ISOMORPHISM OF GRAPHS

- We showed two or more than two different modes for drawing the graph.
- At the time, we focused on the fact that we were dealing with the same set of vertices and verified the edge set was maintained in the new drawings.
- Intuitively, two graphs G and H are considered the 'same' if it is possible to relocate the vertices of one of the graphs, say G, so that these vertices have the same positions as the vertices in H, the result of which is that the two graphs look identical.
- Mathematically, we use a fancier term, isomorphic graphs, to replace 'same graphs'.
- However, two graphs with distinct vertex sets can still produce the same edge relationships; more technically these graphs are called isomorphic if every vertex from G_1 can be paired with a unique vertex from G_2 so that corresponding edges from G_1 are maintained in G_2.

Definition 1.17 Two graphs G_1 and G_2 are ***isomorphic***, denoted $G_1 \cong G_2$, if there exists a bijection $f : V(G_1) \to V(G_2)$ so that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$.

Remarks:

For graph isomorphism we need to check the following.

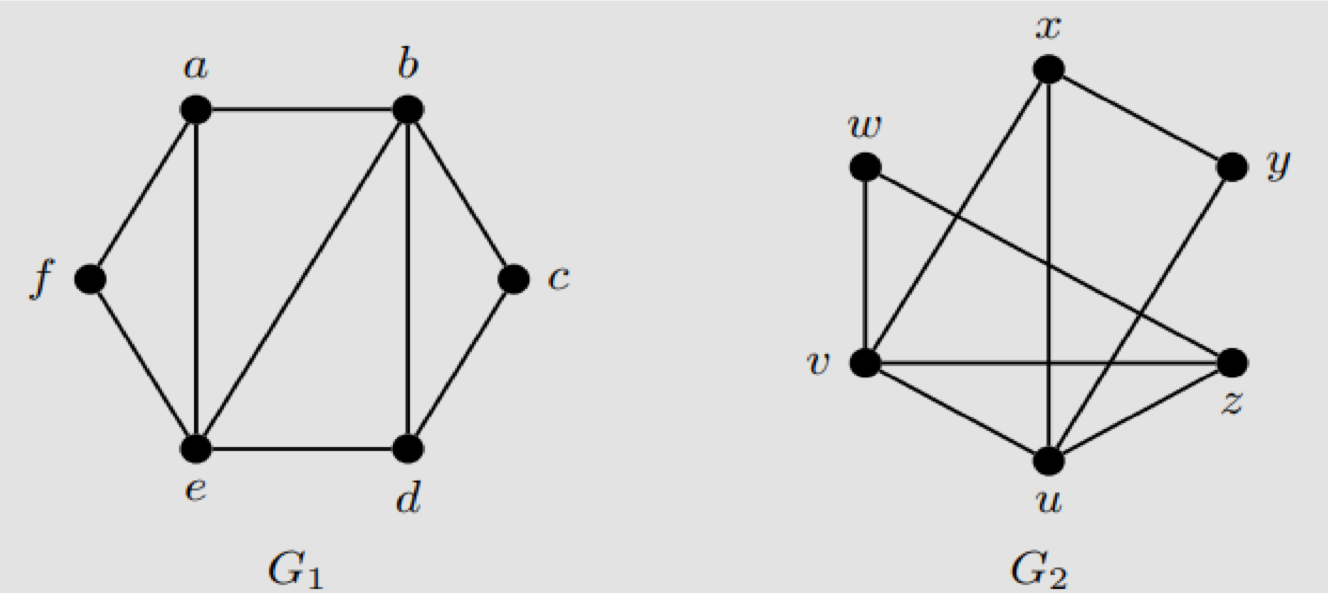
- Number of vertices.
- Number of edges.
- Degree of vertices.
- Bijection (vertex pairing & maintained edge relationship.).

Example:

isomorphism maps w, x, y, z to c, b, d, a, respectively.

$$
\begin{array}{c c}
 & \begin{array}{cccc} w & x & y & z \end{array} \\
\begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c c}
 & \begin{array}{cccc} w & y & z & x \end{array} \\
\begin{array}{c} w \\ y \\ z \\ x \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\end{array}
$$

Example 1.14 Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.

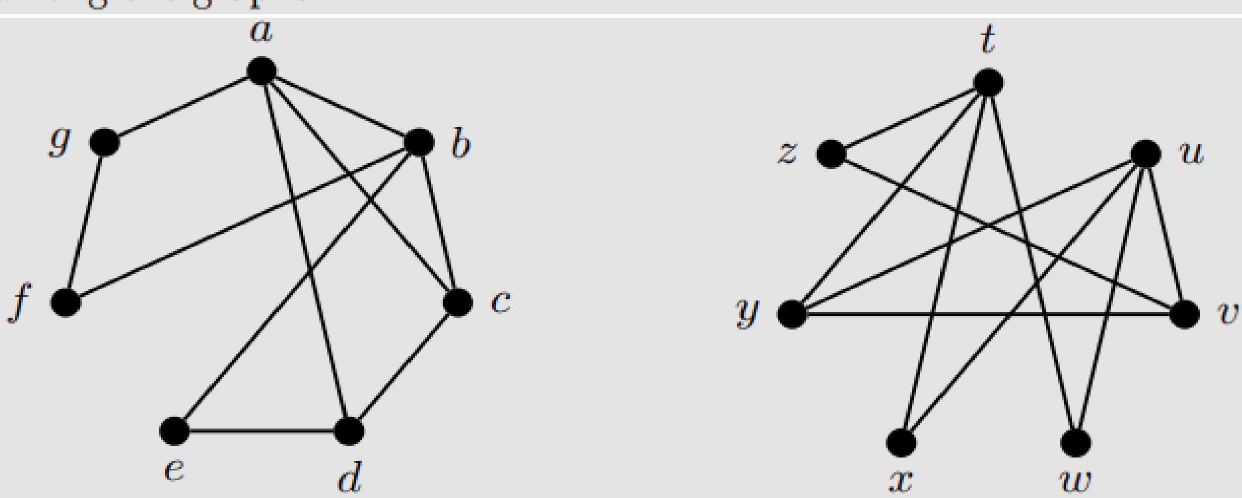


Solution: First note that both graphs have six vertices and nine edges, with two vertices each of degrees 4, 3, and 2. Since corresponding vertices must have the same degree, we know b must map to either u or v. We start by trying to map b to v. By looking at vertex adjacencies and degree, we must have e map to u, c map to w, and a map to x. This leaves f and d, which must be mapped to y and z, respectively. The chart below show the vertex pairings and checks for corresponding edges.

$V(G_1) \longleftrightarrow V(G_2)$	Edges	
$a \longleftrightarrow x$	$ab \longleftrightarrow xv$	✓
$b \longleftrightarrow v$	$ae \longleftrightarrow xu$	✓
$c \longleftrightarrow w$	$af \longleftrightarrow xy$	✓
$d \longleftrightarrow z$	$bc \longleftrightarrow vw$	✓
$e \longleftrightarrow u$	$bd \longleftrightarrow vz$	✓
$f \longleftrightarrow y$	$be \longleftrightarrow vu$	✓
	$cd \longleftrightarrow wz$	✓
	$de \longleftrightarrow zu$	✓
	$ef \longleftrightarrow uy$	✓

Since all edge relationships are maintained, we know G_1 and G_2 are isomorphic.

Example 1.15 Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.



G_3 G_4

Solution: First note that both graphs have seven vertices and ten edges, with two vertices each of degrees 4 and 3, and three vertices of degree 2. As in the previous example, we know corresponding vertices must have the same degree, and so the vertices of degree 4 in G_3, a and b, must map to the vertices of degree 4 in G_4, namely t and u. However, in G_3 the degree 4 vertices (a and b) are adjacent, whereas in G_4 there is no edge between the degree 4 vertices (t and u). Thus G_3 and G_4 are not isomorphic.

Remark:

The previous example illustrates that no one property guarantees two graphs are isomorphic. In fact, simply having the same number of vertices of each degree is not enough.

- The theorem below lists the more useful graph invariants.

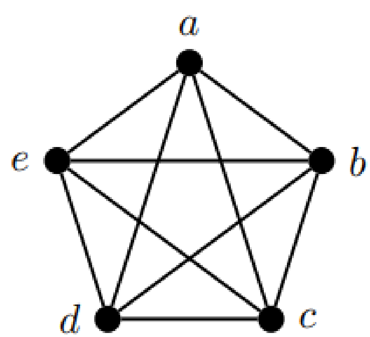
Theorem 1.18 Assume G_1 and G_2 are isomorphic graphs. Then G_1 and G_2 must satisfy any of the properties listed below; that is, if G_1

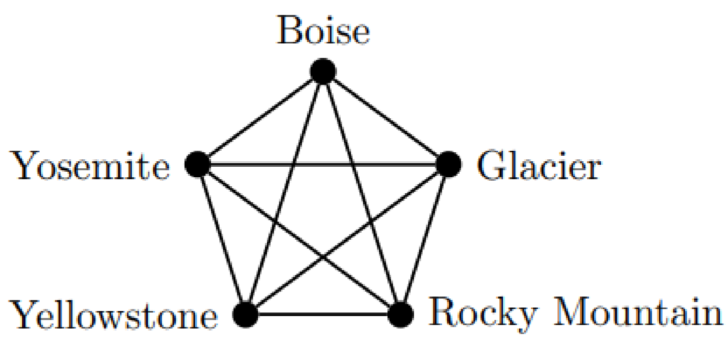
- is connected
- has n vertices
- has m edges
- has m vertices of degree k
- has a cycle of length k (see Section 2.1.2)
- has an eulerian circuit (see Section 2.1.3)
- has a hamiltonian cycle (see Section 2.2)

then so too must G_2 (where n, m, and k are non-negative integers).

Isomorphism Class:

- When two graphs are known to be isomorphic, we say they belong to the same isomorphism class.
- An isomorphism class can be represented by an unlabeled graph and the members of the isomorphism class are those graphs that can be found by labeling the vertices of the representative graph.
- For example, the graph K_5 below is the representative graph for all **complete graphs** on 5 vertices.
- Two (labeled) graphs that belong to this class are shown below it.





PROOF TECHNIQUES:

- In this section we review the basics of mathematical proof and introduce some early graph results that can be proven with little intuition about graphs and their structure.

Direct Proofs:

- Most mathematical statements have an underlying conditional form; that is, they can be written as "If . . ., then . . ..".
- Writing a statement in the standard "if – then" form allows the logical structure to stand out and provides guidance into the format of the argument.
- In logical symbols, conditional statements are given as $p \rightarrow q$.
- A direct proof begins by assuming the premise of the conditional (p) and uses logic, definitions, and previously proven theorems to show the conclusion (q) is true.

Proposition 1.20 The sum of two odd integers is even.

Proof: Assume x and y are odd integers. Then there exist integers n and m such that $x = 2n+1$ and $y = 2m+1$. Thus $x+y = (2n+1)+(2m+1) = 2(n+m+1) = 2k$, where k is the integer given by $n+m+1$. Therefore $x+y$ is even.

- ✓ A proper mathematical proof should be self-contained (all variables are defined), concise (no extra information is included), and complete (the proper conclusion is reached).

The theorem below can be considered one of the first results in graph theory (and in some publications is referred to as "**The First Graph Theorem**"), as it was published in 1736 by **Leonhard Euler**.

Theorem 1.21 (Handshaking Lemma) Let $G = (V, E)$ be a graph and $|E|$ denote the number of edges in G. Then the sum of the degrees of the vertices equals twice the number of edges; that is if $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{n} \deg(v_i) = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

Proof: Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Any edge $e = v_i v_j$ of G will be counted once in the total $|E|$. Since each edge is defined by its two endpoints, this edge will add one to the count of both $\deg(v_i)$ and $\deg(v_j)$. Thus every edge of G will add two to the count of the sum of the degrees. Thus $\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|$.

- ➢ The following is a direct consequence of the Handshaking Lemma.

Corollary 1.22 Every graph has an even number of vertices of odd degree.

Indirect Proofs:

- Direct proofs can be considered the preferable method of proof as their structure models the statement they are proving.
- However, some statements are either impossible or much more difficult to prove in this way and a different technique is needed.
- Classic examples of this include proving there are infinitely many primes or that √2 is irrational.
- There are two main types of indirect proofs:
    1. Proof by contradiction.
    2. Proof by contraposition.

Proof by Contradiction:

- we assume the negation of the statement is true.
- Through logic, definitions, and previous results, we show a contradiction must be occurring, thus proving the original statement must be true.

Example: An example from elementary number theory is shown below.

Proposition 1.23 For any integer n, if n^2 is odd then n is odd.

Proof: Suppose for a contradiction that n^2 is odd but n is even. Then $n = 2k$ for some integer k and $n^2 = (2k)^2 = 4k^2 = 2j$ where j is the integer $2k^2$. Thus n^2 is both even and odd, a contradiction. Therefore if n^2 is odd then n is also odd.

Proof by Contrapositive:

- For a Proof by Contraposition, we use a direct proof on the contrapositive ($\sim q \rightarrow \sim p$) of the original conditional statement ($p \rightarrow q$).

- Since the contrapositive is logically equivalent to the original statement, this shows the intended result to be true.

Example: The statement above can also be proven using the contrapositive, as shown below.

Proposition 1.23 For any integer n, if n^2 is odd then n is odd.

Proof: Suppose n is not odd. Then n is even and $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2 = 2j$ where j is the integer $2k^2$, and so n^2 is even. Thus if n^2 is odd, it must be that n is also odd.

Remark:

Note that both indirect proof techniques can be used on (appropriate) conditional statements, but some statements can only use a contradiction argument.

In graph theory, it is often useful to assume some property of a graph does not hold and then use that assumption to find a contradiction to another known graph property.

Proposition 1.24 For every simple graph G on at least 2 vertices, there exist two vertices of the same degree.

Proof: Suppose for a contradiction that G is a simple graph on n vertices, with $n \geq 2$, in which no two vertices have the same degree. Since there are no loops and each vertex can have at most one edge to any other vertex, we know the maximum degree for any vertex is $n - 1$ and the minimum degree is 0. Since there are exactly n integers from 0 to $n - 1$, we know there must be exactly one vertex for each degree between 0 and $n - 1$. But the vertex of degree $n - 1$ must then be adjacent to every other vertex of G, which contradicts the fact that a vertex has degree 0. Thus G must have at least two vertices of the same degree.

Mathematical Induction:

- The last proof technique we review is quite useful when studying discrete objects, especially objects that can easily be transformed into ones of smaller size.
- The power of induction is that we are proving a statement that holds for an **infinite number of objects** but only need to prove two very specific items.
- Mathematical induction relies on a two-step process.
  1. In the first step (sometimes referred to as the **base case or basis step**) we show the statement to be proved holds for a specific value or size.
  2. In the second step (called the **induction step**) we assume that the statement holds for some unknown value and then show the statement also holds for the next value.

Proposition 1.25 The complete graph K_n has $\dfrac{n(n-1)}{2}$ edges.

Proof: Argue by induction on n. If $n = 1$ then K_1 is just a single vertex and has $0 = \frac{1(0)}{2}$ edges.

Suppose for some $n \geq 1$ that K_n has $\dfrac{n(n-1)}{2}$ edges. We can form K_{n+1} by adding a new vertex v to K_n and adjoining v to all the vertices from K_n. Thus K_{n+1} has n more edges than K_n and so by the induction hypothesis has

$$n + \frac{n(n-1)}{2} = \frac{2n + n(n-1)}{2} = \frac{n(2+n-1)}{2} = \frac{n(n+1)}{2}$$

edges.

Thus by induction we know K_n has $\dfrac{n(n-1)}{2}$ edges for all $n \geq 1$.