Lecture 3

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RECAP

Time derivative of any general vector $\vec{A}$ is defined as

$$\frac{d\vec{A}}{dt} = Lt_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

Here the numerator has to be evaluated vectorially.

In Cartesian system we shall get the following expression of the velocity.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \equiv \dot{x}\hat{i} + \dot{y}\hat{j}$$

In the polar coordinates we shall obtain the following expression.

$$\vec{v} = \frac{d(r\hat{r})}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

We shall have to evaluate the time derivative of the unit vector to find the final expression of the velocity.

The expression of velocity in plane polar coordinates is, therefore, as follows.

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

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Examples:

(1) A motion with constant r. In such a case the velocity is always tangential. This is the well known motion along the arc of a circle. It would be a uniform circular motion, if $\dot{\theta}$ is constant. Otherwise the motion could be non-uniform circular motion.

(2) A motion with a constant θ. In such a case the motion is always along the radial direction, i.e. along a straight line. If $\dot{r}$ is constant the motion is with a constant velocity, otherwise it is accelerated.

Acceleration in polar coordinates: We follow the same prescription to obtain acceleration in the plane polar coordinates

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$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}$$

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt} \\
&= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\end{aligned}$$

We, thus see that there are two terms each in radial and tangential components of the acceleration, unlike Cartesian coordinates.

These terms are called linear radial, centripetal, linear tangential and Coriolis accelerations ***(Revisit)*** respectively.

DIGRESSION

Work-Energy Theorem

- Consider a 1D system confined to move in x direction
- Let the force acting on a system of mass m, be $F(x)$

- Then the work done dW in moving the particle by an infinitesimal amount dx is given by

$$dW = F(x)dx$$

- Thus, the work done W_{ab} in moving the particle from position $x = a$ to $x = b$, will be the integral of the expression above

LINE INTEGRAL ?

$$W_{ab} = \int_a^b F(x)dx$$

$$
\begin{aligned}
W_{ab} &= \int_a^b F(x)dx \\
&= \int_a^b madx \\
&= m\int_a^b \frac{dv}{dt}dx.
\end{aligned}
$$

- But, we can write

$$dx = \frac{dx}{dt}dt = vdt.$$

- Substituting it above, we have

$$W_{ab} = m\int_a^b \frac{dv}{dt}vdt = m\int_a^b \frac{1}{2}\frac{dv^2}{dt}dt = \int_a^b \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt$$

$$\Longrightarrow \; W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

Define Kinetic Energy

$$K = \tfrac{1}{2} m v^2$$

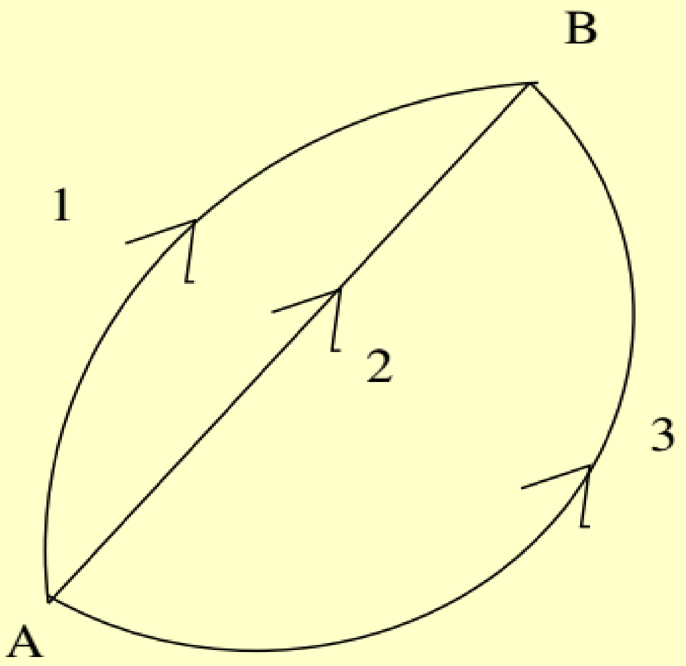
Then,

$$W_{ab} = W_{net} = K_f - K_i = \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2 .$$

$$W_{ab} = W_{net} = \Delta K$$

- Because work done is expressed in terms of a line integral ($W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}$), it will, in principle, depend on the path connecting points A and B.
- For example, for the three paths shown below, the line integral, in general, will have three different values

EXAMPLE ?



- Do we have forces $\mathbf{F}(\mathbf{r})$ for which this line integral is path independent?

YES !!! CONSERVATIVE FORCES

A force is conservative if the work it does on a particle that moves between two points is the same for all paths connecting those points, otherwise, the force is non-conservative.

A force is conservative if the work it does on a particle that moves through a closed path is zero, otherwise, the force is non-conservative.

$$\oint \vec{F} \cdot d\vec{r} = 0$$

Examples: gravitational force, electrostatic force

For such forces work done will not depend on the path of displacement

Rather it will depend only on the positions of the end points (A and B in this case) of the path

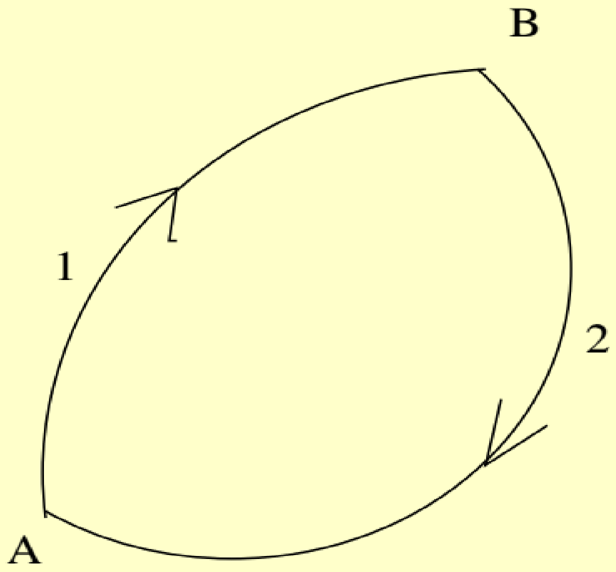
- Thus, for conservative forces, a mathematical function function $V(\mathbf{r})$ exists such that

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = -(V(\mathbf{r}_b) - V(\mathbf{r}_a))$$

  Above negative sign on the RHS is chosen as a matter of convention

The function $V(\mathbf{r})$ has dimensions of energy, and is called the potential energy. $V(\mathbf{r})$ is a scalar field, unlike $\mathbf{F}(\mathbf{r})$, which is a vector field.

- For conservative forces, work done along a closed path is zero
- Consider the closed path shown below



$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = -(V(\mathbf{r}_b) - V(\mathbf{r}_a))$$

- Along the closed path shown above

$$\begin{aligned} \oint \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{r}_b}^{\mathbf{r}_a} \mathbf{F} \cdot d\mathbf{r} \\ &= -(V(\mathbf{r}_b) - V(\mathbf{r}_a)) - (V(\mathbf{r}_a) - V(\mathbf{r}_b)) \\ &= 0 \end{aligned}$$

DONE

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = -\left(V(\mathbf{r}_b) - V(\mathbf{r}_a)\right)$$

V, U Equivalent

$$\Delta V = \Delta U = - W_c$$

$$W_{net} = \Delta K$$

$$W_{ab} =$$

$$W_{net} = W_c$$

Then,

$$\Delta K = - \Delta U$$

$$\Delta K + \Delta U = 0$$

$$\Delta(K + U) = 0$$

Energy Conservation

A consequence of work-energy theorem for conservative forces is that sum of kinetic and potential energies of a system is conserved

V, U Equivalent

$$U_b - U_a = -\int_{x_a}^{x_b} F(x)\,dx. \qquad \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = -(V(\mathbf{r}_b) - V(\mathbf{r}_a))$$

Consider the change in potential energy ΔU as the particle moves from some point x to $x + \Delta x$:

$$\begin{aligned} U(x+\Delta x) - U(x) &\equiv \Delta U \\ &= -\int_x^{x+\Delta x} F(x)\,dx. \end{aligned}$$

For Δx sufficiently small, $F(x)$ can be considered constant over the range of integration and we have

$$\begin{aligned} \Delta U &\approx -F(x)[(x+\Delta x) - x] \\ &= -F(x)\Delta x \end{aligned}$$

or

$$F(x) \approx -\frac{\Delta U}{\Delta x}.$$

V, U Equivalent

$$F(x) \approx -\frac{\Delta U}{\Delta x}.$$

In the limit $\Delta x \to 0$ we have

$$F(x) = -\frac{dU}{dx}.$$

How to generalize it to 3D?

- Symbol ∇V, stands for "gradient of V", defined as **GRADIENT ?**

$$\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

- $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, and $\frac{\partial V}{\partial z}$ are called "partial derivatives", computed by taking the derivative with respect to the given variable (say x), treating other two variables (say y and z) as constants.

- With this Generalizing to 3D

$$V(\mathbf{r}+\Delta\mathbf{r}) - V(\mathbf{r}) = \nabla V \cdot \Delta\mathbf{r} = -\mathbf{F}(\mathbf{r}) \cdot \Delta\mathbf{r}.$$

- Because $\Delta\mathbf{r}$ is an arbitrary displacement, therefore,

$$\nabla V \cdot \Delta\mathbf{r} = -\mathbf{F}(\mathbf{r}) \cdot \Delta\mathbf{r}$$
$$\Longrightarrow \boxed{\mathbf{F}(\mathbf{r}) = -\nabla V}$$

- This is a very important result showing that a conservative force can be written as the gradient of corresponding potential energy.

Consider a scalar function $T(x,y,z)$

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

Let $f(x,y,z) = r^2 = x^2 + y^2 + z^2$, then

$$\nabla f = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 2\mathbf{r}$$

Let $g(x,y,z) = xyz$, then

$$\nabla g = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

Curl (vector) ??

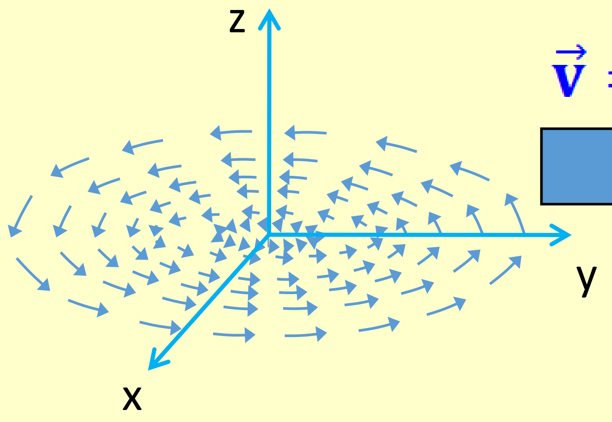
$\vec{\nabla}$ acts on a vector function through cross product

$$\vec{\nabla} \times \vec{v} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Physical meaning:

- **measures the circulation (curl) of a vector at a given point**



$$\vec{v} = (-y\,\hat{i} + x\,\hat{j})$$

Calculate the curl of the vector field $\mathbf{F} = -y\hat{i} + z\hat{j} + x^2\hat{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & z & x^2 \end{vmatrix}$$

$$= -\hat{i} + 2x\hat{j} + \hat{k}$$

Calculate $\nabla \times \mathbf{r}$ where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

STOKES' THEOREM

If a vector field **F** is integrated along a $\boxed{\text{closed loop}}$ of an arbitrary shape, then the line integral is equal to the surface integral of the curl of **F**, evaluated over the area enclosed by the loop

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$$

- For a general vector field $\mathbf{F}$, Stokes theorem states

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}.$$

- If $\mathbf{F}$ is a conservative force, then we know

$$\boxed{\oint \mathbf{F}.\mathbf{dl} = 0.}$$

- The surface area enclosed by a closed loop, in general, is nonzero
- Therefore, for a conservative force $\mathbf{F}$, Stokes theorem implies

$$\boxed{\nabla \times \mathbf{F} = 0.}$$

- Thus, all conservative forces have vanishing curl.

- We also saw that a conservative force can be expressed as

$$\mathbf{F} = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

- Let us calculate the curl of $\mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial y} & -\frac{\partial V}{\partial z} \end{vmatrix}$$

- We obtain

$$\nabla \times \mathbf{F} = \left(-\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^2 V}{\partial z \partial y}\right)\hat{\mathbf{i}} + \left(-\frac{\partial^2 V}{\partial z \partial x} + \frac{\partial^2 V}{\partial x \partial z}\right)\hat{\mathbf{j}}$$
$$+ \left(-\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial x}\right)\hat{\mathbf{k}}$$
$$= 0$$ **DONE**

Requirements/Consequence for/of a Conservative force: Summary

$$\oint \vec{F} \cdot d\vec{r} = 0$$

$$F(x) = -\frac{dV}{dx} = -\frac{dU}{dx}$$

$$\nabla \times \mathbf{F} = 0.$$

$$\mathbf{F}(\mathbf{r}) = -\nabla V$$

$$\frac{1}{2}mv_a^2 + V(\mathbf{r}_a) = \frac{1}{2}mv_b^2 + V(\mathbf{r}_b)$$

A consequence of work-energy theorem for conservative forces is that sum of kinetic and potential energies of a system is conserved

A block with a mass of 5.7 kg slides on a frictionless table with a speed of 1.2 m/s. It is brought to rest by compressing a spring in its path. By how much is the spring compressed if its force constant is 1500 N/m?

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = \int_0^{x_f} (kx\mathbf{i}) \cdot (-dx\mathbf{i}) = \int_0^{x_f} (-kx)\, dx$$

$$W = -\frac{1}{2}kx_f^2$$

$$\Delta K = K_f - K_i = 0 - \frac{1}{2}mv_i^2$$

$$W = \Delta K$$

$$-\frac{1}{2}kx_f^2 = -\frac{1}{2}mv_i^2$$

$$x_f = v_i\sqrt{\frac{m}{k}} = (1.2\, m/s)\sqrt{\frac{5.7\, kg}{1500\, N/m}} = 7.4 \times 10^{-2} m$$

$$x_f = 7.4\, cm$$

- We illustrate the method by a 2D case, where $\mathbf{F}$ is

$$\mathbf{F} = A(x^2\hat{\mathbf{i}} + y\hat{\mathbf{j}})$$

- First we check whether $\nabla \times \mathbf{F} = 0$, or not?
- If $\nabla \times \mathbf{F} \neq 0$, then one cannot find a $V(\mathbf{r})$ which satisfies $-\nabla V = \mathbf{F}$.

- $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2 & Ay & 0 \end{vmatrix} = (0)\hat{\mathbf{i}} + (0)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} = 0$$
- Thus, **F** is a conservative force, and will satisfy
$$\frac{\partial V}{\partial x} = -Ax^2$$
$$\frac{\partial V}{\partial y} = -Ay$$
- On integrating the x equation, we have
$$V(x, y) = -\frac{Ax^3}{3} + f(y),$$
where $f(y)$ is an unknown function of y.

- On substituting this in y equation we have

$$\frac{\partial}{\partial y}\left(-\frac{Ax^3}{3}+f(y)\right)=-Ay$$

- We have

$$\frac{\partial f}{\partial y}=\frac{df}{dy}=-Ay$$

$$\Longrightarrow f(y)=-\frac{Ay^2}{2}+C$$

- Leading to the final result

$$V(x,y)=-\frac{Ax^3}{3}-\frac{Ay^2}{2}+C$$