MA 108-ODE- D3

Lecture 16

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Laplace transform

Laplace transform of Derivatives and Integrals

Laplace Transforms: Recall

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform $\mathcal{L}(f)$ of f is the function defined by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{a \to \infty} \int_0^a e^{-st} f(t) dt,$$

for all values of s for which the integral exists.

Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$ and is of exponential order. Moreover, if the piecewise continuous function f is of exponential order a, for some $a \in \mathbb{R}$, then the $\mathcal{L}(f)(s)$ exists for all $s > a$.

Recall the linearity, scaling, shifting properties of the Laplace transform.

Lerch's Cancellation Law: Recall

Theorem

Suppose f, g are continuous functions and

$$\int_0^\infty e^{-st} f(t)dt \text{ and } \int_0^\infty e^{-st} g(t)dt,$$

converge for some s and that $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for all s for which both integrals converge. Then $f(t) = g(t)$ for all $t > 0$.

Qn. For a continuous function $\phi : [0, \infty) \to \mathbb{R}$, it is given that $\mathcal{L}(\phi)(s) = \frac{c}{s-a}$, for all $s > a$, where c, a are constants. Find ϕ.
Ans. $\phi(t) = ce^{at}, \quad \forall\, t > 0$.

$\mathcal{L}^{-1}$: Notation

Suppose that $f(\cdot)$ has a Laplace transform $F(\cdot)$, i.e., $\mathcal{L}(f)(s) = F(s)$. Then we denote

$$\mathcal{L}^{-1}(F)(t) = f(t).$$

Example.

- For $F(s) = \frac{1}{s-a}, \quad s > a$, $\mathcal{L}^{-1}(F)(t) = e^{at}$.
- For $F(s) = \frac{1}{s^2+1}$, $\mathcal{L}^{-1}(F)(t) = \sin t$.

Laplace Transforms

Example: Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

Note that

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where

$$F(s) = \frac{1}{s^2 + 1} = \mathcal{L}(\sin t).$$

Recall, the shifting property, $\mathcal{L}(e^{2t} \sin t)(s) = F(s-2)$.
Hence,

$$\mathcal{L}^{-1}(G)(t) = e^{2t} \sin t.$$

Laplace Transform of Derivatives and Integrals

Now we derive formulas for

$$\mathcal{L}(f^{(n)})$$

and

$$\mathcal{L}(g)(s), \quad \text{where} \quad g(t) = \int_0^t f(x)dx$$

in terms of $\mathcal{L}(f)$. Notation: $\mathcal{L}(\int_0^t f(x)dx)(s)$ can be used instead of $\mathcal{L}(g)(s)$.

This will be of help in solving differential equations using Laplace transforms.

Laplace Transforms of derivatives

Theorem

Suppose f is differentiable and f', the derivative of f, is piecewise continuous on $[0, \alpha]$ for all $\alpha > 0$. Suppose further that

$$|f(t)| \leq Ke^{at},$$

for $t \geq M > 0$, where $a \in \mathbb{R}$ and $K > 0$. Then $\mathcal{L}(f')(s)$ exists for $s > a$ and

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0).$$

Remark: Note that f' is not assumed to be of exponential order! For instance, $f(t) = \sin e^{t^2}$ satisfies the conditions of the theorem, and hence

$$\mathcal{L}\left(\frac{d}{dt}(\sin e^{t^2})\right) = s\mathcal{L}(\sin e^{t^2}) - \sin 1.$$

Note that $\frac{d}{dt}(\sin e^{t^2})$ is not of exponential order.

Laplace Transforms

Proof: Consider the interval $[0, \alpha]$. Integrating by parts, we get:

$$\int_0^\alpha e^{-st} f'(t) dt \quad = \quad e^{-s\alpha} f(\alpha) - f(0) + s \int_0^\alpha e^{-st} f(t) dt.$$

Taking limit as $\alpha \to \infty$, we get:

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0),$$

for $s > a$. (Why $s > a$?)

Laplace Transforms

Corollary

Suppose f is n-times differentiable and $f, f^{(1)}, \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $[0, \alpha]$, for all $\alpha > 0$. Suppose further that, for all $t \geq M > 0$,

$$|f^{(i)}(t)| \leq Ke^{at},$$

$0 \leq i \leq n-1$, *where $\alpha \in \mathbb{R}$ and $K > 0$. Then, $\mathcal{L}(f^{(n)})(s)$ exists for all $s > a$ and*

$$\begin{aligned}
&\mathcal{L}(f^{(n)})(s) \\
&= s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned}$$

Laplace Transforms

Proof of Corollary: Induction. $n = 1$ is already done. Assume that the result is true for $n - 1$. Then,

$$
\begin{aligned}
\mathcal{L}(f^{(n)})(s) &= \mathcal{L}((f^{(n-1)})')(s) \\
&= s\mathcal{L}(f^{(n-1)}) - f^{(n-1)}(0) \\
&= s\left(s^{n-1}\mathcal{L}(f) - s^{n-2}f(0) - \ldots - f^{(n-2)}(0)\right) - f^{(n-1)}(0) \\
&= s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \ldots - f^{(n-1)}(0).
\end{aligned}
$$

In particular, for $n = 2$,

$$\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s)) - sf(0) - f'(0).$$

Laplace Transforms

Example: Solve the IVP:

$$y'' + y = \sin 2t, y(0) = 2, y'(0) = 1.$$

Take Laplace transform of the DE:

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin 2t);$$

i.e.,

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{2}{s^2+4}.$$

So,

$$\mathcal{L}(y)(s) = \frac{2s + 1 + \frac{2}{s^2+4}}{s^2+1} = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)}.$$

Write rhs as

$$\frac{c_1 s + c_2}{s^2+1} + \frac{c_3 s + c_4}{s^2+4},$$

and solve to get

$$c_1 = 2, c_2 = \frac{5}{3}, c_3 = 0, c_4 = -\frac{2}{3}.$$

Laplace Transforms

Thus,

$$\begin{aligned}\mathcal{L}(y)(s) &= \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4} \\ &= 2\mathcal{L}(\cos t) + \frac{5}{3}\mathcal{L}(\sin t) - \frac{1}{3}\mathcal{L}(\sin 2t),\end{aligned}$$

i.e.,

$$\mathcal{L}(y) = \mathcal{L}\left(2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t\right).$$

Thus,

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

Laplace Transforms of integrals

Theorem

Let f be piecewise continuous on $[0, \alpha]$ for every $\alpha > 0$, and suppose there exist constants $K > 0$ and $a \geq 0$ such that

$$|f(t)| \leq Ke^{at},$$

for $t \geq M > 0$. Then, denoting $g(t) = \int_0^t f(x)dx$,

$$\mathcal{L}(g)(s) = \frac{1}{s}\mathcal{L}(f)(s), \quad \forall\, s > a.$$

Remark: Note that $\int_0^t f(x)dx$ is also of exponential order.

Laplace Transforms

Proof: We need to show that

$$\mathcal{L}\left(\int_0^t f(x)dx\right)(s) = \frac{1}{s}\mathcal{L}(f)(s),$$

for $s > a$, where $|f(t)| \leq Ke^{at}$ for $t \geq M$ and $a \geq 0$. Set

$$g(t) = \int_0^t f(x)dx.$$

Since g is continuous and is of exponential order a, $\mathcal{L}(g)(s)$ exists for all $s > a$.

Recall that f is piecewise continuous on $[0, \alpha]$. Let f be discontinuous at $t_1, t_2, \ldots, t_n$, where $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = \alpha$. Then, on each (t_{i-1}, t_i), g' exists and

$$g'(t) = f(t), \quad \forall\, t \in (t_{i-1}, t_i), \quad i = 1, \cdots, n+1.$$

Then, we have

$$\int_0^\alpha e^{-st} f(t)dt = \int_0^{t_1} e^{-st} g'(t)dt + \int_{t_1}^{t_2} e^{-st} g'(t)dt + \ldots + \int_{t_n}^{\alpha} e^{-st} g'(t)dt.$$

Proof contd..

Integrating by parts, we get:

$$\begin{aligned}\int_0^\alpha e^{-st} f(t)dt &= \sum_{i=1}^{n+1} [e^{-st} g(t)]_{t_{i-1}}^{t_i} + s \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} e^{-st} g(t)dt \\ &= e^{-s\alpha} g(\alpha) - g(0) + s \int_0^\alpha e^{-st} g(t)dt.\end{aligned}$$

Noting $g(0) = 0$, $\lim_{\alpha\to\infty} e^{-s\alpha} g(\alpha) = 0$, and taking limit as $\alpha \to \infty$, we get:

$$\mathcal{L}(f)(s) = s\mathcal{L}(g)(s),$$

for $s > a$.

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Thus,

$$\mathcal{L}(g)(s) = \frac{1}{s}\mathcal{L}(f)(s), \quad \forall\, s > a$$

or denoting $\phi(s) = \frac{1}{s}\mathcal{L}(f)(s), \quad \forall\, s > a$, we get

$$\mathcal{L}^{-1}(\phi)(t) = \int_0^t f(x)dx.$$

In short, we write the above

$$\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(f)\right)(t) = \int_0^t f(x)dx,$$

for $s > a$.

Laplace Transforms

Example: Find f such that

$$\mathcal{L}(f)(s) = \frac{1}{s^2(s^2+a^2)}.$$

Ans. We know that $\frac{1}{s^2+a^2} = \mathcal{L}\left(\frac{\sin at}{a}\right)(s)$. Therefore, using the previous theorem, we get $\frac{1}{s(s^2+a^2)} = \mathcal{L}(g)(s)$, where

$$g(t) = \int_0^t \frac{\sin ax}{a}\,dx = \frac{1}{a^2}(1-\cos at).$$

Thus,

$$\frac{1}{s^2(s^2+a^2)} = \frac{1}{s}\mathcal{L}(g)(s) = \mathcal{L}\left(\int_0^t g(x)\,dx\right).$$

Thus,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right) = \int_0^t \frac{1-\cos ax}{a^2}\,dx = \frac{1}{a^2}(t - \frac{\sin at}{a}).$$

Thus,

$$f(t) = \frac{1}{a^2}(t - \frac{\sin at}{a}).$$