Tut-4 Part 1

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Free particle:

Schroedinger eqⁿ: $-i\hbar \frac{\partial \psi}{\partial x} = \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial t^2}$

($V=0$ for free particle)

Q1

1. *Show that
$$\psi(x) = A\sin(kx) + B\cos(kx)$$
and
$$\psi(x) = Ce^{ikx} + De^{-ikx}$$
are equivalent solutions of TISE of a free particle. A, B, C and D can be complex numbers.

Given A, B, C, D can be complex no.s:

$$A\sin kx + B\cos kx = Ce^{ikx} + De^{-ikx}$$

$$A\left[\frac{e^{ikx} - e^{-ikx}}{2i}\right] + B\left[\frac{e^{ikx} + e^{-ikx}}{2}\right] = Ce^{ikx} + De^{-ikx}$$

$$\Rightarrow \frac{1}{2}(B - iA)e^{ikx} + \frac{1}{2}(B + iA)e^{-ikx} = Ce^{ikx} + De^{-ikx}$$

$$C = \frac{1}{2}(B - iA), \quad D = \frac{1}{2}(B + iA)$$

$\therefore$ $\sin kx, \cos kx$, e^{ikx}, e^{-ikx} form equivalent basis

Q2

2. Show that
$$\Psi(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$
does not obey the time-dependant Schroedinger's equation for a free particle.

Not a eigen funcⁿ of momentum operator

TDSE $\Rightarrow$ $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}$

$$\frac{\partial \psi}{\partial t} = -\omega A\cos(kx - \omega t) + \omega B\sin(kx - \omega t)$$

$$\frac{\partial \psi}{\partial x} = kA\cos(kx - \omega t) - kB\sin(kx - \omega t)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2[\psi]$$

putting into TDSE

$$\frac{\hbar^2 k^2}{2m}[A\sin(kx - \omega t) + B\cos(kx - \omega t)] = i\hbar[\omega][B\sin(kx - \omega t) - A\cos(kx - \omega t)]$$

Sin, cos coeffs on both sides should be equal.

$\therefore$ Both: $\frac{\hbar^2 k^2}{2m}A = i\hbar\omega B$, $\frac{\hbar^2 k^2}{2m}B = -i\hbar\omega A$

Cant hold simultaneously, only if $A = \pm iB$

Q3

3. The wave function for a particle is given by,
$$\phi(x) = Ae^{ikx} + Be^{-ikx}$$
where A and B are real constants. Show that $\phi(x)^*\phi(x)$ is always a positive quantity.

$\phi(x) = Ae^{ikx} + Be^{-ikx}$, $\phi^*(x) = Ae^{-ikx} + Be^{ikx}$

$$\phi^*(x)\phi(x) = A^2 + B^2 + 2AB\cos kx$$

$$(A+B)^2 \geq \phi^*(x)\phi(x) \geq (A-B)^2 \geq 0$$

Q4

4. * A free proton has a wave function given by

free particle

$$\Psi(x,t) = Ae^{i(\underbrace{5.02*10^{11}}_{k}x - \underbrace{8:00*10^{15}}_{\omega}t)}$$

The coefficient of x is inverse meters, and the coeffcient of t is inverse seconds. Find its momentum and energy.

we know, momentum and Energy are eigen value operators for a free particle wave eqⁿ

Momentum operator: $-i\hbar \frac{\partial \psi}{\partial x}$

Energy operator: $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial t^2}$

or momentum $= \frac{h}{\lambda} = \frac{hk}{2\pi}$, Energy $= \frac{h\omega}{2\pi} = h\nu$

Momentum $= 5.29 \times 10^{-23}$ kg m/s

Energy $= 1.342 \times 10^{-19}$ J

Q5

5. A particle moving in one dimension is in a stationary state whose wave function,
$$\Psi(x) = \begin{cases} 0, & x < -a \\ A\left(1 + \cos\frac{\pi x}{a}\right), & -a \leq x \leq a \\ 0, & x > a \end{cases}$$
where A and a are real constants.
(a) Is this a physically acceptable wave function? Explain.
(b) Find the magnitude of A so that $\psi(x)$ is normalized.
(c) Evaluate Δx and Δp. Verify that $\Delta x \Delta p \geq \hbar/2$.
(d) Find the classically allowed region.

a) Acceptable wave funcⁿs:
① Single valued at all x
② ψ, ψ^* are cont at all x
③ Must be square integrable
④ vanishes at $\pm\infty$

From solⁿs, give as hw rn $\quad \frac{1 + \cos\frac{2x\pi}{a}}{2}$

b) Normalisatⁿ:

$$\int_{-\infty}^{\infty} \psi^*\psi = \int_{-a}^{a} A^2\left[1 + \cos\frac{x\pi}{a}\right]^2 \Rightarrow A^2\int_{-a}^{a} 1 + \cos^2\frac{x\pi}{a} + 2\cos\frac{x\pi}{a}$$

$$\Rightarrow 2A^2 \left. \frac{3x}{2} + \frac{2a}{\pi}\sin\frac{x\pi}{a} \right]_0^a + \left. \frac{1}{2}\sin\frac{2x\pi}{a} \times \frac{a}{2\pi} \right)_0^a$$

$$\Rightarrow \cancel{2}A^2\left[\frac{3a}{\cancel{2}}\right] = 1$$

$$\boxed{A = \pm\frac{1}{\sqrt{3a}}}$$

c) $\Delta x, \Delta p \Rightarrow$ $\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$

$\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$

Solve for 4 variables

$$\int_{-a}^{a} A^2\left[1 + \cos^2\frac{x\pi}{a} + 2\cos\frac{x\pi}{a}\right]x^2\,dx$$

$$\Rightarrow \frac{2}{3a}\int_0^a \frac{3}{2}x^2 + 2\cos\frac{x\pi}{a}\,x^2 + \frac{x^2}{2}\cos\frac{2x\pi}{a}$$

$$\Rightarrow \frac{2}{3a}\left[\frac{x^3}{2} + 2\left[-\int_0^a 2x\sin\frac{x\pi}{a} \times \frac{a}{\pi}\right] + \frac{1}{2}\left[-\frac{2a}{2\pi}\int_0^a x\sin\frac{2x\pi}{a}\right]\right]$$

$$\Rightarrow \frac{2}{3a}\left[\frac{a^3}{2} - \frac{4a}{\pi}\left(x\cos\frac{x\pi}{a} \times \left(-\frac{a}{\pi}\right)\right]_0^a - \frac{a}{2\pi}\left(x\cos\frac{2x\pi}{a}\left(-\frac{a}{2\pi}\right)\right]_0^a\right.$$

$$\Rightarrow \frac{2}{3a}\left[\frac{a^3}{2} - \frac{4a}{\pi}\left[\frac{a^2}{\pi}\right] + \frac{a}{2\pi}\left[\frac{a^2}{2\pi}\right]\right] \Rightarrow \frac{2}{3a}\left[\frac{a^3}{2} - \frac{15a^3}{\pi^2}\right)$$

Classically allowed region $V < E$

$\psi(x,t) = \phi(x)T(t) \rightarrow$ TDSE

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = -i\hbar\frac{\partial \psi(x,t)}{\partial t}$$

when $T(t) = e^{-iEt/\hbar}$

$\Rightarrow$ Prob distributⁿ $|\psi(x,t)|^2 \Rightarrow \psi^*(x,t)\psi(x,t)$

$\Rightarrow |\phi(x)|^2$ [exponentials cancel out]

TISE: $-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} + V(x)\phi(x) = E\phi(x)$

$$i\hbar\frac{\partial T(t)}{\partial t} = E\,T(t)$$

Separable solⁿs correspond to stationary states

Q6

6. * Consider the 1-dimensional wave function of a particle of mass m, given by
$$\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$
where, A, n and x_0 are real constants.
(a)Find the potential $V(x)$ for which $\psi(x)$ is a stationary state (It is known that $V(x) \rightarrow 0$ as $x \rightarrow \infty$).
(b)What is the energy of the particle in the state $\psi(x)$?

$V(x) \rightarrow 0$, as $x \rightarrow \infty$

using TISE, assumptⁿ $\Rightarrow$ variable separable

$$-i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi}{\partial x^2} = (E - V)\phi$$

$$\hat{H}\psi(x) = E\psi(x)$$

put $x/x_0 = m$ $\quad dx = x_0\,dm$

$$\phi(m) = Am^n e^{-m}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial m} \times \frac{\partial m}{\partial x} = A\left[nm^{n-1}e^{-m} - m^n e^{-m}\right]$$

$$\frac{\partial \phi}{\partial x} = \frac{A}{x_0}\left[nm^{n-1}e^{-m} - m^n e^{-m}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial m}\left(\frac{\partial \phi}{\partial x}\right) \times \frac{\partial m}{\partial x} = \frac{A}{x_0^2}\left[n(n-1)m^{n-2}e^{-m} - ne^{-m}m^{n-1} - nm^{n-1}e^{-m} - m^n e^{-m}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{A}{x_0^2}\left[n(n-1)\left(\frac{x}{x_0}\right)^{n-2}e^{-(x/x_0)} 2ne^{-(x/x_0)}\left(\frac{x}{x_0}\right)^{n-1} + \left(\frac{x}{x_0}\right)^n e^{-x/x_0}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{xx_0} + \frac{1}{x_0^2}\right]\phi$$

$$E - V(x) = -\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{xx_0} + \frac{1}{x_0^2}\right]$$

$$\boxed{\Rightarrow V(x) = E + \frac{\hbar^2}{2mx_0^2} + \frac{\hbar^2 n(n-1)}{2mx^2} - \frac{2\hbar^2 n}{xx_0}}$$

b) For $V(x) \rightarrow 0$, at $x \rightarrow \infty$

$$\boxed{E = -\frac{\hbar^2}{2mx_0^2}}$$