Tut 3 - Part 1

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Fourier Transform

Q1

1. * If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
(a) Find the Fourier transform for $\phi(k)$
(b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

Q2

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ (for $-\pi \leq x \leq \pi$) and $f(x) = 0$ elsewhere
(a) Plot $f(x)$ versus x.
(b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
(c) At what value of k, $|g(k)|$ attains its maximum value?
(d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
(e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

Q4

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0x)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.
(a) Plot $|f(x)|$ versus x.
(b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx
(c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx$
(d) Plot $g(k)$ versus k.
(e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x.\Delta k$
[Given : $\int_0^\infty e^{-(\alpha-ik)x}dx = \frac{1}{\alpha-ik}$]

All 3 problems discussed in last tut

Q3

3. A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \leq x \leq L$ and $\psi(x) = 0$ otherwise.
(a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
(b) What is the amplitude of the plane wave of wavelength L constituting $\psi(x)$?

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \qquad 0 \leq x \leq L$$

(particle in a box)

Fourier transform: $\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k)e^{ikx}dk$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)e^{-ikx}dx$$

$$\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dx \Rightarrow a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x)e^{-ikx}dx$$

(idea of $1/\sqrt{2\pi}$ absorbed by $a(k)$)

$$\Rightarrow a(k) = \frac{1}{2\pi}\int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-ikx}dx$$

b) For wavelength L, $k = 2\pi/L$

Amplitude of plane wave $[a(k)e^{ikx}]$

$$a(2\pi/L) = -\frac{1}{2\pi}\sqrt{\frac{2}{L}} \times \frac{L}{3\pi}\left[e^{-2\pi i} + 1\right]$$

$$= -\frac{\sqrt{2L}}{3\pi^2}$$