PH 112: Quantum Physics and Applications

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Week 02 Lecture 2: Fourier Series and Fourier transform

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Group velocity: Recap

- Superposition of waves

  When infinitely many waves with different amplitudes and frequencies and wave numbers combine, a pulse, or wave packet is formed.

- Waves localized in space are wave packets. Wave packet moves at a **group velocity**:

$$v_{group} = \frac{\Delta\,\omega}{\Delta\,k}$$

- de Broglie postulated that the group velocity is identical to particle's velocity.

This lecture we will focus the importance of ω and k.

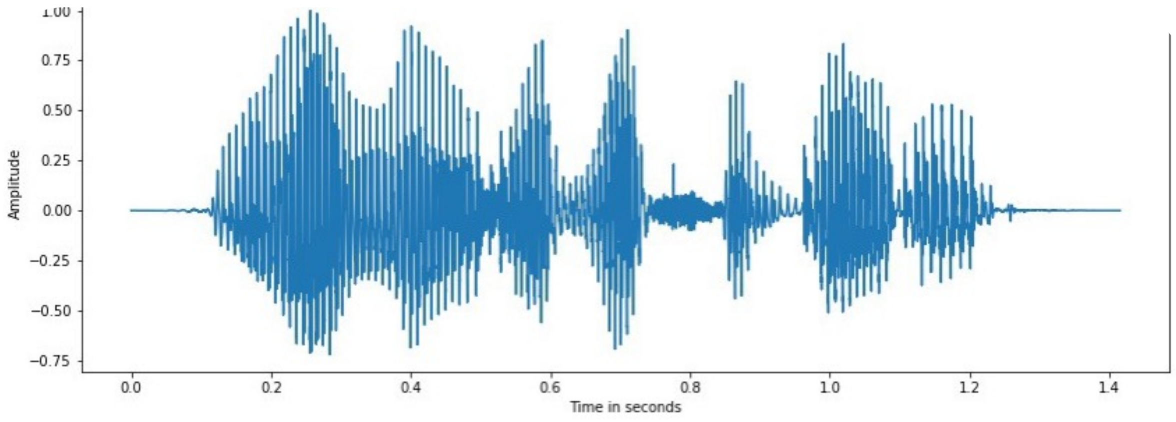
Ever wondered how our ear works?

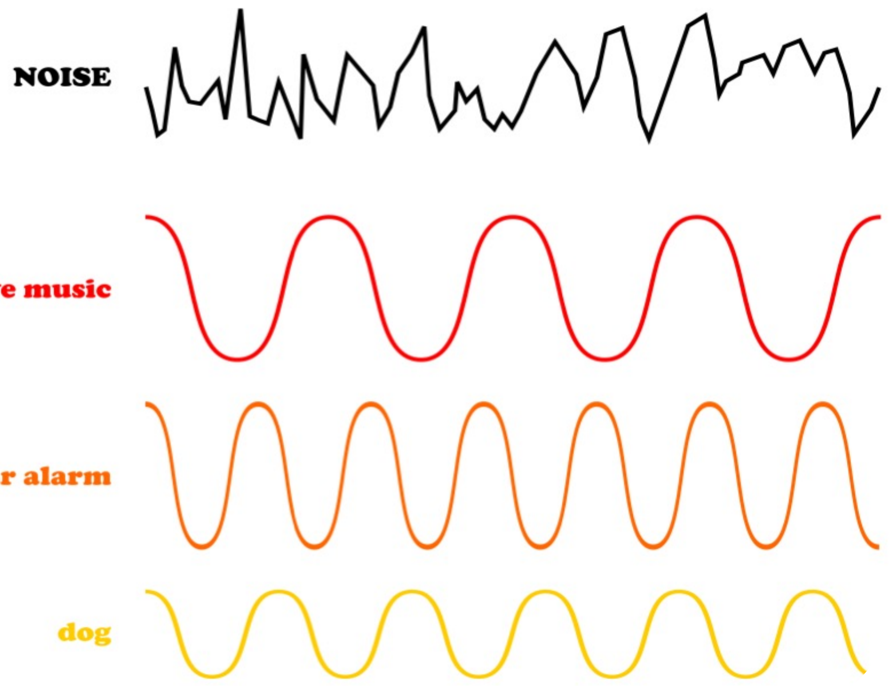
- We never hear only one sound!

- Sound waves from different sources, and different directions make their way into our ear canals.

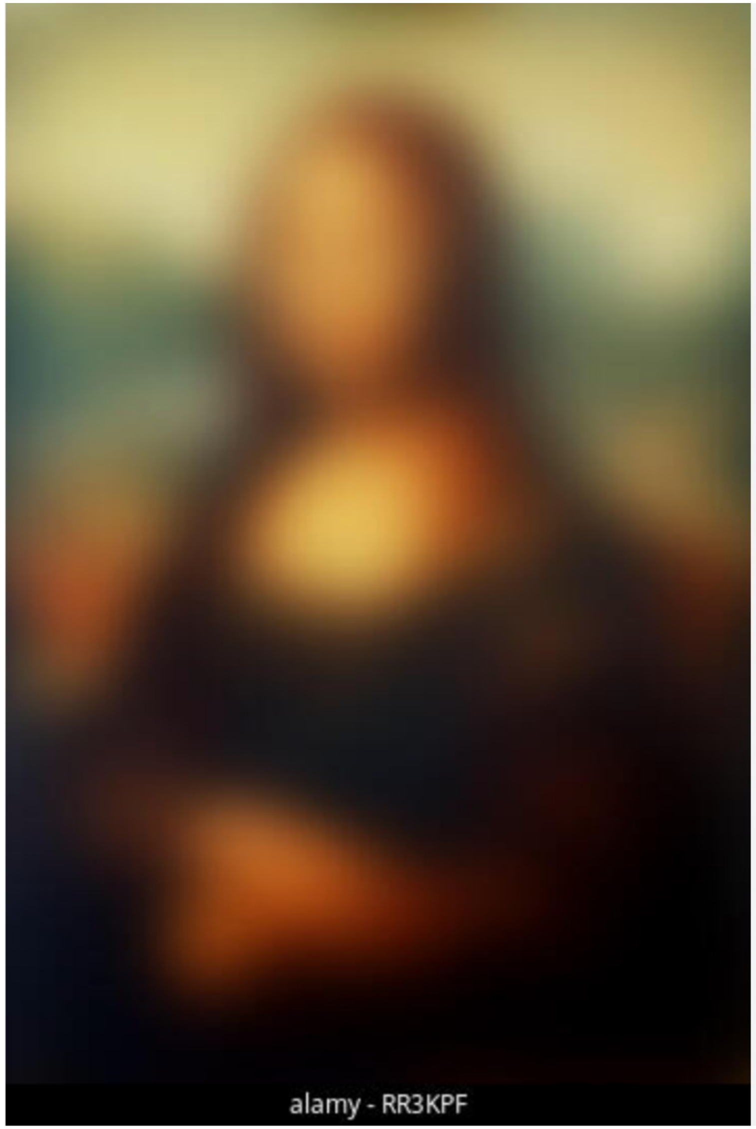
- Ear can distinguish many different frequencies in a sound.

- Ear breaks any sound into a series of sine waves at many different frequencies. Our ear is a great Fourier Analyzer!





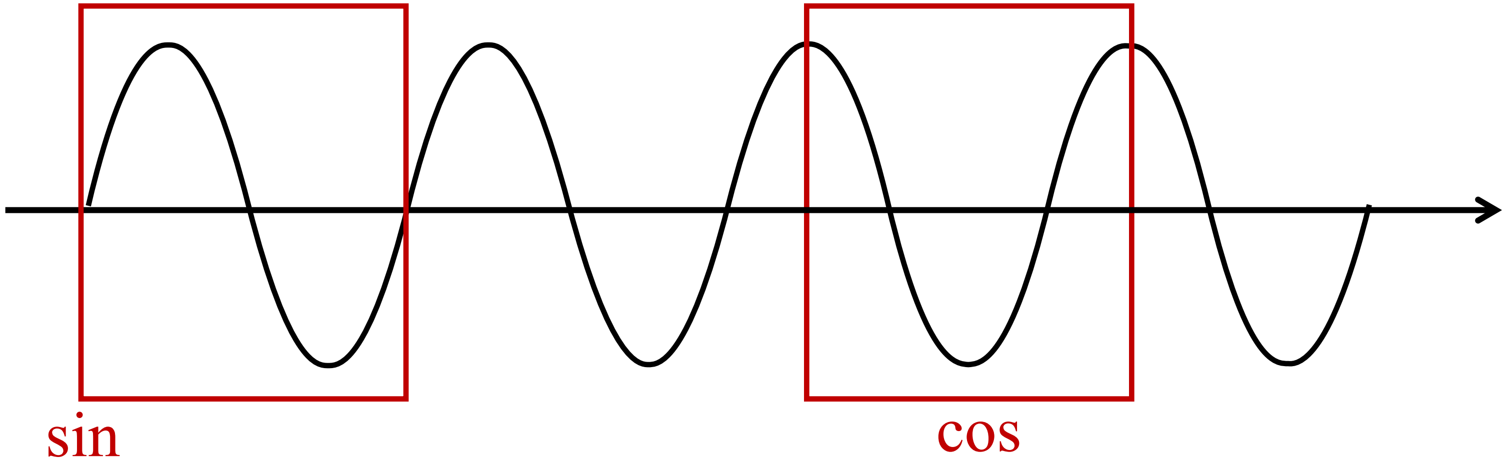
Why does a low-resolution image still makes sense?



- How is that a 10MP mage can be compressed to a few hundred KB without a noticeable change?
- How does our brain process such images?

Language of numbers to the Language of frequencies

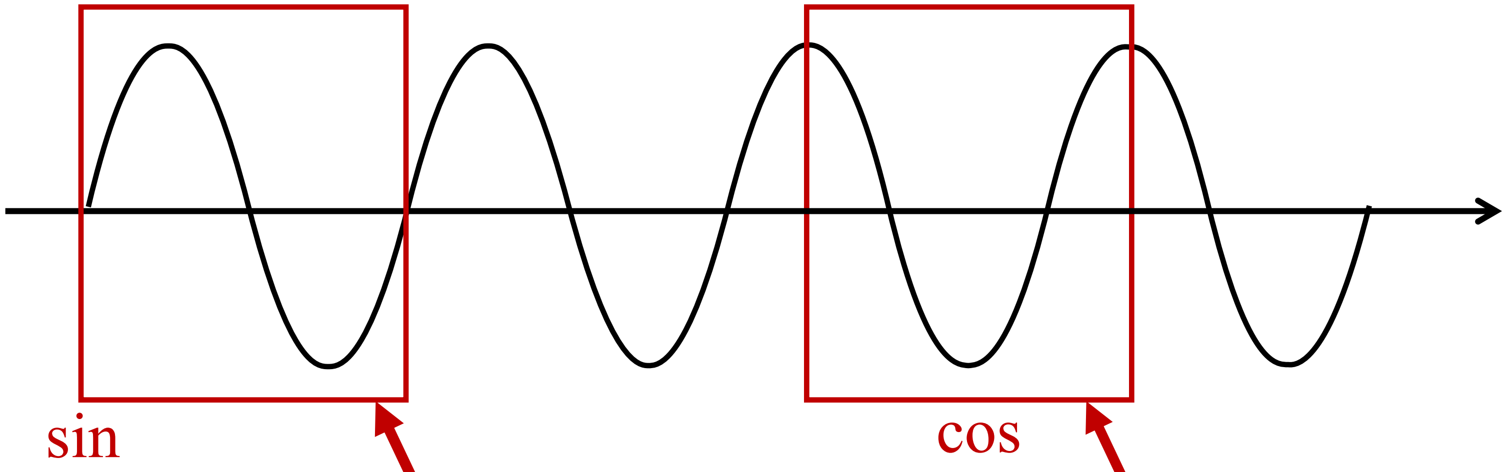
Let us consider a signal $f(t)$ with a single frequency (monochromatic):



Repetition of sines or cosines

Language of numbers to the Language of frequencies

Signal $f(t)$ with a single frequency (monochromatic):



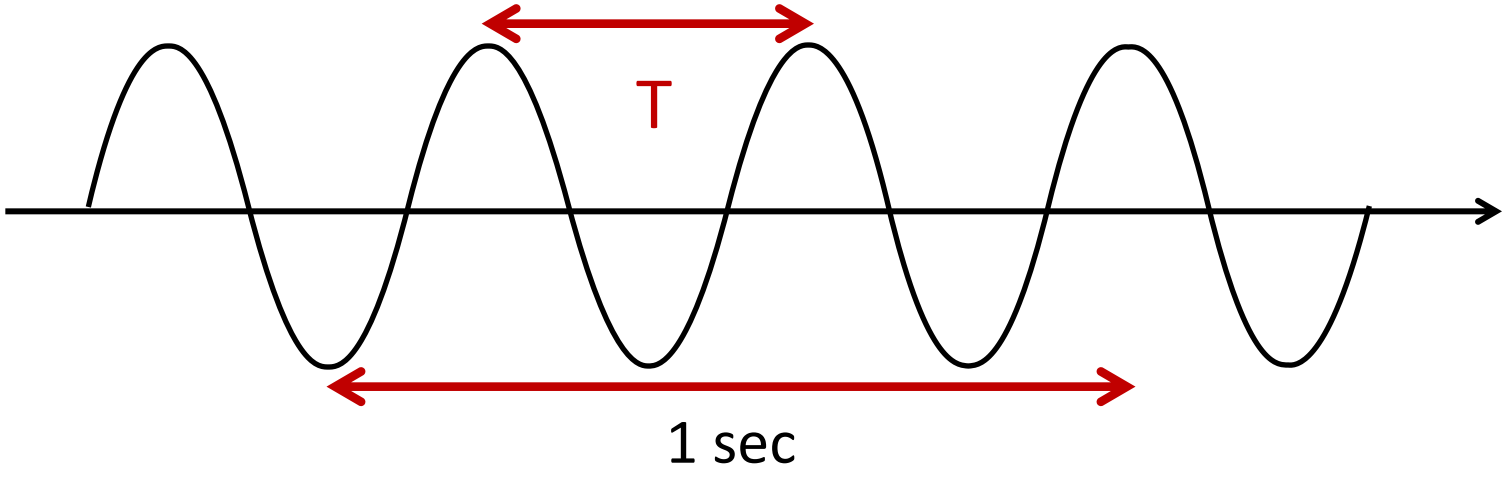
Repetition of sines or cosines

cycle (= 1 repetition) for sin

cycle (= 1 repetition) for cos

Language of numbers to the Language of frequencies

Signal $f(t)$ with a single frequency (monochromatic):



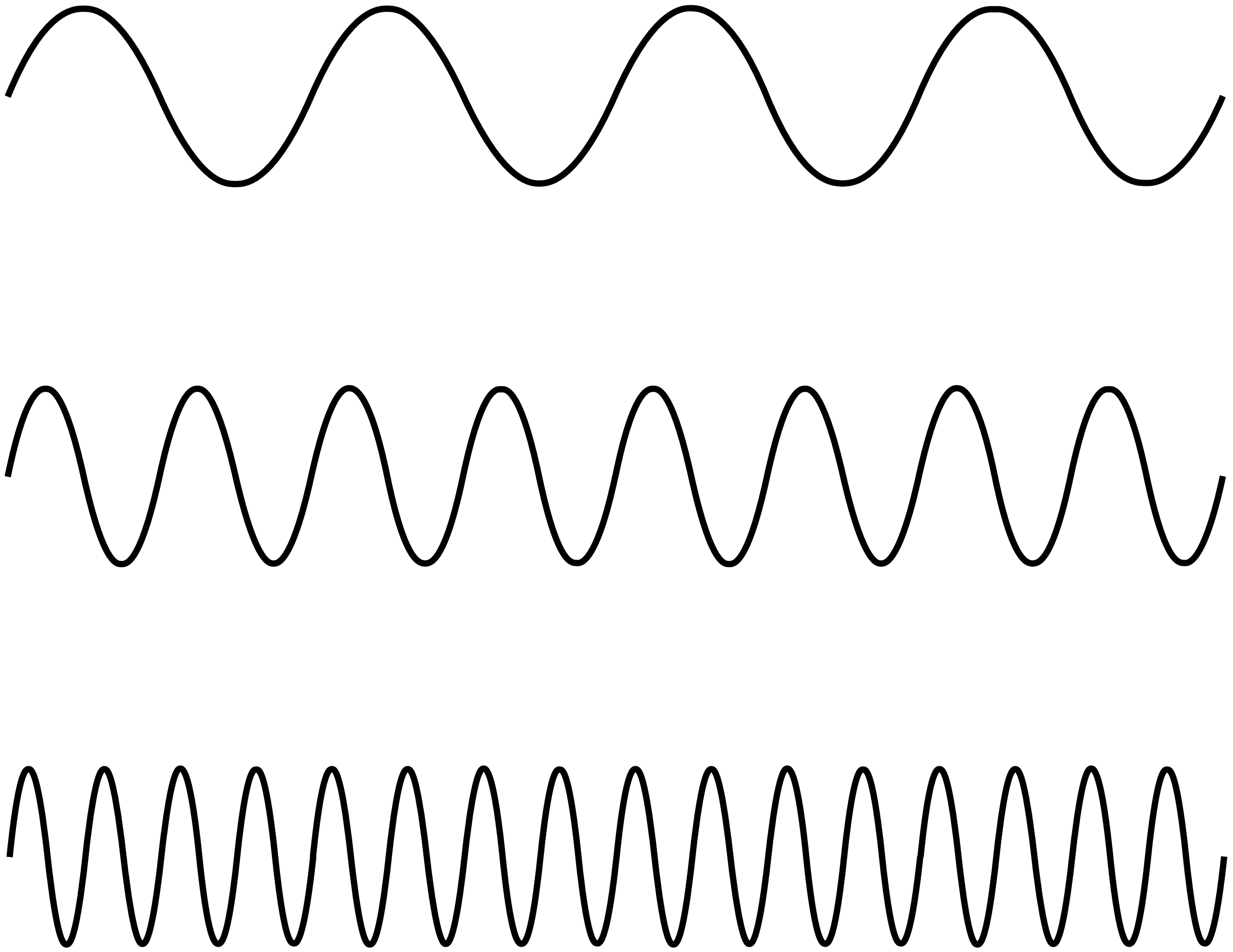
(s = 2.5 cycles per second)

Period T : time needed for the repetition of a cycle

Frequency s : number of cycles in a unit of time (cycles per second)

Wavelength $\lambda = c\,T$: interval covered by a signal travelling with velocity c within one period

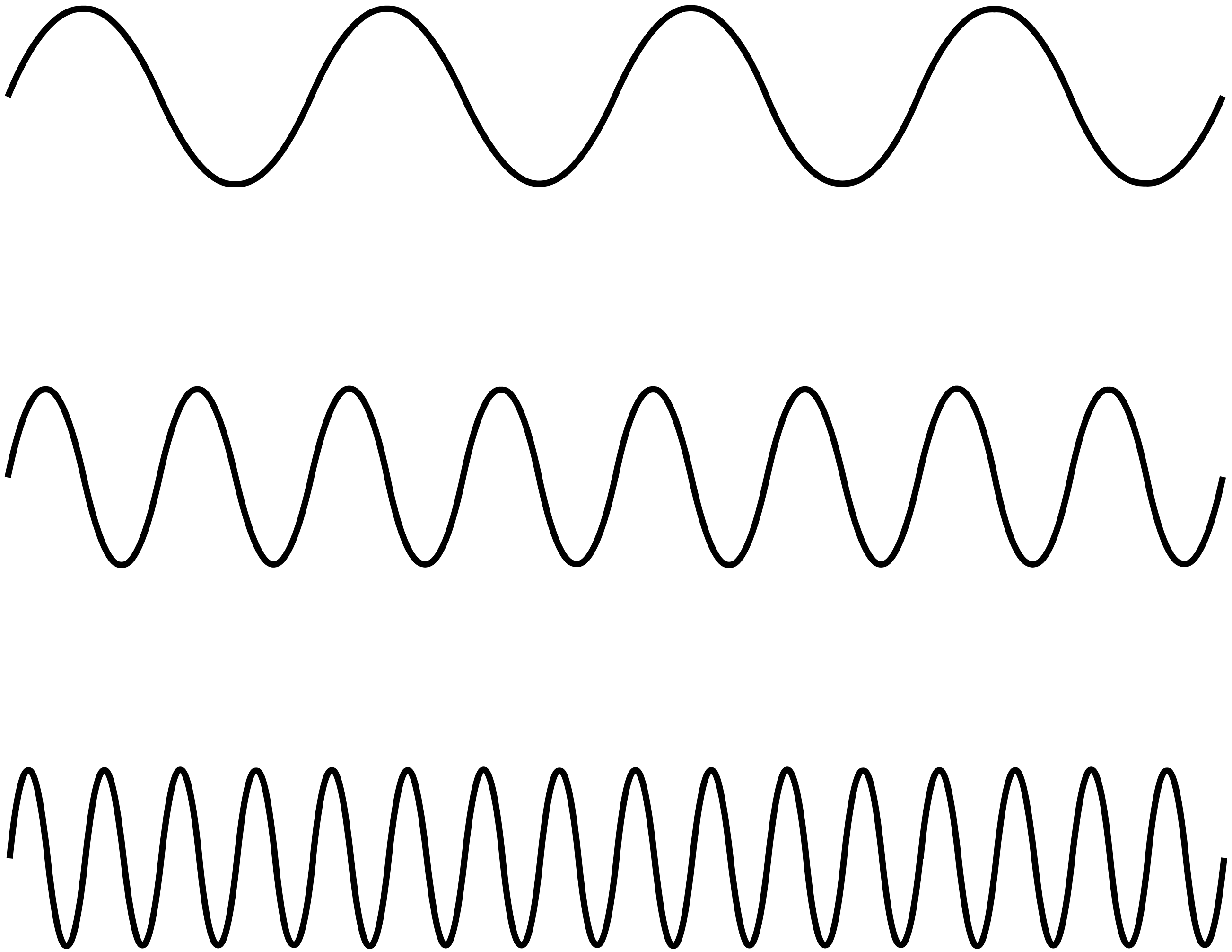
Language of numbers to the Language of frequencies



low frequency

high frequency

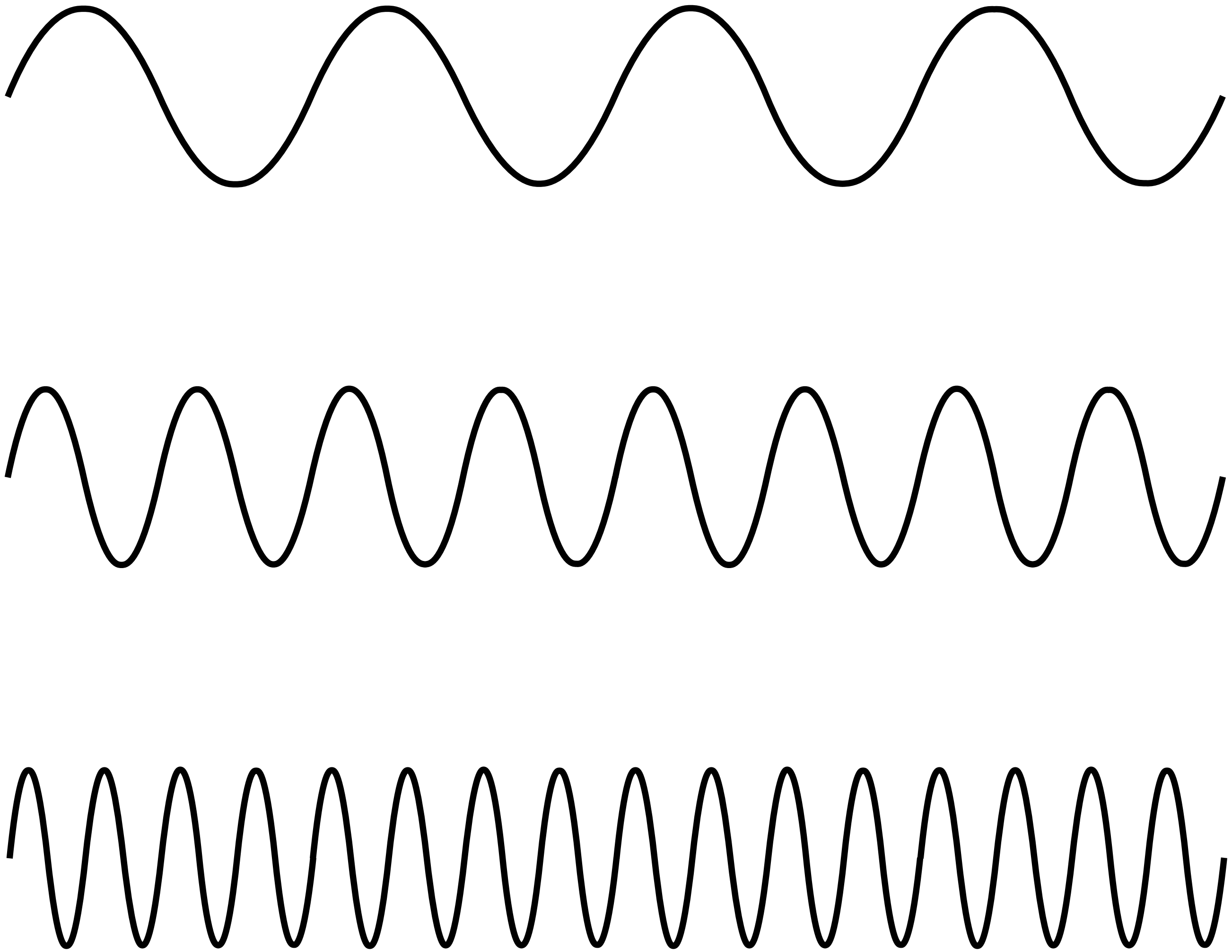
Language of numbers to the Language of frequencies



Large time period

Small time period

Language of numbers to the Language of frequencies



Large Wavelength

Small Wavelength

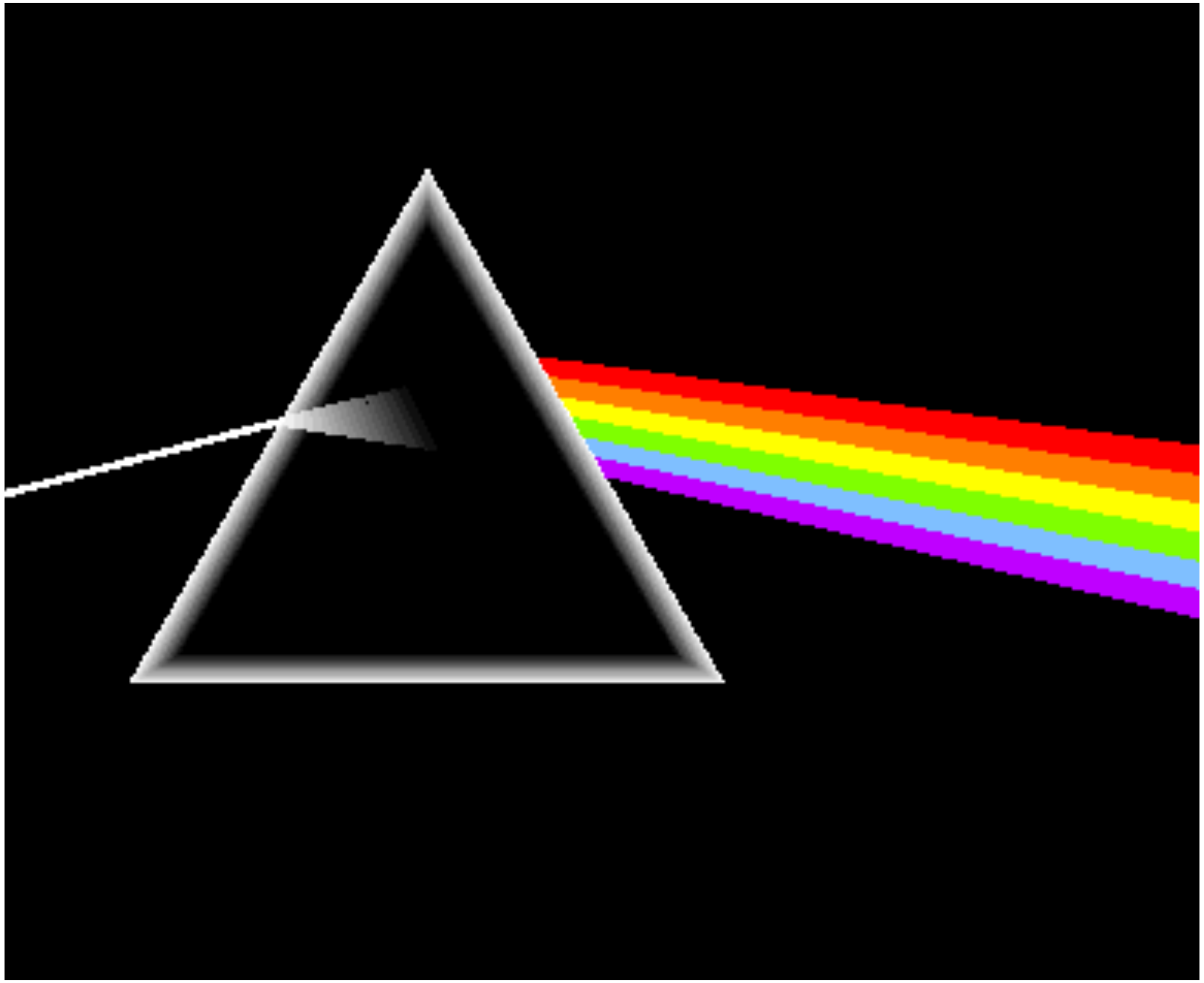
Fourier Analysis

Three equivalent names:

1. Spectral Methods
2. Harmonic Analysis
3. Fourier Analysis

Fourier was the first to realize that “ALL” functions can be expressed through linear combinations of trigonometric functions

The spectral approach



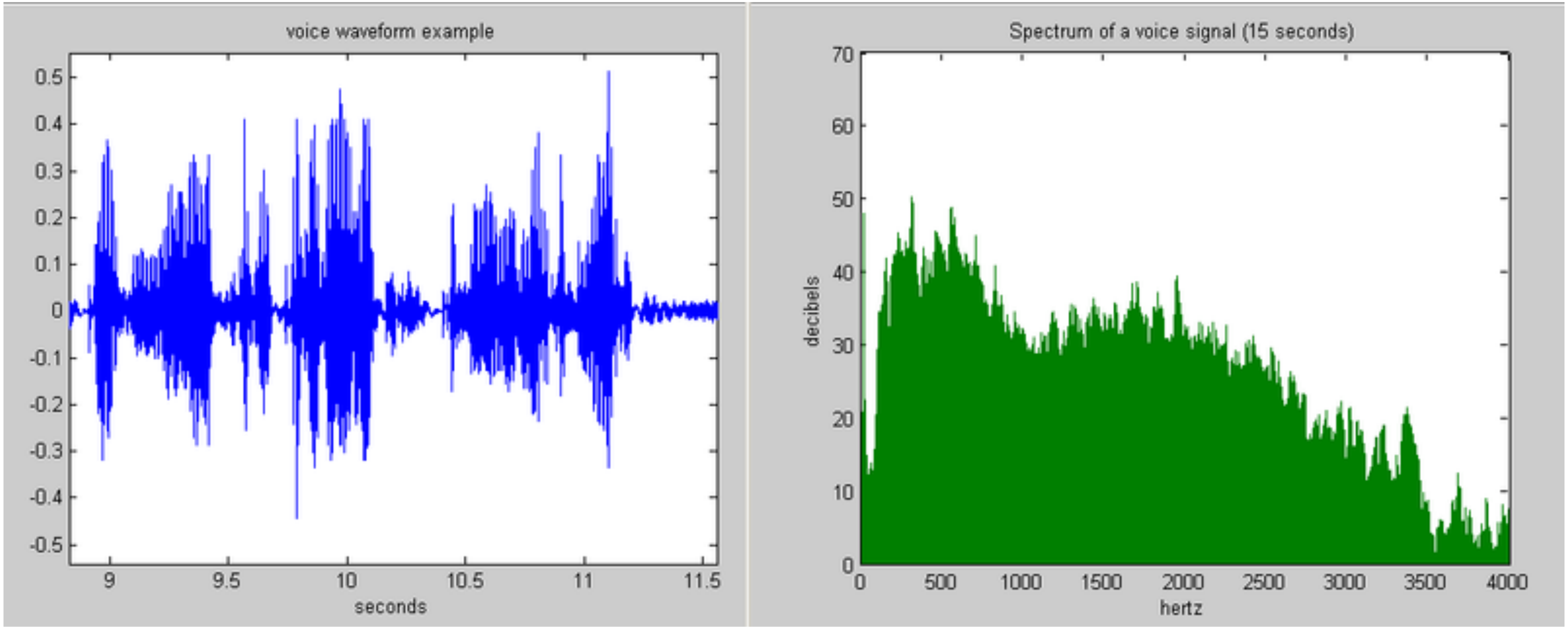
The (white) sunlight is composed of components having different wavelengths.

Refraction of each component depends on its wavelength.

Prism is like a spectrum analyzer!

Harmonic Analysis (Musical harmony and frequencies)

We think of music in terms of frequencies (harmonics) at different magnitudes



Fourier Series

Fourier Series

A periodic waveform *f(t)* could be broken down into an *infinite series of simple sinusoids* which, when added together, would construct the *exact form* of the original waveform.

Consider the periodic function

$$f(t) = f(t+nT) \quad ; \; n = \pm 1, \pm 2, \pm 3, \ldots$$

Fourier proposed in 1807

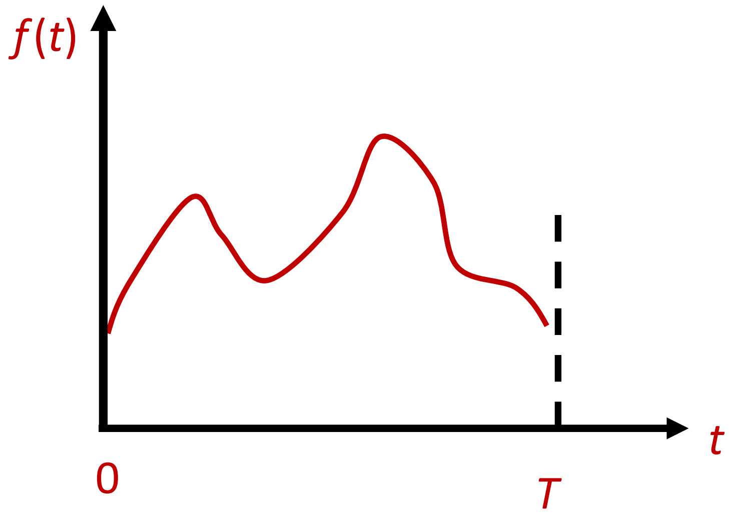
T *is* Period, the smallest value of T that satisfies the above expression.

Fourier Series

The expression for a ***Fourier Series*** is

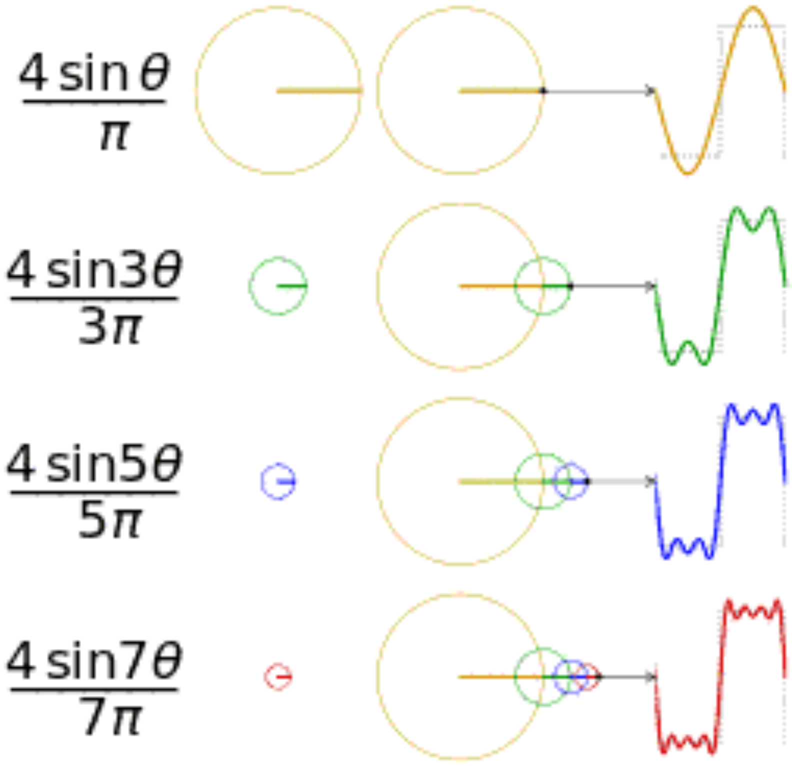
$$\omega_0 = \frac{2\pi}{T}$$

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos n\omega_0 t + \sum_{n=1}^{N} b_n \sin n\omega_0 t$$



a_0, a_n, and b_n are real and are called

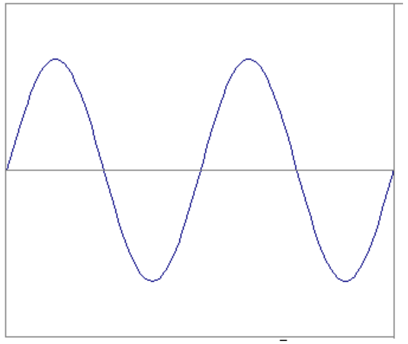
Fourier Trigonometric Coefficients



Fourier Series = a finite (very large N) sum of harmonically related sinusoids

Time and Frequency spectra

$$f(t) = \sin(t) + \frac{1}{3}\sin(3\,t)$$



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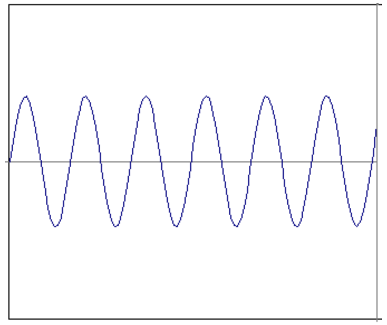
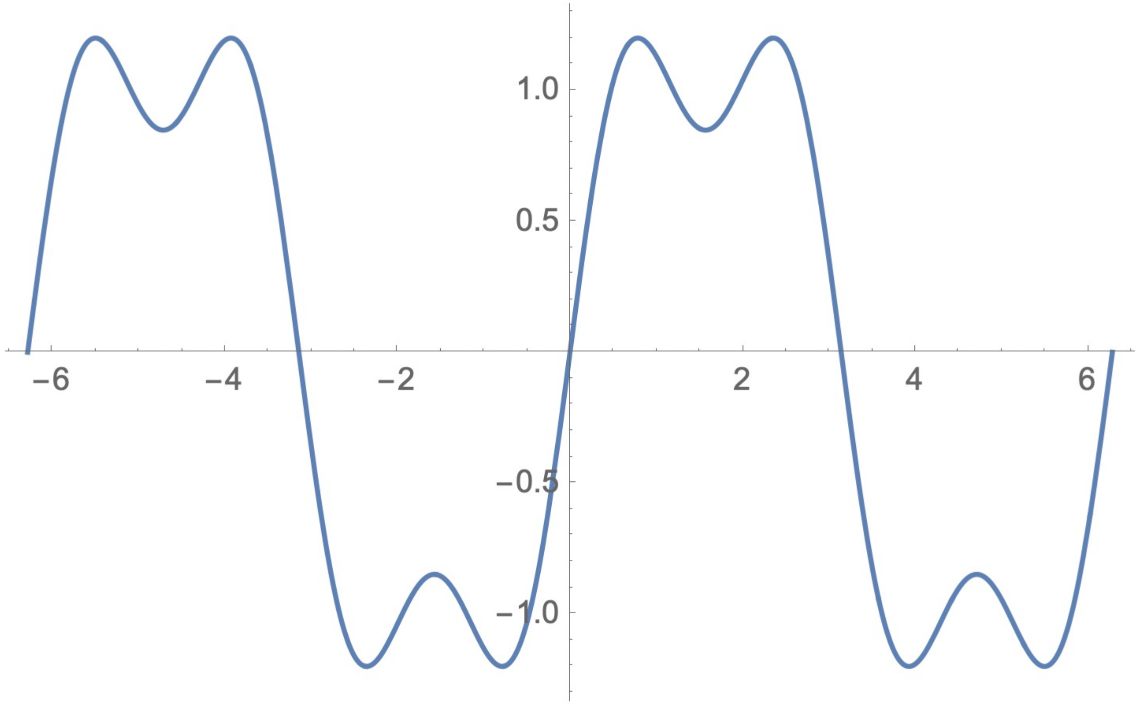


In[3]:= `F[t_] = 4 / Pi * (Sin[t] + Sin[3 * t] / 3)`

Out[3]= $\dfrac{4\left(\mathrm{Sin}[t] + \frac{1}{3}\,\mathrm{Sin}[3\,t]\right)}{\pi}$

In[4]:= `Plot[F[t], {t, -2 * Pi, 2 * Pi}]`

Out[4]=

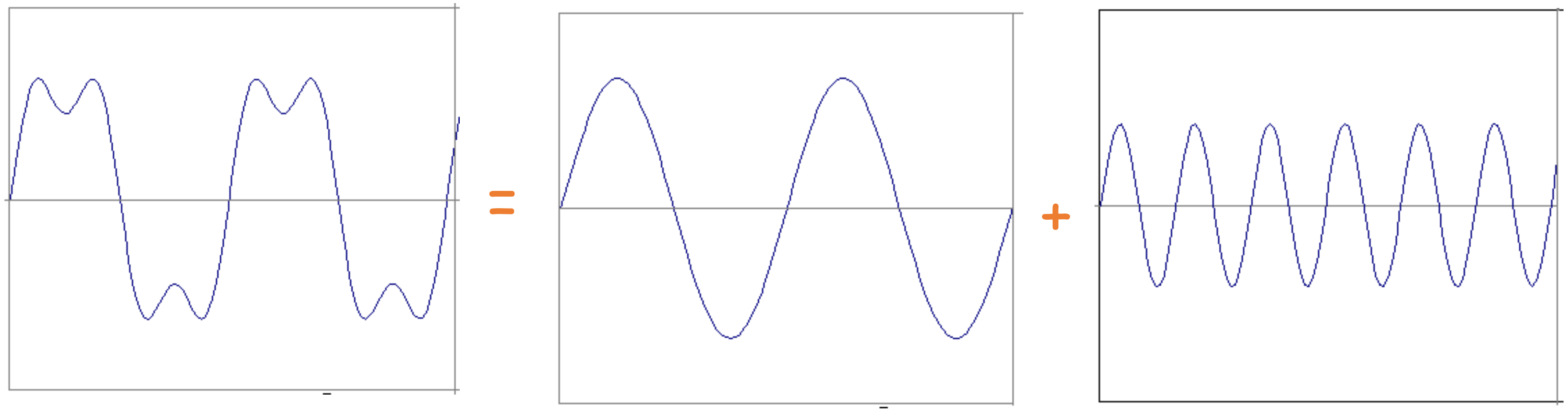


Resultant signal is a mixture of two frequencies

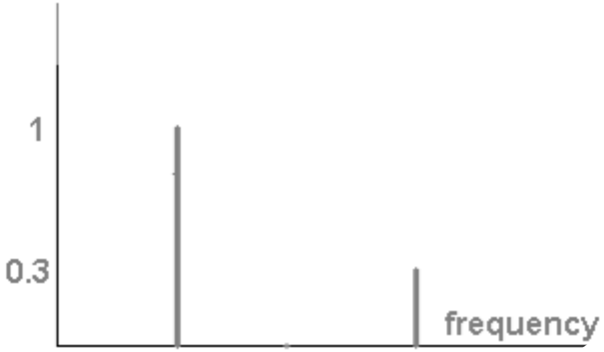
Frequency Spectra

Example: $f(t) = \sin(2\,\pi\,s\,t) + \frac{1}{3}\sin(2\,\pi\,(3\,s)t)$

In time domain



In frequency domain

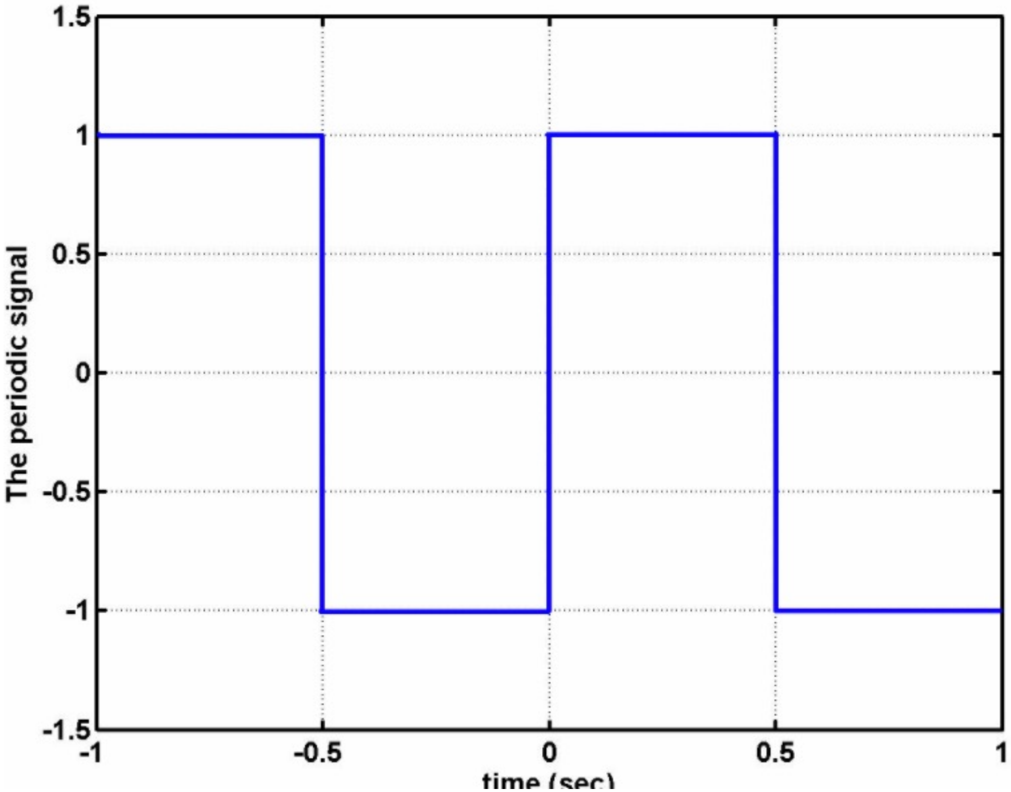


Frequency spectrum of a signal is the range of frequencies contained by a signal.

This signal contains only two frequencies.

Frequency Spectra

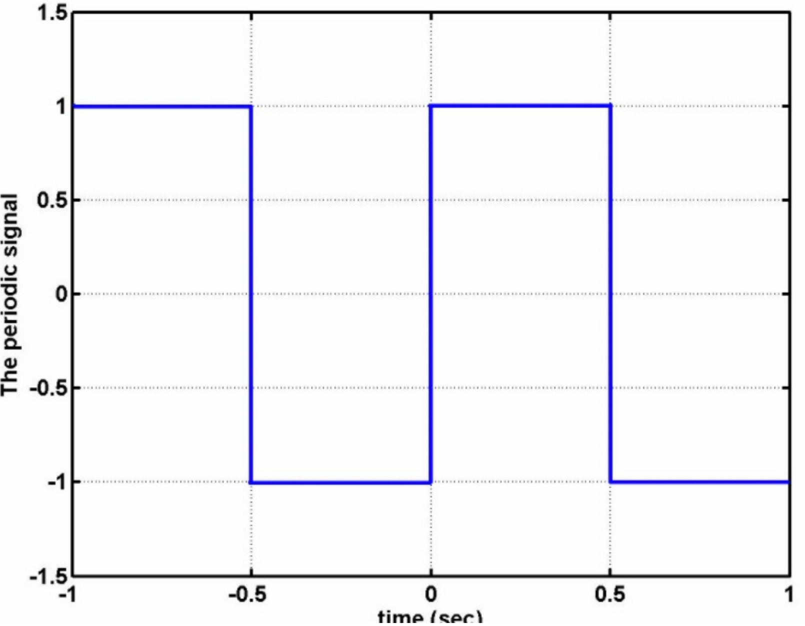
- Usually, frequency is more interesting than the phase
- Let us consider the following periodic signal --- Square wave
- This is a periodic signal with a definite time period (T) and it is continuous.



1. We will decompose it to frequencies,

2. In each slide, we will add one more base function and see how close is the Fourier series to the signal.

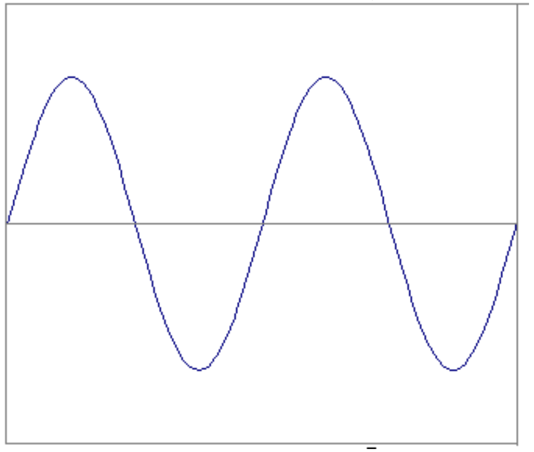
Frequency Spectra: Step 1



First base function

second base function

=



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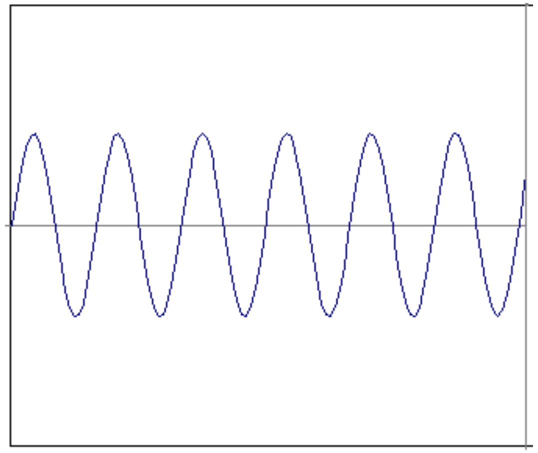

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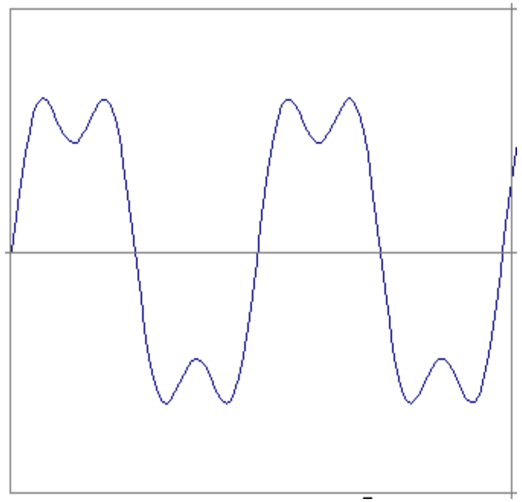
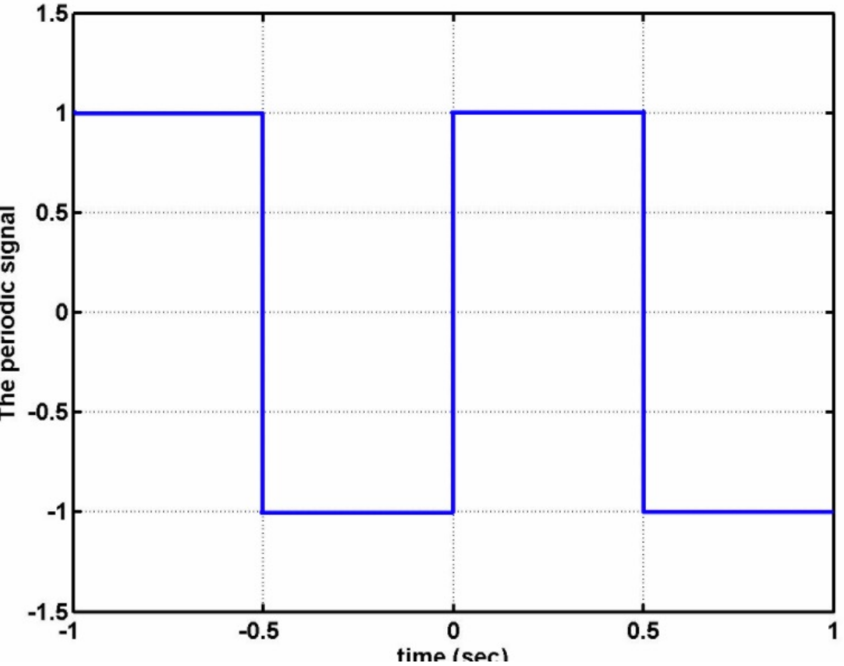


$$\sin(2\,\pi\,s\,t) + \frac{1}{3}\sin(2\,\pi\,(3\,s)t)$$

Sum of first and second base functions is already not bad approximation

Frequency Spectra: Step 2

Sum of first , second and third base functions is even better approximation



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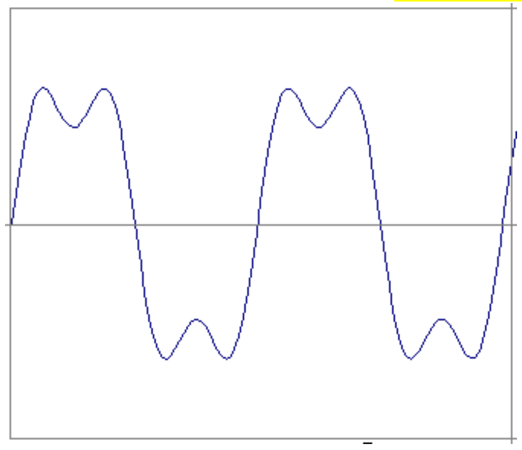

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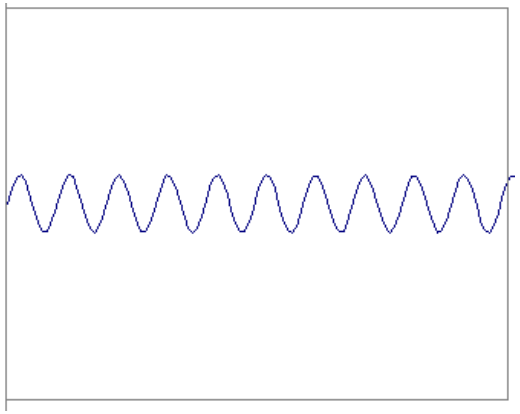

$$\sin(2\,\pi\,s\,t) + \frac{1}{3}\sin(2\,\pi\,(3\,s)t) + \frac{1}{5}\sin(2\,\pi\,(5\,s)t)$$

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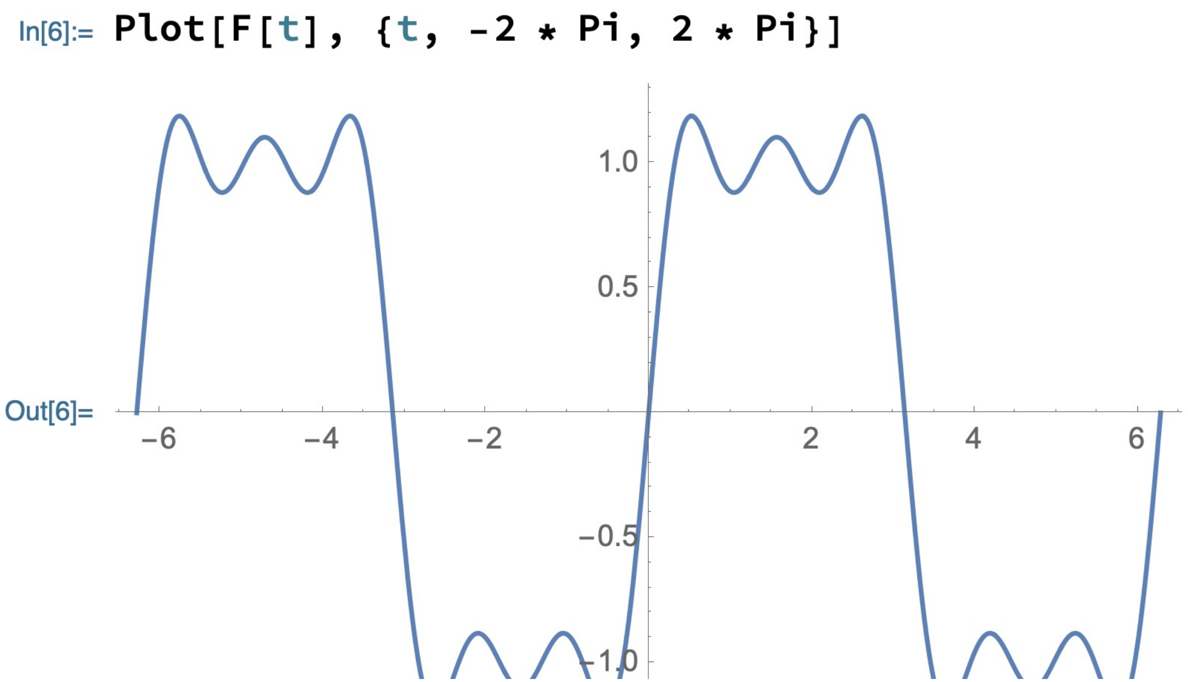
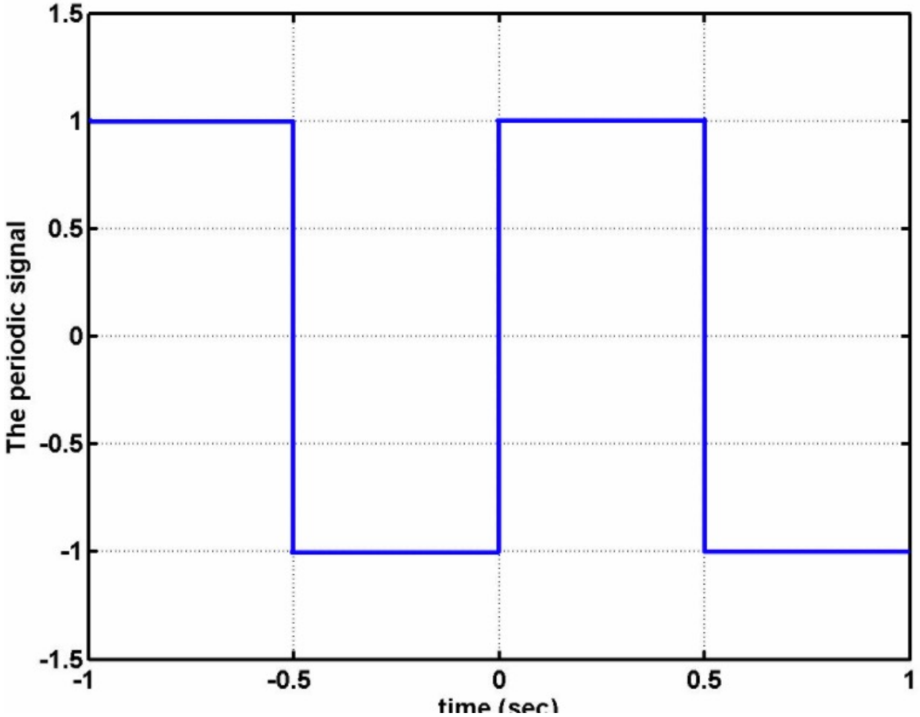


Frequency Spectra: Step 3

Sum of first , second third and fourth base functions is much better approximation



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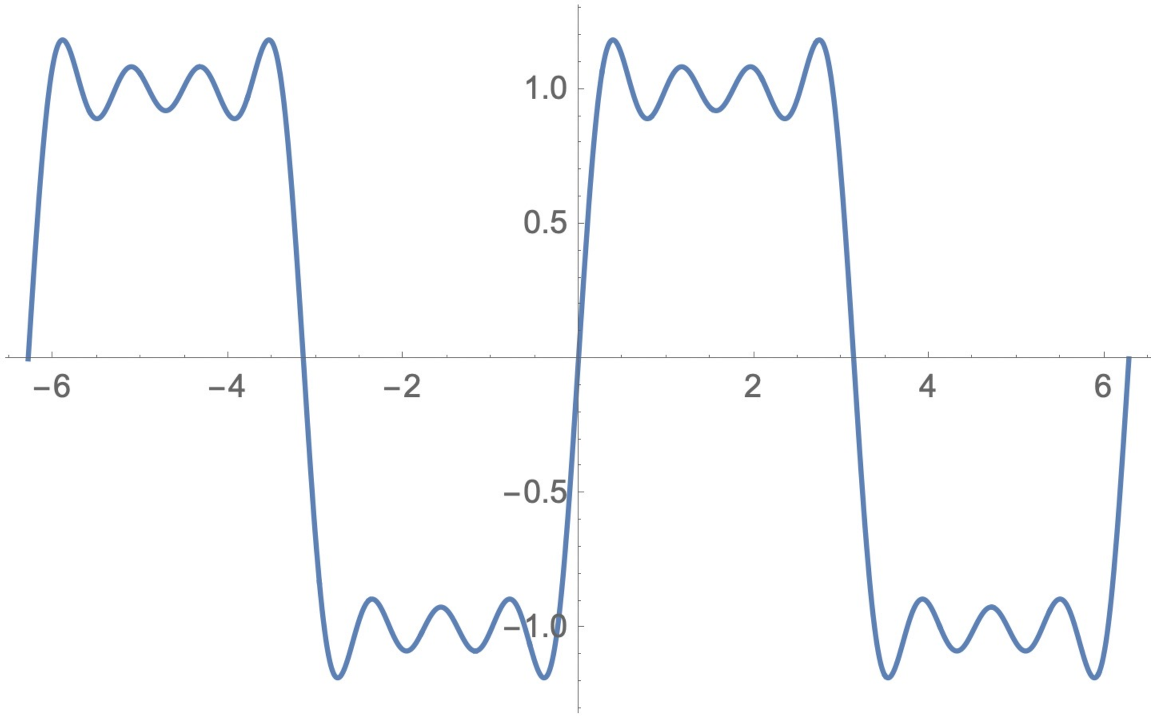
```
In[7]:= F[t_] =
  4 / Pi * (Sin[t] + Sin[3 * t] / 3 + Sin[5 * t] / 5 +
```

Out[7]= $\dfrac{4\left(\text{Sin}[t]+\frac{1}{3}\,\text{Sin}[3\,t]+\frac{1}{5}\,\text{Sin}[5\,t]+\frac{1}{7}\,\text{Sin}[7\,t]\right)}{\pi}$

```
In[9]:= Plot[F[t], {t, -2 * Pi, 2 * Pi}]
```

Out[9]=

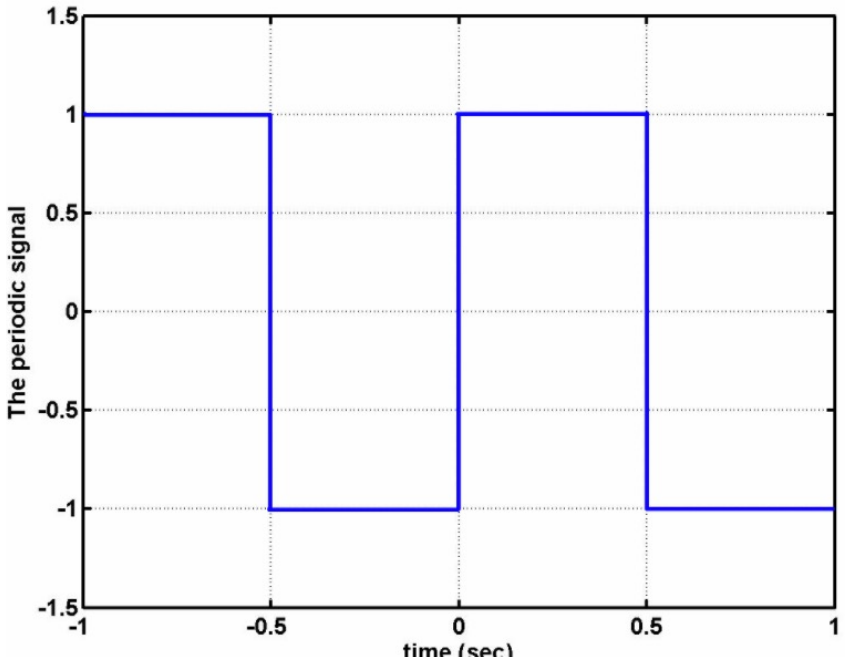


$$\sin(2\,\pi\;s\;t)+\frac{1}{3}\;\sin(2\,\pi\;(3\;s)t)$$
$$+\;\frac{1}{5}\sin(2\,\pi\;(5\;s)t)++\;\frac{1}{7}$$
$$\sin(2\,\pi\;(7\;s)t)$$

=

Frequency Spectra: Step 4

Sum of first 5 base functions makes slight changes



$$\sin(2\,\pi\,s\,t) + \frac{1}{3}\sin(2\,\pi\,(3\,s)t) + \frac{1}{5}\sin(2\,\pi\,(5\,s)t) + \frac{1}{7}\sin(2\,\pi\,(7\,s)t) + \frac{1}{9}\sin(2\,\pi\,(9\,s)t)$$

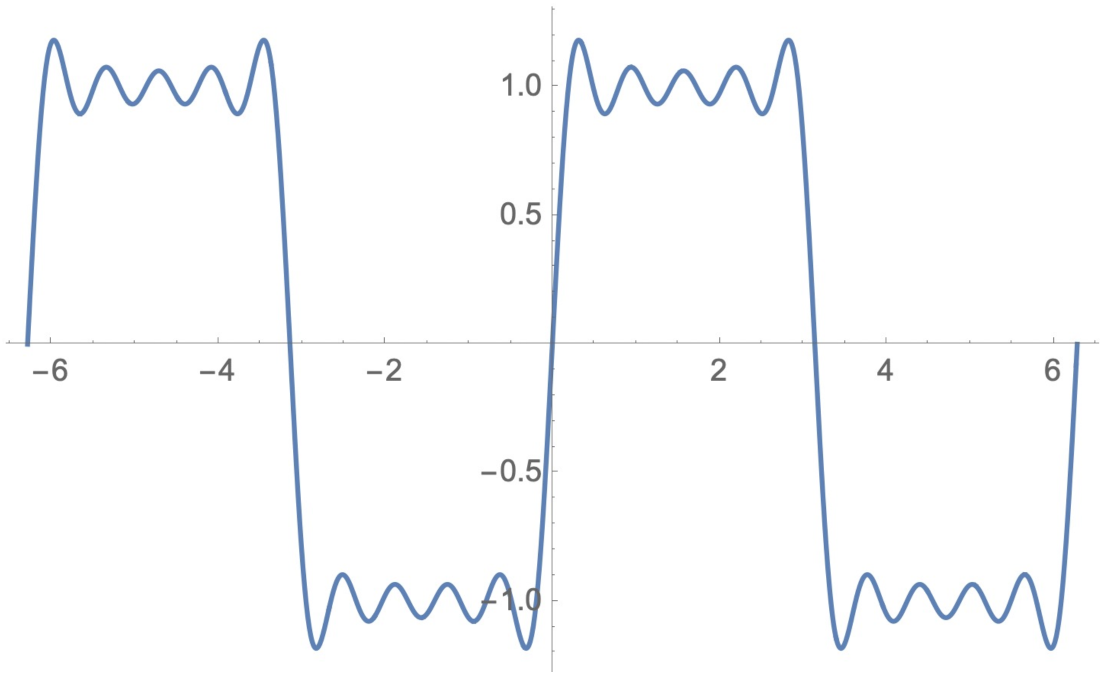
=

Out[10]=

$$\frac{4\left(\text{Sin}[t] + \frac{1}{3}\text{Sin}[3\,t] + \frac{1}{5}\text{Sin}[5\,t] + \frac{1}{7}\text{Sin}[7\,t] + \frac{1}{9}\text{Sin}[9\,t]\right)}{\pi}$$

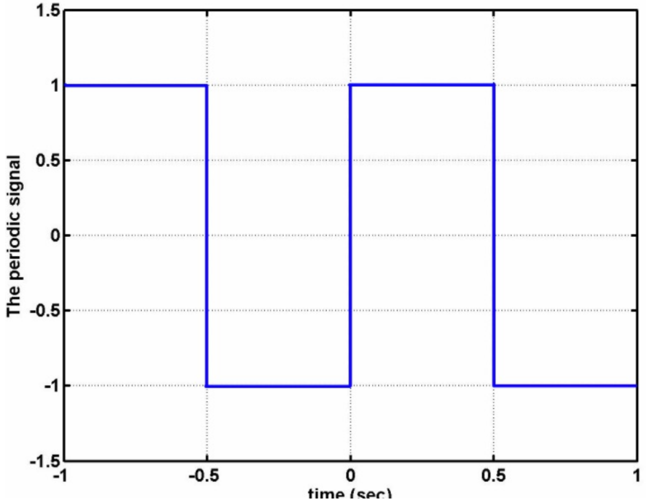
In[11]:= `Plot[F[t], {t, -2 * Pi, 2 * Pi}]`

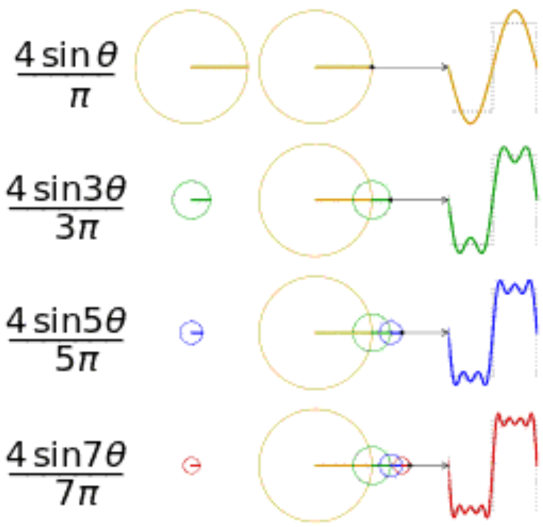
Out[11]=



Frequency Spectra: Step 5

Sum of first 6 base functions makes slight changes





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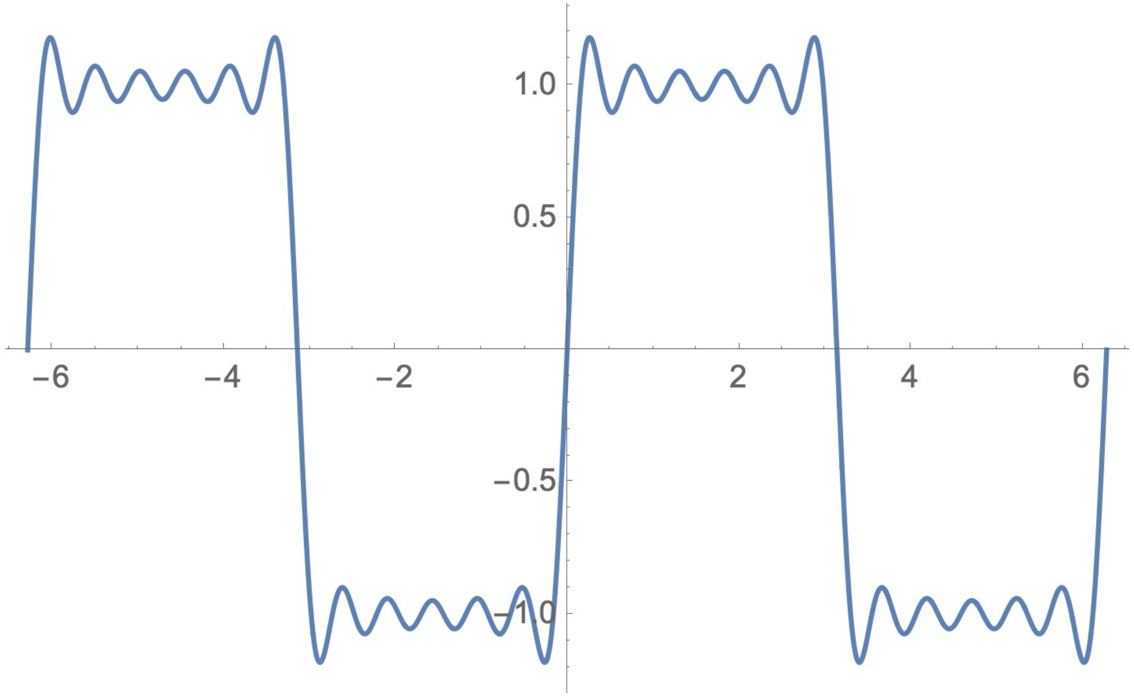
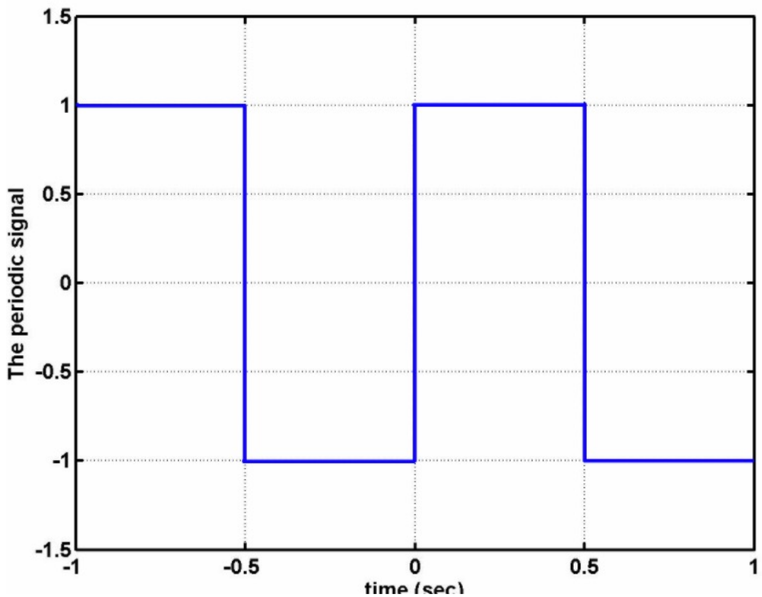
$$\frac{1}{\pi}4\left(\text{Sin}[t] + \frac{1}{3}\text{Sin}[3\,t] + \frac{1}{5}\text{Sin}[5\,t] + \frac{1}{7}\text{Sin}[7\,t] + \frac{1}{9}\text{Sin}[9\,t] + \frac{1}{11}\text{Sin}[11\,t]\right)$$

```
Plot[F[t], {t, -2 * Pi, 2 * Pi}]
```

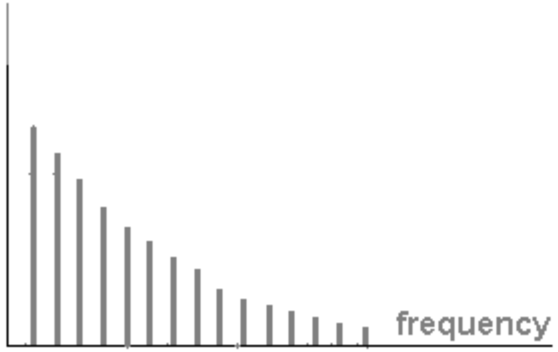


Frequency Spectra



$=$

$$\boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$

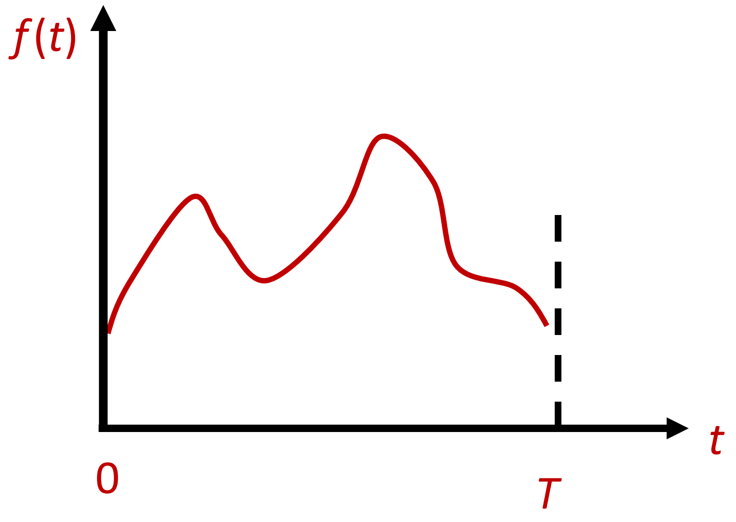


In time domain
(Continuous)

In frequency domain
(Discrete)

Finding Fourier Series for a function

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos n\omega_0 t + \sum_{n=1}^{N} b_n \sin n\omega_0 t$$



Given a function *f(t)*, we can find coefficients (a_n, b_n) that mimic the function.

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cos\left(\frac{2\pi}{T}nt\right) dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \sin\left(\frac{2\pi}{T}nt\right) dt$$

Fourier Series via Linear algebra

1. Looks at functions over an interval as a infinite dimenstional vector space with an inner product

   In the interval [−π,π], we define a vector space V where the scalars are taken from $\mathbb{R}$ and the vectors are functions over the interval [−π,π]

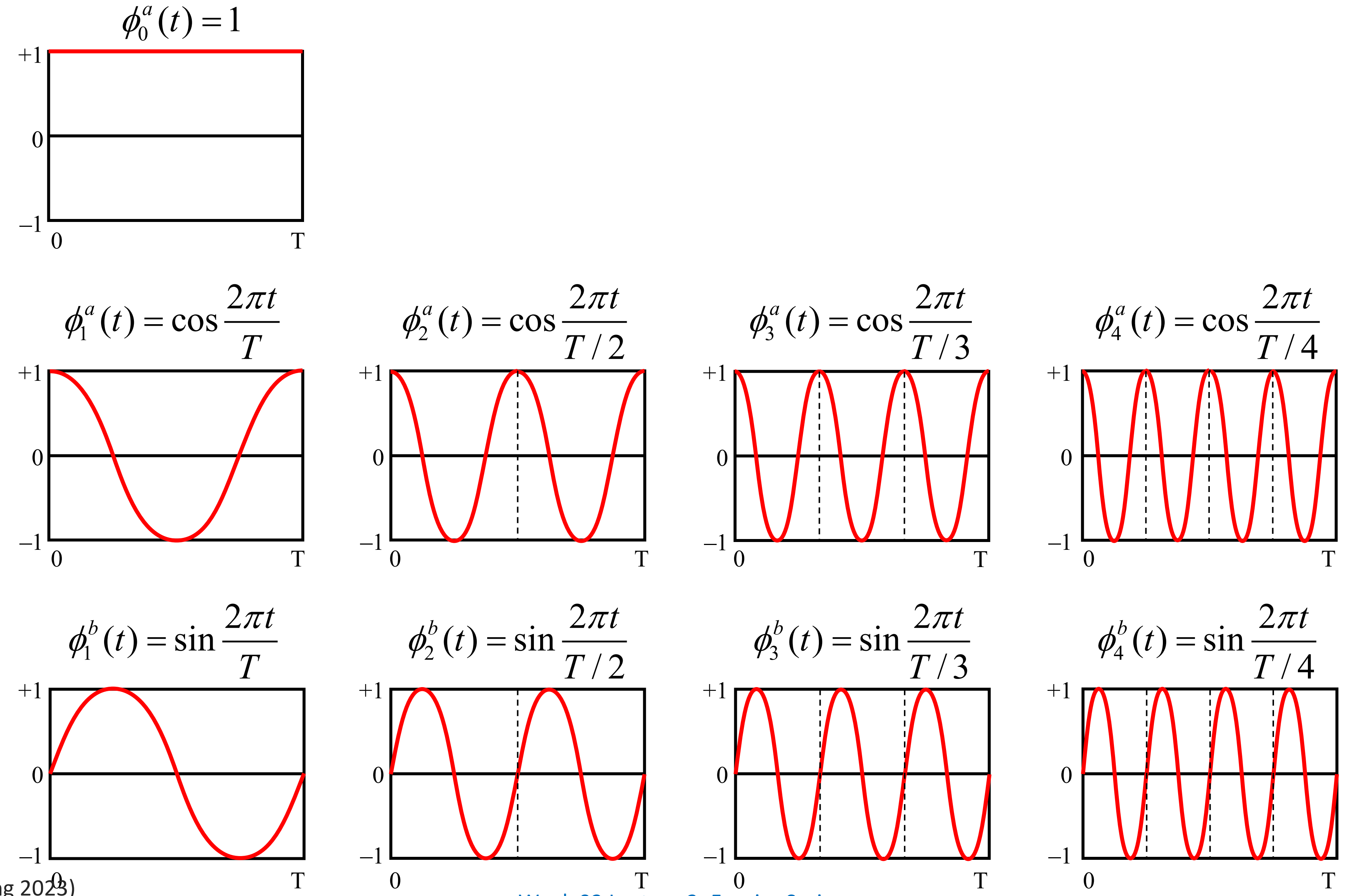
2. Picks an orthonormal basis for the space

$$\sin(t), \sin(2t), \cdots \sin(nt), \cos(t), \cos(2t), \cdots \cos(nt)$$

3. Represents an arbitrary function in this basis by projecting it out on the basis.

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos n\omega_0 t + \sum_{n=1}^{N} b_n \sin n\omega_0 t$$

Base functions of Fourier series



Development of a real function $f(t)$ defined in the interval $[0,T]$ into Fourier series

3 alternative forms:

$$f(t) = a_o + \sum_{k=1}^{\infty}[a_k \boxed{\cos\frac{2\pi kt}{T}} + b_k \boxed{\sin\frac{2\pi kt}{T}}]$$

$$f(t) = a_o + \sum_{k=1}^{\infty}[a_k \boxed{\cos(2\pi s_k t)} + b_k \boxed{\sin(2\pi s_k t)}]$$

$$f(t) = a_o + \sum_{k=1}^{\infty}[a_k \boxed{\cos\omega_k t} + b_k \boxed{\sin\omega_k t}]$$

Every base function has:

period: $\frac{T}{k}$

frequency: $s_k = \frac{k}{T}$

angular frequency:

$$\omega_k = 2\pi s_k = \frac{2\pi k}{T}$$

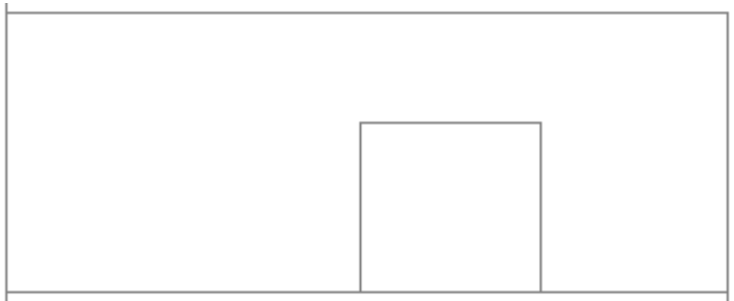
fundamental period	fundamental frequency	fundamental angular frequency
T	$s_T = 1/T$	$\omega_T = 2\pi\, s_T = 2\pi / T$

term periods	term frequencies	term angular frequencies
$T_k = T/k$	$s_k = k\, s_T$	$\omega_k = k\omega_T$

Fourier transform

Fourier Transform

To understand the frequency ω distribution of any continuous signal that is NOT periodic.



$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < T \\ 0 & t > T \end{cases}$$

- Unlike in Fourier Series T does not have importance

$\Longrightarrow$ There is no fundamental frequency s_T

- Hence, term frequencies $s_k = k\, s_T$ do not have importance

$\Longrightarrow$ We need to consider all frequencies!

- In Fourier domain, the signal is not discrete and consists of all frequencies!

f(t) $\longrightarrow$ Fourier Transform $\longrightarrow$ ***F(ω)***

Continuous Function of t

Continuous Function of ω

Fourier Transform (FT): Formal definition

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,2\,\pi\omega\,t}\, dt$$

where $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Given $F(\omega)$, we can obtain $f(t)$ using the Inverse Fourier Transform (IFT):

$$f(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\,2\,\pi\omega\,t}\, d\omega$$

Reason for the prefactor $\frac{1}{2\,\pi}$:

1. FT transforms from time [s] to ω [rad/s], not [Hz = 1/s]!

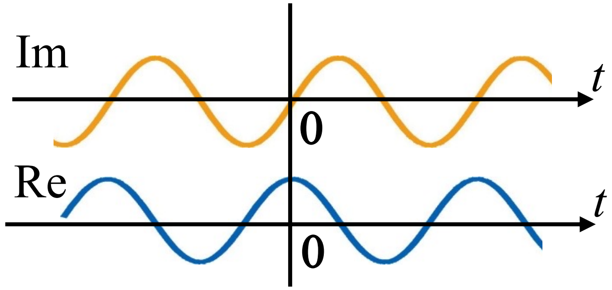
2. It will occur when angular frequencies are used and is not a generic feature of the Fourier transform itself.

Example 1: Fourier transform of $exp(i\,\omega_0 t)$

$$\mathrm{F}\ \{\exp(i\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0\, t)\exp(-i\,\omega t)\, dt$$

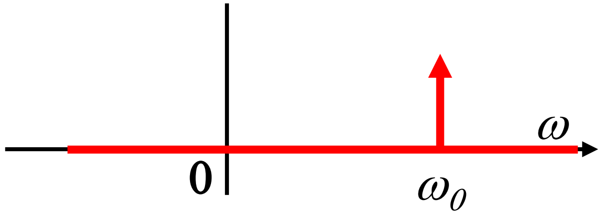
$$= \int_{-\infty}^{\infty} \exp(-i\,[\omega - \omega_0\,]t)\, dt$$

$\exp(i\omega_0 t)$



In the frequency space, the signal has one value at $\omega = \omega_0$.

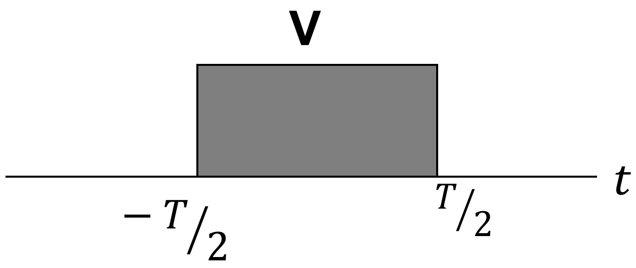
$\mathscr{F}\{\exp(i\omega_0 t)\}$



Fourier transform provides information about the signal.

Example 2: Rectangular Signal

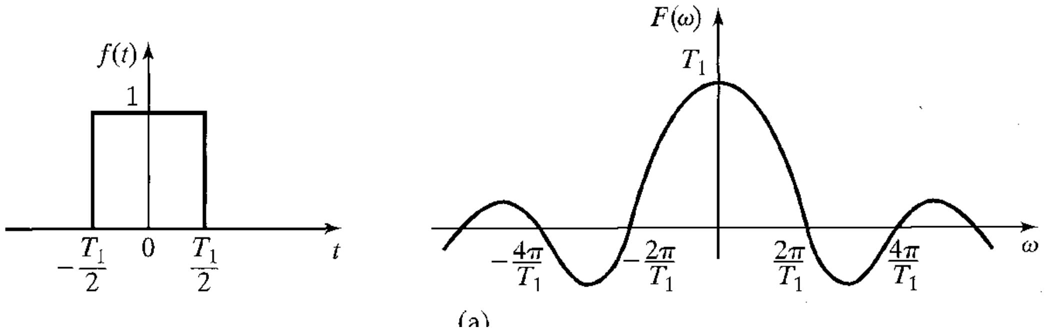
Consider an aperiodic rectangular pulse of T seconds evenly distributed about t=0.



$$f(t) = \begin{cases} V & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,2\,\pi\omega\,t}\, dt$$

$$F(\omega) = 2\frac{\sin(\omega T/2)}{\omega}$$



Diffraction of light shows has such a signal.

All physically realizable signals have Fourier Transforms

Fourier transform: Space and Time

Space

x Space variable

L Spatial wavelength

$k = \frac{2\pi}{\lambda}$ Spatial wavenumber

$F(k)$ wavenumber spectrum

Time

t Time variable

T Time period

$\omega = 2\pi s$ angular frequency

$F(\omega)$ frequency spectrum

Fourier Integrals

With the complex representation of sinusoidal functions e^{ikx} (or e^{iwt}) the Fourier transformation can be written as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{-ikx}\, dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

Another way of writing the prefactor $\frac{1}{2\pi}$: Distribute it to both Fourier and Inverse Fourier

The Fourier Transform: discrete vs. continuous

Whatever we do on the computer (with data) will be based on the discrete Fourier transform

continuous

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{-i k x}\, dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i k x}\, dx$$

discrete

$$F_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-2\pi i k j / N}, k = 0, 1, \ldots, N-1$$

$$f_k = \sum_{j=0}^{N-1} F_j e^{2\pi i k j / N}, k = 0, 1, \ldots, N-1$$

Fourier transform is a change of basis

Key Takeaway

Summary

- Fourier Series

  “ALL” functions can be expressed through linear combinations of trigonometric functions

- Fourier transform is a different representation of a function
  - time vs. frequency
  - position vs. wave number

We will look at the applications to Matter waves in Next lecture.

Recommended Readings

Wave Groups and Dispersion,

section 5.3 in page 164.