PH 112: Quantum Physics and Applications

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Week 03 Lecture 1: Heisenberg Uncertainty Principle

D3, Spring 2023

Heisenberg Uncertainty Principle

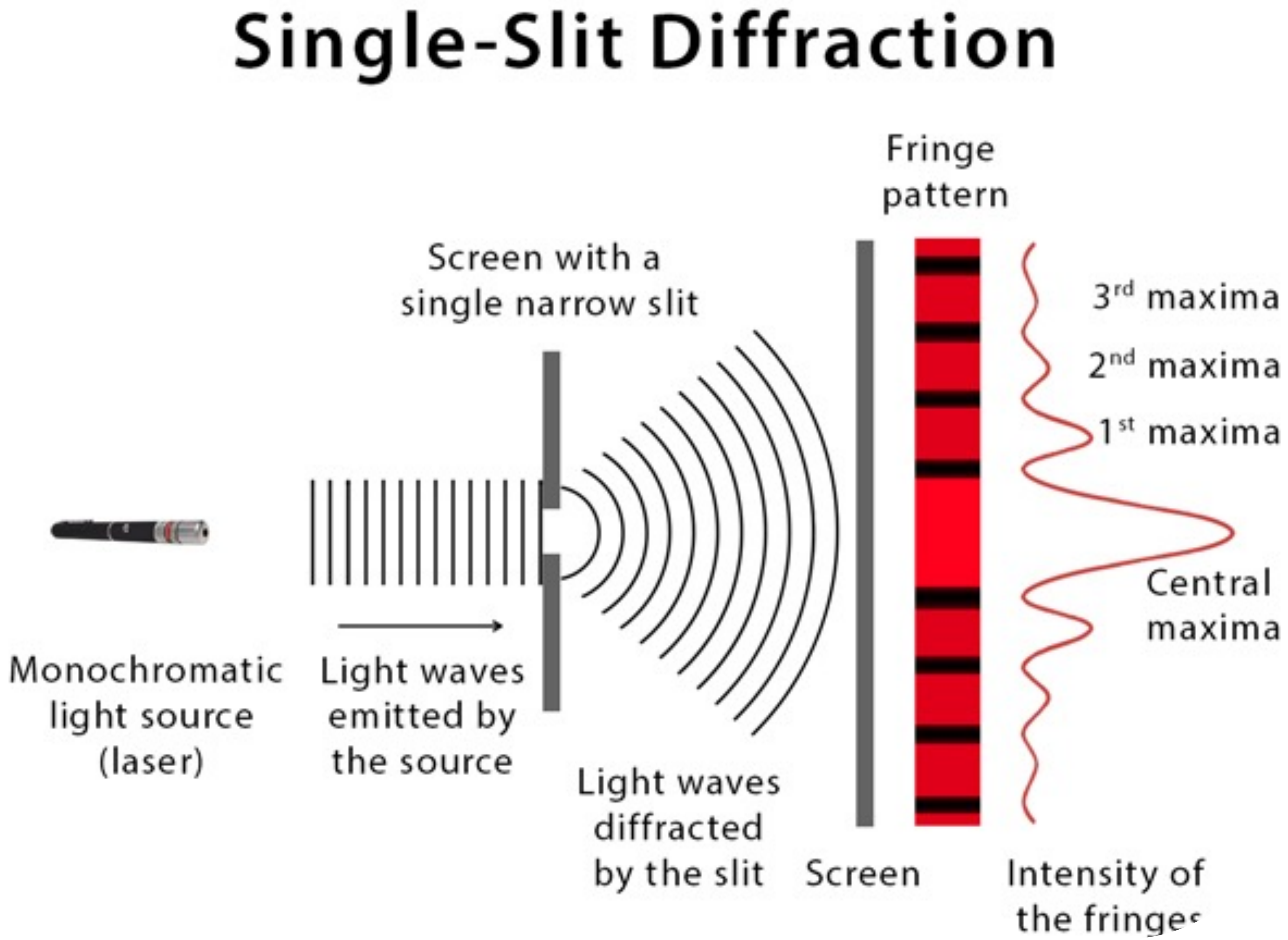
- He realised that in the microscopic world, one can not measure any property of particles without interacting with it in some way.
- This introduces unavoidable uncertainty in the measurement.
- One of the fundamental consequences of quantum mechanics is that it is impossible to simultaneously determine the position and momentum of a particle with complete precision.
- One can never measure complementary variables exactly
  - Position and momentum
  - Energy and time
  - Spin of different axis
  - Entanglement and coherence

Diffraction of light and Uncertainty principle

Single Slit Diffraction: Wave Picture

Geometrical optics picture breaks down when slit width becomes comparable to wavelength (λ).

a is width of the slit,
d is horizontal distance between the screen and slit.



Single Slit Diffraction: Wave Picture

$$\sin\theta = \frac{m\lambda}{a}$$

Position of dark fringes in single-slit diffraction

Let us now make small angle approximation $\sin\theta \approx \tan\theta \approx \theta = z_{min}/R$

$$z_{\min} = \frac{Rm\lambda}{a}$$

Positions of intensity minima (z_{min}) of diffraction pattern on screen, measured from central position.

The above expression is similar to one derived for 2-slit interference experiment:

$$z_{int} = R\,\frac{n\,\lambda}{d}$$

In the Interference experiment, z_{int} are positions of intensity maxima.

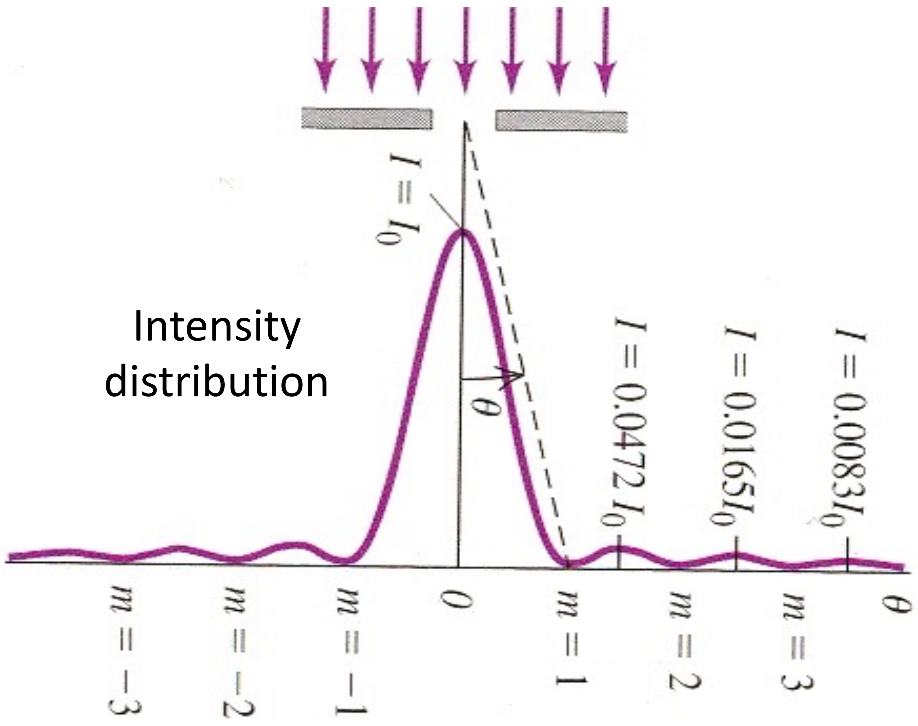
Width of central maximum

The width of the central maximum is the distance between the m = +1 minimum and the m=-1 minimum:

$$\Delta z = \frac{R\lambda}{a} - \left(-\frac{R\lambda}{a}\right) = \frac{2R\lambda}{a}$$

Narrower the slit, the more the diffraction pattern "spreads out"

Intensity distribution



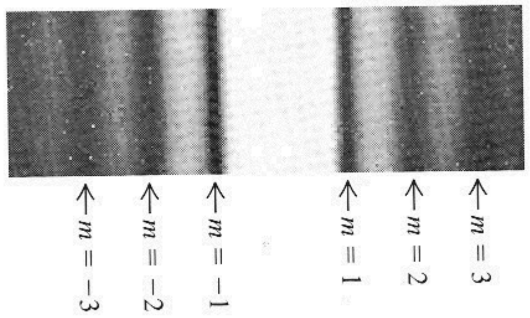


image of diffraction pattern

Single Slit Diffraction: Photon Picture

- In small angle approximation $$\sin\theta = \frac{\lambda}{a} \Rightarrow \theta = \frac{\lambda}{a}$$

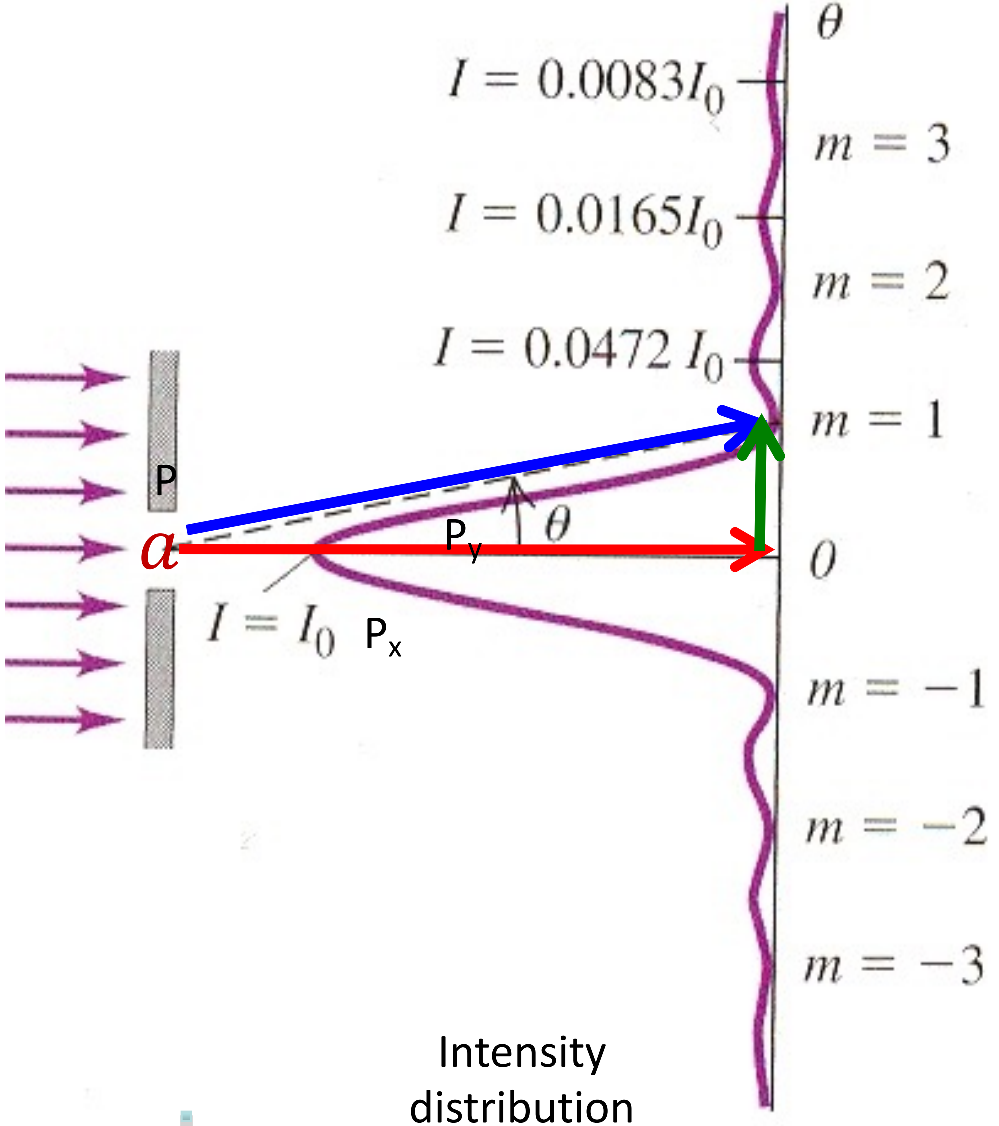
- Photons directed towards outer part of central maximum have momentum

$$\bar{p} = \bar{p}_x + \bar{p}_y$$

$$p_y = \theta p_x = p_x \frac{\lambda}{a} = p_x \frac{h}{p_x a} = \frac{h}{a}$$

- Localizing photons in the y-direction to a slit of width a leads to a spread of y-momenta of at least h/a.

- More we seek to localize a photon (i.e define its position) by shrinking the slit width $(a \sim \Delta y)$ the more spread (uncertainty) we induce in its momentum: $$\Delta p_y \, \Delta y \sim h$$

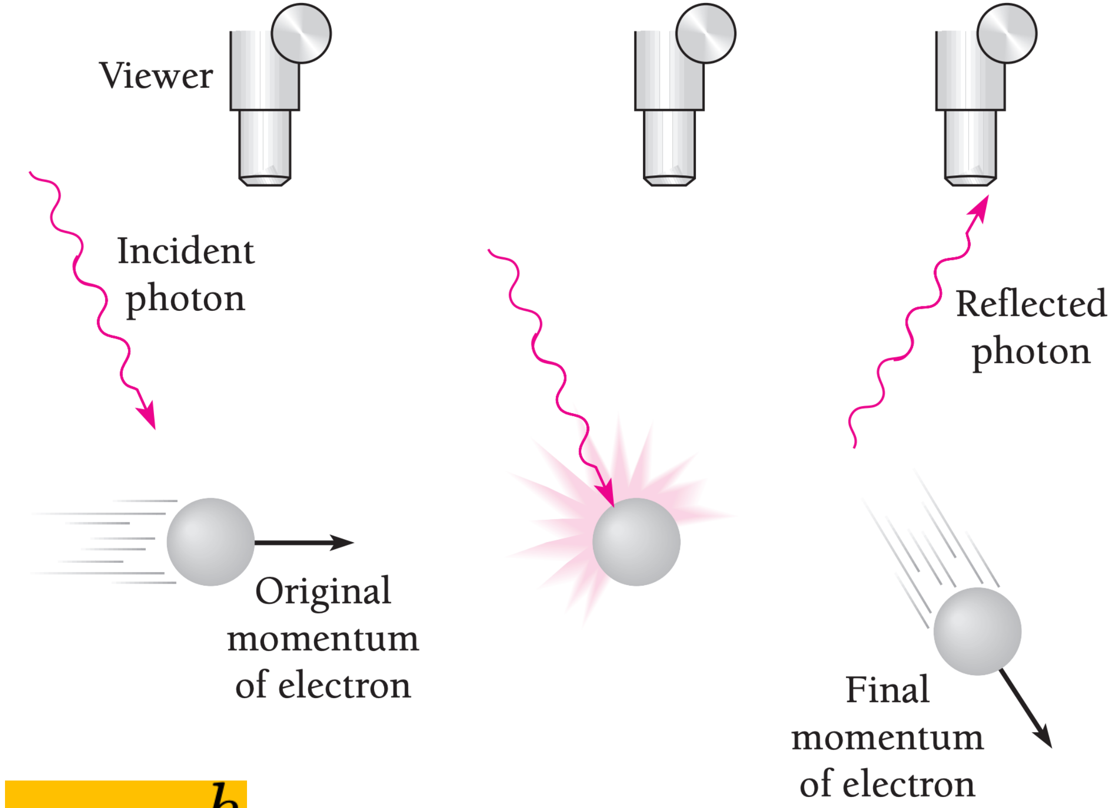


Intensity distribution

Heisenberg Microscope

Uncertainty principle for particles

Consider we want to measure the position and momentum of an electron at a certain time.



Act of measurement may be to shine a light of wavelength (λ) on the electron!

Each photon has momentum $p = h/\lambda$.

When one of these photons bounces off the electron, original momentum of the electron will be changed. Change in electron momentum

$$\Delta p \sim \frac{h}{\lambda}$$

Due to wave nature of light, minimum uncertainty in position of electron is one photon wavelength

$$\Delta x \sim \lambda$$

Uncertainty principle for particles

- By Planck's law $E = hc/\lambda$, $\Longrightarrow$

  A photon with a short wavelength has a large energy $\Longrightarrow$ Imparts a larger kick to electron

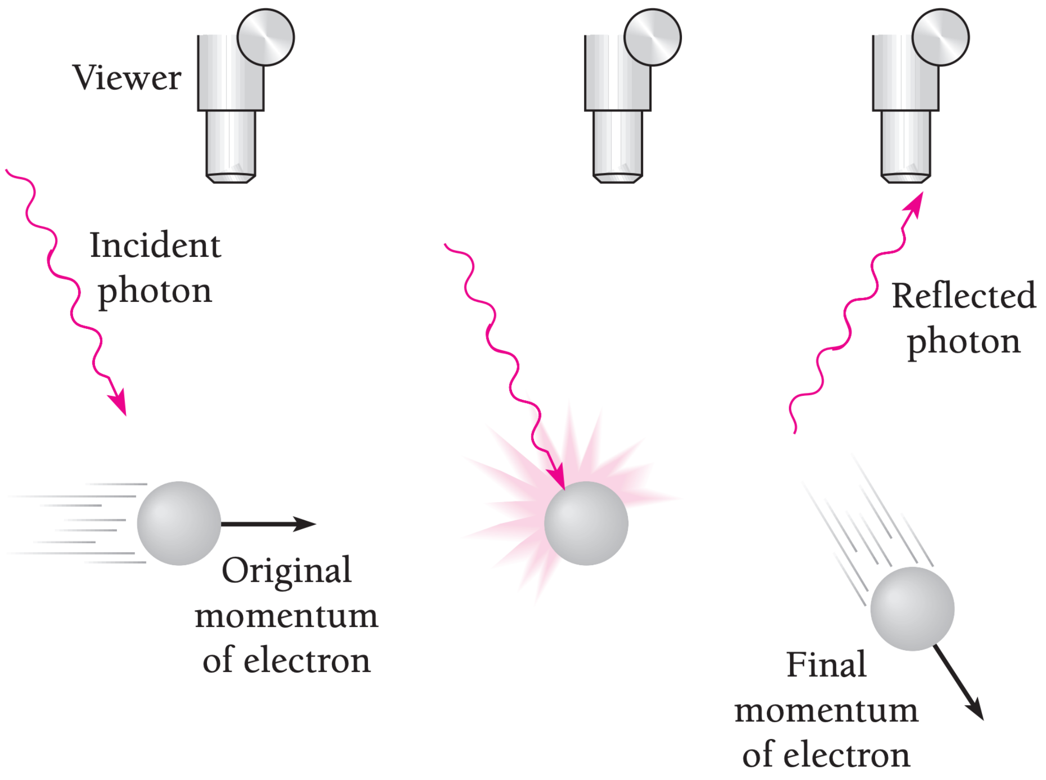
- To determine accurate momentum, electron must only be given a small kick!

- If we use light with short wavelength:

We can accurately measure position not momentum.

- If we use light with long wavelength:

We can accurately measure momentum not position.



Heisenberg microscope can help us understand this.

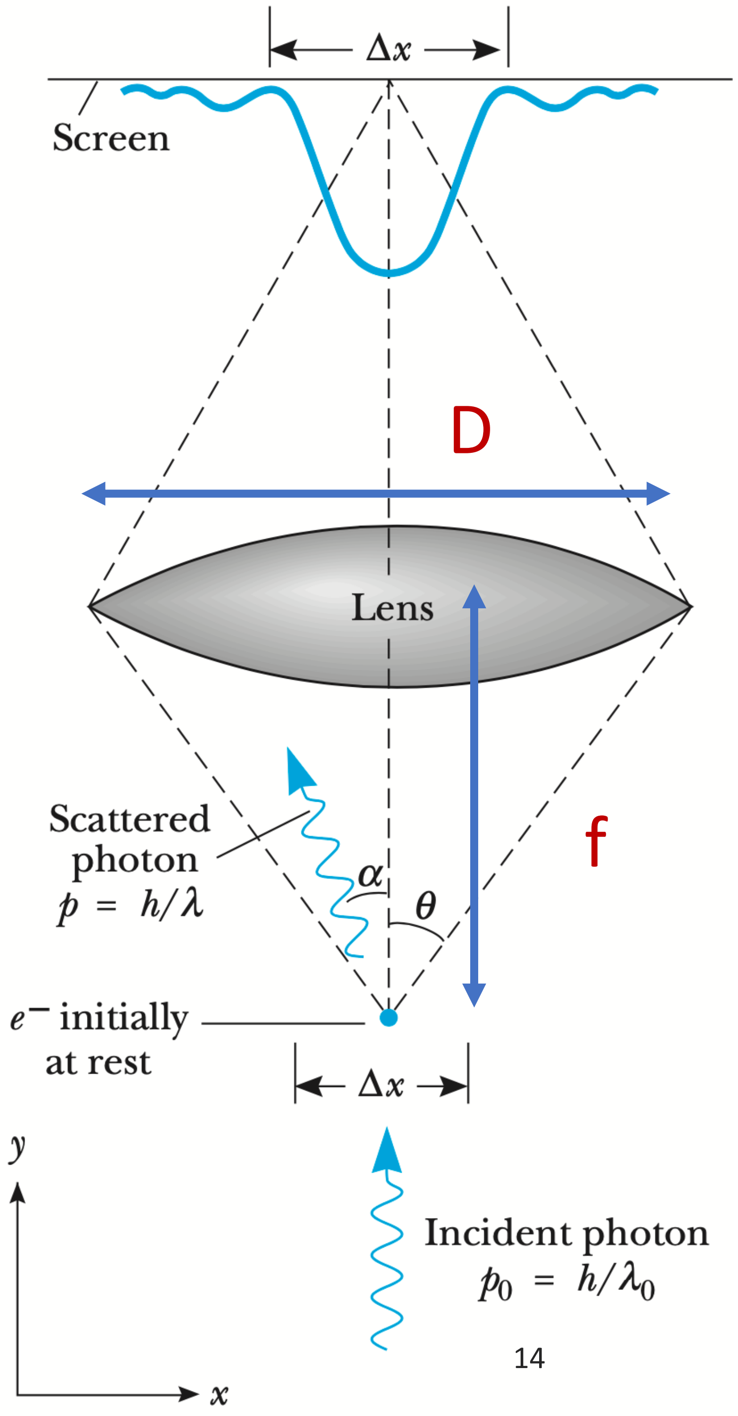
Heisenberg Microscope

Heisenberg demanded operational definitions of physical quantities. For example:

If we want to know the position of an object (like an electron), we need to know what is meant by position of an object. In other words, one needs definite experiments by which "position of electron" can be measured!

Heisenberg came up with a thought experiment that illustrates the trade-off between position of an electron and its momentum.

- Let *D and f* be the diameter and focal-length of the lens.
- We know the position of electron, when we see the photon through the lens.



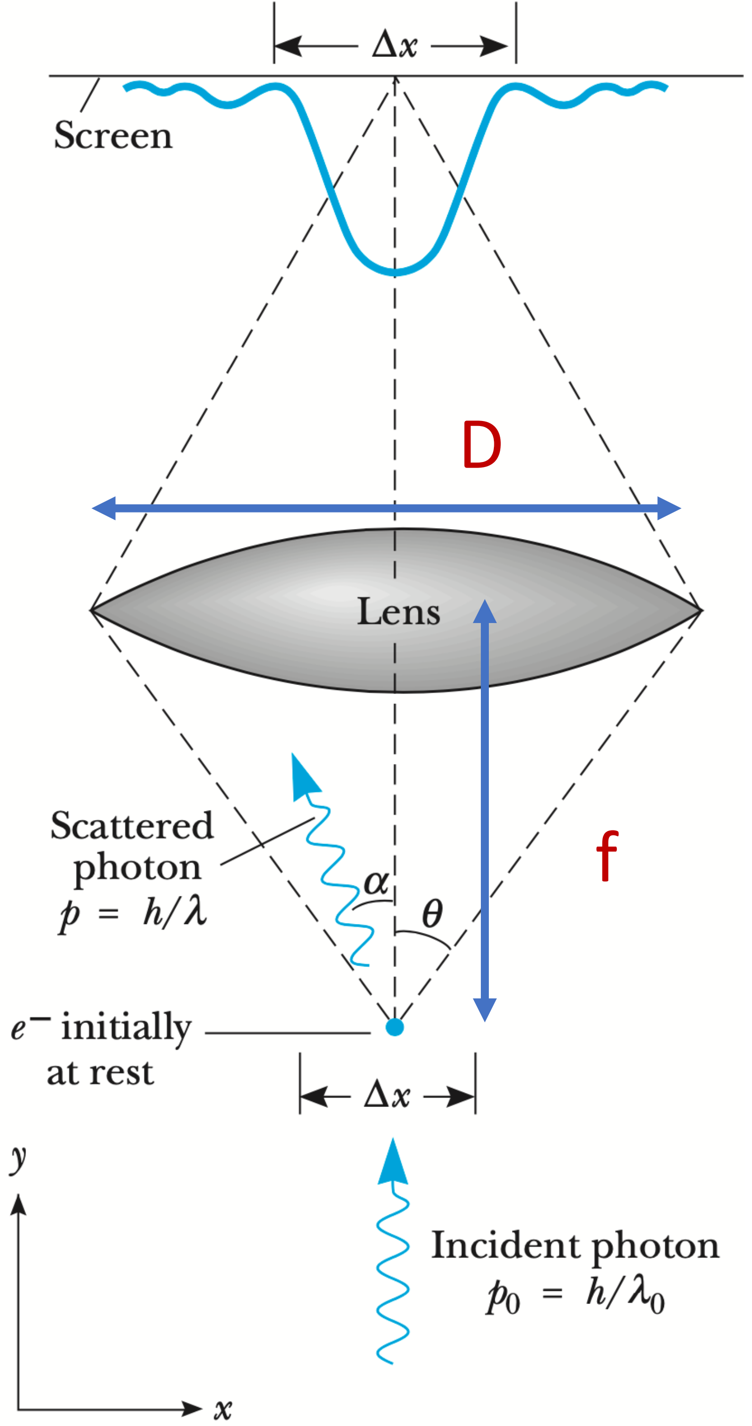
Heisenberg Microscope

Aim: To image an electron using optical system of wavelength (λ) with lens with diameter (D) and focal length (f).

Assume: Electron's initial momentum is precisely known at focal length (f). For convenience assume $p_{initial} = 0$.

Minimum angular resolving power of lens is $\boxed{\sin\theta \approx \frac{\lambda}{D}}$

Minimum resolving power is achieved at f.
Uncertainty in electron's transverse position is $\boxed{\Delta x \sim f\frac{\lambda}{D}}$



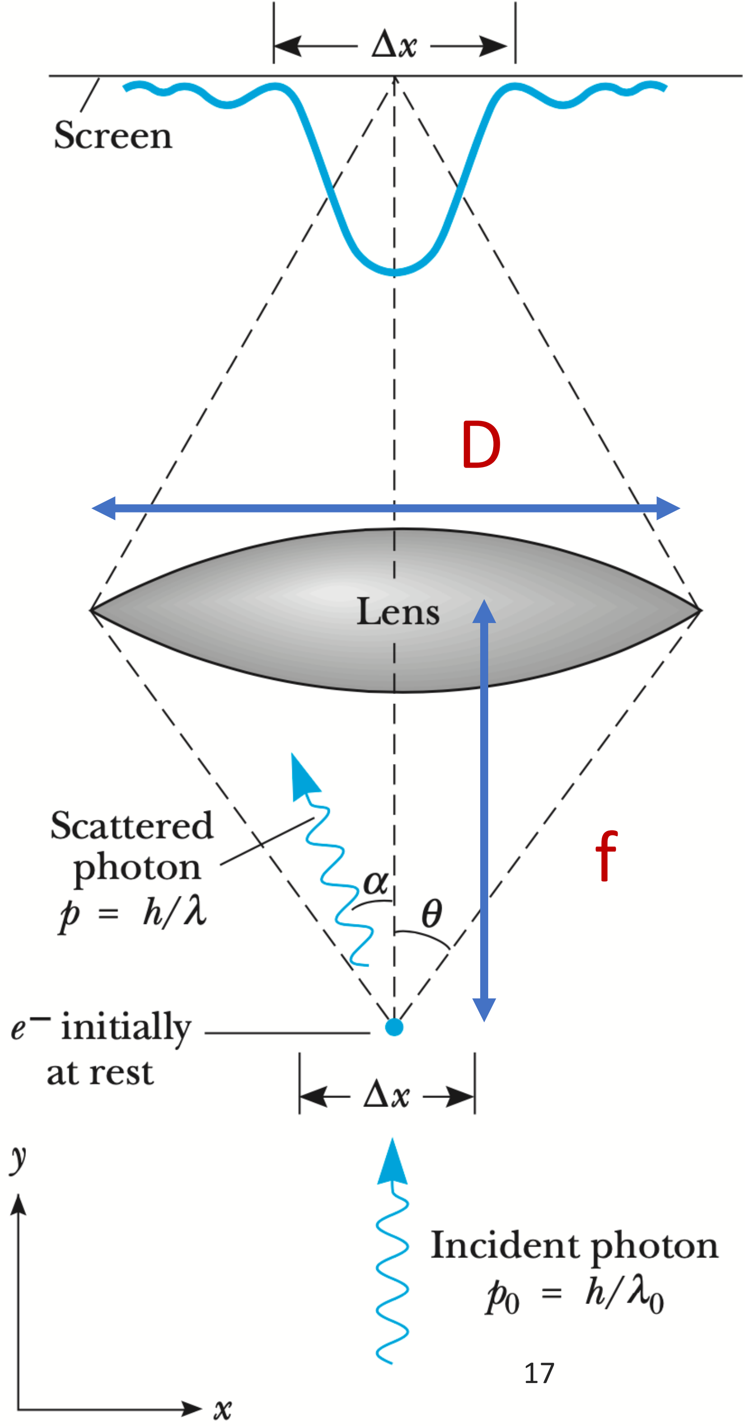
Heisenberg Microscope

We have
$$\tan\theta = \frac{D/2}{f}$$

Under small angle approximation
$$\tan\theta \sim \theta = \frac{D}{2f}$$

Substituting the above expression in Δx expression, we have
$$\Delta x \sim \frac{\lambda}{2\theta}$$

We can reduce Δx uncertainty by minimizing the ratio $\frac{\lambda}{\alpha}$, by using small wavelength and wide-angle lens.



Heisenberg Microscope

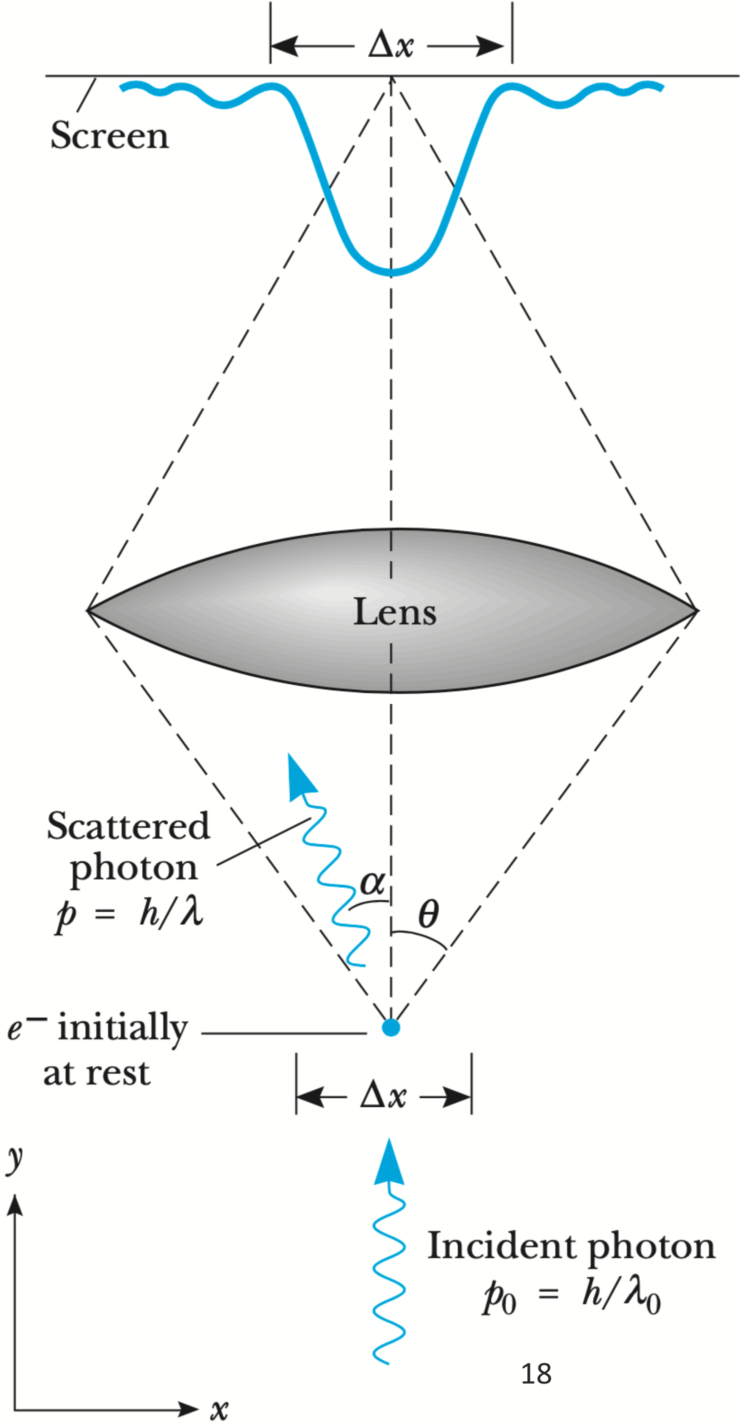
- Suppose we know the initial momentum of electron precisely. For convenience assume $p_{initial} = 0$.

- From Compton scattering, we know that the collision of photon with electron changes the photon wavelength:

$$\lambda - \lambda_0 = \lambda_C(1 - \cos\alpha) = 2\lambda_C \sin^2(\alpha/2)$$

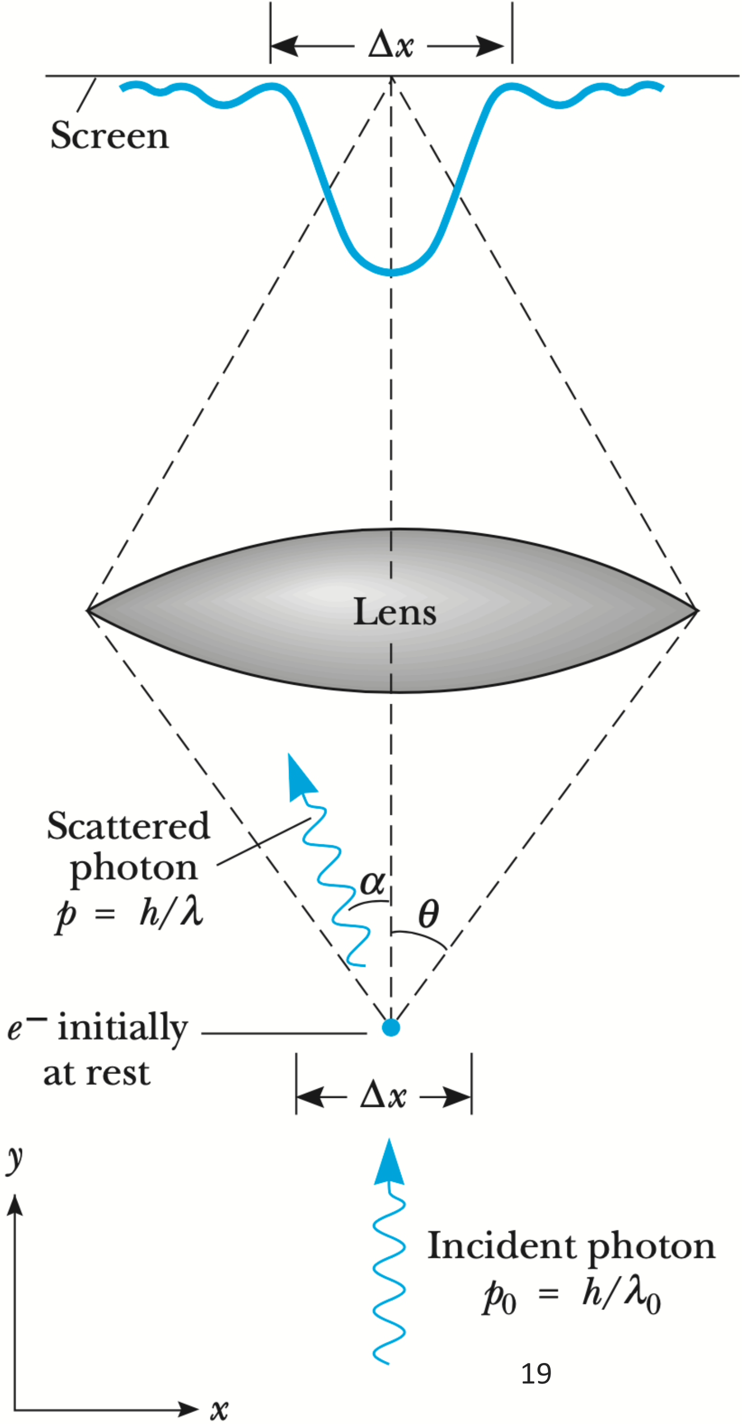
- α is the angle between scattered photon and incident photon. Depending of α, λ is different.

- We will not know how much momentum electron will carry after collision. However, we can estimate the electron momentum (p_e) from the photon through the lens.



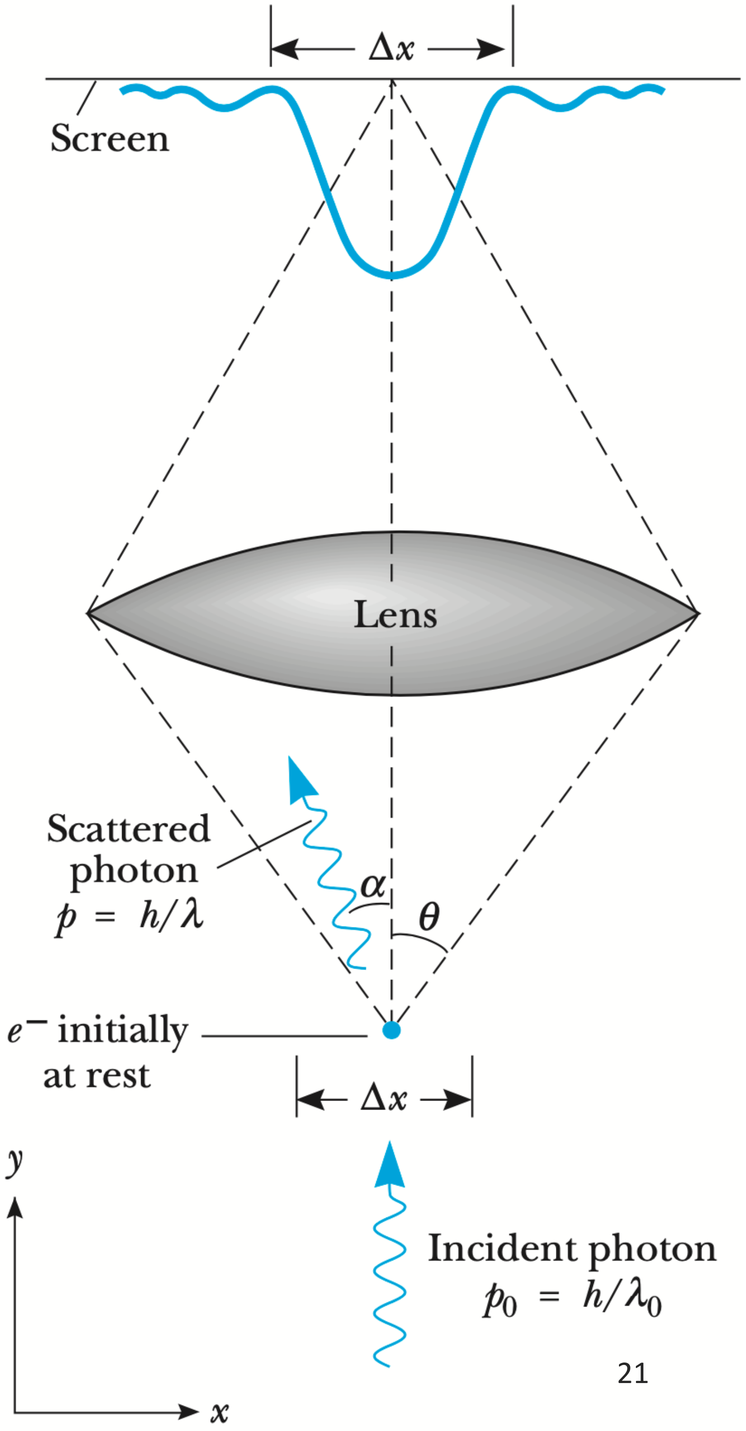
Momentum Uncertainty

- In order to see the Photon, photon has to be within the $\pm\theta$ subtended with the lens.
- Momentum of the electron along x-axis is $p_e \ \sin\theta$
- Minimum and maximum electron momentum along x-axis after collision is $-\frac{h}{\lambda_1} sin\theta \qquad \frac{h}{\lambda_2} sin\theta$
- Consider the angle θ to be very small $\sin\theta \ \approx \theta$
- Assume the change in the momentum is small $\lambda_0 \approx \lambda_1 \approx \lambda_2$
- The range of possible electron momentum is $-\frac{h}{\lambda_0}\theta$ to $\frac{h}{\lambda_0}\theta$



Heisenberg Microscope

- Momentum Uncertainty is $\Delta p_x = \frac{2\,h}{\lambda}\,\theta$
- Position Uncertainty is $\Delta x = \frac{\lambda}{\theta}$
- So, if we attempt to reduce uncertainty in position by decreasing λ, we INCREASE the uncertainty in the momentum of the particle!!!!!!
- From these two equations, we have $\Delta p_x \Delta x = 2\,h$
- The very process of measuring one quantity (position) alters a complementary property (momentum).



The Uncertainty Principle

Our microscope thought experiments give us an estimate for the uncertainties in position and momentum:

$$\Delta x \Delta p \sim h$$

Heisenberg uncertainty principle states: It is impossible to simultaneously determine the position and momentum of a particle with complete precision.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

This also means you can't use classical physics because you can't specify (exactly) the initial conditions!

Energy time Uncertainty

Kinetic Energy Minimum

- Let us consider the situation that we know the approximate position of a particle. Like, an electron somewhere inside an atom.

- Let the uncertainty in particle's position be Δx. Uncertainty in momentum is

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

- Uncertainty in KE of a particle is

$$KE = \frac{p^2}{2m} \implies KE + \Delta(KE) = \frac{(p+\Delta p)^2}{2m} \implies \Delta(KE) = \frac{(\Delta p)^2}{2m}$$

- Hence, the particle can not have zero KE

$$\Delta(KE) = \frac{\hbar^2}{8m\Delta x^2}$$

Energy Uncertainty

- If we are uncertain of the exact position of a particle, for example an electron somewhere inside an atom, the particle can't have zero kinetic energy.

$$K_{\min} = \frac{p_{\min}{}^2}{2m} \geq \frac{(\Delta p)^2}{2m} \geq \frac{\hbar^2}{2m\ell^2}$$

- The energy uncertainty of a Gaussian wave packet is

$$\Delta E = h\,\Delta f = h\frac{\Delta\omega}{2\pi} = \hbar\,\Delta\omega$$

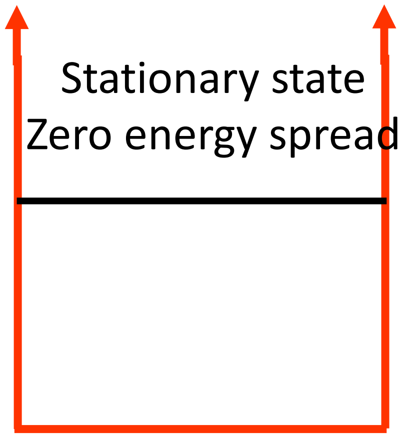
- Combined with the angular frequency relation

$$\Delta\omega\Delta t = \frac{\Delta E}{\hbar}\Delta t = 1$$

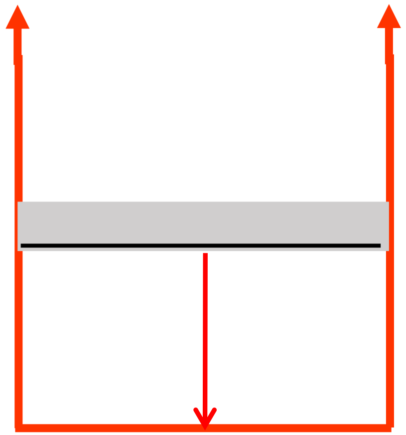
Energy-time Uncertainty

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Uncertainty principle also applies to simultaneous measurements of *energy* and *time*.



The energy can be known with perfect precision ($\Delta E = 0$), only if the measurement is made over an infinite period of time ($\Delta t = \infty$).



The more accurately we know the energy of a body, the less accurately we know how long it possessed that energy!

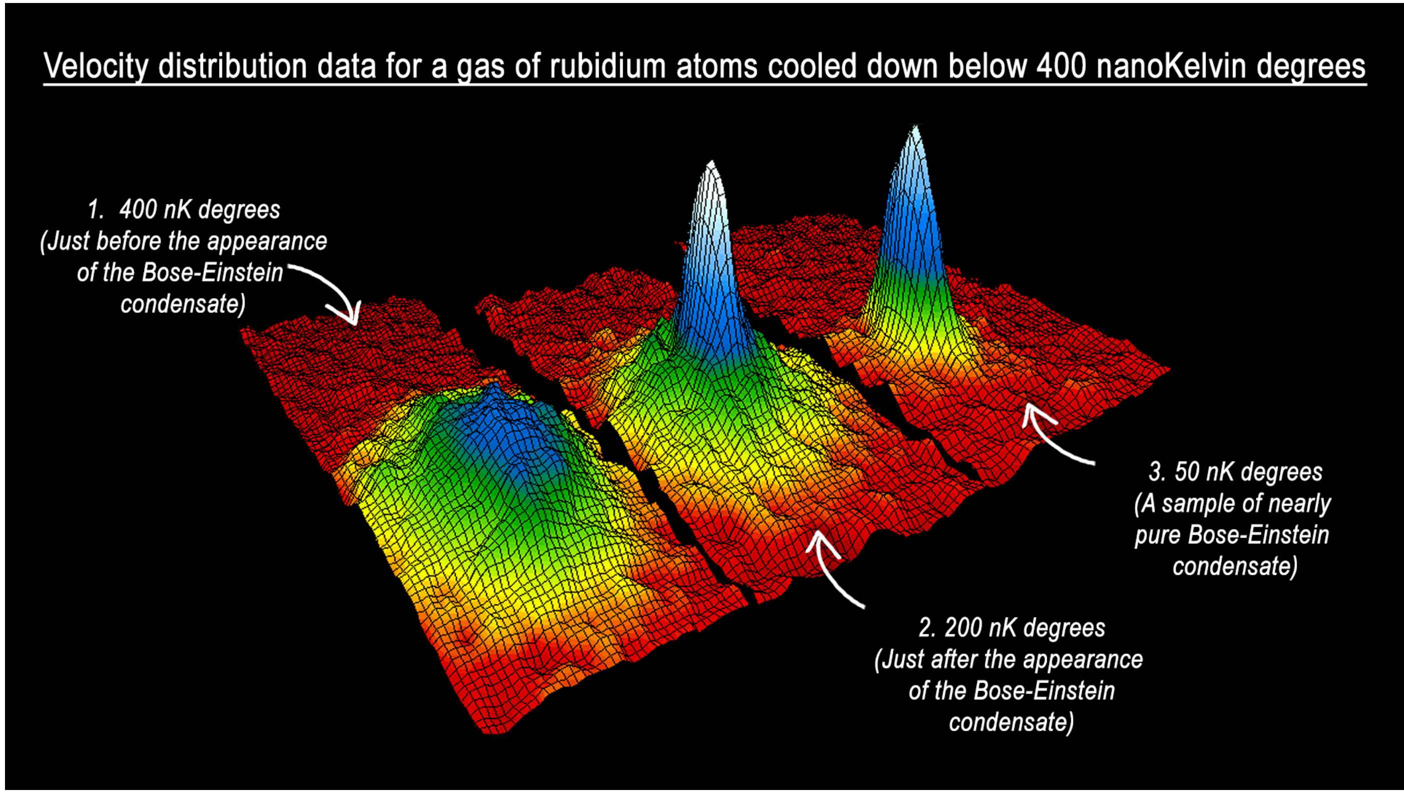
Decay to lower state with finite lifetime Δt: Energy broadening ΔE (explains, for example "natural linewidth" In atomic spectra)

Uncertainty principle in action: BEC

Velocity distribution profile of the cloud of gas as it is cooled down below 400 nK.

Large proportion of the atoms suddenly occupy a single quantum state, and this is precisely where macroscopic quantum phenomena suddenly become apparent.

Spread of momenta around the peak is found to be close to the minimum allowed by the uncertainty principle.



Spatially confined atoms – where their position is known with accuracy – have a minimum spread of momenta distribution, below which the uncertainty principle would be violated.

Summary

- The idea of a perfectly predictable universe cannot be true!

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

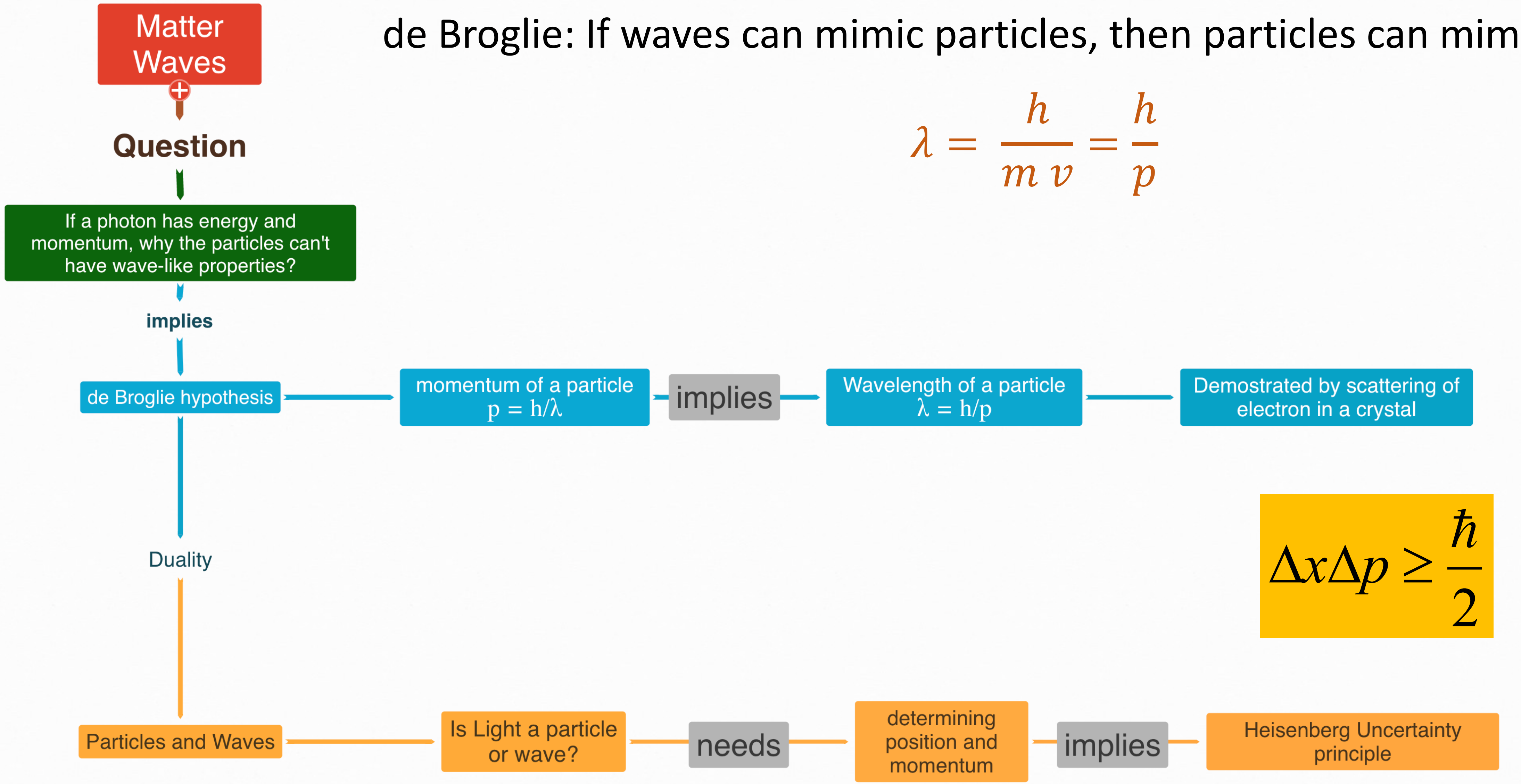
- There is no such thing as an ideal, objective observer! However nature offers probabilities which can be calculated and tested.

We will soon look at the implications of Uncertainty principle for specific cases.

Consequence of Wave-particle duality

de Broglie: If waves can mimic particles, then particles can mimic waves

$$\lambda = \frac{h}{m\,v} = \frac{h}{p}$$



$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Recommended Readings

Heisenberg Uncertainty principle

section 5.5