MA 106 : LINEAR ALGEBRA : SPRING 2023

LINEAR EQUATIONS AND GAUSS ELIMINATION

1. Tutorial Problems

(1) Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM

$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(2) The n^{th} **Hilbert matrix** H_n is defined as the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $\dfrac{1}{i+j-1}$. Obtain H_3^{-1} by the Gauss-Jordan elimination Method.

(3) Find the point in $\mathbb{R}^3$ where the line joining the points $(1,-1,0)$ and $(-2,1,1)$ pierces the plane defined by $3x - y + z - 1 = 0$.

(4) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$. Write $A = EH$ where E is an elementary matrix and H is a symmetric matrix.

(5) Find the null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

(6) Show that an $n \times n$ matrix is invertible if and only if its column vectors are linearly independent.

2. Practice Problems

(7) Find the inverse of the following matrix using elementary row-operations:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}.$$

(8) Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM

$$\begin{array}{lrrrrl} \text{(i)} & & 2x_3 & -2x_4 & +x_5 & = 2 \\ & 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ & x_2 & +3x_3 & & +x_5 & = 8 \end{array} \qquad \begin{array}{lrrrrl} \text{(ii)} & 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & & -2x_2 & +x_3 & -x_4 & = 2 \\ & 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

$$\begin{array}{lrrrrl} \text{(iii)} & & & -2x_4 & +x_5 & = 2 \\ & 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ & 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array} \qquad \begin{array}{lrrrrl} \text{(iv)} & 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & & -2x_2 & +x_3 & -x_4 & = 2 \\ & 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

(9) Find all solutions of the equation $x + y + 2z - u = 3$. Express them as a linear span of linearly independent vectors.

(10) Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

(11) Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_n \ldots E_1 A$ is either the identity matrix or its bottom row is zero.