PH 112: Quantum Physics and Applications

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Week 5 Lecture 2: Step Potential

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Third Application: Finite potential well (Recap)

1. Quantum (discrete) energy states are a typical property of any well-type potential.

2. The corresponding wavefunctions (and probability) are mostly confined inside the potential but exhibit non-zero "tails" in the classically forbidden regions of KE < 0!

(Except when V (x) → ∞ where the tails are not allowed.)

Both properties result from requiring the wavefunction $\psi(x)$ and $\dfrac{d\psi(x)}{dx}$ to be continuous everywhere.

(Except when V (x) → ∞ where $\psi'(x)$ is not continuous.)

Third Application: Finite potential well (Recap)

1. Lowest energy (ground) state is always above the bottom of the potential and is symmetric. [Consequence of Uncertainty Principle.]

2. Wider and/or more shallow the potential, the lower the energies of the quantum states. [Consequence of Uncertainty Principle.]

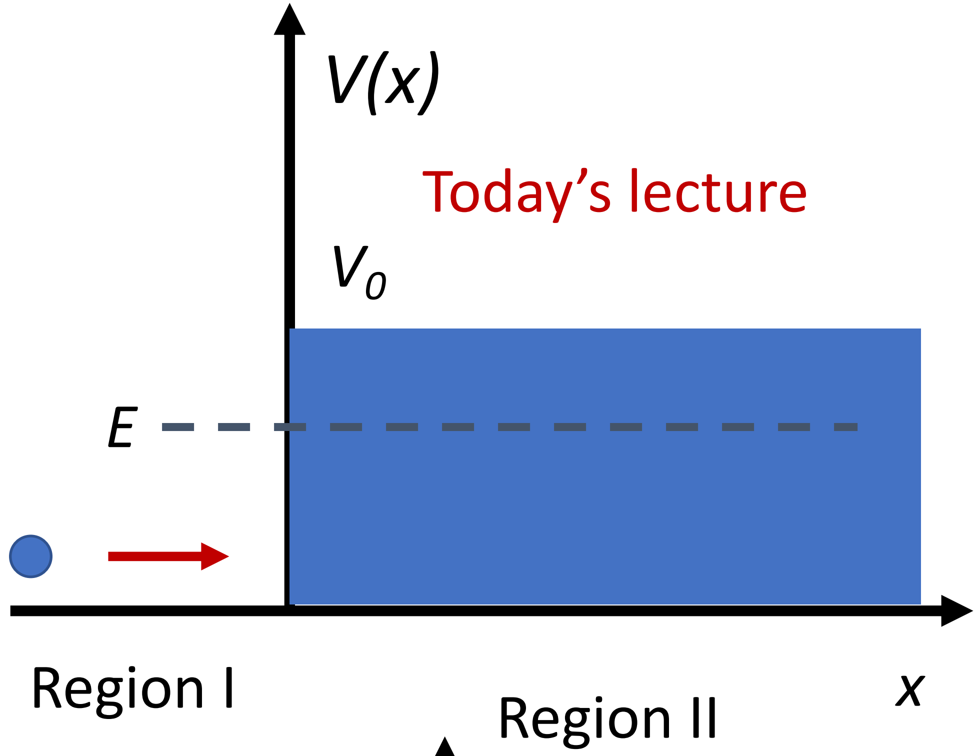
3. Inside “Finite Potential Well” potentials the number of quantum states is finite. When the total energy E is larger than the height of the potential, the energy becomes continuous,

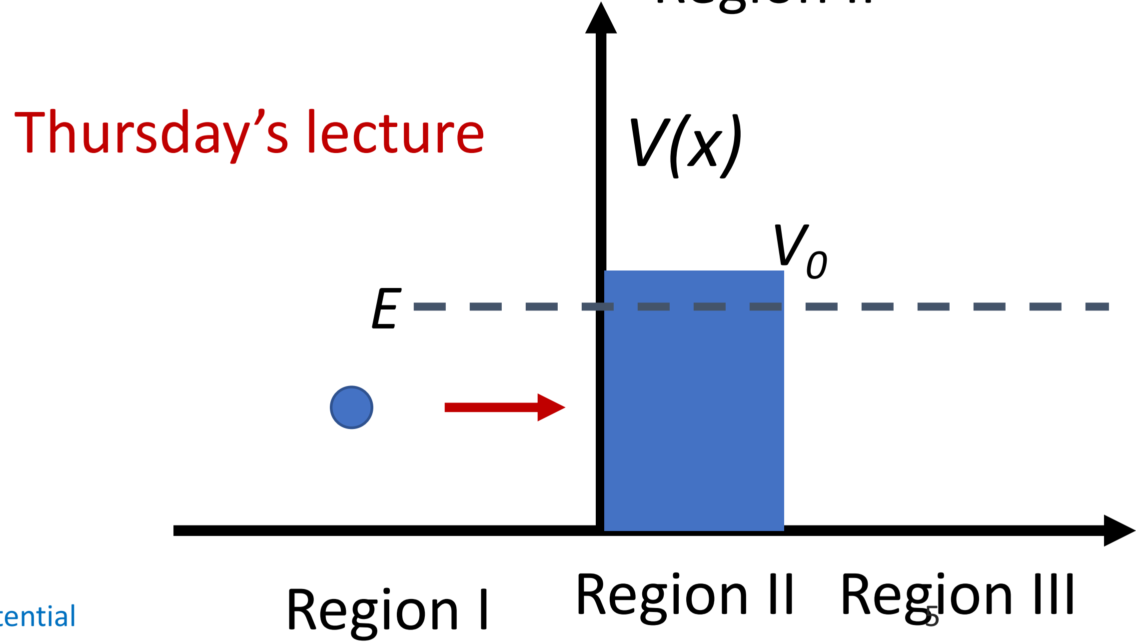
4. When V = V (x), both bound and continuous states are stationary, i.e, the time-dependent wavefunctions are $\Psi(x,t) = \psi(x)\ \exp(-iEt/\hbar)$

Fourth Application: Step potential

Potential Barrier

- A potential barrier is the opposite of Potential well.
- Consider a flux of particles incident from the left on the potential step or barrier with energy E. We assume there is no flux of particles coming from the right.
- Region I and III: Kinetic Energy: E
- Region II: Kinetic energy: $E - V_0$





Potential Barrier: Classical

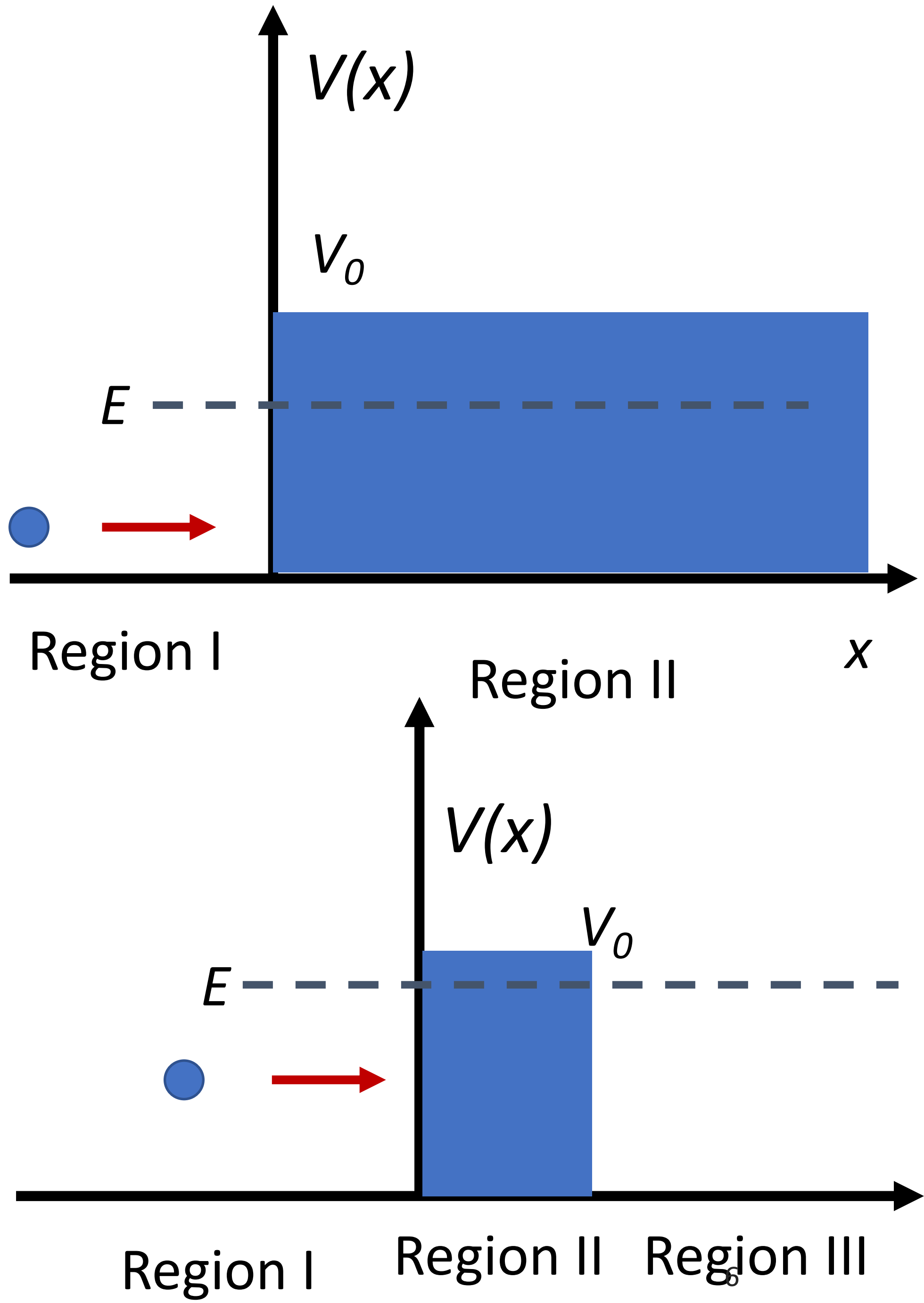
- For $E > V_0$:

    1. all the particles will pass over the step/barrier (they are transmitted).

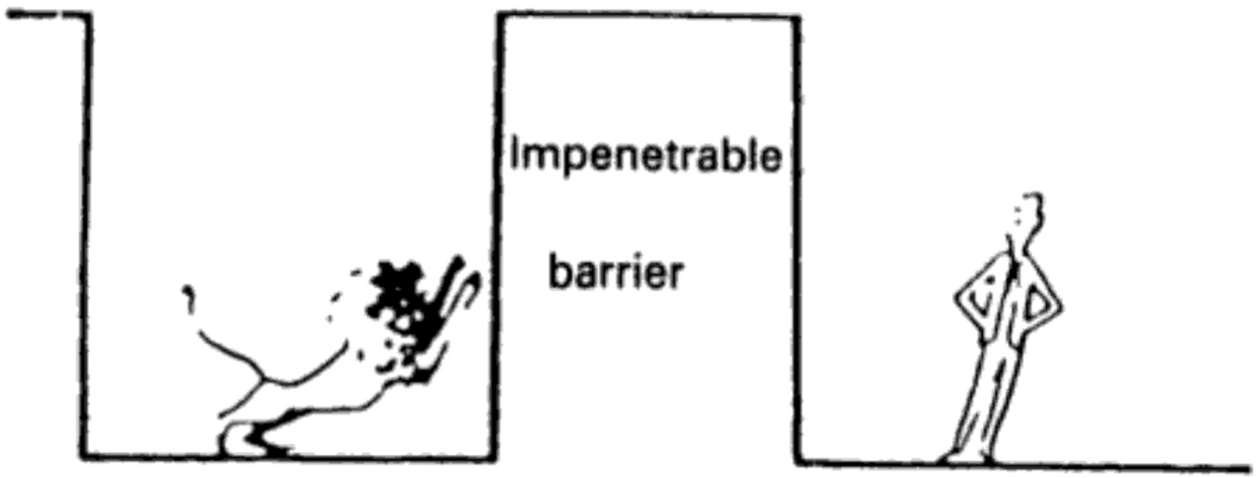
    2. the particles will slow down (smaller momentum).

- For $E < V_0$: all the particles are reflected!

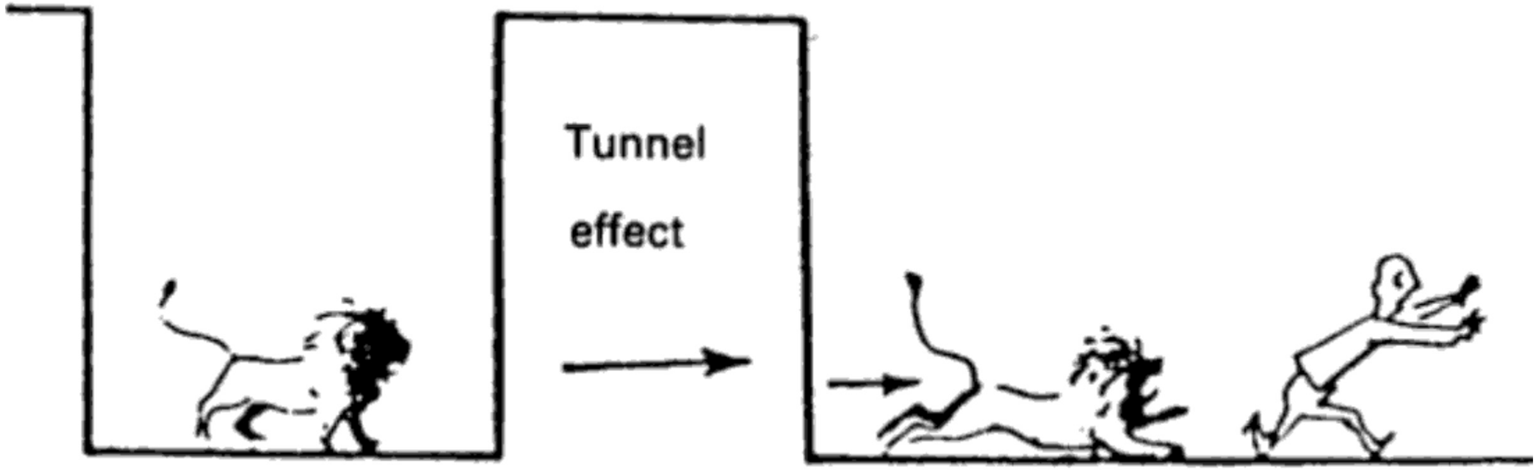
In both cases the flux of particles are the same. (There is no source or sink.)



What does Quantum theory predict?



Credit: Bleaney '84



- Classical world

  Barrier region is forbidden, and this precludes particle motion on the far side of the well.

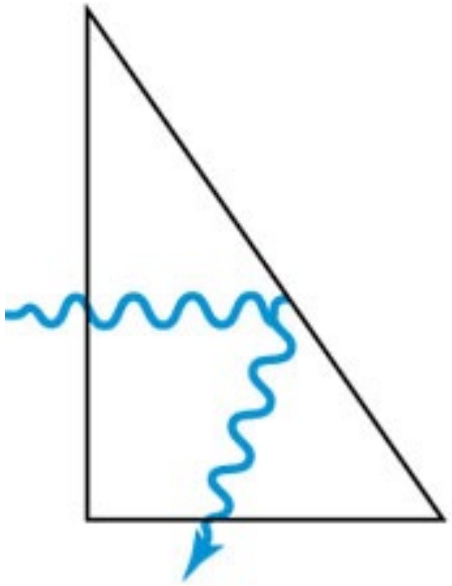
- Quantum World

  There is no inaccessible region to a particle. The matter wave associated with the particle is nonzero everywhere!

Analogous effect in Wave Optics

Experiment 1: Total Internal Reflection

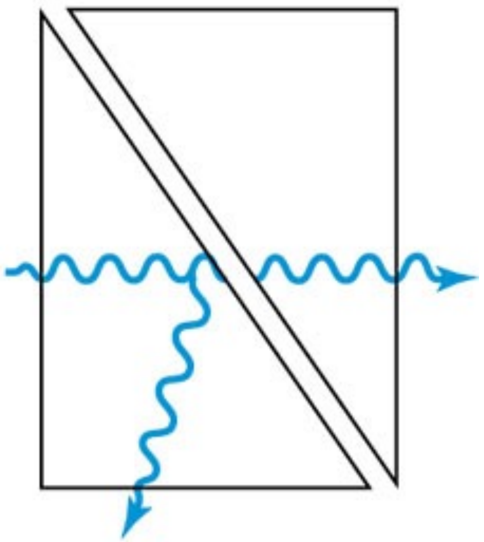
- Consider a light passing through a glass prism.
- Light gets reflected from an internal surface with an angle greater than the critical angle.



Experiment 2: Frustrated total Internal Reflection

YouTube Video 1 YouTube Video 2

- Let us bring another prism very close to the first one.
- Newton first observed that light appears in the second prism.
- The intensity of the second light beam decreases exponentially as the distance between the two prisms increases.

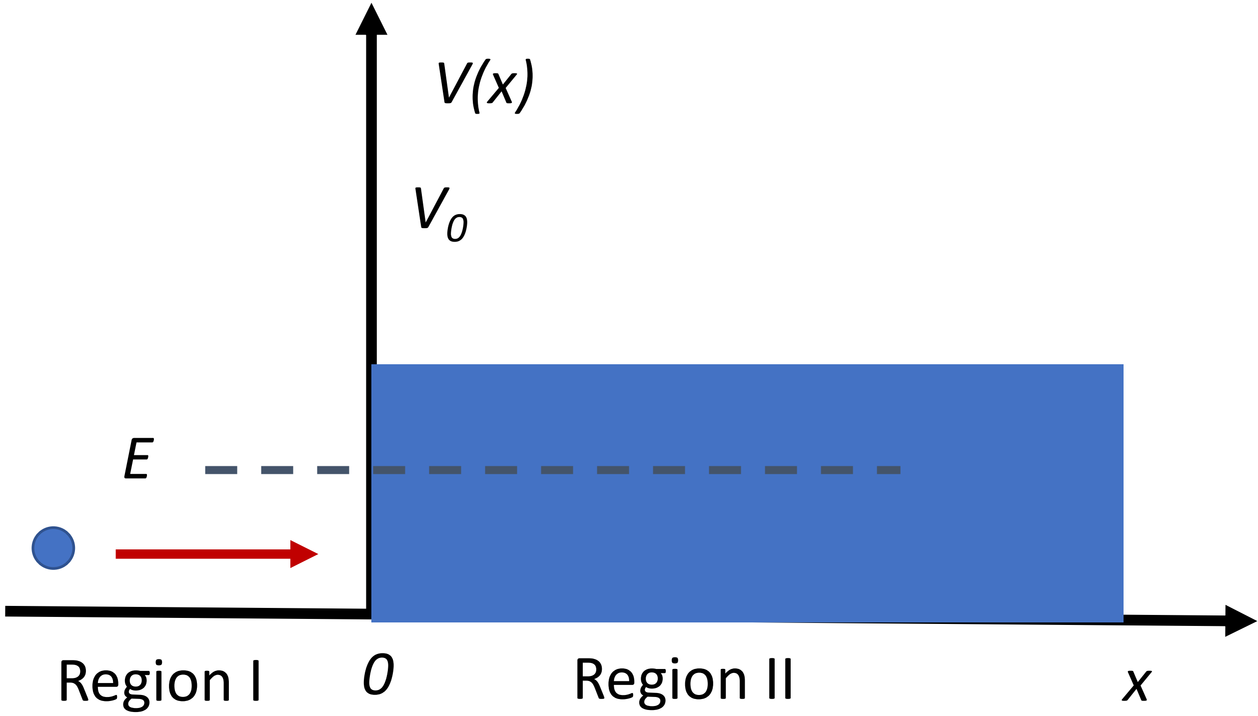


Amplitude of light is not zero just outside the prism.

Finding the solutions

Potential Barrier: Set up

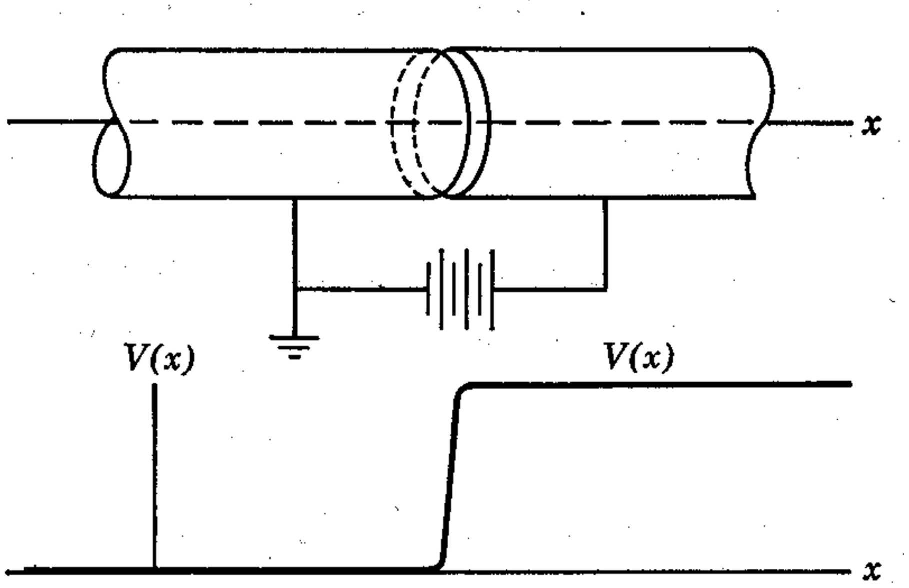
$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x > 0 \end{cases}$$



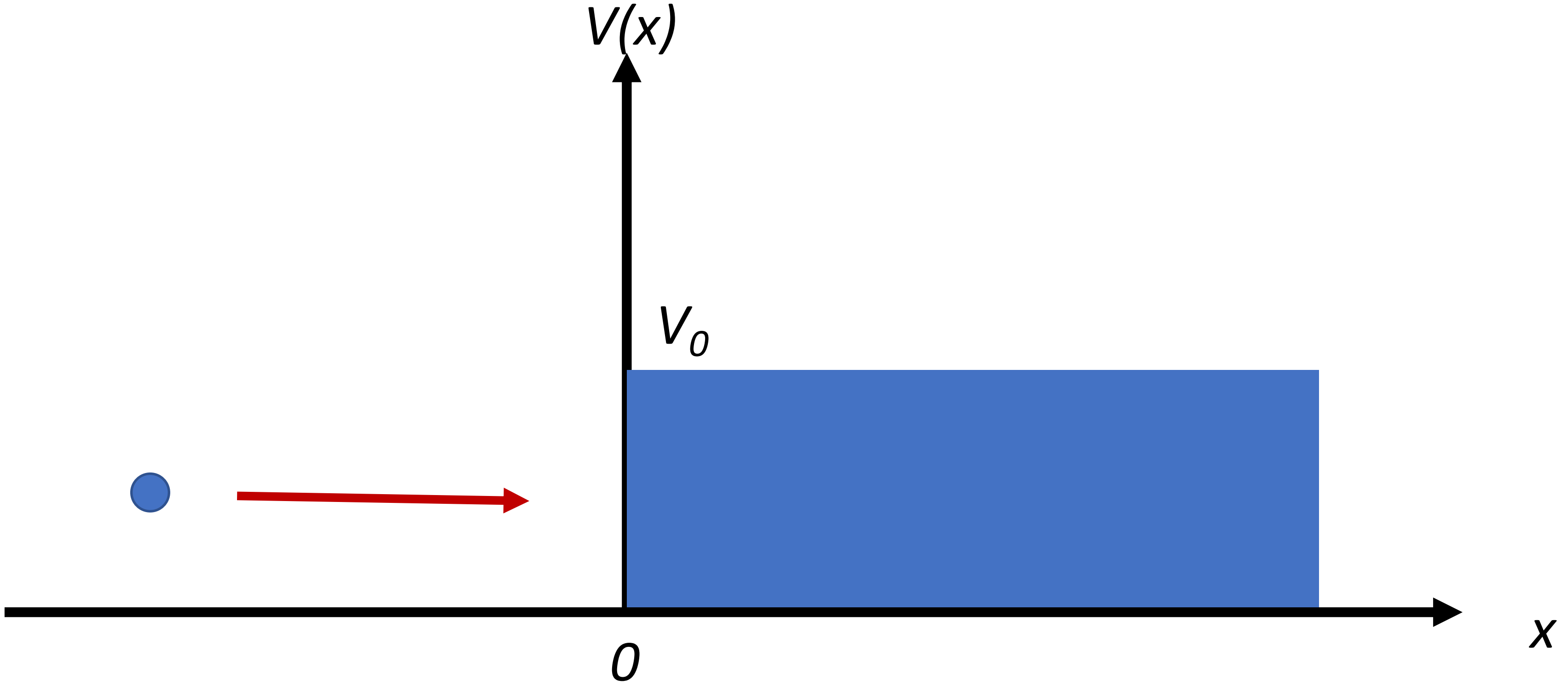
Physical system that can be approximated by a step potential:

1. Charged particle moving along the x-axis of two cylindrical electrodes held at different voltages.

2. PE is constant inside either electrode, but changes rapidly when passing from one to other.



Potential Barrier: Wave-functions



Eigenfunctions of the Schroedinger equation:

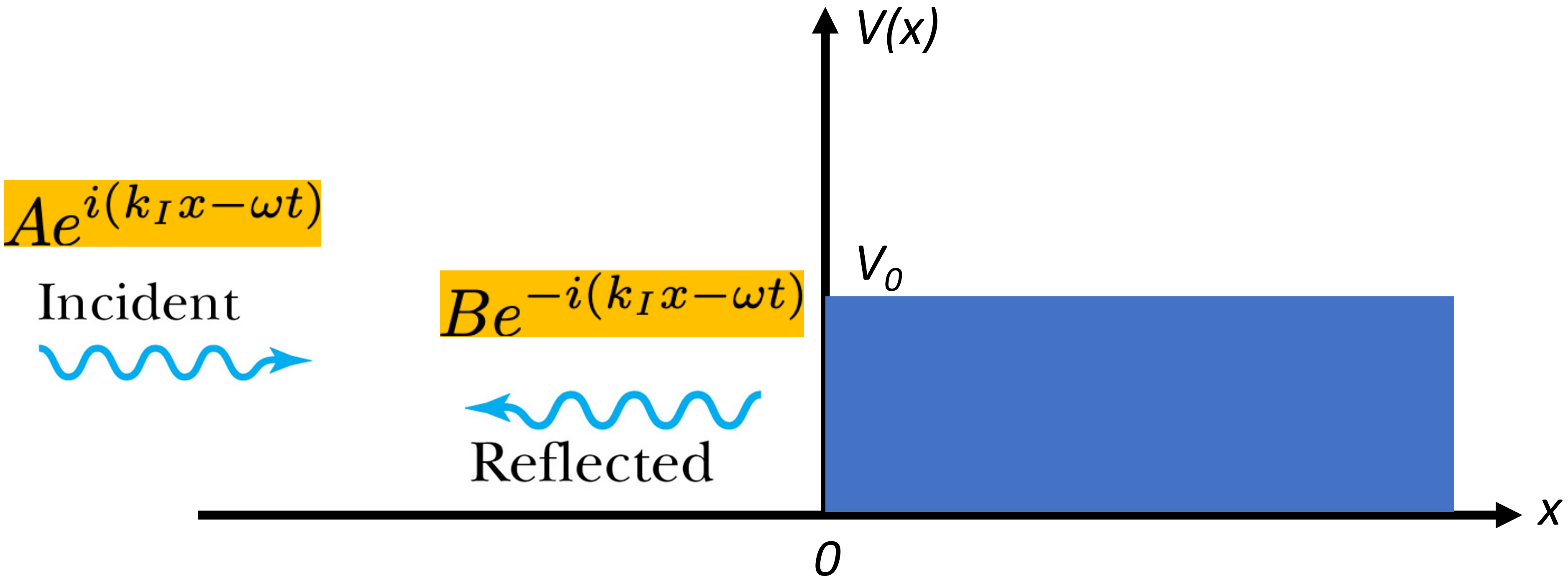
$x < 0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$x > 0$

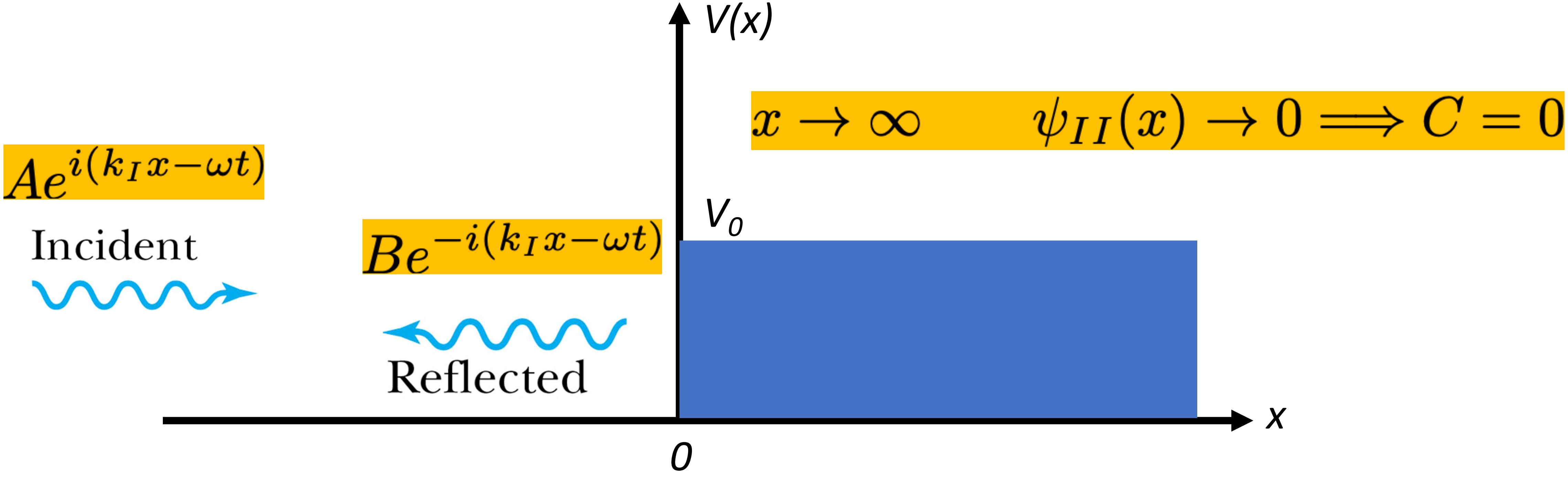
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = \left(E - V_0\right)\psi(x)$$

Potential Barrier: Wave-functions



$x < 0$	$x > 0$
$\psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x}$	$\psi_{II}(x) = Ce^{k_{II} x} + De^{-k_{II} x}$
$k_I = \sqrt{2mE}/\hbar$	$k_{II} = \sqrt{2m(V_0 - E)}/\hbar$

Potential Barrier: Boundary condition as $x \to \infty$

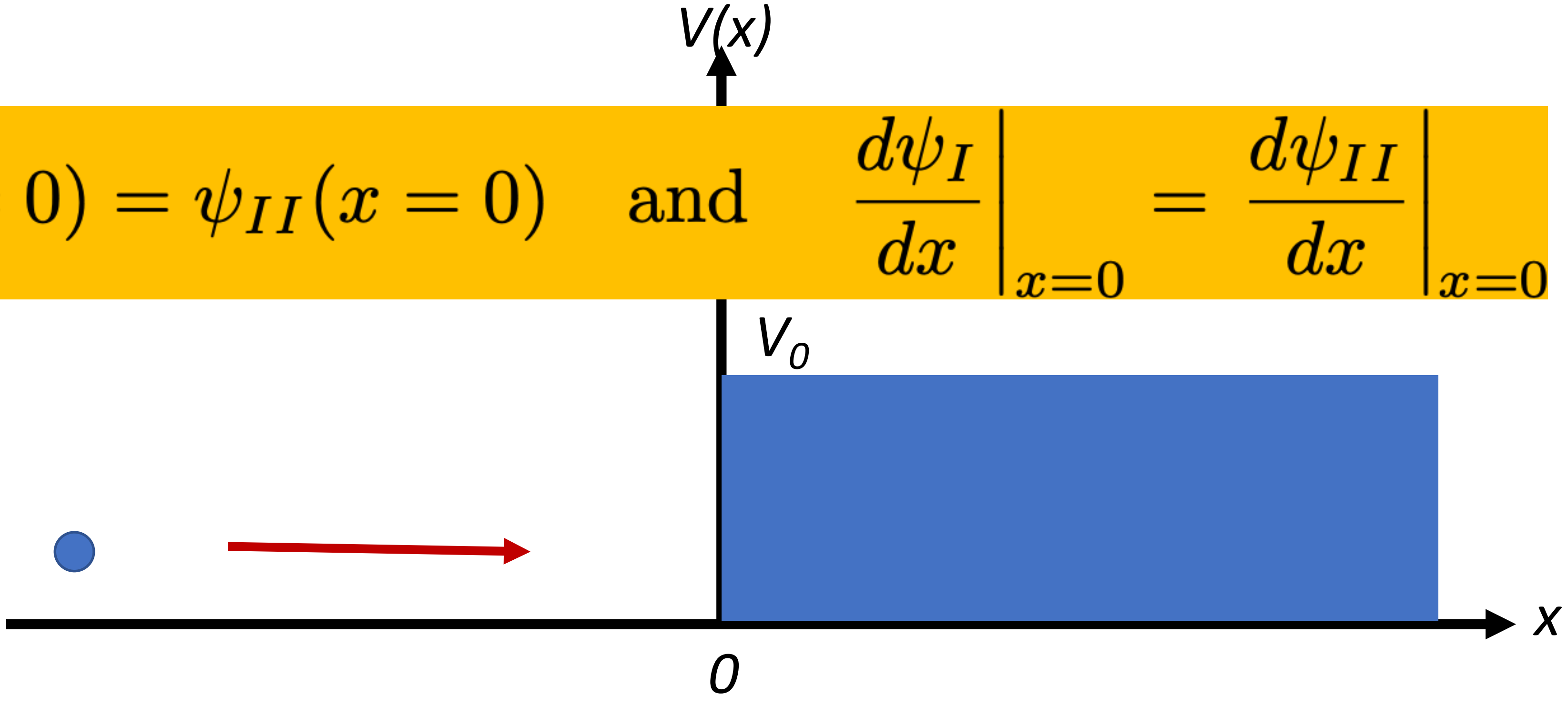


$$x < 0 \qquad\qquad x > 0$$

$$\psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x} \qquad \psi_{II}(x) = De^{-k_{II} x}$$

$$k_I = \sqrt{2mE}/\hbar \qquad k_{II} = \sqrt{2m(V_0 - E)}/\hbar$$

Potential Barrier: Boundary conditions at $x = 0$



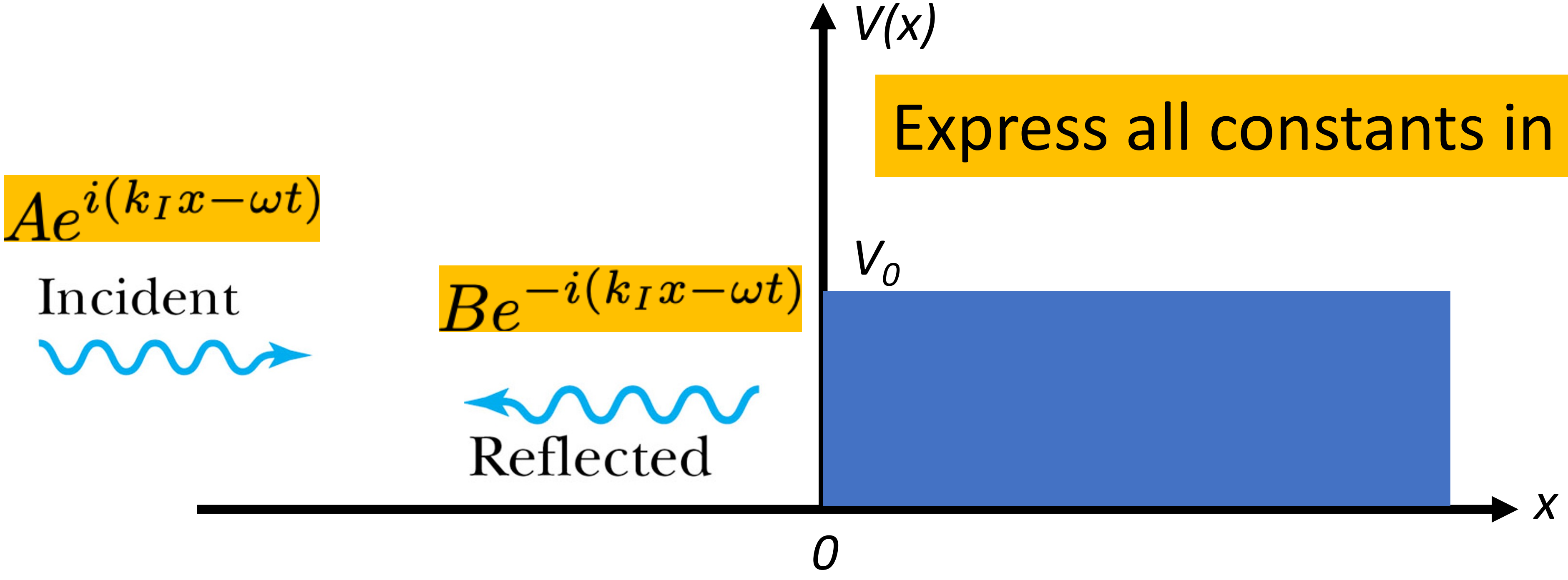
$$\psi_I(x=0) = \psi_{II}(x=0) \quad \text{and} \quad \left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$$

$$\left.De^{-k_{II}x}\right|_{x=0} = \left.Ae^{ik_Ix} + Be^{-ik_Ix}\right|_{x=0} \implies D = A + B$$

$$\left.-k_{II}De^{-k_{II}x}\right|_{x=0} = \left.Aik_Ie^{ik_Ix} - ik_IBe^{-ik_Ix}\right|_{x=0} \implies \frac{ik_{II}}{k_I}D = A - B$$

Potential Barrier: final wavefunction

Express all constants in terms of *D*

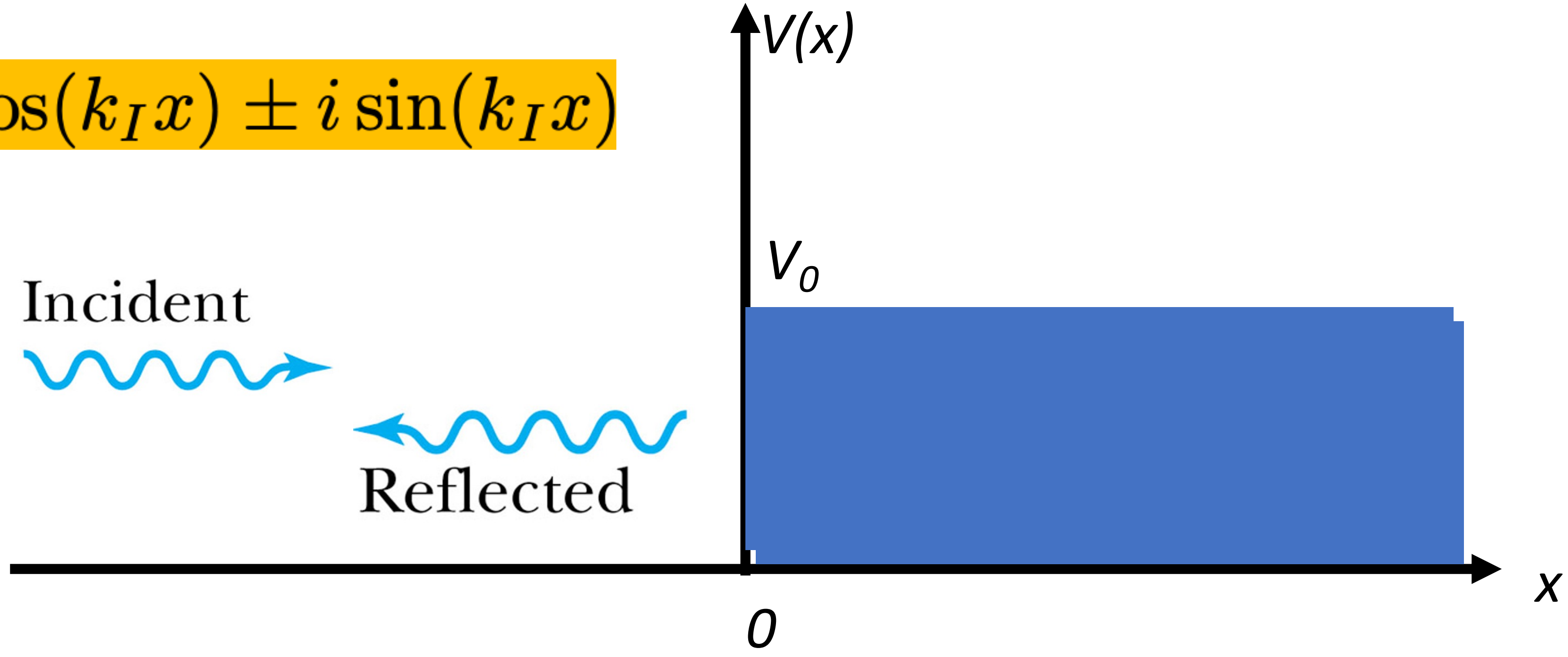


$$A=\frac{D}{2}\left(1+\frac{ik_{II}}{k_I}\right) \qquad \psi_I(x)=\frac{D}{2}\left(1+\frac{ik_{II}}{k_I}\right)e^{ik_Ix}+\frac{D}{2}\left(1-\frac{ik_{II}}{k_I}\right)e^{-ik_Ix}$$

$$B=\frac{D}{2}\left(1-\frac{ik_{II}}{k_I}\right) \qquad \psi_{II}(x)=De^{-k_{II}x}$$

Potential Barrier: Rewriting the wavefunction

$$e^{\pm i k_I x} = \cos(k_I x) \pm i \sin(k_I x)$$



$$\psi(x) = \begin{cases} D\cos(k_I x) - D\frac{k_{II}}{k_I}\sin(k_I x) & x < 0 \\ De^{-k_{II}x} & x > 0 \end{cases}$$

Wavefunction properties

Flux of particles: Definition

Classical context

- Consider *N* particles per unit length and each one has constant speed *v* in +ve *x*–direction.
- In Δt *interval,* all particles within $u\,\Delta t$ will pass a fixed point.
- Particle flux (F) is number of particles passing a fixed point per unit time (like current):

$$F = \frac{N v \Delta t}{\Delta t} = N v$$

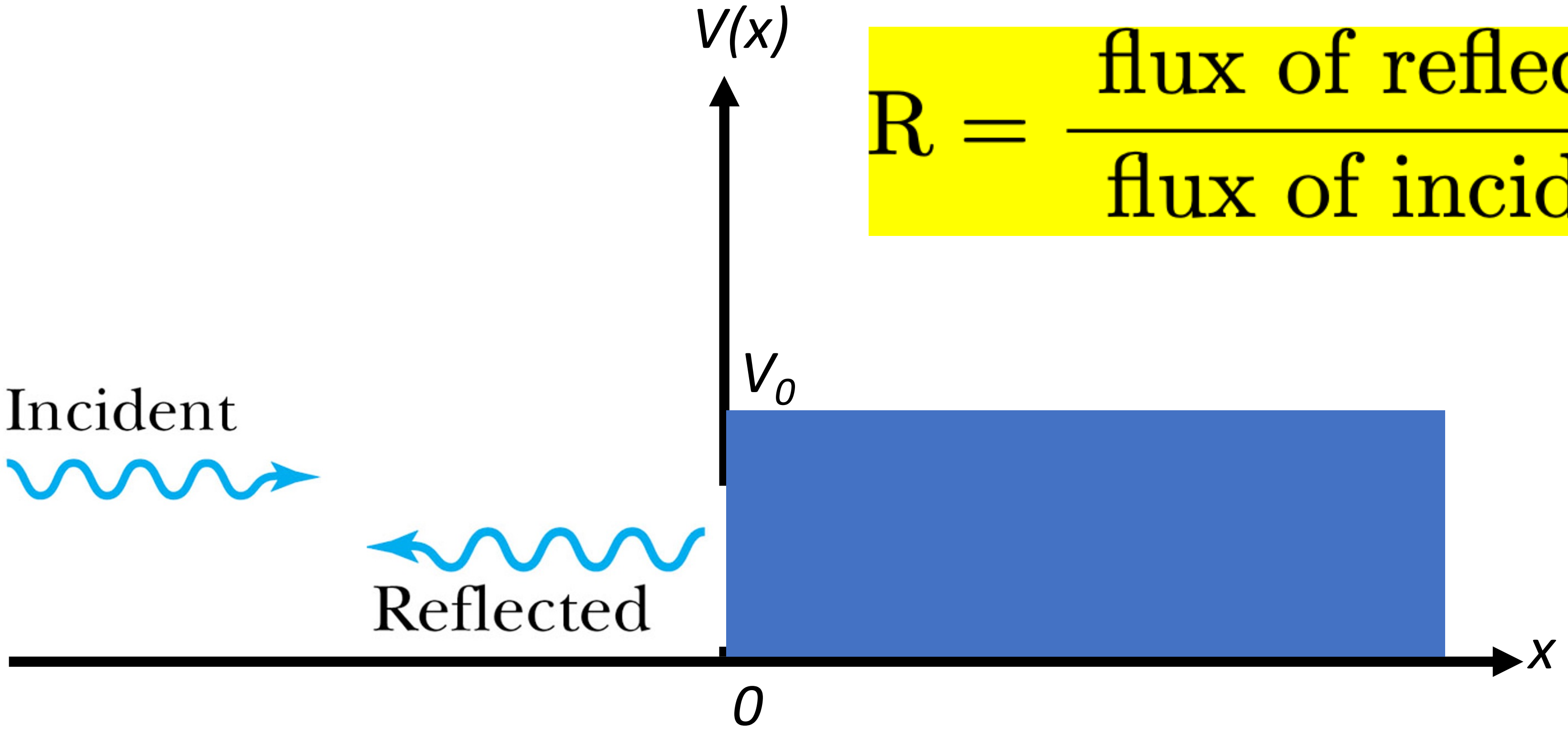
Quantum context

- Wavefunction corresponding to particles moving with definite momentum p is

$$\psi(x) = A \exp(ikx)$$

- We have $p = \hbar k \;\; and \; v = \frac{p}{m}$
- We know probability to find a free particle in same $\sim |A|^2$. Thus, average number of particles per unit length is $\sim |A|^2$.
- Flux of particles is $$F = |A|^2 \frac{\hbar k}{m}$$

Potential Barrier: Reflection coefficient *(R)*



$$R = \frac{\text{flux of reflected particles}}{\text{flux of incident particles}}$$

$$R = \frac{B^*B}{A^*A} = \frac{\left(1 - \frac{ik_{II}}{k_I}\right)^* \left(1 - \frac{ik_{II}}{k_I}\right)}{\left(1 + \frac{ik_{II}}{k_I}\right)^* \left(1 + \frac{ik_{II}}{k_I}\right)} = \frac{\left(1 + \frac{ik_{II}}{k_I}\right)\left(1 - \frac{ik_{II}}{k_I}\right)}{\left(1 - \frac{ik_{II}}{k_I}\right)\left(1 + \frac{ik_{II}}{k_I}\right)} = 1$$

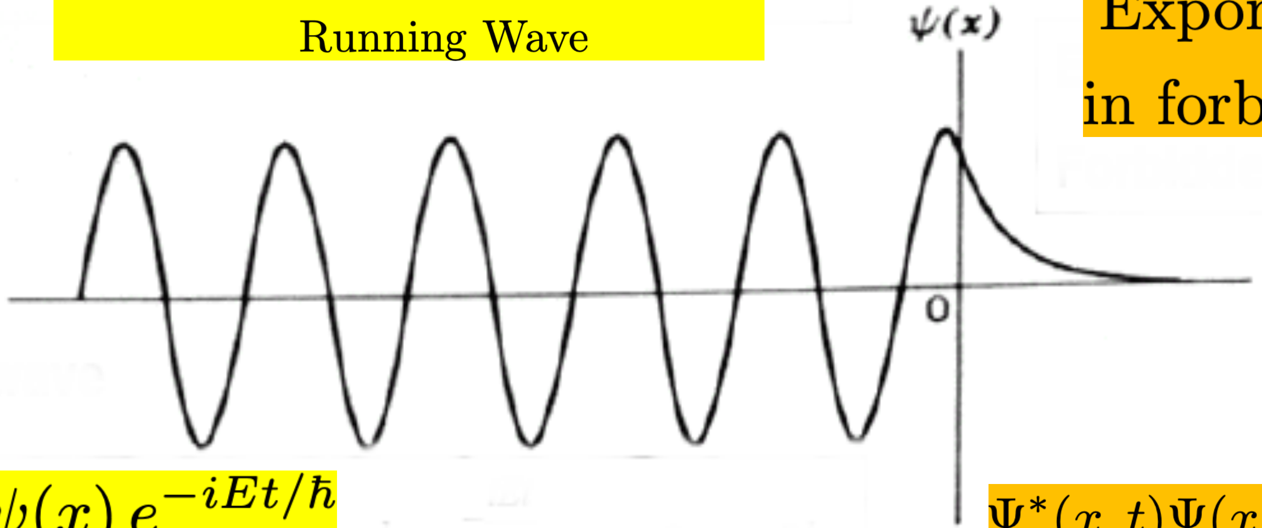
Wavefunction and Probability density

$$\psi_I(x) = D\cos(k_I x) - D\frac{k_{II}}{k_I}\sin(k_I x)$$

Running Wave

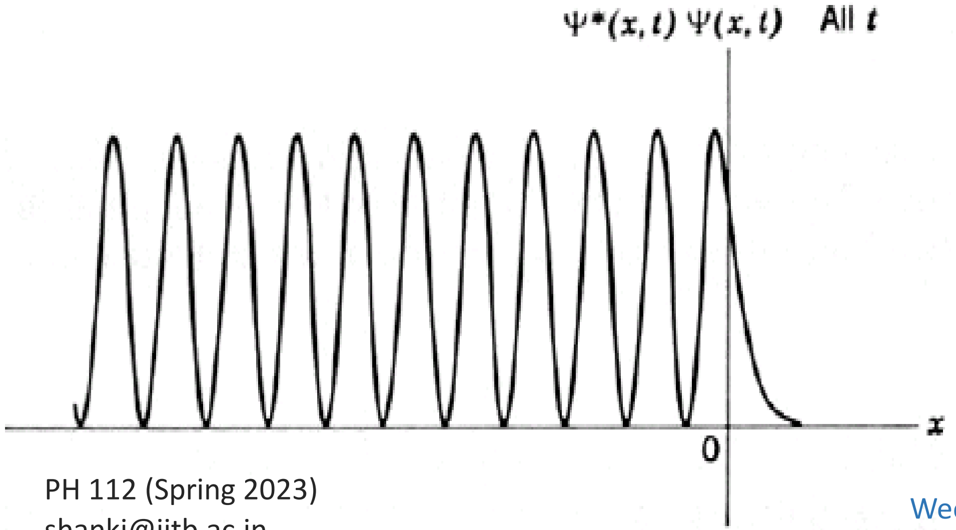
$$\psi_{II}(x) = De^{-k_{II}x}$$

Exponential decay in forbidden region



$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

$$\Psi^*(x,t)\Psi(x,t) = |D|^2\, e^{-k_{II}x}e^{iEt/\hbar}\, e^{-k_{II}x}e^{-iEt/\hbar}$$
$$= |D|^2 e^{-2k_{II}x}$$

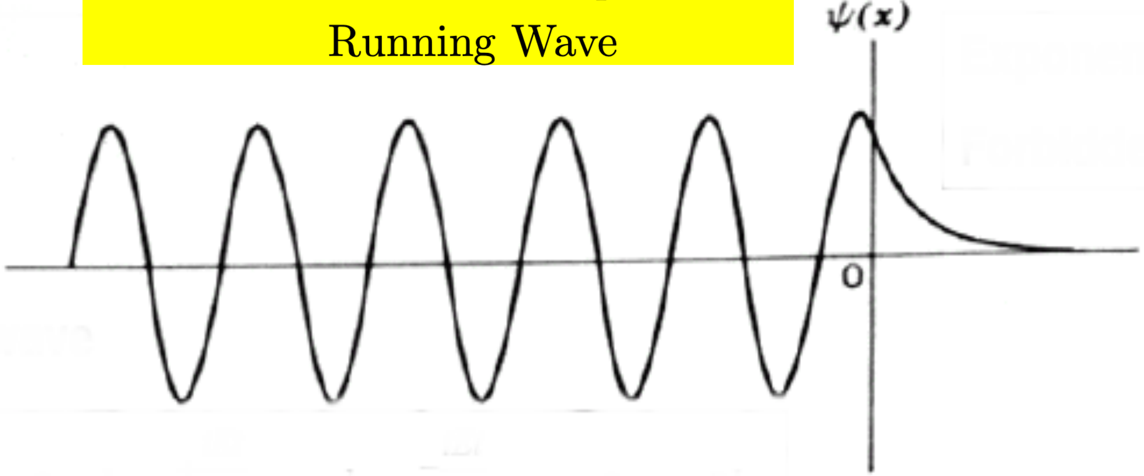


Probability of finding the particle is non-zero in region II.

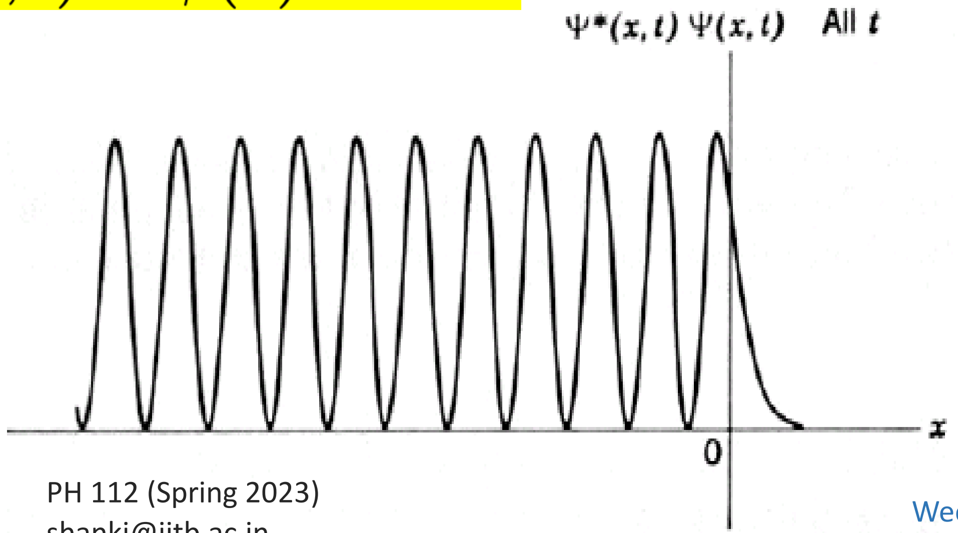
Wavefunction and Probability density

$$\psi_I(x) = D\cos(k_I x) - D\frac{k_{II}}{k_I}\sin(k_I x)$$

Running Wave



$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$



Question: Is there any flux?

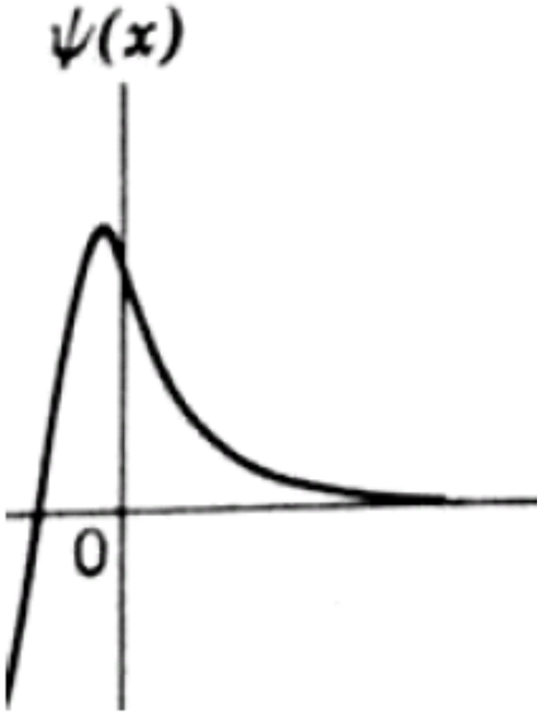
- Since no particles are created or destroyed at the boundary $R + T = 1$. If $R = 1$, then $T = 0$.

  Even though particles are observed in region (II), there is no flux.

- $$\psi_{II} \propto \exp(-k_{II}x)$$

  is not an eigenfunction of momentum, and so does not represent a travelling wave with an associated momentum and an associated flux.

The particle penetrates the walls!



- **This violates classical physics!**
- Penetration depth is the distance outside potential well where the probability decreases to about $1/e$:

$$|\Psi(x,t)|^2 = |D|^2 e^{-2k_{II}x} = |D|^2\left(e^{-1}\right)^2$$

- Note that the penetration depth is proportional to Planck's constant.

$$\delta x \simeq \frac{1}{k_{II}} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$$

- From Uncertainty principle, we have

$$\Delta p \simeq \frac{\hbar}{\delta x} = \sqrt{2m(V_0 - E)} \Longrightarrow \Delta E \simeq \frac{(\Delta p)^2}{2m} = V_0 - E$$

$$\Longrightarrow \text{Uncertainty in the energy of the particle}$$

Summary

- A quantum particle that is incident on an infinite potential barrier and height may cross the barrier. It does not have a classical analog.

- Penetration depth is $$\delta x \simeq \frac{1}{k_{II}} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$$

"According to classical mechanics, it is not possible for a particle to be in a place where its total energy is less than its potential energy. In quantum mechanics, this impossibility is changed into an improbability." Edward Condon

- We will now extend this to a quantum particle that is incident on a potential barrier of a finite width.

Recommended Reading

Tunnelling Chapter 7