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Real Analysis

Main Reference: 1. Walter Rudin – Principles of Mathematical Analysis.

Other references:

2. Apostol – Mathematical Analysis.

3. R.G. Bartle & D.R. Sherbert – Introduction to Real Analysis

- Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$
- Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
- Rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\}$
- Real numbers are defined by the following theorem:

We shall define the unexplained terms in Th below.

- **Theorem 1:** There exists an ordered field $\mathbb{R}$ which has the least upper bound property.

  Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Remark: 1. The set $\mathbb{R}$ is called set of real numbers.

2. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield means $\mathbb{Q} \subseteq \mathbb{R}$ and the operations of addition and multiplication of $\mathbb{R}$ (field) when applied to members of $\mathbb{Q}$, coincide with usual addition and multiplication of $\mathbb{Q}$.

Also, positive rational numbers are positive elements of $\mathbb{R}$.

3. Existance of $\mathbb{R}$ as unique (upto isomorphism).

- Now we shall define the terms of the theorem above:

- **Fields:-**

Defn: A field is a set F with two operations, called addition and multiplication, satisfies the following (1), ~~and~~ (2) and (3):

(1) **Axioms of addition:-**

(a) If $x, y \in F$ then $x+y \in F$

(b) $x+y = y+x, \ \forall x, y \in F$

(c) $(x+y)+z = x+(y+z), \ \forall x, y, z \in F$

(d) F contains an element $0 \in F$ s.t $x+0 = x, \forall x \in F$.

(e) For every $x \in F$, $\exists -x \in F$ s.t. $x+(-x) = 0$.

(2) **Axioms of Multiplication:**

(i) $x, y \in F \Rightarrow x \cdot y \in F$

(ii) $x \cdot y = y \cdot x, \ \forall x, y \in F$

(iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in F$

(iv) F contains an element 1 s.t $1 \cdot x = x \quad \forall x \in F$.

(v) For, $x \in F$ and $x \neq 0$, $\exists \, 1/x \in F$ s.t. $x \cdot \frac{1}{x} = 1$.

(3) The Distributive law:

$$x\cdot(y+z) = x\cdot y + x\cdot z, \ \forall x,y,z \in F.$$

• Example: 1. Is $(\mathbb{Z}, +, \cdot)$ is a field? X

Question: Is $(\mathbb{N}, +, \cdot)$ is a field?

• ~~Ordered set:~~ Ordered set:-

1. Defn:- Let S be a set. An order on S is a relation, denoted by $<$, such that

(i) If $x, y \in S$, then one and only one of the following is true:

either, $x<y$, or, $x=y$ or, $y<x$.

(ii) If $x,y,z \in S$ with $x<y$ and $y<z$ then $x<z$.

2. Defn: An ordered set is a set S, in which an order is defined.

Notation:- "$x \leq y$" means either $x<y$ or $x=y$.

Example (of an ordered set):

$\mathbb{Q}$ with order $<$ means $x<y$ if $y-x$ is a positive rational number.

3. Defn: (Ordered Field):- An ordered field is a field F, which is also an ordered set with the following conditions:-

(i) $x+y<x+z$ if $y<z$ and $x,y,z \in F$

(ii) $x,y \in F$ and $x>0, y>0 \implies x \cdot y>0$.

Example of an ordered field: $(\mathbb{Q}, <)$ is an ordered field.

4. Defn: Let F be an ordered field. If $x>0$, we call x is positive; if $x<0$ we call x is negative.

- So far we have defined ordered field, mentioned in Theorem 1. Now we shall define "least upper bound property".

• Least upper bound property:-

Defn: (Bounded above):- Let S be an ordered set and $E \subseteq S$. We say E is bounded above if $\exists$ a $\beta \in S$ such that $x \leq \beta$, $\forall x \in E$.

Also, we call β is an upper bound of E.

- E is bounded below if $\exists \alpha \in S$ such that $\alpha \leq x$, $\forall x \in E$.

  – Such an α is called lower bound of E.

Defn: (Least upper bound / Supremum):-

Let S be an ordered set, ~~and~~ $E \subseteq S$ and E is bounded above. Suppose $\exists$ an $\alpha \in S$ such that (i) $x \leq \alpha$ $\forall x \in E$,

(ii) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called least upper bound of E, and we write $\alpha = \sup E$.

Defn. (greatest lower bound / infimum) :-

Let S be an ordered set, E ⊆ S and E is bounded below. Suppose there exists $\beta \in S$ such that

(i) $\beta \leq x \quad \forall x \in E$

(ii) if $\gamma > \beta$, then γ is not a lower bound of E.

Then β is called greatest lower bound / infimum of E and we write

$$\beta = \inf E.$$

Exercise: 1. Prove that supremum / infimum of a set (if exist) is unique.

2. Is $\mathbb{N}$ bounded above?

Examples:

1. Let $A = \{ p \in \mathbb{Q} : p \text{ is +ve and } p^2 < 2 \} \subseteq \mathbb{Q}$.

Then A is bounded above and has no supremum in $\mathbb{Q}$. (why?)

2. $E_1 = \{ r \in \mathbb{Q} : r < 0 \} \subseteq \mathbb{Q}$.

- $\sup E_1 = ?$
- $\inf E_1 = ?$
- Does supremum of E_1 belong to E_1 ?.

Remark: Supremum/infimum of a set $E \subseteq S$ may/or may not belong to the set E.

- Defn: (Least upper bound property):-

An ordered set S is said to have the least-upper-bound property if every non-empty bounded above subset E of S has supremum in S.

Now you may go back to Theorem 1, which defines real numbers:

$\mathbb{R}$ — is an ordered field having "least upper bound" property. It also contains $\mathbb{Q}$ as a subfield.

Question: Does $\mathbb{Q}$ satisfies all the properties of $\mathbb{R}$?

Ans: No!

$\mathbb{Q}$ is an ordered field, but do not satisfy least upper bound property (why?).

Consider,

$$A = \{ p \in \mathbb{Q} : p > 0 \text{ and } p^2 < 2 \} \subseteq \mathbb{Q}$$

— bounded above.

Does it have supremum in $\mathbb{Q}$??

Ex: Let $A \subseteq \mathbb{R}$ is bounded above. Then

$\alpha = \sup A$

$\Longleftrightarrow$

(i) $x \leq \alpha \quad \forall x \in A$

(ii) for every $\varepsilon > 0$, $\exists y \in A$ s.t. $\alpha - \varepsilon < y$.