Department of Physics, Indian Institute of Technology Bombay

26-05-2023 **PH 112: Quiz (Model Solutions)** **8.00 - 9:00 hrs**

Instructions:

1. **Weightage is 20 marks.**
2. Non-programmable calculators are permitted.
3. Useful constants:
   (a) Speed of light in vacuum $c = 3 \times 10^8\ \mathrm{m.s^{-1}}$
   (b) Planck constant $h = 6.63 \times 10^{-34}\ \mathrm{J.s}$
   (c) Rest mass of electron $5.1 \times 10^5\ \mathrm{eV}/c^2$

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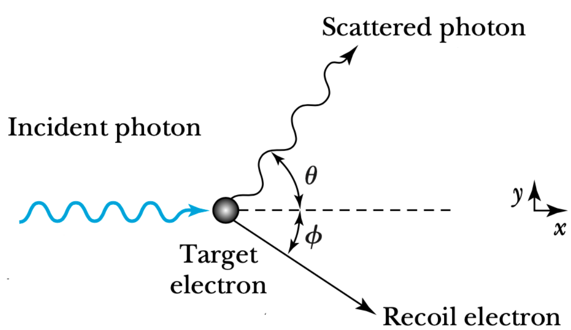
1. Consider Compton scattering where an incoming photon (as shown below) with frequency f (and corresponding wavelength λ) hits an electron at rest. After the collision, the photon is scattered at an angle θ with wavelength λ'. The electron recoils at an angle ϕ with kinetic energy K. Let $\Delta\lambda = \lambda' - \lambda$.



(a) Calculate the kinetic energy of the recoil electron in terms of $\Delta\lambda/\lambda$. **[2 Marks]**

(b) Assuming that the incoming photon has an energy of $200\ \mathrm{keV}$ and the scattered photon is detected at $\theta = 45^o$, calculate the scattering angle ϕ of the recoil electron in degrees. **[3 Marks]**

Answer

1a) By conservation of energy we know the electron's recoil energy equals the energy lost by the photon:

$$K = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{hc\,(\lambda' - \lambda)}{\lambda'\lambda} = \frac{hc\Delta\lambda}{\lambda'\lambda}\,. \tag{1}$$

[Getting above equation 1 mark]

Rewriting $\lambda' = \lambda + \Delta\lambda$ we have,

$$K = \frac{hc\Delta\lambda}{\lambda(\lambda+\Delta\lambda)} = \frac{hf\Delta\lambda}{\lambda+\Delta\lambda} = \frac{hf\Delta\lambda}{\lambda(1+\Delta\lambda/\lambda)} = \frac{(\Delta\lambda/\lambda)}{1+\frac{\Delta\lambda}{\lambda}}hf.$$

[Getting the correction expression 1 mark]

1b) Conservation of p_x

$$p_e\cos\phi + \frac{h}{\lambda'}\cos\theta = \frac{h}{\lambda} \implies p_e\cos\phi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta \quad (2)$$

Conservation of p_y

$$p_e\sin\phi - \frac{h}{\lambda'}\sin\theta = 0 \implies p_e\sin\phi = \frac{h}{\lambda'}\sin\theta \quad (3)$$

[Getting the above two expressions 1 mark]

Dividing the above equation by the momentum conservation equation along x, we have:

$$\tan\phi = \frac{\frac{h}{\lambda'}\sin\theta}{\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta}.$$

Using

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta) \text{ we have: } \tan\phi = \frac{\frac{h\sin\theta}{\lambda+\frac{h}{mc}(1-\cos\theta)}}{\frac{h}{\lambda} - \frac{h\cos\theta}{\lambda+\frac{h}{mc}(1-\cos\theta)}}.$$

Multiplying numerator and denominator by $\lambda\left[\lambda + \frac{h}{mc}(1-\cos\theta)\right]$, we have:

$$\tan\phi = \frac{\lambda h\sin\theta}{\lambda h + \frac{h^2}{mc}(1-\cos\theta) - \lambda h\cos\theta} = \frac{\lambda\sin\theta}{\left(\lambda+\frac{h}{mc}\right)(1-\cos\theta)}.$$

Using the trignometric identity: $\frac{\sin\theta}{(1-\cos\theta)} = \cot\left(\frac{\theta}{2}\right)$, we find:

$$\tan\phi = \frac{\lambda}{\lambda+\frac{h}{mc}}\cot\left(\frac{\theta}{2}\right) = \frac{1}{1+\frac{h}{mc\lambda}}\cot\left(\frac{\theta}{2}\right) = \frac{1}{1+\frac{hf}{mc^2}}\cot\left(\frac{\theta}{2}\right).$$

Inverting the equation gives

$$\cot\phi = \left[1+\frac{hf}{mc^2}\right]\tan\left(\frac{\theta}{2}\right).$$

[Getting any of the above expressions correct 1 mark]

Incoming Photon's energy hf is 200 keV. We know that mass of electron is 5.1×10^5eV. $\theta = 45°$. Substituting these in the above expression, we have $ArcCot[1.39216 * Tan[45/2]] \sim 60.03°$.

[Getting the angle correct 1 mark]

2. "Ultrafast" lasers produce pulses of light that last only for a few femto-seconds. ($1\text{ fs} = 10^{-15}$ s). Such short pulses have large spread in frequency. A particular ultrafast laser produces a 10 fs pulse of light with a central wavelength of $\lambda_0 = 532$ nm.

   (a) Find the minimum spread in frequency, Δf, and the ratio $\frac{\Delta f}{f_0}$. **[2 Marks]**

   (b) Find the corresponding range $\Delta\lambda$ of wavelengths produced. **[2 Marks]**

(c) Calculate the ratios $\dfrac{\Delta\lambda}{\lambda_0}$ and $\dfrac{\Delta\lambda}{L}$ where L is the spatial extent of the light pulse. **[2 Marks]**

Answer

2a) Using $E = hf$, the uncertainty in energy is $\Delta E = h\Delta f$. With this the uncertainty relation $\Delta E \Delta t \geq \hbar/2$ becomes $h\Delta f \Delta t \geq \hbar/2$. This simplifies to $\Delta f \Delta t \geq 1/4\pi$, so the minimum uncertainty in frequency can be computed:

$$\Delta f = \frac{1}{4\pi\Delta t} = \frac{1}{4\pi\,(10 \times 10^{-15}\ \mathrm{s})} = 7.96 \times 10^{12}\ \mathrm{Hz}\,.$$

[Getting the above answer correct 1 mark]

The original frequency is $f = c/\lambda$, so the relative uncertainty is

$$\frac{\Delta f}{f} = \frac{\lambda\Delta f}{c} = \frac{\left(532 \times 10^{-9}\ \mathrm{m}\right)\left(7.96 \times 10^{12}\ \mathrm{Hz}\right)}{3.00 \times 10^{8}\ \mathrm{m/s}} = 0.014$$

over 1% !

[1 Mark for the above answer up to 3 decimal places. NO mark reduction if Δt is taken 1 or 5 fs.]

(b) With $f = c/\lambda, \Delta f/\Delta\lambda \approx -c/\lambda^2$,

[Getting the above expression correct 1 mark]

so the absolute value of the wavelength range is

$$\Delta\lambda = \frac{\lambda^2\Delta f}{c} = \frac{\left(532 \times 10^{-9}\ \mathrm{m}\right)^2\left(7.96 \times 10^{12}\ \mathrm{Hz}\right)}{3.00 \times 10^{8}\ \mathrm{m/s}} = 7.5 \times 10^{-9}\ \mathrm{m} = 7.5\ \mathrm{nm}.$$

[1 Mark for the above answer up to 1 decimal. NO mark reduction if Δt is taken 1 or 5 fs.]

(c) $\frac{\Delta\lambda}{\lambda_0} \sim 0.014$. This is an appreciable fraction of the wavelength.

The pulse has a length $L = ct = \left(3.00 \times 10^{8}\ \mathrm{m/s}\right)\left(10 \times 10^{-15}\ \mathrm{s}\right) = 3.0 \times 10^{-6}\ \mathrm{m} = 3000\ \mathrm{nm}.$

[Getting the above answer correct 1 mark]

Hence, $\frac{\Delta\lambda}{L} \sim 0.0025.$

[Getting the above answer correct upto 3rd decimal place 1 mark]

3. The dispersion relation for water waves in an ocean is given by :

$$\omega^2 = gk\left(\frac{e^{kD} - e^{-kD}}{e^{kD} + e^{-kD}}\right)$$

where g denotes acceleration due to gravity and D is the equilibrium depth. λ and k denote the wavelength and wave-number, respectively.

(a) Calculate the phase and group velocities for these waves in deep water $D \gg \lambda$. **[2 Marks]**

(b) Calculate the phase and group velocities for these waves in shallow water $D \ll \lambda$. **[2 Marks]**

Answer a) Rewrite the above dispersion relation as

$$\omega = \sqrt{gk\frac{(1 - e^{-2kD})}{(1 + e^{-2kD})}}\,.$$

Deep water waves refer to $\lambda \ll D \Longrightarrow kD \gg 1$. Hence, $e^{-kD} \to 0$. We thus have

$$\omega = \sqrt{gk\frac{(1-e^{-2kD})}{(1+e^{-2kD})}} \sim \sqrt{gk}$$

[Getting the above expression 1 mark]

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$
$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

[0.5 mark each for phase and group velocity]

b) Shallow water waves refer to $\lambda \gg D \Longrightarrow kD \ll 1$. We can write $e^{-x} \sim 1 - x$, we then have:

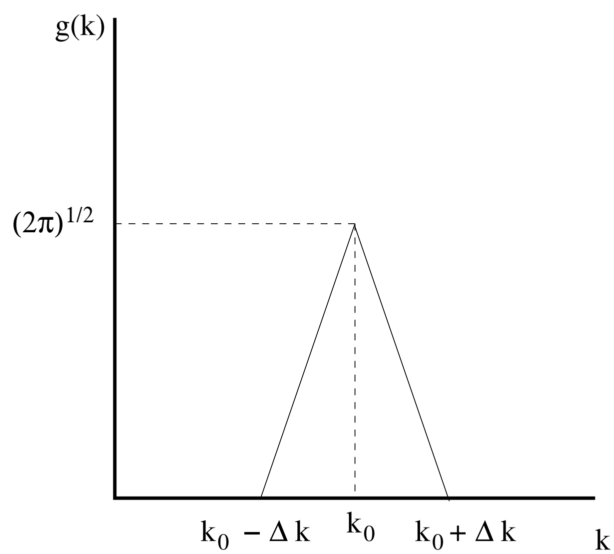
$$\omega = \sqrt{gk\frac{(1-e^{-2kD})}{(1+e^{-2kD})}} \sim k\sqrt{gD}$$

[Getting the above expression 1 mark]

$$V_g = \frac{d\omega}{dk} = \sqrt{gD}$$
$$V_p = \frac{\omega}{k} = \sqrt{gD}$$

[0.5 mark each for phase and group velocity]

4. A free particle, moving in one dimension, has the wave function $\psi(x)$. The Fourier transform of this wave function, $g(k)$, is shown in the figure below.



Calculate $|\psi(x)|^2$ and plot it for an appropriate range. [Note: You can use any convention for the Fourier transform in a consistent manner.] **[5 Marks]**

Answer First we obtain the expressions for $g(k)$

(i) $g_I(k)$, for $k \in (k_0 - \Delta k, k_0)$

$$g_I(k) = \tfrac{\sqrt{2\pi}}{\Delta k}\left(k - (k_0 - \Delta k)\right)$$

$$g_I(k) = \tfrac{\sqrt{2\pi}}{\Delta k}k + \sqrt{2\pi}\left(1 - \tfrac{k_0}{\Delta k}\right)$$

(0.5 marks)

(ii) $g_{II}(k)$ for $k \in (k_0, k_0 + \Delta k)$

$$g_{II}(k) - \sqrt{2\pi} = -\frac{\sqrt{2\pi}}{\Delta k}(k - k_0)$$
$$g_{II}(k) = -\frac{\sqrt{2\pi}}{\Delta k}k + \sqrt{2\pi}\left(1 + \frac{k_0}{\Delta k}\right)$$

(0.5 marks)

Therefore, using the standard convention for Fourier Transform

$$\begin{aligned}\psi(x,0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{ikx}dk \\ &= \frac{1}{\sqrt{2\pi}}\int_{k_0-\Delta k}^{k_0} g_I(k)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{k_0}^{k_0+\Delta k} g_n(k)e^{ikx}dk\end{aligned}$$

$$\begin{aligned}\psi(x,0) =& \frac{1}{\Delta k}\int_{k_0-\Delta_k}^{k_0} ke^{ikx}dk + \left(1 - \frac{k_0}{\Delta k}\right)\int_{k_0-\Delta k}^{k_0} e^{ikx}dk \\ &- \frac{1}{\Delta k}\int_{k_0}^{k_0+\Delta k} ke^{ikx} + \left(1 + \frac{k_0}{\Delta k}\right)\int_{k_0}^{k_0+\Delta k} e^{ikx}dk\end{aligned}$$

using the fact

$$\int e^{ikx}dk = \frac{e^{ikx}}{ix}$$

and

$$\int ke^{ikx}dk = \frac{ke^{ikx}}{ix} - \frac{e^{ikx}}{(ix)^2}$$

we get

$$\begin{aligned}\psi(x,0) =& \frac{1}{\Delta k}\left\{\frac{k_0e^{ik_0x}}{ix} - \frac{e^{ik_0k}}{(ix)^2} - \frac{(k_0-\Delta k)}{ix}e^{i(k_0-\Delta k)x}\right. \\ &\left. + \frac{e^{i(k_0-\Delta k)x}}{(ix)^2}\right\} \\ &+ \left(1 - \frac{k_0}{\Delta k}\right)\left\{\frac{e^{ik_0x} - e^{i(k_0-\Delta k)x}}{ix}\right\} \\ &- \frac{1}{\Delta k}\left\{\frac{(k_0+\Delta k)\,e^{i(k_0+\Delta k)x}}{ix} - \frac{e^{i(k_0+\Delta k)x}}{(ix)^2} - \frac{k_0e^{ik_0x}}{ix} + \frac{e^{ik_0x}}{(ix)^2}\right\} \\ &+ \left(1 + \frac{k_0}{\Delta k}\right)\left\{\frac{e^{i(k_0+\Delta k)x} - e^{ik_0x}}{ix}\right\}\end{aligned}$$

(2 Marks)

Combining the coefficients of $\frac{e^{ik_0x}}{ix}$ and $\frac{e^{ik_0x}}{(ix)^2}$

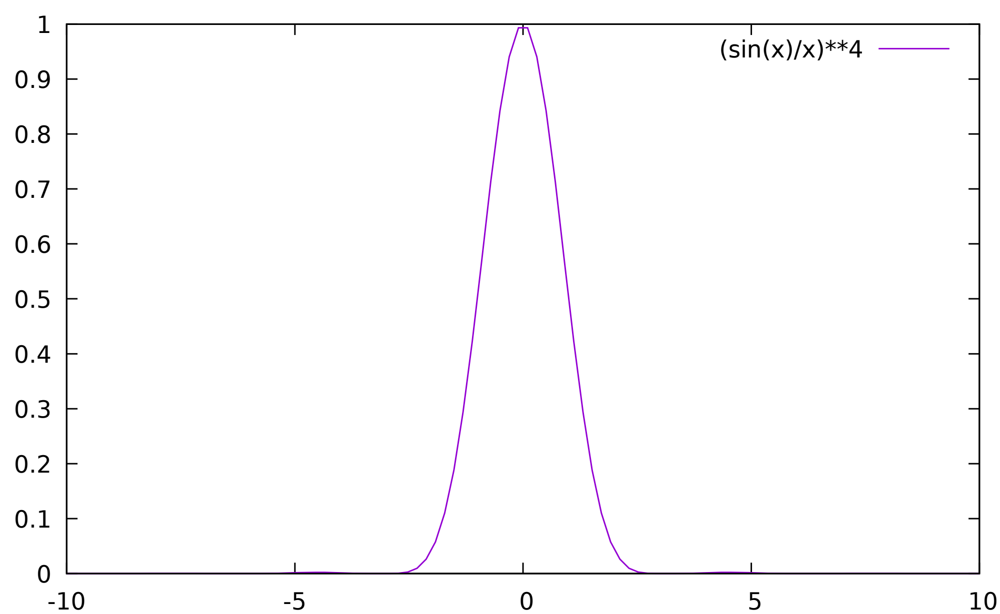
$$\psi(x,0) = \frac{e^{ik_0x}}{ix}\left\{\frac{k_0}{\Delta k} - \frac{k_0}{\Delta k}e^{-i\Delta kx} + e^{-i\Delta kx}\right.$$
$$+1 - \frac{k_0}{\Delta k} + \frac{k_0}{\Delta k}e^{-i\Delta kx} - e^{-i\Delta kx}$$
$$- e^{i\Delta kx} - \frac{k_0}{\Delta k}e^{i\Delta kx} + \frac{k_0}{\Delta k} + e^{i\Delta kx}$$
$$\left.+\frac{k_0}{\Delta k}e^{i\Delta kx} - 1 - \frac{k_0}{\Delta k}\right\}$$
$$+ \frac{e^{ik_0x}}{(ix)^2}\left\{-\frac{1}{\Delta k} + \frac{e^{-i\Delta kx}}{\Delta k} + \frac{e^{i\Delta kx}}{\Delta k} - \frac{1}{\Delta k}\right\}$$

We note that all the terms in the first bracket cancel each other leading to

$$\psi(x,0) = \frac{e^{ik_0x}}{x^2\Delta k}\{2 - 2\cos\Delta kx\} = \frac{e^{ik0x}4\sin^2\Delta kx/2}{x^2\Delta k} = \Delta k e^{ik_0x}\left(\frac{\sin\Delta kx/2}{\Delta kx/2}\right)^2$$
$$\implies |\psi(x,0)|^2 = \Delta k^2\left(\frac{\sin\Delta kx/2}{\Delta kx/2}\right)^4$$

(1 Mark)

The plot of $|\psi(x,0)|^2$ will be sharply peaked at $x = 0$, and look like



(1 Mark for the correct qualitative plot)