MA 108-ODE- D3

Lecture 11

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Method of Variation of Parameters

Linear ODE's of higher order

Method of Variation of Parameters

A method to find a particular solution of a non-homogeneous ODE is the method of variation of parameters. Consider the DE

$$y'' + p(x)y' + q(x)y = r(x),$$

where p, q and r are continuous on an interval I. The associated homogeneous DE is

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

Suppose that we know the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

of equation (2). In the method of variation of parameters, we vary the constants c_1, c_2 by functions $v_1(x), v_2(x)$, (to be suitably determined) so that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Method of Variation of Parameters

Note that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

Let's also demand that

$$v_1' y_1 + v_2' y_2 = 0.$$

Thus,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Method of Variation of Parameters

Substituting y, y', y'' in the given non-homogeneous ODE, and rearranging terms, we get:

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = r(x).$$

Thus,

$$v_1'y_1' + v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1 + v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Method of Variation of Parameters

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)},\ \ v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)}\,dx,\ \ v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)}\,dx.$$

Hence,

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)}\,dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)}\,dx. \end{aligned}$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + y = \csc x.$$

Step I: Find a basis of solutions for the associated homogeneous equation

$$y'' + y = 0.$$

The general solution of this is

$$y(x) = c_1 \sin x + c_2 \cos x.$$

Step II: Calculate the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Method of Variation of Parameters

Now,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{\cos x \csc x}{-1} dx = \ln|\sin x|,$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x.$$

Hence, a particular solution is given by

$$y(x) = \sin x \ln|\sin x| - x \cos x.$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Recall that via the method of undetermined coefficients, you had to modify the proposed initial solution by multiplying it by t, and you got the answer as $\frac{3}{4}t\sin 2t$. Now in variation of parameters,

$$y_1 = \cos 2t,\ \ y_2 = \sin 2t,$$

and

$$v_1 = -\int \frac{\sin 2t \cdot 3\cos 2t}{2}\,dt = \frac{3}{16}\cos 4t,$$

$$v_2 = \int \frac{\cos 2t \cdot 3\cos 2t}{2}\,dt = \frac{3}{16}\sin 4t + \frac{3}{4}t.$$

Thus, a particular solution is

$$v_1 y_1 + v_2 y_2 = \frac{3}{16}\cos 2t + \frac{3}{4}t\sin 2t.$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

Step I: Find a basis of solutions for the associated homogeneous equation

$$y'' + 3y' + 2y = 0.$$

The general solution of the homogeneous DE is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

Step II: We look for a particular solution of the non-homogeneous DE

$$y(x) = v_1 e^{-x} + v_2 e^{-2x}.$$

where

$$v_1' e^{-x} + v_2' e^{-2x} = 0$$

$$-v_1' e^{-x} - 2v_2' e^{-2x} = \frac{1}{1+e^x}.$$

Solve for v_1 and v_2 . . . .

Linear DE's of Higher Order

Consider an n-th order linear DE:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x).$$

Assume that the functions $a_0, a_1, \ldots, a_n, g$ are continuous on an interval I. Also assume that $a_0(x) \neq 0$ for every $x \in I$. Such an equation can be put into standard form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

Note that $p_1, \ldots, p_n, r$ are continuous on I. If $r \equiv 0$ on I, then the above DE is said to be homogeneous.

Linear DE's of Higher Order

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0.$$

can be written as

$$Ly = 0$$

in terms of a differential operator

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I,$$

where $D^k = \frac{d^k}{dx^k}$, $k \geq 0$, and I is the identity operator.

Linear DE's of Higher Order

An IVP in this set-up will be

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1},$$

where $p_1, \ldots, p_n$ are continuous on an interval I, x_0 is a point in I and $k_0, k_1, \ldots, k_{n-1}$ are arbitrary real numbers.

Linear DE's of Higher Order

Theorem (Existence and Uniqueness)

Suppose that $p_1, \ldots, p_n$ are continuous on an interval I, x_0 is a point in I and $k_0, k_1, \ldots, k_{n-1}$ are arbitrary real numbers. Then the IVP

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1},$$

has a unique solution on I.

Note that both existence and uniqueness are guaranteed on the same interval I where the coefficients $p_1, \ldots, p_n$ are known to be continuous.

Linear DE's of Higher Order

Theorem (Dimension Theorem)

Let I be an interval in $\mathbb{R}$, $p_1, p_2, \ldots, p_n$ be continuous on I and

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_{n-1}(x)D + p_n(x)I.$$

Then null space $N(L)$ of L is of dimension n.

Proof of Dimension Theorem

We prove the Dimension theorem using the Existence and Uniqueness theorem for IVPs. We need to show that dimension of $N(L) = n$.

Fix a point x_0 in the interior of the interval I. Define

$$T : N(L) \to \mathbb{R}^n$$

by

$$T(f) = \big(f(x_0), f^{(1)}(x_0), \ldots, f^{(n-1)}(x_0)\big).$$

Then T is a linear transformation (Check).
T is one-one (by the uniqueness of solution to an IVP).
T is onto (using the existence of solution to an IVP).
Hence by the rank-nullity theorem applying to T^{-1}, we get

$$\text{Dimension of } N(L) = \text{ Dimension of } \mathbb{R}^n = n.$$