PH111: Introduction to Classical Mechanics
Chapter 6: Special Theory of Relativity

Reference Books

Two very good books on the subject are:

1. **An Introduction to Mechanics**, by Daniel Kleppner and Robert Kolenkow, Second Edition, Cambridge University Press (the same book which we have used for previous chapters)
2. **Introduction to Special Relativity**, by R. Resnik; Wiley Eastern (New Delhi) 1968

Problems with Newtonian Mechanics

- Austrian physicist/philosopher wrote a book named "Science of Mechanics" in 1883
- In that he carefully examined Newton's explanations of dynamical laws
- According to Mach, there are fundamental weaknesses in Newton's conception of space and time
- Newton wrote in Principia "Absolute, true and mathematical time, of itself and by its own true nature, flows uniformly on, without regard to anything external"
- That is, according to Newton, time flows uniformly and absolutely (i.e. not relatively)

Problems with Newtonian Mechanics...

- Mach argued strongly against this view point
- He wrote "it would appear as though Newton in the remarks cited here still stood under the influence of medieval philosophy, as though he had grown unfaithful to his resolve to investigate only actual facts."
- Mach argued that because time is measured using some physical device such as a pendulum-based clock
- Therefore, time also must depend upon physical laws
- That is, time cannot have an absolute, purely mathematical, definition

Ernst Mach's ideas...

- Mach similarly disagreed with Newton's absolute views of space
- He argued that because spatial distances are measured using meter sticks
- Therefore, the properties of space can be understood by understanding the properties of meter sticks
- Actually, Mach's ideas influenced the young Einstein while he was a student in Zurich.

Examining Newton's law

- We know that the velocity is a frame-dependent quantity
- It is different in different inertial frames
- But, acceleration is independent of the frames
- It is the same in all inertial frames
- The Newton's law

$$\mathsf{F} = ma,$$

  depends on the acceleration and force
- Force, clearly is not frame dependent

Examining Newton's law...

- Therefore, Newton's law should be the same in all inertial frames
- Thus, we can integrate it to obtain the velocity in any inertial frame
- Of course, we must specify the inertial frame to get the velocity w.r.t it
- We also have velocity dependent forces such as the Lorentz force experienced by a charged particle in a magnetic field

$$\mathrm{F} = q\mathrm{E} + q\left(\mathrm{v} \times \mathrm{B}\right)$$

- This force is clearly frame dependent because v is frame dependent
- That means the force is not the same in all inertial frames - a contradiction

Speed of light

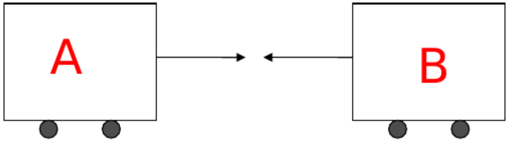
- Speed of light c is related to fundamental constants μ_0 and ε_0

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

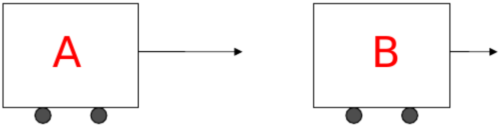
- Does the value of fundamental constants depend on the frame of reference?
- Does the speed of light depend on the speed of the frame of reference with respect to which we measure it?

Relative velocities of Objects

- Suppose, w.r.t. a frame two objects A and B move in opposite directions



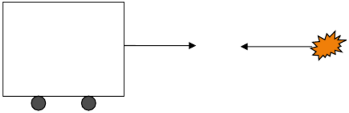
- Clearly, their relative speed will be larger $|v_{rel}| = |v_A + v_B|$
- If the two objects move in the same direction



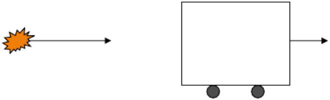
- In this case, the relative velocity is smaller $|v_{rel}| = |v_A - v_B|$

Is the speed of light frame dependent?

- So we agree that velocity is frame dependent
- But, what about the speed of light?
- For the light beam and observer moving in opposite directions



- Is $|v_{rel}| = |c + v_O|$?
- And for the light beam and observer moving in the same direction



- Is $|v_{rel}| = |c - v_O|$?

Speed of light...

- If the speed of light c is frame dependent, then so should be fundamental constants
- Because, as we wrote earlier

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}},$$

in which ε_0 and μ_0 are fundamental constants.

- Is this equation also frame dependent?
- And, if μ_0 and ε_0 are frame dependent, the electromagnetic forces will also be frame dependent
- This will imply that all inertial frames are equivalent from a mechanical point of view
- but, not from an electromagnetic point of view
- Perhaps, there exists a special reference frame called "absolute rest frame" relative to which all motions are defined
- Early physicists believed in this special frame of reference

The notion of ether...

- Sound waves require a medium to travel
- That is sound waves cannot be created in vacuum
- However, light doesn't require a medium to travel
- It also travels without problems through vacuum
- This was considered impossible earlier
- Therefore, scientists believed in the concept of an invisible medium which fills the whole space
- and light travels through that medium
- And that medium was called ether.

Ether continued...

- And scientists believed that ether is present in the entire universe
- All celestial bodies such as stars, planets, and galaxies float in ether
- Ether can be thought of as signifying the absolute rest frame
- And speed of light is c only with respect to ether
- And in other frames of references it could take different values
- And that the speed of light can be determined by measuring it in different frames
- We can also try to determine the absolute speeds of objects like, sun, moon, earth...

Speed of light

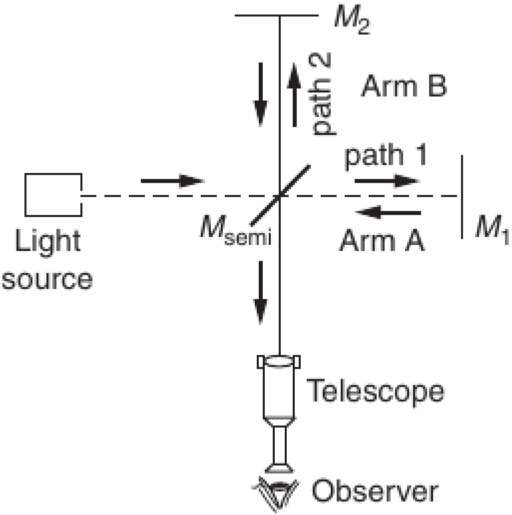
- Therefore, one can try to measure the speed of light in different frames
- Once done, this issue of relative speed of light will be settled once and for all
- But, how to experimentally measure the speed of light in different frames?
- American physicists Michelson and Morley came up with a clever experiment to do so
- That is the famous Michelson-Morley experiment discussed next

Origins of Michelson-Morley Experiment

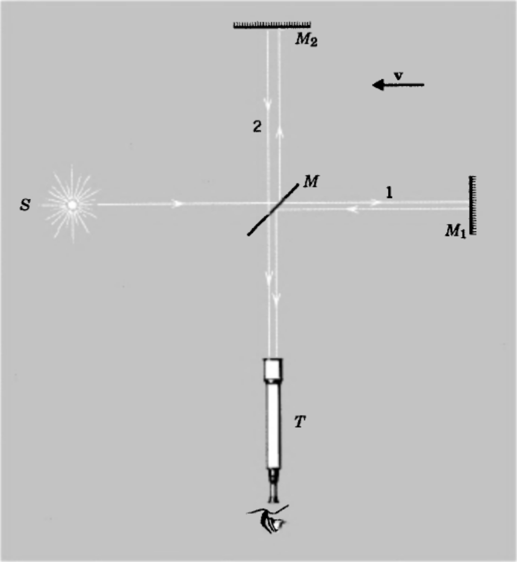
- The initial idea to test the ether hypothesis was given by Maxwell
- The motion of Jupiter with respect to earth should change the speed at which its light reaches earth
- Whenever Jupiter moves towards earth its light should move faster towards earth
- But, when it moves away from earth, its light will move slower towards earth
- The eclipses of moons of Jupiter are periodic
- But eclipses will become aperiodic depending on the relative motion of Jupiter relative to earth
- However, this effect was too tiny and the experiment couldn't be performed
- Nevertheless, it inspired Michelson to develop his interferometer to test similar effects

Michelson-Morley Experiment

- Schematic diagram of Michelson-Morley interferometer



- Another schematic of the interferometer



Michelson Interferometer...

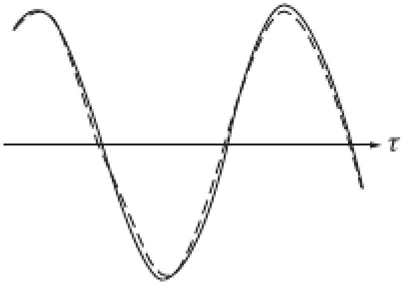
- The instrument was invented by Michelson in 1881
- It is a highly sensitive instrument designed to see the interference of light
- Such an instruments is called an interferometer
- Even though it was invented by Michelson for different purposes
- The instrument is so sensitive that it is used in modern times
- For example, modern LIGO experiment for detecting gravitational waves uses them

Michelson Interferometer...

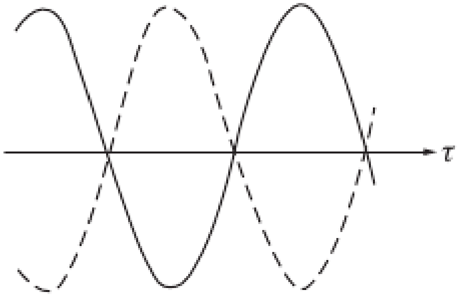
- In the instrument, light from a source is split into two beams using a semi-transparent mirror (M_{semi})
- M_{semi} roughly transmits half the beam along path 1
- While the other half is reflected by it along path 2
- The transmitted beam gets reflected back along reverse of path 1 by mirror M_1
- The reflected beam also gets reflected by mirror M_2 along the reverse of path 2
- The reflected beam 1, again gets partly reflected by M_{semi} and reaches the observer
- The reflected beam 2 gets partly transmitted by M_{semi} and reaches the observer
- Thus both the beams reaching the observer have 1/4 intensity of the original beam

Michelson Interferometer...

- If the two beams reaching the observer travel the same distance, they will be in phase
- And the observer will see a bright spot due to constructive interference
- Beams arriving in phase (i.e. zero path difference)

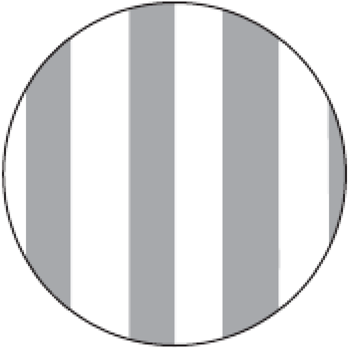


- If they have a path difference which is an odd-integer multiple of $\lambda/2$, i.e., $(m+1/2)\lambda$
- The two beams will be 180° out of phase, and we will see a dark spot due to destructive interference
- Beams arriving out of phase (i.e. $(m+1/2)\lambda$ path difference)



Michelson Interferometer...

- If we slightly misalign the beams by tilting one of the mirrors, we will observe an interference pattern
- It will have alternating bright and dark fringes as shown



- If the length of one of the arms is altered slowly
- The fringe pattern will move...

Michelson-Morley Experiment

- As the earth moves through the ether, the relative velocity of light in one of the arms will be different compared to the other
- Therefore, time taken for the light to travel back and forth through two arms will be different
- This will effectively introduce a small path difference between the two beams
- Which will shift the fringe pattern by a small amount
- Suppose the lab moves through the ether with a speed v
- And arm A of the interferometer is along the direction of motion

Michelson-Morley Experiment...

- Let the length of arm A be l
- Then the time taken by light beam to travel from M_{semi} and back, τ_A will be

$$\tau_A = \frac{l}{c+v} + \frac{l}{c-v} = \frac{2l}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

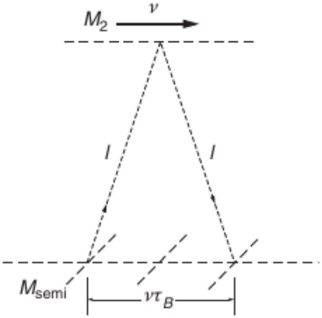
- Because $v \ll c$, we have

$$\tau_A \approx \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right)$$

- Because arm B is perpendicular to earth's motion, so the speed of light will not change along path 2
- But, there will be a time delay as M_{semi} moves through the ether during beam's back and forth journey

Michelson-Morley Experiment...

- The following figure explains the time delay



- Assuming that the length of arm B is also l, clearly now the light travels along the hypotenuse of a right triangle of length

$$l_H = \sqrt{l^2 + \frac{v^2\tau_B^2}{4}}$$

- Thus

$$\tau_B = \frac{2l_H}{c} = \frac{2\sqrt{l^2 + \frac{v^2\tau_B^2}{4}}}{c}$$

$$\implies \tau_B = \frac{2l}{c}\sqrt{\frac{1}{\left(1 - \frac{v^2}{c^2}\right)}}$$

- Or

$$\tau_B \approx \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

- Thus, the time difference between the travel times of light along the two paths is

$$\Delta\tau_1 = \tau_A - \tau_B \approx \frac{l}{c}\left(\frac{v^2}{c^2}\right)$$

Michelson-Morley Experiment

- Now if we rotate the apparatus by 90°, the time difference turns out to be

$$\Delta\tau_2 = -\Delta\tau_1$$

- Therefore, with respect to the previous configuration the time delay will be

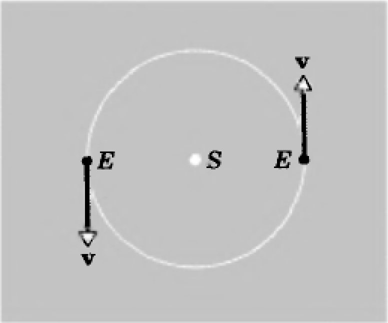
$$\Delta\tau = \Delta\tau_1 - \Delta\tau_2 = \Delta\tau_1 - (-\Delta\tau_1) \approx 2\Delta\tau_1 \approx \frac{2l}{c}\left(\frac{v^2}{c^2}\right)$$

- This extra time difference will lead to an extra path difference
- which should lead to a shift in the fringe patternif ν is the frequency of light, the number of fringe shifts due to this extra path difference is

$$\Delta N = \nu\Delta\tau \approx \frac{c}{\lambda}\frac{2l}{c}\left(\frac{v^2}{c^2}\right) \approx \frac{2l}{\lambda}\left(\frac{v^2}{c^2}\right)$$

Michelson-Morley Experiment...

- Orbital speed of earth around sun v is such that $v/c \approx 10^{-4}$
- And for the sodium light that was used in the experiment $\lambda \approx 590 \times 10^{-9}$m
- By making $l \approx 11$ m, one gets $\Delta N \approx 0.4$ fringes, which was well within the sensitivity of the apparatus
- Alternatively, realizing that every six months, earth reverses its direction of motion



- So the fringe shift should be opposite every six months

Michelson-Morley Experiment

- Various other variations of this experiment were tried by Michelson and Morley
- But in all the cases no fringe shift was observed!
- That is $\Delta N = 0$ in all the cases
- Once, everyone was convinced of this observation
- Physicists of the time were truly puzzled and worried
- It shook the physics world of that time!
- Because this result implies that there is no ether in the universe!
- How to solve this problem? What do we do now?

Lorentz-Fitzgerald Contraction Hypothesis

- To explain the null result of MM experiment, Lorentz and Fitzgerald gave the hypothesis of length contraction
- They said all bodies are contracted when moving through ether
- If the original length of the body is l_0, and if the body moves with speed v through ether, its contracted length l will be

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

- This hypothesis explained the null result of the MM experiment for the wrong reasons
- But, Einstein later on using his special theory of relativity proved that Lorentz contraction is a correct result
- There were many other consequences of Einstein's theory such as time dilation
- Next, we begin the discussion of Einstein's theory by understanding some basic concepts

A note on synchronization of clocks

- Suppose we have two observers, one on Moon and the other one on Earth
- Both the observers want to synchronize their clocks with each other
- They will use light signals to communicate with each other
- It takes about 1 second for the light to travel between Moon to Earth
- Keeping this in mind, the observer on Moon advances their clock by 1 second
- With this, for an observer on Earth, the clock on Earth is synchronized with the clock on Moon
- But, for an observer on Moon, the clock on Earth will always be two seconds behind the Moon clock!
- So what is the correct procedure for synchronizing clocks?

Einstein's way of Clock Synchronization

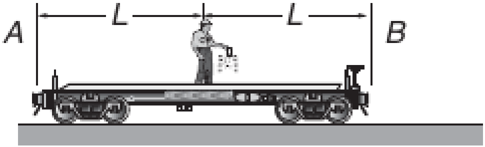
- Note that the problem above wouldn't arise if the light traveled at an infinite speed
- Because then the transmission of signals will be instantaneous
- Keeping the finite nature of the speed of light in mind, Einstein proposed a procedure for clock synchronization
- Let there be two observers A and B
- Observer A sends a light signal to observer B at time T_A
- Observer B immediately sends back a light signal to A, when their clock read T_B
- A receives this signal at time $T'_A = T_A + \Delta T$
- Clearly, the two clocks will be synchronized if $T_B = T_A + \Delta T/2$
- Note that $T_B \neq T_A$ when the clocks are synchronized
- Let us now discuss two fundamental postulates given by Einstein

Einstein's Two Postulates

- Postulate # 1: Laws of physics have the same form with respect to all the inertial frames of references
- Postulate # 2: The speed of light is a universal constant, which is the same for all observers
- Both the laws are quite remarkable
- The first one is also called the Principle of Relativity and was even proposed by Galileo
- However, the mathematical equations of Einstein embodying the first law are quite different from those of Galileo
- Regarding the second postulate, Einstein was possibly inspired by the fact that light needs no medium to propagate
- Therefore, its speed must be same everywhere
- Before exploring the mathematical consequences of Einstein's postulates, let us understand the meaning of simultaneity of events

Simultaneity of Events

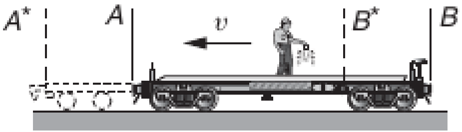
- Consider a railwayman standing in the middle of a flat wagon of length $2L$ as shown



- When he lights his lantern, the light travels in all directions with the speed c
- It will arrive at the two ends after an interval of L/c
- This system, in which the wagon is at rest, the light reaches the ends A and B, simultaneously
- Thus, these two events are simultaneous in this frame.

Simultaneity of Events...

- Let us observe the same event from a frame which moves to the right with speed v



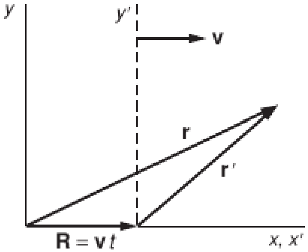
- Clearly, in this frame the wagon moves to the left with speed v, as do its ends A and B
- Light in this frame is also traveling with the speed c
- However, because the wagon is moving, during the transit time of light
- Ends A/B would have reached positions A^*/B^*
- Thus, light will reach B before it reaches A
- So, these events are not simultaneous in this moving frame!

Time as a coordinate?

- Now we realize the importance of time after the previous discussion of simultaneity
- We note that just like space coordinates, it is also frame dependent
- Perhaps we need to take time (t) on the same footing as the space coordinates (x, y, z)
- Let us start treating time t as a coordinate in addition to (x, y, z)
- We will collectively call this quadruplet of (x, y, z, t) as space-time coordinates
- Henceforth, instead of operating in a three-dimensional (3D) space, we will work in a four-dimensional (4D) space-time
- Hence the universe will now become a 4D entity for us
- No absolute and independent flow of time in this approach, everything is relative!

Galilean Relativity

- Now we explore as to how the space-time coordinates related to each other in Newtonian physics, but called Galilean transformations
- Because this was first mathematically explored by Galileo
- Let us consider our usual frames of references S and S' as shown



- S and S' are defined by space-time coordinates (x, y, z, t) and (x', y', z', t'), respectively

Galilean Transformations

- Clearly, the two sets of coordinates are related by the equation, also called transformation

$$\mathrm{r}' = \mathrm{r} - \mathrm{R} = \mathrm{r} - \mathrm{v}t,$$

  where $\mathrm{R} = \mathrm{v}t$.
- Because $\mathrm{v} = v\hat{i}$, is in the x direction, we can write

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

- The fourth equation above is actually a fundamental assumption of Newtonian dynamics as discussed earlier
- These equations are also called Galilean transformation

Lorentz Transformations

- Let us now examine the relationship between space-time coordinates in two inertial frames (S and S') in light of Einstein's two postulates
- This relationship was derived by Albert Einstein
- But, somehow they are known as Lorentz transformations
- The most general Lorentz transformation equations will be

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

- If we use the notations $(x, y, z, t) \equiv (x_1, x_2, x_3, x_4)$ and $(x', y', z', t') \equiv (x'_1, x'_2, x'_3, x'_4)$

Lorentz Transformations...

- The previous transformation equations can be written in the compact form

$$x_i' = \sum_{j=1}^{4} a_{ij} x_j, \qquad \text{for } i = 1, \ldots 4$$

  Clearly, this of the form of matrix multiplication involving 4D vectors (x_i' and x_i) and a 4×4 transformation matrix with matrix elements a_{ij}

- There's another convention called Einstein convention in which it is implied that the repeated indices (j above) are summed over. Under this convention we need to just write

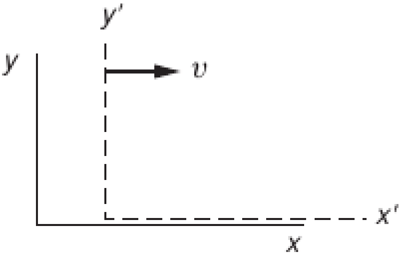
$$x_i' = a_{ij} x_j,$$

  above even $i = 1, \ldots 4$ is implied.

- But we will hardly ever use Einstein's conventions unless mentioned

Lorentz Transformations...

- For the sake of simplicity, we consider again the case when S' frame moves along the x direction with velocity v with respect to S



- We also assume that the origins of S and S' coincide at $t = t' = 0$
- Clearly, for this case y and z coordinates will be unaffected
- Therefore, the following must be true

$$a_{ij} = 0, \text{for } i \neq j, \text{ and } i, j = 2, 3,$$
$$a_{ii} = 1, \text{for } i = 2, 3$$

Lorentz Transformations...

- And we further adopt the notations

$$
\begin{aligned}
a_{11} &= A \\
a_{14} &= B \\
a_{41} &= C \\
a_{44} &= D
\end{aligned}
$$

- Leading to the transformation equations for this case

$$
\begin{aligned}
x' &= Ax + Bt \\
y' &= y \\
z' &= z \\
t' &= Cx + Dt
\end{aligned}
$$

- Equations above have four unknowns A, B, C, and D.
- We need four conditions to determine them

Lorentz Transformation Equations...

- Condition 1: Suppose at time t (according to S), the origin of frame S is observed from frame S', clearly the space-time coordinates in two frames will be
  in S: $(0, t)$, and in S' $(-vt', t')$
  So the first and the fourth equations yield

$$-vt' = Bt$$
$$t' = Dt$$

  we can satisfy both these equations only if

$$B = -vD$$

Lorentz Transformation Equations...

- Now our (x, t) transformation equations are

$$x' = Ax - vDt$$
$$t' = Cx + Dt$$

- Condition 2: The origin of frame S' is observed from frame S, clearly the space-time coordinates in two frames will be in S: (vt, t), and in S' $(0, t')$
Now the transformation equations yield

$$0 = Avt - vDt$$
$$t' = Cvt + Dt$$

from the first equality we obtain

$$A = D$$

Lorentz Transformation Equations...

- Now our (x, t) transformation equations become

$$x' = A(x - vt)$$
$$t' = Cx + At$$

- Condition 3: A light pulse is emitted from the origin at $t = t' = 0$ and is observed later along x and x' directions
In S: (ct, t) and in S': (ct', t'). From above equations we have

$$ct' = A(ct - vt)$$
$$t' = Cct + At$$

which implies

$$\frac{A}{c}(ct - vt) = Cct + At$$
$$\implies C = -(v/c^2)A$$

- Now the (x, t) equations have only one unknown A

$$x' = A(x - vt)$$
$$t' = A\left(-\frac{v}{c^2}x + t\right)$$

- Condition 4: A light pulse is sent at $t = t' = 0$, and is observed along the y/y' axes.
  In S $(0, ct, t)$, in S' $(-vt', y', t')$. From equations above we have

$$x' = -Avt$$
$$y' = y = ct$$
$$t' = At$$

- Now the distance traveled by light in S' is $d' = \sqrt{x'^2 + y'^2} = t\sqrt{A^2v^2 + c^2}$. But, the speed of light is also c in S', therefore,

$$c = \frac{d'}{t'} = \frac{t\sqrt{A^2v^2 + c^2}}{At}$$
$$\implies c^2 = \frac{A^2v^2 + c^2}{A^2}$$
$$\implies A = \pm\frac{1}{\sqrt{1 - v^2/c^2}}$$

Lorentz Transformations

- Only the positive sign is acceptable for A, leading to the final equations called Lorentz transformations

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

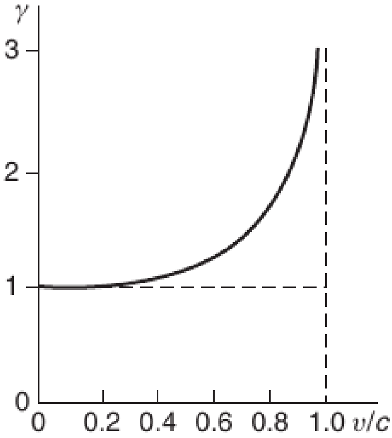
The inverse Lorentz transformation

- The inverse transformation can be obtained by interchanging $(x, y, z, t) \leftrightarrow (x', y', z', t')$ and $v \to -v$
- The inverse transformation is about an observer in the S' frame, observing frame S

$$
\begin{aligned}
x &= \gamma(x' + vt) \\
y &= y' \\
z &= z' \\
t &= \gamma\left(t' + \frac{v}{c^2}x\right)
\end{aligned}
$$

Behavior of γ function

- Note that $\gamma = 1$, for $v = 0$ and ∞ for $v = c$, thus $1 \leq \gamma \leq \infty$, for $0 \leq v \leq c$, or equivalently for $0 \leq v/c \leq 1$.
- If we plot γ as a function of the dimensionless quantity v/c, we obtain



Galilean Relativity as a limiting case...

- Let us assume that the speed of light becomes infinite, i.e., $c = \infty$
- Physically, that means that now observers can communicate with each other instantaneously
- Clearly, $\gamma \to 1$, as $c \to \infty$. In this limit the Lorentz transformation equations become

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned}$$

  which are precisely the equations of Galilean transformation
- Thus, Galilean transformation can be seen as a limiting case ($c \to \infty$) of the Lorentz transormation
- Or the Galilean relativity as a limiting case of Special Theory of Relativity (STR)

Lorentz Contraction

- Let us explore some interesting consequences of STR
- The first one is Lorentz contraction which implies that moving objects as seen from a stationary frame appear shorter
- This effect is also called length contraction
- Let us assume that a rod of length L_0 and ends A and B is at rest in frame S
- We observe the length of the rod from the frame S' moving at speed $-v$ along the direction of the rod
- The length of the rod, L_0 (say), in S can be computed using the space-time coordinates of its ends (x_a, t) and (x_b, t)

$$L_0 = x_b - x_a$$

- How do we determine its length, L_0', as seen by an observer in S'?

Length contraction...

- For that, the observer in S' must measure the coordinates of the two ends simultaneously
- Let the space-time coordinates be (x'_a, t') and (x'_b, t')
- So $L'_0 = x'_b - x'_a$
- Using inverse the Lorentz transformations for velocity $-v$ we have

$$
\begin{aligned}
x_a &= \gamma(x'_a - vt') \\
x_b &= \gamma(x'_b - vt') \\
\implies L_0 &= x_b - x_a = \gamma(x'_b - x'_a) = \gamma L'_0 \\
\implies L'_0 &= \frac{L_0}{\gamma} = L_0\sqrt{1 - v^2/c^2}
\end{aligned}
$$

- Thus for $v > 0$, $L'_0 < L_0$, that is length of the rod as seen from the moving frame is smaller as compared to the one in its rest frame
- This is the famous Lorentz/length contraction phenomenon

Time Dilation

- Now we explore how the time interval between two events as measured from two different frames is related
- Suppose a clock is at rest at position x with respect to the frame S
- and the time interval between its successive ticks is τ_0 measured by an observer in S
- What is the time interval between its ticks measured by an observer in S' moving with speed v with respect to S
- In S' the clock moves with speed $-v$
- With respect to frame S let event 1 be tick $\#$ 1 at time t
- Even 2 is tick $\#$ 2 at time $t + \tau_0$

Time dilation...

- Let the corresponding time coordinates in S' be t' and $t' + \tau_0'$, respectively

$$
\begin{aligned}
t' &= \gamma(t - vx/c^2) \\
t' + \tau_0' &= \gamma(t + \tau_0 - vx/c^2) \\
\implies \tau_0' &= \gamma\tau_0 = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}
$$

- Which implies for $v \geq 0$, $\tau_0' \geq \tau_0$
- This means that time interval as seen from a moving frame is larger as compared to the rest frame
- This phenomenon is called time dilation

Proper Time and Proper Length

- Quantities L_0 and τ_0 in the previous discussion are called proper length and proper time, respectively
- How does define these "proper" quantities?
- The definitions are quite simple
- The length of a body measured in a frame in which it is at rest, is called its proper length
- Similarly, the time interval of an event by a clock in a frame in which it is at rest is called proper time.

Observation of Relativistic Effects

- There are several phenomenon which can be explained only by using the concept of time dilation
- For example, several measurements have revealed that the lifetime of negative muons, μ^- is 2.2×10^{-6}s or 2.2 μs when at rest
- However, fast moving μ^- have been observed with the lifetime of 440 μs
- Such huge difference (200 times) can only be explained in terms of time dilation

Calculation of Relative velocities

- Let us consider the usual two frames S (x, y, z, t) and S'(x', y', z', t), with S' moving with speed v along +ve x direction
- If an object is found to be moving with velocity (u_x, u_y, u_z) with respect to S, and (u'_x, u'_y, u'_z) with respect to S', what is the relationship between the two?
- According to the Galilean relativity, it will be

$$u'_x = u_x - v$$
$$u'_y = u_y$$
$$u'_z = u_z$$

- Let us explore what will be the relationship according to the special theory of relativity.

Relative velocities...

- Clearly the definition of velocity components in S is

$$u_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$
$$u_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$
$$u_z = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

- And in S'

$$u'_x = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'}$$
$$u'_y = \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t}$$
$$u'_z = \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'}$$

Relative velocities

- Using the equations of Lorentz transformation, we have

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

- Using these we have

$$u'_x = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} = \lim_{\Delta t' \to 0} \frac{(\Delta x - v\Delta t)}{\left(\Delta t - \frac{v}{c^2}\Delta x\right)} = \frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)}$$

- Similarly, we obtain for the remaining two components

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

Relative velocities

- In order to obtain the inverse transformation, i.e., (u_x, u_y, u_z) in terms of (u'_x, u'_y, u'_z), we just interchange primed and unprimed quantities and $v \to -v$ in the previous equations

$$u_x = \frac{u'_x + v}{\left(1 + \frac{v}{c^2} u'_x\right)}$$

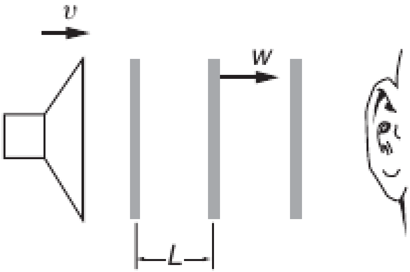
$$u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

- Note that these formulas differ from the Galilean formulas presented earlier by a nontrivial denominator
- This has very important consequences as we shall see
- Noteworthy point is that the relative velocities in the y and z directions also aren't unchanged any more, in complete disagreement with the Galilean relativity

Doppler Effect of Light

- Before deriving the formula for the Doppler effect of light, let us review the two formulas for the Doppler effect of sound
- When the source of sound moves towards the observer with speed v



- If the frequency of the sound is ν_0, and its speed in that medium is w
- Clearly, the sound pulses arrive with gap of time $\tau_0 = 1/\nu_0$
- The distance between successive crests is the wavelength $\lambda = w/\nu_0$

Doppler Effect...

- For the moving source the distance between successive crests, i.e., Doppler shifted wavelength λ_D, reduces

$$\lambda_D = \lambda - v\tau_0 = \lambda - \frac{v}{\nu_0}$$

- If the Doppler shifted frequency is ν_0', we have

$$\frac{w}{\nu_0'} = \frac{w}{\nu_0} - \frac{v}{\nu_0}$$

- Leading to the well-known result

$$\nu_0' = \nu_0 \left(\frac{1}{1 - v/w} \right)$$

- The change in frequency $\Delta\nu = \nu_0' - \nu_0$ is called the Doppler shift

Doppler shift...

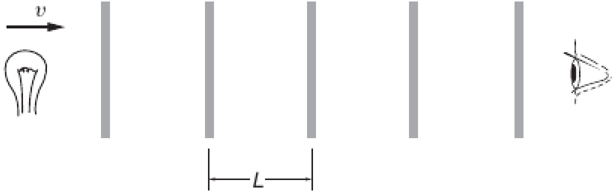
- Suppose, instead, the observer is moving towards the source with speed v
- Now the wave length of the sound wave doesn't change
- But, the effective sound velocity becomes $w + v$
- This causes change in frequency

$$\nu_0' = \frac{w+v}{\lambda} = \left(\frac{w}{\lambda} + \frac{v}{\lambda}\right) = \left(\nu_0 + \frac{v\,\nu_0}{w}\right) = \nu_0\left(1 + \frac{v}{w}\right)$$

- The two results agree to the first order in v/w

Relativistic Doppler Shift of Light

- Suppose the light source is moving towards the observer at a speed v



- If the light has frequency ν_0, then the time period between pulses is $\tau_0 = 1/\nu_0$.
- Because the source is moving, this time period will undergo time dilation with respect to the observer, leading to a new time period

$$\tau = \gamma \tau_0$$

- As in case of the sound wave, the distance between the two successive crests in observer's frame will be

$$\lambda_D = c\tau - v\tau$$

Relativistic Doppler Shift

- Now Doppler-shifted frequency ν_D will be

$$\nu_D = \frac{c}{\lambda_D} = \frac{c}{(c-v)\tau} = \frac{c}{\gamma(c-v)\tau_0}$$
$$= \nu_0 \frac{\sqrt{1-v^2/c^2}}{(1-v/c)} = \nu_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

Some solved problems

- **Problem 1:** Consider a railway wagon of length $2L$, with its two ends A and B, and the origin located at the center O. If a light pulse is emitted at O at time $t = 0$, calculate the time of arrival of the pulses at the two ends with respect to an observer: (a) at rest on the wagon, and (b) in frame S' moving with respect to the wagon with speed v along the direction of the wagon (x direction)
- **Soln 1:** (a) If the x coordinates of the ends A and B of wagon are $\mp L$, we have time of arrival at A $t_1 = L/c = T$, and at B also $t_2 = L/c = T$. So in S frame the two events are simultaneous with $t_1 = t_2 = L/c = T$
  (b) In order to get the times with respect to the S' frame, we use the Lorentz transformation equations

$$
\begin{aligned}
t_1' &= \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \\
&= \gamma\left(\frac{L}{c} + \frac{v}{c^2}L\right)
\end{aligned}
$$

or

$$t_1' = T\frac{1+v/c}{\sqrt{1-v^2/c^2}} = T\sqrt{\frac{1+v/c}{1-v/c}}$$

and

$$\begin{aligned} t_2' &= \gamma\left(t_2 - \frac{v}{c^2}x_2\right) \\ &= \gamma\left(\frac{L}{c} - \frac{v}{c^2}L\right) \\ &= T\frac{1-v/c}{\sqrt{1-v^2/c^2}} = T\sqrt{\frac{1-v/c}{1+v/c}} \end{aligned}$$

Clearly, as expected $t_2' < t_1'$, i.e., according to and observer in S', the pulse arrives at the end B earlier than at A.

- **Problem 2:** A spaceship A is moving with speed $0.9c$ with respect to an observer. Another spaceship B is moving in a direction exactly opposite of A with the speed $0.8c$. Calculate the speed of B with respect to A. What will be the speed of A with respect to B?
- Soln: Here we use the relative velocity formula

$$u'_x = \frac{u_x - v}{\left(1 - \frac{v}{c^2} u_x\right)}$$

Above v is the speed of S' (spaceship A) with respect to S, and u_x is the speed of spaceship B with respect to S. If we take the $+$ve x direction to be that of A, we have

$$v = 0.9c$$

$$u_x = -0.8c$$

so that speed of B with respect to A is

$$u'_x = \frac{-0.8c - 0.9c}{1 - (0.9c)(-0.8c)/c^2} = \frac{-1.7c}{1.72} = -0.988c$$

If we compute the relative velocity of A with respect to B, only the sign will be opposite for obvious reasons. Note that the result is so different compared to the Galilean result of $1.7c$!

- **Problem 3:** With respect to the S frame two events A and B occur at distinct space-time coordinates (x_A, t_A), and (x_B, t_B), respectively. For what conditions in frame S', moving with speed v with respect to S: (a) can the events be made simultaneous, i.e, $t'_A = t'_B$, and (b) made to occur at the same place, i.e., $x'_A = x'_B$?
- **Soln:** Let us define the spatial distance between the events $L = x_B - x_A$, and the temporal distance to be $T = t_B - t_A$. Using the equations of Lorentz transformation, the corresponding quantities with respect to S', L' and T' are

$$L' = \gamma(L - vT)$$
$$T' = \gamma\left(T - \frac{v}{c^2}L\right)$$

(a) if $L > cT$, clearly $L' > 0$, that means that the events cannot be made to occur at the same place in S'.

Solved problems...

However, when S' moves with a speed $v > c^2 T/L$

$$T' < 0$$

if S' moves with speed $v = c^2 T/L < c$

$$T' = 0$$

and if it moves with speed $v < c^2 T/L$

$$T' > 0$$

Thus, for $L > cT$, we can make the two events simultaneous, or we can reverse their order of occurence in S'. But, we cannot make them occur at the same point in space. Such intervals are called space-like intervals.

- (b) if $L < cT$, we have

$$T' > \gamma\left(T - \frac{v}{c^2}(cT)\right) > \gamma T\,(1 - v/c) > 0$$

Therefore, for this case, we can never make the two events simultaneous in S'. However, space interval in S' can take positive, negative or zero values

$$L' = \gamma(L - vT)$$

For $v < L/T$, i.e., $L > vT$ we have $L' > 0$
For $v = L/T < c$, we have $L' = 0$
For $v > L/T$, we have $L' < 0$.
Thus for $L < cT$, by choosing the speed v of S' appropriately, we can make two events occur at the same point in S', which are spatially separated in S. Or even can reverse their spatial order. Such intervals are called time-like intervals.