MA 106 : EIGENVALUES AND EIGENVECTORS : SPRING 2023

Tutorial Problems

(1) Let u be a unit vector in $\mathbb{R}^n$. Define $H = I - 2uu^t$. Find all the eigenvalues and eigenvectors of H. Find a geometric interpretation of $T_H : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_H(v) = Hv$ for all $v \in \mathbb{R}^n$.

(2) If $A, A' \in \mathbb{F}^{n\times n}$ are **similar** , i.e. $A' = P^{-1}AP$ for some invertible $n \times n$ matrix $P \in \mathbb{F}^{n\times n}$. Show that (a) A and A' have same eigenvalues (b) if $\mathbf{v}$ is an eigenvector of A then $P^{-1}\mathbf{v}$ is an eigenvector of A'.

(3) Let A be $n \times n$ matrix. Prove that (i) 0 is an *eigenvalue* of A if and only if A is singular. (ii) if λ is an *eigenvalue* of A then it is also an *eigenvalue* of A^t (where A^t denotes the transpose of A). (iii) If x is an *eigenvector* of A corresponding to λ then x need not be an *eigenvector* of A^t corresponding to λ.

(4) Show that the map $T : C^\infty[0,1] \to C^\infty[0,1]$ given by $T(f)(x) = \int_0^x f(t)dt$ has no eigenvalue while every real number is an eigenvalue of $T(f)(x) = \frac{df(x)}{dx}$.

(5) Let $A \in \mathbb{C}^{n\times n}$ and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a complex polynomial. Suppose that λ is an eigenvalue of A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$. Find all the eigenvalues of $f(A)$.

(6) Find the characteristic polynomial, eigenspaces and their dimensions of the matrix J_n which is the $n \times n$ matrix with each of its entry equal to 1. Is J_n diagonalisable?

(7) Let $\{u, v\}$ be an orthonormal basis of $\mathbb{R}^2$. Let $A = uv^t$. Find all the eigenvalues of A.

(8) Let A be a square matrix. Prove the following statements.
(i) The eigenvalues of A are real if A is Hermitian or real symmetric.
(ii) The eigenvalues of A are either 0 or purely imaginary if A is skew Hermitian.
(iii) The eigenvalues of A are of modulus equal to 1, if A is unitary.
(iv) A^tA has only non negative eigenvalues, if A is real.

(9) A self-adjoint matrix $\mathbf{A}$, i.e. $A^* = A$, is called **positive definite** if $\langle \mathbf{A}\,\mathbf{x}, \mathbf{x}\rangle > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$. Show that a self-adjoint matrix is positive definite if and only if all eigenvalues of $\mathbf{A}$ are positive.

(10) Let $\mathbf{A}$ be a self-adjoint matrix. If $\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, then show that $\mathbf{A} = \mathbf{O}$. Deduce that if $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{A}^*\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then $\mathbf{A}$ is a normal matrix, and if $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then $\mathbf{A}$ is a unitary matrix.

(11) Let a be a nonzero real number and $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.
(a) Find an orthonormal set of eigenvectors of. A.
(b) Find a unitary matrix C such that $C^{-1}AC$ is a diagonal matrix.
(c) Prove: there is no real orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix.

(12) Let C be the locus of the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Using eigenvalues of the symmetric matrix A so that $ax^2 + bxy + cy^2 = [x\ y]A[x\ y]^t$, show that C is ellipse, hyperbola or parabola according as the *discrinimant* $4ac - b^2$ is positive, negative or zero.

Practice Problems

(13) Examine whether the following matrices can be diagonalised. If yes, find P such that $P^{-1}AP$ is diagonal.

(i) $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$

(14) Prove that (a) the *trace* of $A \in \mathbb{C}^{n\times n}$ is equal to the sum of its *eigenvalues.* (b) the determinant of A is equal to the product of its *eigenvalues.*

(15) Let $V = \mathbb{R}^{2\times 2}$. Let $T : V \to V$ be defined by $T(A) = A^t$. Find the eigenvalues and eigenvectors of T.

(16) Let A be a 2×2 real matrix and $p_{_A}(x)$ be its characteristic polynomial. Show that $p_{_A}(A) = 0$. This is called the Cayley-Hamilton Theorem. It is valid for all square matrices.

(17) Find a nonzero matrix so that $N^3 = 0$. Find all the eigenvalues of N. Show that N cannot be symmetric.

(18) Let an $n \times n$ matrix B have n distinct *eigenvalues*. Show that every $n \times n$ matrix A such that $AB = BA$, is diagonalizable.

(19) From the unit vector $u = \frac{1}{6}(1, 1, 3, 5)^t$ construct the rank one projection matrix $P = uu^t$. (a) Show that u is an eigenvector with eigenvalue 1. (b) Show that if $v \perp u$ then $Pv = 0$. Show that the only eigenvalues of P are $0, 1$. What are their algebraic and geometric multiplicities? Is P diagonalizable?

(20) If A is a real skew-Hermitian matrix, prove that $I + A$ and $I - A$ are nonsingular, i.e. invertible and $(I - A)(I + A)^{-1}$ is orthogonal.

(21) Find the values of c for which the graph of $2xy - 4x + 7y + c = 0$ is a pair of lines.

(22) Prove that the *eigenvectors* of a Hermitian (or real symmetric) matrix corresponding to distinct *eigenvalues* are orthogonal.

(23) By a symmetric quadratic form Q of n variables we mean a homogeneous degree 2 polynomial in n variables, say $Q(x) = \sum_{i \leq j} \alpha_{ij} x_i x_j$. Given a quadratic form Q we associate a symmetric matrix $A_Q = (a_{ij})$ to it by taking $a_{ii} = \alpha_{ii}$ and $a_{ij} = \alpha_{ij}/2, i \neq j$. Show that $Q(x) = xA_Q x^t$, where $x = (x_1, \ldots, x_n)$. Write down the associated matrix or the quadratic form from the given data below:
(a) $Q_1(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^3$; (b) $Q_2(x, y) = xy$.
(c) $Q_3(x, y, z) = xy + yz + zx$; (d) $Q_4(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.
(e) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & -5 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(24) Transform the following quadratic equations to a diagonal form and find out what conics they represent:
(a) $41x_1^2 - 24x_1x_2 + 34x_2^2 = 0$ (b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$;
(c) $91x^2 - 24xy + 84y^2 = 25$. (d) $4xy + 3y^2 = 10$.

(25) Let A be a real symmetric matrix with only one eigenvalue 1. Show that $A = I$.

(26) Find all the eigenvalues of a nilpotent matrix A. When is A is diagonalizable?

(27) Find all 2×2 orthogonal and skew symmetric matrices. Also find their eigenvalues.

(28) Does there exist a 3×3 matrix which is orthogonal and skew symmetric?

(29) Prove that if a square complex matrix is unitary and and Hermitian then $A^2 = I$.

(30) Let A be a normal matrix and U be unitary. Prove that U^*AU is normal.

(31) Given an orthogonal matrix A, with -1 as an eigenvalue of multiplicity k then $\det A = (-1)^k$.

(32) If the equation $ax^2 + bxy + cy^2 = 1$ represents an ellipse, prove that the area of the region it bounds is $2\pi/\sqrt{4ac - b^2}$.