ASSIGNMENT 6 : INNER PRODUCT SPACES

MA 106 : SPRING 2023

Tutorial Problems

(1) Find the projection $\mathbf{p}$ of $\mathbf{b}$ onto the column space of A by solving $A^tAx = A^tb$ and $p = Ax$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(2) If P is a real square matrix with $P^2 = P$, show that $(I - P)^2 = I - P$. Suppose P is the matrix of projection onto the columns space of A. Find the space onto which $I - P$ projects.

(3) Let the columns of A be linearly independent and $P = A(A^tA)^{-1}A^t$. Show that P is symmetric and $P^2 = P$.

(4) In the vector space $C[1, e]$, define $\langle f,\, g\rangle = \displaystyle\int_1^e \log x f(x)g(x)\, dx$.

(a) if $f(x) = \sqrt{x}$, compute $\|f\| = \langle f,\, f\rangle^{1/2}$.

(b) Find a linear polynomial $g(x) = ax + b$ that is orthogonal to $f(x) = 1$.

(5) (a) To find the projection matrix onto the plane $x - y - 2z = 0$, choose two linearly independent vectors u, v in the plane and let A be the matrix whose column vectors are u, v. Now find $P = A(A^tA)^{-1}A^t$.

(b) Let e be a vector perpendicular to the plane $L : x - y - 2z = 0$. Find the projection matrix $Q = \frac{ee^t}{e^te}$. Show that $P = I - Q$ is the matrix of projection onto L.

(6) Let $u \in \mathbb{R}^n$ be a unit vector. Let $H_u = I - 2uu^t$. Show that H is an orthogonal matrix. Find $H_u(v)$ for any $v \in L(u)^{\perp}$. Find $H_u(\alpha u)$ for any $\alpha \in \mathbb{R}$. Describe the action of H_u geometrically. Using this find the matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors with respect to the line $y = x \tan\theta$. Find the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ Which reflects vectors with respect to the plane $x + y + z = 0$.

(7) Let $V = C[0, 1] = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$ with inner product given by $\langle f,\, g\rangle = \int_0^{\pi} f(t)g(t)dt$. Let $x_n(t) = \cos nt$ for $n = 0, 1, 2 \ldots$. Prove that the functions $y_0, y_1, y_2, \ldots$ given by

$$y_0(t) = \frac{1}{\sqrt{\pi}} \text{ and } y_n(t) = \sqrt{\frac{2}{\pi}} \cos nt \text{ for } n \geq 1$$

form an orthonormal set spanning the same subspace as $x_0, x_1, x_2, \ldots$.

Practice Problems

(8) In the real vector space $C[0, 2]$ with inner product $\langle f,\, g\rangle = \int_0^2 f(x)g(x)dx$, let $f(x) = e^x$. Show that the constant polynomial g nearest to e^x is $(e^2 - 1)/2$.

(9) Let P be the vector space of all real polynomials in one indeterminate t. Consider the inner product $\langle p,\, q\rangle = \int_{-1}^1 p(t)q(t)dt$. Consider the infinite sequence $x_n(t) = t^n$. The Gram-Schmidt process applied to this sequence gives the polynomials $y_n(t)$ for $n = 0, 1, 2, \ldots$ first encountered by the French mathematician A. M. Legendre (1752-1833). Prove that

$$y_n(t) = \frac{n!}{2n!} \frac{d^n}{dt^n} (t^2 - 1)^n.$$

(10) Let A be an $n \times n$ square matrix with real (or complex) entries. Show that the following are equivalent. (i) A is orthogonal (unitary). (ii) A^t (resp. A^*) is orthogonal (unitary). (iii) The column vectors of A form an orthonormal set. (iv) The row vectors of A form an orthonormal set.

(11) Orthonormalize the set $\{(1,0,0,0),(1,1,0,0),(1,1,1,1)\}$ of vectors in $\mathbb{R}^4$.

(12) On the vector space $C^1[a,b]$ of continuously differentiable real valued functions, examine whether or not $\langle f,\, g\rangle$, defined below is an inner product in each case. Justify your answer.

(a) $\langle f,\, g\rangle = \displaystyle\int_a^b f^{'}(t)g^{'}(t)\,dt$ (b) $\langle f,\, g\rangle = \displaystyle\int_a^b (f(t)g(t) + f^{'}(t)g^{'}(t))\,dt.$

(13) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V. Suppose $\mathbf{u} = \displaystyle\sum_{i=1}^n \alpha_i \mathbf{v}_i$ and $\mathbf{v} = \displaystyle\sum_{i=1}^n \beta_i \mathbf{v}_i \in V$. Prove that

(14) $\alpha_i = \langle \mathbf{u},\, \mathbf{v}_i\rangle$ and $\beta_i = \langle \mathbf{v},\, \mathbf{v}_i\rangle$ for $i = 1, 2, \ldots, n$.

(15) $\langle \mathbf{u},\, \mathbf{v}\rangle = \displaystyle\sum_{i=1}^n \alpha_i\beta_i$ and $\|\mathbf{u}\| = \left(\displaystyle\sum_{i=1}^n \alpha_i^2\right)^{1/2}$.

(16) Prove that in a real inner product space V, the following are equivalent:
(i) $\langle x,\, y\rangle = 0$; (ii) $\|x+y\| = \|x-y\|$; (iii) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(17) In the vector space $\mathcal{P}_n[t]$ of all real polynomials of degree $\leq n$, define

$$\langle f,\, g\rangle = \sum_{j=0}^n f\left(\frac{j}{n}\right) g\left(\frac{j}{n}\right).$$

(a) Prove that $\langle f,\, g\rangle$ is an inner product on P_n.
(b) Compute $\langle f,\, g\rangle$, when $f(t) = t, g(t) = at + b$.
(c) If $f(t) = t$, find all linear polynomials g orthogonal to f.

(18) Write elements of $\mathbb{R}^n$ as column vectors of length n. Let A be an $n \times n$ real matrix and A^t be its transpose. For usual inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, prove that for any two $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\langle A\mathbf{u},\, \mathbf{v}\rangle = \langle \mathbf{u},\, A^t\mathbf{v}\rangle$. State and prove a similar result about $\mathbb{C}^n$.

(19) In a real inner product space V, show that for any $x, y \in V$

$$\langle x+y,\, x-y\rangle = 0 \quad \text{iff } \|x\| = \|y\|.$$

(20) Orthonormalize the following set of vectors in $\mathbb{R}^4$, using the Gram-Schmidt process.
(a) $\{(1,0,0,0),(1,1,0,0),(1,1,1,1)\}$ (b) $\{(1,-1,2,0),(1,0,1,0),(1,0,0,1)\}$.

(21) Let V be an inner product space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set in V.
(a) Show that for all $\mathbf{v} \in V$, $\langle \mathbf{v},\, \mathbf{v}\rangle \geq \sum_{i=1}^n |\langle \mathbf{v},\, \mathbf{v}_i\rangle|^2$, (**Bessel's inequality**)
(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, show $\langle \mathbf{v},\, \mathbf{v}\rangle = \sum_{i=1}^n |\langle \mathbf{v},\, \mathbf{v}_i\rangle|^2$. (**Parseval's identity**)

(22) (a) Find an orthonormal basis of the subspace S of $\mathbb{R}^3$ spanned by the solutions of $x_1 + x_2 + x_3 = 0$.
(b) Find an orthonormal basis of $S^\perp = \{u \in \mathbb{R}^3 \mid \langle u,\, v\rangle = 0 \text{ for all } v \in S\}$.
(c) Find $u \in S$ and $v \in S^\perp$ so that $(1,1,1) = u + v$.

(23) Let $V = \{a + bx \mid a, b \in R\}$ be the real vector space of all polynomials in x of degree at most 1. Show that $B = \{1, x\}$ is an orthogonal basis of V if the inner product is given by $\langle f(x),\, g(x)\rangle = \int_{-1}^1 f(x)g(x)dx$. Find the best approximation of e^x in V.

(24) Let V be an inner product space over $\mathbb{R}$ and $\dim V = n$. Let v be a unit vector in V. What is the dimension of $W = \{u \in V \mid \langle u,\, v\rangle = 0\}$. Justify your answer.