PH 112: Quantum Physics and Applications

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Week 02 Lecture 1 Wave Packet, Phase Velocity and Group Velocity

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Quantum particle according to de Broglie

- de Broglie: If waves can mimic particles, then particles can mimic waves

$$\lambda = \frac{h}{m\,v} = \frac{h}{p}$$

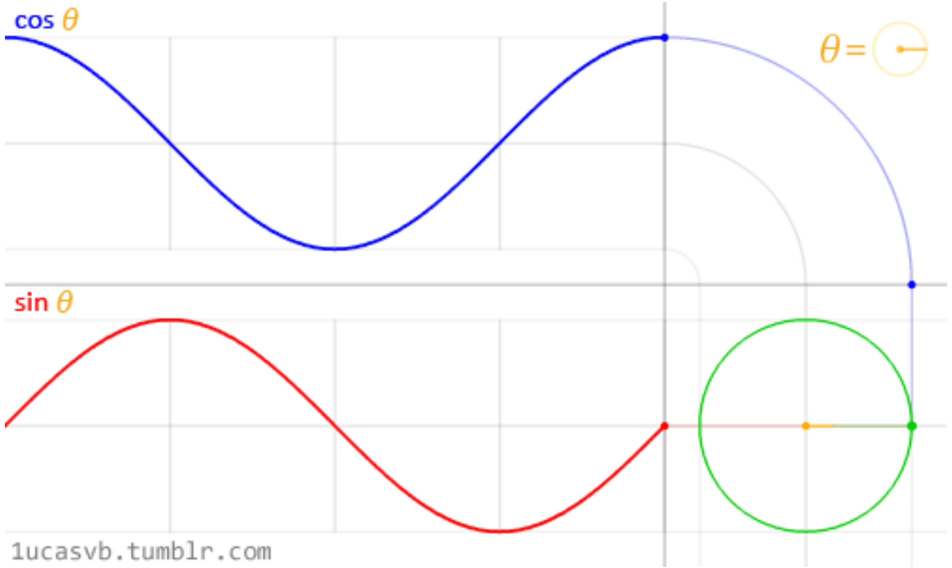
- The quantum particle is a new model that is a result of the recognition of the dual nature.

- Entities have both particle and wave characteristics.

- To understand a particular phenomenon, we must choose an appropriate characteristic .

Ideal Wave vs Ideal Particle

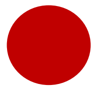
- An ideal wave has a single frequency and is infinitely long.



- Therefore, it is unlocalized in space.

- An ideal particle has zero size

- Therefore, it is localized in space



Describing a entity with both particle and wave characteristics, we need more understanding!

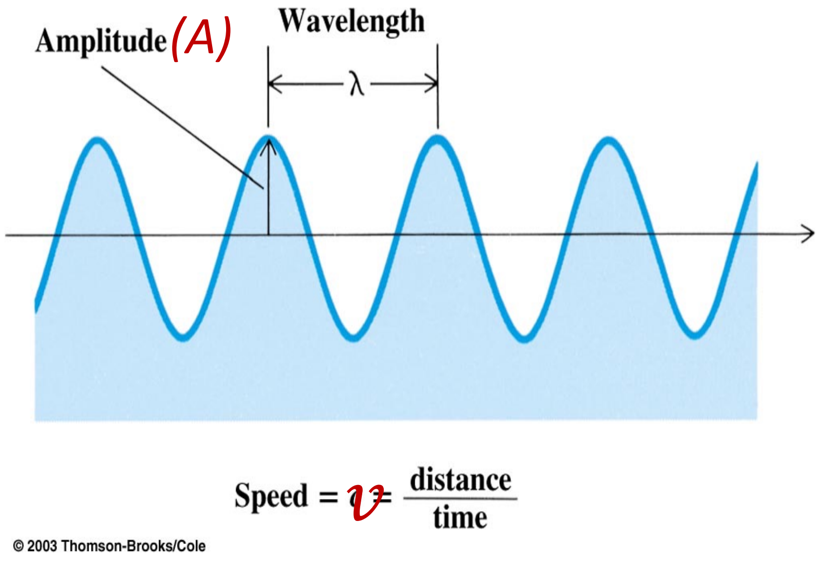
Wave motion

Description of classical waves

- Classical waves and light are characterized by amplitude, wavelength, and speed.

- Displacement of a single, free wave traveling to the right is represented by

$$\Psi(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

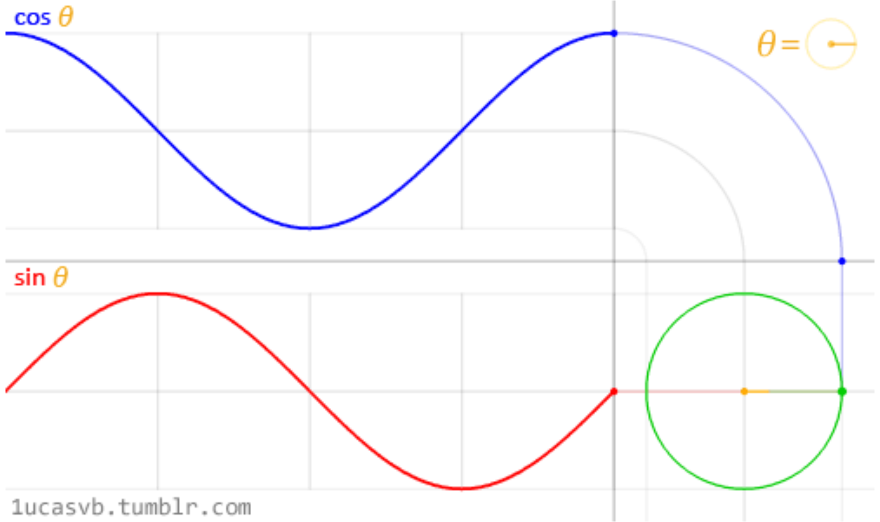


- This is a solution to the 1-D classical wave equation
  Any wave needs to be a solution to this equation.

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

- For classical description, this differential equation can be derived from Newton's laws.

Description of classical waves



- Since the wave has a periodicity (T), we can describe the wave by associating with SHM.

$$\omega = \frac{2\pi}{T}$$

Angular frequency

$$k \equiv \frac{2\pi}{\lambda}$$

Wave-number

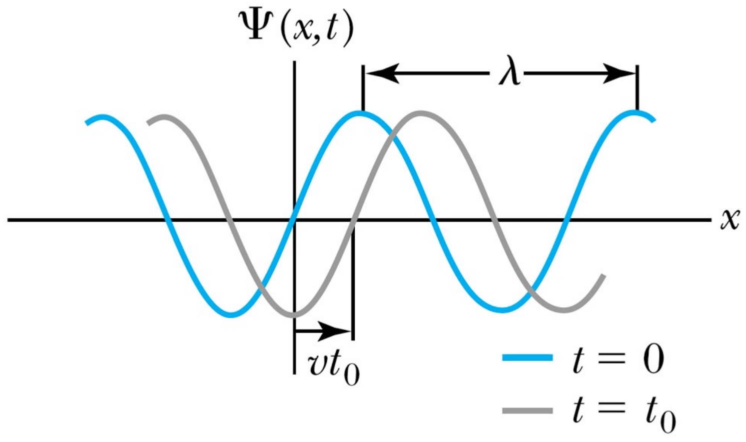
- Phase velocity is the velocity of a point on the wave that has a given phase Example: crest of the wave

$$v_{ph} = \frac{\lambda}{T} = \frac{\omega}{k}$$

$$\omega = 2\pi f$$

- Phase Shift: How far the function is shifted **horizontally** (ϕ) from the usual position.

$$\Psi(x,t) = A\sin(kx - \omega t + \phi)$$



Solutions of the wave equation

- Consider 1-D Wave equation $$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

Right traveling $\Psi_{\mathrm{R}}(x,t) = A\ Sin(kx\ - \omega t)$

- Two Independent solutions:

Left traveling $\Psi_{\mathrm{L}}(x,t) = B\ Sin(kx + \ \omega t)$

- Interestingly, the sum of the above two solutions is also a solution!

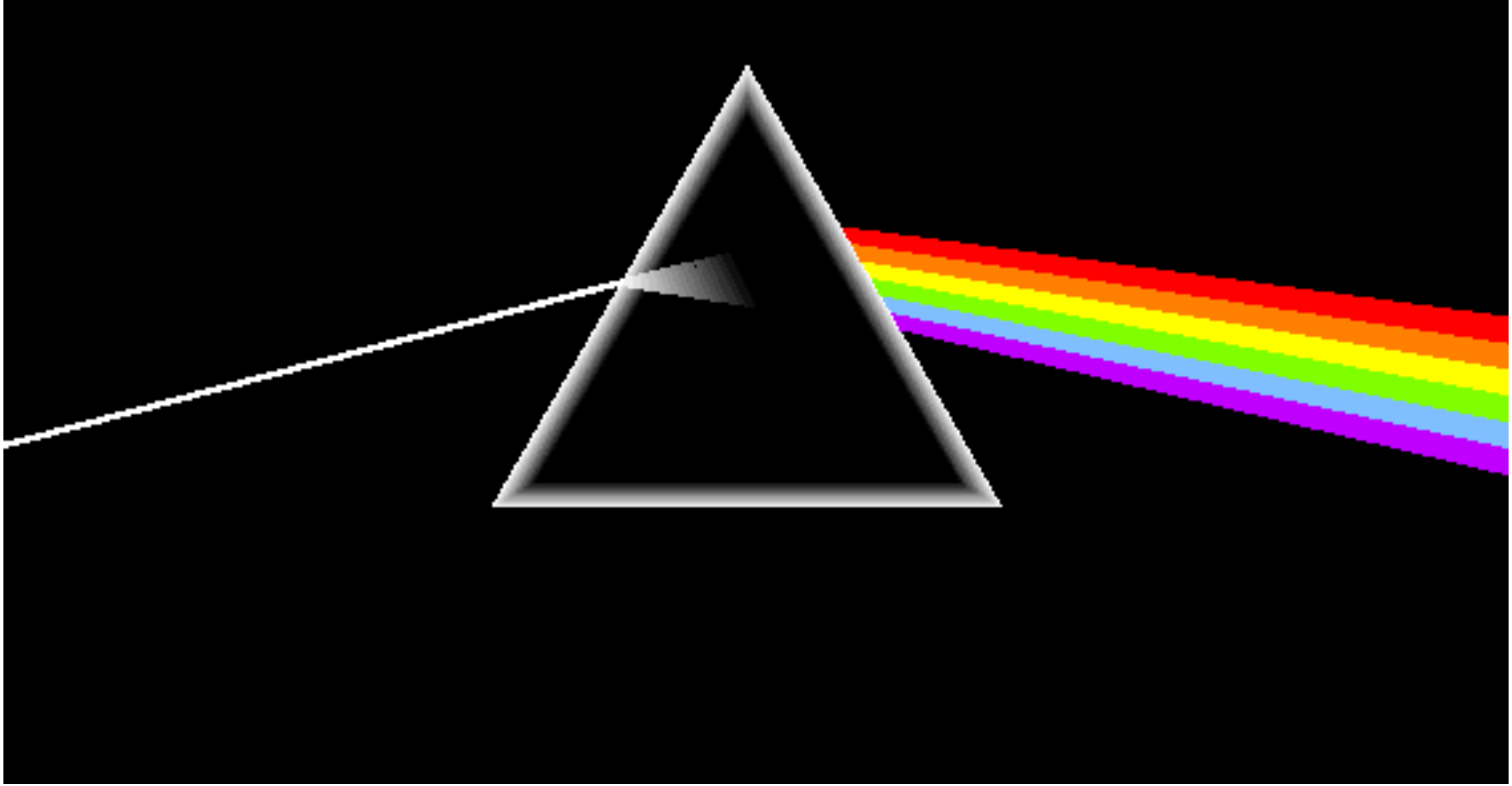
$$\Psi_{L+R}(x,t) = A\ Sin(kx\ - \omega t) + B\ Cos(kx\ - \omega t)$$

- Wave equation is a linear equation, the sum of two solutions will also be a solution.

- This leads to an important concept Principle of superposition of waves

Superposition of Waves

Light spectrum



Credit: http://scv.bu.edu/~aarondf/avgal.html

Example 1: Adding left and right moving waves

$$\cos(kx - \omega t) = \cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t)$$

$$\cos(kx + \omega t) = \cos(kx)\cos(\omega t) - \sin(kx)\sin(\omega t)$$

$$\cos(kx + \omega t) + \cos(kx - \omega t)$$

$$= \cos(\omega t)\cos(kx) - \sin(\omega t)\sin(kx) + \cos(\omega t)\cos(kx) + \sin(\omega t)\sin(kx)$$

$$= 2\cos(\omega t)\cos(kx) = A(t)\cos(kx)$$

Standing wave

a wave whose amplitude is always changing but that does not travel either way

Can think of standing waves as sums of left traveling and right traveling waves

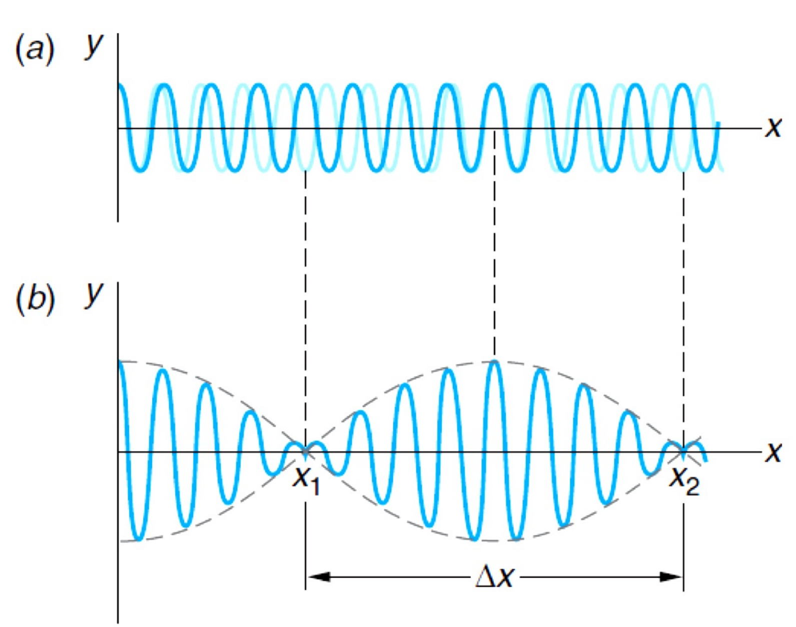
Example 2: Phenomena of beats, two superimposed waves

$$y(x,t) = y_0\cos(k_1x - \omega_1 t) + y_0\cos(k_2x - \omega_2 t) \longrightarrow y(x,t)) = 2y_0\cos\left(\frac{1}{2}\Delta kx - \frac{1}{2}\Delta\omega t\right)\cos(\bar{k}x - \bar{\omega}t)$$

$\bar{k} = (k_1 + k_2)/2$ $\quad$ $\bar{\omega} = (\omega_1 + \omega_2)/2$ $\quad$ Perceived freq. of listener

product of **two** travelling waves

$\Delta k = k_2 - k_1$ $\quad$ $\Delta\omega = \omega_2 - \omega_1$ $\quad$ Controls the amplitude of the wave.



We can define two velocities $\quad v_p = \dfrac{\bar{\omega}}{\bar{k}} \qquad v_g = \dfrac{\Delta\omega}{\Delta k}$

$$v_p = \frac{(\omega_1 + \omega_2)/2}{(k_1 + k_2)/2} \approx \frac{\omega_1}{k_1} = v_1$$

$$v_g = \frac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \frac{\Delta\omega}{\Delta k}$$

Think of a Siren

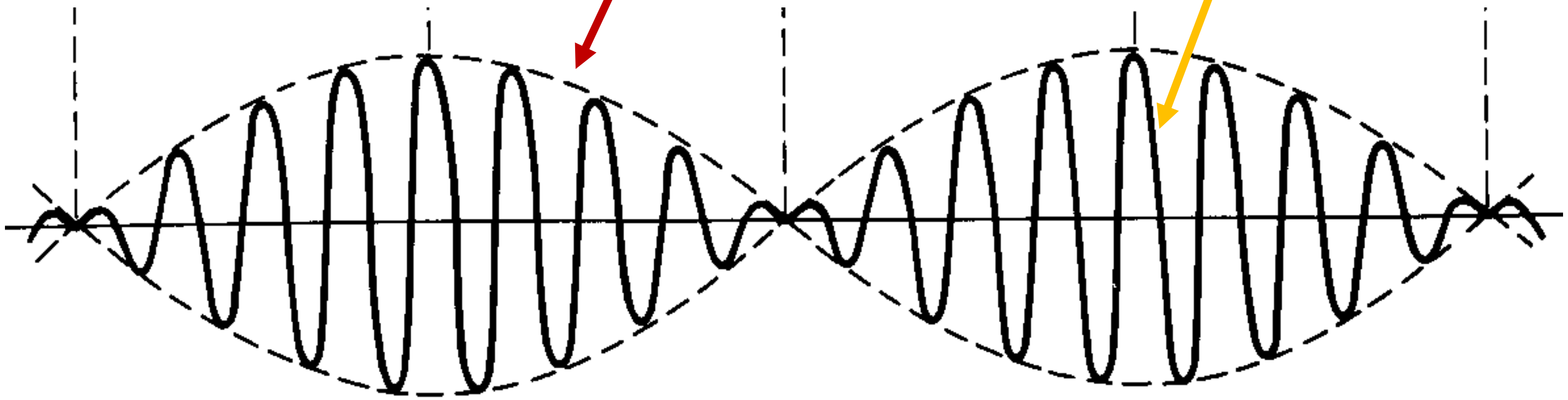
$$y = y_1 + y_2 = \left\{2A\cos\frac{1}{2}(\Delta kx - \Delta\omega t)\right\}\cdot\cos\left\{k_p x - \omega_p t\right\}$$

\`envelop' (group waves)
Also referred 'modulation'

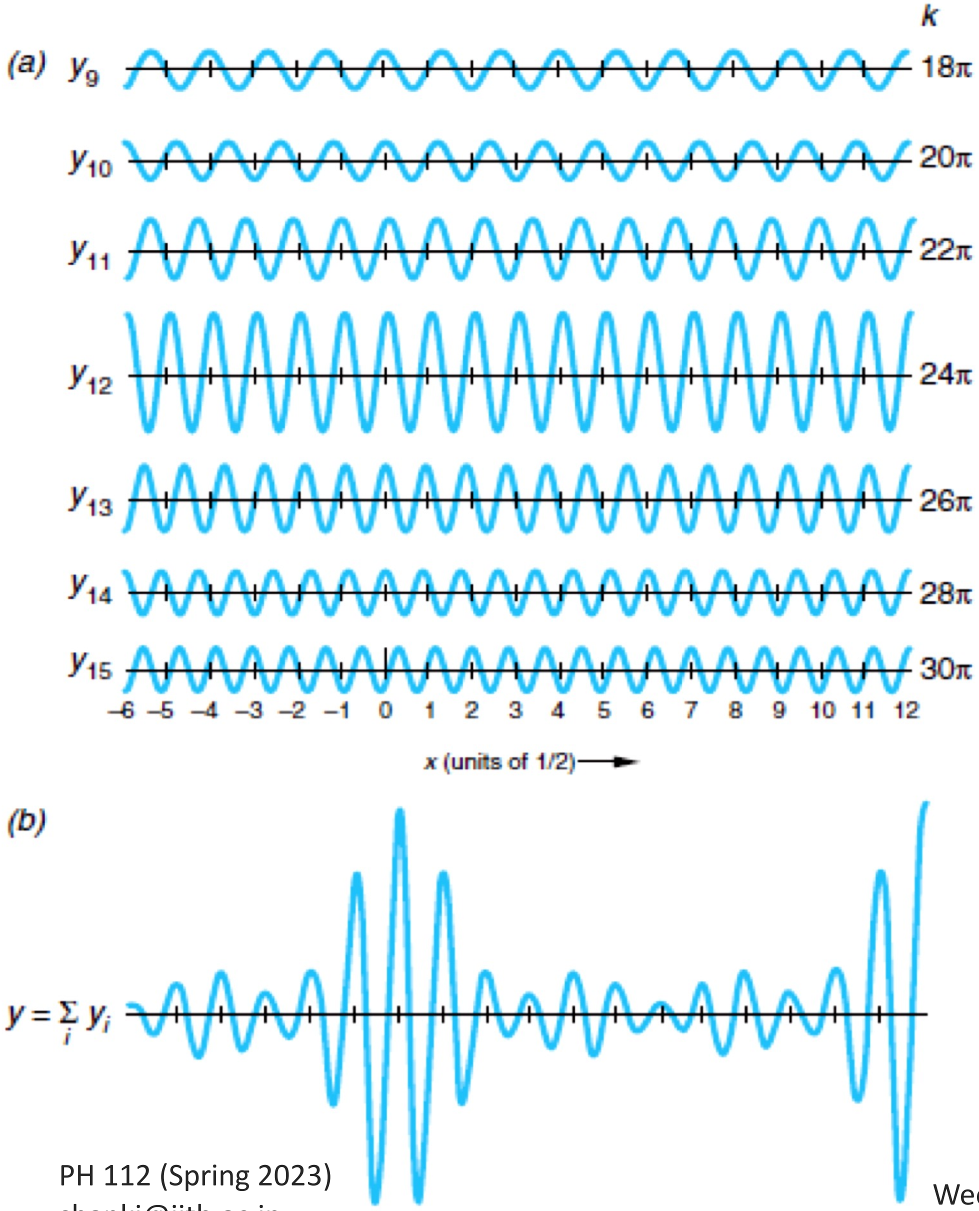
Phase waves

$$v_g = \frac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \frac{\Delta\omega}{\Delta k}$$

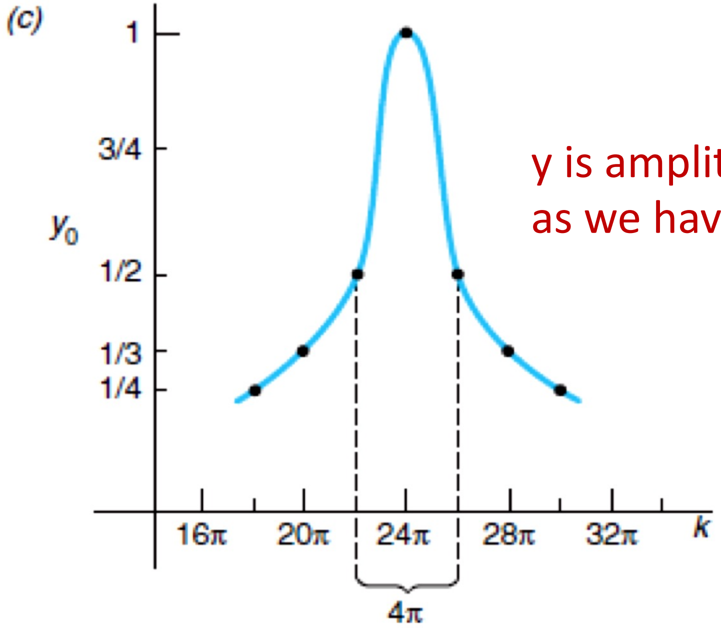
$$v_p = \frac{(\omega_1 + \omega_2)/2}{(k_1 + k_2)/2} \approx \frac{\omega_1}{k_1} = v_1$$



Example 3: Adding many waves with different amplitudes

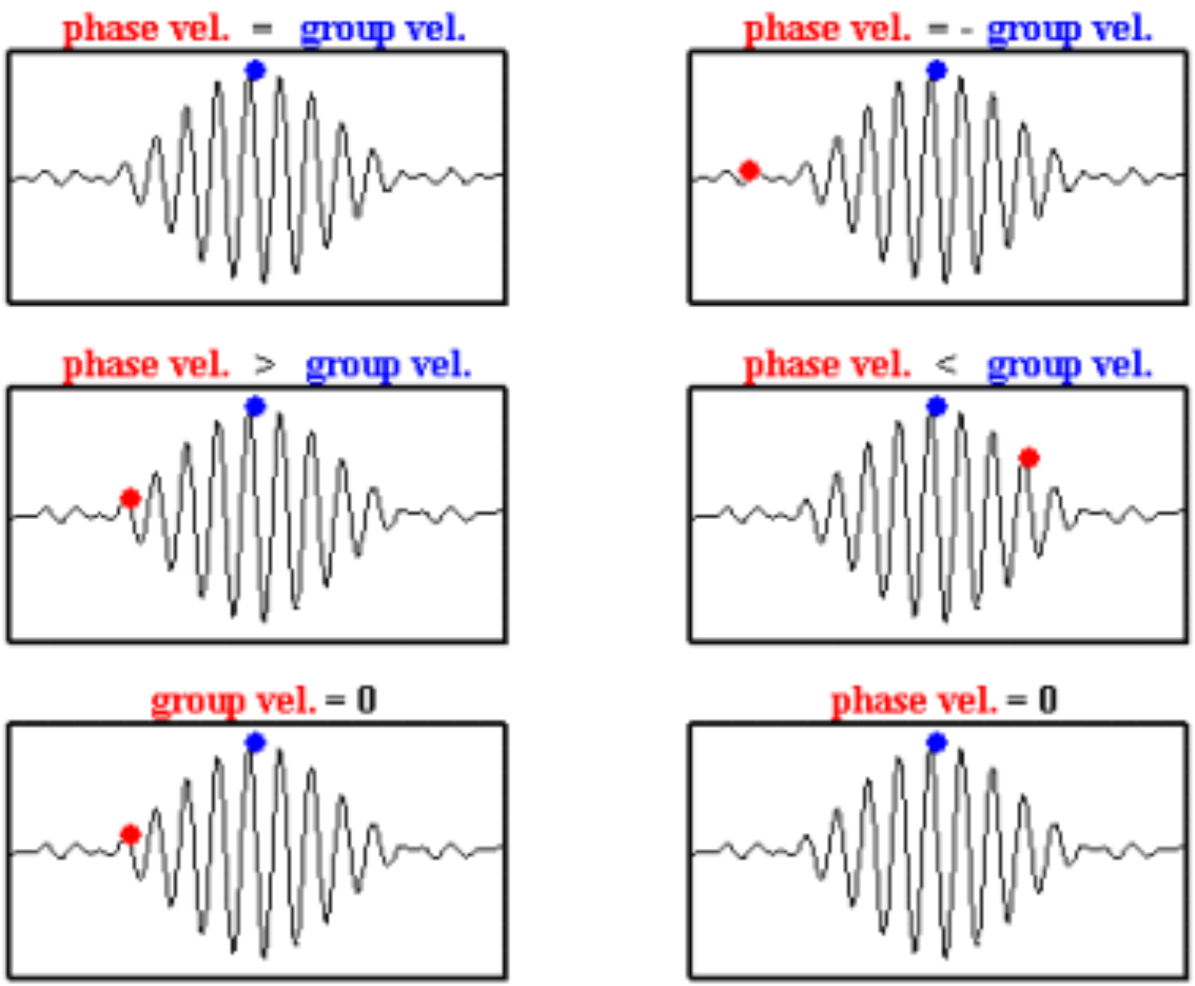


1. Superposition of many different sinusoidal waves results in a blob.
2. The blob represents a particle which travels with the group velocity. =
3. Group velocity is velocity of the particle.



y is amplitude, not space dimension as we have a 1D wave (x,t)

Understanding Phase and Group velocity (Credit: ISVR)



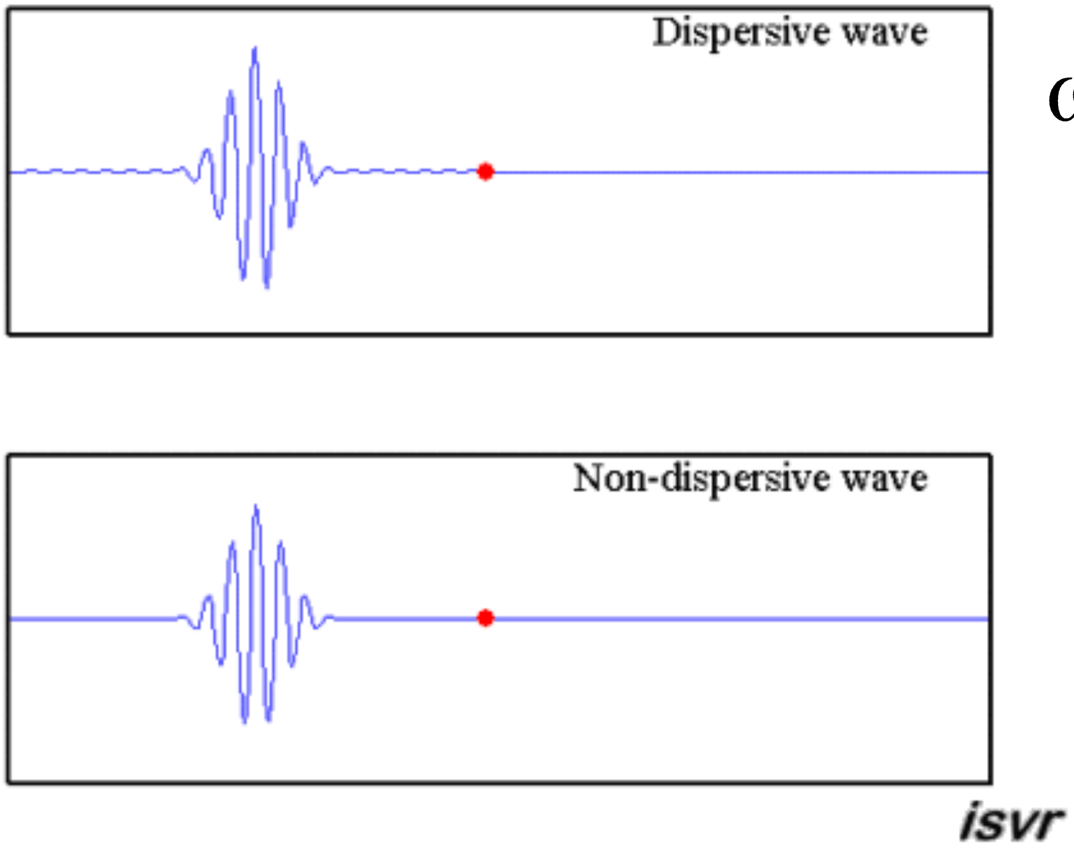
Case 1: Phase velocity = group velocity

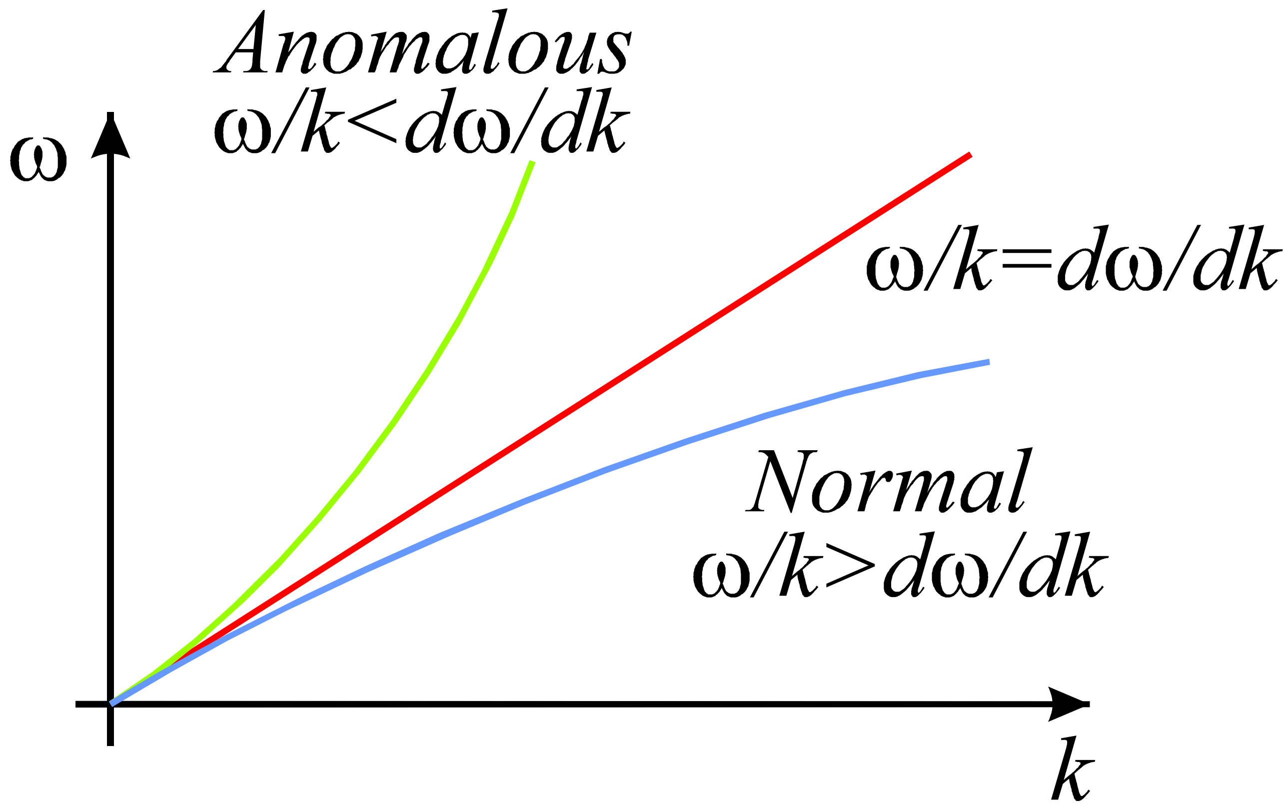
Case 2: Phase velocity ≠ group velocity

Case 3: Phase velocity or group velocity= 0

isvr

Spreading of waves in a medium





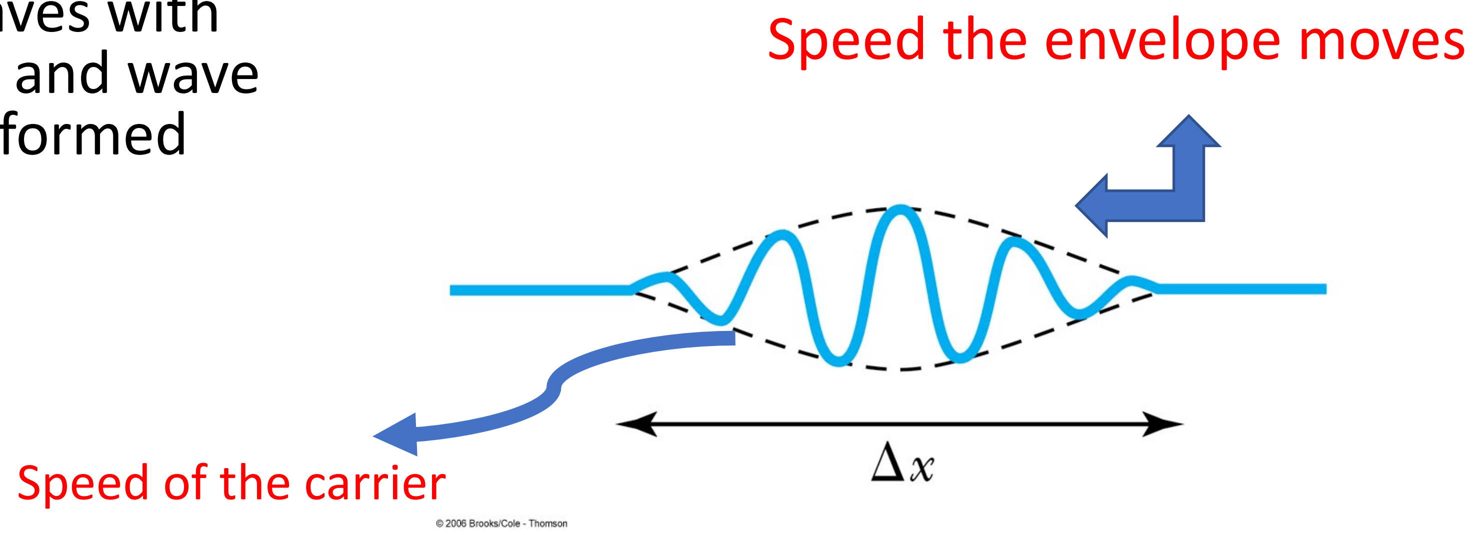
Principle of Superposition of waves

- When two or more waves traverse the same region, they act independently of each other.

- When many more waves are combined, the phase of the wave oscillates within an envelope that denotes the maximum displacement of the combined waves.

- When combining (infinitely many) waves with different amplitudes and frequencies and wave numbers, a pulse, or **wave packet**, is formed which moves at a **group velocity**:

$$v_{group} = \frac{\Delta\, \omega}{\Delta\, k}$$



How the energy of such waves are transported?

- The energy carried by the group wave is concentrated in regions in which the amplitude of the envelope is large.

- The speed with which the waves' energy is transported through the medium is the speed with which the envelope advances, not the phase wave.

- In this sense, the envelop wave is of more ‘physical’ relevance in comparison to the individual phase waves (as far as energy transportation is concerned)

de Broglie waves and Group velocity

Free particle: Phase velocity, Group velocity

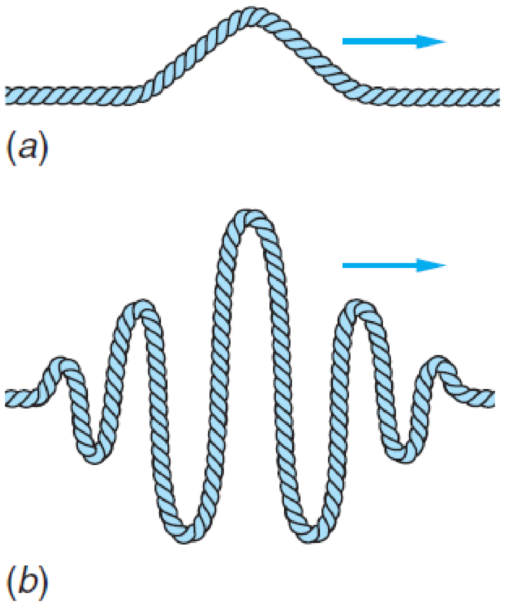
de Broglie relation is $$E = \hbar\omega, p = \hbar k$$

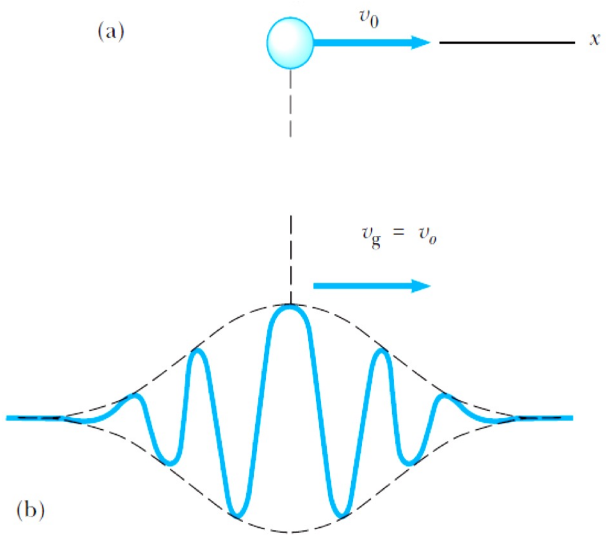
Consider a free particle $$E = \frac{p^2}{2m}, p = mv$$

in terms of ω *and* k $$\omega = \frac{\hbar k^2}{2m}$$

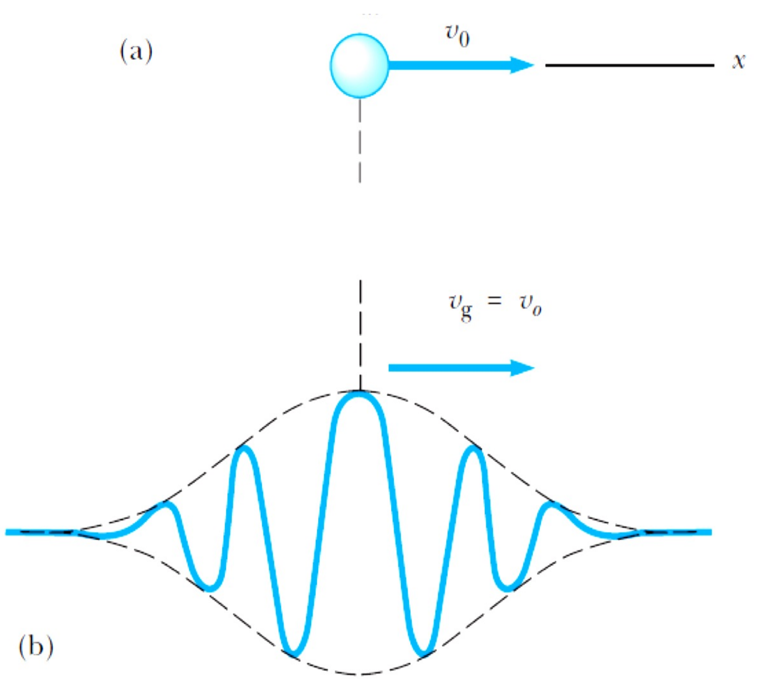
Phase velocity is $$v_{\text{phase}} = \frac{\omega}{k} = \frac{\hbar k}{2m}$$

Group velocity is $$v_{\text{group}} = \frac{d\omega}{dk} = \frac{\hbar k}{m} = v$$

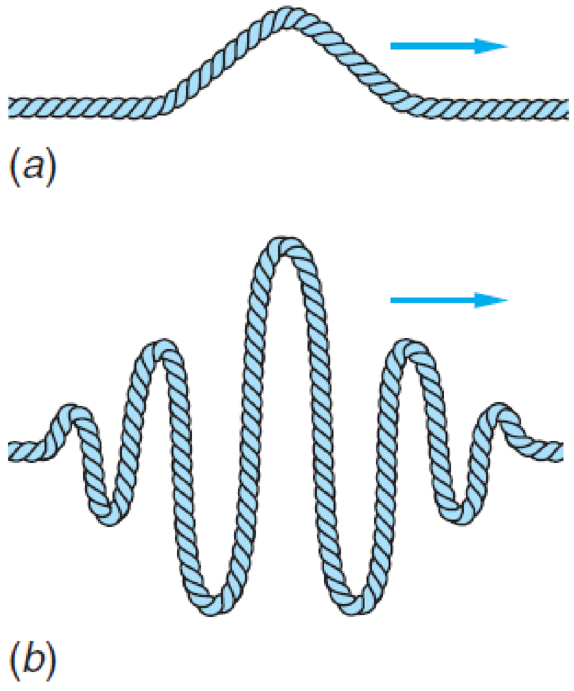




Group velocity and de Broglie waves



1. Group velocity and phase velocity are different.
2. A wave group moves with the group velocity --- which de Broglie showed to be the same as the velocity of the particle v.
3. The waves that form the pulse have a wide range of phase velocities, wave numbers, intensities, angular frequencies.

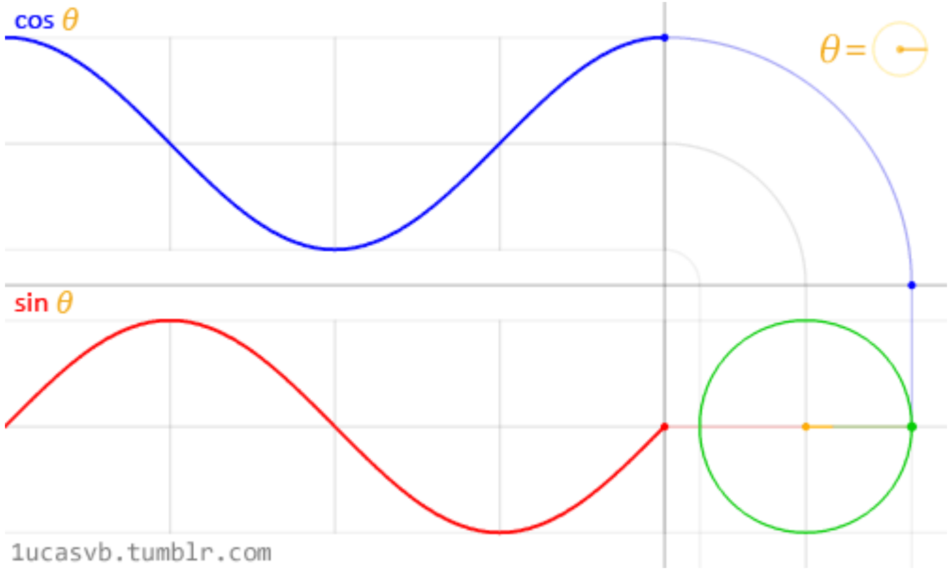


Ideal Waves vs Real waves

- An ideal wave has a single frequency and is infinitely long and unlocalized in space.

- Real waves are always localized in space.

- Waves that are localized in space are wave packets.



Summary

- Superposition of waves

  When infinitely many waves with different amplitudes and frequencies and wave numbers combine, a pulse, or wave packet is formed.

- Group Velocity

  Wave packet moves at a **group velocity**:

$$v_{group} = \frac{\Delta\,\omega}{\Delta\,k}$$

- de Broglie showed that the group velocity is same as the velocity of the particle.

Why are waves and particles important in physics?

- Waves and particles are important in physics because they represent the only modes of energy transport (interaction) between two points.

- E.g we signal another person with a thrown rock (a particle), a shout (sound waves), a gesture (light waves), a telephone call (electric waves in conductors), or a radio message (electromagnetic waves in space).

Recommended Reading

Wave Groups and Dispersion,

section 5.3 in page 164.