Chapter 6 : Linear Transformations

1. Let A be an $m \times n$ matrix with real entries.
2. Then A defines a function
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad T_A(v) = Av.$$
3. By properties of matrix multiplication, T_A satisfies the following properties for all $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.
$$T_A(v + w) = A(v + w) = Av + Aw = T_A(v) + T_A(w)$$
$$T_A(cv) = cT_A(v).$$
4. We say that T_A respects the two operations in the vector space $\mathbb{R}^n$.
5. In this chapter we study such maps between vector spaces.
6. **Definition.** Let V, W be vector spaces over $\mathbb{F}$. A linear transformation $T : V \longrightarrow W$ is a function satisfying
$$T(v + w) = T(v) + T(w) \text{ and } T(cv) = cT(v)$$
where $v, w \in V$ and $c \in \mathbb{F}$.
7. If $T : V \to W$ is a linear transformation, then $T(0) = 0$.

Examples of Linear Transformations

1. Examples: If vector spaces V, W are vector spaces over $\mathbb{F}$, the "zero map" $T_0 : V \to W$ defined as $T_0(v) = 0$ for all $v \in V$, is clearly a linear transformation.
2. The **identity map** $I : V \to V$ defined as $I(v) = v$ for all $v \in V$, is clearly a linear map.
3. Let $c \in \mathbb{R}, V = W = \mathbb{R}^2$. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} = c \begin{bmatrix} x \\ y \end{bmatrix}.$$

   T is a linear transformation since for all $v, w \in \mathbb{R}^2$ and $d \in \mathbb{R}$,

$$\begin{aligned} T(v + w) &= c(v + w) = cv + cw = T(v) + T(w) \\ T(dv) &= c(dv) = d(cv) = dT(v), \end{aligned}$$

4. Rotation: Fix θ and define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

5. Then $T(e_1) = (\cos\theta, \sin\theta)^{\mathsf{t}}$ and $T(e_2) = (-\sin\theta, \cos\theta)^{\mathsf{t}}$.
6. Hence T rotates every vector by an angle θ in anticlockwise direction.

Linear Transformations: Examples

1. **Differentiation.** Let $\mathcal{D}$ be the vector space of differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f^{(n)}$ exists for all n. Define $D : \mathcal{D} \longrightarrow \mathcal{D}$ by

$$D(f) = f^{'}.$$

2. Then D is a linear transformation since for all $a, b \in \mathbb{R}$ and $f, g \in \mathcal{D}$,

$$D(af + bg) = af^{'} + bg^{'} = aD(f) + bD(g).$$

3. **Integration.** Let V be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define $\mathcal{I} : V \longrightarrow V$ by

$$\mathcal{I}(f)(x) = \int_0^x f(t)\, dt.$$

4. By properties of integration, $\mathcal{I}$ is a linear transformation.
5. The map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$ is not linear.
6. Let $V = M_{n\times n}(\mathbb{F})$ be the vector space of all $n \times n$ matrices over $\mathbb{F}$. Fix $A \in V$. The map $T : V \to V$ given by $T(N) = AN$ is linear.

The null space and image of a linear transformation

1. Let $T : V \to W$ be a linear transformation of vector spaces.
2. There are two important subspaces associated with T.
   - Nullspace of T = $\mathcal{N}(T) = \{v \in V \mid T(v) = 0\}$.
   - Image of T = $\mathrm{Im}\,(T) = \{T(v) \mid v \in V\}$.
3. Let V be a finite dimensional vector space. Suppose that α, β are scalars. If $v, w \in \mathcal{N}(T)$ then $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w) = 0$. Hence $\alpha v + \beta w \in \mathcal{N}(T)$. Therefore $\mathcal{N}(T)$ is a subspace of V.
4. The dimension of $\mathcal{N}(T)$ denoted as nullity (T), is called the nullity of T.
5. Suppose that $v, w \in V$. Then
$$\alpha T(v) + \beta T(w) = T(\alpha v + \beta w) \in \mathrm{Im}\,(T).$$
6. Hence $\mathrm{Im}\,(T)$ is a subspace of W.
7. The dimension of $\mathrm{Im}\,(T)$, denoted by $\mathsf{rank}(T)$, is called the rank of T.
8. If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the map $T_A(u) = Au$, Then $\mathrm{Im}\,T = \mathcal{C}(A)$. Thus $\mathsf{rank}(T) = \mathsf{rank} A = \dim \mathcal{C}(A)$.
9. The nullspace of T_A is the null space of A. Hence nullity T_A = nullity (A).

Construction of linear transformations

1. **Proposition.** Let $T : V \to W$ be a linear map of vector spaces. Then T is 1-1 if and only if $\mathcal{N}(T) = \{0\}$.
2. Proof: $(\Longleftarrow)$ $T(u) = T(v) \implies T(u-v) = 0 \implies u = v$.
   $(\Longrightarrow)$ $v \in \mathcal{N}(T) \implies T(v) = 0 = T(0) \implies v = 0$.
3. **Definition.** A $1-1$ and onto linear transformation $T : V \to W$ of vector spaces V and W over the same field of scalars is called an **isomorphism.** In this case, we write $V \simeq W$ and say that V and W are isomorphic.
4. **Proposition.** Let V, W be vector spaces over $\mathbb{F}$. . Assume V is finite dimensional with $\{v_1, \dots, v_n\}$ as a basis. Let $(w_1, \dots, w_n)$ be an arbitrary sequence of vectors in W. Then there is a unique linear map $T : V \to W$ with $T(v_i) = w_i$, for all $i = 1, \dots, n$.
5. **Proof**: (Uniqueness) For any $v \in V$, write $v = a_1v_1 + \cdots + a_nv_n$, for scalars a_i. Then $T(v) = a_1T(v_1) + \cdots + a_nT(v_n) = a_1w_1 + \cdots + a_nw_n$. So T is determined by $(w_1, \dots, w_n)$.
6. **(Existence)** Given $v \in V$ write (uniquely) $v = a_1v_1 + \cdots + a_nv_n$, for scalars a_i and then define $T(v) = a_1w_1 + \cdots + a_nw_n$.
7. Show that T is linear.

Linear Transformations: Rank and Nullity

1. **The rank-nullity Theorem.** Let $T : V \to W$ be a linear transformation of vector spaces where V is finite dimensional. Then
$$\mathsf{rank}(T) + \text{ nullity } (T) = \dim V.$$
2. Proof: Suppose $\dim V = n$. Let $B = \{v_1, v_2, \dots, v_k\}$ be a basis of $\mathcal{N}(T)$.
3. Now extend B to a basis $C = \{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_{n-k}\}$ of V.
4. Now show that a basis of $\operatorname{Im}(T)$ is
$$D = \{T(w_1), T(w_2), \dots, T(w_{n-k})\}.$$
5. Note that any $v \in V$ can be expressed uniquely as
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}.$$
6. This implies that
$$\begin{aligned} T(v) &= \alpha_1 T(v_1) + \cdots + \alpha_k T(v_k) + \beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}) \\ &= \beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}). \end{aligned}$$
7. Hence $\operatorname{Im} T = L(D)$.

Proof of the Rank-Nullity Theorem

1. Now show that D is linearly independent. Suppose there are scalars $\beta_1, \ldots, \beta_{n-k}$ such that
$$\beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}) = T(\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}) = 0.$$
2. Then $\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k} \in \mathcal{N}(T)$. Therefore there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that
$$\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_{n-k} w_{n-k} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$
$$\implies \beta_1 = \beta_2 = \cdots = \beta_{n-k} = 0.$$
3. Hence D is a basis of $\operatorname{Im} T$. Thus
$$\mathsf{rank}(T) = n - k = \dim V - \dim \mathcal{N}(T).$$
4. Therefore $\mathsf{rank}(T) + \text{nullity }(T) = \dim V$.
5. **Corollary.** If there exists an isomorphism $T : V \to W$ of finite dimensional vector spaces V, W then $\dim V = \dim W$.
6. **Proof.** If T is an isomorphism then T is $1-1$. Hence nullity $(T) = 0$. Since T is onto $\mathsf{rank} T = \dim W$. Hence
$$\mathsf{rank} T = \mathsf{rank} T + \text{nullity } T = \dim V = \dim W.$$

Sum of two subspaces and its dimension

1. **Definition.** Let V, W be subspaces of a vector space U. Then the sum of V and W, denoted $V + W$, is the subspace

$$V + W = \{x + y \mid x \in V, y \in W\}.$$

2. **Theorem.** Let V, W be subspaces of a finite dimensional vector space U. Then

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W).$$

3. Proof: We shall give a sketch of a proof leaving you to fill in the details.
4. Consider the set $V \times W = \{(v, w) : v \in V, w \in W\}$. This set is a vector space with component-wise addition and scalar multiplication.
5. Check that the dimension of this space is $\dim V + \dim W$.
6. Define a linear map $T : V \times W \to V + W$ by $T((v, w)) = v - w$.
7. Check that T is onto and that the nullspace of T is $\{(v, v) : v \in V \cap W\}$.
8. The result now follows from the rank nullity theorem for linear maps.

Coordinate vectors with respect to a basis

1. Let V be a finite dimensional vector space of dimension n over $\mathbb{F}$.
2. By an ordered basis of V we mean a sequence $v_1, v_2, \ldots, v_n$ of distinct vectors of V such that the set $B = \{v_1, \ldots, v_n\}$ is linearly independent.
3. Let $u \in V$. Then there are uniquely determined $a_1, a_2, \ldots, a_n \in \mathbb{F}$ so that
$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,\ a_i \in \mathbb{F}.$$
4. Define the coordinate vector of u with respect to the ordered basis B by
$$[u]_B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t.$$
5. Note that for vectors $u, v \in V$ and scalar $a \in \mathbb{F}$, we have
$$[u+v]_B = [u]_B + [v]_B,\ \ [av]_B = a[v]_B.$$
6. Therefore the map $T : V \to \mathbb{F}^n$ defined as $T(v) = [v]_B$ is a linear map.
7. **Theorem.** Let V be a vector space over $\mathbb{F}$ and $\dim V = n$. Then $T : V \to \mathbb{F}^n$ given by $T(v) = [v]_B$ is an isomorphism.
8. **Proof.** Let $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$. T is clearly onto. Let $T(v) = a_1[v_1]_B + a_2[v_2]_B + \cdots + a_n[v_n]_B = (a_1, a_2, \ldots, a_n) = 0$. Hence each $a_j = 0$. Thus $v = 0$. Hence $n = \dim V = \mathsf{rank} T = \dim \mathbb{F}^n = n$.

Matrices and Linear Transformations

1. Let V and W be finite dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Suppose $E = (v_1, v_2, \ldots, v_n)$ is an ordered basis for V and $F = (w_1, w_2, \ldots, w_m)$ is an ordered basis for W.

2. Let $T : V \longrightarrow W$ be a linear transformation.

3. We define $M_F^E(T)$, the matrix of T with respect to the ordered bases E and F, to be the $m \times n$ matrix whose jth column is $[T(v_j)]_F$:

$$M_F^E(T) \quad = \quad [[T(v_1)]_F \; [T(v_2)]_F \; \cdots \; [T(v_n)]_F] \, .$$

4. **Example**: Let A be an $m \times n$ matrix over $\mathbb{F}$ and consider the linear map $T_A : \mathbb{F}^n \to \mathbb{F}^m$ given by $T_A(v) = Av$, for $v \in \mathbb{F}^n$

5. Consider the the standard bases $E = (e_1, \ldots, e_n)$ of $\mathbb{F}^n$ and $F = (e_1, \ldots, e_m)$ of $\mathbb{F}^n$. Then

$$M_F^E(T_A) = [[Ae_1]_F, [Ae_2]_F, \ldots, [Ae_n]_F] = A.$$

Matrices and Linear Transformations

1. Let $\mathcal{L}(V,W)$ denote the set of all linear transformations from V to W. Suppose $S,T \in \mathcal{L}(V,W)$ and c is a scalar.
2. For any $x \in V$, define
$$\begin{aligned}(S+T)(x) &= S(x)+T(x)\\ (cS)(x) &= cS(x)\end{aligned}$$
3. It is easy to show that $\mathcal{L}(V,W)$ is a vector space under these operations.
4. **Proposition.** Fix ordered bases E and F of V and W respectively. For all $S,T \in \mathcal{L}(V,W)$ and scalars c we have
$$M_F^E(S+T) = M_F^E(S)+M_F^E(T) \tag{1}$$
$$M_F^E(cS) = cM_F^E(S) \tag{2}$$
$$M_F^E(S) = M_F^E(T) \iff S=T. \tag{3}$$
5. Proof: We shall prove only (3). Let $E=\{u_1,u_2,\ldots,u_n\}$ be an ordered basis of V and $F=\{v_1,v_2,\ldots,v_m\}$ be an ordered basis of W. Then
$$M_F^E(S)=[[S(u_1)]_F,\ldots,[S(u_n)]_F] \;\; M_F^E(T)=[[T(u_1)]_F,\ldots,[T(u_n)]_F.]$$
6. $M_F^E(S)=M_F^E(T) \iff [S(u_j)]_F=[T(u_j)]_F$ for all $j \iff S=T$.

1. **Proposition.** Suppose V, W are vector spaces of dimensions n, m respectively. Suppose $T : V \longrightarrow W$ is a linear transformation. Let $E = (v_1, \ldots, v_n), F = (w_1, \ldots, w_m)$ be ordered bases of V, W resp. Then

$$[T(v)]_F = M_F^E(T)[v]_E, \quad v \in V.$$

2. Proof: Let $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$. Then $[v]_E = (a_1, a_2, \cdots, a_n)^t$.
3. Therefore

$$T(v) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$$

4. We have

$$\begin{aligned}
[T(v)]_F &= [a_1 T(v_1) + \cdots + a_n T(v_n)]_F \\
&= a_1 [T(v_1)]_F + \cdots + a_n [T(v_n)]_F \\
&= [[T(v_1)]_F \; [T(v_2)]_F \cdots [T(v_n)]_F] \, (a_1, a_2, \cdots, a_n)^t \\
&= M_F^E(T)[v]_E.
\end{aligned}$$

Matrices and Linear Transformations

1. **Proposition.** Suppose U, V, W are vector spaces of dimension n, p, m respectively. Suppose $T : U \longrightarrow V$ and $S : V \longrightarrow W$ are linear transformations. Suppose E, F, G are ordered bases of U, V, W resp. Then

$$M_G^E(S \circ T) = M_G^F(S) M_F^E(T).$$

2. Proof: Let $E = (u_1, u_2, \ldots, u_n)$. Then, the jth column of $M_G^E(S \circ T)$ is

$$= [(S \circ T)(u_j)]_G = [S(T(u_j))]_G.$$

3. Now the jth column of $M_G^F(S) M_F^E(T)$ is

$$\begin{aligned} &= M_G^F(S)(j\text{th column of } M_F^E(T)) \\ &= M_G^F(S)[T(u_j)]_F \\ &= [S(T(u_j))]_G \quad (\text{since } [S(v)]_G = M_G^F(S)[v]_F. \end{aligned}$$

4. Hence $M_G^E(S \circ T) = M_G^F(S) M_F^E(T)$.

Coordinate Vectors: Change of Basis

1. Let $B = (v_1, v_2, \ldots, v_n)$ and $C = (u_1, \ldots, u_n)$ are ordered bases of V.
2. **Question.** Given $u \in V$, what is the relation between $[u]_B$ and $[u]_C$?
3. Define $M_B^C(I)$, the transition matrix from C to B, to be the $n \times n$ matrix of the identity map $I : (V, C) \to (V, B)$.

$$M_B^C(I) \quad = \quad [[u_1]_B \; [u_2]_B \; \cdots \; [u_n]_B] \, .$$

4. Hence for all $u \in V$, we have

$$[u]_B = M_B^C(I)([u]_C).$$

5. **Proposition.** Let V be a finite dimensional vector space and B and C be two ordered bases of V. Then

$$M_B^C(I) = (M_C^B(I))^{-1}.$$

6. Proof: Consider the sequence of identity maps

$$(V, C) \xrightarrow{I} (V, B) \xrightarrow{I} (V, C).$$

7. Then $M_B^C(I) M_C^B(I) = M_C^C(I) = I \implies M_B^C(I) = M_C^B(I)^{-1}$.

Coordinate Vectors: Change of Basis

1. Example: Let $V = \mathbb{R}^3$ and let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2. Consider the ordered bases $B = (v_1, v_2, v_3)$ and $C = (u_1, u_2, u_3)$. Thus

$$M = M^(I) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

3. If $u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, then $[u]_C = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $[u]_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Matrix of a linear operator under change of bases

1. Let V be a finite dimensional vector space. A linear map $T : V \to V$ is said to be a linear operator on V. Let B, C be ordered bases of V.
2. The matrix $M_B^B(T)$ is the matrix of T with respect to the ordered basis B.
3. Recall that the transition matrix $M_B^C(I)$ from C to B is the matrix of the identity map $I : (V, C) \to (V, B)$, and $M_B^C(I) = M_C^B(I)^{-1}$.
4. **Theorem.** Let V be a finite dimensional vector space and B, C be two bases of V. Then

$$M_B^B(T) = (M_C^B(I))^{-1} M_C^C(T) M_C^B(I).$$

5. Proof: Consider the sequence of linear operators where the bases used for computation of matrices of the linear transformations are specified:

$$(V, B) \xrightarrow{I} (V, C) \xrightarrow{T} (V, C) \xrightarrow{I} (V, B).$$

6. Since $T = I \circ T \circ I$, $M_B^B(T) = M_B^C(I) M_C^C(T) M_C^B(I)$. Therefore

$$M_B^B(T) = (M_C^B(I))^{-1} M_C^C(T) M_C^B(I).$$

1. **Example**: Consider the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(e_1) = e_1,\ \ T(e_2) = e_1 + e_2.$$

2. Let $C = (e_1, e_2)$ and $B = (e_1 + e_2,\ \ e_1 - e_2)$ are ordered bases of $\mathbb{R}^2$.

$$M_C^C(T) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ M_C^B(I) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \ M_B^C(I) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

$$M_B^B(T) \quad = \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

Coordinate Vectors: Change of Basis

1. Example: Let M be the $(n+1) \times (n+1)$ matrix, with rows and columns indexed by $\{0, 1, \ldots, n\}$, and with entry in row i and column j, $0 \leq i, j \leq n$, given by $\binom{j}{i}$.
2. We show that M is invertible and find the inverse explicitly.
3. Consider the vector space $\mathcal{P}_n(\mathbb{R})$ of real polynomials of degree $\leq n$.
4. Then $B = (1, x, x^2, \ldots, x^n)$ and $C = (1, x-1, (x-1)^2, \ldots, (x-1)^n)$ are both ordered bases of $\mathcal{P}_n(\mathbb{R})$.
5. We find the change of basis matrix $M = M_C^B(I)$. For $0 \leq j \leq n$ we have

$$\begin{aligned} x^j &= (1 + (x-1))^j \\ &= \sum_{i=0}^{j} \binom{j}{i} (x-1)^i \\ &= \sum_{i=0}^{n} \binom{j}{i} (x-1)^i, \end{aligned}$$

6. where in the last step we have used the fact that $\binom{j}{i} = 0$ for $i > j$.

Coordinate Vectors: Change of Basis

1. Thus $M = \left[\binom{j}{i}\right] = M_C^B(I)$ and hence it is invertible.
2. To find the inverse of M, use the fact that $M^{-1} = (M_C^B)^{-1} = M_B^C(I)$.
3. For $0 \leq j \leq n$, use the Binomial Theorem:

$$\begin{aligned}(x-1)^j &= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} x^i \\ &= \sum_{i=0}^{n} (-1)^{j-i} \binom{j}{i} x^i.\end{aligned}$$

4. Thus the entry in row i and column j of M^{-1} is $(-1)^{j-i}\binom{j}{i}$. Therefore

$$\left[\binom{j}{i}\right]^{-1} = \left[(-1)^{j-i}\binom{j}{i}\right].$$