MA 108-ODE- D3

Lecture 7

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Second order linear ODEs

Summary - First Order Equations

- Linear Equations - Solution
  - Reducible to linear - Bernoulli
- Non-linear equations
  - Variable separable
  - Reducible to variable separable
  - Exact equations - Integrating factors
  - Reducible to Exact
- Existence & Uniqueness results for IVP : $\boxed{y' = f(x, y),\ y(x_0) = y_0}$
  - Peano's existence theorem
  - Picard's existence-uniqueness theorem
- Picard's iteration method

Second Order Linear ODE's

Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

where $a_2(x) \neq 0$ for all x.

Definition

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.

Assume $p(\cdot), q(\cdot), r(\cdot)$ are continuous functions on an interval I of $\mathbb{R}$.

Though there is no formula to find all the solutions of such an ODE, we study the existence and number of linearly independent solutions of such ODE's.

Initial Value Problem

An initial value problem of a second order linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = r(x);\; y(x_0) = a, y'(x_0) = b,$$

where $p(\cdot)$, $q(\cdot)$ and $r(\cdot)$ are assumed to be continuous on an interval I with $x_0 \in I$.

Existence-Uniqueness Theorem for IVP

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = r(x), y(x_0) = a, y'(x_0) = b,$$

where p, q and r are continuous on an interval I, x_0 is any point in I, and a, b are real numbers. Then there is a unique solution to the IVP on I.

Example. $y''(x) + y'(x) + y(x) = \sin x, \quad y(0) = 1, \quad y'(0) = 0.$

Example: The IVP

$$x^2y'' + xy' - 4y = 0, y(x_0) = a, y'(x_0) = b,$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

Second Order Linear ODE's

If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

then the ODE is said to be homogeneous. Otherwise it is called nonhomogeneous.

Solving IVP's

Let

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$
$$C^1(I) = \{f : I \to \mathbb{R} \mid f, f' \text{ are continuous}\}$$
$$C^2(I) = \{f : I \to \mathbb{R} \mid f, f', f'' \text{ are continuous}\}.$$

Check: $C(I), C^1(I), C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f+g)(x) = f(x) + g(x),\ x \in I,$$

$$(k \cdot f)(x) = kf(x),\ k \in \mathbb{R}, x \in I.$$

Solving IVP's

Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

Then L is a linear transformation, i.e.,

$$L(cf + dg) = cL(f) + dL(g),$$

for all $c, d \in \mathbb{R}$ and for all $f, g \in C^2(I)$.
The null space of L, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Qn. How to characterize the space $N(L)$? What is the dimension of the space?

Linearly independent & dependent functions

Definition

The functions f and g are said to be linearly independent on an interval I if

$$c_1 f(x) + c_2 g(x) = 0 \;\; \forall x \in I \Longrightarrow c_1 = c_2 = 0.$$

The functions are said to be linearly dependent on an interval I if they are not linearly independent on I.

Examples : 1. The functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent on $(-\infty, \infty)$.

2. The functions $f(x) = x$ and $g(x) = |x|$ are linearly dependent on $(0, \infty)$ but are linearly independent on $(-\infty, \infty)$.

Wronskian and Linear Independence

Definition

The Wronskian of any two differentiable functions f and g is defined by

$$W(f, g; x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Wronskian and Linear Independence

Proposition

Suppose f and g are linearly dependent and differentiable on an interval I. Then, $W(f, g; x) = 0$ on I.
In other words, if two differentiable functions f and g have $W(f, g; x_0) \neq 0$, for some $x_0 \in I$, then the functions f and g are linearly independent on I.

Proof. As f and g are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0.$$

Thus,

$$cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $W(f, g; x) = f(x)g'(x) - f'(x)g(x) = 0$ for all x (Why?) since $\begin{bmatrix} c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Wronskian and Linear Independence

Example.

1. Let $m_1 \neq m_2$, two real numbers. Are the functions $f(x) = e^{m_1 x}$ and $g(x) = e^{m_2 x}$ linearly independent on $\mathbb{R}$?
2. Let m be a real number. Are the functions $f(x) = e^{mx}$ and $g(x) = xe^{mx}$ linearly independent on $\mathbb{R}$?
3. Let m be a real number. Are the functions $f(x) = \sin(mx)$ and $g(x) = \cos(mx)$ linearly independent on $\mathbb{R}$?

Note: The converse of the Proposition is not true. For instance, if $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

If $x \geq 0$, $W(x^2, x^2; x) = 0$. If $x < 0$, $W(x^2, -x^2; x) = 0$. Hence,

$$W(f, g; x) = 0 \quad \text{forall} \quad x \in \mathbb{R}$$

but f and g are linearly independent on $\mathbb{R}$. (why? Check using definition)

Wronskian and Linear Independence

Theorem (Abel's Formula)

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Let a be any point of I. Then

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}, x \in I$$

Proof. Set $W(f, g; x) = W(x)$. Then,

$$W(x) = (fg' - f'g)(x)$$
$$W'(x) = (fg'' - f''g)(x).$$

Wronskian and Linear Independence

Now,

$$\begin{aligned} f'' &= -p(x)f' - q(x)f \\ g'' &= -p(x)g' - q(x)g. \end{aligned}$$

Thus,

$$\begin{aligned} W'(x) &= (fg'' - f''g)(x) \\ &= \left(-fpg' - fqg + gpf' + gqf\right)(x) \\ &= -p(x)\left(fg' - f'g\right)(x) \\ &= -p(x)W(x), \end{aligned}$$

i.e., W is the solution of the IVP

$$y' + p(x)y = 0, \; y(a) = W(a).$$

Hence,

$$W(x) = W(a)e^{-\int_a^x p(t)dt},$$

i.e.,

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}.$$

Wronskian and Linear Independence

Theorem

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Then,

1. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*
2. *f and g are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*

Thus, f and g are linearly independent on I iff $W(f, g; x) \neq 0$ for all $x \in I$.