CURVES IN $\mathbb{R}^n$

LET $I = [a,b] \subset \mathbb{R}$. A VECTOR FIELD

$$\mathbf{r} : I \to \mathbb{R}^n \quad (n = 2,3)$$

IS CALLED A CURVE IN $\mathbb{R}^n$.

$\mathbf{r}(a) \equiv$ INITIAL POINT OF THE CURVE

$\mathbf{r}(b) \equiv$ FINAL POINT.

$C = \{ \mathbf{r}(t) \mid t \in I \}$ IS THE PATH TRAVERSED BY $\mathbf{r}$.

IF WE WRITE $\mathbf{r}(t) = (x(t), y(t), z(t))$ $(n=3)$ THEN $x(t), y(t), z(t)$ ARE THE COMPONENTS OF $\mathbf{r}$.

EXAMPLE

🚩 $\mathbf{r}(t) = (a\cos t, a\sin t, bt) \quad (t \in \mathbb{R})$

THIS CURVE IS A HELIX.

🚩 SAY $\mathbf{r}$ IS <u>SIMPLE</u> IF $\mathbf{r}$ IS ONE-ONE.

🚩 $\mathbf{r}$ IS SAID TO BE <u>SMOOTH</u> IF IT IS CONTINUOUSLY DIFFERENTIABLE.

🚩 A SMOOTH CURVE $\mathbf{r}$ IS <u>REGULAR</u> IF

$\mathbf{r}'(t) \neq 0 \quad \forall t$

🚩 SUPPOSE $\mathbf{r}_1, \mathbf{r}_2$ ARE CURVES, i.e.

$\mathbf{r}_1 : [a,b] \to \mathbb{R}^n, \quad \mathbf{r}_2 : [c,d] \to \mathbb{R}^n.$

WE SAY $\mathbf{r}_1$ AND $\mathbf{r}_2$ ARE <u>EQUIVALENT</u>

IF THERE EXISTS $\phi : [a,b] \to [c,d]$ WHICH IS

- ONE-ONE, ONTO AND SUCH THAT BOTH

  ϕ, ψ ARE CONTINUOUSLY DIFFERENTIABLE,

  WHERE $\psi : [c,d] \to [a,b], \; \psi(x) = \phi^{-1}(x).$

- $\mathbf{r}_1 = \mathbf{r}_2 \circ \phi$

EXAMPLE

$\mathbf{r}_1 : [0,2] \to \mathbb{R}^3, \qquad \mathbf{r}_2 : [0,1] \to \mathbb{R}^3.$

$\mathbf{r}_1(t) = (t,0,0), \quad \mathbf{r}_2(t) = (t+t^3, 0, 0).$

CONSIDER $\phi(t) = t+t^3, \; \phi : [0,1] \to [0,2]$

CHECK THAT $\mathbf{r}_1, \mathbf{r}_2$ ARE EQUIVALENT.

TANGENT VECTOR

SUPPOSE $\mathbf{r} : I \to \mathbb{R}^3$ IS A SMOOTH CURVE.

$$\mathbf{r}'(t) := \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

IF $\mathbf{r}$ IS A REGULAR CURVE, $\mathbf{r}'(t)$ IS CALLED THE TANGENT VECTOR (OR VELOCITY VECTOR) AT (TIME) t.

$$T_{\mathbf{r}}(t_0) = \frac{1}{\|\mathbf{r}'(t_0)\|} \cdot \mathbf{r}'(t_0)$$

IS CALLED THE UNIT TANGENT VECTOR TO $\mathbf{r}(t)$ AT t_0.

LET $\mathbf{r}_1 : [a,b] \to \mathbb{R}^3$, $\mathbf{r}_2 : [c,d] \to \mathbb{R}^3$ BE EQUIVALENT REGULAR CURVES. THEN

$$T_{\mathbf{r}_2}(t) = \pm T_{\mathbf{r}_1}(\phi(t)) \quad \text{WHERE}$$

$\phi : (c,d) \to (a,b)$ IS THE PARAMETRIZATION.

THE LENGTH OF THE CURVE $\mathbf{r}$ IS GIVEN BY

$$L := \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt$$

IF C IS THE REGULAR CURVE DESCRIBED BY $\mathbf{r}$, THE LENGTH OF THE CURVE BETWEEN TWO POINTS DESCRIBED BY $\mathbf{r}(t_0)$ AND $\mathbf{r}(t)$ IS GIVEN BY

$$s = s(t) = \int_{t_0}^{t} \| \mathbf{r}'(u) \| \, du$$

THIS DEFINITION IS INDEPENDENT OF THE PARAMETRIZATION.

IF FOR A SMOOTH CURVE C WITH PARAMETRIZATION $\mathbf{r}(t)$, WE HAVE $\left\| \frac{d\mathbf{r}}{dt}(t) \right\| = 1 \quad \forall\, t$

THEN FOR EVERY $t_0 \in \mathrm{Dom}(\mathbf{r})$ THE PARAMETER $s = t - t_0$ GIVES AN ARC LENGTH PARAMETRIZATION OF $\mathbf{r}(s)$ FOR C W.R.T THE REFERENCE POINT t_0.

IF $\mathbf{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, THE UNIT $(\mathbf{r}(t) \neq \underset{\sim}{0})$ NORMAL VECTOR $\mathbf{n}$ IS DEFINED AS

$$\mathbf{n} = \pm \frac{y'(t)\,\vec{i} - x'(t)\,\vec{j}}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

EXAMPLE

$\mathbf{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt.$

$t_0 = 0.$

LENGTH OF THE HELIX TILL THE POINT

$\mathbf{r}(t)$ FROM $\mathbf{r}(0)$ IS GIVEN BY

$$s = \int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t \sqrt{a^2+b^2}\,du = t\sqrt{a^2+b^2}$$

(SINCE $\mathbf{r}'(u) = (-a\sin t,\ a\cos t,\ b)$

$\Rightarrow \|\mathbf{r}'(u)\| = \sqrt{a^2+b^2}.$)