PART B

8. Find the general solution of $(x+1)^2y'' + (x+1)y' - y = 2\ln(x+1) + x - 1$. [4]

Solution Set $t = x+1$. The ODE reduces to

$$t^2y'' + ty' - y \;=\; 2\ln t + t - 2, t > 0.$$

Indicial/ auxilliary equation of the corresponding Cauchy-Euler homogeneous equation is

$$m(m-1) + m - 1 = 0 \;\Rightarrow\; m = \pm 1. \qquad [1]$$

Therefore general solution of the homogeneous Cauchy-Euler homogeneous equation is

$$c_1 t + \frac{c_2}{t}, t > 0.$$

Also observe that $(tD-1)(tD)^2$ annihilates $2\ln t + t - 2$. Hence the form of the particular solution is

$$y_p(t) = A + B\ln t + Ct\ln t. \qquad [2]$$

Substitute back into the Cauchy Euler non homogeneous ODE, we get

$$t^2y_p'' + ty_p' - y_p \;=\; 2\ln t + t - 2, t > 0.$$

After simplification and re arranging the terms, we get

$$-A - B\ln t + 2Ct \;=\; 2\ln t + t - 1 \;\Rightarrow\; B = -2, A = 2, C = \frac{1}{2}.$$

Therefore

$$y_p(t) = 2 - 2\ln t + \frac{1}{2}t\ln t. \qquad [1]$$

Hence general solution is

$$y(x) = c_1(x+1) + \frac{c_2}{x+1} + 2 - 2\ln(x+1) + \frac{1}{2}(x+1)\ln(x+1), x > -1.$$

9. Let p, q be continuous functions defined on $\mathbb{R}$ such that $p(x) \neq 0$ for all $x \in \mathbb{R}$. Also let y_1, y_2 be linearly independent solutions of the ODE

$$q(x)y'' + p(x)y' + 2p(x)y = 0$$

satisfying $y_1''(x_0) = y_2''(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Show that $q(x_0) = 0$. [4]
(CANCELLED due to some ambiguity in the question.)

10. Let p be a continuous function defined on $\mathbb{R}$ satisfying $p(x) \leq 0$ for $x \geq 0$. Consider the ODE

$$y'' + (p(x) - 3)y' - 3p(x)y \;=\; 0, x \geq 0.$$

Show that the ODE has a linearly independent set $\mathcal{S}$ of solutions such that $\mathcal{S}$ has two elements and both elements of $\mathcal{S}$ are convex functions on $(0,\ \infty)$. [4]

Solution Observe that

$$y'' + (p(x) - 3)y' - 3p(x)y \;=\; (D + p(x))(D - 3)y.$$

Hence $y_1 = e^{3x}, x \geq 0$ is a solution which is a convex function. [1]

Using Abel's formula, a second linearly independent solution y_2 is given by

$$\begin{aligned} y_2(x) &= y_1(x) \int_0^x \frac{e^{-\int_0^t (p(s)-3)ds}}{y_1^2(t)} dt \\ &= e^{3x} \int_0^x e^{-\int_0^t (p(s)+3)ds} dt. \end{aligned} \quad [1]$$

Differentiate the above, we get

$$y_2'(x) = 3y_2(x) + e^{-\int_0^x p(t)dt} \;\Rightarrow\; y_2'(x) \geq 0 \text{ for all } x > 0. \quad [1]$$

Now using $p(x) \leq 0$ for $x \geq 0$, we get,

$$y_2''(x) \;=\; 3y_2'(x) - p(x)e^{-\int_0^x p(t)dt} \;\geq\; 0, x > 0.$$

Hence y_2 is convex on $(0, \infty)$. [1]

Therefore, $\{y_1, y_1\}$ is a set of linearly independent solutions on $[0,\ \infty)$ which are convex on $(0,\ \infty)$.

Alternate Marking Scheme If a student directly observes that $y_1(x) = e^{3x}$ is a solution, she/he will be awarded ONE mark.

11. Using the method of variation of parameters, solve

$$(x^2 + x)y'' + (2 - x^2)y' - (2 + x)y = x(x + 1)^2, x > 0. \quad [4]$$

Solution Observe that

$$Ly = (x^2 + x)y'' + (2 - x^2)y' - (2 + x)y = ((x^2 + x)D - (2 + x))(D - 1)y.$$

Hence $y_1(x) = e^x$ is a solution to $Ly = 0, x > 0$. [1]

Set

$$p(x) = \frac{2 - x^2}{x^2 + x}, x > 0.$$

Using Abel's formula, a second linearly independent solution y_2 is given by

$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx \\ &= e^x \int \frac{e^{-\int \frac{2-x^2}{x^2+x}dx}}{e^{2x}}dx \\ &= e^x \int \frac{e^{\int (1-\frac{2}{x}+\frac{1}{x+1})dx}}{e^{2x}}dx \\ &= e^x \int e^{-x}(\frac{1}{x}+\frac{1}{x^2})dx \;=\; -\frac{1}{x}, x>0. \end{aligned}$$

Hence, $y_2(x) \;=\; -\frac{1}{x}, x > 0.$ [1]

Using method of variation of parameters, a particular solution is given by

$$y_p = v_1y_1 + v_2y_2.$$

where

$$v_1(x) = -\int \frac{y_2(x)(x+1)}{W(y_1,y_2)(x)}dx,\; v_2(x) = \int \frac{y_1(x)(x+1)}{W(y_1,y_2)(x)}dx.$$

Now $W(y_1,y_2)(x) = \frac{x+1}{x^2}\, e^x, x > 0.$

Therefore

$$v_1(x) = \int xe^{-x}dx = -xe^{-x} - e^{-x}, x > 0. \quad [1]$$

and

$$v_2(x) \;=\; \int x^2 dx = \frac{x^3}{3}, x > 0.$$

Hence $y_p(x) = -(1 + x + \frac{x^2}{3}), x > 0.$ [1]

Therefore the general solution is

$$y(x) = c_1e^x + \frac{c_2}{x} - (1 + x + \frac{x^2}{3}), x > 0.$$

Alternate Marking Scheme If a student directly observes that e^x or $\frac{1}{x}$ is a solution, she/ he gets ONE mark. If the student observes that both $e^x, \frac{1}{x}$ are solutions, then gets TWO marks.

12. Find the general solution of $y'' - 5y' + 4y = (3x+2)e^{-2x}, x \in \mathbb{R}.$ [4]

Solution Characteristic equation of the homogeneous ODE

$$Ly = y'' - 5y' + 4y = 0$$

is $m^2 - 5m + 4 = 0$ and hence the roots are $m = 1, 4$.

Therefore, general solution of $Ly = 0$ is

$$y = c_1 e^{x} + c_2 e^{4x}. \qquad [1]$$

Form of the particular solution is

$$y_p = Ae^{-2x} + Bxe^{-2x}. \qquad [1]$$

Substitute back y_p into $Ly = (3x+2)e^{-2x}$ and simplify, we get

$$(18A - 9B + 18Bx)e^{-2x} \;=\; (3x+2)e^{-2x} \Rightarrow \; B = \frac{1}{6}, \; A = \frac{7}{36}.$$

Therefore

$$y_p = \frac{7}{36}e^{-2x} + \frac{1}{6}xe^{-2x}. \qquad [2]$$

Therefore, general solution is

$$y = c_1 e^{x} + c_2 e^{4x} + \frac{7}{36}e^{-2x} + \frac{1}{6}xe^{-2x}.$$

13. Using Laplace transform technique, solve the initial value problem

$$y'' + y \;=\; \begin{cases} \sin t & 0 \le t < \pi \\ 0 & t \ge \pi, \end{cases}$$

$y(0) = y'(0) = 0.$ [4]

Solution IVP can be rewritten as

$$y'' + y \;=\; (1 - u_\pi(t))\sin t, \; y(0) = y'(0) = 0.$$

Taking Laplace transform, we get

$$L(y'')(s) + L(y)(s) = L(\sin t))(s) + L(u_\pi(t))\sin(t-\pi))(s). \qquad [1]$$

Set $L(y)(s) = Y(s)$ and use the properties (Laplace transform of derivative of function and 2nd Shift theorem), we get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

$$\Rightarrow \; (s^2+1)Y(s) \;=\; \frac{1+e^{-\pi s}}{s^2+1}.$$

Hence

$$Y(s) = \frac{1}{(s^2+1)^2} + \frac{e^{-\pi s}}{(s^2+1)^2}. \qquad [1]$$

Note that

$$L(\sin t - t\cos t)(s) \;=\; \frac{1}{(s^2+1)^2}, s > 0. \qquad [1]$$

Therefore, (using Lerch's theorem)

$$\begin{aligned} y(t) &= \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}u_\pi(t)(\sin(t-\pi) - (t-\pi)\cos(t-\pi)) \\ &= \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}u_\pi(t)(-\sin t + (t-\pi)\cos t), t \geq 0. \end{aligned}$$

Hence

$$y(t) \;=\; \begin{cases} \frac{1}{2}(\sin t - t\cos t), & 0 \leq t < \pi \\ -\frac{\pi}{2}\cos t, & t \geq \pi. \end{cases}$$

[1]

14. Using Laplace transform technique, solve the ODE

$$ty'' + (1-t)y' + ny = 0, t \geq 0,$$

where n is a positive integer. [4]

Solution Take Laplace transform, after using the property of the derivative of Laplace transform, we get

$$\begin{aligned} L(ty'')(s) + L(y')(s) - L(ty')(s) + nL(y)(s) &= 0 \\ \Rightarrow\; -\frac{d}{ds}L(y'') + L(y') + \frac{d}{ds}L(y') + nL(y)(s) &= 0. \end{aligned}$$

Set $L(y)(s) = Y(s)$. Using the property " Laplace transform of derivative of function"

$$-\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) + sY(s) - y(0) + \frac{d}{ds}(sY(s) - y(0)) + nY(s) \;=\; 0 \qquad [1]$$

$$\Rightarrow\; (-s^2+s)Y'(s) + (1+n-s)Y(s) \;=\; 0.$$

Solving the above first order ODE in separable form, we get

$$\ln Y(s) = n\ln(s-1) - (n+1)\ln s + \ln c, s > 1$$

for some constant $c > 0$ [1]

Hence

$$Y(s) \;=\; c\frac{(s-1)^n}{s^{n+1}}, s > 1. \qquad [1]$$

Now

$$\frac{(s-1)^n}{s^{n+1}} = \frac{1}{s}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{-1}{s}\right)^k$$
$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{s^{k+1}} \qquad [1]$$

Now using

$$\frac{1}{k!}L(t^k)(s) = \frac{1}{s^{k+1}},$$

we get

$$\frac{(s-1)^n}{s^{n+1}} = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}\int_0^\infty e^{-st}t^k dt$$
$$= \int_0^\infty e^{-st}\Big(\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k t^k}{k!}\Big)dt.$$

Hence

$$y(t) = c\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k t^k}{k!} \qquad [1]$$