MA 108-ODE- D3

Lecture 4

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Orthogonal Trajectories

Existence and uniqueness theorem

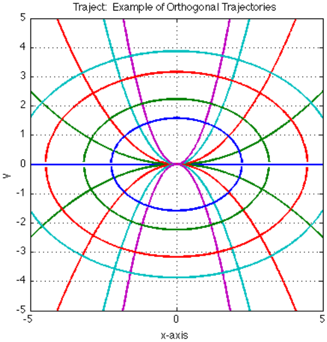
Warm up!

Eample. Solve the IVP: $y'(t) = y(1-y)$, $y(0) = 0$ by finding the solution $y(t;\epsilon)$ of the problem $y' = y(1-y)$, $y(0) = \epsilon$ and then computing $\lim_{\epsilon \to 0} y(t;\epsilon)$.

Ans. $y(t;\epsilon) = \frac{\epsilon e^x}{1-\epsilon+\epsilon e^x}$ and $\lim_{\epsilon \to 0} y(t;\epsilon) = 0$ for all t.

Orthogonal Trajectories

If two families of curves always intersect each other at right angles, then they are said to be orthogonal trajectories of each other.



Working Rule

To find the OT of a family of curves

$$F(x, y, c) = 0.$$

- Find the DE $\frac{dy}{dx} = f(x, y)$.
- Slopes of the OT's are given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

- Obtain a one parameter family of curves $G(x, y, c) = 0$ as solutions of the above DE.

Example

Find the set of OT's of the family of circles $x^2 + y^2 = c^2$.

$$x + y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The slope of OT's are $\frac{dy}{dx} = \frac{y}{x} \Longrightarrow y = kx$.
Hence the orthogonal trajectories are given by $\boxed{y = kx}$.

Existence and Uniqueness

The IVP's that we have considered usually have unique solutions. This need not always be the case.

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\ y(0) = y_0.$$

This is a separable first order DE. First, suppose that $y_0 \neq 0$. Hence, separating the variables, we get:

$$y^{-\frac{1}{3}} dy = dx;$$

i.e.,

$$\frac{3}{2} y^{\frac{2}{3}} = x + c,$$

or

$$y = \left[\frac{2}{3}(x + c)\right]^{\frac{3}{2}}, \quad x \geq -c.$$

The initial condition $y(0) = y_0$ gives $c = \frac{3}{2} y_0^{\frac{2}{3}}$. Hence,

$$y = \left[\frac{2}{3}\left(x + \frac{3}{2} y_0^{\frac{2}{3}}\right)\right]^{\frac{3}{2}}, \quad x \geq -\frac{3}{2} y_0^{\frac{2}{3}}$$

is a solution of the given IVP. **This is also a solution if $y_0 = 0$.**

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\; y(0) = 0.$$

$$y = \phi(x) = \left[\frac{2x}{3}\right]^{\frac{3}{2}};\; x \geq 0$$

is a solution.

$$y = -\phi(x) = -\left[\frac{2x}{3}\right]^{\frac{3}{2}};\; x \geq 0$$

is also a solution of the IVP.

$$y = \psi(x) \equiv 0$$

is also a solution of the IVP.
For any $a > 0$,

$$y = \phi_a(x) = \begin{cases} 0 & \text{if } x \in [0, a) \\ \pm\left[\frac{2}{3}(x-a)\right]^{\frac{3}{2}} & \text{if } x \geq a \end{cases}$$

is continuous, differentiable, and gives a solution of the given IVP.

Existence and Uniqueness

That is, we get infinitely many solutions of the given IVP.

No solution of IVP

It may happen there exists no differentiable function satisfying the ODE and initial value!

Ex. Solve: $y(t)y'(t) = \frac{1}{2}, y(0) = 0$.

Ans. no solution, because if there exists any solution ϕ, then $\phi(t)\phi'(t) = \frac{1}{2}$ and putting $t = 0$, $\phi(0) = 0$ contradicts the equation.

How to determine if an IVP has a solution? In case a solution exists, when it has to be unique? Existence and uniqueness theorem.

Definitions

1. Let f be a real function defined on D, where D is a region in $\mathbb{R}^2$. The function f is said to be bounded in D if there exists a positive number K such that
$$|f(x,y)| \leq K$$
for all (x,y) in D.
2. Let f be defined and continuous on a closed rectangle $R : a \leq x \leq b,\ c \leq y \leq d$. Then, f is bounded in R.
3. Let f be defined on D, where D is either a domain or a closed domain of the xy- plane. The function f is said to satisfy Lipschitz condition (with respect to y) in D if $\exists$ a constant $M > 0$ such that
$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$$
for every $(x,y_1),(x,y_2)$ which belong to D. The constant M is called the Lipschitz constant.

Example. Consider $f(x,y) = |y|$ for all $(x,y) \in \mathbb{R}^2$. The function is Lipschitz in $\mathbb{R}$. What is the Lipschitz constant? Ans. Hint. Use Triangle inequality.

Sufficient condition for Lipschitz continuity

Result : If f, defined on D, a region of $\mathbb{R}^2$ is such that $\frac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, i.e., there exists a positive constant M with $|\frac{\partial f}{\partial y}(x, y)| \leq M, \quad \forall (x, y) \in D$, then f satisfies Lipschitz condition w.r.t. y in D, with M as Lipschitz constant.

Proof : Mean value theorem
$\Longrightarrow f(x, y_1) - f(x, y_2) = (y_1 - y_2)\frac{\partial f}{\partial y}(x, \xi),\ \xi \in (y_1, y_2)$.

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |y_1 - y_2||\frac{\partial f}{\partial y}(x, \xi)| \\ &\leq M|y_1 - y_2|, \end{aligned}$$

using that $|\frac{\partial f}{\partial y}(x, y)| \leq M$ for all $(x, y) \in D$. That is, f satisfies Lipschitz condition.

Example

- Consider $f(x, y) = \sin(y)$, $\forall\,(x, y) \in \mathbb{R}^2$. The function is Lipschitz in $\mathbb{R}$. why? what is the Lipschitz constant?

  Ans. Since $\frac{\partial f}{\partial y}(x, y) = \cos y$ and $\left|\frac{\partial f}{\partial y}(x, y)\right| \leq 1$ for all $(x, y) \in \mathbb{R}^2$. So here the Lipschitz constant is $M = 1$.

- Consider $f(x, y) = y^2$ defined in $D : |x| \leq a,\ |y| \leq b$. The function is Lipschitz in D.

  Ans. $f_y(x, y) = 2y$ is bounded in D and $|\frac{\partial f}{\partial y}(x, y)| \leq 2b$ for all $(x, y) \in D$. The Lipschitz contant is $M = 2b$.

Existence of bounded derivative f_y is a sufficient condition for Lipschitz condition to hold true (not necessary).

Bounded derivative - sufficient condition

Example. Consider

$$f(x,y) = x|y| \text{ defined in } D : |x| \le a,\ |y| \le b.$$

$\frac{\partial f}{\partial y}$ doesn't exist for any point $(x, 0) \in D$. (Why?)
But f satisfies Lipschitz condition :

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \big|x|y_1| - x|y_2|\big| \\ &= |x|\,\big||y_1| - |y_2|\big| \\ &\le |x|\,|y_1 - y_2| \\ &\le a|y_1 - y_2| \end{aligned}$$

Here Lipschitz condition is a.

Example : Consider $f(x,y)=\sqrt{|y|}$ for all $(x,y)\in\mathbb{R}^2$.
f is continuous for all y.
Note that f doesn't satisfy Lipschitz condition in any region that includes $y=0$. Because, note that as for $y_1=0,\ y_2>0$, we have

$$\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|}=\frac{\sqrt{y_2}}{|y_2|}=\frac{1}{\sqrt{y_2}}$$

which can be made as large as we want by making y_2 smaller.

So there cannot exist any positive constant M satisfying

$$|f(x,y_1)-f(y_2)|\le M|y_1-y_2|,\quad \forall\,(x,y_1)\in\mathbb{R}^2,\quad \forall\,(x,y_2)\in\mathbb{R}^2.$$

Continuity w.r.t. second variable does not imply Lipschitz condtn. w.r.t. second variable.

Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0): $R : |x - x_0| < a, \ |y - y_0| < b$.

- $f(x, y)$ be continuous at all points $(x, y) \in R$ in and
- bounded in R, that is, $|f(x, y)| \leq K, \ \forall (x, y) \in R$.

Then, the IVP $y' = f(x, y), \ y(x_0) = y_0$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min \left\{ a, \frac{b}{K} \right\}.$$

In addition to the above conditions, if f satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M |y_1 - y_2| \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then,the IVP admits a unique solution on the interval $(x_0 - \alpha, x_0 + \alpha)$. [1].

[1] Existence - Peano, Existence & uniqueness -Picard

Example

Consider

$$y' = y^{1/3} \qquad y(0) = 0 \text{ in } R : |x| \le a, |y| \le b.$$

$f(x, y)$ is continuous in R and hence existence of solution is guaranteed in an interval containing 0.

But note $\phi_1(x) = 0$ and $\phi_2(x) = \begin{cases} (\frac{2}{3}x)^{3/2} & \text{if } x \ge 0 \\ 0 & \text{if } x \le 0 \end{cases}$ are solutions in $-\infty < x < \infty$.

Does this imply Lipschitz condition won't be satisfied?

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{|y_1^{1/3} - y_2^{1/3}|}{|y_1 - y_2|}.$$

Choosing $y_1 > 0,\ y_2 = 0$, we get

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{1}{y_1^{2/3}},$$

and RHS goes to ∞ as y_1 closes to 0. Hence Lipschitz condition is not satisfied. Solution exists, but not unique.