MULTIPLE INTEGRALS

LET $K = [a,b] \times [c,d]$, AND SUPPOSE

$f : K \to \mathbb{R}$ IS A BOUNDED FUNCTION.

A PARTITION OF K IS A PAIR OF SEQUENCES

$$a = x_0 < x_1 < \cdots < x_n = b$$

AND

$$c = y_0 < y_1 < \cdots < y_m = d$$

THE NORM OF A PARTITION $\mathcal{P}$ IS DEFINED

AS

$$\|\mathcal{P}\| = \max_{i,j} \{ x_i - x_{i-1}, \, y_j - y_{j-1} \}$$

LET $A_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

AND SUPPOSE $(u_{ij}, v_{ij}) \in A_{ij}$. DEFINE

$$S(\mathcal{P}, f) := \sum_{i,j}{}' f(u_{ij}, v_{ij}) \, |A_{ij}| \quad \text{(RIEMANN SUMS)}$$

WHERE

$$|A_{ij}| = (x_i - x_{i-1})(y_j - y_{j-1})$$

$$= \text{AREA OF THE RECTANGLE } A_{ij}.$$

WE SAY f IS INTEGRABLE OVER K IF

$\lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, f)$ EXISTS, AND WE WRITE

$$\iint_K f(x,y) \, dx \, dy = \lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, f)$$

PROPERTIES OF $\iint$

- f IS CONTINUOUS EXCEPT AT FINITELY MANY POINTS IN $K \Rightarrow \iint_K f\, dA$ EXISTS. ($dA = dx\, dy$)

- IF f IS INTEGRABLE ON K AND $K = K_1 \cup K_2$ WHERE K_1, K_2 ARE NONOVERLAPPING, THEN

$$\iint_K f(x,y)\, dA = \iint_{K_1} f(x,y)\, dA + \iint_{K_2} f(x,y)\, dA$$

- g, f IS INTEGRABLE ON $K \Rightarrow \alpha f$, $f \pm g$, fg, $|f|$ ARE INTEGRABLE AND

$$\iint_K (f+g)\, dA = \iint_K f\, dA + \iint_K g\, dA$$

$$\iint_K \alpha f\, dA = \alpha \iint_K f\, dA$$

$$\left| \iint_K f\, dA \right| \leq \iint_K |f|\, dA$$

- f IS INTEGRABLE ON K AND $f(x,y) \geq 0$ $\forall (x,y) \in K \Rightarrow \iint_K f(x,y)\, dx\, dy \geq 0$

FUBINI'S THEOREM

SUPPOSE $f:[a,b]\times[c,d]\to\mathbb{R}$ IS INTEGRABLE.

FOR EACH $x\in[a,b]$ (FIXED), THE FUNCTION $y\mapsto f(x,y)$ IS RIEMANN INTEGRABLE ON $[c,d]$

LET $A(x)=\int_c^d f(x,y)\,dy$, SIMILARLY,

LET $B(y)=\int_a^b f(x,y)\,dx$ FOR EACH FIXED y.

THEN

$$\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_a^b A(x)\,dx=\iint_K f(x,y)\,dx\,dy$$

$$=\int_c^d B(y)\,dy=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

THE INTEGRALS $\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$ AND $\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$ ARE CALLED

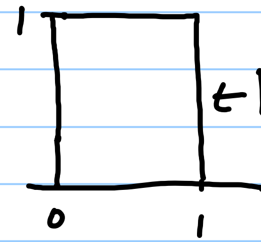
ITERATED INTEGRALS.

EXAMPLE

$$\iint_K (x^2+2y)\,dx\,dy \quad , \quad K=[0,1]\times[0,1].$$

SINCE x^2+2y IS CONTINUOUS, IT IS INTEGRABLE,

SO FUBINI'S THEOREM APPLIES.



FIX $0 \le x \le 1$,

$$A(x) = \int_0^1 (x^2+2y)\,dy$$

$$= x^2+1$$

$$\iint_K = \int_0^1 A(x)\,dx = \int_0^1 (x^2+1)\,dx = \left.\frac{x^3}{3}+x\right|_0^1$$

$$= \frac{4}{3}$$

$$B(y) = \int_0^1 (x^2+2y)\,dx = \frac{1}{3}+2y$$

These are NOT the same function!

So, $$\int_0^1 B(y)\,dy = \int_0^1 \left(\frac{1}{3}+2y\right)dy = \left.\frac{1}{3}y+y^2\right|_0^1$$

$$= \frac{4}{3}$$

FOR VOL. OF THE (UPPER) HEMISPHERE, WE WISH TO 'INTEGRATE' $f(x,y)=\sqrt{1-x^2-y^2}$ (RADIUS 1), OVER THE DOMAIN $K=\{(x,y) \mid x^2+y^2 \le 1\}$

GENERAL DOMAINS

SUPPOSE $K = \dot{\bigcup}_{i=1}^{n} K_i$ WHERE $K_i = [a_i, b_i] \times [c_i, d_i]$

WE DEFINE ($\cdot$ IN $\dot{\bigcup}$ MEANS NON OVERLAPPING)

$$\iint_K f(x,y)\,dx\,dy = \sum_{i=1}^{n} \iint_{K_i} f(x,y)\,dx\,dy$$

PROVIDED EACH INTEGRAL OF RHS EXISTS.

SUPPOSE $K \subseteq R := [a,b] \times [c,d]$ AND SUPPOSE

$f : [a,b] \times [c,d] \to \mathbb{R}$ IS INTEGRABLE.

CONSIDER $f^*(x,y) = f(x,y)$ IF $(x,y) \in K$
$= 0$ OTHERWISE.

IF f^* IS INTEGRABLE ON R, THEN DEFINE

$$\iint_K f(x,y)\,dx\,dy = \iint_R f^*(x,y)\,dx\,dy.$$

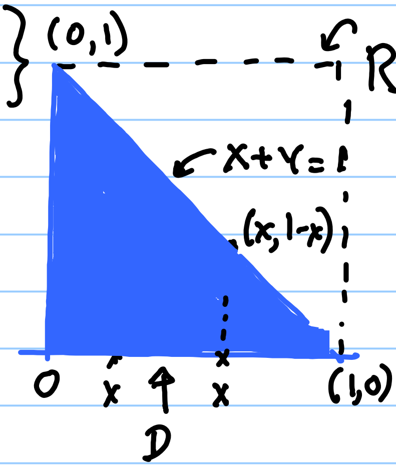
IF $f : R \to \mathbb{R}$ IS CONTINUOUS, THEN, f^* IS INTEGRABLE SO FOR ANY BOUNDED $K \subseteq \mathbb{R}^2$,

$\iint_K f(x,y)\,dA$ EXISTS FOR f CONTINUOUS.

(IT IS NOT NECESSARY FOR f TO BE CONTINUOUS FOR $\iint_K f(x,y)\,dx\,dy$ TO EXIST.)

EXAMPLE

$D = \{(x,y) \mid x \geq 0, y \geq 0, x+y \leq 1\}$



LET $f(x,y) = x^2+y^2$.

LET $R = [0,1] \times [0,1]$

$$f^*(x,y) = x^2+y^2 \quad \text{IF } x+y \leq 1$$
$$= 0 \quad \text{IF } x+y > 1$$

FUBINI $\Rightarrow$

$$A^*(x) = \int_0^1 f^*(x,y)\,dy = \int_0^{1-x} (x^2+y^2)\,dy$$
$$= x^2(1-x) + \frac{(1-x)^3}{3}$$

FUBINI $\Rightarrow$

$$\iint_K f(x,y)\,dA \overset{\text{Def}}{=} \iint_{[0,1]\times[0,1]} f^*(x,y)\,dA$$
$$= \int_0^1 A^*(x)\,dx = \int_0^1 x^2(1-x) + \frac{(1-x)^3}{3}$$

THIS IS NOW A 1-VARIABLE INTEGRAL

WHOSE EVALUATION IS ROUTINE. (EXERCISE)

🚩 THE DEFINITION OF $\iint_D f(x,y)\,dx\,dy$ DOES <u>NOT</u>

DEPEND ON THE ENCLOSING RECTANGLE R.

SPECIAL NON-RECTANGULAR REGIONS

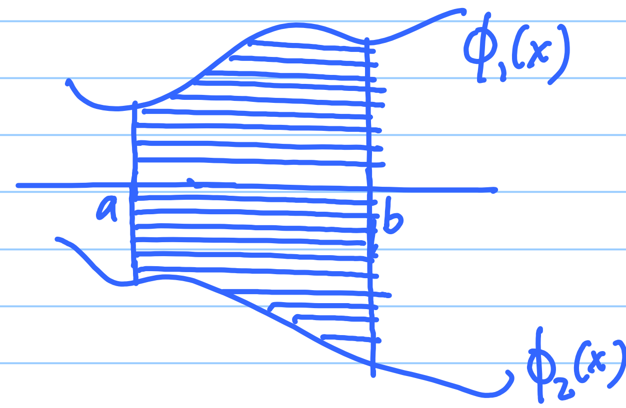
$D \subseteq \mathbb{R}^2$ IS CALLED TYPE-I-ELEMENTARY

IF

$$D = \{ (x,y) \mid a \leq x \leq b,\ \phi_2(x) \leq y \leq \phi_1(x) \}$$

WHERE

$\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ ARE CONTINUOUS.

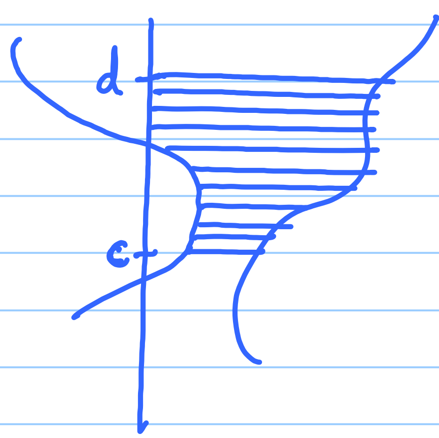


$D \subseteq \mathbb{R}^2$ IS CALLED TYPE-II ELEMENTARY

IF

$$D = \{ (x,y) \mid c \leq y \leq d,\ \psi_1(x) \leq y \leq \psi_2(x) \}$$

WHERE $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ ARE CTS.



FUBINI FOR TYPE I, II DOMAINS

LET $D \subseteq \mathbb{R}^2$ BE CLOSED BOUNDED IN $\mathbb{R}^2$, AND

$f: D \to \mathbb{R}$ IS INTEGRABLE.

🚩 IF D IS A TYPE-I ELEMENTARY REGION,

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$$

WHERE $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ ARE CONTINUOUS.

THEN

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right] dx$$

🚩 IF $D = \{(x,y) \mid c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$

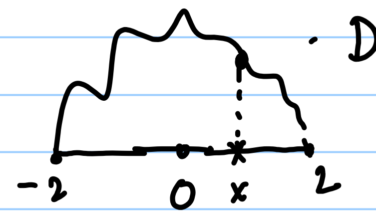
(TYPE-II ELEMENTARY REGION) THEN

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy$$

EXAMPLE 1

$f(x,y) = y$, $D = \{(x,y) \mid y \geq 0, \boxed{x^2+2y^2 \leq 4}\}$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 \leq 1$



THIS IS A TYPE I ELEMENTARY REGION.

FUBINI $\Rightarrow$

$$\iint_D y \, dA = \int_{-2}^{2} \left(\int_{0}^{\sqrt{\frac{4-x^2}{2}}} y \, dy \right) dx$$

$$= \int_{-2}^{2} \frac{4-x^2}{4} \, dx = 4 - \frac{4}{3}$$

$$= \frac{8}{3}.$$

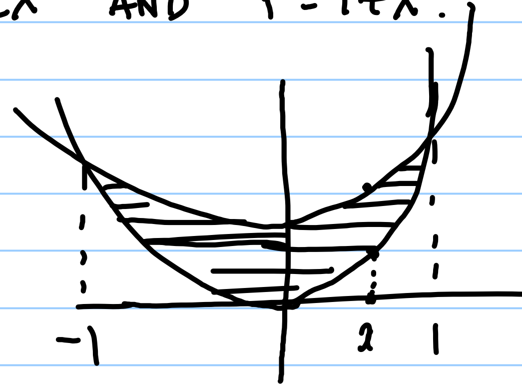
EXAMPLE 2

$\iint_D (x+2y)\, dA$; D = REGION BOUNDED BY

$Y = 2X^2$ AND $Y = 1 + X^2$.

$2x^2 = 1 + x^2$

$\Leftrightarrow \quad x = \pm 1$



TYPE I ELEMENTARY.

So, WE MAY USE FUBINI'S THEOREM.

FUBINI $\Rightarrow$ $\iint_D (x+2y)\, dA$

$$= \int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} (x+2y)\, dy \right) dx$$

$$= \int_{-1}^{1} \left\{ \left[xy + y^2 \right] \Big|_{2x^2}^{(1+x^2)} \right\} dx$$

$$= \int_{-1}^{1} \left[x(1-x^2) + (1+3x^2)(1-x^2) \right] dx$$