Doubt session - 2   10.11.2022

- $\{a_n\} \sim \{b_n\}$ Cauchy in $\mathbb{Q}$.

$$\Leftrightarrow \lim_{n\to\infty} |a_n - b_n| = 0.$$

- equivalence relation.
  $\longrightarrow$ gives equivalence class.

- $$\mathbb{R} := \left(\text{all Cauchy seq in } \mathbb{Q}\right) / \sim$$

$$\mathbb{Q} \subseteq \underline{\underline{\mathbb{R}}}$$

$$q \in \mathbb{Q} \longleftrightarrow \{q, q, q, \ldots\}$$

$\lim \frac{x_{n+1}}{x_n} > 1$

$$\frac{n^2}{n+1} \geq \frac{n^2}{n+n} = \frac{n}{2}$$

$\left\{\frac{n}{2}\right\} \longrightarrow$ is it Bounded? NO!

- $\{x\} \longrightarrow$ not open. $\{x\} = [x, x]$

- $A \subseteq \mathbb{R}$.

open if for any $a \in A$,

$\exists\, \delta > 0$ s.t. $(a-\delta, a+\delta) \subseteq A$.

ex:

$(0,1) \longrightarrow$ open ? ? ✓

$[0,1) \longrightarrow$ open ✗.

$\mathbb{Q} \longrightarrow$ open ✗

$\mathbb{Z} \longrightarrow$ open ✗.

$\emptyset$

$B \subseteq \mathbb{R} \longrightarrow$ closed if B^c – open.

$(0,1] \longrightarrow$ closed ✗.

$[1,2] \longrightarrow$ closed ✓

$\mathbb{Q} \longrightarrow$ closed ✗

$\{x\} \longrightarrow$ closed ✓

$\mathbb{Z} \longrightarrow$ closed ✓

$[a, b]$

$\mathbb{R} \longrightarrow$ open, closed

$\phi \longrightarrow$ open, closed.

only subsets in $\mathbb{R}$ which are both open & closed.

(Connected of $\mathbb{R}$)

- $\lim_{n\to\infty} a_n^{1/n} = \lambda$ $\quad a_n > 0 \ \forall n.$

$0 \leq \lambda < 1$:

$\boxed{\text{Choose } \varepsilon_0 \text{ s.t. } \lambda + \varepsilon_0 < 1}$

$$|a_n^{1/n} - \lambda| < \varepsilon_0 \quad \forall n \geq N.$$

$$\Rightarrow \quad a_n^{1/n} < \lambda + \varepsilon_0, \quad \forall n \geq N.$$

$$\Rightarrow \quad 0 < a_n < (\lambda + \varepsilon_0)^n, \quad \forall n \geq N.$$

- If $\lambda > 1$. ε_1 s.t. $\lambda - \varepsilon_1 > 1$.

$$\lambda - \varepsilon_1 < a_n^{1/n}, \quad \forall n \geq N.$$

$$\Rightarrow \quad (\lambda - \varepsilon_1)^n < a_n, \quad \forall n \geq N$$

unbounded.

Thm: Every Cauchy seq. conv. in $\mathbb{R}$.

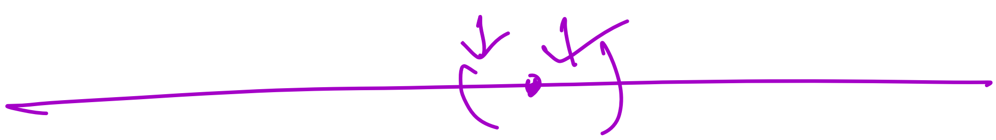
Pf. $\{a_n\} \to$ Cauchy seq.

$\longrightarrow$ **Bounded**.

$\{a_n : n = 1, 2, \ldots\}$ bounded subset of $\mathbb{R}$.

Bolzano Weierstrass Thm: $\longrightarrow$ has a limit pt. p

Claim: $\{a_n\} \longrightarrow p$.



- $\left\{ 1+\frac{1}{n}, -1+\frac{1}{n} : n \in \mathbb{N} \right\}$

$\left\{ 2, 0, 1+\frac{1}{2}, -1+\frac{1}{2}, \ldots \right\}$

- $\lim_{x \to c} \sin x = \sin c.$

$$|\sin x - \sin c| = \left| 2\cos\frac{x+c}{2} \sin\frac{x-c}{2} \right|$$

$$\leq 2\left|\sin\frac{x-c}{2}\right|$$

$$\leq |x-c|.$$

<u>Fix $\varepsilon > 0$:</u>

If $|x-c| < \delta (= \varepsilon)$, then

$$|\sin x - \sin c| < \varepsilon.$$

$$a_{n+1} = a_n + a_{n-1}^2 .$$

Suppose, $\{a_n\}$ $a_n \rightarrow$ conv.

$$\boxed{\lim a_n = l}.$$

$$\lim a_{n+1} = \lim a_n + (\lim a_n)^2$$

$$l = l + l^2$$

Construction of real no. from $\mathbb{Q}$.

$$\sum_{n=1}^{\infty} a_n \qquad \boxed{\lim s_n}$$

$$\{s_n\} = \underbrace{a_1 + \cdots + a_n}.$$

- $\lim\limits_{x\to c} f(x) = l_1$, $\lim\limits_{x\to c} g(x) = l_2$.

$$\lim_{x\to c} \left[f(x)g(x)\right] = l_1 l_2 .$$

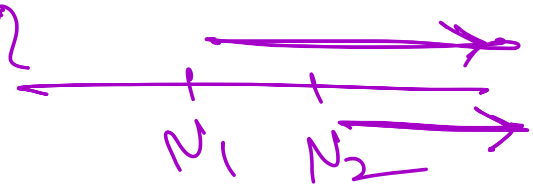
- $\lim a_n = l_1$, $\lim b_n = l_2$

Then $\lim a_n b_n = l_1 l_2$.

To prove: Fix $\epsilon > 0$, to get N s.t.

$$|a_n b_n - l_1 l_2| < \epsilon \quad \forall n \geq N.$$

Given:
$$\begin{cases} |a_n - l_1| < \epsilon/(2|l_2|) & \forall n \geq N_1 \\ |b_n - l_2| < \epsilon/2M & \forall n \geq N_2. \end{cases}$$



Let $N = \max\{N_1, N_2\}$.

Then $n \geq N \Rightarrow \underline{|a_n - \ell_1|} < \varepsilon/2|\ell_2|$

& $\underline{|b_n - \ell_2| < \varepsilon/2M}$

$$|a_n b_n - \ell_1 \ell_2|$$

$$= |a_n b_n - a_n \ell_2 + a_n \ell_2 - \ell_1 \ell_2|$$

$$\leq |a_n|\,|b_n - \ell_2| + |\ell_2|\,|a_n - \ell_1|$$

$$\leq M\,|b_n - \ell_2| + |\ell_2|\,|a_n - \ell_1|.$$

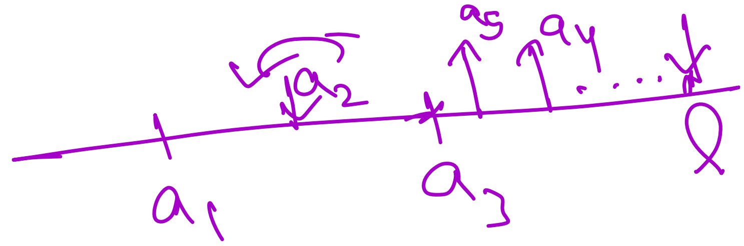
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\forall n \geq N$.

$\{0, 1, \frac{1}{2}, 1-\frac{1}{3}, \frac{1}{3}, \ldots \}$

$a_n \to l$. if $l = \sup a_n$.

If $a_n \uparrow$ & $\downarrow$.



$\frac{n}{n+1} \longrightarrow 1$.

$x > 0$, $\exists n \in \mathbb{N}$ s.t.

$nx > 1$.

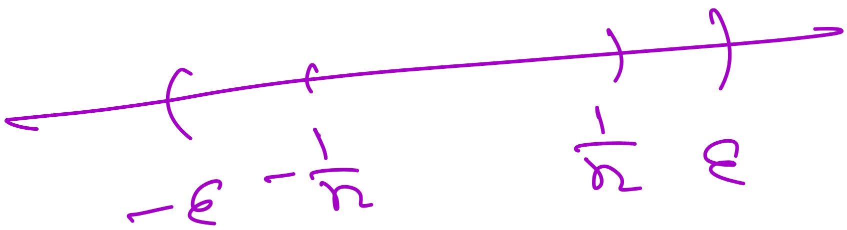
so, $n > \frac{1}{x}$.

• For every real no. $\exists$ a bigger natural no.

Fix $\varepsilon > 0$.

Then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n$.

$$\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq (-\varepsilon, \varepsilon).$$



1. Dedekind's cut.
2. From cauchy seq.
3. axiomatic defn.

- Rudin $\longrightarrow$ Principle of Apostol - Analysis.