Tut 2 - Part 1

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wave packets: Group and phase velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

Q1 $\psi_1(y,t) = 5y\cos 7t$, $\psi_2(y,t) = -5y\cos 9t$

$y \to m$, $t \to s$

Superposn $\Rightarrow$ $\psi_1 + \psi_2$

$\Rightarrow 5y(\cos 7t - \cos 9t)$

imp $\Rightarrow$ dont mess up order

$\Rightarrow 5y\left[-2\sin\left(\frac{9t+7t}{2}\right)\sin\left(\frac{7t-9t}{2}\right)\right]$

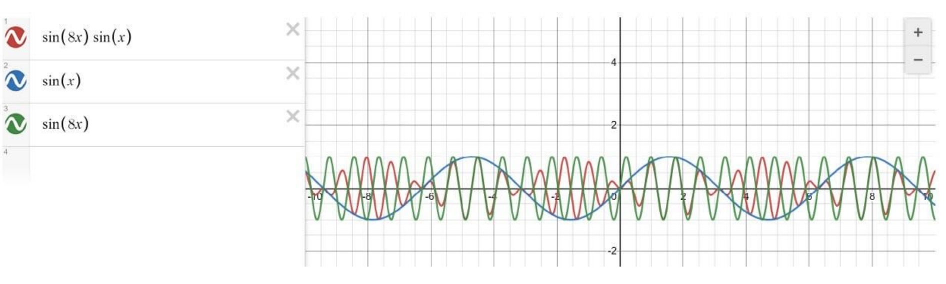
$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\Rightarrow 5y\left[2\sin 8t \sin t\right]$

modulating wave

$\Rightarrow 10y\sin t \sin 8t$ modulated wave

wave packet faster oscillating term



2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

where x, y are in meters and t is in sec.

(a) Find the resultant wave and its phase and group velocities

(b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ?

Phase vel $\Rightarrow$ vel of pt of const phase on wave

Group vel $\Rightarrow$ vel of wave packet

Q2 $y_1 = 0.002\cos(8x - 400t)$, $y_2 = 0.002\cos(7.6x - 380t)$

$x, y \Rightarrow m$, $t \Rightarrow s$

a) Resultant wave $\Rightarrow y_1 + y_2$

fast oscillating slow oscillating (envelop)

$\boxed{\Rightarrow 0.004\cos(7.8x - 390t)\cos(0.2x - 10t)}$

k ω Δk $\Delta\omega$

$v_g = \frac{\Delta\omega}{\Delta k} = \left(\frac{d\omega}{dk}\right)_{k_0} = \frac{10}{0.2} = 50$ m/s

$v_p = \frac{\omega}{k} = \frac{390}{7.3} = 50$ m/s

For Δx, $0.2\Delta x = \pi \Rightarrow \boxed{\Delta x = 5\pi}$

$\Delta k \Rightarrow$ diff bet k_1 & k_2 of resp waves $= 0.4$

$\boxed{\Delta k \Delta x = 2\pi}$

$p = \hbar k$, $\Delta p = \hbar\Delta k$ $k = \frac{2\pi}{\lambda}$

$\Rightarrow \Delta p \frac{\Delta\lambda}{\hbar} = 2\pi$

$\Delta\lambda\Delta p \approx \Delta x\Delta p = 2\pi\hbar > \frac{\hbar}{2}$ $\left[\frac{h}{4\pi}\right]$

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
(a) very large wavelengths and
(b) very small wavelengths.

Q3 $\omega = \left(gk + \frac{Tk^3}{\rho}\right)^{1/2}$ $\boxed{k = \frac{2\pi}{\lambda}}$

$T \Rightarrow$ surface tension

(a) very large $\lambda \Rightarrow$ small k

$v_p = \frac{\omega}{k} = \left(\frac{g}{k} + \frac{Tk}{\rho}\right)^{1/2}$

$\Rightarrow \frac{Tk}{\rho} < \frac{g}{k} \Rightarrow v_p = (g/k)^{1/2}$

$v_g = \frac{d\omega}{dk} = \frac{1}{2}\frac{(g + \cancel{3k^2T/\rho})}{[gk + \cancel{Tk^3/\rho})^{1/2}} = \frac{g}{2\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}}$

(b) small $\lambda \Rightarrow$ large k

$v_p = (Tk/\rho)^{1/2}$ $v_g = \frac{3k^2T/\rho}{2(Tk^3/\rho)^{1/2}} = \frac{3}{2}\sqrt{\frac{kT}{\rho}}$

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

imp

Q4 Dispersion relatns $\Rightarrow$ curves betn ω and k.

which of v_p or v_g is vel of particle to which wave packet corresponds:

$v_p = \omega/k = \lambda\nu$

$p = \frac{h}{k} = \hbar k = \gamma m v \Rightarrow \lambda = \frac{h}{\gamma m_0 v}$

$\lambda = h/mv$, $\nu = E/h = mc^2/h = \frac{\gamma m_0 c^2}{h}$

$= \frac{(p^2c^2 + m_0^2c^4)^{1/2}}{h}$

$v_p = \nu\lambda = \frac{h}{mv} \times \frac{mc^2}{h} = \frac{c^2}{v} > c$

not possible $\Rightarrow$ $\therefore v_p \neq v$

wave of particle travels faster than particle $\Rightarrow$ conflicting duality.

Dispersion relatn: $E^2 = p^2c^2 + m_0^2c^4$

$\hbar^2\omega^2 = \hbar^2k^2c^2 + m_0^2c^4$

$\Rightarrow \omega^2 - k^2c^2 = \frac{m_0^2c^4}{k^2} \Rightarrow \omega = \left(k^2c^2 + \frac{m_0^2c^4}{\hbar^2}\right)^{1/2}$

$\therefore 2\omega d\omega - 2kc^2dk = 0$

$\Rightarrow \frac{d\omega}{dk} = \frac{kc^2}{\omega} = v$

vel of particle

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

(a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ε and permeability μ.

(b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

Q5 EM wave: $Ae^{i(kx - \omega t)}$

$c = \frac{\omega}{k} = \frac{1}{(\epsilon_0\mu_0)^{1/2}}$ $\epsilon_0 \Rightarrow$ electric permittivity, $\mu_0 \Rightarrow$ magnetic permeability (free space)

(a) $v = \frac{1}{\sqrt{\epsilon\mu}}$

plasma freq $\Rightarrow$ const

(b) $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ $\mu = \mu_0$

dispersion relatn

$\Rightarrow \frac{1}{\epsilon\mu} = \left(\frac{\omega}{k}\right)^2 = \frac{1}{\mu_0\epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)} = \frac{\omega^2}{k^2}$

$\Rightarrow (\omega^2 - \omega_p^2)\mu_0\epsilon_0 = k^2$

$1/c^2$

(c) $\omega = 3\omega_p$

$v_p = \frac{\omega}{k} = \frac{c}{(1 - 1/9)^{1/2}} = \frac{3c}{2\sqrt{2}} = 1.06c$

note $\Rightarrow$ greater than c

$v_g = \frac{d\omega}{dk} \Rightarrow$

$\frac{1}{c^2}(\cancel{2}\omega d\omega) = \cancel{2}k dk \Rightarrow \frac{d\omega}{dk} = \frac{kc^2}{\omega} = 0.94c$

$v_p v_g = c^2$

6. A wave packet describes a particle having momentum p. Starting with the relativistic relationship $E^2 = p^2c^2 + E_0^2$, show that the group velocity is βc and the phase velocity is c/β (where $\beta = v/c$). How can the phase velocity physically be greater than c?

Q6 Same as Q4

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x(1 - \cos k_x a_x) + 2\beta_y(1 - \cos k_y a_y) = 0$$

where $\vec{k} = k_x\hat{i} + k_y\hat{j}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

Q7 Dispersion relatn:

$-m\omega^2 + 2\beta_x(1 - \cos k_x a_x) + 2\beta_y(1 - \cos k_y a_y) = 0$

imp $\Rightarrow$ partial differentials in 2 direcns

$\vec{k} = k_x i + k_y j$, $a_x, a_y \Rightarrow$ natural disp

$v_g = \frac{d\omega}{dk} = \left(\frac{\partial\omega}{\partial k_x}\right)i + \left(\frac{\partial\omega}{\partial k_y}\right)j$

Differentiate eqn wrt x:

$-m\left(2\omega\frac{\partial\omega}{\partial x}\right) + a_x 2\beta_x \sin k_x a_x \frac{\partial k_x}{\partial x} = 0$

$\frac{\partial\omega}{\partial k_x} = \frac{(\sin k_x a_x)\beta_x a_x}{m\omega}$

|||ly $\frac{\partial\omega}{\partial k_y} = \frac{(\sin k_y a_y)\beta_y a_y}{m\omega}$

$\boxed{\theta = \frac{\partial\omega/\partial k_y}{\partial\omega/\partial k_x} = \frac{\beta_y a_y}{\beta_x a_x}\left[\frac{\sin k_y a_y}{\sin k_x a_x}\right]}$