ASSIGNMENT 5 : LINEAR TRANSFORMATIONS
MA 106 : LINEAR ALGEBRA : SPRING 2023

1. Tutorial Problems

(1) Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by

$$f((x_1, x_2, x_3, x_4, x_5)^t) = (2x_3 - 2x_4 + x_5, 2x_2 - 8x_3 + 14x_4 - 5x_5, x_2 + 3x_3 + x_5)^t.$$

Find bases for the null-space and the range of f, using the REF of the matrix of f with respect to standard bases of $\mathbb{R}^5$ and $\mathbb{R}^3$.

(2) Find the range and null-space of the following linear transformations. Also find the rank and nullity wherever applicable.

(a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2)^t = (x_1 + x_2, x_1)^t$.

(b) $T : C^1(0,1) \longrightarrow C(0,1)$ defined by $T(f)(x) = f'(x)e^x$.

(3) Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors $(x_1, x_2, x_3)^t \in \mathbb{R}^3$ satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T. (ii) the range of T.

(4) Let $\mathcal{P}[x]$ denote the space of all real polynomials in one variable. Consider the subspace

$$V = \{p(x) \in \mathcal{P}[x] \ : p(0) = 0\}.$$

Prove that taking the derivative defines a one-to-one linear transformation from $D : V \longrightarrow \mathcal{P}[x]$ and $D^{-1}(p)(x) = \displaystyle\int_0^x p(t)\,dt$.

(5) Let $f : V \longrightarrow W$ be a linear transformation.

(a) Suppose f is injective and $S \subset V$ is linearly independent. Then show that $f(S)$ is linearly independent.

(b) Suppose f is onto and S spans V. Then show that $f(S)$ spans W.

(c) Suppose S is a basis for V and f is an isomorphism then show that $f(S)$ is a basis for W.

(6) Let V be a finite dimensional vector space and $f : V \longrightarrow V$ be a linear map. Prove that the following are equivalent:

(i) f is an isomorphism.

(ii) f is injective.

(iii) f is surjective.

2. **Practice Problems**

(7) Consider the linear transformations $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$. If T_2 is one-one then show that $rank(T_2 \circ T_1) = rank(T_1)$.

(8) Let $f, g : V \to V$ be two linear maps which commute with each other, i.e., $f \circ g = g \circ f$. Show that $f(\text{Image}(g)) \subset \text{Image}(g)$, and $f(\text{Null}(g)) \subset \text{Null}(g)$.

(9) Show that each of the following linear transformations is 1-1 and onto and find its inverse.

(i) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (5x_1, 3x_2)$

(ii) $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3 + x_2, x_3)$.

(10) Find the range and null-space of the following linear transformations. Also find the rank and nullity wherever applicable.

(a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2)^t = (x_1, x_1 + x_2, x_2)^t$.

(b) $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^t = (x_1 - x_4, x_2 + x_3, x_3 - x_4)^t$.

(c) $T : C(0,1) \longrightarrow C(0,1)$ defined by $T(f)(x) = f(x) \sin x$.

(11) Let f be a bijective linear map. Show that the inverse is also linear.

(12) Let $T : U \longrightarrow V$ be a linear transformation. Show that

(i) T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.

(ii) Let $\mathcal{R}(T)$ denote the image of a linear map T. If $L(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = U$ then $\mathcal{R}(T) = L(\{T(\mathbf{u}_1), T(\mathbf{u}_2), \cdots, T(\mathbf{u}_n)\})$.

(13) Find all 2×2 real matrices A so that $T_A(L_1) = L_2$ where L_1 is the line $y = x$ and L_2 is the line $y = 3x$.

(14) Let $T : V \to V$ be a linear transformation where $\dim V = 2$. Assume that T is not multiplication by a scalar. Prove that there is a vector $v \in V$ so that $B = \{v, T(v)\}$ is a basis of V. Find the matrix of T with respect to B.

(15) Let A be an $m \times n$ matrix where $m \leq n$. Use the Rank-Nullity theorem to show that the space of solutions of $Ax = 0$ has dimension at least $n - m$.