Lecture 2

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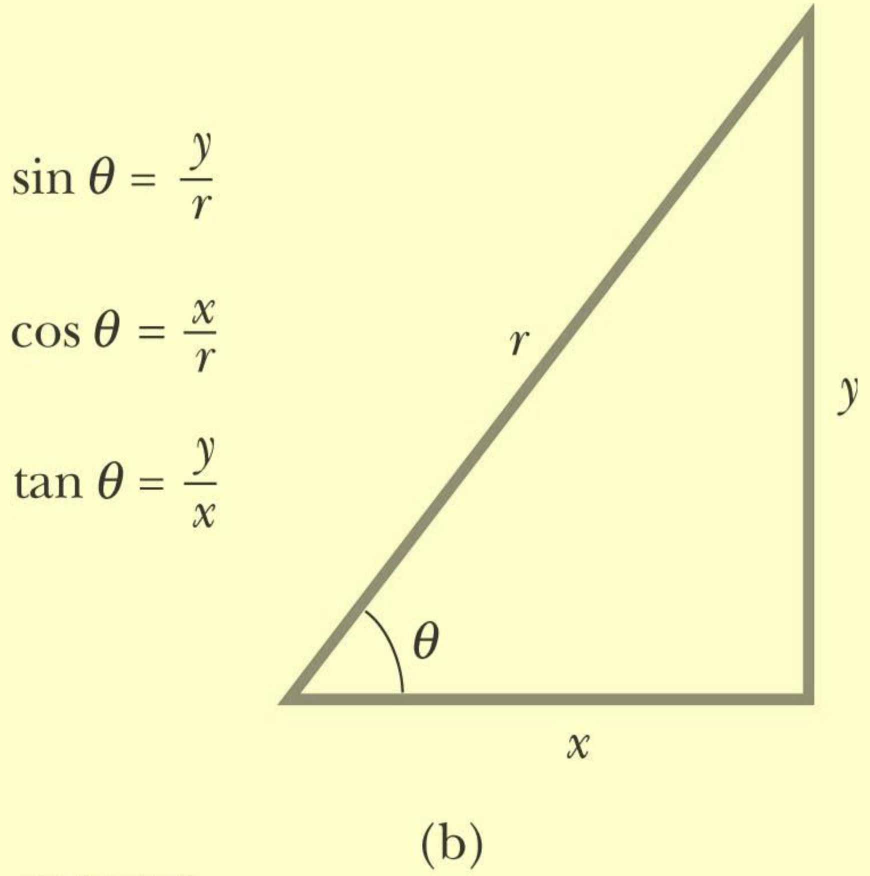
RECAP Polar to Cartesian Coordinates

- Based on forming a right triangle from r and θ
- $x = r \cos \theta$
- $y = r \sin \theta$

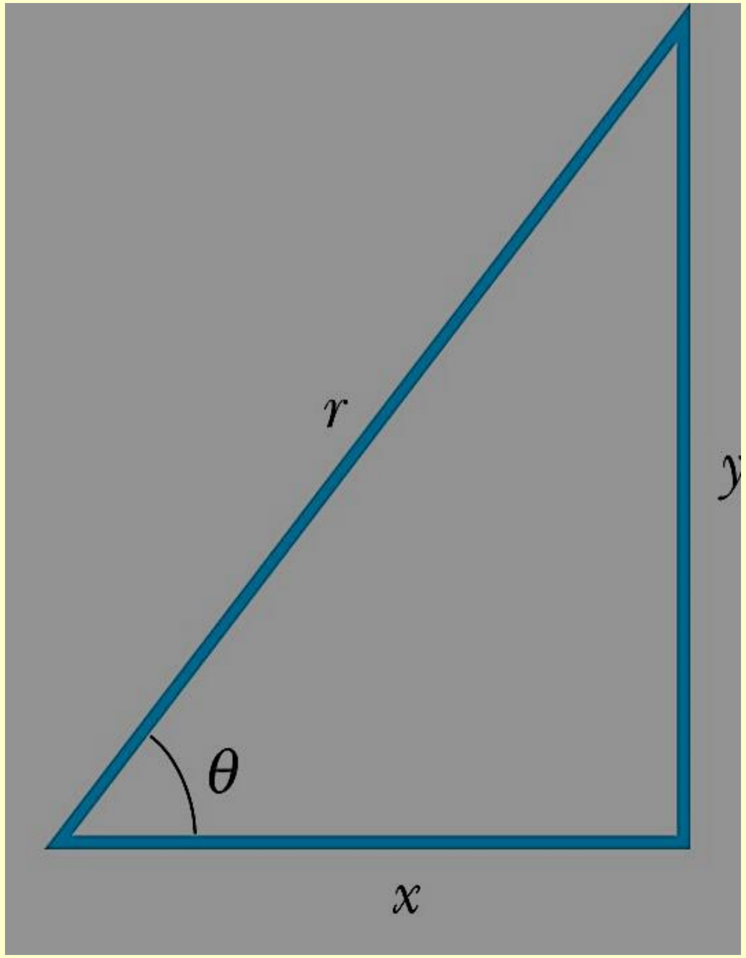


RECAP Cartesian to Polar Coordinates

- r is the hypotenuse and θ an angle

$$\tan\theta = \frac{y}{x}\ ; \quad \theta = \tan^{-1}\frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$



θ must be counter clockwise from positive x axis for these equations to be valid

Relation between plane polar and Cartesian unit vectors

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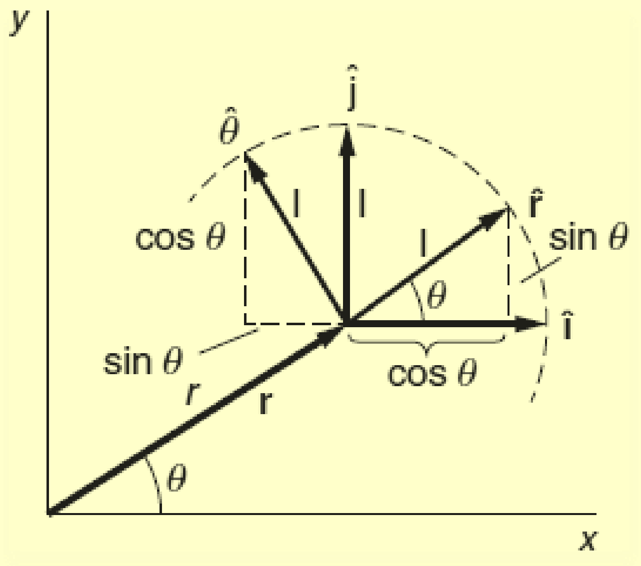
- Consider the figure below



- From above, it is easy to derive the relationship between two sets of unit vectors

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$$

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Polar-Cartesian Relationship

- And, the inverse relationship

$$\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{j}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$$

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Polar-Cartesian Comparison

- Position vector of an arbitrary point P in two coordinate systems is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

$$\mathbf{r} = r\hat{\mathbf{r}}$$

- Infinitesimal displacement **dr** is given by

$$\mathbf{dr} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$$

$$\mathbf{dr} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}}$$

Velocity Vector: Take derivative of position vector. We have to take care that in Cartesian co-ordinates unit vector would be time independent but in polar coordinates they would be time dependent.

Time derivative of any general vector $\vec{A}$ is defined as

$$\frac{d\vec{A}}{dt} = Lt_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

Here the numerator has to be evaluated vectorially.

In Cartesian system we shall get the following expression of the velocity.

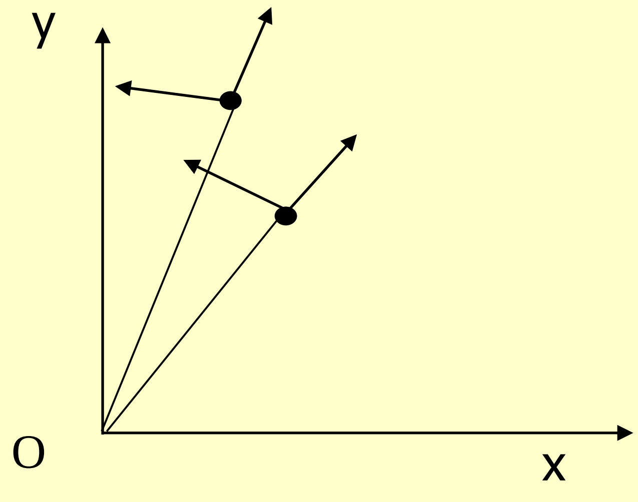
$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \equiv \dot{x}\hat{i} + \dot{y}\hat{j}$$

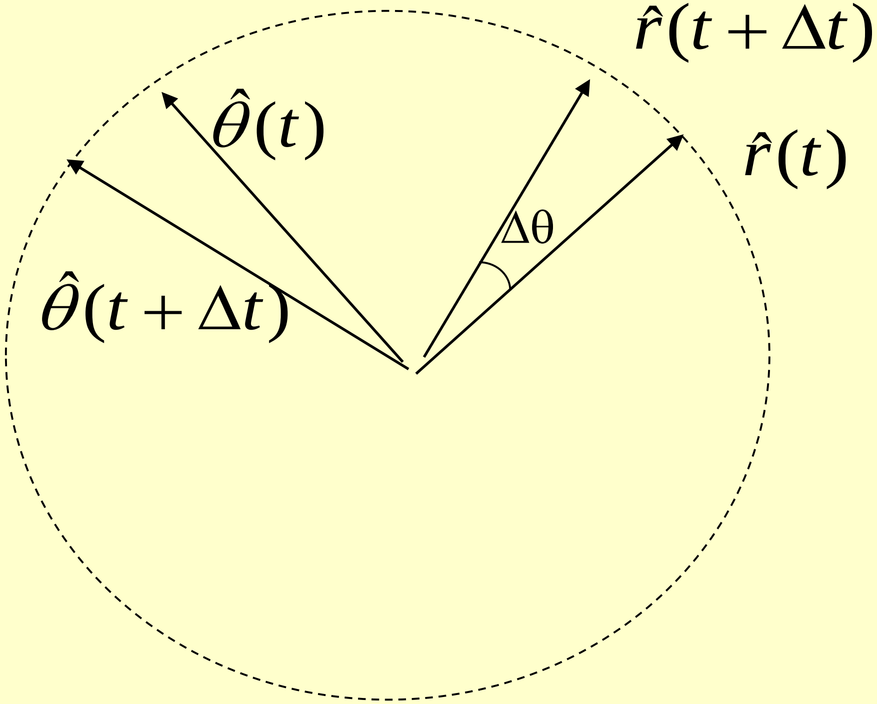
In the polar coordinates we shall obtain the following expression.

$$\vec{v} = \frac{d(r\hat{r})}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

We shall have to evaluate the time derivative of the unit vector to find the final expression of the velocity.

Evaluation of derivative of polar unit vectors: As the vectors are of unit magnitude, they can only change direction. Hence if drawn from the same point, its tip can only move on a circle as time changes.





Recall:

$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d}{dt}(\cos\theta)\,\hat{\mathbf{i}} + \frac{d}{dt}(\sin\theta)\,\hat{\mathbf{j}}\\ &= -\sin\theta\,\dot{\theta}\,\hat{\mathbf{i}} + \cos\theta\,\dot{\theta}\,\hat{\mathbf{j}}\\ &= (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\,\dot{\theta}.\end{aligned}$$

$$\begin{aligned}\frac{d\hat{\boldsymbol{\theta}}}{dt} &= (-\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}})\,\dot{\theta}\\ &= -\dot{\theta}\,\hat{\mathbf{r}}.\end{aligned}$$

$$\boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}}$$

$$\boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}}$$

The expression of velocity in plane polar coordinates is, therefore, as follows.

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\vec{v} = \frac{d(r\hat{r})}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

Examples:

(1) A motion with constant ***r***. In such a case the velocity is always tangential. This is the well known motion along the arc of a circle. It would be a uniform circular motion, if $\dot{\theta}$ is constant. Otherwise the motion could be non-uniform circular motion.

(2) A motion with a constant $\boldsymbol{\theta}$. In such a case the motion is always along the radial direction, i.e. along a straight line. If $\dot{r}$ is constant the motion is with a constant velocity, otherwise it is accelerated.

Note : *In general motion could be more involved. Remember that tangential direction would not mean a direction tangent to the curve. It would mean a direction along* $\hat{\theta}$.

Acceleration in polar coordinates: We follow the same prescription to obtain acceleration in the plane polar coordinates

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\boxed{= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}$$

We, thus see that there are two terms each in radial and tangential components of the acceleration, unlike Cartesian coordinates.

These terms are called linear radial, centripetal, linear tangential and Coriolis accelerations ***(Revisit)*** respectively.

Differences with Cartesian System:

1. *If radial velocity is constant, it does not mean that radial acceleration is zero,* **unlike Cartesian system.** *Similarly if tangential velocity is constant, it does not mean that tangential acceleration is zero. In fact even if both radial and tangential velocities are constant, there could still be an acceleration.*

2. *If $\ddot{r}$ is non zero, still the radial acceleration could be zero, if the centripetal term balances it. Similarly if $\ddot{r}$ is zero, the radial acceleration could be non-zero if the centripetal term is non zero. (Example: a uniform circular motion)*

$$
\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt} \\
&= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\end{aligned}
$$

Similarity with Cartesian System:

One can resolve the forces in r and θ directions and equate them to radial and tangential accelerations.

Example : Circular (Radius R) Motion with constant angular acceleration α

Radial velocity $v_r = 0$?

Radial acceleration $a_r = -R\,[\omega(t)]^2$?

Tangential velocity $v_t = R\,\omega(t)$?

Tangential acceleration $a_t = R\alpha$?

Where $\omega(t) = \omega_o + \alpha t$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt} \\
&= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\end{aligned}$$

For writing equations of motion we can equate radial and tangential forces to mass times respective acceleration in polar coordinates. In Cartesian on the other hand the forces have also to be written as a function of time.

Calculate a_x and a_y

In Cartesian system

$a_x = -R\cos\theta\,\omega^2 - \alpha R\sin\theta$

$a_y = -R\sin\theta\,\omega^2 + \alpha R\cos\theta$

Here both ω and θ are functions of time.

Tangential acceleration $a_t = R\alpha$

Radial acceleration $a_r = -R\,[\omega(t)]^2$

- And, the inverse relationship

$$\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$
$$\hat{\mathbf{j}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$$

Example 2: Uniform motion in a straight line

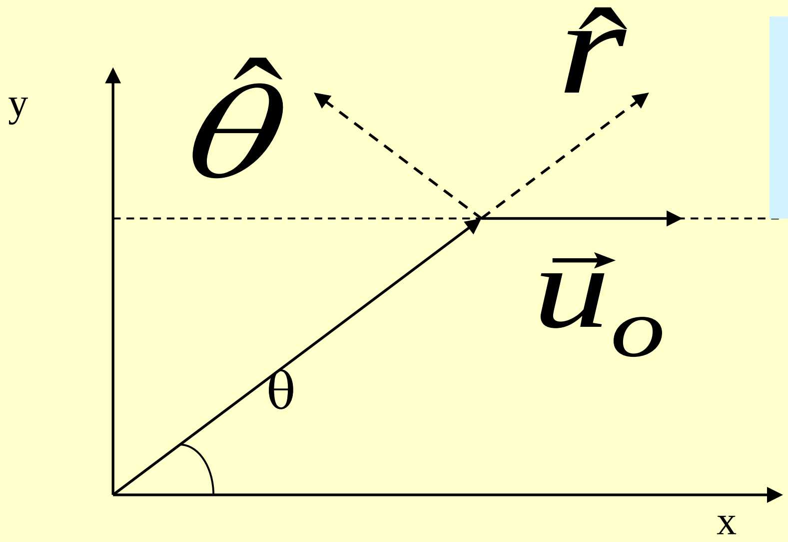
Let a particle move along a line y = a with constant speed u_o directed along positive x-axis.

In Cartesian system

$v_x = u_o$; $a_x = a_y = 0$

Velocity In Polar coordinates ?

$v_r = u_o\cos\theta$; $v_t = -u_o\sin\theta$



$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$
$$\hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$

Let us try to evaluate the radial and tangential component of acceleration in polar co-ordinates and show that both would be individually zero for this case.

We have the following equations :

$$r = \frac{a}{\sin\theta};$$

$$\dot{r} = u_o \cos\theta$$

$$r\dot{\theta} = -u_o \sin\theta$$

Therefore, we get

$$\dot{\theta} = -\frac{u_o \sin\theta}{r} = -\frac{u_o \sin^2\theta}{a}$$

$$\ddot{r} = -u_o \sin\theta\dot{\theta} = \frac{u_o{}^2 \sin^3\theta}{a}$$

$$\ddot{\theta} = -\frac{2u_o \sin\theta\cos\theta\dot{\theta}}{a}$$

$$= \frac{2u_o{}^2 \sin^3\theta\cos\theta}{a^2}$$

In Cartesian system

$v_x = u_o$; $a_x = a_y = 0$

In Polar coordinates,

$v_r = u_o \cos\ \theta$; $v_t = -\ u_o \sin\ \theta$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

We have the following equations:

$$r = \frac{a}{\sin\theta};$$

$$\dot{r} = u_o \cos\theta$$

$$r\dot{\theta} = -u_o \sin\theta$$

Therefore, we get

$$\dot{\theta} = -\frac{u_o \sin\theta}{r} = -\frac{u_o \sin^2\theta}{a}$$

$$\ddot{r} = -u_o \sin\theta\dot{\theta} = \frac{u_o{}^2 \sin^3\theta}{a}$$

$$\ddot{\theta} = -\frac{2u_o \sin\theta\cos\theta\dot{\theta}}{a}$$

$$= \frac{2u_o{}^2 \sin^3\theta\cos\theta}{a^2}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$= \frac{u_o{}^2 \sin^3\theta}{a} - \frac{a}{\sin\theta}\frac{u_o{}^2 \sin^4\theta}{a^2}$$

$$= 0$$

$$a_t = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= \frac{a}{\sin\theta}\frac{2u_o{}^2 \sin^3\theta\cos\theta}{a^2}$$

$$+ 2u_o\cos\theta\left(-\frac{u_o \sin^2\theta}{a}\right)$$

$$= 0$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

PHEW !! We thus get the expected result, but in a complicated way.

What is interesting that in this case none of $\dot{r}, \dot{\theta}, \ddot{r}$ and $\ddot{\theta}$ ***are zero, but the tangential and radial accelerations turn out to be zero.***

Example : A particle with constant $\dot{\theta}$ and $r = r_0 e^{\beta t}$

Radial and Tangential accelerations ?

$$\dot{\theta} = \omega; \ddot{\theta} = 0$$

$$\dot{r} = r_0 \beta e^{\beta t}; \ddot{r} = r_0 \beta^2 e^{\beta t}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{v} = r_0 \beta e^{\beta t} \hat{\boldsymbol{r}} + \omega r_0 e^{\beta t} \hat{\boldsymbol{\theta}}$$

$$\vec{a} = (r_0 \beta^2 e^{\beta t} - \omega^2 r_0 e^{\beta t}) \hat{\boldsymbol{r}} + 2 r_0 \omega \beta e^{\beta t} \hat{\boldsymbol{\theta}}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

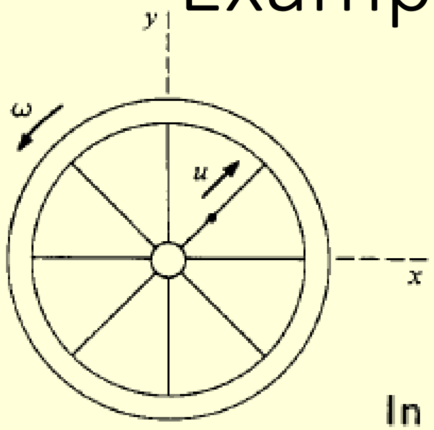
Note: *1. If $\beta = \pm\omega$; the radial acceleration is zero at all times.* ***It is interesting because in this case $\ddot{r}$ is not equal to zero.***

2. One can find the force required to sustain such type of motion by multiplying acceleration by mass.

Sometime Polar coordinates are good

Example:

Velocity of a Bead on a Spoke



A bead moves along the spoke of a wheel at constant speed u meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis, and the bead is at the origin. Find the velocity at time t

In polar coordinates : $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$. Hence

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

$$\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{j}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$$

To specify the velocity completely, we need to know the direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. This is obtained from $\mathbf{r} = (r,\theta) = (ut,\omega t)$.

In cartesian coordinates. $\quad v_x = v_r \cos\theta - v_\theta \sin\theta \qquad v_y = v_r \sin\theta + v_\theta \cos\theta.$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$,

$$\mathbf{v} = (u\cos\omega t - ut\omega\sin\omega t)\hat{\mathbf{i}} + (u\sin\omega t + ut\omega\cos\omega t)\hat{\mathbf{j}}$$

Note how much simpler the result is in plane polar coordinates.

Symmetry is important.

Acceleration of a Bead on a Spoke

A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the velocity and acceleration.

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

We are given that $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position is given by $r = ut$, and we have

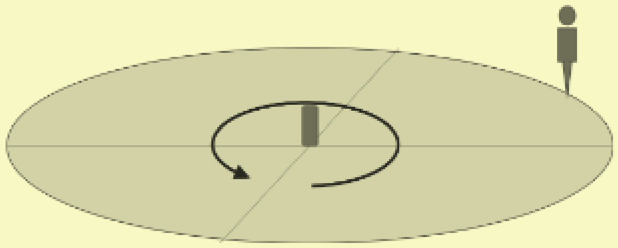
$$\mathbf{v} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.\end{aligned}$$

Cartesian: Be my guest

Suppose you are standing on a rotating platform of radius 5 meters that is rotating at a constant angular velocity of magnitude 0.5 radian per second.



(a) What will be your speed when you are standing at the edge?

(b) What will be your velocity at an instant you are in the direction of 30° with respect to the positive x axis?

(c) What will be your acceleration at the instant in (b)?

(d) What will be your speed and magnitude of acceleration when you are standing 2 meters from the center.

(a) 2.5 m/s, (b) 2.5 m/s $\hat{u}_\theta$, $\hat{u}_\theta - \dfrac{1}{2}\,\hat{\imath} + \dfrac{\sqrt{3}}{2}\,\hat{\jmath}$, (c) -1.25 m/s $\hat{u}_r$, $\hat{u}_r = \dfrac{\sqrt{3}}{2}\,\hat{\imath} + \dfrac{1}{2}\,\hat{\jmath}$,
(d) 1.0 m/s, 0.5 m/s^2.

Consider a motion described by the following velocities of polar coordinates of a particle.

$$\frac{d\theta}{dt} = \omega, \text{ constant}$$
$$\frac{dr}{dt} = \omega\, r$$

(a) Prove that radial component of acceleration, $a_r = 0$. (b) Let $r = r_0$ and $\theta = 0$ at $t = 0$, show that $r(t) = r_0 e^{\omega t}$. (c) A small element of the trajectory can be given by $r d\theta$ by using the arc-angle formula. Use this to find the total distance on the spiral traveled in time T.

(c) $r_0 \left(e^{\omega T} - 1\right)$.

- Exploit the natural symmetry of the problem
- Make the equations simpler to handle

Simple Pendulum

Cartesian

$\ddot{x} = -T\left(\frac{x}{l}\right)$

$\ddot{y} = \mathrm{mg} - T\left(\frac{y}{l}\right)$

$x^2 + y^2 = l^2$

Plane Polar

$\ddot{\theta} = -\left(\frac{g}{l}\right)\theta$

Final Note

Convenience!

Cartesian Coordinates are useful for

1. Measuring the distance of a point from the x or y axis
2. Translations

But inconvenient for handling rotations, or measuring distances from the origin

Example: Air traffic controllers need to know how far the aeroplanes are from the airport and what direction they are coming in from.