Chapter 3: Systems of linear equations

1. General principles for the solutions of linear equations with examples.
2. Computational aspects of systems of linear equations.
3. The **Gauss Elimination Method** for solving linear equations.
4. We will introduce elementary matrices and show how they can be used in solving linear equations.
5. Any invertible matrix is a product of elementary matrices.
6. We will discuss a practical algorithm called the Gauss-Jordan algorithm for computation of the inverse of an invertible matrix.

A pair of homogeneous linear equations

1. Let us consider a pair of linear equations
$$\begin{aligned} 2x + 3y - z &= 0 \\ x + y + z &= 0. \end{aligned}$$
2. These are equations of two planes in $\mathbb{R}^3$ passing through the origin. Their normal vectors are $n_1 = (2, 3, -1)$ and $n_2 = (1, 1, 1)$.
3. Since the normal vectors are not parallel, we expect that the set of solutions constitute a line passing through the origin that is the intersection of these planes.
4. Eliminate x from the 1^{st} equation by subtracting 2 times the 2^{nd} equation:
$$\begin{aligned} y - 3z &= 0 \\ x + y + z &= 0. \end{aligned}$$
5. Substitute $y = 3z$ in the second equation $x + y + z = 0$ to get $x = -4z$.
6. Hence any solution vector is $(x, y, z) = (-4z, 3z, z) = z(-4, 3, 1)$ for any $z \in \mathbb{R}$.
7. Therefore the solution vectors constitute a line passing through the origin and parallel to the vector $w = (-4, 3, 1)$. Note that $w \perp n_1$ and $w \perp n_2$.

General facts about systems of linear equations

1. A general system of linear equations with coefficients in $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

2. The entries $a_{ij}, b_k \in \mathbb{F}$ for all i, j and k.
3. A compact way of writing the above m equations in n variables is to use matrices. Let $A = (a_{ij}) \in \mathbb{F}^{m\times n}$, $b = (b_i) \in \mathbb{F}^m$ and $x = (x_j)$.
4. Then the above system of equations can be written as $Ax = b$.
5. Let A^j denote the j^{th} column of A for $j = 1, 2, \ldots, n$.
6. A useful way of writing $Ax = b$ is $x_1A^1 + x_2A^2 + \cdots + x_nA^n = b$.
7. Hence $Ax = b$ has a solution $\iff$ b is a linear combination of the column vectors of A.

Four fundamental spaces associated to $Ax = b$.

1. Let $u_1, u_2, \dots, u_m$ be vectors in $\mathbb{F}^n$. The vector

$$a_1u_1 + a_2u_2 + \cdots + a_mu_m$$

where $a_1, a_2, \dots, a_m \in \mathbb{F}$ is called a **linear combination** of $u_1, \dots, u_m$.

2. The set $L(u_1, u_2, \dots, u_m)$ of all linear combinations of $u_1, u_2, \dots, u_m$ is called the **linear span** of $u_1, u_2, \dots, u_m$.
3. The linear span $C(A)$ of the column vectors of A, is called the **column space of** A. Hence $C(A) \subset \mathbb{F}^m$.
4. The linear span $R(A)$, of row vectors of A, is called the **row space of** A.
5. A vector $c = (c_1\ c_2\ \dots,\ c_n)^t \in \mathbb{F}^n$ is called a **solution of** $Ax = b$ if $Ac = b$. Therefore if $Ax = b$ has a solution if and only if $b \in C(A)$.
6. If $b = 0$ then $Ax = 0$ is called a **homogeneous system.** If $b \neq 0$ then $Ax = b$ is called an **inhomogeneous system.**
7. The set of vectors $c \in \mathbb{F}^n$ that are solutions of $Ax = 0$ is called the **null space of** A. It is denoted by $N(A)$.
8. The fourth space associated to $Ax = b$ is the space $N(A^t)$.

General facts about linear equations

1. The solutions of $Ax = b$ and the corresponding homogeneous system $Ax = 0$ are closely related. If c, d are solutions of $Ax = 0$ then for any scalars α, β, $\alpha c + \beta d$ is also a solution of $Ax = 0$ since
$$A(\alpha c + \beta d) = \alpha Ac + \beta Ad = 0.$$
2. Recall that the set of solutions of $Ax = 0$, denoted by $N(A)$, is called the **null space of** A.
3. **Theorem.** Let $s \in \mathbb{F}^n$ be any solution of $Ax = b$. Then the set of solutions of $Ax = b$ is given by
$$s + N(A) = \{s + c \mid c \in N(A)\}.$$
4. **Proof.** Let $c \in N(A)$. Then $A(s + c) = As + Ac = As = b$.
5. Conversely let d be a solution of $Ax = b$. Then $A(d - s) = Ad - As = b - b = 0$.
6. Hence $d - s \in N(A)$. Therefore $d \in s + N(A)$.

Basic facts about system of linear equations

1. **Proposition.** A system of linear equations $Ax = b$ has either no solution, or one solution or infinitely many solutions.
2. **Proof.** Suppose that $Ax = b$ has two distinct solutions $c, d \in \mathbb{F}^n$. Then for $z \in \mathbb{F}$,
$$A(zc - zd) = zAc - zAd = zb - zb = 0.$$
3. Hence $z(c - d)$ for any nonzero $z \in \mathbb{F}$ is a nonzero vector in $N(A)$.
4. Therefore $c + z(c - d)$ is a solution of $Ax = b$ for all $z \in \mathbb{F}$.
5. Hence $Ax = b$ has infinitely many solutions.
6. **Proposition.** Let A be a $m \times n$ matrix over $\mathbb{F}$, $b \in \mathbb{F}^m$, and E an invertible $m \times m$ matrix over $\mathbb{F}$. Then $Ax = b$ has the same solutions as $EAx = Eb$.
7. **Proof.** If $Ax = b$ then $EAx = Eb$.
8. If $EAx = Eb$ then $E^{-1}(EAx) = E^{-1}(Eb)$ or $Ax = b$.
9. **Remark.** We will introduce invertible matrices called the elementary matrices which can be used to simplify a system of linear equations so that the solutions of the new system are easy to find.

Linear independence and dependence of vectors

1. A set of vectors $u_1, u_2, \ldots, u_m \in \mathbb{F}^n$ are called linearly dependent if there are scalars $x_1, x_2, \ldots, x_m \in \mathbb{F}^n$ not all zero so that

$$x_1u_1 + x_2u_2 + \cdots + x_mu_m = 0.$$

2. The vectors $u_1, u_2, \ldots, u_m$ are called **linearly independent** if they are not linearly dependent.
3. Consider the row vectors

$$a = (-3, 2, 1, 4),\ b = (4, 1, 0, 2),\ c = (-10, 3, 2, 6).$$

4. We wish to find whether a, b, c are linearly dependent.
5. This is same as solving for x, y, z so that $xa + yb + zc = 0$.
6. This is a homogeneous system of linear equations.
7. If we can find a nontrivial solution then a, b, c are linearly dependent.

Elementary row operations

1. We shall perform the following three operations on these vectors which do not change the linear span of a, b, c.
2. **Exchange one vector for another.** Suppose we exchange a and b.
3. Then it is clear that $L(a, b, c) = L(b, a, c)$.
4. **Replace a vector u by $u + \alpha v$ where v is another vector and $\alpha \neq 0$.** For example replace a by $a - 2b$. Then $L(a, b, c) = L(a - 2b, b, c)$.
5. In fact It is clear that any linear combination of $a - 2b, b, c$ is also a linear combination of a, b, c. Let $x, y, z \in \mathbb{F}$. Then

$$\begin{aligned} xa + yb + zc &= x(a - 2b) + 2xb + yb + zc \\ &= x(a - 2b) + (2x + y)b + zc \\ &\in L(a - 2b, b, c). \end{aligned}$$

   Therefore $L(a, b, c) = L(a - 2b, b, c)$.
6. **Replace a vector u by αu where $\alpha \neq 0$.** It is clear that this operation does not change linear span of a given set of vectors.
7. The above three operations are called the **elementary row operations.**

Elementary row operations

1. Rather than carrying out these operations on a, b, c we can introduce a 3×4 matrix whose row vectors are a, b, c :

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{bmatrix}.$$

2. Exchanging two vectors amounts to exchanging two rows of A. We indicate this row operation as follows:

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} A_1 = \begin{bmatrix} 4 & 1 & 0 & 2 \\ -3 & 2 & 1 & 4 \\ -10 & 3 & 2 & 6 \end{bmatrix}.$$

3. Replacing a by $a - 2b$ is described by the notation:

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_1 \longrightarrow R_1 - 2R_2} A' = \begin{bmatrix} -11 & 0 & 1 & 0 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{bmatrix}.$$

Elementary row operations

1. We can perform these three types of row operations on A to simplify A without changing the row space of A until we get a matrix in which it is easy to find linearly independent row vectors.
2. One way to do this is to use the *Gauss Elimination Method.* This method consists of performing a series of row operations of the three kinds described above so that the matrix A is transformed into a simpler matrix that is in **row echelon form** or **row canonical form.**
3. If a matrix A' is obtained from a matrix A by applying a sequence row operations to A then A' is called **row equivalent** to A.
4. An $m \times n$ matrix M is said to be in **row echelon form (REF)** if it satisfies the following conditions:

(a) Suppose M has k nonzero rows and $m - k$ zero rows. Then the last $m - k$ rows of M are the zero rows.

(b) The first nonzero entry in a nonzero row is called a **pivot**. For $i = 1, 2, \ldots, k$, suppose that the pivot in row i occurs in column j_i.

The row echelon form of a matrix

1. Then we have $j_1 < j_2 < \cdots < j_k$. The columns $\{j_1, \ldots, j_k\}$ are called the set of **pivotal columns** of M.
2. The columns $\{1, \ldots, n\} \setminus \{j_1, \ldots, j_k\}$ are the **nonpivotal columns**.
3. An $m \times n$ matrix M that is in row echelon form is said to be in **row canonical form (RCF)** if it satisfies the following conditions:

(a) The first nonzero entry in every nonzero row is 1.

(b) The only nonzero entry in a pivotal column is the pivot 1.

4. Note that a matrix in row canonical form is in row echelon form. Also note that, in both the definitions above, the number k of pivots is $\leq m, n$.
5. Let us find the REF of the matrix A.

$$A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ -10 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_2 \longrightarrow R_2 + \frac{4}{3}R_1} A(1) = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ -10 & 3 & 2 & 6 \end{bmatrix}$$

$$A(1) = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ -10 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_3 \longrightarrow R_3 - \frac{10}{3} R_1} A(2) = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & \frac{-11}{3} & \frac{-4}{3} & \frac{-22}{3} \end{bmatrix}$$

$$A(2) = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & \frac{-11}{3} & \frac{-4}{3} & \frac{-22}{3} \end{bmatrix} \xrightarrow{R_3 \longrightarrow R_3 + R_2} A(3) = \begin{bmatrix} \text{-3} & 2 & 1 & 4 \\ 0 & \frac{11}{3} & \frac{4}{3} & \frac{22}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. The matrix $A(3)$ is row equivalent to A. In fact $A(3)$ is in REF.
2. The circled numbers are the two pivots. Hence $R(A) = R(A(3))$.
3. The row space of $A(3)$ is the linear span of the first two row vectors of $A(3)$.
4. Check that the first two row vectors are linearly independent.

The Gauss elimination method for solving linear equations

1. Note that each solution of $Aw = 0$ is perpendicular to each vector in $R(A)$.
2. Since $R(A) = R(A(3))$, solutions to $Aw = 0$ are solutions to $A(3)w = 0$.
3. Let $w = (x, y, z, u)$. Solutions of $A(3)w = 0$ are solutions to

$$\begin{aligned} -3x + 2y + z + 4u &= 0 \\ 0.x + (11/3)y + (4/3)z + (22/3)u &= 0. \end{aligned}$$

4. Therefore $y = -(4/11)z - 2u$ and $x = z/11$.
5. Hence a general solution to $Aw = 0$ is given by

$$w = (x, y, z, u) = (z/11, -4/11z - 2u, z, u)$$

$$= z(1/11. - 4/11, 1, 0) + u(0, -2, 0, 1).$$

6. Note that w is a linear combination of two linearly independent vectors.

Elementary matrices for the elementary row operations

1. We have seen how elementary row operations on A can be used to find an REF A' of A. The equations $A'x = 0$ and $Ax = 0$ have same set of solutions.
2. Now we show that this method is useful for solving inhomogeneous systems of linear equations also.
3. To justify this, we show that elementary row operations on A can be carried out by multiplying A by invertible matrices on the left of A.
4. These matrices are called **elementary matrices.**
5. Let e_j be the column vector in $\mathbb{F}^m$ whose *jth* component is 1 and others are zero.
6. The column vectors $e_1, e_2, \ldots, e_m$ are linearly independent.
7. In fact $\mathbb{R}^m$ is the linear span of $e_1, e_2, \ldots, e_m$.
8. The corresponding row vectors will be denoted by $e_1^t, e_2^t, \ldots, e_m^t$.
9. Elementary row operations of three kinds can be performed by premultiplying A by certain invertible matrices.

Elementary row operations

1. **Exchanging row vectors.** Let $1 \leq i < j \leq m$ and E be the $m \times m$ matrix whose row vectors are

$$E_1 = e_1^t, E_2 = e_2^t, \ldots, E_i = e_j^t, \ldots, E_j = e_i^t, \ldots, E_m = e_m^t.$$

2. Then for any $m \times n$ matrix A, EA is the matrix obtained from A by exchanging the row vectors A_i and A_j. Since $E^2 = E$, $E^{-1} = E$.
3. We call E as the **elementary matrix of the first kind.** Note that E is obtained from the identity matrix by exchanging the i^{th} and the j^{th} row.
4. **Adding a scalar multiple of a row vector to another row vector.** Let $1 \leq i < j \leq m$ and F be the $m \times m$ matrix whose row vectors are

$$F_1 = e_1^t, \ldots, F_j = e_j^t + xe_i^t, \ldots, F_m = e_m^t.$$

5. Note that F is obtained from the identity matrix by adding x times the i^{th} row vector to the j^{th} row vector and FA is the matrix obtained from A by adding xA_i to A_j. Check that $F^{-1} = [e_1^t, \ldots, e_j^t - xe_i^t, \ldots, e_m^t]^t$. The matrix F is called the **elementary matrix of the second kind.**

Elementary row operations

1. **Multiply a row vector by a nonzero scalar.** Let x be a nonzero scalar. Let $G_i(x)$ be the $m \times m$ matrix whose row vectors are

$$G_i(x) = e_1^t, \ldots, G_i = xe_i^t, \ldots, G_m = e_m^t.$$

2. $G_i(x)$ is obtained by multiplying the i^{th} row of the identity matrix by x.
3. Check that $G_i(x)A$ is the matrix obtained from A by multiplying A_i by x.
4. $G_i(x)$ is invertible and $G^{-1} = G_i(1/x)$.
5. We say that $G_i(x)$ is the **elementary matrix of the third kind.**
6. **The Gauss elimination method for solving linear equations.**
7. If A' is an REF of A then there are invertible matrices $E_1, E_2, \ldots, E_r$ corresponding to elementary row operations so that

$$A' = E_r E_{r-1} \ldots E_2 E_1 A.$$

The Gauss elimination method

1. In order to solve the system of linear equations $Ax = b$ we apply elementary row operations to the **augmented matrix** $[A, b]$ so that $[EA, Eb]$ is in REF for some invertible matrix E.
2. The system $Ax = b$ are transformed into the system $EAx = Eb$.
3. This system and the original system $Ax = b$ have the same set of solutions. The system $EAx = Eb$ can be solved by back-substitutions.
4. The non-pivotal variables can be assigned any values and the values of pivotal variables can then be determined.
5. **Example.** Consider the system

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b.$$

6. Apply elementary row operations to the augmented matrix $[A, b]$ we get

A linear system with unique solution

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \underset{R_3+R_1}{\overset{R_2-2R_1}{\longrightarrow}} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

$$\overset{R_3+R_2}{\longrightarrow} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{R_2+2R_3}{\overset{R_1-R_3}{\longrightarrow}} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\overset{R_1+\frac{1}{8}R_2}{\longrightarrow} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{\frac{-1}{8}R_2}{\overset{\frac{1}{2}R_1}{\longrightarrow}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Since there are no non-pivotal columns there a unique solution: $x_1 = x_2 = 1$ and $x_3 = 2$.

A linear system with infinitely many solutions

Consider the system of linear equations:

$$Ax = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix} = b.$$

Apply elementary row operations to A and b we get

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \begin{array}{c} R_2 - 2R_1 \\ \longrightarrow \\ R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \end{array} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \begin{array}{c} R_3 - 5R_2 \\ \longrightarrow \\ R_4 - 4R_2 \end{array} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

A linear system with infinitely many solutions

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1/6)R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - 3R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now check that every solution to $Ax = b$ is of the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

for any choice of the scalars s, t, r.

A linear system with no solution

Consider the system

$$Ax = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 6 \\ 6 \end{bmatrix} = b.$$

Apply elementary row operations to A and b to get

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \xrightarrow[R_4 - 2R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \xrightarrow[R_4 - 4R_2]{R_3 - 5R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

A linear system with no solution

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1/6)R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - 3R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore the system has no solution.

1. **Theorem.** A system $Ax = 0$ of m homogeneous equations in n unknowns where $m < n$ has infinitely many solutions.
2. **Proof.** Let A' be the REF of A. Then $Ax = 0$ and $A'x = 0$ have the same solutions. Since $m < n$, A' has at most m pivots. Therefore $n - m$ variables are free variables which can be assigned any values to get values of the pivotal variables.
3. Thus there are infinitely many solutions to $Ax = 0$.

Calculation of A^{-1} by Gauss-Jordan method

1. We describe an efficient method to find the inverse of an invertible matrix using the Gauss-Jordan method.
2. The method also shows that that all invertible matrices are products of elementary matrices.
3. **Theorem.** Let A be a square matrix. Then the following are equivalent:
   (a) A can be reduced to the identity matrix I by a sequence of elementary row operations.
   (b) A is a product of elementary matrices.
   (c) A is invertible.
   (d) The system $Ax = 0$ has only the trivial solution $x = 0$.
4. **Proof.** (a) $\Rightarrow$ (b). Let $E_1, \ldots, E_k$ be elementary matrices so that $E_k \ldots E_1 A = I$. Therefore,
$$A = E_1^{-1} \ldots E_k^{-1}.$$
5. (b) $\Rightarrow$ (c) Since elementary matrices are invertible, A is also invertible.
6. (c) $\Rightarrow$ (d) Suppose A is invertible and $Ax = 0$. Hence $A^{-1}(Ax) = x = 0$.
7. (d) $\Rightarrow$ (a) First observe that a square matrix in RCF is either the identity matrix or its bottom row is zero.

Calculation of A^{-1} by Gauss-Jordan method

1. If A can't be reduced to I by elementary row operations then U = the RCF of A has a zero row at the bottom.
2. Hence $Ux = 0$ has at most $n - 1$ nontrivial equations. which have a nontrivial solution. This contradicts (d).
3. This proposition provides us with an algorithm to calculate inverse of a matrix.
4. If A is invertible then there exist invertible matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_1 A = I$. Multiply by A^{-1} on both sides to get $E_k \cdots E_1 I = A^{-1}$.
5. **Proposition.** [**The Gauss-Jordan Algorithm**] Let A be an invertible matrix. To compute A^{-1}, apply elementary row operations to A to reduce it to an identity matrix. The same operations when applied to I, produce A^{-1}.

The Gauss-Jordan method to find the inverse of a matrix

1. **Example.** We find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
2. Construct the following the 3×6 matrix
$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$
3. Now perform row operations to reduce the matrix A to I. The same row operations when applied to I give A^{-1}.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]. \qquad \text{Hence} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$