Tut 5 - Part 1

Tuesday, 6 June 2023 2:55 PM

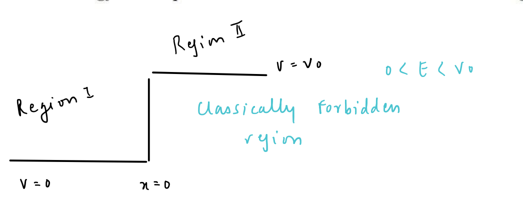
Scattering problems

Q1

1. * A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy E ($< V_0$) approaches the barrier from left.

(a) Find the value of $x = x_0$ ($x_0 > 0$), for which the probability density is $1/e$ times the probability density at $x = 0$.

(b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.



$0 < E < V_0$

Classically forbidden region

a) For $x > 0$, Schroedinger eqn

$$-\frac{h^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi$$

$V = 0$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2} = E\psi_1$$

$$\Rightarrow \frac{\partial^2\psi_1}{\partial x^2} + k_1^2\psi_1 = 0 \qquad k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$$

For Region - 2, $V = V_0$

$E < V_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} + V_0\psi_2 = E\psi_2$$

$$\frac{\partial^2\psi_2}{\partial x^2} - \underbrace{\frac{2m}{\hbar^2}(V_0 - E)}_{k_2^2}\psi_2 = 0$$

$$\psi_2 = Ce^{-k_2x} + De^{k_2x}$$

Now, $\psi_1(0) = \psi_2(0)$

and $\psi_2(x \to \infty) = 0 \Rightarrow D = 0$

$\Rightarrow A + B = C$

$$\therefore \Psi(x,t) = \begin{cases} Ae^{i(k_1x - \omega t)} + Be^{-i(k_1x + \omega t)} & x < 0 \\ Ce^{-k_2x}e^{-i\omega t} & x \geq 0 \end{cases}$$

$$\frac{|\Psi(x_0)|^2}{|\Psi(0)|^2} = \frac{1}{e}$$

$$\frac{C^2e^{-2k_2x_0}}{C^2} = \frac{1}{e} \Rightarrow 2k_2x_0 = 1$$

$$\boxed{x_0 = 1/2k_2}$$

b) Δx = not given in ques)

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

$$\Delta E \geq \frac{(\Delta p)^2}{2m} \Rightarrow \Delta E = \frac{\hbar^2k_2^2}{2m} \geq (V_0 - E)$$

$\therefore E + \Delta E \geq V_0$

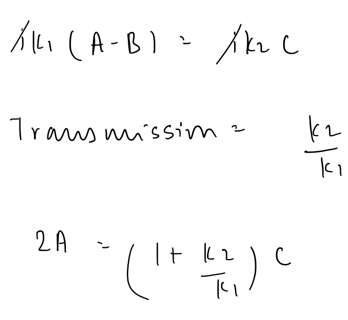
$\therefore$ Can't be sure that energy is lesser than V_0

Q2

2. Consider a potential

$$V(x) = 0 \text{ for } x < 0, \quad = -V_0 \text{ for } x > 0$$

Consider a beam of non-relativistic particles of energy $E > 0$ coming from $x \to -\infty$ and being incident on the potential. Calculate the reflection and transmission coefficients.



$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi_2 = Ce^{ik_2x} + \cancel{De^{-ik_2x}}^{\,0}$$

$A + B = C$, $D = 0$, no incidence from right

$$\cancel{i}k_1(A - B) = \cancel{i}k_2C$$

$$\text{Transmission} = \frac{k_2}{k_1}\left|\frac{C}{A}\right|^2, \quad \text{Reflection} = \left|\frac{B}{A}\right|^2$$

$$2A = \left(1 + \frac{k_2}{k_1}\right)C \Rightarrow \frac{C}{A} = \frac{2}{1 + k_2/k_1}$$

$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1}$$

Q3

3. A potential barrier is defined by $V = 0$ eV for $x < 0$ and $V = 7$ eV for $x > 0$. A beam of electrons with energy 3 eV collides with this barrier from left. Find the value of x for which the probability of detecting the electron will be half the probability of detecting it at $x = 0$.

Same as Q1, Just with no.s

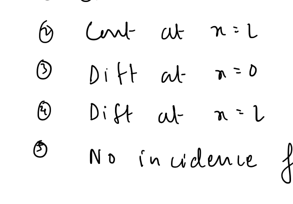
Final ans $\Rightarrow$ $\boxed{x = 2.127\,\text{Å}}$

Q4

4. * A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.

(a) Obtain an expression for the transmission coefficient.

(b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.



De Broglie wavelength = λ

Energy = E

* Can check movement directn using momentum operator.

$$\Psi(x) = \begin{cases} Ae^{ikx} + Be^{ikx} & x < 0 \\ Cx + D & 0 < x < L \\ Ge^{ikx} + Fe^{-ikx} & x > L \end{cases} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

Boundary condns:

1. Cont at $x = 0 \Rightarrow A + B = D$
2. Cont at $x = L \Rightarrow CL + D = Ge^{ikL}$
3. Diff at $x = 0 \Rightarrow (B - A)ik = C$
4. Diff at $x = L \Rightarrow (G)ike^{ikL} = C$
5. No incidence from right $\Rightarrow F = 0$

Transmissn coeff $= |G/B|^2$

$$\frac{1}{2}\left(\frac{C}{ik} + D\right) = B, \qquad CL + D = \frac{C}{ik}$$

$$D = \frac{C}{ik} - CL$$

$$\Rightarrow B = \frac{1}{2}\left(\frac{2C}{ik} - CL\right), \qquad G = \frac{Ce^{-ikL}}{ik}$$

$$\Rightarrow \left|\frac{G}{B}\right|^2 = \frac{4(C/ik)^2}{\left(\frac{2Ci}{k} + CL\right)^2} = \frac{4/k^2}{L^2 + 4/k^2} \Rightarrow \frac{4}{4 + k^2L^2}$$

when $k = \sqrt{\frac{2mE}{\hbar^2}}$

Now, $R + T = 1$

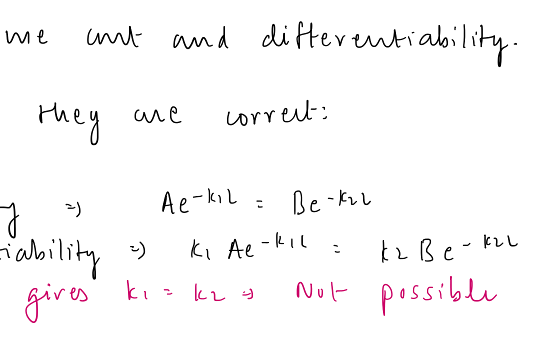
If $T = 1/2$, $R = 1/2$

$$\frac{1}{2} = \frac{4}{4 + k^2L^2} \Rightarrow kL = 2$$

$$\boxed{\lambda = \pi L}$$

Q5

5. A beam of particles of energy $E < V_0$ is incident on a barrier (see figure below) of height $V = 2V_0$. It is claimed that the solution is $\psi_I = A\exp(-k_1x)$ for region I ($0 < x < L$) and $\psi_{II} = B\exp(-k_2x)$ for region II ($x > L$), where $k_1 = \sqrt{\frac{2m(2V_0 - E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$. Is this claim correct? Justify your answer.



$V(x) = 0 \quad V(x) = 2V_0 \quad V(x) = V_0$

Simply use cont and differentiability.

Assuming they are correct:

Continuity $\Rightarrow Ae^{-k_1L} = Be^{-k_2L}$

Differentiability $\Rightarrow k_1Ae^{-k_1L} = k_2Be^{-k_2L}$

Which gives $k_1 = k_2 \Rightarrow$ Not possible

Q6

check

6. * A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0$, $V = 5V_0$ for $x \leq d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \pi h/\sqrt{8mV_0}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.

(a) Find n. Are there more than one possible values for n?

(b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

(c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used

correctn

mass m, $E = 9V_0$, $d = \frac{\pi\hbar}{\sqrt{8mV_0}}$, $T = 0.75$

For RI : $\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$ $\quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$ $\quad x < 0$

RII : $\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}$ $\quad k_2 = \sqrt{\frac{2m(E - 5V_0)}{\hbar^2}}$ $\quad 0 < x < d$

RIII : $\psi_3(x) = Ee^{ik_3x}$ $\quad k_3 = \sqrt{\frac{2m(E - nV_0)}{\hbar^2}}$ $\quad x > d$

(No incidence from right)

$$\because T = 0.75 = \frac{k_3}{k_1}\left|\frac{E}{A}\right|^2, \qquad \frac{k_3}{k_1} = \left(\frac{9-n}{9}\right)^{1/2}$$

Boundary condns:

1. Cont at $x = 0 \Rightarrow A + B = C + D$
2. Diff at $x = 0 \Rightarrow k_1(A - B) = k_2(C - D)$
3. Cont at $x = d \Rightarrow Ce^{ik_2d} + De^{-ik_2d} = Ee^{ik_3d}$ (and $k_2d = \pi$)
   $\Rightarrow C + D = -Ee^{ik_3d}$
4. Diff at $x = d \Rightarrow -k_2(C - D) = Ek_3e^{ik_3d}$

$$\text{Now, } A + B = -Ee^{ik_3d}, \qquad A - B = \frac{\cancel{k_2}}{k_1}\left(-\frac{Ek_3}{\cancel{k_2}}e^{ik_3d}\right)$$

$$2A = -Ee^{ik_3d} - \frac{Ek_3}{k_1}e^{ik_3d}$$

$$2A = -Ee^{ik_3d}\left(1 + \frac{k_3}{k_1}\right)$$

$$\Rightarrow \frac{E}{A} = \frac{2e^{-ik_3d}}{(1 + k_3/k_1)} \Rightarrow \frac{3}{4} = \frac{k_3}{k_1}\left[\frac{4}{(1 + k_3/k_1)^2}\right]$$

$$k_3/k_1 = t \Rightarrow \frac{3}{16}(1+t)^2 = t$$

$$\Rightarrow 3t^2 - 10t + 3 = 0 \Rightarrow 3t^2 - 9t - t + 3 = 0$$

$$\Rightarrow 3t(t - 3) - 1(t - 3) = 0$$

$$\Rightarrow \frac{k_3}{k_1} = 3, 1/3$$

$$\frac{9-n}{9} = 9, \qquad \frac{9-n}{9} = \frac{1}{9}$$

$$\boxed{n = -72}, \qquad \boxed{n = 8}$$

b)

$$2A = -Ee^{ik_3d}\left(1 + \frac{k_3}{k_1}\right)$$

$$2B = -Ee^{ik_3d}\left(1 - \frac{k_3}{k_1}\right)$$

$$2C = -Ee^{ik_3d}\left(1 + \frac{k_3}{k_2}\right)$$

$$2D = -Ee^{ik_3d}\left(1 - \frac{k_3}{k_2}\right)$$

and solve each in terms of A.

Q7

7. A scanning tunneling microscope (STM) can be approximated as an electron tunneling into a step potential [$V(x) = 0$ for $x \leq 0$, $V(x) = V_0$ for $x > 0$]. The tunneling current (or probability) in an STM reduces exponentially as a function of the distance from the sample. Considering only a single electron-electron interaction, an applied voltage of 5 V

and the sample work function of 7 eV, calculate the amplification in the tunneling current if the separation is reduced from 2 atoms to 1 atom thickness (take approximate size of an atom to be 3 Å).

$\phi \Rightarrow$ Potential Barrier = 7eV

E of particle = 5 V

Probability density over two atoms $\Rightarrow I \propto |\Psi(x)|^2$

$$\Rightarrow \frac{|\psi_2(2x_0)|^2}{|\psi_2(x_0)|^2}$$

$x_0 = 3\,\text{Å}$