MA 108-ODE- D3

Lecture 13

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nth order linear DE: Constant coefficient

Constant coefficient Differential Operators: Recall

Set

$$D^k = \frac{d^k}{dx^k}, k = 0, 1, 2, \ldots$$

and let

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n I,$$

and

$$M = b_0 D^m + b_1 D^{m-1} + \ldots + b_{m-1} D + b_m I$$

be constant coefficient linear differential operators, i.e., $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m \in \mathbb{R}$. Since,

$$D^r \cdot D^s = D^s \cdot D^r,$$

for $r, s \geq 0$, it follows that

$$L(M(\cdot)) = M(L(\cdot)).$$

Homogeneous n th order ODE with constant coefficients

If

$$L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n I,$$

the characteristic polynomial of the differential operator L is defined by:

$$P_L(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$

Constant Differential Operators

Theorem

Let L and M be two constant coefficient linear differential operators. Then,

1. $L = M$ *if and only if* $P_L = P_M$.
2. $P_{L+M} = P_L + P_M$.
3. $P_{LM} = P_L \cdot P_M$.
4. $P_{\lambda L} = \lambda \cdot P_L$, *for every* $\lambda \in \mathbb{R}$.

Proof: (2),(3) and (4) are straightforward from the definition of the characteristic polynomial.

Constant coefficient Differential Operators

Proof of (1): Suppose

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \; M = \sum_{i=0}^{m} b_{m-i} D^i.$$

Then,

$$P_L(x) = \sum_{i=0}^{n} a_{n-i} x^i, \; P_M(x) = \sum_{i=0}^{m} b_{m-i} x^i.$$

Thus, $P_L = P_M$ iff $n = m$ and $a_i = b_i$ for $0 \leq i \leq n$ and hence $L = M$.
Conversely, suppose $L = M$. In particular,

$$L(e^{rx}) = M(e^{rx}) \text{ for every } r \in \mathbb{R},$$

It follows that

$$\sum_{i=0}^{n} a_{n-i} r^i e^{rx} = \sum_{i=0}^{m} b_{m-i} r^i e^{rx}.$$

Conclude that $P_L(r) = P_M(r)$ for every $r \in \mathbb{R}$ and hence $P_L = P_M$.

Constant Differential Operators

Corollary

Let L, M, N be constant coefficient linear differential operators such that

$$P_L = P_M \cdot P_N.$$

Then,

$$L = MN.$$

In particular, if

$$P_L(x) = a_0(x - r_1)\dots(x - r_n),$$

then,

$$L = a_0(D - r_1)\dots(D - r_n).$$

Proof.

$$P_L = P_M \cdot P_N = P_{MN} \implies L = MN.$$

Example:

$$D^2 - 5D + 6 = (D - 3)(D - 2).$$

Constant Differential Operators

Theorem

Let $L = D^n + a_1 D^{n-1} + \ldots + a_n I$*. Suppose*

$$L = A_1 A_2 \ldots A_k$$

where A_i *are linear differential operators with constant coefficients. Then,*

$$N(A_i) \subseteq N(L),$$

for $1 \leq i \leq k$*.*

Proof: Let $f \in N(A_i)$. Thus, $A_i(f) = 0$. Now,

$$L(f) = [(A_1 A_2 \ldots\ldots A_k) \cdot A_i](f) = 0.$$

(Why?) This means that $f \in N(L)$.
In the right hand side above, $(A_1 A_2 \ldots\ldots A_k)$ does not contain A_i.

Constant Differential Operators

Example: Find a basis of solutions of the DE:

$$y^{(3)} - 7y' + 6y = 0.$$

Here,

$$L = D^3 - 7D + 6I,$$

and hence

$$P_L(x) = x^3 - 7x + 6 = (x-1)(x-2)(x+3).$$

Therefore,

$$L = (D-1)(D-2)(D+3).$$

Note that

$$e^x \in \mathrm{Ker}(D-1),\ \ e^{2x} \in \mathrm{Ker}(D-2),\ \ e^{-3x} \in \mathrm{Ker}(D+3).$$

Thus, $e^x, e^{2x}, e^{-3x} \in \mathrm{Ker}(L)$ and are linearly independent. Hence, $\{e^x, e^{2x}, e^{-3x}\}$ is a basis of Ker L (why?). Thus, the general solution is of the form

$$c_1 e^x + c_2 e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Constant Differential Operators

Remark: The above example illustrates the general case of P_L having distinct real roots. If

$$P_L(x) = a_0(x - r_1)\ldots(x - r_n),$$

with $r_i \in \mathbb{R}$ distinct, then a basis for Ker L is

$$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$$

Aim. Give a formulation for the basis of *ker L*, where

$$L = D^n + a_1 D^{n-1} + \ldots + a_n I,$$

with constant coefficients $a_1, a_2, \cdots a_n$.

Constant Differential Operators

Constant differential operator: $L = D^n + a_1 D^{n-1} + \ldots + a_n I$, with constant coefficients $a_1, a_2, \cdots a_n$.

Case I: P_L has distinct real roots:

Theorem

Let L be a constant coefficient linear differential operator of order n such that

$$P_L(x) = (x - r_1) \ldots (x - r_n)$$

where $r_1, r_2, \ldots, r_n$ are distinct real numbers. Then the general solution of $L(y) = 0$ is given by

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Proof: We have

$$L = (D - r_1) \ldots (D - r_n)$$

and the Null-space

$$N(D - r_i) = \{ce^{r_i x} : c \in \mathbb{R}\}.$$

Constant Differential Operators

It follows that

$$e^{r_1 x}, \ldots, e^{r_n x} \in N(L).$$

Check! $\{e^{r_1 x}, \ldots, e^{r_n x}\}$ are linearly independent over $\mathbb{R}$ as $r_1, r_2, \ldots, r_n$ are distinct real numbers.

As dimension of $N(L)$ is n by the Dimension Theorem, we get

$$N(L) = \{c_1 e^{r_1 x} + \ldots + c_n e^{r_n x} : c_1, \ldots, c_n \in \mathbb{R}\}.$$

Constant Differential Operators

Case II: $P_L(x)$ has some repeated real roots: What happened in the $n = 2$ case? $m_1 = m_2 = m$ gave us only one solution - $f(x) = e^{mx}$. The other solution was obtained using the method of looking for a solution of the form vf. This method yielded xe^{mx} as the other solution.

Proposition

For any real number r, the functions,

$$u_1(x) = e^{rx}, u_2(x) = xe^{rx}, \ldots, u_m(x) = x^{m-1}e^{rx}$$

are linearly independent and

$$u_1(x), \ldots, u_m(x) \in \mathrm{Ker}((D - r)^m).$$

Constant Differential Operators

Proof. Since $\{1, x, x^2, \ldots, x^m\}$ is linearly independent, it follows that $\{e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}\}$ is also linearly independent. We need to show that these functions are in Ker $(D-r)^m$. Let's first verify this when $m = 1$. We need to show that

$$u_1(x) = e^{rx} \in \mathrm{Ker}((D-r)),$$

which is true, since

$$(D-r)(e^{rx}) = re^{rx} - re^{rx} = 0.$$

Suppose $m = 2$. Since u_1 is in Ker of $(D-r)$, it's in Ker of $(D-r)^2$ (why?). What about u_2?

$$\begin{aligned}
(D-r)^2(xe^{rx}) &= (D-r)(D-r)(xe^{rx}) \\
&= (D-r)(xre^{rx} + e^{rx} - rxe^{rx}) \\
&= (D-r)(e^{rx}) = 0.
\end{aligned}$$

Constant Differential Operators

So how do we prove this in general? Induction. Assume

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^{m-1}),$$

and we need to show that

$$u_1, u_2, \ldots, u_m \in \mathrm{Ker}((D-r)^m).$$

That

$$u_1, u_2, \ldots, u_{m-1} \in \mathrm{Ker}((D-r)^m)$$

is easy since

$$\mathrm{Ker}((D-r)^{m-1}) \subseteq \mathrm{Ker}((D-r)^m).$$

To show that u_m is also in $\mathrm{Ker}((D-r)^m)$, consider

$$\begin{aligned}
(D-r)^m (u_m(x)) &= (D-r)^m (x^{m-1}e^{rx}) \\
&= (D-r)^{m-1}(D-r)(x^{m-1}e^{rx}) \\
= (D-r)^{m-1}\big(x^{m-1}re^{rx} &+ (m-1)x^{m-2}e^{rx} - rx^{m-1}e^{rx}\big) \\
&= (D-r)^{m-1}((m-1)x^{m-2}e^{rx}) \\
&= 0.
\end{aligned}$$

Constant Differential Operators

Thus, if

$$P_L(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \dots (x - r_\ell)^{m_\ell},$$

where $\sum_{i=1}^{\ell} m_i = n$, a basis of Ker L is given by

$$e^{r_1 x}, \dots, x^{m_1 - 1}e^{r_1 x}, e^{r_2 x}, \dots, x^{m_2 - 1}e^{r_2 x}, \dots, e^{r_\ell x}, \dots, x^{m_\ell - 1}e^{r_\ell x}.$$

Note that that the above functions are linearly independent and since dim Ker $L = n$, these form a basis.

Exercise: Check that the above functions are linearly independent.