1. Selected Tutorial Problems: Matrix operations

(1) A matrix is called *symmetric* if $A^t = A$ and *skew-symmetric* if $A^t = -A$. Let A and B be symmetric matrices of same size. Show that AB is a symmetric matrix iff $AB = BA$. Show also that any square matrix can be written as sum of symmetric and skew symmetric matrices in a unique way.

Solution: Suppose $AB = BA$. Then $(AB)^t = (BA)^t = A^tB^t = AB$ and so AB is symmetric. Conversely, if AB is symmetric, then $BA = B^tA^t = (AB)^t = AB$. If A is any square matrix then $A = (1/2)[(A + A^t) - (A - A^t)]$. Note that $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric. If $A = B + C = D + E$ where B, D are symmetric and C, E are skew-symmetric then $B - D = E - C$. Note that $B - D$ is symmetric and $E - C$ is skew-symmetric. But the only metric that is symmetric and skew-symmetric is the zero matrix. Thus the decomposition is unique.

(2) A square matrix A is said to be *nilpotent* if $A^n = 0$ for some $n \geq 1$. Let A, B be nilpotent matrices of the same order. (i) Show by an example that $A + B, AB$ need not be nilpotent. (ii) However, prove that this is the case if A and B commute with each other, i.e. if $AB = BA$. (Show that if $AB = BA$ then the binomial theorem holds for expansion of $(A + B)^n$.)

Solution: (i) Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ Verify that $A^2 = 0 = B^2$. Verify that $C = AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then check that $C^2 = C$ and hence $C^n = C \neq 0$ for any n. Also see that if $C' = A + B$ then $C'^2 = I_2$. Hence the sum is also sot nilpotent.

(ii) Assume that A, B commute with each other. We have proved that every power of A commutes with every power of B. Thus

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2.$$

Inductively, it follows that binomial formula is true for $(A + B)^n$.

Now suppose $A^m = 0$ and $B^n = 0$. Then in the binomial expansion of $(A + B)^{m+n}$ each term will have a factor A^m or a factor B^n and hence vanishes. Therefore $(A + B)^{n+m} = 0$. Finally $(AB)^n = A^nB^n = 0$ shows that AB is also nilpotent.

(3) If A and B are square matrices, show that $I - AB$ is invertible iff $I - BA$ is invertible. [Hint: Start from $B(I - AB) = (I - BA)B$.]

Solution: Let C be the inverse of $I - AB$ Then we have $I = (I - AB)C$. Multiply both sides by B on the left to get $B = B(I - AB)C = (I - BA)BC$. Multiply both sides by A on the right to get $BA = (I - BA)BCA$. Therefore $I = I - BA + BA = I - BA + (I - BA)BCA = (I - BA)(I + BCA)$. Likewise one can show that $(I + BCA)(I - BA) = I$. So, $I - BA$ is invertible. By symmetry 'if' part also follows.

(4) Let $N = \{1, 2, \ldots, n\}$. By a permutation on n letters we mean a bijective mapping $\sigma : N \longrightarrow N$. Given a permutation $\sigma : N \longrightarrow N$ define the permutation matrix P_σ to be the

$n \times n$ matrix $((p_{ij}))$ where

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $P_{\sigma\circ\tau} = P_\tau P_\sigma$. Deduce that all permutation matrices are invertible and

$$P_\sigma^{-1} = P_{\sigma^{-1}} = P_\sigma^T.$$

Solution: Put $P = P_\sigma,\ Q = P_\tau$ and $R = P_{\tau\circ\sigma}$. Then $r_{ij} = 1$ iff $j = \tau\circ\sigma(i)$. (Other entries are zero of course.) On the other hand $(PQ)_{ij} = \sum_k p_{ik}q_{kj} = q_{\sigma(i),j} = 1$ iff $j = \tau\circ\sigma(i)$. Hence $P_{\tau\circ\sigma} = PQ$. It follows that $P_\sigma P_{\sigma^{-1}} = I_n$. Hence all permutation matrices are invertible. Alternatively, directly we can verify that $P_\sigma P_\sigma^t = I_n$, which proves the last assertion also.

(5) The matrix $A = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$, where $i^2 = -1, a = \frac{1}{2}(1+\sqrt{5})$, and $b = \frac{1}{2}(1-\sqrt{5})$, has the property $A^2 = A$. Describe completely all 2×2 matrices A with complex entries such that $A^2 = A$.

Solution: Put $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and obtain the four equations from $A^2 = A$:

$$a^2 + bc = a;\ ab + bd = b;\ ac + cd = c;\ bc + d^2 = d.$$

In particular, from the first and the last equations it follows that a, d are solutions of the equation $x^2 - x + bc = 0$. If $b = c = 0$, then the other two equations do not matter and we get diagonal matrices with diagonal entries equal to 0 or 1. which are solutions as seen in an earlier exercise.

Now suppose, $b \neq 0$ then we must also have $a + d = 1$. This is the case even if $c \neq 0$. Thus in this case, starting from arbitrary b, c we solve the equation $x^2 - x + bc = 0$ call the two solutions a, d respectively. Then automatically, the middle two equations are satisfied. By interchanging a, d we get all the solutions.

2. Selected Tutorial Problems : Linear equations and Gauss Elimination

(a) Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM

$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

Solution: The Gauss-Jordan form of the augmented matrix is

$$J[A|\mathbf{b}] = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 7/12 & 42/5 \\ 0 & 0 & 1 & 0 & -1/2 & -1 \\ 0 & 0 & 0 & 1 & -1/2 & -1 \end{array}\right].$$

The solution can thus be represented in the form:

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42/5 & 0 & -7/2 \\ 2/5 & 0 & -1/2 \\ -1 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ s \end{bmatrix}; x_1 = t, x_5 = s.$$

(b) The n^{th} **Hilbert matrix** H_n is defined as the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $\dfrac{1}{i+j-1}$. Obtain H_3^{-1} by the Gauss-Jordan elimination Method.

Solution: $[H|I] = \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 1 & 0 \\ 1/3 & 1/4 & 1/5 & 0 & 0 & 1 \end{array}\right] \quad R_2 - R_1/2 \;\&\; R_3 - R_1/3 \longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 0 & 1/12 & 1/12 & -1/2 & 1 & 0 \\ 0 & 1/12 & 4/5 & -1/3 & 0 & 1 \end{array}\right] R_3 - R_2 \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 0 & 1/12 & 1/12 & -1/2 & 1 & 0 \\ 0 & 0 & 1/180 & 1/6 & -1 & 1 \end{array}\right]$$

$$\left.\begin{array}{l} 12 \times R_2 \\ 180 \times R_3 \end{array}\right\} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] \quad \left.\begin{array}{l} R_1 - R_3/3 \\ R_2 - R_3 \end{array}\right\} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] R_1 - R_2/2 \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & -30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

Therefore $H^{-1} = \begin{bmatrix} 9 & -36 & -30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$.

(c) Find the point in $\mathbb{R}^3$ where the line joining the points $(1,-1,0)$ and $(-2,1,1)$ pierces the plane defined by $3x - y + z - 1 = 0$.

Solution: The points on the line joining $(1,-1,0)$ and $(-2,1,1)$ are of the form $a(1,-1,0) + b(-2,1,1) = (a-2b, b-a, b)$ where $a, b \in \mathbb{R}$ and $a + b = 1$. If such a point pierces the plane defined by $3x - y + z - 1 = 0$ then $3(a-2b) - (b-a) + b = 4a - 6b = 1$. As $a + b = 1$, we put $b = 1 - a$ in the equation $4a - 6b = 1$ to get the equation $4a - 6(1-a) = 1$. Thus $10a = 7$ and $a = 7/10, b = 3/10$. Therefore the required point is $(1/10, -4/10, 3/10)$.

(d) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$. Write $A = EH$ where E is an elementary matrix and H is a symmetric matrix.

Solution: Subtract two times the first row from the second row. The resulting matrix is a symmetric matrix. This row operation is indicated below.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

Hence the required factorisation is

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

(e) Find the null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution: The null space of A consists of all $u = (x, y)^t$ so that $Au = 0$. This is same as solving the equations $x + 2y = 0 = 3x + 6y = 0$. Hence $x = -2y$. Hence $(-2y, y)^{\in} N(A)$. The null space is the line passing through the origin and the point$(-2, 1)$.

(f) Show that an $n \times n$ matrix is invertible if and only if its column vectors are linearly independent.
Solution: Let $A^1, A^2, \ldots, A^n$ b the column vectors of A. Suppose that A is invertible and $Ax = 0$ for a vector $x = (x_1, x_2, \ldots, x_n)^t$. This means $Ax = x_A^1 + x_2A^2 + \ldots + x_nA^n = 0$. If A is invertible then $A^{-1}Ax = Ix = x = 0$. Hence the column vectors are linearly independent. Conversely if the column vectors are linearly independent. Then The row vectors of A^t are linearly independent. The RCF of such a matrix is the $n \times n$ identity matrix I_n. Hence there are elementary matrices $E_1E_2 \ldots E_rA^t = I_n$. Taking transpose on both sides proves that A is a product of elementary matrices. But elementary matrices are invertible. Therefore A is also invertible.

3. **Selected Tutorial Problems : Determinants**

(6) Compute the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

using the Gauss-Jordan Elimination Method and cofactors and compare the results.

Solution: Use EROs to transform $[A|I]$ to $[I|C]$ where C is A^{-1}

$$[A|I] = \left[\begin{array}{ccc|ccc} 5 & -1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -5 & 3 & -15 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3\to R3-R2]{R3\to R3+R1} \left[\begin{array}{ccc|ccc} 5 & -1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -10 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R3\to R3/-10]{R2\to R2/2}$$

$$\left[\begin{array}{ccc|ccc} 5 & -1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/10 & 1/10 & -1/10 \end{array}\right] \xrightarrow{R1\to R1+R2, R1-5R3, -1/5R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/10 & 0 & 1/10 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/10 & 1/10 & -1/10 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 3/10 & 0 & 1/10 \\ 0 & 1/2 & 0 \\ -1/10 & 1/10 & -1/10 \end{bmatrix}$. Now Find inverse of $A = \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$ using the adjoint matrix. The cofactors of A are:
cof a_{11} = -30, cof a_{12} = 0, cof a_{13} = 10, cof a_{21} = 0, cof a_{22} = -50, cof a_{23} = -10, cof a_{31} = -10, cof a_{32} = 0, cof a_{33} = 10. We know that $Adj(A)$ = (Cof A)^t. So $Adj(A) = \begin{bmatrix} -30 & 0 & -10 \\ 0 & -50 & 0 \\ 10 & -10 & 10 \end{bmatrix}$ and also $A^{-1} = \frac{1}{det(A)}$Adj(A)

Computing det(A) = -100, So, $A^{-1} = \begin{bmatrix} 3/10 & 0 & 1/10 \\ 0 & 1/2 & 0 \\ -1/10 & 1/10 & -1/10 \end{bmatrix}$

Observe that the value sof A^{-1} calculated using both the methods are equal.

(7) Calculate the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \dots & n \end{bmatrix}.$$

Solution: Perform row operations in the following order: $R_n - R_{n-1}, R_{n-1} - R_{n-2}, \cdots, R_2 - R_1$ to obtain:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

Now expand along the last column and use the fact that the determinant of an upper triangular matrix with diagonal entries all equal to 1 is equal to 1 to conclude that the given determinant is equal to $(-1)^{n+1}n$.

(8) (Vandermonde determinant): (a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

(b) Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column.

Solution: (a) Expand directly and factorise.

(b) Let $V(x_1, \ldots, x_n)$ denote the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

We prove the formula

$$V(x_1, \ldots, x_n) = \prod_{i>j} (x_i - x_j) \tag{1}$$

by induction on n. Observe that for $n = 1$ by definition, the RHS is equal to 1. For $n = 2$ and 3 the formula is easily verified. So assume formula (**??**) for $n-1$. Perform the following row operations on the matrix in that order:

$$R_n - x_1 R_{n-1}, R_{n-1} - x_1 R_{n-2}, \ldots, R_2 - x_1 R_1.$$

We get

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{bmatrix}$$

After pulling out the factors $(x_i - x_1)$ from i^{th} column, $i \geq 2$, we get

$$V(x_1, \ldots, x_n) = \left(\prod_{i>1} (x_i - x_1) \right) V(x_2, \ldots x_n).$$

The conclusion follows from induction.

(9) Solve the following systems by Cramer's rule:

$$\text{(i)} \quad \begin{aligned} -x + 3y - 2z &= 7 \\ 3x + y + 3z &= -3 \\ 2x + y + 2z &= -1 \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} 4x + y - z &= 3 \\ 3x + 2y - 3z &= 1 \\ -x + y - 2z &= -2 \end{aligned}$$

Solution: (i) The system of linear equations is equivalent to $AX = b$, where

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ b = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}.$$

Now $\det(A) = -1, \det(C1) = 6, \det(C2) = -3, \det(C3) = -4$. Let Ck be the matrix obtained from A by replacing the kth column of A by b. Then by Cramer's rule,

$$x = \frac{\det(C1)}{\det(A)} = \frac{6}{-1} = -6, \ y = \frac{\det(C2)}{\det(A)} = \frac{-3}{-1} = 3, z = \frac{\det(C3)}{\det(A)} = \frac{-4}{-1} = 4.$$

(ii) The system of linear equations is equivalent to $AX = b$, where

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & -3 \\ -1 & 1 & -2 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

Note that $\det(A) = 0$. Hence A is not invertible. Thus we can't use Cramer's Rule to find the solution to this system of linear equations. In fact, $\det(C1) = \det(C2) = \det(C3) = 0$. Thus the system has infinitely many solutions.

(10) Let A be an $n \times n$ and B be an $m \times m$ matrix. Show that $\det \begin{bmatrix} A & O \\ O & B \end{bmatrix} = \det A \det B$.

Solution: Note that $\begin{bmatrix} A & O \\ O & B \end{bmatrix} = \begin{bmatrix} A & O \\ O & I_m \end{bmatrix} \begin{bmatrix} I_n & O \\ O & B \end{bmatrix}$. Now regard the function $f(A) = \det \begin{bmatrix} A & O \\ O & I_m \end{bmatrix}$ as a function of columns of A. Show that $f(A)$ is a determinant function. Hence $f(A) = \det A$.