MA 108-ODE- D3

Lecture 1

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Welcome

Lecture Hours	Monday 8:30-9:25 (1A),
	Tuesday 9:30-10:25 (1B),
	Thursday 10:35-11:30 (1C)
Venue	LA 202
Tutorial Hour	Wednesday 16:00-17:00 (X3)
Course website	http://moodle.iitb.ac.in

1. Please use the doubt forum on moodle to ask questions.
2. Attendance in the lectures and tutorials is strongly encouraged.

Academic Honesty
Any form of academic dishonesty, including, but not limited to cheating, will not be tolerated, and will invite the harshest possible penalties as per institute norms.

Basic Concepts

Let y be a function defined from I to $\mathbb{R}$, where I is a subset of $\mathbb{R}$ or $I = \mathbb{R}$, i.e., $x \mapsto y(x) \in \mathbb{R}, \quad \forall x \in I$. Here x is the independent variable and $y(\cdot)$ is the dependent variable.

An ordinary differential equation is an equation containing the derivatives of an unknown function y, the unknown function y itself, and known functions of x including constants.

Notation. The n-th order derivative of y with respect to the independent variable x, $\frac{d^n}{dx^n}y(x)$ will be denoted by $y^{(n)}(x)$.

In other words, an ODE is a relation between the derivatives $y, y^{(1)}, \ldots, y^{(n)}$ and functions of x:

$$F(x, y, y^{(1)}, \ldots, y^{(n)}) = 0.$$

DE's occur naturally in physics, engineering and so on.

Can you give some obvious examples? Velocity and acceleration being derivatives, often give rise to DE's.

Order of an ODE

Definition
The order of an ODE is n (a natural number) if the nth derivative of the unknown function y is the highest derivative of y in the equation.

Examples :

1. $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ (ODE, 2nd order)
2. $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$ (ODE, 4th order)

Examples

Example 1: A falling object.
A body of mass m falls under the force of gravity. The drag force due to air resistance is $c \cdot v^2$ where v is the velocity and c is a constant. Then

$$m\frac{dv}{dt} = mg - c \cdot v^2.$$

An ODE of first order.

Examples

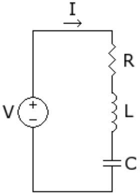
Example 2: Radioactive decay.
A radioactive substance decomposes at a rate proportional to the amount present. Let $y(t)$ be the amount present at time t. Then

$$\frac{dy}{dt} = k \cdot y$$

where k is a physical constant whose value is found by experiments ($-k$ is called the decay constant). ODE of first order.

Examples

Example 3: Electrical circuits.
Consider a basic RLC circuit:
resistance of the resistor - R ohms,
inductance of the inductor - L henrys,
capacitance of the capacitor - C farads.
These are wired and connected to an electromotive force $V(t)$ volts.



Let $Q(t)$ (coulombs) be the total charge in the capacitor at time t.

$$I(t) = \frac{dQ}{dt} = \text{ current.}$$

By Kirchhoff's voltage law, $L\frac{dI}{dt} + RI + \frac{Q(t)}{C} = V(t)$, i.e.,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C} \cdot Q = V(t).$$

ODE of second order.

Linear equations

Definition (Linear ODE)

The ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ is called linear if F is a linear function of the variables $y, y', \ldots, y^{(n)}$. A linear ODE of order n is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x)$$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Check list : If the dependent variable is y, no products of y and/or its derivatives are there.

Examples :

1. $y'' + 5y' + 6y = 0$ Ans. 2nd order, linear
2. $y^{(4)} + x^2y^{(3)} + x^3y' = xe^x$ Ans. 4th order, linear
3. $y'' + 5(y')^3 + 6y = 0$ Ans. 2nd order, non-linear.
4. $y'(t) = y^2(t)$ Ans. 1st order, non-linear.
5. $y'(t) = t\sin(y(t))$ Ans. 1st order, non-linear.

Can we solve it?

Given an equation, you would like to solve it.

Questions:

1. What is a solution?
2. Does an equation always have a solution? If so, how many?
3. Can the solutions be expressed in a 'nice form' (representation formula)? If not, how to get a feel for it?
4. How much can we proceed in a systematic manner?

order - first, second, ..., n^{th}, ...
linear or non-linear?

What is a solution?

Consider a n-th order ODE: $F(x, y, y', \dots, y^{(n)}) = 0$.

Definition (Solution of a n-th order ODE)

A solution of the n-th order ODE $F(x, y, y', \dots, y^{(n)}) = 0$ on an interval (α, β) is a real-valued function ϕ defined on the interval (α, β) such that all n-derivatives of ϕ, i.e., $\phi'(\cdot)$, $\phi''(\cdot), \cdots, \phi^{(n)}(\cdot)$ exist on the interval (α, β) satisfying

$$F(x, \phi(x), \phi'(x), \cdots, \phi^{(n)}(x)) = 0, \quad \forall\, \alpha < x < \beta.$$

Solution curve: If $\phi(\cdot)$ is a solution of the ODE on an interval (α, β), then the solution curve is given by the set in xy plane

$$\{(x, \phi(x)) \mid \alpha < x < \beta\}.$$

Examples

Example 5: Given an amount of a radioactive substance, say 1 gm, find the amount present at any later time.
The relevant ODE is

$$\frac{dy}{dx} = k \cdot y.$$

By inspection, $y(x) = ce^{kx}$, for an arbitrary constant c, is a solution of the above ODE.
Now, initial amount given is 1 gm at time $x = 0$, i.e.,

$$y(0) = 1.$$

The initial condition determines $c = 1$. Hence

$$y(x) = e^{kx}$$

is a particular solution to the above ODE with the given initial condition.

Examples

Example 6: Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $-\frac{y}{x}$.
The relevant ODE is

$$\frac{dy}{dx} = -\frac{y}{x}.$$

By inspection,

$$y(x) = \frac{c}{x}$$

is its general solution for an arbitrary constant c; i.e., a family of hyperbolas.
The initial condition given is

$$y(1) = 1,$$

which implies $c = 1$. Hence a particular solution for the above problem is

$$y(x) = \frac{1}{x}.$$

Geometric Meaning of Solutions of $\frac{dy}{dx} = f(x, y)$

Consider the first order ODE

$$\frac{dy}{dx} = f(x, y).$$

Suppose that $f(x, y)$ is defined in a region $D \subseteq \mathbb{R}^2$. If $y = \phi(x)$ is a solution curve and (x_0, y_0) is a point on it, then the slope of the tangent line to the curve at (x_0, y_0) is $f(x_0, y_0)$.
At each point $(a, b) \in D$, assign a unit vector with slope $f(a, b)$. The vector field $H : D \to \mathbb{R}^2$ given by

$$H(a, b) = (1, f(a, b)), \quad \forall\, (a, b) \in D$$

is called <u>the direction field</u>. A drawing of the vector field H at a large number of points of D gives us approximate solution curves.

Along the curves $f(x, y) = c$, where c is a constant, the slopes are constant. These curves are called <u>isoclines</u>.

Geometric Meaning of Solutions of $y' = f(x, y)$

Example 7: Consider the first order linear ODE

$$y' = xy.$$

The isoclines are the hyperbolas $xy = k, \; k \neq 0$, and the coordinate axes. The direction fields are given by

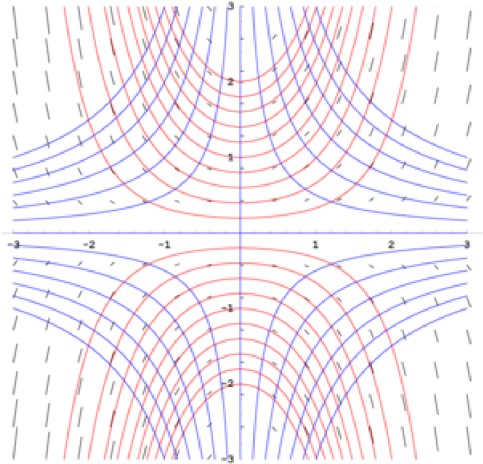
$$H(x, y) = (1, xy).$$

Check that the solutions for the above ODE are

$$y(x) = c \cdot e^{x^2/2},$$

where c is any constant.

Isoclines & Direction Fields



Isoclines are in blue, direction fields are in black, and solution curves are in red. Note that solution curves are approximated by direction fields.

Solving first order ODE's

Let's now start building up a systematic way of attacking ODE's. As we remarked at the very start, there is no general method to solve an arbitrary first order ODE:

$$y' = f(x, y).$$

We consider special forms of the above ODE to find solutions.

Separable ODE's

Definition (Separable ODE)

An ODE of the form
$$M(x) + N(y)y' = 0 \tag{1}$$
is called a separable ODE.

Let's first assume that $y(\cdot)$ is a solution. Let $\widetilde{M}(\cdot)$ and $\widetilde{N}(\cdot)$ be antiderivatives of $M(\cdot)$ and $N(\cdot)$, i.e.,
$$\widetilde{M}'(x) = M(x) \text{ and } \widetilde{N}'(z) = N(z) \tag{2}$$
Then, from the chain rule
$$\frac{d}{dx}\widetilde{N}\left(y(x)\right) = \widetilde{N}'\left(y(x)\right)y'(x) = N\left(y(x)\right)y'(x)$$
Then, (1) is equivalent to
$$\frac{d}{dx}\widetilde{N}\left(y(x)\right) = -\frac{d}{dx}\widetilde{M}(x)$$
Integrating both sides with respect to x gives
$$\widetilde{N}(y) + \widetilde{M}(x) = c, \tag{3}$$
where c is a constant.

Separable ODE's

Example: Solve the differential equation:

$$y' = -2xy.$$

Note $y(x) = 0$ is a solution of the ODE.
Separating the variables, we get: for $y \neq 0$,

$$\frac{dy}{y} = -2xdx.$$

Integrating both sides, we get:

$$\ln |y| = -x^2 + c_1.$$

Thus, the solutions are

$$y(x) = ce^{-x^2}.$$

How do they look?

Separable ODE's

The solutions are

$$y(x) = ce^{-x^2}.$$

If we are given an initial condition

$$y(x_0) = y_0,$$

then we get:

$$c = y_0 e^{x_0^2}$$

and

$$y = y_0 e^{x_0^2 - x^2}.$$

Separable ODE's

Example: Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2};\ y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\frac{1 + 2y^2}{y} dy = \cos x \ dx.$$

Integrating,

$$\ln |y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$. Hence a particular solution to the IVP is

$$\ln |y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE but it is not a solution to the given IVP.