ADDENDUM: SETS OF MEASURE ZERO

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In this short note, we define the notion of sets of MEASURE ZERO in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ specifically, and see their relevance in the theory of Riemann integration.

DEFINITION: Sup. $K \subseteq \mathbb{R}$. K is a set of 1-dimensional MEASURE ZERO, or 1-MEASURE ZERO if there exist open intervals (a_i, b_i) s.t

- $K \subseteq \bigcup_i (a_i, b_i)$
- $\sum_i (b_i - a_i) < \varepsilon$

for any given $\varepsilon > 0$.

The collection of open intervals $\{(a_i, b_i)\}$ can be finite or countably infinite.

DEFINITION: Sup. $K \subseteq \mathbb{R}^2$. K is a set of 2-dimensional MEASURE ZERO or 2-MEASURE ZERO if: Given $\varepsilon > 0$, there exist (open) intervals $R_i := (a_i, b_i) \times (c_i, d_i)$ s.t

- $K \subseteq \bigcup_i R_i$
- $\sum_i \text{Area}(R_i) < \varepsilon$.

We leave the definition of 3-MEASURE ZERO sets as an exercise.

A set of Measure Zero is, intuitively speaking, a very sparse set. Loosely speaking, any meaningful assignment of "content of the set" (Length, Area, Volume) must be ZERO.

PROPOSITION: Any countable set has measure zero (in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$)

Proof Sketch: If $K = \{a_1, a_2, \ldots\}$, then for a given $\varepsilon > 0$ find open sets (intervals, rectangles, cuboids, as the case may be) V_i s.t $a_i \in V_i$ and $a(V_i) < \frac{\varepsilon}{2^{i+3}}$, where $a(V_i)$ is the length, or area, or volume accordingly as $K \subseteq \mathbb{R}, \mathbb{R}^2$ or $\mathbb{R}^3$. Then, clearly, $K \subseteq \bigcup_i V_i$ and $\sum_i a(V_i) < \sum_{i \geq 1} \frac{\varepsilon}{2^{i+3}} < \varepsilon$. ▨

The main theorem concerning Riemann-integrability is the following:

THEOREM:

- Sup. $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable iff the set of discontinuities of f is a set of 1-Measure Zero.
- Sup $f : [a, b] \times [c, d] \to \mathbb{R}$ is bounded. Then f is Riemann integrable iff the set of discontinuities of f is a set of 2-Measure Zero.
- $f : [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ is Riemann integrable (f is bounded) iff [illegible]

The set of discontinuities of f is a set of 3-Measure Zero.

Proof: OMITTED. You can see it in any text on Real Analysis.

In the example in the lecture, $f(x,y) = x^2+y^2$ on the domain $0 \le x+y \le 1$
if we set $R = [0,1]\times[0,1]$, then $f^*(x,y) = x^2+y^2$ if $0 \le x+y \le 1$

$= 0$ if $x+y > 1$

has the set $D = \{(1-x, x) : x \in [0,1]\}$ as the set of discontinuities.

It is easy to see that this is a set of 2-Measure Zero. Indeed, for the rectangular partition induced by the partition $x_i = \frac{i-1}{n}$ $(i=0,\dots,n+1)$, and $y_i = \frac{i-1}{n}$ $(i=0,1,\dots,n+1)$ then the block of "diagonal rectangles" covers the set D and the total area of the block rectangles equals $n \cdot \frac{1}{n^2} = \frac{1}{n} < \varepsilon$ for $n > \lceil 1/\varepsilon \rceil$.

REMARK: It is tempting to think that for any continuous $f : [0,1] \to \mathbb{R}^2$, the set $\operatorname{Im}(f) \subseteq \mathbb{R}^2$ must be a set of 2-Measure Zero. However, this is NOT TRUE: there exist $f : [0,1] \to \mathbb{R}^2$ s.t. $\operatorname{Im}(f) = [0,1]\times[0,1]$! (SPACE-FILLING CURVES)

However, it is true that if $f : [0,1] \to \mathbb{R}$ is continuous, then the graph of f has 2-Measure Zero.