MA108 QUIZ 19-05-2023 **8:30-9:15AM** **Maximum Marks: 20**

Name: Yellow **Division:**
Roll No: **Tutorial Batch:**

1. Write your Name, Roll No., Division, Tutorial Batch.
2. This a question paper cum answer booklet. At the end of the quiz, ***only*** *this booklet will be collected for evaluation. Write the answers in the space provided against each question. Separate sheets will be provided for rough work.*
3. There are ***nine*** *questions.*
4. No books, notes, calculators, mobile phones, electronic devices are permitted.
5. There is ***no*** *negative marking.*

No partial credits for Qn 1- 8.

1. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and $f(0) = 0$. The function f such that the ODE [1+1]

$$y \cos x + f(x)y' = 0$$

is exact, is given by

$$f(x) = \sin x.$$

(ii) For the function f as in (i), the implicit solution of the ODE is $u(x, y) =$ constant, where

$$u(x, y) = y \sin x (or \quad y \sin x + constant, \quad or - y \sin x, \quad -y \sin x + constant).$$

(Each of the above mentioned functions is a correct answer).

2. The ODE $(2y + 6y^3x^{-1}) - (3x + 10y^2)\frac{dy}{dx} = 0$ has an integrating factor of the form $x^a y^b$. Find [1+1]

$$a = -3, \quad b = 2.$$

3. Let $\{\phi_n\}_{n=0}^{\infty}$ be the sequence of functions given by the Picard's iteration method for the IVP $y' = x + y^2 - 1,\ y(0) = 1$, starting with $\phi_0 \equiv 1$. Then the first two Picard iterates are [1+1]

$$\phi_1(x) = 1 + \frac{x^2}{2},$$

$$\phi_2(x) = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^5}{20}.$$

4. The solution of the IVP $xy' + y = x^4y^3,\ y(1) = 2$, is given by [2]

$$y(x) = \frac{1}{x}\frac{1}{\sqrt{\frac{5}{4} - x^2}}, \quad 0 < x < \sqrt{\frac{5}{4}}.$$

Update. Full 2 **marks for** $y(x) = \frac{1}{x}\frac{1}{\sqrt{\frac{5}{4}-x^2}}$ **or** $y(x) = \frac{1}{|x|}\frac{1}{\sqrt{\frac{5}{4}-x^2}}$ **or** $y(x) = \frac{1}{x}\frac{1}{\sqrt{|\frac{5}{4}-x^2|}}$ **or** $y(x) = \frac{1}{|x|}\frac{1}{\sqrt{|\frac{5}{4}-x^2|}}$**, or** $y(x) = \frac{1}{\sqrt{x^2(\frac{5}{4}-x^2)}}$**.**

5. Let $f(x) = x^3$ and $g(x) = x^2|x|$ for all $x \in \mathbb{R}$. Are the functions f and g linearly dependent on $\mathbb{R}$? Ans.No [1]

6. Possibly multiple correct answers. Let f and g be two distinct solutions of $y' + p(x)y = q(x)$, where p, q are continuous on $\mathbb{R}$. **Circle** the correct option(s). [2]

   a. The solution curves associated to f and g can never intersect.
   b. The solution curves associated to f and g intersect exactly once.
   c. The solution curves associated to f and g intersect at least twice.
   d. None of the above.

   Here 2 marks for full set of correct option(s). Otherwise 0.

7. Possibly multiple correct answers. Consider the IVP: $(x-1)y' - y = 0,\ y(1) = y_0$. **Circle** the correct option(s). [2]

   a. The ODE is linear and in homogeneous form.
   b. The IVP has no solution for $y_0 \neq 0$.
   c. The IVP has a unique solution for each $y_0 \in \mathbb{R}$.
   d. The IVP has infinitely many solutions for $y_0 = 0$.

   Here 2 marks for full set of correct option(s). Otherwise 0.

   Update: Here $\{(a),(b),(d)\}$ **is also considered as full set of correct options and** 2 **marks for the choice** $\{(a),(b),(d)\}$ **.**

8. Let f and g be two linearly independent solutions of [2]

$$(x+1)y'' + y' + y\cos x = 0,$$

   on $(-1,\infty)$. Let $W(f,g;x)$ be the Wronskian of f and g at a point $x \in (-1,\infty)$. Given $W(f,g;0) = 9$, compute

$$W(f,g;2) = 3.$$

9. Consider the IVP for $x \neq 0$

$$y' = \sqrt{\frac{|y|}{|x|}},\ y(1) = y_0,$$

(a) Find all y_0 such that the IVP is guaranteed to have a solution in an interval containing the point 1. Justify your answer. [2]

Ans. The IVP has a solution for every $y_0 \in \mathbb{R}$ in an interval containing 1.

Step 1[1 mark] Let $y_0 \in \mathbb{R}$ be any real number. Consider $a > 0, b > 0$ such that $1-a > 0$ and set $R := \{(x,y) \in \mathbb{R}^2 : |x-1| < a, |y-y_0| < b\}$. Now for any $(x,y) \in R$, since $0 < 1-a < x < 1+a$, the function $f(x,y) := \sqrt{\frac{|y|}{|x|}}, \quad \forall\,(x,y) \in R$ is well-defined and f is continuous on R. Moreover, there exists $M > 0$ such that

$$|f(x,y)| \leq \sqrt{\frac{|y_0|+b}{1-a}} := M, \quad \forall\,(x,y) \in R.$$

Step 2[1 mark] Since for any $y_0 \in \mathbb{R}$, there exists a rectangle R containing $(1, y_0)$ such that f is continuous and bounded on R, from the existence theorem, it is guaranteed that the IVP has a solution for every $y_0 \in \mathbb{R}$ in an interval of 1.

Full 2 marks if explicit solution is **computed correctly for all** $y_0 \in \mathbb{R}$**.**

(b) Find all y_0 such that the IVP is guaranteed to have a unique solution in an interval containing the point 1. Justify your answer.

Ans. The IVP admits a unique solution for every $y_0 \in \mathbb{R}\backslash\{0\}$ in an interval containing 1.

Step 1[1 mark] Let $y_0 \neq 0$. Without loss of generality $y_0 > 0$. Then there exists a rectangle

$$R_1 := \{(x,y) \in \mathbb{R}^2 : |x-1| < a, |y-y_0| < \delta\},$$

that does not contain points $\{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$. So that f is Lipschitz with respect to y on R_1.

Step 2[1 mark] because: the partial derivative of f w.r.to y exists on R_1 and

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \frac{1}{2}\sqrt{\frac{1}{xy}} \leq M_2, \forall (x,y) \in R_1$$

for some $M_2 > 0$. OR it can be shown directly from the definition of the Lip cont. Thus the IVP has a unique solution for $y_0 \in \mathbb{R}\backslash\{0\}$.

Step 3[1 Mark] For $y_0 = 0$, the function f is not Lip w.r.to y on any rectangle containing $(1,0)$. Thus, we cannot apply the 'uniqueness Theorem' and the uniqueness of the solution to IVP with $y_0 = 0$ is not guaranteed.

Or, one can give multiple solutions for $y(1) = 0$:

for any $c \geq 1$, $y(x) = \begin{cases} (\sqrt{x}-c)^2, & x \geq c \\ 0, & x < c \end{cases}$