SURFACE INTEGRALS

$F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$. THE SET

$$S = S_c = \{ (x,y,z) \in D \mid F(x,y,z) = c \} \quad \left(\begin{matrix}\text{TYPICALLY} \\ c=0\end{matrix}\right)$$

IS CALLED A SURFACE IN D.

$\mathbf{r} : \tilde{D} \subseteq \mathbb{R}^2 \to \mathbb{R}^3$:

$$\mathbf{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

IS CALLED A PARAMETRIZATION OF S IF

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x = x(u,v),\ y = y(u,v),\ z = z(u,v) \text{ FOR } (u,v) \in D \right\}$$

WE SAY S IS A SMOOTH SURFACE IF $x(\cdot,\cdot)$, $y(\cdot,\cdot)$, $z(\cdot,\cdot)$ ALL HAVE CONTINUOUS PARTIAL DERIVATIVES.

FOR A SURFACE S WITH PARAMETRIZATION $\mathbf{r}(u,v)$ WITH $(u,v) \in D$, THE CURVES

$$v \longmapsto \mathbf{r}(u,v) \ (\text{FOR FIXED } u), \quad u \longmapsto \mathbf{r}(u,v) \ (\text{FIXED } v)$$

ARE CALLED COORDINATE CURVES OF S.

EXAMPLES

🚩 LINEAR SURFACES: $F(x,y,z) = aX + bY + cZ$

$$S = \{(x,y,z) \mid F(x,y,z) = d\} \quad (\text{PLANES})$$

🚩 $F(x,y,z) = Ax^2 + BY^2 + CZ^2 + DXY + EXZ + GYZ$
$+ HX + KY + JZ + L$

$$S = \{(x,y,z) \mid F(x,y,z) = \alpha\} \quad (\text{QUADRATIC SURFACES})$$

🚩 $$S = \{(x,y,z) \mid x^2 + y^2 = a^2\} \quad (\text{CIRCULAR CYLINDER})$$

🚩 $$S = \left\{(x,y,z) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\} \quad (\text{ELLIPSOID})$$

🚩 $$S = \left\{(x,y,z) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\right\}$$

(HYPERBOLOID OF 1 SHEET)

PARAMETRIZATIONS

🚩 $S = \{ax + bY + cZ = d\}$:

$$\mathbf{r}(x,y) = x\vec{i} + y\vec{j} + \left[\frac{d}{c} - \left(\frac{a}{c}x + \frac{b}{c}y\right)\right]\vec{k}$$

(IF $c \neq 0$)

🚩 $S = \{x^2 + y^2 + z^2 = a^2\}$:

$$\mathbf{r}(\theta,\phi) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$$
$$\theta \in [0, 2\pi], \quad \phi \in [0, \pi]$$

$$S = \{(x,y,z) \in D \,/\, F(x,y,z) = c\}$$

SUPPOSE S IS SMOOTH AND HAS PARAMETRIZATION

$$\mathbf{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

THEN THE UNIT NORMAL VECTOR TO S AT (x,y,z) IS GIVEN BY

$$\mathbf{n} = \pm \frac{\nabla F}{\|\nabla F\|}, \quad \text{IF } \nabla F \neq (0,0,0).$$

$$= \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \quad \text{IF } \mathbf{r}_u \times \mathbf{r}_v \neq (0,0,0).$$

(IDEA OF) PROOF:

$$F(x(u,v), y(u,v), z(u,v)) = c$$

CHAIN RULE $\Rightarrow$

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial u} = 0$$

$$\Leftrightarrow \quad \nabla F \cdot \mathbf{r}_u = 0$$

SIMILARLY, $\nabla F \cdot \mathbf{r}_v = 0$

HENCE ∇F, $\mathbf{r}_u \times \mathbf{r}_v$ ARE PARALLEL VECTORS

HENCE $\mathbf{n} = \pm \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$

EXAMPLE

SUPPOSE $z = f(x,y)$ IS THE SURFACE, i.e.,

$$S = \{(x,y,z) \mid \underbrace{f(x,y) - z}_{F(x,y,z)} = 0\}.$$

$$\nabla F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right).$$

FOR THE PARAMETRIZATION

$$\mathbf{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$$

$$\mathbf{r}_x = \vec{i} + \frac{\partial f}{\partial x}\vec{k}, \quad \mathbf{r}_y = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right), \text{ SO,}$$

$$\mathbf{n} = \pm \frac{(f_x, f_y, -1)}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

SURFACE AREA

LET S BE A SURFACE WITH PARAMETRIZATION $\mathbf{r} : R \to \mathbb{R}^3$, WHERE $R \subseteq \mathbb{R}^2$.

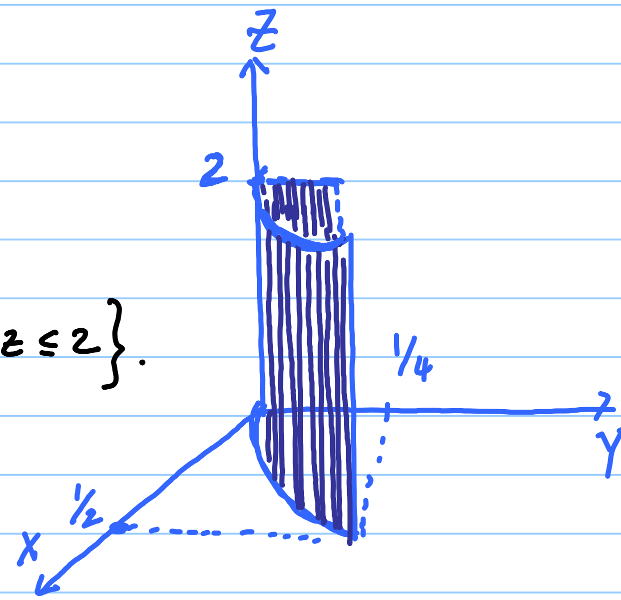
$$\text{SURFACE AREA}(S) := \iint_R \| \mathbf{r}_u \times \mathbf{r}_v \| \, du\, dv$$

🚩 A SURFACE S WITH PARAMETRIZATION $\mathbf{r}$ IS CALLED **REGULAR** IF $\forall (u,v) \in \mathrm{Dom}(\mathbf{r})$ WE HAVE $(\mathbf{r}_u \times \mathbf{r}_v)(u,v) \neq (0,0,0)$

THE DEFINITION OF SURFACE AREA IS WHEN $\| \mathbf{r}_u \times \mathbf{r}_v \| \neq 0$. IN OTHER WORDS, IF $\mathbf{r}_u \times \mathbf{r}_v = 0$, THIS DEFINITION MAY BE MISLEADING. HENCE WE USE THIS ONLY WHEN $\mathbf{r}_u \times \mathbf{r}_v \neq 0$.

EXAMPLES

FIND AREA OF THE PARABOLIC CYLINDER $Y = X^2$ IN THE FIRST OCTANT BOUNDED BY $Z = 2$ AND $Y = \frac{1}{4}$.



$$\mathbf{r}(x,z) = (x, x^2, z)$$

$$R = \left\{ (x,z) \,\middle|\, 0 \le x \le \tfrac{1}{2},\ 0 \le z \le 2 \right\}.$$

HENCE,

$$\text{SURFACE AREA} = \iint_R \| \mathbf{r}_x \times \mathbf{r}_z \| \, dx\, dz$$

$$\mathbf{r}_x = (1, 2x, 0)$$

$$\mathbf{r}_z = (0, 0, 1)$$

$$\begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \iint_R \| 2x\vec{i} - \vec{j} \| \, dx\, dz$$

$$= \int_0^2 \int_0^{1/2} \sqrt{1+4x^2} \, dx\, dz$$

$$= 2 \int_0^{1/2} \sqrt{1+4x^2} \, dx$$

SURFACE AREA OF A SPHERE

$$\mathbf{r}(\theta,\phi) = (a\cos\theta\cos\phi,\ a\cos\theta\sin\phi,\ a\sin\theta)$$

$$0 \le \phi \le 2\pi, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(GEOGRAPHICAL PARAMETRIZATION)

$$\mathbf{r}_\theta = (-a\sin\theta\cos\phi,\ -a\sin\theta\sin\phi,\ a\cos\theta)$$

$$\mathbf{r}_\phi = (-a\cos\theta\sin\phi,\ a\cos\theta\cos\phi,\ 0)$$

$$\mathbf{r}_\theta \times \mathbf{r}_\phi =$$

INTEGRATE TO GET $4\pi a^2$ (DO IT!)

IF $S = \{(x,y,z) \mid z = f(x,y),\ (x,y) \in R\}$ FOR SOME $R \subseteq \mathbb{R}^2$, THEN

$$\text{SURFACE AREA}(S) = \iint_R \|\mathbf{r}_x \times \mathbf{r}_y\| \, dx\, dy$$

WHERE

$$\mathbf{r}(x,y) = (x, y, f(x,y)).$$

THEN

$$\mathbf{r}_x = (1, 0, f_x)$$

$$\mathbf{r}_y = (0, 1, f_y)$$

SO $\underline{\mathbf{r}_x \times \mathbf{r}_y = -f_x \vec{i} - f_y \vec{j} + \vec{k}}$

SO

$$\text{SURFACE AREA}(S) = \iint_R \|\mathbf{r}_x \times \mathbf{r}_y\| \quad dx\, dy$$

IF γ IS THE ANGLE BETWEEN $\mathbf{r}_x \times \mathbf{r}_y$ AND THE POSITIVE z-AXIS, THEN

$$(\mathbf{r}_x \times \mathbf{r}_y) \cdot \vec{k} = \underset{''}{\|\mathbf{r}_x \times \mathbf{r}_y\|} \cdot \cos\gamma$$

BUT $(\mathbf{r}_x \times \mathbf{r}_y) \cdot \vec{k} = 1 \qquad \Rightarrow \|\mathbf{r}_x \times \mathbf{r}_y\| = \sec\gamma$

HENCE

$$\text{SURFACE AREA}(S) = \iint_R \sec\gamma \, dx\, dy$$

EXAMPLE

FIND THE SURFACE AREA OF THE CIRCULAR CONE $z^2 = x^2 + y^2$, $0 \le z \le 1$

$F(x,y,z) = x^2 + y^2 - z^2$, SO THE SURFACE IS GIVEN BY $F(x,y,z) = 0$, $R = \{(x,y) \mid x^2+y^2 \le 1\}$.

HENCE

SURFACE AREA OF THE CONE

$= \iint_R \sec\gamma \, dx\,dy$, γ = ACUTE ANGLE BETWEEN Z-AXIS AND n.

So, $|n \cdot \vec{k}| = \cos\gamma \Rightarrow \sec\gamma = \dfrac{1}{|n \cdot k|}$, $n = \dfrac{\nabla F}{\|\nabla F\|}$

$\nabla F = (2x, 2y, -2z)$. CHECK THAT $\sec\gamma = \sqrt{2}$

So, $\iint_R \sec\gamma \, dx\,dy =$

EXAMPLE

FIND THE AREA OF THE PORTION OF $(x-a)^2+y^2=a^2$

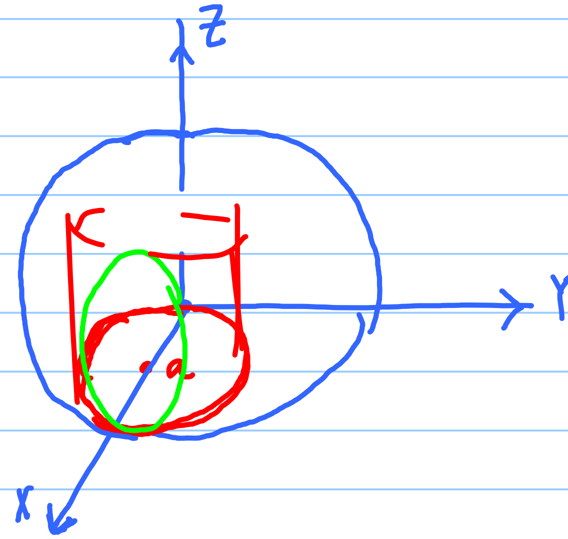
THAT LIES INSIDE $x^2+y^2+z^2=4a^2$

$F(x,y,z) = (x-a)^2+y^2-a^2 = x^2+y^2-2ax,$

THE SURFACE IS GIVEN BY $F(x,y,z)=0$.

$\nabla F = (2x-2a, 2y, 0)$

$\Rightarrow \nabla F \cdot \vec{k} = 0$



CONSIDER THE SURFACE

$(x-a)^2+z^2=a^2$, OR

$G(x,y,z)=0$, WHERE

$G(x,y,z) = x^2+z^2-2ax.$

$\nabla G = (2x-2a, 0, 2z) \Rightarrow \nabla G \cdot \vec{k} \neq 0$

SURFACES DESCRIBED IMPLICITLY

SUPPOSE $S = \{(x,y,z) \mid F(x,y,z) = 0\}$

SUPPOSE S_{xy} DENOTES ITS PROJECTION ONTO THE

XY PLANE.

SUPPOSE $z = h(x,y)$ $\quad (x,y) \in S_{xy}$, IS AN (IMPLICIT)

EXPLICIT DESCRIPTION OF S.

THEN

$$\text{SURFACE AREA} = \iint_{S_{xy}} \sqrt{1 + h_x^2 + h_y^2}\, dx\, dy.$$

$$F(x,y,h(x,y)) = 0 \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot h_x = 0$$

SIMILARLY

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} h_y = 0 \qquad \text{(WHY?)}$$

HENCE

$$\text{SURFACE AREA} = \iint_R \left\{ \sqrt{\frac{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}{\left(\frac{\partial F}{\partial z}\right)^2}} \right\} dx\, dy$$

SURFACE OF REVOLUTION

IF $z = f(x)$ $a \le x \le b$ IS ROTATED ABOUT THE Z-AXIS TO GET THE SURFACE S, THEN WE HAVE SEEN

$$\text{SURFACE AREA }(S) = \int_a^b 2\pi x \sqrt{1+(f'(x))^2}\, dx$$

WRITE $\mathbf{r}(u,v) = \left(u\cos v,\ u\sin v,\ f(u)\right)$ $0 \le a \le u \le b$, $0 \le v \le 2\pi$

THEN THE SURFACE AREA EQUALS

$$\int_a^b\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dx\, dy$$

$$\mathbf{r}_u = \left(\cos v,\ \sin v,\ f'(u)\right);\ \mathbf{r}_v = \left(-u\sin v,\ u\cos v,\ 0\right)$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & f'(u) \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \left(-u\cos v\, f'(u)\right)\vec{i} - \left(u\sin v\, f'(u)\right)\vec{j} + (u)\vec{k}$$

$$\Rightarrow \|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(f'(u))^2 + 1}$$

So $$\int_a^b\int_0^{2\pi} u\sqrt{1+(f'(u))^2}\, dv\, du = 2\pi\int_a^b u\sqrt{1+(f'(u))^2}\, du$$