PH111: Introduction to Classical Mechanics
Quiz (2023): Model Solution

Maximum Marks: 20
Date: March 31, 2023
Time: 8:15 AM - 9:15 AM

Note: This quiz contains three questions, and marks of each question are written next to it

1. A particle is moving in the $r-\theta$ plane such that $r(t) = a+bt+ct^2$, and $\theta(t) = 2\pi\cos\omega t$, where a, b, c, and ω constants. It is also given that $r(0) = 1$ m, $\dot{r}(0) = 1$ m/s, and $\ddot{r}(0) = 2$ m/s^2.

   (a) Determine the values of a, b, and c (2 marks)
   Soln: Clearly $\dot{r}(t) = b + 2ct$ and $\ddot{r} = 2c$. Using the initial conditions we have

$$
\begin{aligned}
r(0) &= a = 1 \\
\dot{r}(0) &= b = 1 \\
\ddot{r}(0) &= 2c = 2
\end{aligned}
$$

   From these we obtain $a = b = c = 1$, so that

$$r(t) = 1 + t + t^2$$

   (b) Calculate the radial and tangential components of the velocity of the particle as functions of time (2 marks)
   Soln: Using $\mathbf{v}(t) = \dot{r}(t)\hat{\mathbf{r}} + r(t)\dot{\theta}\hat{\boldsymbol{\theta}}$ for which

$$
\begin{aligned}
\dot{r}(t) &= 2t + 1 \\
\dot{\theta}(t) &= -2\pi\omega\sin\omega t
\end{aligned}
$$

$$\mathbf{v}(t) = (2t+1)\hat{\mathbf{r}} - 2\pi\omega(1+t+t^2)\sin\omega t\hat{\boldsymbol{\theta}}$$

   (c) Calculate the radial and tangential components of acceleration of the particle as functions of time. (4 marks)
   Soln: We use the formula $\mathbf{a}(t) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}}$, for which

$$
\begin{aligned}
\ddot{r}(t) &= 2 \\
\ddot{\theta} &= -2\pi\omega^2\cos\omega t
\end{aligned}
$$

$$\left\{2 - 4\pi^2\omega^2(1+t+t^2)\sin^2\omega t\right\}\hat{\mathbf{r}} + \left\{-4\pi\omega(2t+1)\sin\omega t - 2\pi\omega^2(1+t+t^2)\cos\omega t\right\}\hat{\boldsymbol{\theta}}$$

2. Consider the force field

$$(2xy + 2z^2)\hat{\mathbf{i}} + (x^2 + yz)\hat{\mathbf{j}} + (\frac{y^2}{2} + 4zx)\hat{\mathbf{k}}.$$

(a) Is this force conservative? Justify your answer using mathematical arguments. (2 Marks)
Soln: First we compute the curl of the vector field

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 2z^2 & x^2 + yz & \frac{y^2}{2} + 4zx \end{vmatrix} \\
&= (y - y)\,\hat{\mathbf{i}} + (4z - 4z)\hat{\mathbf{j}} + (2x - 2x)\hat{\mathbf{k}} \\
&= 0
\end{aligned}$$

(b) If your answer to part (a) is yes, find the potential energy function which gives rise to this force. If no, how should we modify the force to make it conservative? (4 Marks)
Soln: Because, the force field has a vanishing curl, there must exist a potential energy function $V(x, y, z)$ satisfying

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

which implies

$$-\frac{\partial V}{\partial x} = 2xy + 2z^2 \tag{1}$$
$$-\frac{\partial V}{\partial y} = x^2 + yz \tag{2}$$
$$-\frac{\partial V}{\partial z} = \frac{y^2}{2} + 4zx \tag{3}$$

On integrating Eq. (1), we obtain

$$V(x, y, z) = -x^2 y - 2xz^2 + f(y, z) \tag{4}$$

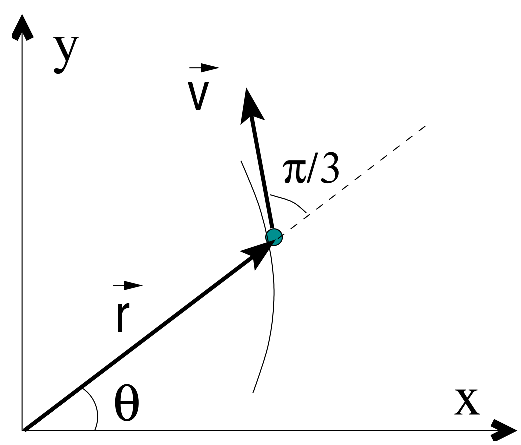
Substituting Eq. (4) in LHS of Eq. (2), we have

$$\begin{aligned}
x^2 - \frac{\partial f}{\partial y} &= x^2 + yz \\
\implies \frac{\partial f}{\partial y} &= -yz \\
\implies f(y, z) &= -\frac{1}{2}y^2 z + g(z) \\
\implies V(x, y, z) &= -x^2 y - 2xz^2 - \frac{1}{2}y^2 z + g(z) \qquad (5)
\end{aligned}$$

Substituting Eq. (5) on the RHS of Eq. (3), we obtain

$$
\begin{aligned}
\frac{y^2}{2} + 4zx + \frac{dg}{dz} &= \frac{y^2}{2} + 4zx \\
\implies \frac{dg}{dz} &= 0 \\
\implies g(z) &= C \quad \text{(a constant)} \\
\implies V(x, y, z) &= -x^2 y - 2xz^2 - \frac{1}{2} y^2 z + C
\end{aligned}
$$

3. A particle moves along a path so that its velocity vector always makes an angle of 60° with respect to its position vector as shown in the figure. Obtain the equation of its trajectory in plane polar coordinates, $r = r(\theta)$, given the condition $r(\theta = 0) = r_0$, where r_0 is a constant. (6 Marks)



Soln: Method 1: The given condition clearly implies

$$
\begin{aligned}
\frac{v_\theta}{v_r} &= \tan\frac{\pi}{3} \\
\implies \frac{r\dot{\theta}}{\dot{r}} &= \sqrt{3} \\
\implies \frac{\dot{r}}{r} &= \frac{\dot{\theta}}{\sqrt{3}} \\
\implies \frac{dr}{r} &= \frac{d\theta}{\sqrt{3}} \\
\implies r &= r_0 e^{\theta/\sqrt{3}}
\end{aligned}
$$

Method 2: The given condition amounts to

$$
\begin{aligned}
\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{|\mathbf{v}|} &= \cos\frac{\pi}{3} \\
\implies \frac{v_r}{\sqrt{v_r^2+v_\theta^2}} &= \frac{1}{2} \\
\implies \frac{\dot{r}}{\sqrt{\dot{r}^2+r^2\dot{\theta}^2}} &= \frac{1}{2} \\
\implies \frac{\dot{r}^2}{\dot{r}^2+r^2\dot{\theta}^2} &= \frac{1}{4} \\
\implies 3\dot{r}^2 &= r^2\dot{\theta}^2 \\
\implies \frac{r\dot{\theta}}{\dot{r}} &= \pm\sqrt{3}
\end{aligned}
$$

Above, only the + sign is acceptable because the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}$ is acute. After that, the solution is identical to the method 1.