MA 106 : LINEAR ALGEBRA : SPRING 2023

SOLUTIONS OF TUTORIAL PROBLEMS ASSIGNMENTS 4–7

1. Tutorial Problems about vector spaces

(1) Obtain the REF of the following matrices. Use them to find rank and nullity of the matrix. Also write down a basis for the range. Finally obtain the RCF and use to write down a basis for the null space.

(i) $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$.

Solution: (ii) $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{bmatrix} \left.\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 5R_1 \end{matrix}\right\} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & -5 & 3 \\ 0 & -4 & 6 \end{bmatrix}$

$\left.\begin{matrix} R_3 + 5R_2 \\ R_4 + 4R_2 \end{matrix}\right\} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 18 \\ 0 & 0 & 18 \end{bmatrix} R_4 - R_3 \longrightarrow B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 18 \\ 0 & 0 & 0 \end{bmatrix}$.

As there are three pivots in B rank $A = 3$. The three columns of A form a basis for $\mathcal{C}(A)$. The nullity is zero by the rank-nullity theorem. The row canonical form is the 3×4 matrix whose first three rows and the first 3 columns form I_3 and the last row is a zero vector.

(2) Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through the origin, planes passing through the origin and the whole space.

Solution: Clearly the above mentioned spaces are subspaces. Since the dimension of $\mathbb{R}^3$ is 3, any subspace V has dimension ≤ 3. If the dimension is zero then V has no nonzero elements and hence $V = (0)$. If the dimension is 1 then $V = L(\{\mathbf{v}\})$ where $\mathbf{v}$ is a non zero vector. This consists of precisely all scalar multiples of $\mathbf{v}$ and hence is a line passing through the origin. If the dimension is two, then $V = L(v, u)$. So, we get the set of points of the form $\alpha v + \beta u$ for

$\alpha, \beta \in \mathbb{R}$. This is precisely the plane through the origin containing the two vectors v, u. Finally if the dimension is 3, then the subspace must be the whole of $\mathbb{R}^3$, for otherwise, there will be four linearly independent elements in $\mathbb{R}^3$.

(3) A **hyperplane** in $\mathbb{R}^n$ is defined to be the set $u + W$ where $u \in \mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ having dimension $n-1$. Prove that a hyperplane in $\mathbb{R}^n$ is the set of solutions of a single linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_1, \ldots, a_n, b \in \mathbb{R}$.

Solution: Let $B = \{u_1, u_2, \ldots, u_{n-1}\}$ be a basis of W. Let $x_1, x_2, \ldots, x_n$ be indeterminates. Let A be the $n \times (n-1)$ matrix whose column vectors are $u_1, u_2, \ldots, u_{n-1}$. Then the homogeneous system of linear equations $[x_1\ x_2\ \ldots\ x_n]A = 0$ has a nontrivial solution, say $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$. Then $u_1, u_2, \ldots, u_{n-1}$ are solutions to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. Since $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ has a nontrivial solution say u. Hence the set of all solutions is $u + W$.

(4) Consider the following subsets of the space $M_n(\mathbb{C})$ of $n \times n$ complex matrices :
(a) $\mathrm{Sym_n}(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^T\}$ of **symmetric matrices**.
(b) $\mathrm{Herm_n}(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^*\}$ of **Hermitian matrices**.
(c) $\mathrm{Skew_n}(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*\}$ of **skew-Hermitian Matrices**.
Show that each of them is an $\mathbb{R}$-vector subspace of $M_n(\mathbb{C})$ and compute their dimension by explicitly writing down a basis for each of them.

Solution: :(a) This is a complex vector subspace with basis

$$\{E_{ii} : 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} : 1 \leq i < j \leq n\}.$$

Therefore its complex dimension is $n(n+1)/2$ and its real dimension is $n(n+1)$.
(b) This is defined by linear equations over real numbers and hence is a real subspace. The set

$$\{E_{ii}\} \cup \{E_{ij} + E_{ji} : i < j\} \cup \{\imath(E_{ij} - E_{ij}) : i < j\}$$

is a basis. Hence the dimension is n^2. It is not a complex subspace, because $\imath(E_{12} + E_{21})$ is not Hermitian.
(c) This is also defined by real linear equations and hence is a real subspace. The set

$$\{\imath E_{ii}\} \cup \{\imath(E_{ij} + E_{ji}) : i < j\} \cup \{E_{ij} - E_{ij} : i < j\}$$

is a basis and hence its dimension is also n^2. It is not a complex subspace.

(5) Let $P_n[x]$ denote the vector space consisting of the zero polynomial and all real polynomials of degree $\leq n$, where n is fixed. Let S be a subset of all polynomials $p(x)$ in $P_n[x]$ satisfying the following conditions. Check whether S is a subspace; if so, find the dimension of S. (i) $p(0) = 0$; (ii) p is an odd function; (iii) $p(0) = p''(0) = 0$.

Solution: (i) Yes. $\{x, x^2, \ldots, x^n\}$ is basis. So, the dimension is n.

(ii) Recall that p is odd means $p(-x) = -p(x)$. By comparing coefficients on either side we see that all even degree terms vanish. This set is then spanned by $1, x^3, \ldots, x^k$ where $k =$ largest odd number $\leq n$.

(iii) Yes. The given condition is equivalent to say that the constant term and the degree 2 term are missing. $\{x, x^3, x^4, \ldots, x^n\}$ is basis. So, the dimension is $n-1$, $(n \geq 2)$.

(6) Examine whether the following sets are linearly independent.

(a) $\{(a,b), (c,d)\} \subset \mathbb{R}^2$, with $ad - bc \neq 0$.

(b) For $\alpha_1, \ldots, \alpha_k$ distinct real numbers, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where $\mathbf{v}_i = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1})$.

(c) $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$.

(d) $\{1, \sin x, \sin 2x, \ldots, \sin nx\}$.

(e) $\{e^x, xe^x, \ldots, x^n e^x\}$.

Solution: (a) The 2×2 matrix A whose row vectors are (a,b) and (c,d) is invertible as its determinant is nonzero. Hence the row space is 2-dimensional and rank $A = 2$.

(b) Suppose $\sum_{i=1}^k \beta_i \mathbf{v}_i = 0$. This is the same as the matrix equation $V\mathbf{b} = 0$ where $V = V(\alpha_1, \ldots, \alpha_k)$ is the Vadermonde matrix and $\mathbf{b} = (\beta_1, \ldots, \beta_k)^t$. Since we know that the Vandermonde determinant is nonzero for distinct $\alpha_i's$, it follows that the matrix V is invertible. Hence the equation has only the zero as solution. Therefore $\mathbf{b} = 0$ which means $\beta_i = 0$ for all i. Hence $v_1, v_2, \ldots, v_k$ are linearly independent.

(c) Let $\sum_{r=0}^n \beta_r \cos rx = 0$. Differentiating $2k$ times and putting $x = 0$, for $k = 0, \ldots, n-1$ we get,

$$\sum_{r=0}^{n} \beta_r (r)^{2k} = 0.$$

Now take $\alpha_r = r^2$ for $r = 0, 1, \ldots, n$, we get $\sum_{r=0}^n \beta_r \mathbf{v}_r = 0$. Hence by (b), $\beta_r = 0$ for all r.

(d) Here differentiate once and use (c).

(e) Suppose $\sum_{i=0}^n \beta_i x^i e^x = 0$ Since e^x is never zero this yields $\sum_{i=0}^n \beta_i x^i = 0$. Since we know that $\{1, x, \ldots, x^n\}$ are linearly independent, it follows that $\beta_i = 0$ for all i.

(7) Find a basis for the subspace $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0\}$. Let P be the xy-plane. Find a basis of $W \cap P$. Find a basis of the subspace of all vectors in $\mathbb{R}^3$ which are perpendicular to the plane W.

Solution: $W = \{(x, y, z) \mid x - 2y + 3z = 0.\}$. We write $x = 2y - 3z$. Hence $(x, y, z) = (2y - 3z, y, z) = y(2, 1, 0) + z(-3, 0, 1)$. This shows that W is a 2-dimensional subspace spanned by the linearly independent vectors $u = (2, 1, 0)$ and $v = (-3, 0, 1)$. The vectors in $W \cap P$ Have their z-component zero. Hence $W \cap P = \{(2y, y, 0)\}$. Thus $\{(2, 1, 0)\}$ is a basis of $W \cap P$. It is clear that $(1, -2, 3)$ is perpendicular to the plane W. The subspace of vectors that are perpendicular to W is one-dimensional and $(2, 1, 0)$ is a basis.

2. Tutorial problems about linear transformations

(1) Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by

$$f((x_1, x_2, x_3, x_4, x_5)^t) = (2x_3 - 2x_4 + x_5, 2x_2 - 8x_3 + 14x_4 - 5x_5, x_2 + 3x_3 + x_5)^t.$$

Find bases for the null-space and the range of f, using the row echelon form of the matrix of f with respect to standard bases.

Solution: We first write down the associated matrix and then perform row operations on it to bring it to an REF:

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & -8 & 14 & -5 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix} R_1 \sim R_2 \longrightarrow \begin{bmatrix} 0 & 2 & -8 & 14 & -5 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{l} R_1/2 \\ R_3 - R_1 \end{array}\right\} \longrightarrow \begin{bmatrix} 0 & 1 & -4 & 7 & -5/2 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 7 & -7 & 7/2 \end{bmatrix}$$

$$\left.\begin{array}{l} R_2/2 \\ R_3 - 7R2 \end{array}\right\} \longrightarrow B = \begin{bmatrix} 0 & 1 & -4 & 7 & -5/2 \\ 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the pivotal columns of B are the 2nd, and 3rd. Accordingly, the columns $(0, 2, 1)^2, (2, -8, -1)^t$ give a basis for the range of f. Hence the rank of f is 2. The nullity is therefore equal to 3. We continue to perform row operations on B above to obtain

$$J(A) = \begin{bmatrix} 0 & 1 & 0 & 3 & -1/2 \\ 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now the method for writing down the general solution of $J(A)x = 0$ tells us how to write down a basis for the null space also, viz., consider the problem for homogeneous equation, i.e., with $\mathbf{b} = 0$. We know the general solution is given by $x_2 = -3x_4 + x_5/2; x_3 = x_4 - x_5/2$. Here x_1, x_4 and x_5 are free variables. Therefore, by putting special values for them we obtain $(1,0,0,0,0)^t, (0,-3,1,1,0)^t, (0,1/2,-1/2,0,1)^t$ belonging to $\mathcal{N}(A)$. Since these are linearly independent they give a basis for the null space.

(2) Find the range and null-space of the following linear transformations. Also find the rank and nullity wherever applicable.

(a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2)^t = (x_1 + x_2, x_1)^t$.

(b) $T : C^1(0,1) \longrightarrow C(0,1)$ defined by $T(f)(x) = f'(x)e^x$.

Solution: (a) The associated matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The range is $\mathbb{R}^2$ and null$(T) = (0)$. (b) If $f \in$ null(T) then $f'(x) = 0$. As f is continuous, it is a constant function. Conversely all constant functions are mapped to the zero function by T. Thus the null space of T consists of all constant functions.

(3) Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T. (ii) the range of T.

Solution: We first observe that the vectors which satisfy the given equation form plane in $\mathbb{R}^3$. So, we pick up to independent vectors in it, say, $(3,4,0,)^t, (0,1,3)^t$. We then pick up another vector which does not lie in the plane, say a vector perpendicular to it, viz. $(4,-3,1)^t$. These three vectors then form a basis for $\mathbb{R}^3$. So, a linear map on $\mathbb{R}^3$ will be completely determined if we know its value on these three vectors.

(i) Take $T(3,4,0)^t = 0 = T(0,1,3)^t$ and $T(4,-3,1)^t = e_1$. Then the null space of T will be precisely the given plane.

(ii) Take $T(\mathbf{e}_1) = (3,4,0)^t, T(\mathbf{e}_2) = (0,1,3)^t$ and $T(\mathbf{e}_3) = 0$. Then the range of T will be precisely the given plane.

(4) Let $\mathcal{P}[x]$ denote the space of all real polynomials in one variable. Let

$$V = \{p(x) \in \mathcal{P}[x] \ : p(0) = 0\}.$$

Prove that taking the derivative defines a one-to-one linear transformation from $D : V \longrightarrow \mathcal{P}$ and $D^{-1}(p)(x) = \int_0^x p(t)\, dt$.

Solution: [Hint.] Use the fundamental theorem of Calculus.

(5) Let $f : V \longrightarrow W$ be a linear transformation.

(a) Suppose f is injective and $S \subset V$ is linearly independent. Then show that $f(S)$ is linearly independent.

(b) Suppose f is onto and S spans V. Then show that $f(S)$ spans W.

(c) Suppose S is a basis for V and f is an isomorphism then show that $f(S)$ is a basis for W. **Solution:** (a) Let $\sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i) = 0$ where $\mathbf{v}_i$ distinct elements of S. Then $f(\sum_i \alpha_i \mathbf{v}_i) = 0$ and since f is injective we have $\sum_i \alpha_i \mathbf{v}_i = 0$. But since S is linearly independent it follows that $\alpha_1 = \cdots = \alpha_k = 0$.

(b) Given $\mathbf{w} \in W$ take $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \mathbf{w}$. Write $\mathbf{v} = \sum_i \alpha_i \mathbf{v}_i \in L(S)$. Then $\mathbf{w} = f(\sum_i \alpha_i \mathbf{v}_i) = \sum_i \alpha_i f(\mathbf{v}_i) \in L(f(S))$.

(c) Combine (a) and (b).

(6) Let V be a finite dimensional vector space and $f : V \longrightarrow V$ be a linear map. Prove that the following are equivalent:

(i) f is an isomorphism.

(ii) f is injective.

(iii) f is surjective.

(iv) there exist $g : V \longrightarrow V$ such that $g \circ f = Id_V$.

(v) there exists $h : V \longrightarrow V$ such that $f \circ h = Id_V$.

Solution: Clearly (i) implies all the other statements. So it remains to show that each one of the other statements implies (i). Let S be a basis for V. Clearly, S has n elements where $n = dim\, V$.

(ii) $\Longrightarrow$ (i) Let f be injective. Then by the above exercise, $f(S)$ is linearly independent. If $L(f(S)) \neq V$ then there exists an element $\mathbf{v} \in V \setminus f(S)$. But then $f(S) \cup \{\mathbf{v}\}$ will be a L.I. set with more elements than the dimension of V which is a contradiction. Hence $L(f(S)) = V$. This in turn means that $f(V) = f(L(S)) = V$.

(iii) $\Longrightarrow$ (i) Let f be surjective. Assume $\mathcal{N}(f) \neq \{0\}$. Pick a basis $\{\mathbf{v}_1 \ldots, \mathbf{v}_k\}$ for it and complete it to a basis S for V. Since f is surjective, $f(S)$ spans V. But $f(S)$ has at most $n - k$ non zero elements. This means $k = 0$. Hence $\mathcal{N}(f) = 0$. That means f is injective.

(iv) $\implies$ (ii) Suppose $f(\mathbf{v}) = 0$. Then $0 = g(0) = g(f(\mathbf{v})) = \mathbf{v}$. Hence $\mathcal{N}(f) = \{0\}$ and this means f is injective.

(v) $\implies$ (iii) Let $\mathbf{w} \in V$ be any. Then $f(h(\mathbf{w})) = \mathbf{w}$. This implies that f is onto.

(7) Consider the linear transformations $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$. If T_2 is one-one then show that $rank(T_2 \circ T_1) = rank(T_1)$.

Solution: Recall that by the rank of a linear map we mean the dimension of its image. Now $\mathcal{R}(T_2 \circ T_1) = T_2(\mathcal{R}(T_1))$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $\mathcal{R}(T_1)$. Then $\{T_2(\mathbf{v}_1), \ldots, T_2(\mathbf{v}_k)\}$ is L.I. But clearly it also spans the image of $T_2 \circ T_1$. Hence it is a basis for $T_2(T_1(U))$. So, the dimension of the image of $T_2 \circ T_1$ is equal to k.

3. Tutorial problems about Inner product spaces

(1) Find the projection $\mathbf{p}$ of $\mathbf{b}$ onto the column space of A by solving $A^t Ax = A^t b$ and $p = Ax$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution: Clearly, $A^t A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Hence $x_2 = 3$ and $x_1 = -1$ and $p = Ax = (2, 3, 0)^t$.

(2) If P is a real square matrix with $P^2 = P$, show that $(I - P)^2 = I - P$. Suppose P is the matrix of projection onto the columns space of A. Find the space onto which $I - P$ projects.

Solution: Let $W = C(A)$. As $(I - P)(u) = u - P(u)$, $\langle u - P(u), P(u) \rangle = 0$. Thus $I - P$ projects vectors into $N(A^t)$.

(3) Let the columns of A be linearly independent and $P = A(A^t A)^{-1} A^t$. Show that P is symmetric and $P^2 = P$.

Solution: Note that rank$A$ = rank$A^t A = n$. Thus $A^t A$ is invertible. Since the normal equations are $A^t Ax = A^t b$, we have $x = (A^t A)^{-1} A^t b$. Thus $P_{C(A)}(b) = Ax = A(A^t A)^{-1} A^t b$.

This shows that the matrix of projection map $P : \mathbb{R}^n \to C(A)$ is $P = A(A^tA)^{-1}A^t$. Check that $P^2 = A(A^tA)^{-1}(A^tA(A^tA)^{-1})A^t = P$.

(4) In the vector space $C[1,e]$, define $\langle f, g\rangle = \int_1^e \log x f(x)g(x)\,dx$.
   (a) if $f(x) = \sqrt{x}$, compute $\|f\| = \langle f, f\rangle^{1/2}$.
   (b) Find a linear polynomial $g(x) = ax + b$ that is orthogonal to $f(x) = 1$.

Solution: (a) $\|f\|^2 = \int_1^e x\log x\,dx = (e^2+1)/4$
(b)

$$\begin{aligned} 0 = \langle f, ax+b\rangle &= \int_1^e (ax+b)\log x\,dx \\ &= a\int_1^e x\log x\,dx + b\int_1^e \log x\,dx \\ &= \frac{a(e^2+1)}{4} + b. \end{aligned}$$

Thus $b = -\frac{1}{4}a(e^2+1)$ and $ax + b = ax - \frac{1}{4}a(e^2+1)$ is orthogonal to 1.

(5) (a) To find the projection matrix onto the plane $x - y - 2z = 0$, choose two linearly independent vectors u, v in the plane and let A be the matrix whose column vectors are u, v. Now find $P = A(A^tA)^{-1}A^t$.
(b) Let e be a vector perpendicular to the plane $L : x - y - 2z = 0$. Find the projection matrix $Q = \frac{ee^t}{e^te}$. Show that $P = I - Q$ is the matrix of projection onto L.

Solution: We view the plane as the null space of the matrix $\begin{bmatrix} 1 & -1 & -2 \end{bmatrix}$. The vectors in the plane are $(y+2z, y, z) = y(1,1,0) + z(2,0,1)$. Then a basis of this space is given by $(1,1,0)^t, (2,0,1)^t$. Hence we take the matrix A to be

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $A^tA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ and $(A^tA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$. Then the projection matrix is

$$P = A(A^tA)^{-1}A^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}.$$

(b) Since the row space of the matrix $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ is orthogonal to the nullspace, as above, we may take $e = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Thus we have

$$Q = \frac{ee^t}{e^te} = 1/6 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/3 & -1/6 \\ 1/3 & 2/3 & -1/3 \\ -1/6 & -1/3 & 1/6 \end{bmatrix}.$$

Hence $P = I - Q = \begin{bmatrix} 5/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 5/6 \end{bmatrix}$.

(6) Let $u \in \mathbb{R}^n$ be a unit vector. Let $H_u = I - 2uu^t$. Show that H is an orthogonal matrix. Find $H_u(v)$ for any $v \in L(u)^\perp$. Find $H_u(\alpha u)$ for any $\alpha \in \mathbb{R}$. Describe the action of H_u geometrically. Using this find the matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors with respect to the line $y = x \tan\theta$. Find the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ Which reflects vectors with respect to the plane $x + y + z = 0$.

Solution: Write $H = H_u$. Then $H^tH = (I - 2uu^t)(I - 2uu^t) = I - 4uu^t + 4uu^tuu^t = I$. Hence H is orthogonal. Let $v \perp u$. Then $H(v) = v - 2uu^tv = v$. Let $\alpha \in \mathbb{R}$. Then $H(\alpha u) = \alpha u - 2\alpha uu^tu = -\alpha u$. This show that H is a reflection with respect to the hyperplane perpendicular to u. Now we find the matrix that induces reflection across the line $L : y = \tan\theta x$. The vector $u = (-\sin\theta, \cos\theta)^t \perp L$. Hence $H = I - 2uu^t = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

(7) Let $V = C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ with inner product given by $\langle f, g \rangle = \int_0^\pi f(t)g(t)dt$. Let $x_n(t) = \cos nt$ for $n = 0, 1, 2 \ldots$. Prove that the functions $y_0, y_1, y_2, \ldots$

given by

$$y_0(t) = \frac{1}{\sqrt{\pi}} \text{ and } y_n(t) = \sqrt{\frac{2}{\pi}} \cos nt \text{ for } n \geq 1$$

form an orthonormal set spanning the same subspace as $x_0, x_1, x_2, \ldots$.

Solution: See the lecture slides.

4. Tutorial problems about eigenvalues and eigenvectors

(1) Let u be a unit vector in $\mathbb{R}^n$. Define $H = I - 2uu^t$. Find all the eigenvalues and eigenvectors of H. Find a geometric interpretation of $T_H : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_H(v) = Hv$ for all $v \in \mathbb{R}^n$.

Solution: Note that H is a real symmetric matrix, since $H^t = I - (2uu^t)^t = I - 2uu^t$. Thus it is diagonalizable. Now $H(u) = u - 2uu^t u = -u$. Hence u is an eigenvector for the eigenvalue -1. If $v \perp u$ then $H(v) = v - 2uu^t v = v$. Thus all the nonzero vectors in the space $P = u^\perp = \{v \in \mathbb{R}^n \mid u \perp v\}$ are eigenvectors with eigenvalue 1. Since $\dim P = n - 1$, a basis of P along with u is a basis of eigenvectors for H. In fact H is a reflection with respect to the hyperplane P.

(2) If $A, A' \in \mathbb{F}^{n\times n}$ are **similar** , i.e. $A' = P^{-1}AP$ for some invertible $n \times n$ matrix $P \in \mathbb{F}^{n\times n}$. Show that (a) A and A' have same eigenvalues (b) if $\mathbf{v}$ is an eigenvector of A then $P^{-1}\mathbf{v}$ is an eigenvector of A'.

Solution: For a nonzero vector $\mathbf{v}$ we have $P^{-1}\mathbf{v} \neq 0$. Now $A\mathbf{v} = \lambda\mathbf{v}$ iff $P^{-1}AP(P^{-1}\mathbf{v}) = \lambda P^{-1}\mathbf{v}$. This proves both (i) and (ii).

(3) Let A be $n \times n$ complex matrix. Prove that (i) 0 is an *eigenvalue* of A if and only if A is singular. (ii) if λ is an *eigenvalue* of A then it is also an *eigenvalue* of A^t (iii) If x is an *eigenvector* of A corresponding to λ then x need not be an *eigenvector* of A^t corresponding to λ.

Solution: (i) 0 is an eigenvalue iff 0 is a root of the characteristic polynomial $\chi_A(\lambda) = det\,(A - \lambda I)$. Putting $\lambda = 0$, we get $det\, A = 0$. This implies that A is singular.
(ii) $\chi_A(\lambda) = det\,(A - \lambda I) = \det(A - \lambda)^t = det\,(A^t - \lambda I) = \chi_{A^t}(\lambda)$. Since the eigenvalues are nothing but roots of the characteristic polynomial, the conclusion follows.
(iii) Take $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues are $\pm\imath$. Corresponding to the eigenvalue $\imath$ we have

$(1, \imath)^t$ is an eigenvector for A but not for A^t.

(4) Show that the map $T : C^\infty[0,1] \to C^\infty[0,1]$ given by $T(f)(x) = \int_0^x f(t)dt$ has no eigenvalue while every real number is an eigenvalue of $T(f)(x) = \frac{df(x)}{dx}$.

Solution: If T has an eigenvector f with eigenvalue α then $T(f) = \int_0^x f(t)dt = \alpha f(x)$. By the fundamental theorem of Calculus, $f(x) = \alpha f(x)$. As $f(x)$ is nonzero, $\alpha = 1$. But then $f'(x) = f(x)$ For all x. Hence $f(x) = e^x$. But $T(e^x) = e^x - 1 \neq e^x$. If $T(f)(x) = \frac{df(x)}{dx}$ then $T(e^{rx}) = re^{rx}$ for all $r \in \mathbb{R}$. Thus every real number is an eigenvalue of T.

(5) Let $A \in \mathbb{C}^{n \times n}$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a complex polynomial of degree n.. Suppose that λ is an eigenvalue of A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$. Find all the eigenvalues of $f(A)$.

Solution: Let $Au = zu$ for a nonzero vector u and a complex number z. Then $f(A)u = f(z)u$. Thus $f(z)$ is an eigenvalue of $f(A)$ with u as an eigenvector. Conversely, if z is an eigenvalue of $f(A)$ with eigenvector u then $f(A)u = zu$. Consider the complex polynomial $f(x) - z$. Let $z_1, z_2, \ldots, z_n$ be all the roots of $f(x) - z$. Then $f(x) - z = a_n(x - z_1)\ldots(x - z_n)$. Hence $f(A) - zI = a_n \prod_{i=1}^n (A - z_i I)$. Take determinant on both sides to get $\det(f(A) - zI) = 0 = a_n \prod_{i=1}^n \det(A - z_i I)$. Hence for some j, $\det(A - z_j I) = 0$. Hence $z = f(z_j)$.

(6) Find the characteristic polynomial, eigenspaces and their dimensions of the matrix J_n which is the $n \times n$ matrix with each of its entry equal to 1. Is J_n diagonalisable?

Solution: Note that J_n is a real symmetric matrix. Thus it is diagonalizable. As J_n is a rank one matrix, $det J_n = 0$. Hence 0 is an eigenvalue of J_n. The eigenspace E_0 is the solution vectors of the equation $x_1 + \ldots + x_n = 0$. Thus the $\dim E_0 = n - 1$. Hence the algebraic multiplicity of 0 is $n - 1$. Note that $J_n((1,1,\ldots,1)^t = n(1,1,\ldots,1)^t$. Hence n is an eigenvalue of J_n. It follows that $\chi_{J_n}(x) = x^{n-1}(x - n)$.

(7) Let $\{u, v\}$ be an orthonormal basis of $\mathbb{R}^2$. Let $A = uv^t$. Find all the eigenvalues of A.

Solution: Let $w \perp v$. Then $Aw = uv^t w = 0$. So E_0 contains the 1-dimensional subspace $v^\perp$. If $u = (a,b)^t$ and $v = (c,d)^t$ then $\text{tr}A = ac + bd = 0$. Hence the only eigenvalue of A is 0.

(8) Let A be a square matrix. Prove the following statements.

(i) The eigenvalues of A are real if A is Hermitian or real symmetric.

(ii) The eigenvalues of A are either 0 or purely imaginary if A is skew Hermitian.

(iii) The eigenvalues of A are of modulus equal to 1, if A is unitary.

(iv) A^tA has only non negative eigenvalues, if A is real.

Solution: Let $\mu \in \mathbb{K}$, $\mathbf{v} \neq 0$ be such that $A\mathbf{v} = \mu\mathbf{v}$.

(i) Suppose A is Hermitian, i.e., $A = A^*$. Then $\mu\|\mathbf{v}\|^2 = \mu(\mathbf{v}^*\mathbf{v}) = \mathbf{v}^*(\mu\mathbf{v}) = \mathbf{v}^*(A\mathbf{v}) = (\mathbf{v}^*A^*)\mathbf{v} = (A\mathbf{v})^*\mathbf{v} = (\mu\mathbf{v})^*\mathbf{v} = \bar{\mu}\mathbf{v}^*\mathbf{v} = \bar{\mu}\|\mathbf{v}\|^2$. Hence $\mu = \bar{\mu}$ and so, μ is real. Since a real symmetric matrix is hermitian, the second case follows.

(ii) In the above proof, if A were skew-Hermitian, we get $\mu\|\mathbf{v}\|^2 = -\bar{\mu}\|\mathbf{v}\|^2$. Hence $\mu = -\bar{\mu}$ which means $\mu = 0$ or purely imaginary.

(iii) Since A is unitary, $\langle \mathbf{v}, \mathbf{v}\rangle = \langle A\mathbf{v}, A\mathbf{v}\rangle = \langle \mu\mathbf{v}, \mu\mathbf{v}\rangle = \mu\bar{\mu}\langle \mathbf{v}, \mathbf{v}\rangle$ which means that $|\mu|^2 = \mu\bar{\mu} = 1$.

(iv) Take $\mathbf{v} = \sum_{i=1}^n \mathbf{e}_i$. If A_i denotes the columns of A then it follows that $A\mathbf{v} = \sum_{i=1}^n A_i = \mathbf{v}$ (since A is Markov). This shows that 1 is an eigenvalue of A.

(v) Since A^tA is real symmetric, its eigenvalues are real. Let $A^tA\mathbf{u} = \lambda\mathbf{u}$. Then $\lambda\|\mathbf{u}\|^2 = \lambda\mathbf{u}^t\mathbf{u} = \mathbf{u}^t(\lambda\mathbf{u}) = \mathbf{u}^t(A^tA\mathbf{u}) = (\mathbf{u}^tA^t)A\mathbf{u} = (A\mathbf{u})^t(A\mathbf{u}) = \|A\mathbf{u}\|^2$.
Therefore $\lambda \geq 0$.

(9) A self-adjoint matrix A, i.e. $A^* = A$, is called **positive definite** if $\langle A\,x, x\rangle > 0$ for all nonzero $x \in \mathbb{C}^n$. Show that a self-adjoint matrix is positive definite if and only if all eigenvalues of A are positive.

Solution: A real symmetric matrix is congruent to a diagonal matrix. Since congruence does not change the positivity (check this), and since the eigenvalues are the diagonal entries of the diagonal form, the result follows.

(10) Let A be a self-adjoint matrix. If $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{C}^n$, then show that $A = O$. Deduce that if $\|Ax\| = \|A^*x||$ for all $x \in \mathbb{C}^n$, then A is a normal matrix, and if $\|Ax\| = ||x||$ for all $x \in^n$, then A is a unitary matrix.

Solution: Since A is self-adjoint, $A^* = A$ and A has an orthonormal basis of eigenvectors. Let u be a unit eigenvector with eigenvalue a. As a is real, $\langle Au, u\rangle = u^*au = a = 0$. Thus all eigenvalues are zero. Thus $A = 0$. Now let $||Ax|| = ||A^*x||$ for all $x \in \mathbb{C}^n$. Therefore, $x^*A^*Ax = x^*AA^*x$. Since A^*A, AA^* are self-adjoint, so is there difference. Hence $x^*(A^*A - AA^*)x = 0$ for all x. Hence $AA^* = A^*A$. Thus A is normal. Now let $||Ax|| = ||x||$ for all x. This means that $x^*A^*Ax = x^*x$. Hence $x^*(AA^* - I)x = 0$ for all x. But $AA^* - I$ is self-adjoint. Hence $AA^* = I$. Thus A is unitary.

(11) Let a be a nonzero real number and $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

(a) Find an orthonormal set of eigenvectors of A.

(b) Find a unitary matrix C such that $C^{-1}AC$ is a diagonal matrix.

(c) Prove: there is no real orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix.

Solution: (a) The characteristic polynomial of A is $f(x) = x^2 + a^2$. Hence $x = \pm ia$. If $u = (x, y)^t$ is an eigenvector for the eigenvalue ia then $A(x, y)^t = (ya, -ax)^t = (iax, iay)^t$. Thus $(i, 1)^t$ is an eigenvector for the eigenvalue ia Similarly, $(1, i)^t$ is an eigenvector for the eigenvalue $-ia$.

(b) The columns of the unitary matrix C consists of unit eigenvectors for the eigenvalues. Hence $C = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$.

(12) Let C be the locus of the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Using eigenvalues of the symmetric matrix A so that $ax^2 + bxy + cy^2 = [x\ y]A[x\ y]^t$, show that C is ellipse, hyperbola or parabola according as the *discriminant* $4ac - b^2$ is positive, negative or zero.