PH111: Introduction to Classical Mechanics Chapter 7: Four-vector formalism of Special Theory of Relativity

Four-vector Formalism

- We mentioned earlier that in order to describe an event, we need a 4D space characterized by the space-time coordinates (x, y, z, t)
- We will put this on more rigorous footing and generalize it
- But, first let us understand the notion of vectors and their transformations in the Cartesian 3D space
- The position vector r in this space is represented as

$$\mathrm{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

- Or equivalently we can express r as a column vector

$$\mathrm{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vectors...

- Similarly, a general Cartesian vector A can be expressed as

$$\mathsf{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

which can be represented as

$$\mathsf{A} \equiv \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

- A vector is characterized by its transformation laws when the axes in terms of which it is represented are transformed
- One example of a transformation is the rotation

Rotation of the coordinate system

- Suppose, we rotate the coordinate system, i.e., perform the transformation

$$\hat{i} \to \hat{i}'$$
$$\hat{j} \to \hat{j}'$$
$$\hat{k} \to \hat{k}'$$

- in such a way that the coordinate system remains orthogonal

$$\hat{i}' \cdot \hat{i}' = \hat{j}' \cdot \hat{j}' = \hat{k}' \cdot \hat{k}' = 1$$
$$\hat{i}' \cdot j' = \hat{j}' \cdot \hat{k}' = \hat{k}' \cdot \hat{i}' = 0$$

- One can show that the representation of a vector in the new coordinate system

$$\mathrm{r} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$
$$\mathrm{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$$

is related to the old one by an orthogonal transformation

$$\mathrm{r}' = O\mathrm{r}$$
$$\mathrm{A}' = O\mathrm{A}$$

such that

$$O^T O = O O^T = I,$$

where T denotes the transpose operation, and I is the 3×3 identity matrix.

- where

$$\mathrm{r}' \equiv \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\mathrm{A}' \equiv \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

Orthogonal transformations...

- and

$$O = \begin{pmatrix} O_{xx} & O_{xy} & O_{xz} \\ O_{yx} & O_{yy} & O_{yz} \\ O_{zx} & O_{zy} & O_{zz} \end{pmatrix}$$

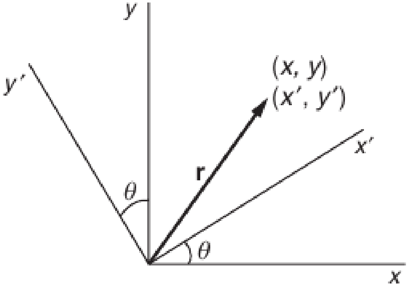
- Orthogonal transformations (OTs) are norm conserving, i.e., length of a vector under an OT is an invariant (remains unchanged)

$$r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 = r^2$$

$$|A'|^2 = A_x'^2 + A_y'^2 + A_z'^2 = A_x^2 + A_y^2 + A_z^2 = |A|^2$$

Orthogonal Transformation: an example

- Let us consider a counter-clockwise rotation about the z-axis, by an angle θ



- Clearly

$$
\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -x\sin\theta + y\cos\theta \\
z' &= z
\end{aligned}
$$

An orthogonal transformation

- Transformation equations are identical for a general vector A

$$
\begin{aligned}
A'_x &= A_x \cos\theta + A_y \sin\theta \\
A'_y &= -A_x \sin\theta + A_y \cos\theta \\
A'_z &= A_z
\end{aligned}
$$

- Obviously, the transformation matrix is

$$
O = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

  which can be easily verified to be orthogonal
- Clearly, the norm (or length) of a vector is an invariant under an orthogonal transformation

Four Vectors

- We have already discussed that in Special Theory of Relativity (STR) time coordinate t is to be treated on the same footing as the space coordinates (x, y, z)
- That is space-time is a 4D space, and any vector in that space is called a "four vector"
- Can we call (x, y, z, t) a four vector?
- This is problematic because t doesn't have the dimensions of space coordinates (x, y, z)
- This problem is solved by taking the fourth coordinate as ct instead of t
- Thus, we can define a position vector R in the 4D space as

$$\mathrm{R} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

Four vectors...

- A general four vector A in this 4D space will be given by

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

- In physics this 4D space, or four-space, is called the Minkowski space
- The Lorentz transformation equations of the previous chapter can be seen as “rotations” in the Minkowski space
- It is easy to verify that the Lorentz transformation equations can denoted by the following matrix equation

$$R' = LR,$$

where

$$\mathrm{R}' = \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}$$

and the rotation matrix L is given by

$$L = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

where $\beta = v/c$.

Invariants of a Lorentz Transformation

- In a 3D space under a orthogonal transformation we saw that the invariant quantity was the norm of a vector

$$r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 = r^2$$
$$|A'|^2 = A_x'^2 + A_y'^2 + A_z'^2 = A_x^2 + A_y^2 + A_z^2 = |A|^2$$

- The question arises what is the counterpart of this under a Lorentz transformation
- That is what quantity associated with a 4-vector stays invariant under a Lorentz transformation?
- Let us define the norm of a "position" vector in the Minkowski space as

$$R^2 = x^2 + y^2 + z^2 - (ct)^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

note the negative sign associated with the fourth component.

Distance between two points in Minkowski space

- We can also denote the 4-vector corresponding to a point in the Minkowski space as

$$\mathrm{R} \equiv (\mathrm{r}, ct),$$

where r has its usual meaning, i.e., $\mathrm{r} = x\hat{i} + y\hat{j} + z\hat{k} \equiv (x, y, z)$

- And a general 4-vector A as

$$\mathrm{A} \equiv (\vec{A}, a_4),$$

where $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

Lorentz Invariants

- We show below that R^2 is an invariant under the Lorentz transformation

$$
\begin{aligned}
R'^2 &= x'^2 + y'^2 + z'^2 - (ct')^2 \\
&= \gamma^2 (x - \beta ct)^2 + y'^2 + z'^2 - \gamma^2 (-\beta x + ct)^2 \\
&= \gamma^2 \left\{ x^2 - 2\beta xct + \beta^2 c^2 t^2 - \beta^2 x^2 + 2\beta xct - c^2 t^2 \right\} + y^2 + z^2 \\
&= \gamma^2 (1 - \beta^2) \left\{ x^2 - (ct)^2 \right\} + y^2 + z^2 \qquad \text{using } \gamma^2 (1 - \beta^2) = 1 \\
&= x^2 + y^2 + z^2 - (ct)^2 \\
&= R^2
\end{aligned}
$$

Lorentz invariants...

- Let us consider two space time points $\mathrm{R} = (x, y, z, ct)$ and $\mathrm{R} + \Delta\mathrm{R} = (x + \Delta x, y + \Delta y, z + \Delta z, c(t + \Delta t))$, which are infinitesimally apart from each other
- We define the distance between these two points as

$$ds^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

- According to an observer in the frame S', the coordinates of these two points are related to each other by Lorentz transformation

$$\mathrm{R}' = L\mathrm{R}$$
$$(\mathrm{R} + \Delta\mathrm{R})' = L(\mathrm{R} + \Delta\mathrm{R})$$

- One can show similarly that the distance between two points is also a Lorentz invariant

$$\begin{aligned} ds'^2 &= \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2 \\ &= \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = ds^2 \end{aligned}$$

Lorentz Invariants

- Similarly, for a general 4-vector in the Minkowski space also the same transformation rules will apply

$$\mathrm{A}' = L\mathrm{A},$$

where

$$\mathrm{A}' = \begin{pmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{pmatrix}$$

And again one can easily show that the norm of the 4-vector A is conserved under a Lorentz transformation

$$A'^2 = a_1'^2 + a_2'^2 + a_3'^2 - a_4'^2 = a_1^2 + a_2^2 + a_3^2 - a_4^2 = A^2$$

- Let us define the four vectors corresponding to some familiar quantities from classical physics

Four Velocity

- Let us define the 4-velocity vector U as

$$\mathrm{U} = \frac{d\mathrm{R}}{d\tau} = \left(\frac{d\mathrm{r}}{d\tau}, \frac{d(ct)}{d\tau}\right)$$

- Above the derivative has been taken with respect to the proper time τ because all observers agree on it
- However, in Newtonian physics the velocity u is defined as

$$\mathrm{u} = \frac{d\mathrm{r}}{dt}$$

- To connect the 4-velocity to the 3-velocity vector u, we write the spatial component of U as

$$\frac{d\mathrm{r}}{d\tau} = \frac{d\mathrm{r}}{dt}\frac{dt}{d\tau},$$

using the relations $\tau = t/\gamma$ and $d\tau = dt/\gamma$, we immediately obtain

$$\mathrm{U} = \gamma(\mathrm{u}, c)$$

The Energy-Momentum 4-Vector

- The norm of U will be given by

$$U^2 = \gamma^2(u^2 - c^2)$$

- Because U is a 4-vector, this norm will be a Lorentz invariant.
- We define the energy-momentum 4-vector, or in short, 4-momentum of a particle by simply multiplying the 4-velocity by m_0, the rest mass of the particle

$$\mathrm{P} = m_0\gamma(\mathrm{u}, c).$$

- The rest mass m_0 is defined as the mass of the particle measured by an observer with respect to whom the particle is at rest
- Clearly, this concept is similar to that of proper length and time
- We introduce another mass m of the particle, called its relativistic mass

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-\beta^2}}$$

4-momentum

- With this, we have

$$\mathrm{P} = m_0\gamma(\mathrm{u}, c) = m(\mathrm{u}, c) = (\mathrm{p}, mc),$$

  where $\mathrm{p} = m\mathrm{u}$ is the relativistic 3-momentum of the particle

- Clearly, P^2 will be a Lorentz invariant

$$P^2 = \mathrm{p}\cdot\mathrm{p} - m^2c^2 = C,$$

  where C is a constant.

- But, in the rest frame of the particle, $\mathrm{p} = 0$ and $m = m_0$, which means

$$p^2 - m^2c^2 = -m_0^2c^2 = C$$

- This leads to

$$p^2 = (mc)^2 - (m_0c)^2$$

- We speculate that the fourth component of the this 4-vector mc is related to the energy E of the particle by relation

$$mc = E/c,$$

  which implies $E = mc^2$, the most famous formula in the world!

4-momentum

- This leads to the final definition of the 4-momentum

$$\mathrm{P} = (\mathrm{p}, E/c)$$

- which on substitution above leads to

$$p^2 = E^2/c^2 - m_0^2 c^2$$
$$E = \sqrt{m_0^2 c^4 + p^2 c^2}$$

- Therefore, the energy of particle at rest is obtained by setting $p = 0$

$$E_0 = m_0 c^2$$

- Also noteworthy is that because of the relation

$$p^2 - m^2 c^2 = -m_0^2 c^2$$
$$\implies E^2 - p^2 c^2 = (m_0 c^2)^2$$

the rest energy of a particle is a Lorentz invariant quantity.

Relativistic Kinetic Energy....

- The relativistic definition of the kinetic energy K of a particle is difference of its total energy and the rest energy

$$K = E - m_0 c^2 = \sqrt{p^2 c^2 + m_0^2 c^4} - m_0 c^2$$

- One can easily show that in the non-relativistic limit of $\beta = u/c \ll 1$, where u is the speed of the particle

$$K \approx \frac{1}{2} m_0 u^2$$

- which is a familiar expression for the kinetic energy that we learned in the Newtonian physics