Chapter 5: Vector Spaces

1. A nonempty set V of objects (called elements or vectors) is called a vector space over the scalars $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) if the following axioms are satisfied.
2. Closure axioms: For every pair of elements $x, y \in V$ there is a unique element $x + y \in V$ called the sum of x and y.
3. For every $x \in V$ and every scalar $\alpha \in \mathbb{F}$ there is a unique element $\alpha x \in V$ called the product of α and x.
4. Axioms for vector addition: $x + y = y + x$ for all $x, y \in V$.
5. $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$.
6. There exists 0 in V such that $x + 0 = 0 + x = x$ for all $x \in V$.
7. For $x \in V$ there exists an element written as $-x$ such that $x + (-x) = 0$.

Vector Spaces: Definition

1. Axioms for scalar multiplication:
2. (associativity) For all $\alpha, \beta \in \mathbb{F},\ x \in V$,
$$\alpha(\beta x) = (\alpha\beta)x.$$
3. (distributive law for addition in V) For all $x, y \in V$ and $\alpha \in \mathbb{F}$,
$$\alpha(x + y) = \alpha x + \alpha y.$$
4. (distributive law for addition in $\mathbb{F}$) For all $\alpha,\ \beta \in \mathbb{F}$ and $x \in V$,
$$(\alpha + \beta)x = \alpha x + \beta x$$
5. (existence of identity for multiplication) For all $x \in V$, $1x = x$.
6. When $\mathbb{F} = \mathbb{R}$, we say that V is called a real vector space.
7. When $\mathbb{F} = \mathbb{C}$, we say that V is called a complex vector space.

Examples of vector spaces:

1. In the examples below we leave the verification of the axioms for vector addition and scalar multiplication as exercises.
2. Let $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ with ordinary addition and multiplication as vector addition and scalar multiplication. Then V is a real vector space.
3. Let $V = \mathbb{C}$, $\mathbb{F} = \mathbb{C}$ with ordinary addition and multiplication as vector addition and scalar multiplication. Then V is a complex vector space.
4. Let $V = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ with ordinary addition and multiplication as vector addition and scalar multiplication. Then V is a real vector space.
5. Let $V = \mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) | a_1, \ldots, a_n \in \mathbb{R}\}$ and $\mathbb{F} = \mathbb{R}$ with addition of row vectors as vector addition and multiplication of a row vector by a real number as scalar multiplication. So $\mathbb{R}^n$ a real vector space.
6. We can similarly define a real vector space of real column vectors.
7. Depending on the context $\mathbb{R}^n$ could refer to either the set of all row vectors or all column vectors with n real components.

Vector Spaces: Examples

1. Let $V = \mathbb{C}^n = \{(a_1, a_2, \ldots, a_n) | a_1, \ldots, a_n \in \mathbb{C}\}$ and $\mathbb{F} = \mathbb{C}$ with addition of row vectors as vector addition and multiplication of a row vector by a complex number as scalar multiplication. Then V is a complex vector space.
2. We can similarly define a complex vector space of column vectors with n complex components.
3. Depending on the context $\mathbb{C}^n$ could refer to either row vectors or column vectors with n complex components.
4. Let $a < b$ be real numbers and set $V = \{f : [a, b] \longrightarrow \mathbb{R}\}$, $\mathbb{F} = \mathbb{R}$.
5. If $f, g \in V$ then we set $(f + g)(x) = f(x) + g(x)$ for all $x \in [a, b]$.
6. If $a \in \mathbb{R}$ and $f \in V$ then $(af)(x) = af(x)$ for all $x \in [a, b]$.
7. V is a real vector space denoted by $\mathbb{R}^{[a,b]}$.
8. Let t be an indeterminate. The set
$$\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 t + \cdots + a_n t^n | a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$
is a real vector space under usual addition of polynomials and multiplication of polynomials with real numbers.

Vector Spaces: Examples

1. $C[a,b] := \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$ is a real vector space under addition of functions and scalar multiplication.
2. $V = \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is differentiable at } x \in [a,b], x \text{ fixed}\}$ is a real vector space under addition and scalar multiplication of functions.
3. The set of all solutions to the differential equation $y^{''} + ay^{'} + by = 0$ where $a, b \in \mathbb{R}$ form a real vector space.
4. Let $V = M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. Then V is a real vector space under usual matrix addition and multiplication of a matrix by a real number.
5. The above examples indicate that the notion of a vector space is quite general.
6. A result proved for vector spaces will simultaneously apply to all the above different examples.

Subspace of a Vector Space

1. **Definition.** Let V be a vector space over $\mathbb{F}$.
2. A nonempty subset W of V is called a subspace of V if
3. (a) $0 \in W$ (b) If $u, v \in W$ then $\alpha u + \beta v \in W$ for all $\alpha, \beta \in \mathbb{F}$.
4. **Definition.** Let V be a vector space over $\mathbb{F}$.
5. Let $x_1, \ldots, x_n$ be vectors in V and let $c_1, \ldots, c_n \in \mathbb{F}$.
6. The vector $\sum_{i=1}^{n} c_i x_i \in V$ is called a linear combination of x_i's and c_i are called the coefficients of x_i in this linear combination.
7. **Definition.** Let S be a subset of a vector space V over $\mathbb{F}$.
8. The **linear span** of S is the subset of all vectors in V expressible as linear combinations of finite subsets of S, i.e.,

$$L(S) = \left\{ \sum_{i=1}^{n} c_i x_i \mid n \geq 1,\ x_1, x_2, \ldots, x_n \in S \text{ and } c_1, c_2, \ldots, c_n \in \mathbb{F} \right\}.$$

9. We say that $L(S)$ is spanned by S.

Subspace of a Vector Space: Linear Span

1. **Proposition.** Let S be a subset of a vector space V. Then $L(S)$ is the smallest subspace of V containing S.
2. **Proof.** Note that $L(S)$ is a subspace.
3. If $S \subset W \subset V$ and W is a subspace of V then $L(S) \subset W$.
4. Let A be an $m \times n$ matrix over $\mathbb{F}$. **The row space** of A, denoted $\mathcal{R}(A)$, is the subspace of $\mathbb{F}^n$ spanned by the row vectors of A.
5. **The column space of a A,** denoted $\mathcal{C}(A)$, is the subspace of $\mathbb{F}^m$ spanned by the column vectors of A.
6. **The null space of A** denoted $\mathcal{N}(A)$, is defined by

$$\mathcal{N}(A) = \{x \in \mathbb{F}^n : Ax = 0\}.$$

7. The null space of A is the set of all solutions of the homogeneous linear equations $Ax = 0$ and so $\mathcal{N}(A)$ is a subspace of $\mathbb{F}^n$.

Linear Span

1. Different sets may span the same subspace. For example,
$$L(\{e_1, e_2\}) = L(\{e_1, e_1 + e_2\}) = \mathbb{R}^2.$$
2. The vector space $\mathcal{P}_n(\mathbb{R})$ is spanned by $\{1, t, t^2, \ldots, t^n\}$ and also by
$$\{1, (1+t), \ldots, (1+t)^n\}.$$
3. We have introduced the notion of linear span of a subset S of a vector space. This raises some natural questions:
4. Which spaces can be spanned by finite number of elements?
5. If V is a vector space, $S \subset V$ and $V = L(S)$ then what is the minimum number of elements can S have?
6. To answer these questions we use the notions of linear dependence and independence, basis and dimension of a vector space.
7. **Definition.** Let V be a vector space. A subset $S \subset V$ is called **linearly dependent** if there exist distinct $v_1, v_2, \ldots, v_n \in S$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ **not all zero** such that
$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$$

Linearly Dependent and Independent subsets

1. **Definition.** A set S is called linearly independent (L.I.) if it is not linearly dependent, i.e., for all $n \geq 1$ and for all distinct $v_1, v_2, \ldots, v_n \in S$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Longrightarrow \alpha_i = 0, \text{ for all } i.$$

2. **Convention.** The empty set is linearly independent.
3. **Proposition.** (a) Any subset of V containing a linearly dependent set is linearly dependent.
   (b) Any subset of a linearly independent set in V is linearly independent.
   (c) Let $|S| \geq 2$. Then S is linearly dependent $\iff$ either $0 \in S$ or a vector in S is a linear combination of other vectors in S.
   (d) If $S = \{v\}$ then S is linearly independent $\iff$ $v \neq 0$.
4. **Example.** Consider the vector space $\mathbb{R}^n$ and let $S = \{e_1, e_2, \ldots, e_n\}$. Then S is linearly independent. Indeed, if for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = 0$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. So each $\alpha_j = 0$ and hence S is a linearly independent set.

L.D. and L.I. subsets : Remarks and Examples

1. **Example.** Let S denote the subset of $\mathbb{R}^4$ consisting of the row vectors
2. $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$.
3. Then S is linearly independent. To see this, let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and $\alpha_1 \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} + \alpha_2 \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix} + \alpha_3 \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} + \alpha_4 \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = 0.$
4. Then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_3 + \alpha_4 = 0$ and $\alpha_4 = 0$, that is, $\alpha_4 = \alpha_3 = \alpha_2 = \alpha_1 = 0$.
5. **Example.** Let V be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $S = \{1, \cos^2 t, \sin^2 t\}$.
6. Then the relation $\cos^2 t + \sin^2 t - 1 = 0$ shows that S is linearly dependent.

L.D. and L.I. subsets : Examples

1. **Example.** Let $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ be real numbers. Let
$$V = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is continuous } \}.$$
2. Consider the set $S = \{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\}$.
3. We show that S is linearly independent by induction on n.
4. Let $n = 1$ and $\beta e^{\alpha_1 x} = 0$. Since $e^{\alpha_1 x} \neq 0$ for any x, we get $\beta = 0$.
5. Now assume that the assertion is true for $n - 1$ and
$$\beta_1 e^{\alpha_1 x} + \ldots + \beta_n e^{\alpha_n x} = 0.$$
6. Then $\beta_1 e^{(\alpha_1 - \alpha_n)x} + \cdots + \beta_n e^{(\alpha_n - \alpha_n)x} = 0$.
7. Let $x \longrightarrow \infty$ to get $\beta_n = 0$.
8. Now apply induction hypothesis to get $\beta_1 = \ldots = \beta_{n-1} = 0$.

L.D. and L.I. subsets : Examples

1. **Example.** Let $\mathcal{P}$ denote the vector space of all polynomials $p(t)$ with real coefficients. Then the set $S = \{1, t, t^2, \ldots\}$ is linearly independent. Suppose that $0 \leq n_1 < n_2 < \ldots < n_r$ and

$$\alpha_1 t^{n_1} + \alpha_2 t^{n_2} + \ldots + \alpha_r t^{n_r} = 0$$

2. for certain real numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$. Differentiate n_1 times to get $\alpha_1 = 0$. Continuing this way we see that all $\alpha_1, \alpha_2, \ldots, \alpha_r$ are zero.
3. **Bases and dimension of a vector space.** A vector space may be realized as linear span of several sets of different sizes.
4. We shall now study properties of the smallest sets whose linear span is a given vector space.
5. **Definition.** A subset S of a vector space V is called a **basis** of V if elements of S are linearly independent and $V = L(S)$. A vector space V possessing a finite basis is called **finite dimensional**.
6. Otherwise V is called **infinite dimensional.**

1. **Proposition.** Let $\{v_1, \ldots, v_n\}$ be a basis of a finite dimensional vector space V. Then every $v \in V$ can be uniquely expressed as
$$v = a_1 v_1 + \cdots + a_n v_n, \text{ for scalars } a_1, \ldots, a_n.$$
2. **Proof.** Let $v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$ for some scalars $b_1, b_2, \ldots, b_n \in \mathbb{F}$. Then $v - v = 0 = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$. by the linear independence of $v_1, v_2, \ldots, v_n$, $a_j - b_j = 0$ for all j.
3. Hence $a_1, a_2, \ldots, a_n$ are uniquely determined.
4. **Theorem.** All bases of a finite dimensional vector space have same number of elements.
5. For this we prove the following result.
6. **Lemma.** Let $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of a vector space V. Then any $k+1$ elements in $L(S)$ are linearly dependent.
7. **Proof**. Let $T = \{u_1, \ldots, u_{k+1}\} \subseteq L(S)$. Write
$$u_i = \sum_{j=1}^{k} a_{ij} v_j, \quad i = 1, \ldots, k+1.$$
8. Consider the $(k+1) \times k$ matrix $A = (a_{ij})$.

Bases and Dimension

1. Since A has more rows than columns there exists a nonzero row vector $c = [c_1, \ldots, c_{k+1}]$ such that $cA = 0$, i.e., for $j = 1, \ldots k$
$$\sum_{i=1}^{k+1} c_i a_{ij} = 0.$$
2. Therefore
$$\sum_{i=1}^{k+1} c_i u_i = \sum_{i=1}^{k+1} c_i \left(\sum_{j=1}^{k} a_{ij} v_j \right) = \sum_{j=1}^{k} \left(\sum_{i=1}^{k+1} c_i a_{ij} \right) v_j = 0,$$
3. This shows that $u_1, u_2, \ldots, u_{k+1}$ are linearly dependent.
4. **Theorem.** Any two bases of a finite dimensional vector space have same number of elements.
5. **Proof.** Suppose $|S| < |T|$. Since $T \subset L(S) = V$, T is linearly dependent. This is a contradiction.
6. **Definition.** The number of elements in a basis of a finite-dimensional vector space V is called the **dimension** of V. It is denoted by $\dim V$.

Bases and Dimension: Examples

1. **Examples:** The set $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$ is a basis.
2. The columns of $A \in \mathbb{F}^{n\times n}$ form a basis of $\mathbb{F}^n \iff A$ is invertible.
3. $\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 t + \ldots + a_n t^n |\ a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ is spanned by $S = \{1, t, t^2, \ldots, t^n\}$. Since S is LI, $\dim \mathcal{P}_n(\mathbb{R}) = n + 1$.
4. Let e_{ij} denote the $m \times n$ matrix with 1 in $(i, j)^{\text{th}}$ position and 0 elsewhere. If $A = (a_{ij}) \in \mathbb{F}^{m\times n}$ then $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} e_{ij}$.
5. It is easy to see that the mn matrices E_{ij} are linearly independent. Hence $\mathbb{F}^{m\times n}$ is an $mn-$dimensional vector space.
6. What is the dimension of $M_{n\times n}(\mathbb{C})$ as a real vector space?
7. **Proposition.** Let S be a linearly independent subset of a finite dimensional vector space V. Then S can be enlarged to a basis of V.
8. **Proof.** Suppose that $\dim V = n$ and S has less than n elements.
9. Let $v \in V \setminus L(S)$. Then $S \cup \{v\}$ is a linearly independent subset of V.
10. Continuing this way we can enlarge S to a basis of V.

Gauss elimination, row space, and column space

1. **Proposition.** Let $A \in \mathbb{F}^{m\times n}$ and $E \in \mathbb{F}^{m\times m}$ be invertible. Then

(1) $\mathcal{R}(A) = \mathcal{R}(EA)$. Hence $\dim \mathcal{R}(A) = \dim \mathcal{R}(EA)$.

(2) Let $1 \le i_1 < i_2 < \cdots < i_k \le n$. The Columns $\{i_1, \ldots, i_k\}$ of A are linearly independent $\iff$ the columns $\{i_1, \ldots, i_k\}$ of EA are linearly independent. In particular, $\dim \mathcal{C}(A) = \dim \mathcal{C}(EA)$.

2. **Proof.** (1) Note that $\mathcal{R}(EA) \subseteq \mathcal{R}(A)$ since every row of EA is a linear combination of the rows of A. Similarly,

$$\mathcal{R}(A) = \mathcal{R}(E^{-1}(EA)) \subseteq \mathcal{R}(EA).$$

3. To prove (2), observe that

$$\begin{aligned}
& \alpha_1(EA)_{i_1} + \alpha_2(EA)_{i_2} + \cdots + \alpha_k(EA)_{i_k} = 0 \\
\iff \quad & E(\alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k}) = 0 \\
\iff \quad & E^{-1}(E(\alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k})) = 0 \\
\iff \quad & \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k} = 0
\end{aligned}$$

4. Hence $\dim \mathcal{C}(A) = \dim \mathcal{C}(EA)$.

1. **Theorem.** Let A be an $m \times n$ matrix. Then $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.
2. **Proof.** Apply row operations to reduce A to the RCF U.
3. Therefore $A = EU$, where E is a product of elementary matrices.
4. Let the first k rows of U be nonzero. Then U has k pivotal columns.
5. Then the first k rows of U are a basis of $\mathcal{R}(A)$.
6. Suppose that $j_1, \ldots, j_k$ are the pivotal columns of U.
7. Then columns $j_1, \ldots, j_k$ of A form a basis of $\mathcal{C}(A)$.
8. **Example:** Let A be a 4×6 matrix whose RCF is

$$U = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

9. It follows that $\{A_1, A_4, A_6\}$ a basis of $\mathcal{C}(A)$ and the first 3 rows of U is a basis of $\mathcal{R}(A)$.

1. **Definition.** The **rank** of a matrix A, denoted by rank (A), is $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. The **nullity** of A is $\dim \mathcal{N}(A)$.
2. **The Rank-Nullity Theorem**: Let $A \in \mathbb{F}^{m\times n}$. Then
$$\text{rank } A + \text{ nullity } A = n.$$
3. **Proof.** Let $V = \mathbb{F}^n$. Let $B = \{v_1, v_2, \ldots, v_k\}$ be a basis of $\mathcal{N}(A)$.
4. Extend B to a basis $C = \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{n-k}\}$ of V.
5. We show that $D = \{A(w_1), A(w_2), \ldots, A(w_{n-k})\}$ is a basis of $\mathcal{C}(A)$.
6. Any $v \in V$ can be expressed uniquely as
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}.$$
$$\begin{aligned} \implies Av &= \alpha_1 A(v_1) + \cdots + \alpha_k A(v_k) + \beta_1 A(w_1) + \cdots + \beta_{n-k} A(w_{n-k}) \\ &= \beta_1 A(w_1) + \cdots + \beta_{n-k} A(w_{n-k}). \end{aligned}$$
7. Hence D spans $\mathcal{C}(A)$. It remains to show that D is linearly independent.

The rank-nullity theorem for matrices

1. Suppose $\beta_1 A(w_1) + \cdots + \beta_{n-k} A(w_{n-k}) = 0$.
2. Then $A(\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}) = 0$.
3. Hence $\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k} \in \mathcal{N}(A)$.
4. Therefore there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_{n-k} w_{n-k}.$$

5. By linear independence of $\{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{n-k}\}$ we conclude that $\beta_1 = \beta_2 = \cdots = \beta_{n-k} = 0$.
6. Therefore D is a basis of $\mathcal{C}(A)$. Hence

$$\boxed{\text{rank } \text{ A} + \text{ nullity } \text{ A} = \text{n.}}$$

Rank in terms of determinants

1. **Definition.** An $r \times r$ submatrix of A is called minor of order r of A.
2. **Theorem.** A matrix A has rank $r \geq 1 \iff \det M \neq 0$ for some order r minor M of A and $\det N = 0$ for all order $r+1$ minors N of A.
3. **Proof.** Let rank $A = r \geq 1$. Then some r columns of A are L. I.
4. Let B be the $m \times r$ matrix consisting of these r columns of A.
5. Then rank $(B) = r$ and thus some r rows of B are be linearly independent. Let C be the $r \times r$ matrix having these r rows of B.
6. Then $\det(C) \neq 0$, since C is invertible, hence $Cx = 0 \implies x = 0$.
7. Let N be a $(r+1) \times (r+1)$ minor of A.
8. Without loss of generality we may take N to consist of the first $r+1$ rows and columns of A, since the interchanges of rows or interchanges of columns does not change the rank of the matrix.
9. Suppose $\det(N) \neq 0$. Then the $r+1$ rows of N, and hence the first $r+1$ rows of A, are linearly independent, a contradiction.
10. The converse is left as an exercise. $\square$