Tut - 4 Part 2

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Particle in a box (Potential barrier)

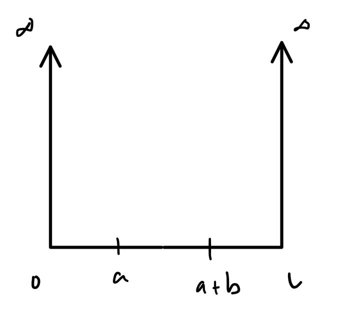
For particle in a box of side L:

$$\phi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E = \frac{n^2h^2}{8mL^2}$$

Q1

1. * For a particle in a 1-D box of side L, show that the probability of finding the particle between $x = a$ and $x = a + b$ approaches the classical value b/L, if the energy of the particle is very high.

1-D box of side L.



Prob of particle from $a \rightarrow a+b$

$$\int_a^{a+b} \phi^* \phi = \frac{2}{L} \int_a^{a+b} \sin^2 \frac{n\pi x}{L} dx$$

$$= \frac{1}{L} \int_a^{a+b} 1 - \cos 2\frac{n\pi x}{L} dx$$

$$= \frac{1}{L} \left[x - \sin \frac{n\pi x}{L} \times \frac{L}{n\pi} \right]_a^{a+b}$$

$$= \frac{1}{L} \left[b - \frac{L}{n\pi}\left(\sin \frac{n\pi(a+b)}{L} - \sin \frac{a n\pi}{L}\right)\right]$$

$\rightarrow 0$ when $n \rightarrow \infty$

when $n \rightarrow \infty$

$\therefore$ prob $= b/L$ [as expected classically]

Q2

2. Consider a particle confined to a 1-D box. Find the probability that the particle in its ground state will be in the central one-third region of the box.

Required prob $\Rightarrow \frac{2}{L} \int_{L/3}^{2L/3} \sin^2 \frac{\pi x}{L}$

$$\boxed{\text{Ans} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}}$$

$$\frac{1}{L}\left[\int_{L/3}^{2L/3} 1 - \cos 2\frac{\pi x}{L} dx\right] = \frac{1}{L}\left(\frac{L}{3} - \sin\frac{2\pi x}{L} \times \frac{L}{2\pi}\Big|_{L/3}^{2L/3}\right)$$

$$= \frac{1}{3} + \frac{\sqrt{3}}{2\pi}$$

Q3

3. Consider a particle of mass m moving freely between $x = 0$ and $x = a$ inside an infinite square well potential.
(a) Calculate the expectation values $\langle \hat{X} \rangle_n, \langle \hat{P} \rangle_n, \langle \hat{X}^2 \rangle_n$, and $\langle \hat{P}^2 \rangle_n$, and compare them with their classical counterparts.
(b) Calculate the uncertainties product $\Delta x_n \Delta p_n$.
(c) Use the result of (b) to estimate the zero-point energy.

Infinite square well $\Rightarrow$ particle in a box

Explanatn from the solⁿ.

Q4

4. Consider a one dimensional infinite square well potential of length L. A particle is in $n = 3$ state of this potential well. Find the probability that this particle will be observed between $x = 0$ and $x = L/6$. Can you guess the answer without solving the integral? Explain how.

From the solⁿ $\Rightarrow$ good method [nodes]

Q5

5. * Consider a one-dimensional particle which is confined within the region $0 \leq x \leq a$ and whose wave function is $\Psi(x,t) = \sin(\pi x/a)\exp(-i\omega t)$.
(a) Find the potential $V(x)$.
(b) Calculate the probability of finding the particle in the interval $a/4 \leq x \leq 3a/4$.

↳ stationary state with energy $\hbar\omega$

Use TDSE

a) $\Psi(0,t) = 0, \quad \Psi(a,t) = 0$

for $0 < x < a$, TDSE

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x)\phi = i\hbar\frac{\partial\phi}{\partial t}$$

$$\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\phi + V\phi = \hbar\omega\phi$$

$$V = \hbar\omega - \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$$

Normalisatⁿ const $= \sqrt{2/a}$

$$\text{prob} = \frac{2}{a}\int_{a/4}^{3a/4} \sin^2\frac{\pi x}{a} dx = \frac{1}{a}\left[x - \sin\frac{2\pi x}{a} \times \frac{a}{2\pi}\right]_{a/4}^{3a/4}$$

$$= \frac{1}{a}\left[\frac{a}{2} + \frac{a}{\pi}\right] = \frac{1}{2} + \frac{1}{\pi} > \frac{1}{2}$$

Classical prob

Q6

V imp

6. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. If the electron is initially in the ground state ($n = 1$) of the box and if we suddenly quadruple the size of the box (i.e., the right-hand side wall is moved instantaneously from $x = a$ to $x = 4a$), calculate the probability of finding the electron in:
(a) the ground state of the new box and
(b) the first excited state of the new box.

The total wave funcn of new box stays same as old one

$\Psi(x)$ in $= \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$ $(n=1)$

$$\Rightarrow \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \sum_{i=1}^{\infty} c_i \sqrt{\frac{2}{4a}} \sin\left(\frac{i\pi x}{4a}\right)$$

integrate with conjugate of ith funcn to obtain c_is

Q7

7. Solve the time independent Schrodinger equation for a particle in a 1-D box, taking the origin at the centre of the box and the ends at $\pm L/2$, where L is the length of the box.

Q8

8. * Consider a particle of mass m in an infinite potential well extending from $x = 0$ to $x = L$. Wave function of the particle is given by

$$\psi(x) = A\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

where A is the normalization constant

(a) Calculate A
(b) Calculate the expectation values of x and x^2 and hence the uncertainty Δx.
(c) Calculate the expectation values of p and p^2 and hence the uncertainty Δp.
(d) What is the probability of finding the particle in the first excited state, if an energy measurement is made?

(given, $\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = 0$, $\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = 0$, for all n)

From solns.