MA 109, Calculus 1, Week-1

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Aims of the course

Welcome to IIT Bombay. Wish you a very happy, fruitful and memorable stay here at IITB.

Few Suggestions

- Try to understand each and every statement in class only. If there is any doubt please ask in the class.
- There will be a separate doubt clearing session by myself and by TA's also.
- Stay upto date.

Beware of

There are lots of such Aedes mosquitoes in IITB.

Course information

For details about the syllabus, tutorials, assignments, quizzes, exams and procedures for evaluation please refer to the course booklet which is available on moodle:
http://moodle.iitb.ac.in/login/index.php and also on my webpage
https://sites.google.com/site/spusti
The emphasis of this course will be on the underlying ideas and methods rather than intricate problem solving (though there will be some of that too). The aim is to get you to think about calculus, in particular, and mathematics in general.

Basic Preliminaries

- The set of natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$.
- The set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$.
- The set of rational numbers $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$.

Exercise: Prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ are not rational numbers.

Exercise: Let $m \in \mathbb{N}$ is not a perfect square. Prove that $\sqrt{m}$ is irrational.

- The set of real numbers $\mathbb{R}$. Definition ? $\sqrt{2}, \sqrt{3}$, etc. are in $\mathbb{R}$.

Basic Preliminaries

Definition

Let $A, B \subseteq \mathbb{R}$ or $\mathbb{C}$.

1. A function $f : A \to B$ is injective/one-one if
$$f(x) = f(y) \text{ implies } x = y, \text{ for all } x, y \in A.$$
2. A function $f : A \to B$ is surjective/onto if
$$f(A) = B.$$
That is for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.
3. A function $f : A \to B$ is bijective if it is both injective and surjective.

Definition

A set $B \subseteq \mathbb{R}$ is said to be countable if there exists a bijective map $f : \mathbb{N} \to B$.

Basic Preliminaries

Example: The set of integers $\mathbb{Z}$ is countable. Define a bijective map

$$f(n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ is even} \\ -\frac{n-1}{2} & \text{if} \quad n \text{ is odd} . \end{cases}$$

Results (Without Proof):

1. Every infinite subset of a countable set is countable.
2. Countable union of countable set is countable.
3. The set of rational numbers $\mathbb{Q}$ is countable.
4. The set of real numbers $\mathbb{R}$ is uncountable. In fact any interval $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$ is uncountable.

- What about set of irrational numbers?

Negation:

- If p is statement then negation of p is denoted by $\neg p$.
- $\neg(p \cup q) = \neg p \cap \neg q$, $\neg(p \cap q) = \neg p \cup \neg q$
- $\neg\neg p = p$.
- $p \implies q \Leftrightarrow \neg q \implies \neg p$. (Eating sugar gives taste sweet is equivalent to not some thing sweet means not sugar).
- Negation of "for all" is "there exists".

What is $\neg$ (All the students of this class room are Boys)?
What is $\neg$ (There exists an exceptional student in this class room)?

A story

One day the set of all irrational numbers got very angry because of their name "irrational" numbers, which they don't like. But their complement set $\mathbb{Q}$ got a good name "rational" numbers. So all the irrational numbers went to GOD and started shouting. Then GOD asked them to stand in a line. Only then he would think about the change of their name. What will happen ?

Sequences

Definition: A sequence in a set X is a function $f : \mathbb{N} \to X$, that is, a function from the natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occassion to consider sequences of functions sometimes. In later mathematics courses X may be the complex numbers $\mathbb{C}$, vector spaces (whatever those maybe) the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions.

Rather than denoting a sequence by a function, it is often customary to describe a sequence by listing the first few elements

$$a_1, a_2, a_3, \dots$$

or, more generally by describing the n^{th} term a_n. Note that we write a_n rather than $a(n)$. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want and f is just the identity function).
2. $a_n = 1/n$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \sin\left(\frac{1}{n}\right)$ (here the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).
4. $a_n = \frac{n!}{n^n}$.
5. $a_n = n^{1/n}$.
6. $s_n = \sum_{i=0}^{n} r^i$, for some r such that $0 \leq r < 1$.
7. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).
8. $f_n(x) = \cos nx$ (here X is the space of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).
9. $s_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, or writing it out

   $$s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

   Once again X is a space of functions, for instance the space of continuous functions on $\mathbb{R}$.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 ($a_n = n$) is a monotonically increasing sequence, Example 2 ($a_n = 1/n$) is a monotonically decreasing sequence, while Example 3 ($a_n = \sin\left(\frac{1}{n}\right)$) is also monotonically decreasing. How about Examples 4 and 5?

In Example 4 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 5 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$!). However, $3^{1/3} > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonic sequence, that is, the sequence becomes monotonic(ally decreasing) after some stage. One can similarly define eventually monotonically increasing sequences.

For any fixed non-negative value of r, Example 6 ($s_n = \sum_{i=0}^{n} r^i$) gives a monotonic increasing sequence, while for any fixed non-negative value of x, the sequence $s_n(x) = \sum_{i=0}^{n} \frac{x^n}{n!}$ in Example 9 also gives a monotonically increasing sequence.

Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigourous definition. We will be more interested in limits of functions (which is what arise in the differential calculus), but limits of sequences are closely related to the former, and occur in their own right in the theory of Riemann integration.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001 (= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigourous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = l.$$

Equivalently, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit l. If we just want to say that the sequence has a limit without specifying what that limit is, we simply say $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.

A sequence that does not converge is said to diverge, or to be divergent.

Definition:(Recall) A sequence a_n tends to a limit l/ converges to l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

Negation of convergence:
A sequence a_n does not converge to l if there exists $\epsilon_0 > 0$ such

that for each natural number n, there exist $m_n \in \mathbb{N}$ with $m_n > n$ satisfies

$$|a_{m_n} - l| \geq \epsilon_0.$$

Remarks on the definition

1. Note that the N will (of course) depend on ϵ, as it did in our example, so it would have been more correct to write $N(\epsilon)$ in the definition of the limit. However, we usually omit this extra bit of notation.
2. We have already shown that $\lim_{n\to\infty} 1/n^2 = 0$. The same argument works for $\lim_{n\to\infty} 1/n^\alpha$, for any real $\alpha > 0$. We just take N to be any integer bigger than $1/\epsilon^{1/\alpha}$ for a given ϵ. Recall that for $x > 0$, x^α is defined as $e^{\alpha \log x}$.
3. For a given ϵ, once one N works, any larger N will also work. In order to show that a sequence tends to a limit l we are not obliged to find the best possible N for a given ϵ, just some N that works. Thus, for the sequence $1/n^2$ and $\epsilon = 0.1$, we took $N = 3$, but we can also take $N = 2023$, or any other number bigger than 3.
4. Showing that a sequence converges to a limit l is not easy. One first has to guess the value l and then prove that l satisfies the definition. We will see how to get around this in various ways.

More examples of limits

Let us show that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

For this we note that for $x \in [0, \pi/2]$, $0 \leq \sin x \leq x$ (try to remember why this is true).
Hence,

$$|\sin 1/n - 0| = |\sin 1/n| \leq 1/n.$$

Thus, given any $\epsilon > 0$, if we choose some $N > 1/\epsilon$, $n > N$ implies $1/n < 1/N < \epsilon$. It follows that $|\sin 1/n - 0| < \epsilon$.

Let us consider Exercise 1.1.(ii) of the tutorial sheet. Here we have to show that $\lim_{n\to\infty} 5/(3n+1) = 0$. Once again, we have only to note that

$$\frac{5}{3n+1} < \frac{5}{3n},$$

and if this is to be smaller than ϵ, we must have $n > N > 5/3\epsilon$.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above we have

$$\lim_{n\to\infty}(c \cdot b_n) = c \cdot \lim_{n\to\infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

The Sandwich Theorem(s)

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n.$$

Remark If $a_n < b_n$ for all $n \geq N$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

A second version of the theorem is especially useful:

Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that we do not assume that b_n converges in this version of the theorem - we get the convergence of b_n for free.

Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

An example using the theorems above

Consider Exercise 1.2.(iii) on the tutorial sheet. We have to show that

$$\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$$

exists and to evaluate it.

It is clear that

$$0 < \frac{n^3+3n^2+1}{n^4+8n^2+2} \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}.$$

(How do we get this?)
Note that $n^3/(n^4+8n^2+2) < n^3/n^4 = 1/n$, and the other two terms can be handled similarly.)

Hence, applying the Sandwich Theorem (Theorem 2) to the sequences

$$a_n = 0, \quad b_n = \frac{n^3+3n^2+1}{n^4+8n^2+2} \quad \text{and} \quad c_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

we see that the limit we want exists provided $\lim_{n\to\infty} c_n$ exists, so this is what we must concentrate on proving.

The limit $\lim_{n\to\infty} c_n$ exists provided each of the terms appearing in the sum has a limit and in that case it is equal to the sum of the limits (by the first formula). But each of these limits is quite easy to evaluate.

We already know that

$$\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} 1/n^4,$$

while

$$\lim_{n\to\infty} 3/n^2 = 3 \cdot \lim_{n\to\infty} 1/n^2 = 0$$

where we have used the special case of the second formula (limit of the product is the product of the limits) for the first equality in the equation above. Since all three limits converge to 0, it follows the given limit is $0 + 0 + 0 = 0$.

Bounded Sequences

The formulæ and theorems stated above can be easily proved starting from the definitions. We will prove the second formulæ and leave the other proofs as exercises.

Definition: A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. A sequence that is not bounded is called unbounded.

In our list of examples, Example 1 ($a_n = n$) is an example of an unbounded sequence, while Examples 2 - 5 ($a_n = 1/n, \sin(1/n), n!/n^n, n^{1/n}$) are examples of bounded sequences.

Bounded sequence don't necessarily converge - for instance $a_n = (-1)^n$. However,

Convergent sequences are bounded

Lemma: Every convergent sequence is bounded.

Proof: Suppose a_n converges to l. Choose $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that $|a_n - l| < 1$ for all $n > N$. In other words, $l - 1 < a_n < l + 1$, for all $n > N$, which gives $|a_n| < |l| + 1$ for all $n > N$. Let

$$M_1 = \max\{|a_1|, |a_2|, \ldots, |a_N|\}$$

and let $M = \max\{M_1, |l| + 1\}$. Then $|a_n| < M$ for all $n \in \mathbb{N}$. $\square$

We will use this Lemma to prove the product rule for limits.

The proof of the product rule

We wish to prove that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

Suppose $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$. We need to show that $\lim_{n\to\infty} a_n b_n = l_1 l_2$.

Fix $\epsilon > 0$. We need to show that we can find $N \in \mathbb{N}$ such that $|a_n b_n - l_1 l_2| < \epsilon$, whenever $n > N$. Notice that

$$\begin{aligned} |a_n b_n - l_1 l_2| &= |a_n b_n - a_n l_2 + a_n l_2 - l_1 l_2| \\ &= |a_n(b_n - l_2) + (a_n - l_1) l_2| \\ &\leq |a_n||b_n - l_2| + |a_n - l_1||l_2|, \end{aligned}$$

where the last inequality follows from the triangle inequality. So in order to guarantee that the left hand side is small, we must ensure that the two terms on the right hand side together add up to less than ϵ. In fact, we make sure that each term is less than $\epsilon/2$.

The proof of the product rule, continued

Since a_n is convergent, it is bounded by the lemma we have just proved. Hence, there is an M such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Given the quantities $\epsilon/2|l_2|$ and $\epsilon/2M$, there exist N_1 and N_2 such that

$$|a_n - l_1| < \epsilon/2|l_2| \quad \text{and} \quad |b_n - l_2| < \epsilon/2M.$$

Let $N = \max\{N_1, N_2\}$. If $n > N$, then both the inequalities above hold. Hence, we have

$$|a_n||b_n - l_2| \leq M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \quad \text{and} \quad |a_n - l_1||l_2| \leq l_2 \cdot \frac{\epsilon}{2l_2} = \frac{\epsilon}{2}.$$

Now it follows that

$$|a_n b_n - l_1 l_2| \leq |a_n||b_n - l_2| + |a_n - l_1||l_2| < \epsilon,$$

for all $n > N$, which is what we needed to prove. $\square$

The proofs of the other rules for limits are similar to the one we proved above. Try them as exercises.

Few examples

1. Let $x \in \mathbb{R}$ such that $|x| < 1$. Then $\lim_{n\to\infty} x^n = 0$. This is left as an exercise (use definition).
2. Let $x > 0$ real number. Then $\lim_{n\to\infty} x^{1/n} = 1$.
   Let $x > 1$. We write $x^{1/n} = 1 + k_n, k_n > 0$. So
   $$x = (1 + k_n)^n \geq 1 + nk_n \text{ (by binomial).}$$
   Hence $0 < k_n \leq \frac{x-1}{n}$. Therefore by Sandwich theorem $\lim_{n\to\infty} k_n = 0$. If $x < 1$, then take $y = \frac{1}{x}$ and then $y > 1$. Therefore $\lim_{n\to\infty} y^n = 1$. That is $\frac{1}{\lim_{n\to\infty} x^n} = 1$. Hence $\lim_{n\to\infty} x^{1/n} = 1$.
3. Prove that $\lim_{n\to\infty} n^{1/n} = 1$.
   Let $n^{1/n} = 1 + k_n, k_n > 0$. So $n = (1 + k_n)^n \geq 1 + \frac{n(n-1)}{2} k_n^2$ (by binomial). Hence $0 < k_n \leq \sqrt{\frac{2}{n}}$. Therefore by Sandwich theorem $\lim_{n\to\infty} k_n = 0$.

A result

Theorem

Let $\{x_n\}$ be a sequence of real numbers such that $x_n > 0$ for all n. Let $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. Then

1. *if $\lambda < 1$, then $\{x_n\}$ is convergent and converges to 0.*
2. *if $\lambda > 1$, then $\{x_n\}$ is divergent.*

The result is inconclusive if $\lambda = 1$. Consider the following two sequences $x_n = n$ and $x_n = \frac{1}{n}$, one is divergent whereas other is convergent but $\lambda = 1$ in both the cases.

A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition. The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

Definition: A sequence a_n is said to be bounded above (resp. bounded below) if $a_n < M$ (resp. $a_n > M$) for some $M \in \mathbb{R}$.

A sequence that is bounded both above and below is obviously bounded.

Theorem 3: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence a_n bounded above might be?

It will be the supremum or least upper bound (lub) of the sequence. This is the number, say M which has the following properties:

1. $a_n \leq M$ for all n and
2. If M_1 is such that $a_n < M_1$ for all n, then $M \leq M_1$.

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence $1 - 1/n$. Clearly there is no maximal element in the sequence, but 1 is its supremum.

Monotonic sequence

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

$$\begin{aligned} a_{n+1} < a_n \quad &\iff \quad \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) < a_n \\ &\iff \quad \sqrt{2} < a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2}, \quad \text{(Why is this true?)}$$

so $a_{n+1} \geq \sqrt{2}$ for all $n \geq 1$ and $a_1 > \sqrt{2}$ is given.
Hence, $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence, bounded below by $\sqrt{2}$. By Theorem 3, it converges.
Exercise 1. What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

More remarks on limits

Exercise 2. More generally, what is the limit of a monotonically decreasing sequence bounded below? How can you describe it? This number is called the infimum or greatest lower bound (glb) of the sequence.

An important remark

If we change finitely many terms of a sequence it does not affect the convergence and boundedness properties of a sequence.

If it is convergent, the limit will not change.

If it is bounded, it will remain bounded though the supremum/infimum may change.

Thus, an eventually monotonically increasing sequence bounded above will converge (formulate the analogue for decreasing sequences).

Bottomline: From the point of view of the limit, only what happens for large N matters.

Cauchy sequences

As we saw last time, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, things are slightly better since we only need to bound the sequence.

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the elements of the sequence itself. We describe this below.

Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges.

Cauchy sequences: Some Remarks

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

One can easily check the converse:
Theorem 5: Every convergent sequence is Cauchy.

Remark 2: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, subsets of $\mathbb{R}$.

For instance, if we take $X = \mathbb{R} \setminus \{0\}$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X.

If we take $X = \mathbb{Q}$, the example given in 1.5.(i) ($a_1 = 3/2$ and $a_{n+1} = (a_n + 2/a_n)/2$) is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

Thus Theorem 4 is really a theorem about real numbers.

Supremum of a set

Let $A \subseteq \mathbb{R}$. We say M is the supremum or least upper bound of A if

1. $x \leq M$ for all $x \in A$ and
2. If M_1 is such that $x < M_1$ for all $x \in A$, then $M \leq M_1$,

and we write $\sup A = M$.
What is

1. $\sup\{1, 2, 3, \cdots, 2023\}$.
2. $\sup(0, 1]$.
3. $\sup(0, 1)$.
4. $\sup\left\{1 - \frac{1}{2^n} \mid n \in \mathbb{N}\right\}$.
5. $\sup\left\{1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, \cdots, 1 + \frac{1}{1!!} + \cdots + \frac{1}{n!}, \cdots\right\}$.

Supremum of a set

Answers

1. 2023
2. 1
3. 1
4. 1
5. e

Observation: Supremum of every subset of $\mathbb{Q}$ is not in $\mathbb{Q}$.

Theorem:(Least upper bound axiom) If a set of real numbers is bounded above, it has a supremum (or least upper bound). If a set of real numbers is bounded below it has an infimum (or greatest lower bound).

Properties of real numbers

Theorem

(Archimedean Property) For any positive $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$nx > 1$$

That is, for any positive real number, there exists a bigger natural number.

Theorem

Between any two distinct real numbers there is a rational number.

Few exercises

Exercise: Between any two distinct rational numbers there is an irrational number.
Exercise: Let β be an irrational number. There there exist a sequence of rational numbers $\{r_n\}$ such that $\{r_n\}$ converges to β.
Exercise: Let $A \subseteq \mathbb{R}$ be bounded above and let $\alpha = \sup A$. Then for any $\epsilon > 0$, there exists $a_\epsilon \in A$ such that $\alpha - \epsilon < a_\epsilon$.
Consequently there exists a sequence $\{a_n\}$ in A such that $\{a_n\}$ converges to $\alpha = \sup A$.

The definition of a real number

Two sequence $\{a_n\}$ and $\{b_n\}$ will be related to each other (and we write $a_n \sim b_n$) if

$$\lim_{n \to \infty} |a_n - b_n| = 0$$

You can check that this is an equivalence relation and it is a fact that it *partitions* the set S into disjoint classes. The set of disjoint classes is denote $S/\sim$.

You can easily see that if two sequences converge to the same limit, they are necessarily in the same class.

Definition: A real number is an equivalence class in $S/\sim$.

So a real number should be thought of as the collection of all rational sequences which converge to it.

Series

Given a sequence a_n of real numbers, we can construct a new sequence, namely the sequence of partial sums s_n:

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

which is also called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

For example, we can define $a_n = r^{n-1}$, for some $r \in \mathbb{R}$ and in this case we obtain a geometric progression $\sum_{k=0}^{\infty} r^k$ for which the n-th partial sum $s_n = \sum_{k=0}^{n-1} r^k$.

Infinite series - a more rigourous treatment

Let us recall what we mean when $|r| < 1$ and we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} ar^{k-1} = \sum_{k=0}^{n-1} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.

So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Theorem

If $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof.

Since $\sum_{n=0}^{\infty} a_n$ is convergent, so is the sequence $\{s_n\}$ of partial sums and hence the sequence $\{s_n\}$ is Cauchy. Therefore $a_n = s_{n+1} - s_n \to 0$ as $n \to \infty$. $\square$

Corollary: If $|x| > 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Theorem (Comparison test)

Let $0 \leq a_n \leq b_n$ for all $n \geq k$ for some k. Then

1. *The convergence of $\sum b_n$ implies convergence of $\sum a_n$.*
2. *The divergence of $\sum a_n$ implies divergence of $\sum b_n$.*

Exercise:

1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.
2. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.

Convergence test

Theorem (Ratio test)

Let $\sum_{n=0}^{\infty} a_n$ be a series of positive real numbers. Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$.

1. *If $\lambda < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.*

Theorem (Root test)

Let $\sum_{n=0}^{\infty} a_n$ be a series of positive real numbers. Let $\lim_{n\to\infty} a_n^{1/n} = \lambda$.

1. *If $\lambda < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.*

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - l_1)^2 + a(n)_2 - l_2)^2} < \epsilon$$

whenver $n > N$. A similar defintion can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$.

Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

The completeness of other spaces

Theorem 4, however, makes perfect sense - one can define Cauchy sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$ exactly as before, using the distance functions - and indeed, remains valid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

So $\mathbb{R}^2$ and $\mathbb{R}^3$ are complete sets too (but $\mathbb{Q}^2$ and $\mathbb{Q}^3$ are not). That is every Cauchy sequence in $\mathbb{R}^2$ and $\mathbb{R}^3$ is convergent but every Cauchy sequence in $\mathbb{Q}^2$ and $\mathbb{Q}^3$ may not converge in $\mathbb{Q}^2$ (or $\mathbb{Q}^3$).