ASSIGNMENT 7 : EIGENVALUES AND EIGENVECTORS

MA 106 : LINEAR ALGEBRA : SPRING 2023

Tutorial Problems

(1) Let u be a unit vector in $\mathbb{R}^n$. Define $H = I - 2uu^t$. Find all the eigenvalues and eigenvectors of H. Find a geometric interpretation of $T_H : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_H(v) = Hv$ for all $v \in \mathbb{R}^n$.

Solution: Note that H is a real symmetric matrix, since $H^t = I - (2uu^t)^t = I - 2uu^t$. Thus it is diagonalizable. Now $H(u) = u - 2uu^tu = -u$. Hence u is an eigenvector for the eigenvalue -1. If $v \perp u$ then $H(v) = v - 2uu^tv = v$. Thus all the nonzero vectors in the space $P = u^\perp = \{v \in \mathbb{R}^n \mid u \perp v\}$ are eigenvectors with eigenvalue 1. Since $\dim P = n-1$, a basis of P along with u is a basis of eigenvectors for H. In fact H is a reflection with respect to the hyperplane P.

(2) If $A, A' \in \mathbb{F}^{n\times n}$ are **similar** , i.e. $A' = P^{-1}AP$ for some invertible $n \times n$ matrix $P \in \mathbb{F}^{n\times n}$. Show that (a) A and A' have same eigenvalues (b) if $\mathbf{v}$ is an eigenvector of A then $P^{-1}\mathbf{v}$ is an eigenvector of A'.

Solution: For a nonzero vector $\mathbf{v}$ we have $P^{-1}\mathbf{v} \neq 0$. Now $A\mathbf{v} = \lambda\mathbf{v}$ iff $P^{-1}AP(P^{-1}\mathbf{v}) = \lambda P^{-1}\mathbf{v}$. This proves both (i) and (ii).

(3) Let A be $n \times n$ matrix. Prove that (i) 0 is an *eigenvalue* of A if and only if A is singular. (ii) if λ is an *eigenvalue* of A then it is also an *eigenvalue* of A^t (where A^t denotes the transpose of A). (iii) If x is an *eigenvector* of A corresponding to λ then x need not be an *eigenvector* of A^t corresponding to λ.

Solution: (i) 0 is an eigenvalue iff 0 is a root of the characteristic polynomial $\chi_A(\lambda) = det\,(A - \lambda I)$. Putting $\lambda = 0$, we get $det\,A = 0$. This implies that A is singular.
(ii) $\chi_A(\lambda) = det\,(A - \lambda I) = \det\,(A - \lambda)^t = det\,(A^t - \lambda I) = \chi_{A^t}(\lambda)$. Since the eigenvalues are nothing but roots of the characteristic polynomial, the conclusion follows.
(iii) Take $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues are $\pm\imath$. Corresponding to the eigenvalue $\imath$ we have $(1, \imath)^t$ is an eigenvector for A but not for A^t.

(4) Show that the map $T : C^\infty[0,1] \to C^\infty[0,1]$ given by $T(f)(x) = \int_0^x f(t)dt$ has no eigenvalue while every real number is an eigenvalue of $T(f)(x) = \frac{df(x)}{dx}$.

Solution: If T has an eigenvector f with eigenvalue α then $T(f) = \int_0^x f(t)dt = \alpha f(x)$. By the fundamental theorem of Calculus, $f(x) = \alpha f(x)$. As $f(x)$ is nonzero, $\alpha = 1$. But then

$f'(x) = f(x)$ For all x. Hence $f(x) = e^x$. But $T(e^x) = e^x - 1 \neq e^x$. If $T(f)(x) = \frac{df(x)}{dx}$ then $T(e^{rx}) = re^{rx}$ for all $r \in \mathbb{R}$. Thus every real number is an eigenvalue of T.

(5) Let $A \in \mathbb{C}^{n\times n}$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a complex polynomial of degree n.. Suppose that λ is an eigenvalue of A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$. Find all the eigenvalues of $f(A)$.

Solution: Let $Au = zu$ for a nonzero vector u and a complex number z. Then $f(A)u = f(z)u$. Thus $f(z)$ is an eigenvalue of $f(A)$ with u as an eigenvector. Conversely, if z is an eigenvalue of $f(A)$ with eigenvector u then $f(A)u = zu$. Consider the complex polynomial $f(x)-z$. Let $z_1, z_2, \ldots, z_n$ be all the roots of $f(x)-z$. Then $f(x)-z = a_n(x-z_1)\ldots(x-z_n)$. Hence $f(A)-zI = a_n \prod_{i=1}^{n}(A-z_iI)$. Take determinant on both sides to get $\det(f(A)-zI) = 0 = a_n \prod_{i=1}^{n} \det(A - z_iI)$. Hence for some j, $\det(A - z_jI) = 0$. Hence $z = f(z_j)$.

(6) Find the characteristic polynomial, eigen spaces and their dimensions of the matrix J_n which is the $n \times n$ matrix with each of its entry equal to 1. Is J_n diagonalisable?

Solution: Note that J_n is a real symmetric matrix. Thus it is diagonalizable. As J_n is a rank one matrix, $det J_n = 0$. Hence 0 is an eigenvalue of J_n. The eigenspace E_0 is the solution vectors of the equation $x_1 + \ldots + x_n = 0$. Thus the $\dim E_0 = n - 1$. Hence the algebraic multiplicity of 0 is $n - 1$. Note that $J_n((1, 1, \ldots, 1)^t = n(1, 1, \ldots, 1)^t$. Hence n is an eigenvalue of J_n. It follows that $\chi_{J_n}(x) = x^{n-1}(x - n)$.

(7) Let $\{u, v\}$ be an orthonormal basis of $\mathbb{R}^2$. Let $A = uv^t$. Find all the eigenvalues of A.

Solution: Let $w \perp v$. Then $Aw = uv^tw = 0$. So E_0 contains the 1-dimensional subspace $v^\perp$. If $u = (a, b)^t$ and $v = (c, d)^t$ then $\mathrm{tr}A = ac + bd = 0$. Hence the only eigenvalue of A is 0.

(8) Let A be a square matrix. Prove the following statements.
(i) The eigenvalues of A are real if A is Hermitian or real symmetric.
(ii) The eigenvalues of A are either 0 or purely imaginary if A is skew Hermitian.
(iii) The eigenvalues of A are of modulus equal to 1, if A is unitary.
(iv) A^tA has only non negative eigenvalues, if A is real.

Solution: Let $\mu \in \mathbb{K},\ \mathbf{v} \neq 0$ be such that $A\mathbf{v} = \mu\mathbf{v}$.
(i) Suppose A is hermitian, i.e., $A = A^*$. Then $\mu\|\mathbf{v}\|^2 = \mu(\mathbf{v}^*\mathbf{v}) = \mathbf{v}^*(\mu\mathbf{v}) = \mathbf{v}^*(A\mathbf{v}) = (\mathbf{v}^*A^*)\mathbf{v} = (A\mathbf{v})^*\mathbf{v} = (\mu\mathbf{v})^*\mathbf{v} = \bar{\mu}\mathbf{v}^*\mathbf{v} = \bar{\mu}\|\mathbf{v}\|^2$. Hence $\mu = \bar{\mu}$ and so, μ is real. Since a real symmetric matrix is hermitian, the second case follows.
(ii) In the above proof, if A were skew hermitian, we get $\mu\|\mathbf{v}\|^2 = -\bar{\mu}\|\mathbf{v}\|^2$. Hence $\mu = -\bar{\mu}$ which means $\mu = 0$ or purely imaginary.
(iii) Since A is unitary, $\langle \mathbf{v}, \mathbf{v}\rangle = \langle A\mathbf{v}, A\mathbf{v}\rangle = \langle \mu\mathbf{v}, \mu\mathbf{v}\rangle = \mu\bar{\mu}\langle \mathbf{v}, \mathbf{v}\rangle$ which means that

$|\mu|^2 = \mu\bar{\mu} = 1$.
(iv) Take $\mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i$. If A_i denotes the columns of A then it follows that $A\mathbf{v} = \sum_{i=1}^{n} A_i = \mathbf{v}$ (since A is Markov). This shows that 1 is an eigenvalue of A.
(v) Since A^tA is real symmetric, its eigenvalues are real. Let $A^tA\mathbf{u} = \lambda\mathbf{u}$. Then $\lambda\|\mathbf{u}\|^2 = \lambda\mathbf{u}^t\mathbf{u} = \mathbf{u}^t(\lambda\mathbf{u}) = \mathbf{u}^t(A^tA\mathbf{u}) = (\mathbf{u}^tA^t)A\mathbf{u} = (A\mathbf{u})^t(A\mathbf{u}) = \|A\mathbf{u}\|^2$.
Therefore $\lambda \geq 0$.

(9) A self-adjoint matrix A, i.e. $A^* = A$, is called **positive definite** if $\langle A\,x,\,x\rangle > 0$ for all nonzero $x \in \mathbb{C}^n$. Show that a self-adjoint matrix is positive definite if and only if all eigenvalues of A are positive.

Solution: A real symmetric matrix is congruent to a diagonal matrix. Since congruence does not change the positivity (check this), and since the eigenvalues are the diagonal entries of the diagonal form, the result follows.

(10) Let A be a self-adjoint matrix. If $\langle Ax,\,x\rangle = 0$ for all $x \in \mathbb{C}^n$, then show that $A = O$. Deduce that if $\|Ax\| = \|A^*x||$ for all $x \in \mathbb{C}^n$, then A is a normal matrix, and if $\|Ax\| = ||x||$ for all $x \in^n$, then A is a unitary matrix.

Solution: Since A is self-adjoint, $A^* = A$ and A has an orthonormal basis of eigenvectors. Let u be a unit eigenvector with eigenvalue a. As a is real, $\langle Au,\,u\rangle = u^*au = a = 0$. Thus all eigenvalues are zero. Thus $A = 0$. Now let $||Ax|| = ||A^*x||$ for all $x \in \mathbb{C}^n$. Therefore, $x^*A^*Ax = x^*AA^*x$. Since A^*A, AA^* are self-adjoint, so is there difference. Hence $x^*(A^*A - AA^*)x = 0$ for all x. Hence $AA^* = A^*A$. Thus A is normal. Now let $||Ax|| = ||x||$ for all x. This means that $x^*A^*Ax = x^*x$. Hence $x^*(AA^* - I)x = 0$ for all x. But $AA^* - I$ is self-adjoint. Hence $AA^* = I$. Thus A is unitary.

(11) Let a be a nonzero real number and $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.
(a) Find an orthonormal set of eigenvectors of A.
(b) Find a unitary matrix C such that $C^{-1}AC$ is a diagonal matrix.
(c) Prove: there is no real orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix.

Solution: (a) The characteristic polynomial of A is $f(x) = x^2 + a^2$. Hence $x = \pm ia$. If $u = (x, y)^t$ is an eigenvector for the eigenvalue ia then $A(x, y)^t = (ya, -ax)^t = (iax, iay)^t$. Thus $(i, 1)^t$ is an eigenvector for the eigenvalue ia Similarly, $(1, i)^t$ is an eigenvector for the eigenvalue $-ia$.
(b) The columns of the unitary matrix C consists of unit eigenvectors for the eigenvalues. Hence $C = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$.

(12) Let C be the locus of the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Using eigenvalues of the symmetric matrix A so that $ax^2 + bxy + cy^2 = [x\ y]A[x\ y]^t$, show that C is ellipse, hyperbola or parabola according as the *discriminant* $4ac - b^2$ is positive, negative or zero.

Practice Problems

(13) Examine whether the following matrices can be diagonalised. If yes, find P such that $P^{-1}AP$ is diagonal. $ox(i)$ $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ $ox(ii)$ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $ox(iii)$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$

Solution: (i) Let us find out the eigenvalues and their algebraic and geometric multiplicities. Here $\chi_A(\lambda) = \det(A - \lambda I) = (\lambda - 4)^2$. This means that 4 is the only eigenvalue and it has algebraic multiplicity $= 2$. Now consider the matrix $A - 4I$ and see that its rank is 1. This means its nullity is 1 which is equal to the geometric multiplicity of the eigenvalue 4. Since these two multiplicities do not match, we conclude that A cannot be diagonalised.
(ii) The eigenvalues are $3, 1, -1$ corresponding to which we have eigen vectors $\mathbf{v}_1 = (1, 0, 0)^t$, $\mathbf{v}_2 = (0, 1/\sqrt{2}, 1/\sqrt{2})^t$ and $\mathbf{v}_3 = (0, -1/\sqrt{2}, 1/\sqrt{2})^t$. Therefore, by putting them side by side, we get a matrix $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ which is as required.
(iii) This is a lower triangular matrix and hence the eigenvalues are the diagonal entries. Corresponding eigenvectors are respectively $(1, -1, -1)^t, (0, 1, 1)^t, (0, 0, 1)^t$. Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.
(iv) In this case we have $\chi_A(\lambda) = (\lambda - 3)(\lambda - 1)(\lambda + 1)(\lambda - 1/2)$. Therefore all the eigenvalues are distinct. Therefore, the matrix is diagonalizable. We now proceed to find the eigenvectors corresponding to these eigenvalues, which we denote by $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ respectively.
$\underline{\lambda = 3}$: The form of the matrix immediately gives that $A\mathbf{e}_1 = 3\mathbf{e}_1$. Hence $\mathbf{u} = \mathbf{e}_1$ is an eigenvector for $\lambda = 3$.
$\underline{\lambda = 1}$: We want to solve $A\mathbf{v} = \mathbf{v}$. This is easily seen to have a solution $\mathbf{v} = (-3, 2, 2, 0)^t$.
$\underline{\lambda = -1}$: Here we have to solve for $A\mathbf{w} = -\mathbf{w}$. It follows that $w_2 = w_4 = 0$ and $4w_1 + 2w_2 + w_3 = 0$. Therefore $\mathbf{w} = (1, 0, -4, 0)^t$ is a solution.
$\underline{\lambda = -1/2}$: This time we have to solve

$$\begin{bmatrix} 5/2 & 2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 2 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Check that $\mathbf{x} = (-8, 6, 8, 3)^t$ is a solution. Thus a transformation matrix is obtained by writing these four eigen vectors side by side as columns. $P = [\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}]$. Check that $AP = PD$ where $D = oxdiag(3, 1, -1, 1/2)$. (Observe that in each of the above three cases, there are plenty of choices for P.)

(14) Prove that (a) the *trace* of $A \in \mathbb{C}^{n\times n}$ is equal to the sum of its *eigenvalues*. (b) the determinant of A is equal to the product of its *eigenvalues*.

Solution: (a) Let $f(x) = \det(xI - A) = (x - a_1)(x - a_2)\dots(x - a_n)$ for some $a_i \in \mathbb{C}$ for all $I = 1, 2, \dots, n$. Equate the coefficient of $-x^{n-1}$ to get that the trace of A is the sum of all the roots of $f(x)$.
(b) Put $x = 0$ to see that $\det A = \prod_{j=1}^{n} a_j$.

(15) Let $V = \mathbb{R}^{2\times 2}$. Let $T : V \to V$ be defined by $T(A) = A^t$. Find the eigenvalues and eigenvectors of T.

Solution: Let A be a nonzero matrix and $r \in R$ so that $T(A) = A^t = rA$. Hence $A = rA^t = r^2A$. As A is nonzero, $r = 1, -1$. The nonzero symmetric matrices are the eigenvectors with eigenvalue 1 and the nonzero skew-symmetric matrices are the eigenvectors with eigenvalue -1.

(16) Let A be a 2×2 real matrix and $p_{_A}(x)$ be its characteristic polynomial. Show that $p_{_A}(A) = 0$. This is called the Cayley-Hamilton Theorem. It is valid for all square matrices.

(17) Find a nonzero matrix so that $N^3 = 0$. Find all the eigenvalues of N. Show that N cannot be symmetric.

Solution: If a is an eigenvalue of N then $a^3 = 0$. Thus $a = 0$. If N is symmetric then there exists an invertible matrix A so that $A^1NA = oxdiag(0, 0 \dots, 0)$. Hence $N = 0$. This is a contradiction. Hence N is not symmetric.

(18) Let an $n \times n$ matrix B have n distinct *eigenvalues*. Show that every $n \times n$ matrix A such that $AB = BA$, is diagonalizable.

Solution: Let $AB = BA$. Let $B\mathbf{v} = \lambda\mathbf{v}$. Then $B(A\mathbf{v}) = AB\mathbf{v} = \lambda A\mathbf{v}$. This implies that $A\mathbf{v} \in E_B(\lambda)$, the eigenspace of B corresponding to λ. Now if B has distinct eigenvalues λ_i then this means that each $E_B(\lambda_i)$ is one dimensional. Let it be spanned by $\mathbf{v}_i$ say. Then we have $A\mathbf{v}_i = \mu_i\mathbf{v}_i$ for each i. Therefore, with respect to the basis $\{\mathbf{v}_i : 1 \le i \le n\}$ both B and A are diagonal.

(19) From the unit vector $u = \frac{1}{6}(1, 1, 3, 5)^t$ construct the rank one projection matrix $P = uu^t$. (a) Show that u is an eigenvector with eigenvalue 1. (b) Show that if $v \perp u$ then $Pv = 0$. Show that the only eigenvalues of P are $0, 1$. What are their algebraic and geometric multiplicities? Is P diagonalizable?

Solution: Note that $P(u) = uu^tu = u$. Hence u is an eigenvector with eigenvalue 1. Let $v \perp u$. Then $P(v) = uu^tv = 0$. Thus all nonzero vectors in the subspace $u^\perp$ are eigenvectors with eigenvalue 0. As P is symmetric, it is diagonalizable. Hence the algebraic multiplicity of 0 is at least $n-1$ and that of 1 is at least one. But $\dim E_0 + \dim E_1 = n$. Hence P has no other eigenvalue. Thus the characteristic polynomial of P is $x^{n-1}(x-1)$.

(20) If A is a real skew-Hermitian matrix, prove that $I + A$ and $I - A$ are nonsingular, i.e. invertible and $(I - A)(I + A)^{-1}$ is orthogonal.

Solution: Let r be an eigenvalue of A with eigenvector u. Then

$$\langle Au,\, u\rangle = \langle ru,\, u\rangle = \bar{r}\langle u,\, u\rangle = \langle u,\, A^*u\rangle = \langle u,\, -Au\rangle = -r\langle u,\, u\rangle.$$

Thus $\bar{r} = -r$. Hence r is purely imaginary. The eigenvalues of $I + A$ are for the form $1 + r$ where r is an eigenvalue of A. Hence all the eigenvalues of $= I + A$ are nonzero. Similarly all eigenvalues of $I - A$ are also nonzero. Thus $I + A$ and $I - A$ are invertible. Since $I + A$ and $I - A$ commute with each other, $I + A$ and $(I - A)^{-1}$ also commute with each other. The matrix $(I + A)(I - A)^{-1}$ is orthogonal as

$$\begin{aligned}(I+A)(I-A)^{-1}((I+A)(I-A)^{-1})^t &= (I+A)(I-A)^{-1}((I-A)^t)^{-1}(I+A^t) \\ &= (I+A)(I-A)^{-1}(I+A)^{-1}(I-A) = I.\end{aligned}$$

(21) Find the values of c for which the graph of $2xy - 4x + 7y + c = 0$ is a pair of lines.

(22) Prove that the *eigenvectors* of a Hermitian (or real symmetric) matrix corresponding to distinct *eigenvalues* are orthogonal.

Solution: See the lecture slides.

(23) By a symmetric quadratic form Q of n variables we mean a homogeneous degree 2 polynomial in n variables, say $Q(x) = \sum_{i\leq j} \alpha_{ij}x_ix_j$. Given a quadratic form Q we associate a symmetric matrix $A_Q = (a_{ij})$ to it by taking $a_{ii} = \alpha_{ii}$ and $a_{ij} = \alpha_{ij}/2, i \neq j$. Show that $Q(x) = xA_Qx^t$, where $x = (x_1, \ldots, x_n)$. Write down the associated matrix or the quadratic form from the given data below:
(a) $Q_1(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^3$; (b) $Q_2(x, y) = xy$.
(c) $Q_3(x, y, z) = xy + yz + zx$; (d) $Q_4(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.
(e) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & -5 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: (a) $A_1 = \operatorname{diag}(1, 2, 3)$; (b) $A_2 = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) $3x^2 - 5y^2 + z^2 + 4xy + 4xz + 8yz$ (f) $x^2 + y^2 + z^2$.

(24) Transform the following quadratic equations to a diagonal form and find out what conics they represent:
(a) $41x_1^2 - 24x_1x_2 + 34x_2^2 = 0$
(b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$;
(c) $91x^2 - 24xy + 84y^2 = 25$.
(d) $4xy + 3y^2 = 10$.

Solution: (a) $\begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}$. Eigen values are: $25, 50$. Transformation matrix is $\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$. The principal axis form is $25x_1^2 + 50x_2^2 = 0$; this is a degenerate ellipse.
(b) $\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$; eigenvalues are: $0, 10$. The transformation matrix is $\frac{1}{\sqrt{10}} \begin{bmatrix} 1/ & -3 \\ 3 & 1 \end{bmatrix}$. The p.a. form is: $10y^2 = 40$ which represents a pair of parallel lines.
(c) The matrix here is $\begin{bmatrix} 91 & -12 \\ -12 & 84 \end{bmatrix}$ with eigenvalues $75, 100$. The transformation matrix is $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$. The p.a. form is $3x^2 + 4y^2 = 1$ which is an ellipse.
(d) $\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$. The eigen values are: $4, -1$. The p.a. form is: $4x_1^1 - x_2^2 = 10$ which is a hyperbola. The transformation matrix is $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

(25) Let A be a real symmetric matrix with only one eigenvalue 1. Show that $A = I$.

Solution: Since a real symmetric matrix is orthogonally diagonalizable, the eigenspace $E_0 = \mathbb{R}^n$ if A is an $n \times n$ matrix. Thus $Ae_j = e_j$ for all j. This means $A = I$.

(26) Find all the eigenvalues of a nilpotent matrix A. When is A is diagonalizable?

Solution: If r is an eigenvalue of A then r^m is an eigenvalue of A^m for all m. As A is nilpotent there is an n so that $A^n = o$. Thus $r = 0$. If A were diagonalizable, there is an invertible matrix P so that $P^{-1}AP = D = 0$. Hence $A = 0$.

(27) Find all 2×2 orthogonal and skew- symmetric matrices. Also find their eigenvalues.

Solution: Let A be such a matrix. Then $I = A^tA = -A^2$. Hence $A^2 + I = 0$. if r is an eigenvalue of A then $r^2 + 1 = 0$. Thus $i, -i$ are the eigenvalues of A. Let $D = oxdiag(i, -i)$. Since A has diagonal entries 0, it follows that

$$A = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(28) Does there exist a 3×3 matrix which is orthogonal and skew symmetric?

Solution: No. For a skew symmetric matrix A, we have

$$\det\, A = \det\, A^t = \det(-A) = (-1)^n \det\, A.$$

Thus if n is odd, then $\det\, A = 0$. On the other hand for any orthogonal matrix the determinant is ± 1.

(29) Prove that if a square complex matrix is unitary and and Hermitian then $A^2 = I$.

Solution: If A is unitary then its eigenvalues have absolute value 1. As A is also Hermitian, its eigenvalues are real. Hence ± 1 are the only eigenvalues of A. But $A = PDP^t$ where the entries of the diagonal matrix are ± 1 and P is unitary. Hence $A^2 = PDP^P DP^t = I$.

(30) Let A be a normal matrix and U be unitary. Prove that $U^* AU$ is normal.

Solution: Hint: Use the facts:$UU^* = I$ and $AA^* = A^* A$.

(31) Given an orthogonal matrix A, with -1 as an eigenvalue of multiplicity k then $\det A = (-1)^k$.

Solution: If z is an eigenvalue of A then so is $\bar{z}$. Moreover $|z| = 1$. Therefore $\det A = (-1)^k$.

(32) If the equation $ax^2 + bxy + cy^2 = 1$ represents an ellipse, prove that the area of the region it bounds is $2\pi/\sqrt{4ac - b^2}$.

Solution: Let $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Then the equation of the ellipse is given by $[x, y]A \begin{bmatrix} x \\ y \end{bmatrix} = 1$. Let λ, μ be the eigenvalues of A. Then $\lambda\mu = \det A = ac - b^2/4$. The equation in the diagonal form is $\lambda x^2 + \mu y^2 = 1$. Hence the lengths of the axes are $1/\sqrt{\lambda}$ and $1/\sqrt{\mu}$. Using calculus we have

$$Area \quad = \quad 4\int_0^{1/\sqrt{\lambda}} y dx = 4\int_0^{1/\sqrt{\lambda}} \sqrt{\frac{1 - \lambda x^2}{\mu}} dx.$$

Now substitute $\sqrt{\lambda}x = \sin\theta$.