Lecture 7

Punit Parmananda

PH111

Email : punit@phy.iitb.ac.in

Office: Room 203, 2nd Floor, Physics Department

$$\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + \left[\frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}\right]_{rot}$$

RECAP

$$+\vec{\Omega}\times(\vec{v}_{rot}+\vec{\Omega}\times\vec{r})$$

Let us assume a constant angular velocity. Then the above equation can be written as

$$\vec{a}_{in} = \vec{a}_{rot} + \boxed{2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

Having calculated the accelerations, we can calculate the pseudo forces as before. In *S* frame:

$$\vec{F} = m\vec{a}_{in}$$

where $\vec{F}$ is sum of all real forces

Coriolis Force

RECAP

- The magnitude of the Coriolis force increases from zero at the Equator to a maximum at the poles.
- The Coriolis force acts at right angles to the direction of motion, so as to cause deflection to the *right* in the Northern Hemisphere and to the *left* in the Southern Hemisphere.

Foucault Pendulum RECAP

The Foucault Pendulum was conceived by Léon Foucault in the middle of the 19th century, with the goal of proving Earth's rotation through the effect of the *Coriolis Force*. In essence, the *Foucault Pendulum* is a Pendulum with a long enough damping rate such that the precession of its plane of oscillations can be observed after typically an hour or more. A whole revolution of the plane of oscillation takes anywhere between a day if it is at the pole, or longer at lower latitudes. At the equator the plane of oscillation does not rotate at all. (Note that if a precession period is defined as a rotation of 180° of the plane of oscillation, then the period at the pole is 12 hrs).

Foucault's Pendulum, *(the Panthéon, Paris, Photograph taken by Michael Reeve, 30/1/04)*

Consider a pendulum of mass m that is swinging with frequency $\omega = \sqrt{g/l}$, where l is the length of the pendulum. If we describe the position of the pendulum's bob in the horizontal plane by coordinates r, θ, then

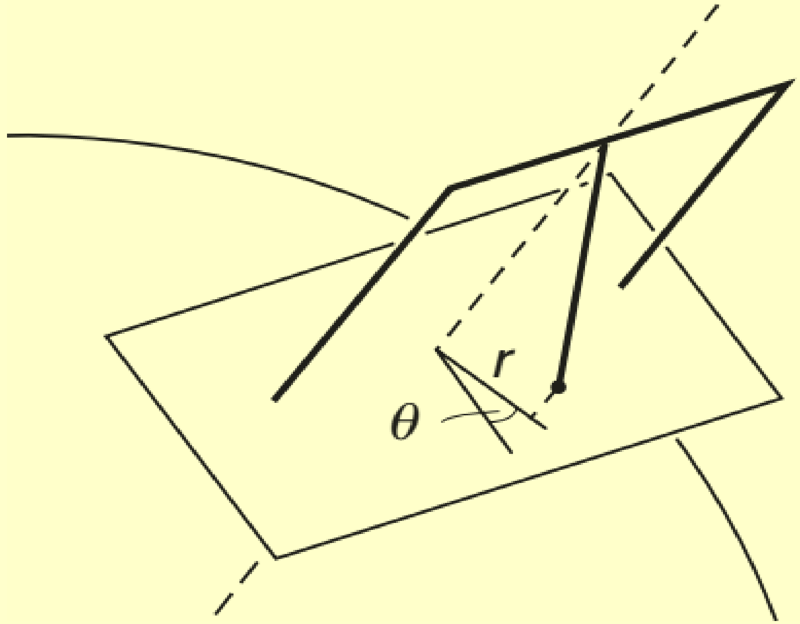
$$r = r_0 \sin\omega t,$$

where r_0 is the amplitude of the motion. In the absence of the Coriolis force, there are no tangential forces and θ is constant.

RECAP

$$r = r_o \sin \omega t; \omega = \sqrt{\frac{g}{l}}$$

θ (Precession angle along the horizontal plane)



As the pendulum oscillates, a tangential Coriolis force is felt by the bob, being zero at the end and maximum at *O*. This causes the pendulum to continuously change the plane of oscillation.

$$\boxed{= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

RECAP

The tangential equation of motion $ma_\theta =$

$$\vec{F}_{coriolis} = -2m\Omega \frac{dr}{dt}\hat{\theta}$$

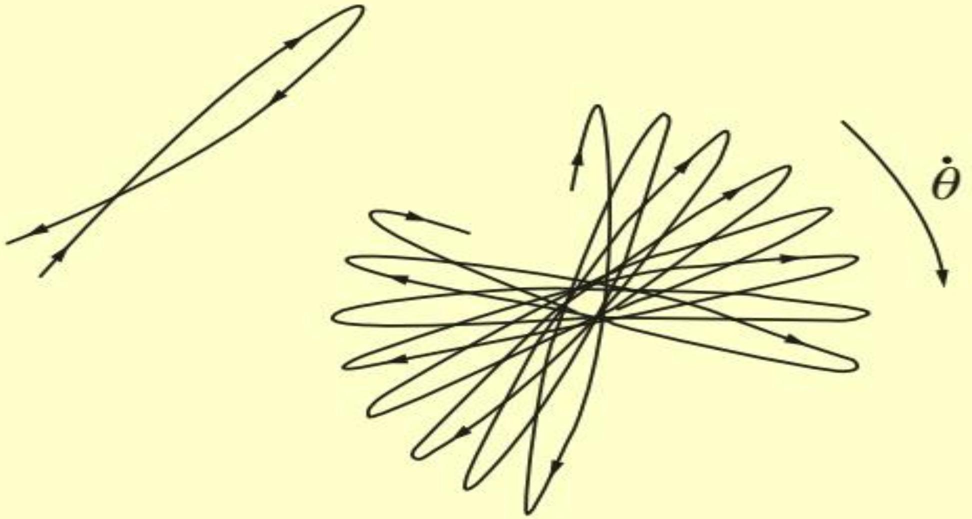
We thus get

$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right) = -2m\Omega \frac{dr}{dt}$$

The simplest solution correspond s to $\frac{d^2\theta}{dt^2} = 0;$

giving $\frac{d\theta}{dt} = -\Omega$

RECAP



The pendulum precesses uniformly in a clockwise direction. The time for the plane of oscillation to rotate once is

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\Omega}$$

RECAP

If T is the time period to complete one revolution, this would be 24 hrs. This result is obvious from the point of view of an inertial observer, with respect to whom the plane is fixed and the earth is rotating below it.

For a latitude λ, only the vertical component of angular velocity would cause tangential coriolis force. Hence, T=24hr /sin λ.

Central Force

Question: What is a central force?

Answer: Any force which is directed towards a center, and depends only on the distance between the center and the particle in question $f(r)\hat{r}$.

Question: Any examples of central forces in nature?

Answer: Two fundamental forces of nature, gravitation, and Coulomb forces are central forces.

Central Force Properties

(1) Central Force is conservative: The work done is given by

$$\int_c \vec{F}\cdot\overrightarrow{ds} = \int_c \vec{F}\cdot(dr\,\hat{r} + rd\theta\,\hat{\theta})$$

$$= \int_c f(r)\,\hat{r}\cdot(dr\,\hat{r} + rd\theta\,\hat{\theta}) \qquad \boxed{f(r)\hat{r}}$$

$$= \int f(r)dr$$

As in the final expression there is no reference of θ, the work done does not depend on the path.

(2) Angular Momentum is conserved: The torque about the origin can be written as follows.

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times f(r)\hat{r} = 0$$

As the torque is zero the angular momentum is constant of motion

(3) The motion is confined to a plane: As the angular momentum is constant and its direction is always perpendicular to the velocity direction, the motion is always confined to a plane.

(4) The Kepler's second law (Law off equal areas) is obeyed: The expression for the tangential expression can be written as follows.

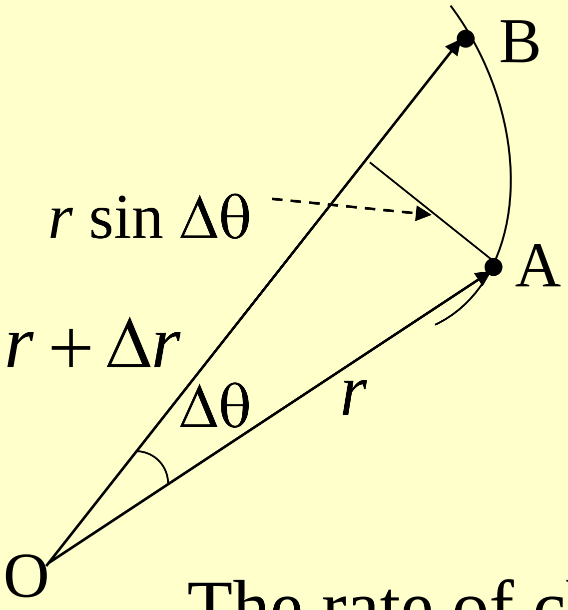
Recall:

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

$$a_t = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

If there is no force in the tangential direction, the tangential acceleration would be zero. This would imply that $(r^2\dot{\theta})$ would be constant. How ?



If a particle moves between positions A and B in time Δt, then area ΔA subtended by it at the origin is given by

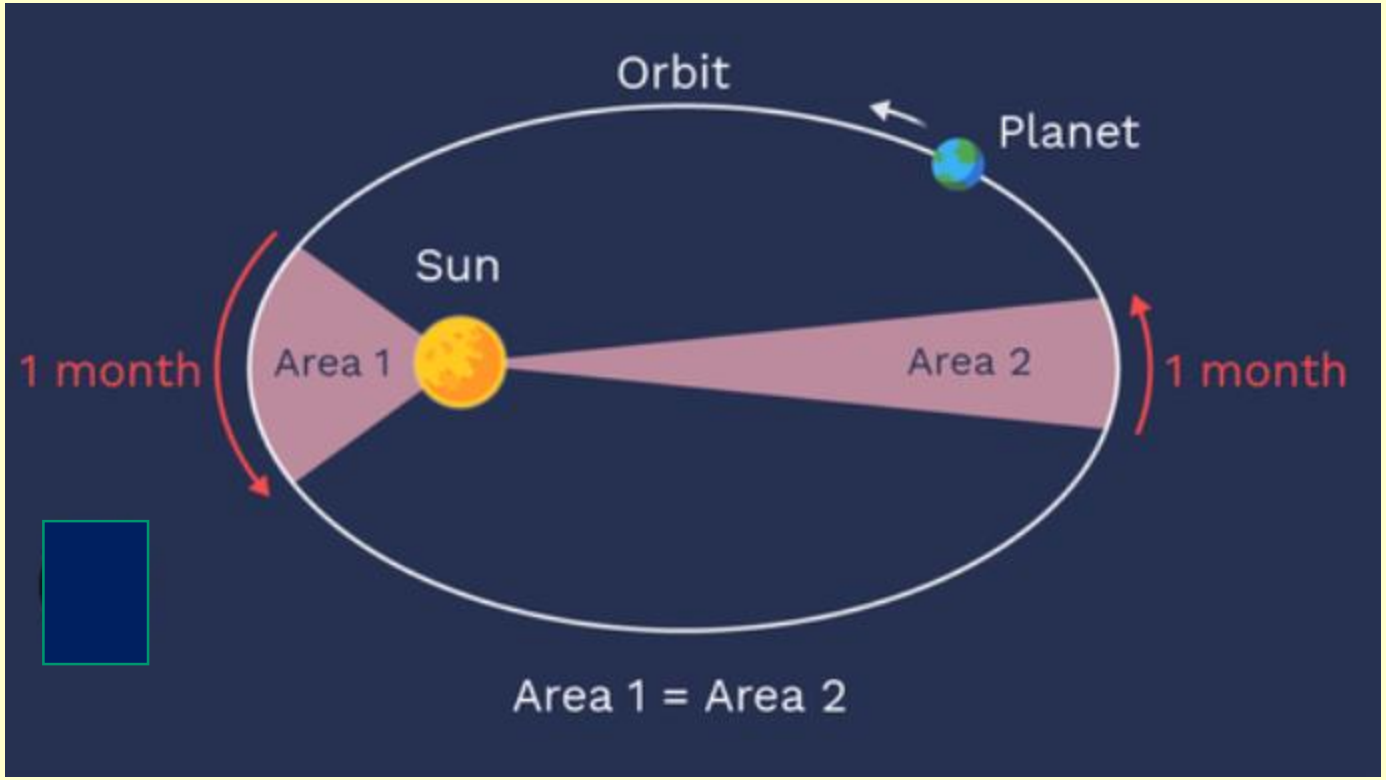
$$\Delta A = \frac{1}{2}(r + \Delta r) r \sin \Delta\theta$$

$$\approx \frac{1}{2} r^2 \Delta\theta$$

The rate of change of area is, therefore, given by the following equation

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

We, thus, get that if the tangential acceleration is zero, which would happen if the tangential force is zero, ***then the rate of change of area at the center would be constant. This is known as Kepler's second law,*** observed for the motion of the planets. Clearly the gravitational force is always radial in the case of planets, hence the tangential acceleration is zero.



Kepler's Second Law says that a line running from the sun to the planet sweeps out equal areas of the ellipse in equal times. This would imply that the planet speeds up as it is closer to the sun and slows down if is far from it

5) **Energy is conserved**

- Kinetic energy in plane polar coordinates can be written as

$$\begin{aligned} K &= \frac{1}{2}\mu \mathbf{v}\cdot\mathbf{v} \\ &= \frac{1}{2}\mu\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right).\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 \end{aligned}$$

$$m = \mu$$

- Potential energy $V(r)$ can be obtained by the basic formula

$$V(r) - V(r_O) = -\int_o^r f(r)dr,$$

where r_O denotes the location of a reference point.

- Total energy E from work-energy theorem

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V(r) = \text{constant}$$

- We have

$$L = \mu r^2 \dot{\theta}$$

$$\Longrightarrow \frac{1}{2}\mu r^2 \dot{\theta}^2 = \frac{L^2}{2\mu r^2}$$

- So that

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

- We can write

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

$$\boxed{\text{with } V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)}$$

- This energy is similar to that of a 1D system, with an effective potential energy $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$
- In reality $\frac{L^2}{2\mu r^2}$ is kinetic energy of the particle due to angular motion
- But, because of its dependence on position, it can be treated as an effective potential energy

(6) The equations of motion: For a particle under the influence of central force the equation of motion in radial direction would be given as follows.

$$m(\ddot{r}-r\dot{\theta}^2)=f(r) \qquad \mathbf{a}=\ddot{\mathbf{r}}=(\ddot{r}-r\dot{\theta}^2)\hat{\mathbf{r}}+(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

The tangential equation can be written as: $r^2\dot{\theta}=constant=k\text{(say)}$

This yields:

$$m\left(\ddot{r}-\frac{k^2}{r^3}\right)=f(r)$$

$$\frac{d\dot{r}}{dt}=\ddot{r} \qquad m\left(\frac{d\dot{r}}{dr}\frac{dr}{dt}-\frac{k^2}{r^3}\right)=f(r)$$

$$m\left(\dot{r}d\dot{r}-\frac{k^2}{r^3}dr\right)=f(r)dr$$

Integrating the above equation we get

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = -(V(\mathbf{r}_b) - V(\mathbf{r}_a))$$

$$\frac{1}{2}m\dot{r}^2 - \int \frac{k^2}{r^3}dr = \int f(r)dr$$

l=L=angular momentum

$$r^2\dot{\theta} = constant = k(\text{say})$$

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{k^2}{r^2}\right) + U(r) = E$$

Here $\boxed{U(r)/V(r)}$ is potential energy function and E is constant of integration, which can easily be recognized as the total energy, after substituting the value of k.

k, on the other hand, can be related to the magnitude of the angular momentum of the particle about the origin as follows.

$$|\vec{l}| = |r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})|$$

$$= mr^2\dot{\theta}$$

$$= mk$$

$$= l(\text{say})$$

Recall: $r^2\dot{\theta} = constant = k(\text{say})$

$$m = \mu$$

For a central force motion the torque is zero.

$$\vec{\tau} = r\hat{r} \times f(r)\hat{r}$$
$$= 0$$

As the total energy and the angular momentum are constants of motion One can, then solve the equation for $\dot{r}$ and $\dot{\theta}$, to evaluate r and θ as a function of time. The equation of the orbit can be obtained by dividing the two expressions.

$$\frac{1}{2}m\dot{r}^2 - \int \frac{k^2}{r^3}dr = \int f(r)dr$$
$$\frac{1}{2}m\left(\dot{r}^2 + \frac{k^2}{r^2}\right) + U(r) = E$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

$$\left|\vec{l}\right| = \left|r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\right|$$
$$= mr^2\dot{\theta}$$
$$= mk$$
$$= l(\text{say})$$

$$\frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\frac{dr}{d\theta} = \frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

U(r)=V(r) **= Potential Energy**

Problem: A particle of mass m is moving under the influence of an inverse cubic central force of the type *(– Cr* $^{-3}$*)*. Find the orbit of the particle, if its total energy is zero and the angular *l* momentum is given by $l^2 = mC/2$. For what values of energy and the angular momentum, the orbit would be one, in which r would be proportional to θ.

First calculate *U(r)* and substitute in the differential equation of the orbit, along with the energy and angular momentum values

$$\boxed{\frac{dr}{d\theta} = \frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}}$$

$$U(r) = -\frac{C}{2r^2}$$

$$\frac{dr}{d\theta} = r^2\sqrt{\frac{2m}{C}}\sqrt{\frac{2}{m}\left(\frac{C}{2r^2} - \frac{C}{4r^2}\right)}$$

$$= \pm r$$

Integrating we get $$r = r_o \exp(\pm\theta)$$

Going back to the general differential equation of orbit and substituting only the value of *U(r)*.

$$\frac{dr}{d\theta} = \frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E + \frac{C}{r^2} - \frac{l^2}{2mr^2}\right)}$$

In case the term under the square root sign is constant, we shall get the desired orbit. Hence we must have the following.

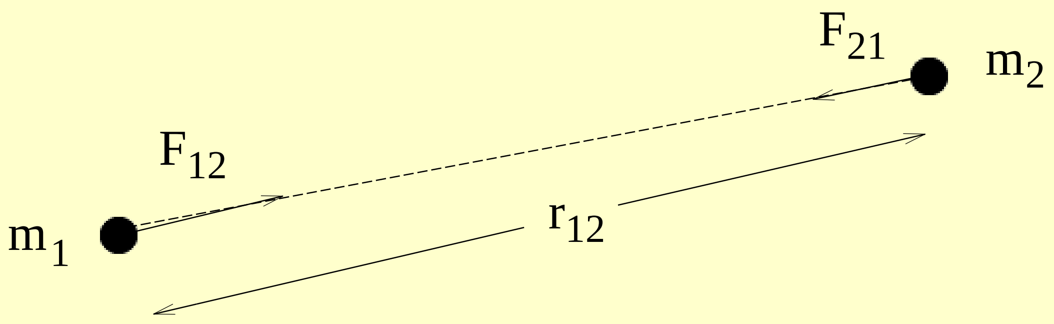
$$C = \frac{l^2}{2m}$$

Revisiting Central force

Gravitational force acting on mass m_1 due to mass m_2 is

$$\mathbf{F}_{12} = -\frac{G m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12},$$

i.e., it acts along the line joining the two masses



Analogously, the Coulomb force between two charges q_1 and q_2 is given by

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\hat{\mathbf{r}}_{12}.$$

Reduction of a two-body central force problem to a one-body problem

An ideal central force is of the form

$$\mathbf{F}(r) = f(r)\hat{\mathbf{r}},$$

i.e., it is a one-body force depending on the coordinates of only the particle on which it acts

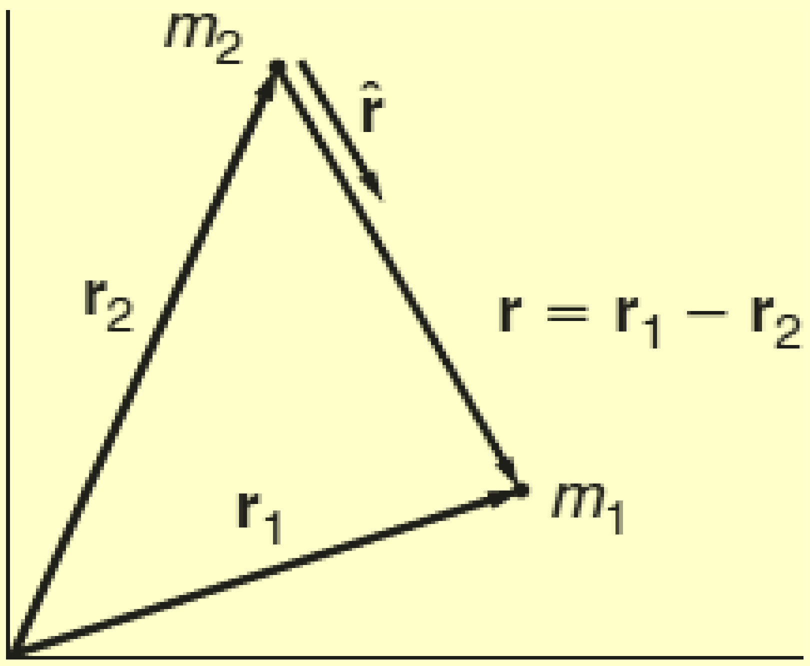
But gravity and Coulomb forces are two-body forces, of the form

$$\mathbf{F}(r_{12}) = f(r_{12})\hat{\mathbf{r}}_{12}$$

But, fortunately, they can be reduced to a pure one-body form

Reduction of a two-body central force problem to a one-body problem

Relevant coordinates are shown in the figure



Define:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\Rightarrow \; r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$$

Given $\mathbf{F}_{12} = f(r)\hat{\mathbf{r}}$, we have

$$m_1 \ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$

$$m_2 \ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

Decoupling equations of motion

Both the equations above are coupled, because both depend upon $\mathbf{r}_1$ and $\mathbf{r}_2$.

In order to decouple them, we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (called relative coordinate), and center of mass coordinate $\mathbf{R}$

$$\mathsf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

$$\ddot{\mathsf{R}} = \frac{m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2}{m_1 + m_2} = \frac{f\hat{\mathbf{r}} - f\hat{\mathbf{r}}}{m_1 + m_2} = 0$$

$$\Longrightarrow \mathsf{R} = \mathsf{R}_0 + \mathbf{V}t,$$

above $\mathbf{R}_0$ is the initial location of center of mass, and $\mathbf{V}$ is the center of mass velocity.

Decoupling equations of motion

This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.

$$\boxed{\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= f(r)\hat{\mathbf{r}} \\ m_2 \ddot{\mathbf{r}}_2 &= -f(r)\hat{\mathbf{r}} \end{aligned}}$$

Moreover, $\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = f(r)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{\mathbf{r}}$

$$\Longrightarrow \ddot{\mathbf{r}} = \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(r)\hat{\mathbf{r}}$$

$$\mu \ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}},$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, is the reduced mass

OBSERVATIONS

Note that this final equation is entirely in terms of relative coordinate **r**

It is an effective/equivalent equation of motion for a single particle of mass μ, moving under the influence of force $f(r)\hat{\mathbf{r}}$.

There is just one coordinate (**r**) involved in this equation of motion

Thus the two body problem has been effectively reduced to a one-body problem

This separation was possible only because the two-body force is central, i.e., along the line joining the two particles

In order to solve this equation, we need to know the nature of the force, i.e., $f(r)$.

Apply to the case of gravitational problem such as planetary orbits