MA 108-ODE- D3

Lecture 9

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Second order Homogeneous ODEs

Finding linearly independent solutions

We've been looking at the second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p, q are continuous on I, an interval of $\mathbb{R}$.

As we remarked earlier, there is no general method to find a basis of solutions.
However, if we know one no-where zero solution f then we have a method to find another solution g such that f and g are linearly independent.

To find such a g, set

$$g(x) = v(x)f(x).$$

We'll choose v such that $\{f, g\}$ are linearly independent.

Can v be a constant? No.

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Now for $g(x) = v(x)f(x)$ to be a solution of the given ODE

$$g'' + p(x)g' + q(x)g = 0.$$

i.e.,

$$(vf)'' + p(x)(vf)' + q(x)(vf) = 0.$$

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Thus,

$$\begin{aligned} 0 &= (v'f + vf')' + p(v'f + vf') + qvf \\ &= v''f + 2v'f' + vf'' + p(v'f + vf') + qvf \\ &= v(f'' + pf' + qf) + v'(2f' + pf) + v''f. \end{aligned}$$

Thus, get $v'' + \left[\frac{2f'+pf}{f}\right] v' = 0$, and set $u(x) = v'(x)$, get a linear first order ODE

$$u' + \left[\frac{2f' + pf}{f}\right] u = 0.$$

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Therefore, using integrating factor and solving, get

$$u(x) = \frac{e^{-\int p(x)dx}}{f^2(x)} \quad \forall \quad x \in I,$$

i.e., $v'(x) = \frac{e^{-\int p(x)dx}}{f^2(x)}$ on I, and thus

$$v(x) = \int \frac{e^{-\int p(x)dx}}{f^2(x)} dx, \quad \text{on} \quad I.$$

Claim: f and vf are linearly independent.
Proof. Enough to check Wronskian!

$$\begin{aligned} W(f, vf) &= f(v'f + f'v) - f'vf \\ &= f^2 v' \\ &= f^2 \frac{e^{-\int pdx}}{f^2} \\ &= e^{-\int pdx} \\ &\neq 0. \end{aligned}$$

Second Order Linear ODE's

Example. Find all solutions of

$$y'' - 2y' + y = 0.$$

Note $f(x) = e^x$ is a solution. How do you find another linearly independent solution on $\mathbb{R}$? Let $g(x) = v(x)f(x) = v(x)e^x$ be another solution. Then as shown in the method above $v(x)$ can be obtained as

$$v(x) = \int \frac{e^{-\int p dx}}{f^2} dx,$$

where $p(x) = -2$ and $f(x) = e^x$. Thus, $v(x) = x$.

Two linearly independent solution of the second order ODE: $f(x) = e^x$ and $g(x) = xe^x$ on $\mathbb{R}$.
General solution: $y(x) = c_1 e^x + c_2 x e^x$, for any real numbers c_1 and c_2.

Second Order Linear Homogeneous DE's with constant coefficients

We have developed enough theory to now find all solutions of

$$y'' + py' + qy = 0,$$

where p and q are in $\mathbb{R}$, i.e., a second order homogeneous linear ODE with constant coefficients. Suppose e^{mx} is a solution of this equation. Then,

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0,$$

and this implies

$$m^2 + pm + q = 0.$$

This is called the characteristic equation of the linear homogeneous ODE with constant coefficients. The roots of this equation are

$$m_1, m_2 = -\frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

Second Order Linear ODE's

Case I: $m_1, m_2 \in \mathbb{R}, m_1 \neq m_2$.
When $p^2 - 4q > 0$, m_1 and m_2 are distinct real numbers. Moreover,

$$W(e^{m_1 x}, e^{m_2 x}; x) \neq 0, \quad x \in \mathbb{R}.$$

Hence, $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent on $\mathbb{R}$. So the general solution of

$$y'' + py' - qy = 0$$

is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

where $c_1, c_2 \in \mathbb{R}$.

Second Order Linear ODE's

Case II: $m_1 = m_2 \in \mathbb{R}$.

$$m_1 = m_2 \iff p^2 - 4q = 0,$$

and in this case $m_1 = m_2 = -\frac{p}{2}$. Hence $e^{-\frac{px}{2}}$ is one solution. To find the other solution, let

$$g(x) = v(x)e^{-\frac{px}{2}}.$$

Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{e^{-px}} dx \\ &= ax + b, \end{aligned}$$

(note: $p \in \mathbb{R}$) for some $a, b \in \mathbb{R}$. Choose $v(x) = x$. Then, $g(x) = xe^{-\frac{px}{2}}$. Hence the general solution is

$$ae^{-\frac{px}{2}} + bxe^{-\frac{px}{2}},$$

with $a, b \in \mathbb{R}$.

Second Order Linear ODE's

Case III: $m_1 \neq m_2 \in \mathbb{C}\backslash\mathbb{R}$.
$m^2 + pm + q = 0$ has distinct complex roots if and only if $p^2 - 4q < 0$.
In this case, let

$$m_1 = a + \imath b, m_2 = a - \imath b.$$

(Why not $a_1 + \imath b_1, a_2 + \imath b_2$?) Thus,

$$e^{m_1 x} = e^{(a+\imath b)x} = e^{ax}(\cos bx + \imath \sin bx),$$

and

$$e^{m_2 x} = e^{(a-\imath b)x} = e^{ax}(\cos bx - \imath \sin bx).$$

As we are only interested in real valued functions, we take

$$f(x) = \frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx,$$

and

$$g(x) = \frac{e^{m_1 x} - e^{m_2}}{2\imath} = e^{ax} \sin bx.$$

Second Order Linear ODE's

Now, check

$$W(f, g; x) \neq 0 \quad x \in \mathbb{R}.$$

Thus the general solution is of the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx),$$

with $c_1, c_2 \in \mathbb{R}$.

Example

Example: Find the general solution of $y'' + 4y' + 13y = 0$.

Characteristic polynomial is $m^2 + 4m + 13 = (m+2)^2 + 9$.

Roots of the characteristic equation are $-2+3i$ and $-2-3i$.

It is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions. To obtain real valued solutions, consider the sum and difference given by $e^{-2x}\cos 3x$ and $e^{-2x}\sin 3x$ respectively.

Hence the general solution is

$$y(x) = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x),$$

where $c_1, c_2 \in \mathbb{R}$.

Example: Umdamped Simple Harmonic Vibrations

Consider a cart of mass M attached to a wall by means of a spring. At the equilibrium position $x = 0$, the spring exerts no force. If the center of the cart is at a distance x from the spring, the spring exerts a force

$$F_s = -kx$$

where $k > 0$ is a constant. By Newton's second law of motion,

$$M\frac{d^2x}{dt^2} = -kx.$$

Set

$$a = \sqrt{\frac{k}{M}}.$$

Then the characteristic equation of the above differential equation is

$$m^2 + a^2 = 0,$$

i.e., $m = \pm \imath a$. Hence, the general solution of the above DE is

$$x = c_1 \sin at + c_2 \cos at$$

where $c_1, c_1 \in \mathbb{R}$.

Example continued

Suppose the cart is pulled aside to the initial position $x = x_0$ with initial velocity $v = \frac{dx}{dt} = 0$ at $t = 0$. Then

$$c_2 = x_0 \text{ and } c_1 = 0.$$

Hence,

$$x = x_0 \cos at.$$

where x_0 is the amplitude of the simple harmonic vibration and T is the time required for one complete cycle. Then

$$aT = 2\pi,$$

i.e.,

$$T = \frac{2\pi}{a} = 2\pi\sqrt{\frac{M}{k}}.$$

Example: Damped Vibrations

Suppose, in the previous example, the cart of mass M experiences a damping force F_d, due to the viscosity of the medium through which it moves, such a air or water. Assume that

$$F_d = -c\frac{dx}{dt},$$

where c is a positive constant that measures the resistance of the medium. Then

$$M\frac{d^2x}{dt^2} = F_s + F_d$$

i.e.,

$$\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = 0.$$

Set

$$2b = \frac{c}{M} \text{ and } a = \sqrt{\frac{k}{M}}.$$

Then the characteristic equation of the above DE is

$$m^2 + 2bm + a^2 = 0.$$

The roots m_1, m_2 are given by $-b \pm \sqrt{b^2 - a^2}$.

Example continued

Case I: $b > a$. Then m_1 and m_2 are distinct negative real numbers.

The general solution of the DE is

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

The initial conditions are

$$x(0) = x_0 \text{ and } \frac{dx}{dt}(0) = 0.$$

Since

$$\frac{dx}{dt} = m_1 c_1 e^{m_1 t} + m_2 c_2 e^{m_2 t},$$

it follows that

$$\begin{aligned} c_1 + c_2 &= x_0 \\ m_1 c_1 + m_2 c_2 &= 0. \end{aligned}$$

Solving the above two equations for c_1 and c_2,

$$c_1 = \frac{m_2 x_0}{m_2 - m_1} \text{ and } c_2 = \frac{m_1 x_0}{m_1 - m_2}.$$

Hence

$$x(t) = \frac{x_0}{m_1 - m_2}\left(m_1 e^{m_2 t} - m_2 e^{m_1 t}\right)$$

Example continued

Case II: $b = a$. Then $m_1 = m_2 = -b = -a$.

Hence

$$x(t) = c_1 e^{-at} + c_2 t e^{-at}$$

is the general solution to the given DE. The initial conditions are

$$x(0) = x_0 \text{ and } \frac{dx}{dt}(0) = 0.$$

It follows that

$$\begin{aligned} x_0 &= c_1, \\ 0 &= -ac_1 + c_2. \end{aligned}$$

Hence,

$$x(t) = x_0 e^{-at}(1 + at).$$

Example continued

Case III: $b < a$. Then m_1, m_2 are complex conjugates of each other and are given by

$$-b \pm \sqrt{b^2 - a^2}.$$

Set $\alpha = \sqrt{a^2 - b^2}$. Then m_1, m_2 are given by $-b \pm \imath\alpha$. The general solution of the given DE is

$$x(t) = e^{-bt}(c_1 \cos \alpha t + c_2 \sin \alpha t).$$

Recall the initial conditions

$$x(0) = x_0 \text{ and } \frac{dx}{dt}(0) = 0.$$

It follows that $x_0 = c_1$ and

$$\frac{dx}{dt} = -be^{-bt}(c_1 \cos \alpha t + c_2 \sin \alpha t) + e^{-bt}(c_2 \alpha \cos \alpha t - c_1 \alpha \sin \alpha t)$$

so that at $t = 0$, we get

$$c_2 = \frac{bc_1}{\alpha} = \frac{bx_0}{\alpha}.$$

Hence

$$x(t) = e^{-bt}(x_0 \cos \alpha t + \frac{bx_0}{\alpha} \sin \alpha t).$$

Example continued

Recall that

$$x(t) = e^{-bt}(x_0 \cos \alpha t + \frac{bx_0}{\alpha} \sin \alpha t).$$

Let $\tan \theta = \frac{b}{\alpha}$. Then

$$x(t) = \frac{x_0 e^{-bt}\sqrt{b^2 + \alpha^2}}{\alpha} \cos(\alpha t - \theta).$$

This function oscillates with an amplitude which decreases exponentially. The graph passes the equilibrium position $x = 0$ at regular intervals. Let T be the time required for one complete cycle. Then

$$\alpha T = 2\pi,$$

i.e.,

$$T = \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}}.$$

Cauchy-Euler Equations

The equation

$$x^2 y'' + axy' + by = 0$$

where $a, b \in \mathbb{R}$ is called a Cauchy-Euler equation. Assume $x > 0$. Suppose $y = x^m$ is a solution to this DE. Then,

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0.$$

We get:

$$m(m-1) + am + b = 0.$$

i.e.,

$$m^2 + (a-1)m + b = 0.$$

This is called the auxiliary equation of the given Cauchy-Euler equation. The roots are

$$m_1, m_2 = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

Cauchy-Euler Equations

Case I: Distinct real roots.
Are x^{m_1} and x^{m_2} linearly independent? Yes. Hence the general solution is given by

$$y = c_1 x^{m_1} + c_2 x^{m_2},$$

for $c_1, c_2 \in \mathbb{R}$.

Cauchy-Euler Equations

Case II: Equal real roots.
i.e.,

$$m_1 = m_2 = \frac{1-a}{2}.$$

Hence

$$y = f(x) = x^{\frac{1-a}{2}}$$

is a solution. To get a solution g linearly independent from f, set

$$g(x) = v(x)f(x).$$

Now,

$$g' = v'f + vf',$$

and

$$g'' = v''f + 2v'f' + f''v.$$

Thus,

$$x^2(v''f + 2v'f' + f''v) + ax(v'f + vf') + bvf = 0;$$

Cauchy-Euler Equations

$$x^2(v''f + 2v'f' + f''v) + ax(v'f + vf') + bvf = 0;$$

i.e.,

$$v(x^2f'' + axf' + bf) + x^2(v''f + 2v'f') + axv'f = 0.$$

Thus,

$$x^2v''f + v'x(2f'x + af) = 0.$$

Now, for $f(x) = x^{\frac{1-a}{2}}$, we have:

$$2f'x + af = 2x \cdot \frac{1-a}{2} x^{-\frac{1}{2}(a+1)} + ax^{\frac{1-a}{2}}$$

$$= (1-a)x^{\frac{1-a}{2}} + ax^{\frac{1-a}{2}} = x^{\frac{1-a}{2}} = f(x).$$

Hence, we get:

$$x^2v''f + v'xf = 0.$$

i.e.,

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Cauchy-Euler Equations

Hence,

$$\ln|v'| = -\ln x = \ln\left(\frac{1}{x}\right).$$

Take,

$$v' = \frac{1}{x}.$$

Set

$$v(x) = \ln x, \quad x > 0.$$

Hence,

$$g(x) = (\ln x)x^{\frac{1-a}{2}}.$$

Thus the general solution is given by

$$y = c_1 x^{\frac{1-a}{2}} + c_2 x^{\frac{1-a}{2}} \ln x,$$

$c_1, c_2 \in \mathbb{R}$.

Cauchy-Euler Equations

Case III: Complex roots.

Roots are $m_1 = \mu + \imath\nu,\ m_2 = \mu - \imath\nu$.

$$x^{m_1} = x^{\mu} x^{\imath\nu} = x^{\mu} \left(e^{\ln x}\right)^{\imath\nu} = x^{\mu} e^{\imath\nu \ln x} = x^{\mu}(\cos(\nu \ln x) + \imath \sin(\nu \ln x)),$$

Similarly,

$$x^{m_2} = x^{\mu} e^{-\imath\nu \ln x} = x^{\mu}(\cos(\nu \ln x) - \imath \sin(\nu \ln x)),$$

It follows that

$$\frac{x^{m_1} + x^{m_2}}{2} = x^{\mu} \cos(\nu \ln x),$$
$$\frac{x^{m_1} - x^{m_2}}{2\imath} = x^{\mu} \sin(\nu \ln x)$$

These are linearly independent solutions of the given DE and hence the general solution is given by

$$y = x^{\mu}(c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)),$$

$c_1, c_2 \in \mathbb{R}$.