Lecture 5

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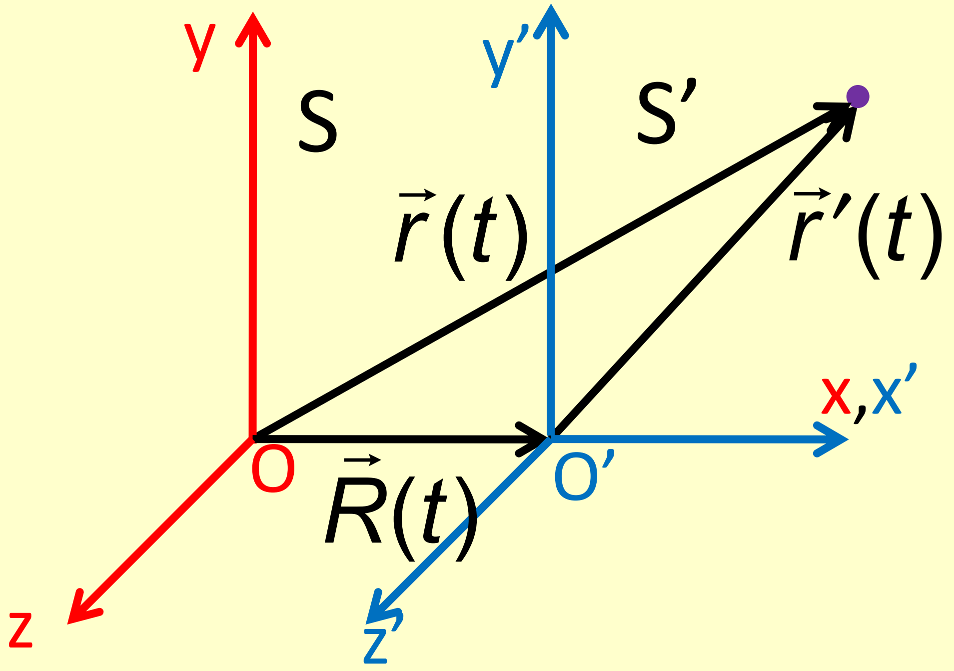
RECAP

Two Inertial Frames

Tutorials ??

constant velocity



$$\boxed{\vec{r}(t) = \vec{R}(t) + \vec{r}'(t)}$$

RECAP Velocity and Acceleration

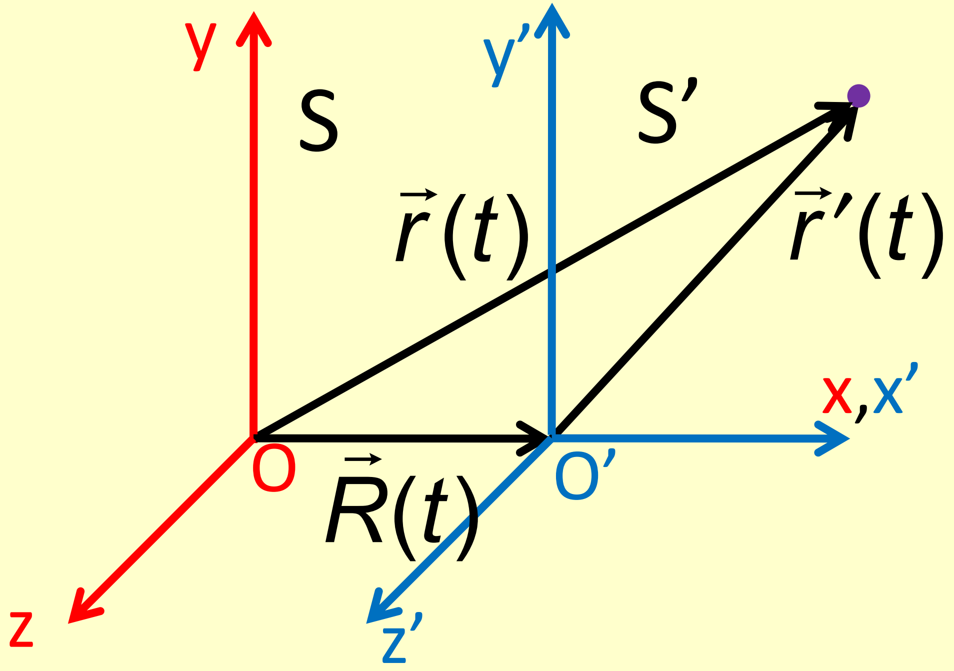
$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t)$$

$$\vec{v}(t) = \vec{v}_o + \vec{v}'(t)$$

$$\vec{a} = \vec{a}'$$

Thus acceleration of a particle in any inertial frame of reference is same, so we do not have to specify the frame, ***as long as it is inertial (constant velocity).***

RECAP Linearly accelerating N.I.F.O.R



Assume S is an inertial frame and S' is accelerating, with a constant acceleration $\vec{A}$

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Acceleration

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t)$$

$$\vec{v}(t) = \vec{v}_o + \vec{v}'(t)$$

$$\vec{a} = \vec{A} + \vec{a}'$$

According to S, acceleration of the particle is due to some real force

$$\vec{F}_{real} = m\vec{a}$$

RECAP Newton's Law in S'

Acceleration of the particle observed in S' is $\vec{a}'$

If Newton's law has to be applied in S'

$$\vec{F} = m\vec{a}'$$

$$= m\left(\vec{a} - \vec{A}\right)$$

$$= \vec{F}_{real} - m\vec{A}$$

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Newton's Law in S'

One can apply Newton's law if we assume that in addition to real forces an additional force of $-m\vec{A}$ is also applied to the particle.

Such a force is called ***Fictitious*** or ***Pseudo*** Force.

Rotating Frame of Reference

- Intrinsically non-inertial. Pseudo forces are must for describing dynamics.
- Pseudo Forces do not have such simple form as in uniformly accelerating frames.

Rotating Coordinate System

We denoted the position of a particle as a vector

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Can we similarly specify the angular position of a particle

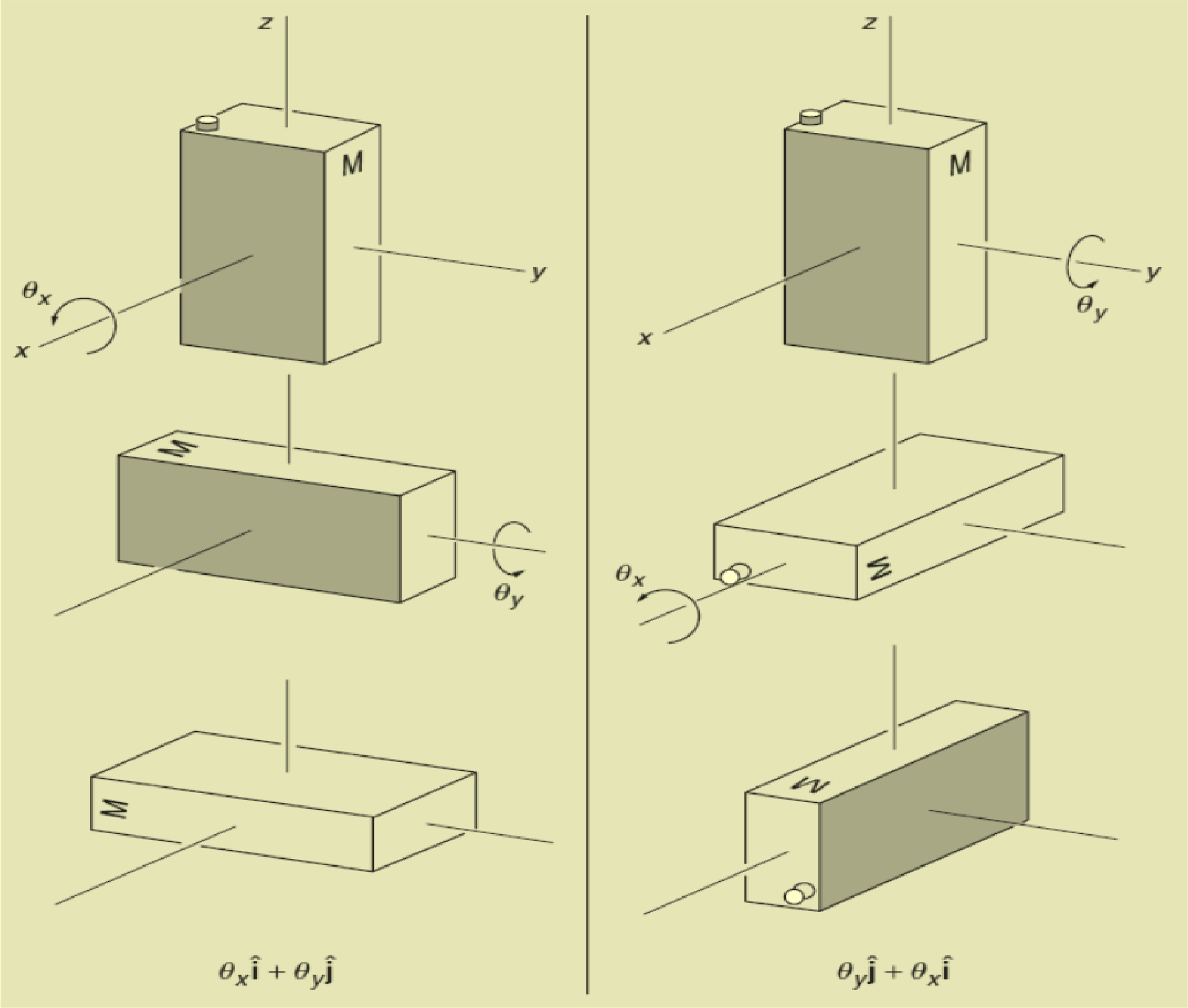
$$\boldsymbol{\theta} = \theta_x\hat{\mathbf{i}} + \theta_y\hat{\mathbf{j}} + \theta_z\hat{\mathbf{k}}\,?$$

The answer is no because such an expression does not satisfy commutative law of vector addition

$$\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$$

Let us rotate a block first around the x axis, and then around the y axis. Compare that to the same operations performed in the reverse order

Consider those two rotations, with each one of them being $\boxed{\pi/2}$



Clearly $\theta_x \hat{\mathbf{i}} + \theta_y \hat{\mathbf{j}} \neq \theta_y \hat{\mathbf{j}} + \theta_x \hat{\mathbf{i}}$

On the other hand, one can verify that infinitesimal rotations commute to first order terms

$$\Delta\theta_x\hat{\mathbf{i}} + \Delta\theta_y\hat{\mathbf{j}} \approx \Delta\theta_y\hat{\mathbf{j}} + \Delta\theta_x\hat{\mathbf{i}}$$

Thus, infinitesimal rotations can be represented as vectors

Because angular velocity is defined in terms of infinitesimal rotations

$$\boldsymbol{\omega} = \lim_{\Delta t\to 0}\frac{\boldsymbol{\Delta\theta}}{\Delta t},$$

Relation between v and ω

Angular velocity can be denoted as a vector

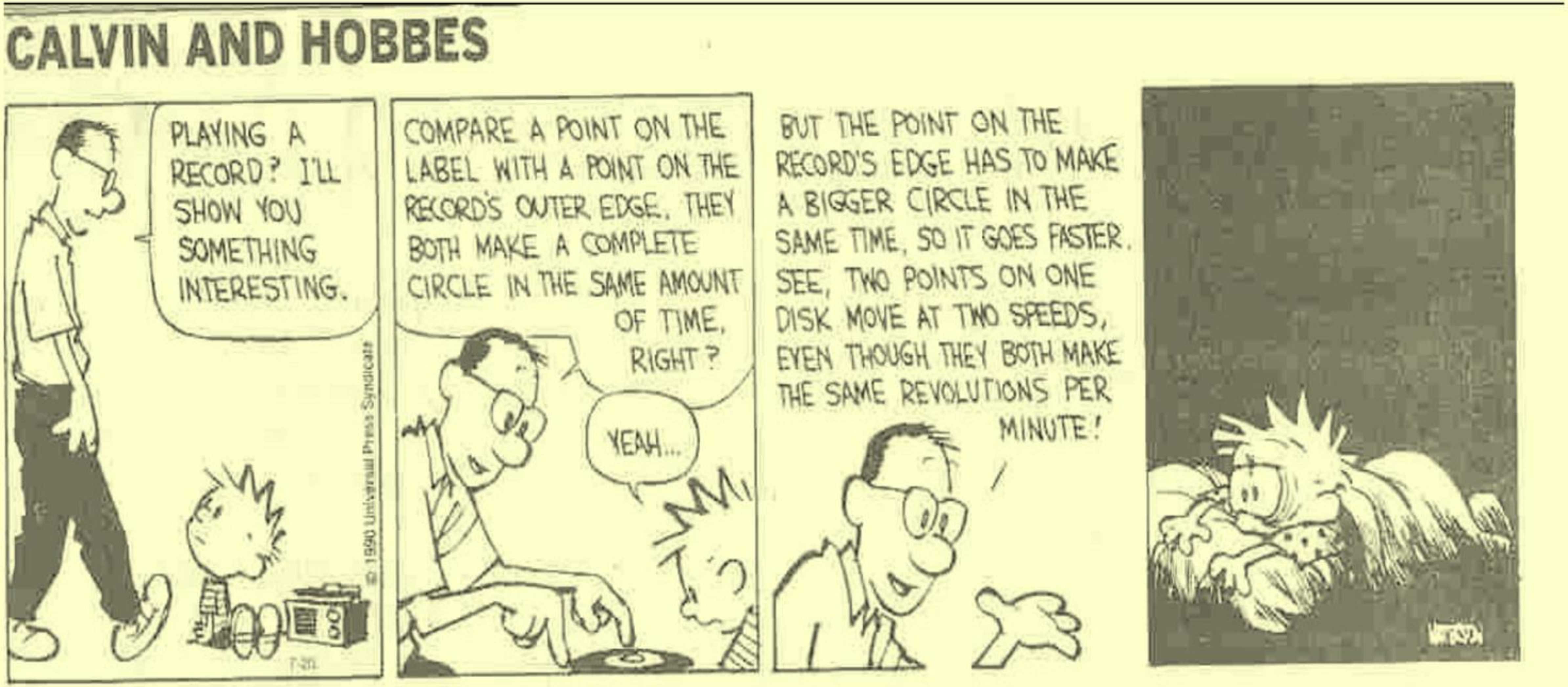
$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \omega \times \mathbf{r}$$

And, in general,

$$\boldsymbol{\omega} = \omega\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the direction of the axis of rotation, and ω is the magnitude of the angular velocity.


CALVIN AND HOBBES
PLAYING A RECORD? I'LL SHOW YOU SOMETHING INTERESTING.
COMPARE A POINT ON THE LABEL WITH A POINT ON THE RECORD'S OUTER EDGE. THEY BOTH MAKE A COMPLETE CIRCLE IN THE SAME AMOUNT OF TIME, RIGHT?
YEAH...
BUT THE POINT ON THE RECORD'S EDGE HAS TO MAKE A BIGGER CIRCLE IN THE SAME TIME, SO IT GOES FASTER. SEE, TWO POINTS ON ONE DISK MOVE AT TWO SPEEDS, EVEN THOUGH THEY BOTH MAKE THE SAME REVOLUTIONS PER MINUTE!
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Total Differentiation of a Vector in a Rotating Frame of Reference

- Before we can write Newton’s second law of motion for a reference frame rotating with the earth, we need to develop a relationship between the total derivative of a vector in an inertial reference frame and the corresponding derivative in a rotating system.

Let $\vec{A}$ **be an arbitrary vector with Cartesian components**

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ **in an inertial frame of reference, and**

$\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$ **in a rotating frame of reference.**

If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ in an inertial frame of reference, then

$$\frac{d\vec{A}}{dt} = \left(A_x\frac{d\hat{i}}{dt} + \hat{i}\frac{dA_x}{dt}\right) + \left(A_y\frac{d\hat{j}}{dt} + \hat{j}\frac{dA_y}{dt}\right) + \left(A_z\frac{d\hat{k}}{dt} + \hat{k}\frac{dA_z}{dt}\right)$$

Since the coordinate axes are in an inertial frame of reference,

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0$$

$$\frac{d\vec{A}}{dt} = \left(\cancel{A_x\frac{d\hat{i}}{dt}} + \hat{i}\frac{dA_x}{dt}\right) + \left(\cancel{A_y\frac{d\hat{j}}{dt}} + \hat{j}\frac{dA_y}{dt}\right) + \left(\cancel{A_z\frac{d\hat{k}}{dt}} + \hat{k}\frac{dA_z}{dt}\right)$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} \qquad \text{(Eq. 1)}$$

If $\vec{A} = A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}'$ in a rotating frame of reference, then

$$\frac{d\vec{A}}{dt} = \left(A'_x \frac{d\hat{i}'}{dt} + \hat{i}' \frac{dA'_x}{dt} \right) + \left(A'_y \frac{d\hat{j}'}{dt} + \hat{j}' \frac{dA'_y}{dt} \right) + \left(A'_z \frac{d\hat{k}'}{dt} + \hat{k}' \frac{dA'_z}{dt} \right) \quad \text{(Eq. 2)}$$

Because the left hand sides of Eq. 1 and Eq. 2 are identical,

$$\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} = \left(A'_x \frac{d\hat{i}'}{dt} + \hat{i}' \frac{dA'_x}{dt} \right) + \left(A'_y \frac{d\hat{j}'}{dt} + \hat{j}' \frac{dA'_y}{dt} \right) + \left(A'_z \frac{d\hat{k}'}{dt} + \hat{k}' \frac{dA'_z}{dt} \right)$$

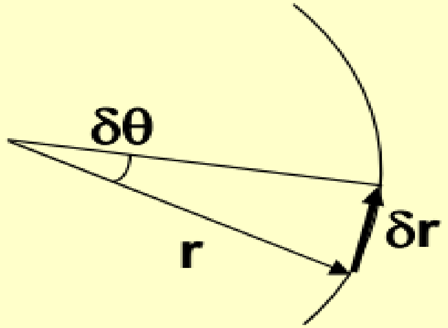
Regrouping the terms

$$\underbrace{\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x \frac{d\hat{i}'}{dt} + A'_y \frac{d\hat{j}'}{dt} + A'_z \frac{d\hat{k}'}{dt}}_{\text{effects of rotation}}$$

$$\underbrace{\frac{dA_x}{dt}\hat{i}+\frac{dA_y}{dt}\hat{j}+\frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}}=\underbrace{\frac{dA'_x}{dt}\hat{i}'+\frac{dA'_y}{dt}\hat{j}'+\frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}}+\underbrace{A'_x\frac{d\hat{i}'}{dt}+A'_y\frac{d\hat{j}'}{dt}+A'_z\frac{d\hat{k}'}{dt}}_{\text{effects of rotation}}$$

To interpret $\frac{d\hat{i}'}{dt}, \frac{d\hat{j}'}{dt}, \frac{d\hat{k}'}{dt}$ **think of each unit vector as a position vector.**

linear velocity = angular velocity x position vector $\rightarrow$ $\vec{V}=\vec{\Omega}\times\vec{r}=\omega\times\mathbf{r}$



Because $\vec{V}=\frac{d\vec{r}}{dt}$, $\frac{d\vec{r}}{dt}=\vec{\Omega}\times\vec{r}$

Thus $\frac{d\hat{i}'}{dt}=\vec{\Omega}\times\hat{i}'$, $\frac{d\hat{j}'}{dt}=\vec{\Omega}\times\hat{j}'$, $\frac{d\hat{k}'}{dt}=\vec{\Omega}\times\hat{k}'$

$$\underbrace{\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x\left(\vec{\Omega}\times\hat{i}'\right) + A'_y\left(\vec{\Omega}\times\hat{j}'\right) + A'_z\left(\vec{\Omega}\times\hat{k}'\right)}_{\vec{\Omega}\times\vec{A}}$$

(effects of rotation)

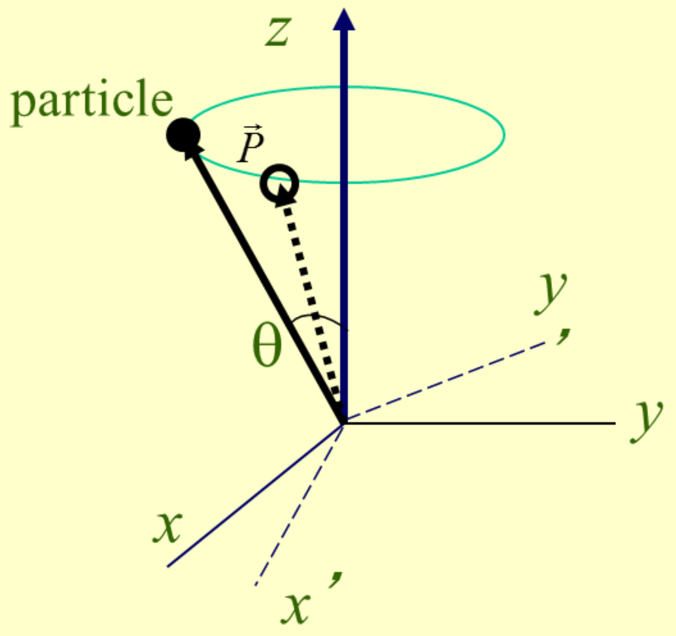
$$\boxed{\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{rotating} + \vec{\Omega}\times\vec{A}}$$

Taking the limit $\Delta t \to 0$, we get

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{r}$$

No subscript has been attached to the vector $\vec{r}$ as this is instantaneously same in both the frames of references. This is very different from the case of uniformly accelerating frames.

Example: The vector $\vec{r}$ is constant in frame S'.
We can assume the particle to be fixed on one of the chairs of merry go round. This implies



$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = 0$$

This gives

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = \vec{\Omega} \times \vec{r} \neq 0;$$

as expected

In order to find the direction of velocity, an instantaneous value of position vector is chosen.

The velocities are different in the two frames.

Example: Assume the particle to be at rest in S frame. In this case

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = 0$$

This gives

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = -(\vec{\Omega}\times\vec{r}\,) \neq 0;$$

again as expected

Note: As before the value of position vector is chosen instantaneously.

Rotating Coordinate System

Taking $\mathbf{A} = \mathbf{r}$, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$

$$\implies \mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$

On taking $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{v}_{in}$$

$$\implies \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r})$$

$$\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + \left[\frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}\right]_{rot}$$
$$+\vec{\Omega}\times(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})$$

Let us assume a constant angular velocity. Then the above equation can be written as

$$\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Having calculated the accelerations, we can calculate the pseudo force as before. In *S* frame:

$$\vec{F} = m\vec{a}_{in}$$

$$\text{where } \vec{F} \text{ is sum of all real forces}$$

In S', on the other hand, the observed acceleration would be

$$\vec{a}_{rot} = \vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Thus the force found by an observer in S' would be

$$\vec{F}_{rot} = m\vec{a}_{rot} = m(\vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}))$$

$$\vec{F}_{rot} = \vec{F} + \vec{F}_{fict}$$

where

$$\vec{F}_{fict} = -2m\vec{\Omega} \times \vec{v}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

The first term in above expression is called **Coriolis force**. This force is present only when a particle is observed to move in the rotating frame of reference.

The second term on the other hand is called **Centrifugal force**. This force is present whenever the particle is at a non zero distance from origin.

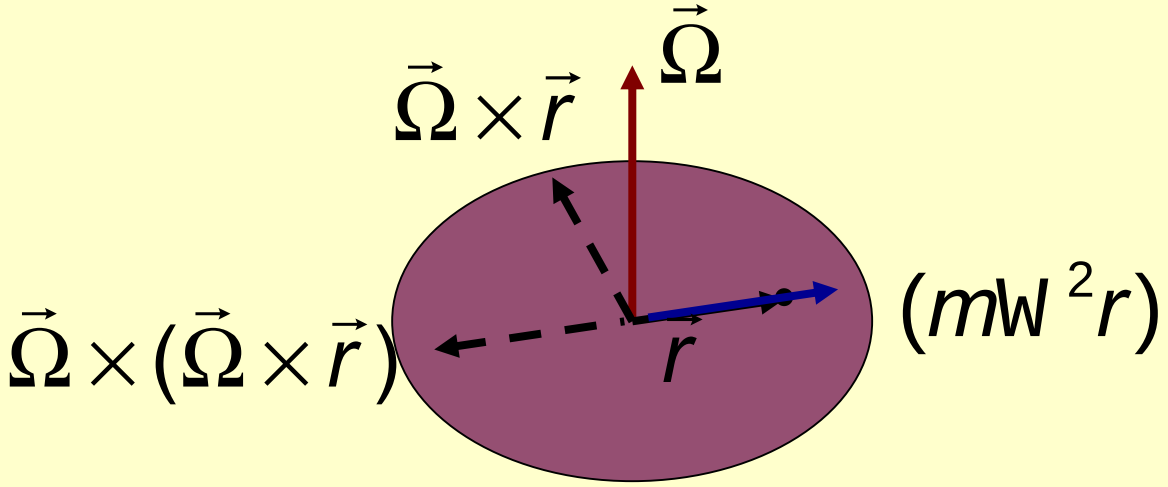
Note: ***In order to apply these forces we must know the angular speed of S′ relative to S like we have to know the acceleration in the case of uniformly accelerating frames of reference.***

Centrifugal force is the only Pseudo force acting, in addition to real forces **if** particle is at rest in a rotating frame.

This pseudo force always acts **radially outwards**.

Centrifugal Force

$$\boxed{\vec{F}_{cf} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

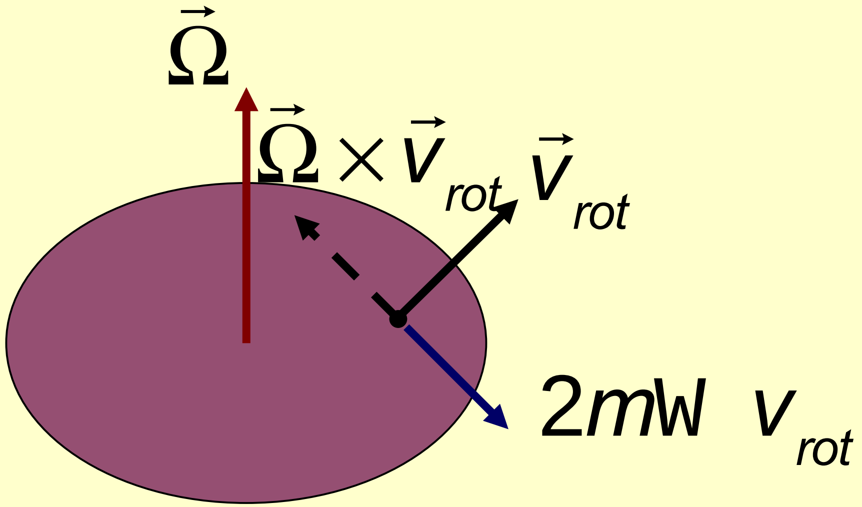


Coriolis force is a Pseudo force which is present only when the particle is moving in a rotating frame and is in addition to real forces and centrifugal force.

This pseudo force is always **perpendicular to the instantaneous velocity of the particle** in the rotating frame of reference.

Coriolis Force

$$\vec{F}_{cor} = -2m\vec{\Omega} \times \vec{v}_{rot}$$



Example

1. A particle of mass *m* stationary in a merry go round at distance *r* from the centre.

In an inertial frame the mass is rotating with the same angular velocity as merry go round. Let this velocity be $\vec{\Omega}$.

This is only possible if a real force like friction provides the centripetal force. Hence a real force has to act on the mass radially inwards with the following value.

$$\vec{F}_{real} = m\Omega^2 r$$

In the rotating frame the mass is in equilibrium. Hence Net Force is zero.

$$\vec{F}_{net} = \vec{F}_{real} + \vec{F}_{cf}$$
$$= m\Omega^2 r - m\Omega^2 r$$
$$= 0$$

2. An object of mass *m* stationary on ground at a distance *r* from axis of rotation of merry go round.

In an inertial frame the mass is in equilibrium hence net real force must be zero.

$$\vec{F}_{real} = 0$$

In the rotating frame the mass is rotating in a circle with angular velocity $-\vec{\Omega}$.

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = -(\vec{\Omega} \times \vec{r}) \neq 0;$$

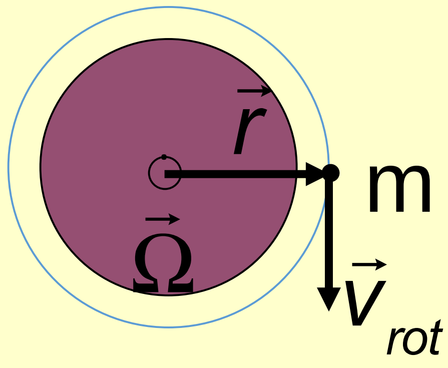
In the rotating frame the net force must take account of pseudo forces also.

The net radially inward force is

$$\vec{F}_{cf} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{F}_{cor} = -2m\vec{\Omega}\times\vec{v}_{rot}$$

$$\begin{aligned}\vec{F}_{net} &= \vec{F}_{real} + \vec{F}_{cf} + \vec{F}_{cor} \\ &= 0 - m\Omega^2 r + 2m\Omega^2 r \\ &= m\Omega^2 r\end{aligned}$$



It is these pseudo forces which provide the centripetal force to the mass necessary for a circular motion.

Problem: Find the effect of centrifugal force on acceleration due to gravity at equator (Effective Gravity)

Solution: The angular velocity of earth's rotation Ω is given by the following

$$\Omega = \frac{2\pi}{24 \times 60 \times 60} = 7.27 \times 10^{-5} rad/s$$

The value of g at equator would be given by, (R is radius of earth, while M_e is the mass of earth)

$$mg = \frac{GmM_e}{R^2} - m\Omega^2 R;\ \text{giving}$$

$$g = \frac{GM_e}{R^2} - \Omega^2 R$$

$$\text{Taking } G = 6.67\times10^{-11}\,Nm^2/kg^2$$

$$M_e = 5.98\times10^{24}\,kg;\ R = 6.37\times10^{6}\,m \text{ gives}$$

$$g = 9.83 - 0.033$$

$$= 9.797\,m/s^2$$

The change is around 0.3%.

Observations

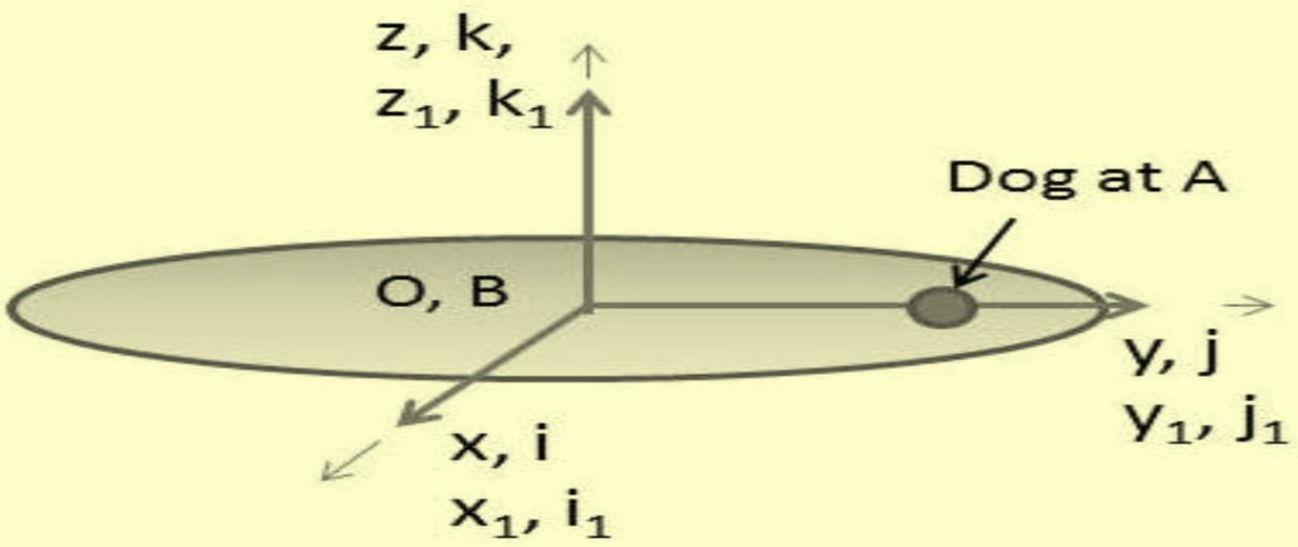
The centrifugal force is proportional to the tangential speed of the rotating reference frame.

The equator is moving quickly as the earth's spins, so it has a lot of centrifugal force.

In contrast, the poles are not spinning at all, so they have zero centrifugal force.

Since centrifugal force points outwards from the center of rotation, it tends to cancel out a little bit of earth's gravity.

Since there is more centrifugal force at the equator to cancel gravity, your overall weight at the equator versus at the poles is less.



A dog runs outwards in a straight radial line on a rotating platform. At the instant shown the dog is at position 'A'at a radius of r=5.0 feet form B. At this time the dog has an outward radial speed of 2.0ft/s.

The platform rotates at a rate of $\vec{\omega} = 0.5 \dfrac{\text{rad}}{s} \hat{k}$. Find the speed of the dog relative to the O_{xyz} fixed inertial frame of reference. The B_{x1y1z1} frame is attached to the platform.

The velocity equation for use when working with rotating and translating coordinate systems is appropriate to use in this case.

$\vec{V}_{A/O} = \vec{V}_{B/O} + \vec{V}_{A/B} + \vec{\omega} \times \vec{r}_{A/B}$, where

$\vec{V}_{A/O}$ = unknown velocity of point A(dog) in the O_{xyz} inertial frame.

$\vec{V}_{B/O} = 0$ = velocity of translation of the moving coordinate system B_{x1y1z1}, which is attached to the platform at O, but rotates.

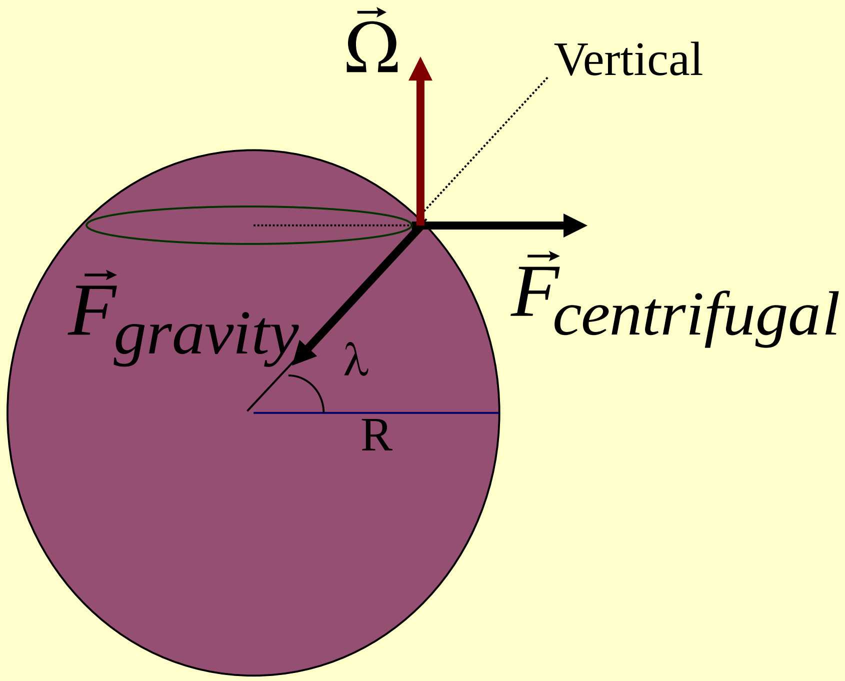
$\vec{V}_{A/B} = 2\ \mathrm{ft/s}\ \hat{\mathrm{j}}_1$ = velocity of point A(dog) relative to the rotating B_{x1y1z1} frame.

$\vec{\mathrm{r}}_{A/B} = 5\ \mathrm{ft}\ \hat{\mathrm{j}}_1$ = position vector of the dog at A in the B_{x1y1z1} frame.

$\vec{\omega} \times \vec{r}_{A/B} = 0.5\mathrm{rad/s}\ \hat{\mathrm{k}} \times 5\ \mathrm{ft}\ \hat{\mathrm{j}}_1 = -2.5\mathrm{ft/s}\ \hat{\mathrm{i}}_1$

$\vec{V}_{A/O} = 0 + 2\ \mathrm{ft/s}\ \hat{\mathrm{j}}_1 - 2.5\mathrm{ft/s}\ \hat{\mathrm{i}}_1 = 2\ \mathrm{ft/s}\ \hat{\mathrm{j}} - 2.5\mathrm{ft/s}\ \hat{\mathrm{i}}$

Find the effect of centrifugal force on acceleration due to gravity at a latitude λ, assuming earth to be spherical.



$$\vec{F}_{centrifugal} = m\Omega^2 R\cos\lambda$$

$$\vec{F}_{gravity} = \frac{GmM_e}{R^2} = mg_o(\text{say})$$

Adding the two vectors we get the magnitude of the net force represented as *mg*.

$$(mg)^2 = (m\Omega^2 R\cos\lambda)^2 + (mg_o)^2 +$$

$$2(m\Omega^2 R\cos\lambda)(mg_o)\cos(\pi - \lambda)$$

This gives

$$g = \sqrt{g_o{}^2 + (\Omega^4 R^2 - 2\Omega^2 R g_o)\cos^2\lambda}$$