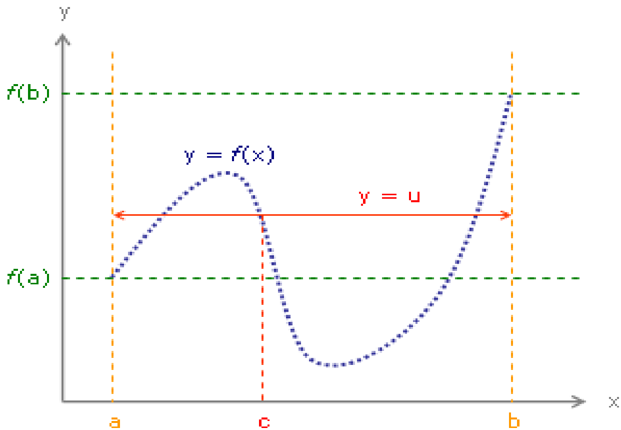

Image created by Enoch Lau see
http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png

Zeros of functions

One of the most useful applications of the intermediate value property is to find roots of polynomials, or, more generally, to find zeros of continuous functions, that is to find points $x \in \mathbb{R}$ such that $f(x) = 0$.

Theorem 10: Every polynomial of odd degree has at least one real root.

Proof: Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial of odd degree. We can assume without loss of generality that $a_n > 0$. It is easy to see that if we take $x = b > 0$ large enough, $P(b)$ will be positive. On the other hand, by taking $x = a < 0$ small enough, we can ensure that $P(a) < 0$. Since $P(x)$ is continuous, it has the IVP, so there must be a point $x_0 \in (a, b)$ such that $P(x_0) = 0$. $\square$

The IVP can often be used to get more specific information. For instance, it is not hard to see that the polynomial $x^4 - 2x^3 + x^2 + x - 3$ has a root that lies between 1 and 2.

Continuous functions on closed, bounded intervals

The other major result on continuous functions that we need is the following. A closed bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

Again, we will not prove this, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0, 1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1, \infty)$ does not attain its minimum, although, it is bounded below.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a, b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a, b)$?

The function $\sin \frac{1}{x}$

Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. The question asks if this function is continuous at $x = 0$. How about $x \neq 0$? Why is $f(x)$ continuous? Because it is a compostion of the sin function and a rational function $1/x$. Since both of these are continuous when $x \neq 0$, so is $f(x)$.

Let us look at the sequence of points $x_n = 2/n\pi$. Clearly $x_n \to 0$ as $n \to \infty$.
For these points $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$. But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0. The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.

Sequential continuity

The preceeding example showed that in order to demonstrate that a function, say $f(x)$, is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large. Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? The following theorem answers the question affirmatively.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$.

Proof of sequential criteria for continuity

Proof: $(\Rightarrow)$. For a given $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Since the sequence (x_n) converges to a, for the above δ, $\exists N \in \mathbb{N}$ such that $|x_n - a| < \delta$ whenever $n \geq N$.

Hence $|f(x_n) - f(a)| < \epsilon$ whenever $n \geq N$.

$(\Leftarrow)$. We will show this part by using the method of contradiction.

For, let if possible the function f is not continuous at a, that is, there exists $\epsilon > 0$ such that $\forall\ \delta > 0$, $\exists\ x_\delta \in I$ with $|x_\delta - a| < \delta$ and $|f(x_\delta) - f(a)| \geq \epsilon$.

Now we find a sequence (x_n) converging to a, for which, $(f(x_n))$ does not converge to $f(a)$.

Proof of sequential criterion for continuity

For $n \in \mathbb{N}$ and for the same ϵ, if we take $\delta_n = \frac{1}{n}$ then $\exists\ x_n \in I$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \epsilon$.

It is now clear that the sequence (x_n) converges to a but for the ϵ above, there does not exist $N \in \mathbb{N}$ such that $|f(x_n) - f(x)| < \epsilon$, whenever $n \geq N$, that is, the sequence $(f(x_n))$ does not converge to $f(a)$ which is a contradiction.

Hence the function f is continuous. $\square$

Limits of functions of several variables

Just like we did for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can define the notion of the limit of a function for functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The function $f(x_1, x_2)$ is said to tend to a limit l as $(x_1, x_2) \to (a_1, a_2)$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_1, x_2) - l| < \epsilon$$

whenever $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$. Notice, that one can now approach the point (a_1, a_2) from any direction in the plane. Our definition requires that the limits from the different directions all exist and be equal. This is quite a powerful condition.

If we have functions from $\mathbb{R}^2 \to \mathbb{R}^2$ we can make exactly the same definition. But this time $l = (l_1, l_2)$ will be in $\mathbb{R}^2$ and so will $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, so we will have to replace the modulus function by the distance between these two quantities:

$$\sqrt{[f_1(x_1, x_2) - l_1]^2 + [f_2(x_1, x_2) - l_2]^2}.$$

Limits of functions of several variables

The definitions we have made go through for functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, where m and n may be different. For instance, we have considered the case when $m = 2$ and $n = 1$ and also the case $m = 2$ and $n = 2$ above. But we could allow m and n to take any of the values 1, 2 or 3 (in fact, we can allow values greater than 3 as well!).

Exercise 1: Show that

$$\lim_{y \to x} f(y) = l$$

for $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \iff$

$$\lim_{y \to x} f_1(y) = l_1 \quad \text{and} \quad \lim_{y \to x} f_2(y) = l_2,$$

where $l = (l_1, l_2)$ In other words, when dealing with limits of functions which are vector-valued, it is enough to study the limits of the coordinate functions.

Continuous functions of several variables

Once the definition of the limit is clear it makes sense to talk of continuity as well. All the definitions remain the same, only the definition of the distance function changes depending on the domain and the range.

For instance, provided we know what "closed and bounded sets" are in $\mathbb{R}^2$ or $\mathbb{R}^3$, Theorem 11 goes through for continuous functions $f : \mathbb{R}^m \to \mathbb{R}$. $(m = 2, 3)$. For functions with more than one variable in the range the first part of Theorem 11 still works, but for the second part things are more compacted (again there is no "ordering" in $\mathbb{R}^2$ or $\mathbb{R}^3$).

While it is easy to see what a bounded set in $\mathbb{R}^m$ should be, closed is a little more complicated and we will not give the definition here. However, a rectangle of the form $[a, b] \times [c, d]$ in $\mathbb{R}^2$ is an example of a closed and bounded set (also called "compact sets" of this form).
Theorem 12 goes through without any problems even when the range is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

The defintion

For now, if you did not understand the rigourous definition of the limit, forget about it. You will be able to understand what follows as long as you remember your 11th standard treatment of limits. Recall that $f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c. The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y = f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x-coordinate, then $x'(t)$ is the velocity of the particle. If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle. If the function we are studying is the population of India, then the derivative measures the rate of change of the population.

The slope of the tangent

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{y \to c} \frac{f(y) - f(c)}{y - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(y, f(y))$, and as y approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Another way of thinking of the derivative

Another way of thinking of the derivative of the function f at the point x_0 is as follows. If f is differentiable at x_0 we know that

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \to 0$$

as $h \to 0$. Since we are keeping x_0 fixed, we can treat the above quantity as a function of h. Thus we can write

$$\frac{f(x_0 + h) - f(x)}{h} - f'(x_0) = o(h)$$

for some function $o(h)$ with the property that $o(h) \to 0$ as $h \to 0$. Taking a common denominator,

$$\frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{h} = o(h) \tag{1}$$

We can use the above equality to give an equivalent definition for the derivative. A function f is said to be differentiable at the point x_0 if there exists a real number (denoted $f'(x_0)$) such that (1) holds for some function $o(h)$ such that $o(h) \to 0$ as $h \to 0$.

The derivative as a linear map

We can rewrite equation (1) as

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)h$$

Thus, the derivative of $f(x)$ at a point x_0 can be viewed as that real number (if it exists) by which you have to multiply h so that the resulting remainder goes to 0 faster than h (that is, the remainder divided by h goes to 0 as h goes to 0).

A function $f : \mathbb{R} \to \mathbb{R}$ which has the property that $f(x + y) = f(x) + f(y)$ is called a linear function (or linear map). All such functions are given by multiplication by a real number, that is, every linear function has the form $f(x) = \lambda x$, for some real number λ. Thus the derivative may be regarded as a linear function (in the variable h). This point of view will be particularly useful in multivariable calculus.

Calculating derivatives

As with limits all of you are already familiar with the rule for calculating the sums, differences, products and quotients of derivatives. You should try and remember how to prove these.

You should also recall the chain rule for calculating the derivative of the composition of functions and try to prove it as an exercise using the $\epsilon - \delta$ definition of a limit.

Note that the proof of the chain rule given in some books involves writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}.$$

and then taking limits as $\Delta x \to 0$. This is not quite correct since Δu could be 0 even for infinitely many values of u.

Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Once again, I remind you that, in general, f may not attain a maximum or minimum at all on the set X. The standard example being $X = (0, 1)$ and $f(x) = 1/x$ (can you find an example on the closed interval $[0, 1]$?).

However, if X is a closed bounded interval and f is a continuous function Theorem 11 tells us that the maximum and minimum are actually attained. Theorem 11 is sometimes called the Extereme Value Theorem.

Maxima and minima and the derivative

If f has a maximum at the point x_0 and if it is also differentiable at x_0, we can reason as follows. We know that $f(x_0 + h) - f(x_0) \leq 0$ for every $h > 0$ such that $x + h \in X$. Hence, we see that (one half of the Sandwich Theorem!)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

On the other hand, when $h < 0$, we get

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because f is assumed to be differentiable at x_0 we know that left and right hand limits must be equal. It follows that we must have $f'(x_0) = 0$. A similar argument shows that $f'(x_0) = 0$ if f has a minimum at the point x_0.

Local maxima and minima

The preceeding argument is purely local. Before explaining what this means, we give the following defintion.

Definition: Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is a sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Sometimes we use the terms global maximum or global minimum instead of just maximum or minimum in order to emphasize the points are not just local maxima or minima. The argument of the previous slide actually proves the following

Theorem 13: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X, f'(x_0) = 0$.