Chapter 4 : Determinant of matrices

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Axiomatic approach for the Determinant Function

1. Recall the formula for determinants of square matrices.

$$\det[a] = a, \ \ \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\text{and} \ \ \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei - ahf - bdi + bgf + cdh - ceg.$$

2. We will explain how these formulas and similar formulas for all square matrices can be derived using properties of determinant of matrices.
3. Our approach to determinants of square matrices is via their properties rather than via an explicit formulas as above.
4. Let $\mathbb{F}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.
5. The set of $n \times n$ matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{n\times n}$.

Axioms for determinant functions

1. Suppose that the columns of $A \in \mathbb{F}^{n \times n}$ are $A_1, A_2, \dots, A_n$,
2. Define $d : \mathbb{F}^{n \times n} \to \mathbb{F}$ by $d(A) = d(A_1, A_2, \dots, A_n)$.
3. The function d is called a **multilinear** function if for each $k = 1, 2, \dots, n$; scalars α, β and column vectors $A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_n, B, C \in \mathbb{F}^{n \times 1}$
$d(A_1, \dots, A_{k-1}, \alpha B + \beta C, A_{k+1}, \dots, A_n) =$
$\alpha\, d(A_1, \dots, A_{k-1}, B, A_{k+1}, \dots, A_n) + \beta\, d(A_1, \dots, A_{k-1}, C, A_{k+1}, \dots, A_n)$.
4. d is called an **alternating** function if for some $i \neq j$ and $A_i = A_j$, then
$$d(A_1, A_2, \dots, A_n) = 0$$
5. If $d(I) = d(e_1, e_2, \dots, e_n) = 1$ then d is called a **normalized** function.
6. **Definition.** A normalized, alternating, and multillinear function d on $n \times n$ matrices is called a **determinant function** of order n.

Properties of determinant function

1. **Lemma:** Suppose that $d(A_1, A_2, \dots, A_n)$ is a multilinear alternating function on columns of $n \times n$ matrices. Then
   (a) If some $A_k = 0$ then $d(A_1, A_2, \dots, A_n) = 0$.
   (b) $d(A_1, \dots, A_k, A_{k+1}, \dots A_n) = -d(A_1, \dots, A_{k+1}, A_k, \dots, A_n)$.
   (c) $d(A_1, \dots, A_i, \dots, A_j, \dots, A_n) = -d(A_1, \dots, A_j, \dots, A_i, \dots, A_n)$.
2. **Proof:** (a) If $A_k = 0$ then by multilinearity
$$d(A_1, \dots, 0 \cdot A_k, \dots, A_n) = 0 \cdot d(A_1, \dots, A_k, \dots, A_n) = 0.$$
3. (b) Put $A_k = B, A_{k+1} = C$. By the alternating property
$$\begin{aligned} 0 &= d(A_1, \dots, B + C, B + C, \dots, A_n) \\ &= d(A_1, \dots, B, B + C, \dots, A_n) + d(A_1, \dots, C, B + C, \dots, A_n) \\ &= d(A_1, \dots, B, C, \dots, A_n) + d(A_1, \dots, C, B, \dots, A_n) \end{aligned}$$
4. Hence $d(A_1, \dots, B, C, \dots, A_n) = -d(A_1, \dots, C, B, \dots, A_n)$.
5. (c) can be proved similarly.

Formula for the determinant of a 2×2 matrix

1. Suppose $d(A_1, A_2)$ is an alternating multilinear normalized function on 2×2 matrices $A = (A_1, A_2)$. Then

$$d \begin{bmatrix} x & y \\ z & u \end{bmatrix} = xu - yz.$$

2. Write $A_1 = xe_1 + ze_2$ and $A_2 = ye_1 + ue_2$.
3. Then using the axioms for determinant function we get

$$\begin{aligned} d(A_1, A_2) &= d(xe_1 + ze_2, ye_1 + ue_2) \\ &= d(xe_1 + ze_2, ye_1) + d(xe_1 + ze_2, ue_2) \\ &= d(xe_1, ye_1) + d(ze_2, ye_1) \\ &\quad + d(xe_1, ue_2) + d(ze_2, ue_2) \\ &= yzd(e_2, e_1) + xud(e_1, e_2) \\ &= (xu - yz)d(e_1, e_2) = xu - yz \end{aligned}$$

Uniqueness of the determinant function

1. **A Vanishing Lemma for multilinear functions:** Suppose f is a multilinear alternating function on $n \times n$ matrices and $f(e_1, e_2, \ldots, e_n) = 0$. Then $f = 0$.
2. **Proof:** Let $A = (a_{ij}) = (A_1, \ldots, A_n)$ be an $n \times n$ matrix. Write
$$A_j = a_{1j} e_1 + a_{2j} e_2 + \cdots + a_{nj} e_n.$$
3. Since f is multilinear we have
$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1} a_{h(2)2} \cdots a_{h(n)n} \, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$
4. Here the sum is over all functions $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
5. Since f is alternating we have
$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1} a_{h(2)2} \cdots a_{h(n)n} \, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$
6. where the sum is over all bijections $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Uniqueness of the determinant function

1. By using part (c) of the lemma above we see that we can write

$$f(A_1, \ldots, A_n) = \sum_h \pm a_{h(1)1} a_{h(2)2} \cdots a_{h(n)n} \, f(e_1, e_2, \ldots, e_n),$$

2. Here the sum is over all bijections $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$.
3. Therefore $f(A) = 0$.
4. **Theorem:** Let f be an alternating multilinear function on $\mathbb{F}^{n\times n}$ and d a determinant function on $\mathbb{F}^{n\times n}$.

$$f(A_1, \ldots, A_n) = d(A_1, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$

5. In particular, if f is also a determinant function then

$$f(A_1, A_2, \ldots, A_n) = d(A_1, A_2, \ldots, A_n).$$

Proof of uniqueness of determinant function

1. **Proof:** Consider the function
$$g(A_1, \ldots, A_n) = f(A_1, \ldots, A_n) - d(A_1, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$
2. Since f, d are alternating and multilinear so is g. Since $g(e_1, e_2, \ldots, e_n) = 0$ the result follows from the previous Lemma.
3. **Notation:** We denote the determinant of A by $\det A$.
4. Setting $\det[a] = a$ shows existence for $n = 1$.
5. Assume that we have shown existence of determinant function on $\mathbb{F}^{(n-1)\times(n_1)}$.
6. The determinant of an $n \times n$ matrix A can be computed in terms of $(n-1) \times (n-1)$ determinants.
7. Let A_{ij} = the $(n-1) \times (n-1)$ matrix obtained from A by deleting the $i\mathrm{th}$ row and $j\mathrm{th}$ column of A.

Existence of determinant function

1. **Theorem.** Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the function

$$f(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

   is the determinant function on $n \times n$ matrices.
2. **Proof:** Denote the function by $f(A_1, A_2, \ldots, A_n)$.
3. Suppose that the columns A_j and A_{j+1} of A are equal.
4. Then A_{1i} have equal columns except when $i = j$ or $i = j + 1$.
5. By induction $f(A_{1i}) = 0$ for $i \neq j, j+1$. Therefore

$$f(A) = a_{1j} \left[(-1)^{j+1} \det(A_{1j})\right] + \left[(-1)^{j+2} \det(A_{1j+1})\right] a_{1j+1}.$$

6. Since $A_j = A_{j+1}, a_{1j} = a_{1j+1}$ and $A_{1j} = A_{1j+1}$.
7. Therefore $f(A) = 0$ and hence $f(A_1, A_2, \ldots, A_n)$ is alternating.
8. Multilinearity of f is left as an exercise. If $A = I$ then by induction
$$f(A) = 1 \det(A_{11}) = f(e_1, e_2, \ldots, e_{n-1}) = 1.$$

Determinant of elementary and upper triangular matrices

1. **Theorem:** (i) Let U be an upper triangular or a lower triangular matrix. Then $\det U$ is the product of diagonal entries of U.
2. (ii) If $E = [e_1, \ldots, e_i + me_j, \ldots, e_n]$, for some $i \neq j$. Then $\det E = 1$.
3. (iii) If $F = [e_1, e_2, \ldots, e_j, \ldots, e_i, \ldots, e_n]$, for some $i \neq j$. Then $\det F = -1$.
4. (iv) If $G = [e_1, e_2, \ldots, me_i, \ldots, e_n]$ then $\det G = m$.
5. **Proof:** (i) Let $U = (u_{ij})$ be upper triangular. Use induction on n. The case $n = 1$ is clear. For $n \times n$ upper triangular matrix U, use the formula

$$f(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

   To see that $\det U = u_{11} u_{22} \ldots u_{nn}$.
6. (ii) Follows from part (i).
7. (iii) As E is obtained from the identity matrix by exchanging columns i and j and the det is alternating, the result follows.
8. (iv) Follows form part (i).

$\det(AB) = \det A \det B$

1. **Theorem:** Let A, B be two $n \times n$ matrices. Then
$$\det(AB) = \det A \det B.$$
2. **Proof:** Let D_i be the $i\mathrm{th}$ column of a matrix D. Then $(AB)_i = AB_i$.
3. Therefore we prove that
$$\det(AB_1, AB_2 \dots, AB_n) \quad = \quad \det(A_1, A_2, \dots, A_n) \det(B_1, \dots, B_n)$$
4. Keep A fixed and define $f(B_1, B_2, \dots, B_n) = \det(AB_1, AB_2, \dots, AB_n)$.
5. We show that f is alternating and multilinear.

$\det(AB) = \det A \det B$

1. Let C be an $n \times 1$ column vector. For any scalars x, y we get

$$
\begin{aligned}
0{=}f(B_1, \ldots, B_i, \ldots, B_i, \ldots, B_n) &= \det(AB_1, \ldots, AB_i, \ldots, AB_i, \ldots, AB_n) \\
f(B_1, \ldots, xB_k + yC, \ldots, B_n) &= \det(AB_1, \ldots, A(xB_k + yC), \ldots, AB_n) \\
&= \det(AB_1, \ldots, xAB_k + yAC, \ldots, AB_n) \\
&= \det(AB_1, \ldots, xAB_k, \ldots, AB_n) \\
&+ \det(AB_1, \ldots, yAC, \ldots, AB_n) \\
&= xf(B_1, \ldots, B_n) + yf(B_1, \ldots, C, \ldots, B_n).
\end{aligned}
$$

2. Therefore $f(B_1, B_2, \ldots, B_n) = \det(B_1, \ldots, B_n) f(e_1, e_2, \ldots, e_n)$.
3. As $f(e_1, e_2, \ldots, e_n) = \det(Ae_1, \ldots, Ae_n) = \det(A_1, \ldots, A_n) = \det A$,
4. It follows that $\det(AB) = \det A \det B$.

Determinant and invertibility

1. **Proposition:** (i) If A is invertible then $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$.
2. (ii) If $\det A \neq 0$ then A is invertible.
3. (iii) If $AB = I$ then A is invertible and $B = A^{-1}$.
4. **Proof:** (i) Since $AA^{-1} = I, \det A^{-1} \det A = \det I = 1$.
5. (ii) Suppose A is not invertible.
6. Then there is a nonzero column vector x such that $Ax = 0$.
7. So some column of A is a linear combination of other columns of A.
8. By multilinearity and alternating properties we have $\det A = 0$.
9. (iii) Let $AB = I$. Taking determinants we have $\det A \det B = 1$. So $\det A \neq 0$ and A is invertible. Now $B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}$.

Determinant of transpose of a matrix

1. **Theorem:** For any $n \times n$ matrix A,
$$\det A = \det A^t.$$
2. **Proof:** Since $(A^t)^{-1} = (A^{-1})^t$, A is invertible $\iff$ A^t is invertible.
3. Therefore if A is not invertible then A^t is also not invertible and
$$\det A = 0 = \det A^t.$$
4. So we may assume that A is invertible. Now we write $A = E_1 E_2 \cdots E_k$, where $E_1, \ldots, E_k$ are elementary matrices.
5. Now transpose of an elementary matrix is also an elementary matrix of the same type and has the same determinant.
6. The result follows by multiplicativity of the determinant function.
7. **Theorem:** Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq k \leq n$. Then
$$\det A = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} \det A_{ik}.$$

Computation of determinant by Gauss-Jordan elimination

1. Let $E =$ the $n \times n$ elementary matrix for the row operation $A_i + cA_j$
2. $F =$ the $n \times n$ elementary matrix for the row operation $A_i \sim A_j$
3. $G =$ the $n \times n$ elementary matrix for the row operation $A_i \sim cA_i$.
4. Suppose that A be an $n \times n$ matrix and U is the RCF of A.
5. If $c_1, c_2, \ldots, c_p$ are the multipliers used for the row operations
6. $A_i \sim cA_i$ and r row exchanges have been used to get U from A then for any alternating multilinear function d,
$$d(U) = (-1)^r c_1 c_2 \ldots c_p \, d(A).$$
7. Note that $d(FA) = -d(A), \; d(EA) = d(A)$ and $d(GA) = cd(A)$.
8. Suppose that $u_{11}, u_{22}, \ldots, u_{nn}$ are the diagonal entries of U then
$$d(A) = (-1)^r (c_1 c_2, \ldots c_p)^{-1} u_{11} u_{22} \ldots u_{nn} d(e_1, e_2, \ldots, e_n).$$
9. Therefore $\det(A) = (-1)^r (c_1 c_2, \ldots c_p)^{-1} u_{11} u_{22} \ldots u_{nn}$.

Matrix inverse and the cofactor matrix

1. **Definition:** Let $A = (a_{ij})$ be an $n \times n$ matrix. The **cofactor** of a_{ij}, denoted by $\operatorname{cof} a_{ij}$ is defined as

$$\operatorname{cof} a_{ij} = (-1)^{i+j} \det A_{ij}.$$

2. The **cofactor matrix** of A is defined as the matrix $\operatorname{cof} A = (\operatorname{cof} a_{ij})$.
3. **Theorem:** For any $n \times n$ matrix A,

$$A(\operatorname{cof} A)^t = (\det A)I = (\operatorname{cof} A)^t A.$$

4. Therefore if $\det A$ is nonzero then $A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^t$.
5. **Proof:** The (i, j) entry of $(\operatorname{cof} A)^t A$ is :

$$a_{1j} \operatorname{cof} a_{1i} + a_{2j} \operatorname{cof} a_{2i} + \cdots + a_{nj} \operatorname{cof} a_{ni}.$$

6. If $i = j$, it is easy to see that it is $\det A$. When $i \neq j$ consider the matrix B obtained by replacing i^{th} column of A by j^{th} column of A.
7. Then B has a repeated column. Therefore $\det B = 0$.
8. The other equation $A(\operatorname{cof} A)^t = (\det A)I$ is proved similarly.

Cramer's Rule for solving linear equations

1. **Cramer's Rule:** Let $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

   be a system of n linear equations in n unknowns, $x_1, x_2, \ldots, x_n$.

   Suppose the coefficient matrix $A = (a_{ij})$ is invertible.

   Let C_j be the matrix obtained from A by replacing the j^{th} column of A by $b = (b_1, b_2, \ldots, b_n)^t$. Then for $j = 1, 2, \ldots, n$, $x_j = \frac{\det C_j}{\det A}$.

2. **Proof:** Let $A_1, \ldots, A_n$ be the columns of A. Write

$$b = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n.$$

3. Then $\det(b, A_2, A_3, \ldots, A_n) = x_1 \det A$ and each $x_j = \frac{\det C_j}{\det A}$.

4. Cramer's rule gives a compact formula for the solutions. But it requires too many computations. The Gauss-Jordan method is preferred over this method.