usage: teamwork for week 15

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```
clear all; close all; clc
```

1. 求解初值问题

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t^2}{y} \\ y(0) = 2, \quad t \in [0, 7.0] \end{cases}$$

(1) 设置步长$h = 0.7$，采用前向欧拉算法。

(2) 设置步长$h = 0.7$，采用**Heun**方法(即改进的欧拉公式)。

(3) 设置步长$h = 0.7$，采用经典**RK4**方法。

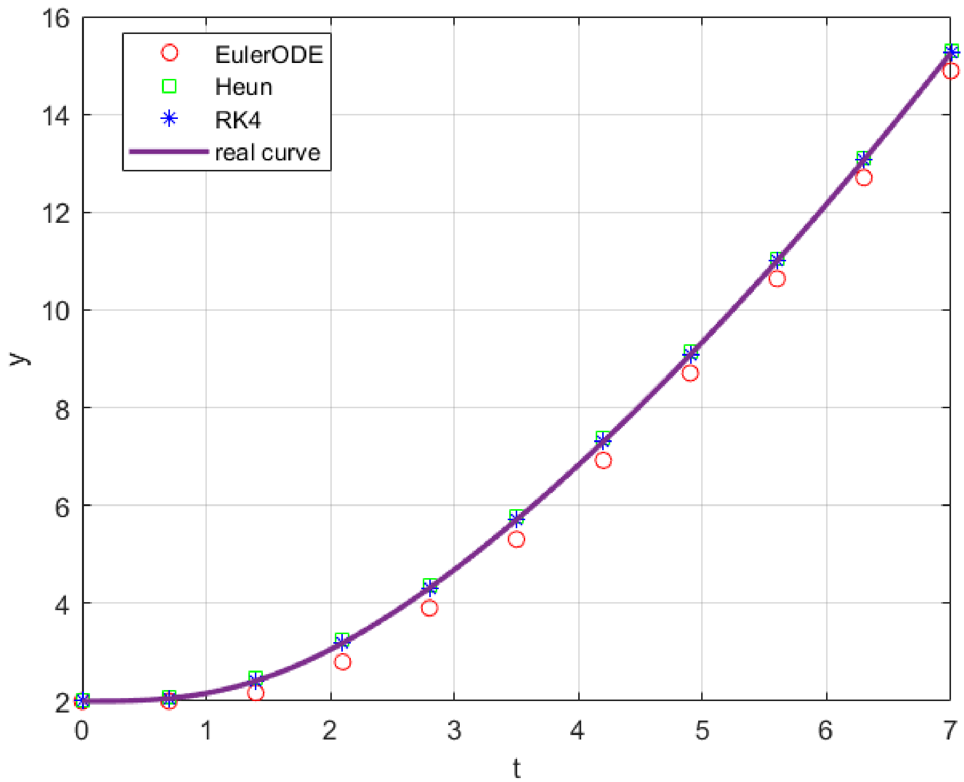
(4) 精确解为$y = \sqrt{\dfrac{2x^3}{3} + 4}$，请将上述数值计算结果与之比较， 讨论误差。并将结果画在一张图上.

解:

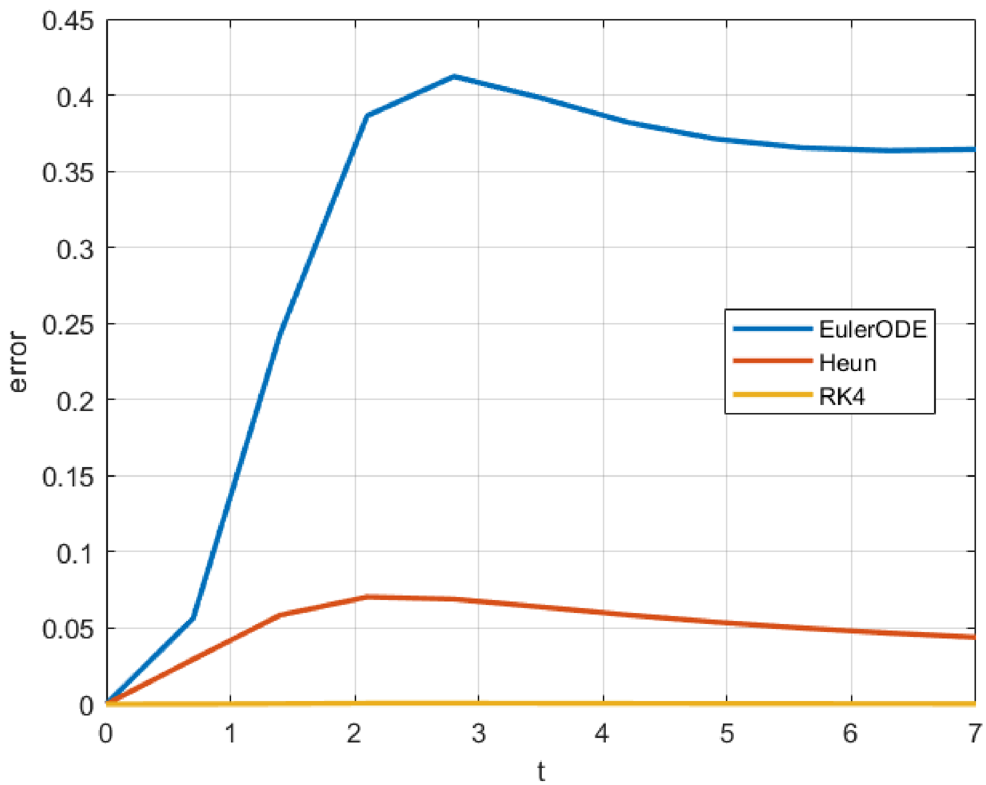
```
clear all; close all; clc
f_1 = @(t, y) t.^2./y;
h_1 = 0.7;
t_1 = 0:h_1:7;
y0_1 = 2;
% (1) 前向欧拉算法
y_1_EulerODE = EulerODE(f_1, t_1, y0_1);
% (2) Heun方法(即RK2中alpha=0.5)
y_1_Heun = RK2(f_1, t_1, y0_1, 0.5);
% (3) 经典RK4方法
y_1_RK4 = RK4(f_1, t_1, y0_1);
% (4) 精确解
y_1_check = sqrt(2*t_1.^3./3+4);

% 计算数值方法和精确解的误差
E_1_EulerODE = abs(y_1_EulerODE - y_1_check);
E_1_Heun = abs(y_1_Heun - y_1_check);
E_1_RK4 = abs(y_1_RK4 - y_1_check);
```

```
% 做出函数图像
plot(t_1, y_1_EulerODE, 'ro', t_1, y_1_Heun, 'gs', t_1, y_1_RK4, 'b*')
hold on
plot(0:0.1:7, sqrt(2*(0:0.1:7).^3./3+4), 'LineWidth', 2)
grid on
xlabel('t')
ylabel('y')
hold off
```

```
legend("EulerODE", "Heun", "RK4", "real curve", 'Location', "best")
```




```
% 做出三种方法的误差图像
plot(t_1, E_1_EulerODE, t_1, E_1_Heun, t_1, E_1_RK4, 'LineWidth', 2)
grid on
xlabel('t')
ylabel('error')
legend("EulerODE", "Heun", "RK4", 'Location', "best")
```



从图中可以直观的看出前向欧拉方法的误差最大，Heun方法次之，经典RK4方法的误差最小。

2. 求解初值问题

$$\begin{cases} \dfrac{dy}{dx} = x - \dfrac{xy}{2} \\ y(1) = 1, \quad x \in [1,5] \end{cases}$$

(1) 设置步长$h = 0.4$，采用前向欧拉算法。

(2) 设置步长$h = 0.4$，采用**Heun**方法(即改进的欧拉公式)。

(3) 设置步长$h = 0.4$，采用经典**RK4**方法。

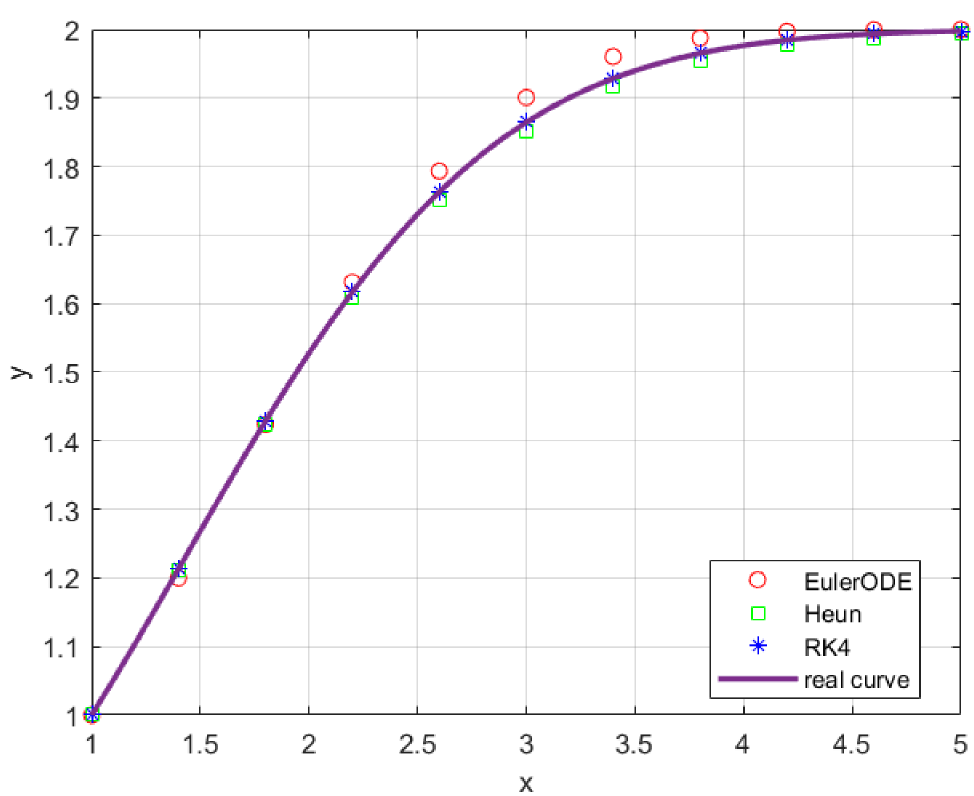
(4) 精确解为 $y = 2 - e^{\frac{1-x^2}{4}}$，请将上述数值计算结果与之比较， 讨论误差。并将结果画在一张图上.

解:

```
clear all; close all; clc
f_2 = @(x, y) x - x.*y/2;
h_2 = 0.4;
x_2 = 1:h_2:5;
y0_2 = 1;
% (1) 前向欧拉算法
y_2_EulerODE = EulerODE(f_2, x_2, y0_2);
% (2) Heun方法(即RK2中alpha=0.5)
```

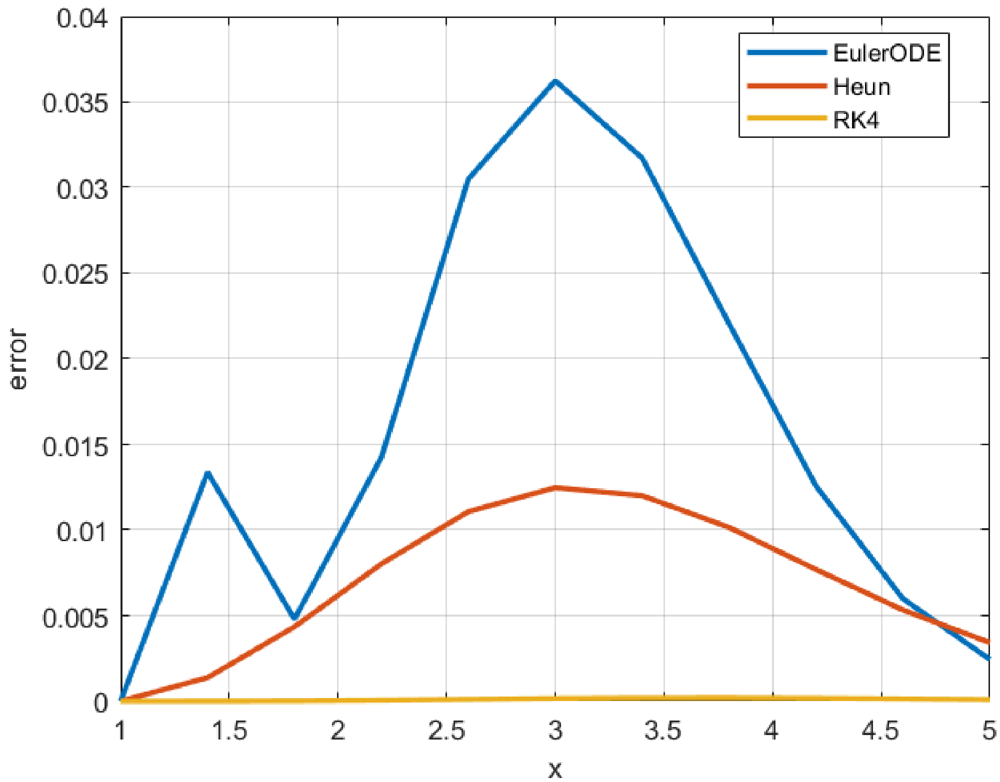
```
y_2_Heun = RK2(f_2, x_2, y0_2, 0.5);
% (3) 经典RK4方法
y_2_RK4 = RK4(f_2, x_2, y0_2);
% (4) 精确解
y_2_check = 2-exp((1-x_2.^2)/4);

% 计算数值方法和精确解的误差
E_2_EulerODE = abs(y_2_EulerODE - y_2_check);
E_2_Heun = abs(y_2_Heun - y_2_check);
E_2_RK4 = abs(y_2_RK4 - y_2_check);
```

```
% 做出函数图像
plot(x_2, y_2_EulerODE, 'ro', x_2, y_2_Heun, 'gs', x_2, y_2_RK4, 'b*')
hold on
plot(1:0.1:5, 2-exp((1-(1:0.1:5).^2)/4), 'LineWidth', 2)
grid on
xlabel('x')
ylabel('y')
hold off
legend("EulerODE", "Heun", "RK4", "real curve", 'Location', "best")
```




```
% 做出三种方法的误差图像
plot(x_2, E_2_EulerODE, x_2, E_2_Heun, x_2, E_2_RK4, 'LineWidth', 2)
grid on
xlabel('x')
ylabel('error')
```

```
legend("EulerODE", "Heun", "RK4", 'Location', "best")
```



从误差曲线中可以看出，仍然是前向欧拉算法的误差最大，Heun方法的误差次之，经典RK方法的误差最小。

3. 求解初值问题

$$
\begin{cases} \dfrac{dy}{dt} = y + t^3 \\ y(0) = 1, \quad x \in [0, 5] \end{cases}
$$

(1) 设置步长 $h = 0.5$，采用前向欧拉算法。

(2) 设置步长 $h = 0.5$，采用中心公式。

(3) 设置步长 $h = 0.5$，采用经典**RK4**方法。

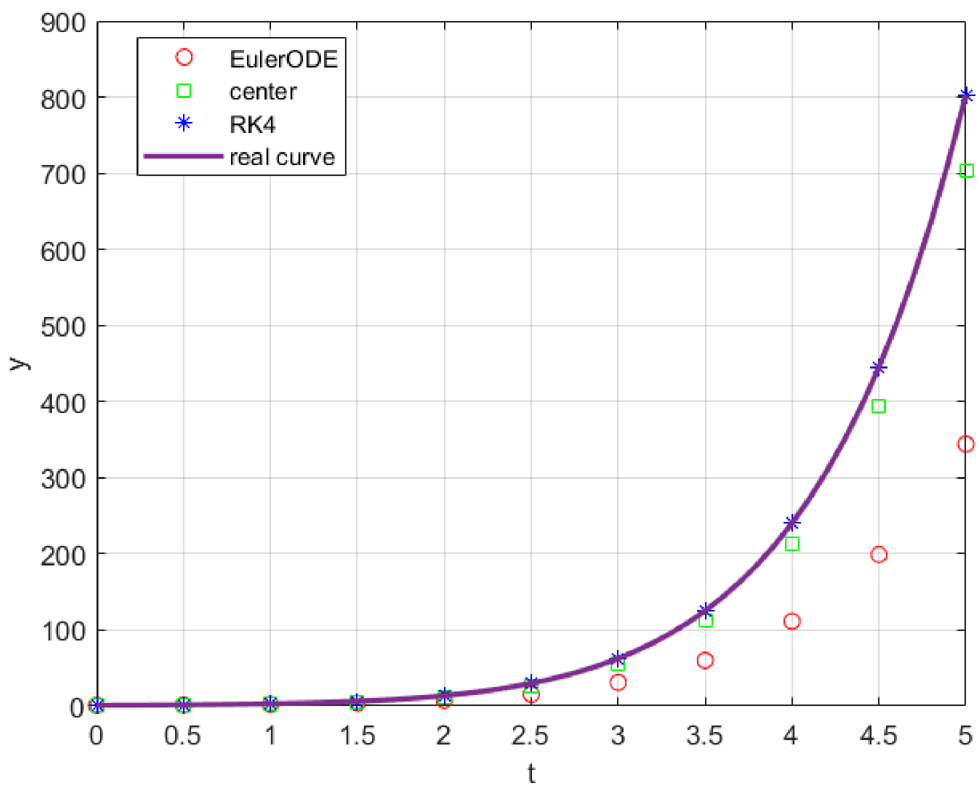
(4) 精确解为 $y = 7e^t - t^3 - 3t^2 - 6t - 6$，请将上述数值计算结果与之比较， 讨论误差。并将结果画在一张图上.

解:

```
clear all; close all; clc
f_3 = @(t, y) y + t.^3;
h_3 = 0.5;
t_3 = 0:h_3:5;
y0_3 = 1;
% (1) 前向欧拉算法
y_3_EulerODE = EulerODE(f_3, t_3, y0_3);
```

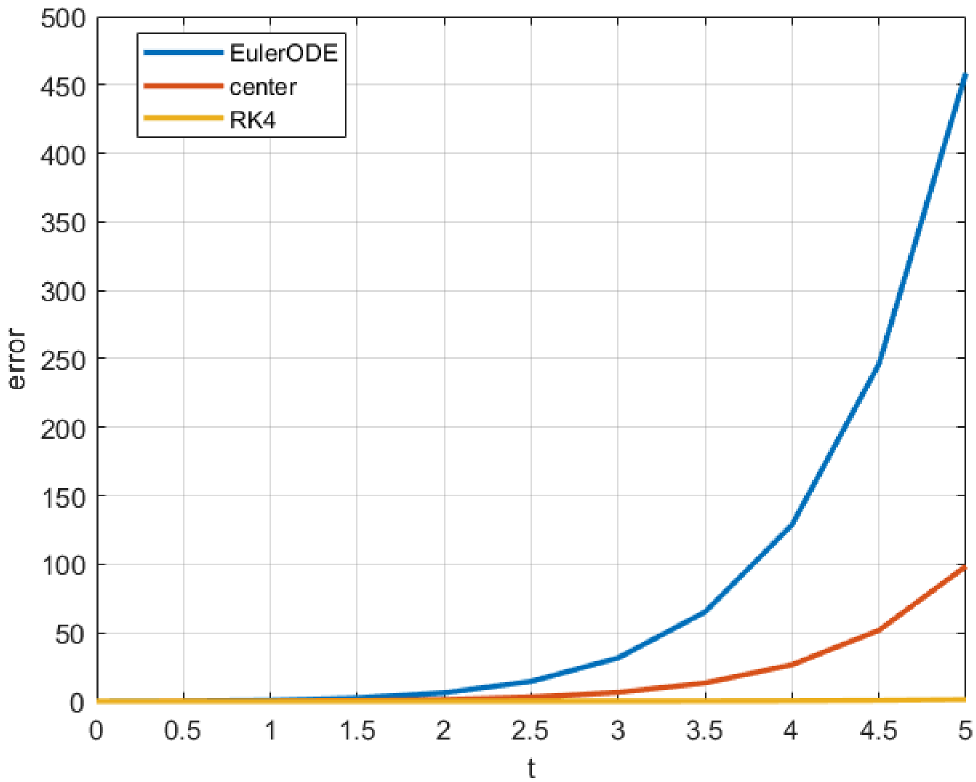
```
% (2) 中心方法(即RK2中alpha=0)
y_3_Heun = RK2(f_3, t_3, y0_3, 0);
% (3) 经典RK4方法
y_3_RK4 = RK4(f_3, t_3, y0_3);
% (4) 精确解
y_3_check = 7.*exp(t_3) - t_3.^3 - 3*t_3.^2 - 6*t_3 - 6;

% 计算数值方法和精确解的误差
E_3_EulerODE = abs(y_3_EulerODE - y_3_check);
E_3_Heun = abs(y_3_Heun - y_3_check);
E_3_RK4 = abs(y_3_RK4 - y_3_check);
```

```
% 做出函数图像
plot(t_3, y_3_EulerODE, 'ro', t_3, y_3_Heun, 'gs', t_3, y_3_RK4, 'b*')
hold on
plot(0:0.1:5, 7.*exp(0:0.1:5) - (0:0.1:5).^3 - 3*(0:0.1:5).^2 - 6*(0:0.1:5) - 6, 'LineWidth', 2
grid on
xlabel('t')
ylabel('y')
hold off
legend("EulerODE", "center", "RK4", "real curve", 'Location', "best")
```




```
% 做出三种方法的误差图像
plot(t_3, E_3_EulerODE, t_3, E_3_Heun, t_3, E_3_RK4, 'LineWidth', 2)
grid on
xlabel('t')
ylabel('error')
```

```
legend("EulerODE", "center", "RK4", 'Location', "best")
```



从误差曲线中可以看出，前向欧拉算法的误差随*t*的增大增长最大，中心方法的误差次之，经典RK方法的误差最小。

4. 求解变系数的一阶微分方程

$$\begin{cases} y'(t) + f(t)y(t) = g(t) \\ y(1) = 1, \quad t \in [1,5] \end{cases}$$

已知$f(t)$在时间节点`ft`的函数值为`f`. $g(t)$在时间节点`gt`的函数值为`g`

```
clear all; close all; clc
ft = linspace(0,5,25);
f = ft.^2 - ft - 3;

gt = linspace(1,6,25);
g = 3 * sin(gt - 0.25);
```

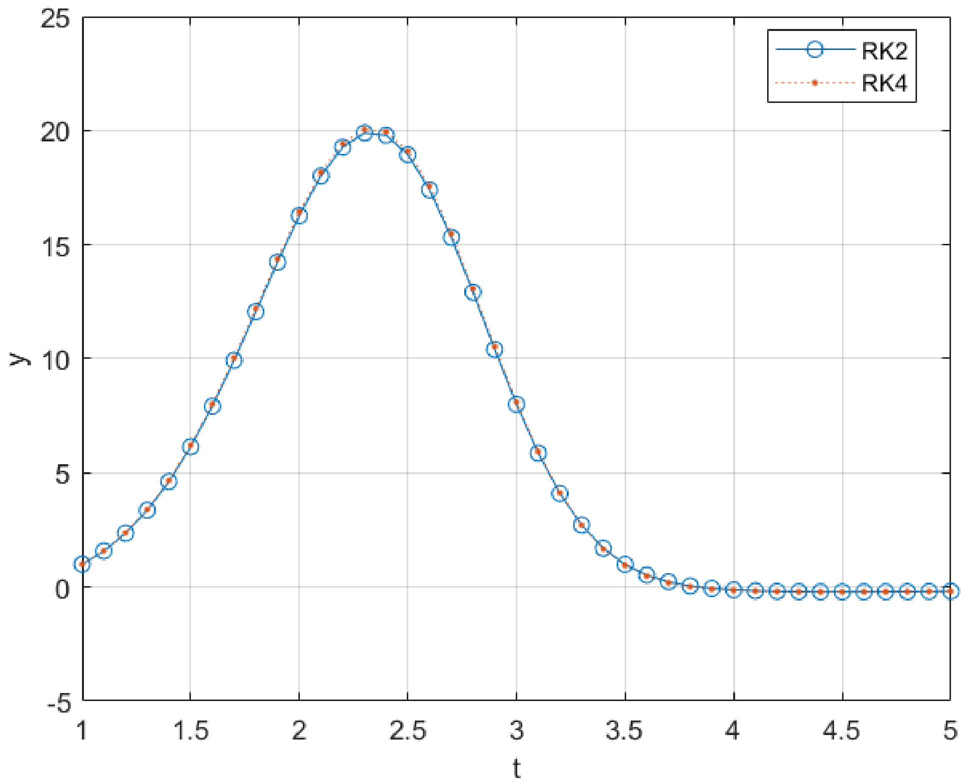
(1) 设置步长$h = 0.1$，采用**RK2**的中心公式， 计算$y(t)$的近似值。

(2) 设置步长$h = 0.1$，采用经典**RK4**公式， 计算$y(t)$的近似值，并与**(1)**题结果比较。请将结果画在一张图上。

提示： 可调用**matlab**的内置函数`interp1`对$f(t)$与$g(t)$进行插值。

解:

```
F_4 = @(t) interp1(ft, f, t);
G_4 = @(t) interp1(gt, g, t);
f_4 = @(t, y) G_4(t) - F_4(t).*y;
h_4 = 0.1;
t_4 = 1:h_4:5;
y0_4 = 1;
% (1) RK2中心公式(alpha=0)
y_4_RK2 = RK2(f_4, t_4, y0_4, 0);
% (2) 经典RK4公式
y_4_RK4 = RK4(f_4, t_4, y0_4);

% 作出两个方法的结果图像
plot(t_4, y_4_RK2, 'o-', t_4, y_4_RK4, ':.');
xlabel("t")
ylabel("y")
grid on
legend("RK2", "RK4", 'Location', "best")
```



两种方法得到的函数图像基本相同，在$[1.5, 3]$的区间内两种方法得到近似值有些偏差。

5. 误差函数$erf(x)$由以下积分形式定义

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

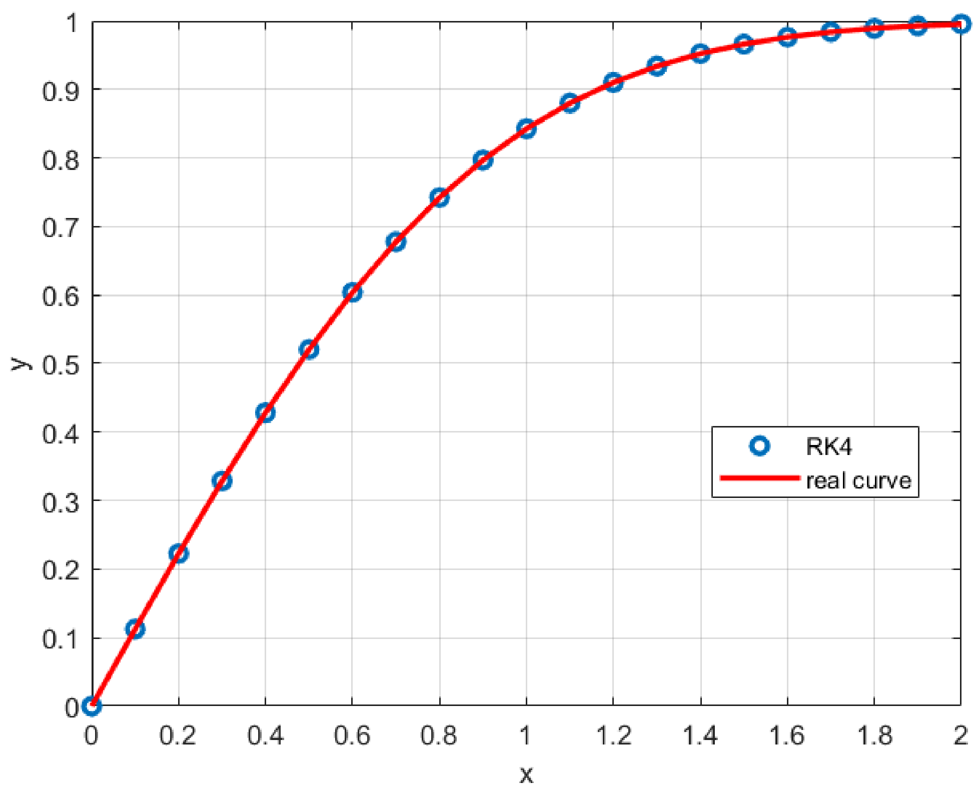
也可以通过求解以下常微分方程求得近似值

$$\begin{cases} y' = \frac{2}{\sqrt{\pi}} e^{-x^2} \\ y(0) = 0 \end{cases}$$

选择合适的步长，采用**RK4**方法在$0 \le x \le 2$上求解，并与**matlab**内置函数**`erf(x)`**比较结果.

解:

```
clear all; close all; clc
f_5 = @(x, y) 2*exp(-x.^2)/sqrt(pi);
h_5 = 0.1;
x_5 = 0:h_5:2;
y0_5 = 0;
% 采用经典RK4方法求解
y_5_RK4 = RK4(f_5, x_5, y0_5);
% matlab内置函数 erf(x)
y_5_check = erf(x_5);
% 计算误差
E_5_RK4 = abs(y_5_check - y_5_RK4)./y_5_check;
% 作出近似值和精确值的比较
plot(x_5, y_5_RK4, 'o', x_5, y_5_check, 'r-', 'LineWidth', 2);
xlabel("x")
ylabel("y")
grid on
legend("RK4", "real curve", 'Location', "best")
```



RK4方法迭代得到的近似值与erf(x)精确值基本相等。

6. 请将以下**2**阶常微分方程改写成两个一阶常微分方程组

$$\frac{d^2y}{dt^2} + 5\left(\frac{dy}{dt}\right)^2 - 6y + e^{\sin t} = 0$$

解:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ 6y_1 - e^{sin(t)} - 5y_2^2 \end{pmatrix}$$

7. 请将以下两个**2**阶常微分方程改写成四个一阶常微分方程组

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{m}\left(\frac{dx}{dt}\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$\frac{d^2y}{dt^2} = -g - \frac{\gamma}{m}\left(\frac{dy}{dt}\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

解:

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{\gamma}{m}x_2\sqrt{(x_2)^2 + (y_2)^2} \\ y_2 \\ -g - \frac{\gamma}{m}y_2\sqrt{(x_2)^2 + (y_2)^2} \end{pmatrix}$$

functions*

前向欧拉算法

```
function y = EulerODE(f,x,y0)
%
% EulerODE uses Euler's method to solve a first-order
% ODE given in the form y' = f(x,y) subject to initial
% condition y0.
%
% y = EulerODE(f,x,y0) where
%
```

```
% f is an inline function representing f(x,y),
% x is a vector representing the mesh points,
% y0 is a scalar representing the initial value of y,
%
% y is the vector of solution estimates at the mesh
% points.
%
% Ramin S. Esfandiari, Numerical Methods for Engineers and Scientists Using
% Matlab,
% Section 7.1 p.331
%
y = zeros(1,length(x));
y(1) = y0; h = x(2) - x(1);
for n = 1:length(x) - 1
    y(n+1) = y(n) + h*f(x(n),y(n));
end
end
```

RK2方法

$$\begin{cases} \alpha + \beta = 1 \\ \beta q = \frac{1}{2} \end{cases}$$

```
function y = RK2(f,x,y0,alpha)
    beta = 1 - alpha;
    q = 0.5/beta;
    y = zeros(1, length(x));
    y(1) = y0;
    h = x(2) - x(1);
    for j = 1:length(x) - 1
        k1 = h*f(x(j),y(j));
        k2 = h*f(x(j) + q*h,y(j) + q*k1);
        y(j+1) = y(j) + alpha * k1 + beta * k2;
    end
end
```

经典龙格-库塔法RK4

```
function y = RK4(f,x,y0)
%
% RK4 uses the classical RK4 method to solve a first-
% order ODE given in the form y' = f(x,y) subject to
% initial condition y0.
%
% y = RK4(f,x,y0) where
%
% f is an inline function representing f(x,y),
% x is a vector representing the mesh points,
% y0 is a scalar representing the initial value of y,
%
% y is the vector of solution estimates at the mesh
% points.
```

```
%
% Ramin S. Esfandiari, Numerical Methods for Engineers and Scientists Using
% Matlab,
% Section 7.3.3.1 p.347
%
y = zeros(1,length(x));
y(1) = y0; h = x(2) - x(1);
for n = 1:length(x) - 1
    k1 = f(x(n),y(n));
    k2 = f(x(n) + h/2,y(n) + h*k1/2);
    k3 = f(x(n) + h/2,y(n) + h*k2/2);
    k4 = f(x(n) + h,y(n) + h*k3);
    y(n+1) = y(n) + h*(k1 + 2*k2 + 2*k3 + k4)/6;
end
end
```