MA 106 : LINEAR ALGEBRA : SPRING 2023

DETERMINANTS

1. Tutorial Problems

(1) Compute the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

using the Gauss-Jordan Elimination Method and cofactors and compare the results.

(2) Calculate the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \dots & n \end{bmatrix}.$$

(3) (Vandermonde determinant): (a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

(b) Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column.

(4) Solve the following systems by Cramer's rule:

(i)
$$\begin{aligned} -x + 3y - 2z &= 7 \\ 3x + y + 3z &= -3 \\ 2x + y + 2z &= -1 \end{aligned}$$

(ii)
$$\begin{aligned} 4x + y - z &= 3 \\ 3x + 2y - 3z &= 1 \\ -x + y - 2z &= -2 \end{aligned}$$

(5) Let A be an $n \times n$ and B be an $m \times m$ matrix. Show that $\det \begin{bmatrix} A & O \\ O & B \end{bmatrix} = \det A \det B$.

Hint. Note that $\begin{bmatrix} A & O \\ O & B \end{bmatrix} = \begin{bmatrix} A & O \\ O & I_m \end{bmatrix} \begin{bmatrix} I_n & O \\ O & B \end{bmatrix}$. Now regard the function $f(A) = \det \begin{bmatrix} A & O \\ O & I_m \end{bmatrix}$ as a function of columns of A. Show that $f(A)$ is a determinant function. Hence $f(A) = \det A$.

2. **Practice Problems**

(6) Compute the inverse of the following matrices using Gauss-Jordan Method and using the cofactors and compare the results.

$$\text{(i)} \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

(7) Calculate the determinant of the following matrices :

$$\text{(a)} \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \\ 1 & & & & \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \dots & n \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \cdot & & \\ & & & \cdot & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

(8) Prove that the equation of the line in the plane through the points $(a, b), (c, d)$ is given by

$$\det \begin{bmatrix} x & y & 1 \\ a & b & 1 \\ c & d & 1 \end{bmatrix} = 0.$$

(9) Show that the area of the triangle in the plane with vertices $(a, b), (c, d), (e, f)$ is given by

$$\frac{1}{2} \det \begin{bmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{bmatrix}.$$

(10) Show that the volume of the tetrahedron with vertices (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) and (d_1, d_2, d_3) is given by

$$\frac{1}{6} \det \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{bmatrix}$$

(11) Let A be a 2×2 matrix. Show that $\det(A + I) = 1 + \det A$ if and only if trace $(A) = 0$.

(12) Let A be an $n \times n$ matrix having the block form

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_k \end{bmatrix}$$

where A_j is an $r_j \times r_j$ matrix for $j = 1, 2, \dots, k$. Show that $\det A = \det A_1 \det A_2 \dots \det A_k$.

(13) Prove: Let A be an $n \times n$ real matrix with the property that the entries in each row add up to 0. Show that $\det A = 0$.

(14) Recall that for a square matrix adj$A = (\mathrm{cof}(a_{ij}))^T$. Show that if A is an $n \times n$ invertible matrix then $\det(\mathrm{adj}(A)) = \det(A^{n-1})$.

(15) Given n^2 functions $f_{ij}(x)$ each differentiable on the interval (a, b), define $f(x) = \det(f_{ij(x)})$ for each $x \in (a, b)$. Let $A(x) = (f_{ij}(x))$. Let $A_i(x)$ be the matrix obtained from $A(x)$ by differentiating the functions in the i^{th} row of $A(x)$. Prove that $f'(x) = \sum_{i=1}^{n} \det A_i(x)$.