ASSIGNMENT 4 : VECTOR SPACES
MA 106 : LINEAR ALGEBRA
SPRING 2023

1. **Tutorial Problems**

(1) Obtain the REF of the following matrices. Use them to find the rank and the nullity of the matrix. Also write down a basis for the range. Finally obtain the RCF and use it to write down a basis for the null space.

$$\text{(i)} \begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{bmatrix}.$$

(2) Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through the origin, planes passing through the origin and the whole space.

(3) A **hyperplane** in $\mathbb{R}^n$ is defined to be the set $u + W$ where $u \in \mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ having dimension $n-1$. Prove that a hyperplane in $\mathbb{R}^n$ is the set of solutions of a single linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_1, \ldots, a_n, b \in \mathbb{R}$.

(4) Consider the following subsets of the space $M_n(\mathbb{C})$ of $n \times n$ complex matrices :
(a) $Sym_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^T\}$ of **symmetric matrices**.
(b) $Herm_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^*\}$ of **Hermitian matrices**.
(c) $Skew_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*\}$ of **skew-Hermitian Matrices**.
Show that each of them is an $\mathbb{R}$-vector subspace of $M_n(\mathbb{C})$ and compute their dimension by explicitly writing down a basis for each of them.

(5) Let $P_n[x]$ denote the vector space consisting of the zero polynomial and all real polynomials of degree $\leq n$, where n is fixed. Let S be a subset of all polynomials $p(x)$ in $P_n[x]$ satisfying the following conditions. Check whether S is a subspace; if so, find the dimension of S. (i) $p(0) = 0$; (ii) p is an odd function; (iii) $p(0) = p''(0) = 0$.

(6) Examine whether the following sets are linearly independent.
(a) $\{(a, b), (c, d)\} \subset \mathbb{R}^2$, with $ad - bc \neq 0$.
(b) For $\alpha_1, \ldots, \alpha_k$ distinct real numbers, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where $\mathbf{v}_i = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1})$.
(c) $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$.
(d) $\{1, \sin x, \sin 2x, \ldots, \sin nx\}$.
(e) $\{e^x, xe^x, \ldots, x^ne^x\}$.

(7) Find a basis for the subspace $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0\}$. Let P be the xy-plane. Find a basis of $W \cap P$. Find a basis of the subspace of all vectors in $\mathbb{R}^3$ which are perpendicular to the plane W.

2. **Practice Problems**

(8) Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with entries in $\mathbb{C}$. Show that $M_n(\mathbb{C})$ is a vector space over $\mathbb{C}$ and $\{E_{(i,j)}, 1 \leq i,j \leq n\}$, where $E_{(i,j)}$ denote $n \times n$ matrix with 1 at $(i,j)^{\text{th}}$ place and 0 elsewhere is a basis of it.

(9) Examine whether the following subsets of the set of real valued functions on $\mathbb{R}$ are linearly dependent or independent. Compute the dimension of the subspace spanned by each set (a) $\{1+t, (1+t)^2\}$; (b) $\{x, |x|\}$.

(10) Examine whether the following sets of vectors constitute a vector space. If so, find the dimension and a basis of that vector space.
(a) The set of all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that (i) $x_4 = 0$; (ii) $x_1 \leq x_2$; (iii) $x_1^2 - x_2^2 = 0$; (iv) $x_1 = x_2 = x_3 = x_4$; (v) $x_1 x_2 = 0$.
(b) The set of all real functions of the form $a\cos x + b\sin x + c$, where $a, b, c \in \mathbb{R}$. (c) Cubic homogeneous polynomials together with the zero polynomial.
(d) The set of all $n \times n$ real matrices $((a_{ij}))$ which are:
(i) diagonal; (ii) upper triangular; (iii) having zero trace; (iv) symmetric; (v) anti-symmetric (i.e., those satisfying $A^t = -A$;) (vi) invertible.
(e) The set of all real polynomials of degree 5 together with the zero polynomial.
(f) The set of all complex polynomials of degree ≤ 5 with $p(0) = p(1)$ together with the zero polynomial.
(g) The real functions of the form $(ax+b)e^x, \quad a, b \in \mathbb{R}$.

(11) Let $\alpha_1, \alpha_2, \alpha_3$ be fixed real numbers. Show that the vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_4 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ forms a subspace, which is spanned by $(1, 0, 0, \alpha_1)$, $(0, 1, 0, \alpha_2)$ and $(0, 0, 1, \alpha_3)$. Find the dimension of this subspace.

(12) Let A be a 10×10 matrix with $A^2 = 0$. Show that rank $A \leq 5$.

(13) Let A be a 5×4 matrix having rank 4. Show that $Ax = b$ has no solution when the augmented matrix $[A|b]$ is invertible. Show that $Ax = b$ is solvable then $[A|b]$ is singular.