UNIT 4 CURVE TRACING

Structure

4.1 INTRODUCTION

A picture is worth a thousand words. A curve which is the visual image of a functional relation gives us a whole lot of information about the relation. Of course, we can also obtain this information by analysing the equation which defines the functional relation. But studying the associated curve is often easier and quicker. In addition to this, a curve which represents a relation between two quantities also helps us to easily find the value of one quantity corresponding to a specific value of the other. In this unit we shall try to understand what is meant by the picture or the graph of a relation like $f(x, y) = 0$, and how to draw it. We shall be using many results from the earlier units here. With this unit we come to the end of Block 2, in which we have studied various geometrical features of functional relations with the help of differential calculus.

Objectives

After studying this unit, you should be able to

- list the properties which can be used for tracing a curve
- trace some simple curves whose equations are given in Cartesian, parametric or polar forms.

4.2 GRAPHING A FUNCTION AND CURVE TRACING

Recall that by the graph of a function $f : D \to \mathbf{R}$ we mean the set of points $\{(x, f(x)) : x \in D\}$. Similarly, the set of points $\{(x, y) : f(x, y) = 0\}$ is known as the graph of the functional relation $f(x, y) = 0$. Graphing a function or a functional relation means showing the points of the corresponding set in a plane. Thus, essentially curve tracing means plotting the points which satisfy a given relation. However, there are some difficulties involved in this. Let's see what these are and how to overcome them.

It is often not possible to plot all the points on a curve. The standard technique is to plot some suitable points and to get a general idea of the shape of the curve by considering tangents, asymptotes, singular points,extreme points, inflection points, concavity, monotonicity, periodicity etc. Then we draw a free hand curve as nearly satisfying the various properties as is possible.

The curves or graphs that we draw have a limitation. If the range of values of either (or both) variable is not finite, then it is not possible to draw the complete graph. In such cases the graph is not only approximate, but is also incomplete. For example, consider the simplest curve, a straight line. Suppose we want to draw the graph of $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = c$. We know that this is in line parallel to the x-axis. But it is not possible to draw a complete graph as this line extends infinitely on both sides. We indicate this by arrows at both ends as in Fig. 1.

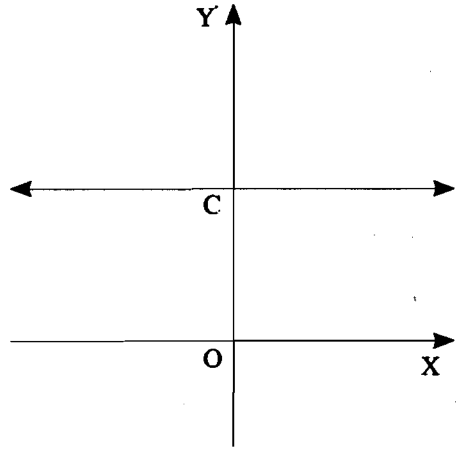


Fig. 1

In the next section we shall take up the problem of tracing of curves when the equation is given in the Cartesian form.

4.3 TRACING A CURVE : CARTESIAN EQUATION

Suppose the equation of a curve is f(x, y) = 0. We shall now list some steps which, when taken, will simplify our job of tracing this curve.

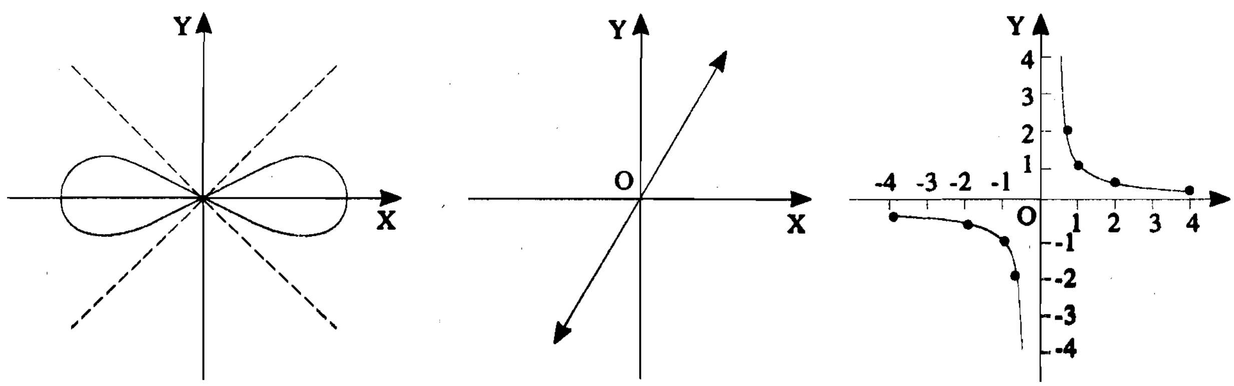
1) The first step is to determine the **extent** of the curve. In other words we try to find a region or regions of the plane which cannot any point of the curve. For example, no point on the curve $y^2 = x$, lies in the second or the third quadrant, as the x-coordinate of any point on the curve has to be non-negative. This means that our curve lies entirely in the first and the fourth quadrants.

   A point to note here is that it is easier to determine the extent of a curve if its equation can be written explicitly as y = f(x) or x = f(y).

A curve is symmetrical about a line if, when we fold the curve on the line, the two portions of the curve exactly coincide.

2) The second step is to find out if the curve is symmetrical about any line, or about the origin. We have already discussed symmetry of curves in Unit 1. Fig. 2, shows you some examples of symmetric curves.

A curve is symmetrical about the origin if we get the same curve after rotating it through 180°



(a) Symmetric about the x-axis. **(b) Symmetric about the origin.** **(c) Symmetric about the line y = x.**

Fig. 2

Here we give you some hints to help you determine the symmetry of a curve.

a) If all the powers of x occurring in f(x, y) = 0 are even, then f(x, y) = f(– x, y) and the curve is symmetrical about the y-axis.

In this case we need to draw the portion of the graph on only one side of the y-axis. Then we can take its reflection in the y-axis to get the complete graph. We can similarly text the symmetry of a curve about the x-axis.

b) If $f(x, y) = 0 \Leftrightarrow f(-x, -y) = 0$, then the curve is symmetrical about the origin is such cases, it is enough to draw the part of the graph above the x-axis and rotate it through 180° to get the complete graph.

c) If the equation of the curve does not change when we interchange x and y, then the curve is symmetrical about the line $y = x$. Table 1 illustrates the application of these criteria for different curves.

Table 1

Equation	Symmetry
$x^3 + y^2 + y^4 = 0$	About the x-axis (even powers of y)
$x^4 + y^3 + y^2 = 0$	About the y-axis (even powers of x)
$x^4 + x^2y^2 + y^4 = 0$	About the origin $(f(-x, -y) = 0 \Leftrightarrow f(x,y) = 0)$ About both axes $(f(x, y) = f(-x, y), f(x, y) = f(x, -y))$ About the line $y = x$ $(f(x, y) = f(y, x))$
$x^2 + y^4 = 10$	About both axes, (even powers of x and y) but not about $y = x$. $f(x, y) \neq f(y, x)$

3) The next step is to determine the points where the curve intersects the axes. If we put $y = 0$ in $f(x, y) = 0$, and solve the resulting equation for x, we get the points of inersection with the x-axis. Similarly, putting $x = 0$ and solving the resulting equation for y, we can find the points of intersection with the y-axis.

4) Try to locate the points where the function is discontinuous.

5) Calculate dy/dx. This will help you in locating the portions where the curve is rising $(dy/dx > 0)$ or falling $(dy/dx < 0)$ or the points where it has a corner (dy/dx does not exitst).

6) Calculate d^2y/dx^2. This will help you in locating maxima $(dy/dx = 0, d^2y/dx^2 < 0)$ and minima $(dy/dx = 0, d^2y/dx^2 > 0)$. You will also be able to determine the points of inflection $(d^2y/dx^2 = 0)$. These will give you a good idea about the shape of the curve.

7) The next step is to find the asymptotes, if there are any. They indicate the trend of the branches of the curve extending to infinity.

8) Another important step is to determine the singular points. The shape of the curve at these points is, generally, more complex, as more than one branch of the curve passes through them.

9) Finally, plot as many points as you can, around the points already plotted. Also try to draw tangents to the curve at some of these plotted points. For this you will have to calculate the derivative as these points. Now join the plotted points by a smooth curve (except at points of discontinuity). The tangents will guide you in this, as they give you the direction of the curve.

We shall now illustrate this procedure through a number of examples. You will notice, that it may not be necessary to take all the nine steps mentioned above, in each case. We begin by tracing some functions which were introduced in Unit 1.

Example 1 Consider the function $y = |x|$. Here y can take only positive values. Thus, the graph lies above the x-axis. Further, the function $y = |x|$ is symmetric about the y-axis. On the right of the y-axis, $x > 0$ and so $|x| = x$. Thus the graph reduces to that of $y = x$ and you know that this is a straight line equally inclined to the axes (Fig. 3(a) below).

The curve meets the y-axis only at the origin. Taking its reflection in the y-axis, we get the complete graph as shown in Fig. 3(b). We have drawn arrows at the end of the line segment to indicate that the graph extends indefinitely.

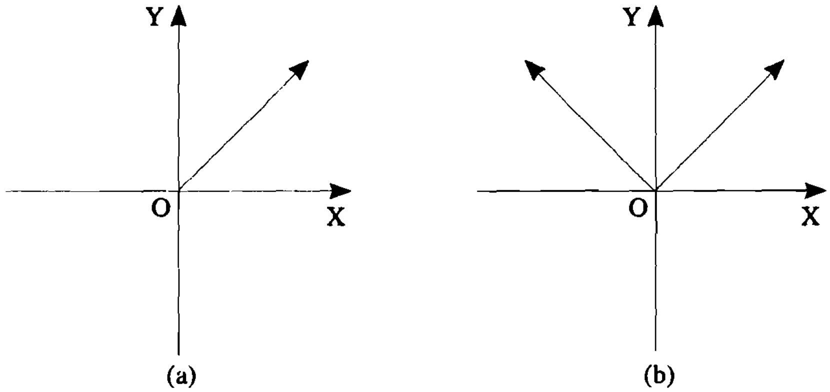


Fig. 3 : (a) Graph to the right of the y-axis. (B) Complete graph.

Example 2 The greatest integer function $y = [x]$ is discontinuous at every integer point. Hence there is a break in the graph at every integer point n. In every interval $[n, n + 1[$ its value is constant, namely n. Hence the graph is as shown in Fig. 4. Note that a hollow circle around a point indicates that the point is not included in the graph.

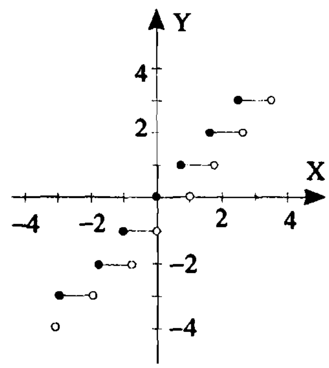


Fig. 4 : Graph of (x) = [x].

Example 3 Consider the curve $y = x^3$. Now (x, y) lies on the curve
$\Leftrightarrow y = x^3 \Leftrightarrow -y = (-x)^3 \Leftrightarrow (-x, -y)$ is on the curve. This means that the curve is symmetric about the origin. Thus, it is sufficient is draw the graph above the x-axis and join to it the portion obtained by rotating is through 180°.

Above the x-axis, y is positive. Hence $x = \sqrt[3]{y}$ must be positive. Thus, there is no portion of the graph in the second quadrant. The curve meets the axes of coordinates only at the origin and the tangent there, is the x-axis.

$\frac{dy}{dx} = 3x^2$ which is always non-negative. This means that as x increases, so does y. Thus the graph keeps on rising.

$\frac{dy}{dx} = 0$ at $(0, 0)$ and $\frac{d^2y}{dx^2} = 6x$ is 0 at $(0, 0)$.

$$\frac{d^2y}{dx^2} = 6x \begin{cases} > 0 \text{ for } x > 0 \\ < 0 \text{ for } x < 0 \end{cases}$$

This implies that there are no extreme points, and that (0, 0) is a point of inflection. The graph has no asymptotes parallel to the axes. Further $\lim\limits_{x \to \infty} \frac{y}{x} = \lim\limits_{x \to \infty} x^2$ and obviously, this does not exist. This means that the curve does not have any oblique asymptotes. You can also verify that it has no singular points. The graph is shown in Fig. 5.

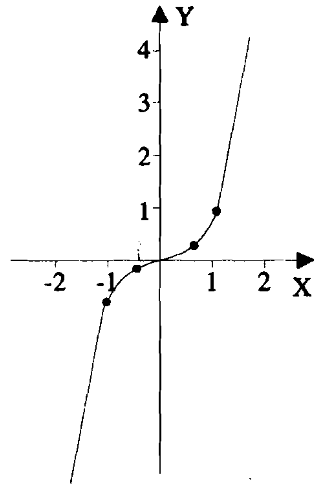


Fig. 5: Graph of $y = x^3$

Example 4 Consider $y = \frac{1}{x^2}$. The y-coordinates of any point on the curve cannot be negative. So the curve must be above the x-axis. The curve is also symmetric about the y-axis. Hence we shall draw the graph of the right of the y-axis first.

The curve does not intersect the axes of coordinates at all.

$\frac{dy}{dx} = -\frac{2}{x^3}$ and $\frac{d^2y}{dx^2} = \frac{6}{x^4}$. Since $\frac{dy}{dx} < 0$ for all $x > 0$, the function is non-increasing in $]0, \infty[$, that is, the graph keeps on falling as x increases. Further, since $\frac{dy}{dx}$ is non-zero for all x, there are no extreme points.

Similarly, since $\frac{d^2y}{dx^2}$ is non-zero, there are no points of inflection. Writing the equation of the curve as $x^2y = 1$, we see that both the axes are asymptotes of the curve.

There are no singular points. Therefore, the curve does not fold upon itself. The curve is shown in Fig. 6.

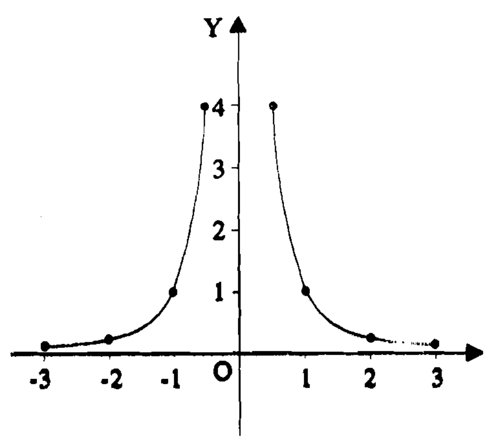


Fig. 6 : Graph of $y = 1/x^2$

Example 5 Let us try to trace the curve given by the equation $xy = 1$.

Here we can see that either x and y both will be positive or both will be negative. This means that the curve lies in the first and the third quadrants.

Further, it is symmetric about the origin and hence, it is sufficient to trace it in the first quadrant and rotate this through 180° to get the portion of the curve in the third quadrant.

(1, 1) is a point on the curve and $x = 1/y$ means that as x increases in the first quadrant, y decreases.

Now the distance of any points (x, y) on the curve from the x-axis $= |y| = y = \frac{1}{x} \to 0$ as $x \to \infty$. This means that the x-axis is an asymptote. Arguing on the same lines we see that the y-axis is also an asymptote.

$\frac{dy}{dx} = \frac{-1}{x^2} \neq 0$ for any x. That is, there are no extrema.

At the point (1, 1) we have, $\frac{dy}{dx} = -1$, which implies that the tangent at (1, 1) makes an angle of 135° with the x-axis. Considering all these points we can trace the curve in the first quadrant (see Fig. 7 (a)). Fig. 7 (b) gives the complete curve.

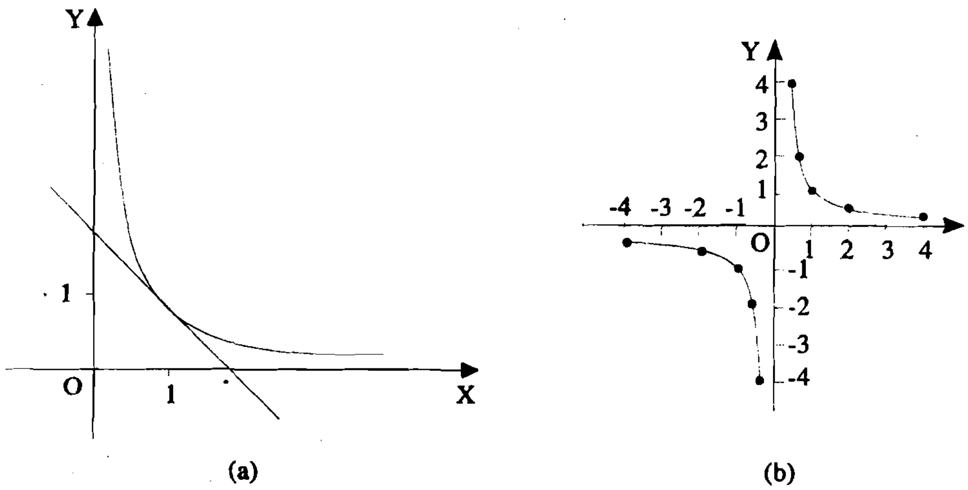


Fig. 7 (a) Graph of xy = 1 in the first Quadrant (b) complete graph.

The curve traced in Example 5 is a **hyperbola**. If we cut a double cone by a plane as in Fig. 8(a), we get a hyperbola. It is a section of a cone. For this reason, it is also called a **conic section**. Figs. 8(b), (c) (d) and (e) show some other conic sections. You are already familiar with the circle in Fig. 8(d) and the pair of intersecting lines in Fig. 8 (e). The curve in Fig. 8(b) is called a **parabola** and that in Fig. 8(c) is called an **ellipse**.

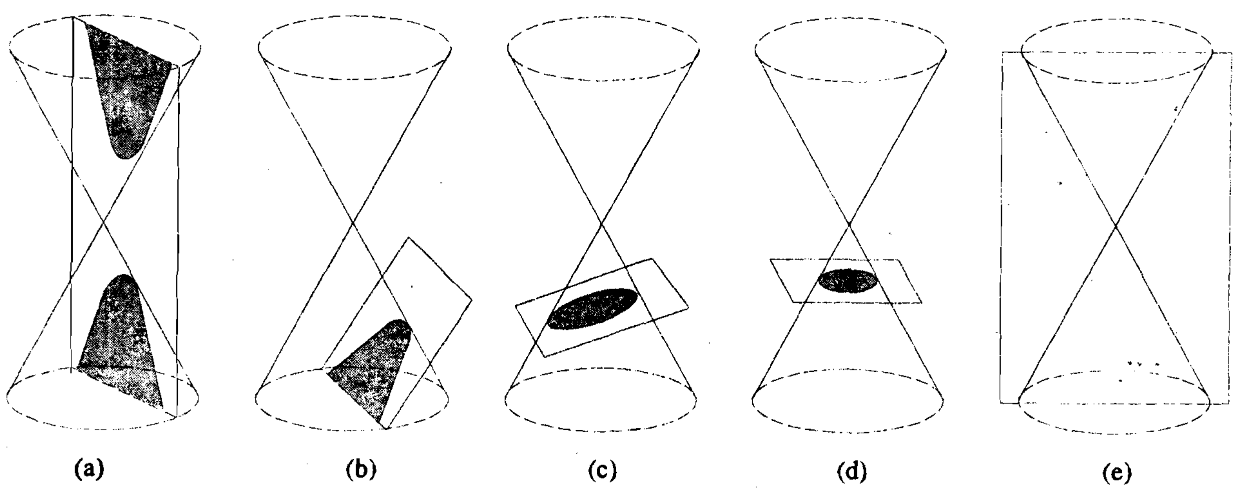


Fig. 8

The earliest mention of these curves is found in the works of a Greek mathematician Menaechmas (fourth century B.C.). Later Apollonius (third century B.C.) studied them extensively and gave them their current names.

In the seventeenth century Rene' Descartes discovered that the conic sections can be characterised as curves which are governed by a second degree equation in two variables. Blaise Pascal (1623-1662) presented them as projections of a circle. (Why don't you try this? Throw the light of a torch on a wall at different angles and watch the different conic sections on the wall). Galileo (1564-1642) showed that the path of a projectile thrown obliquely

(Fig. 9) is a parabola. Paraboloid curves are also used in arches and suspension bridges (Fig. 10). Paraboloids surfaces are used in telescopes, search lights, solar heaters and radar receivers.

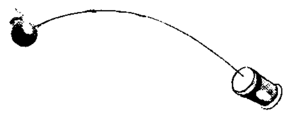


Fig. 9

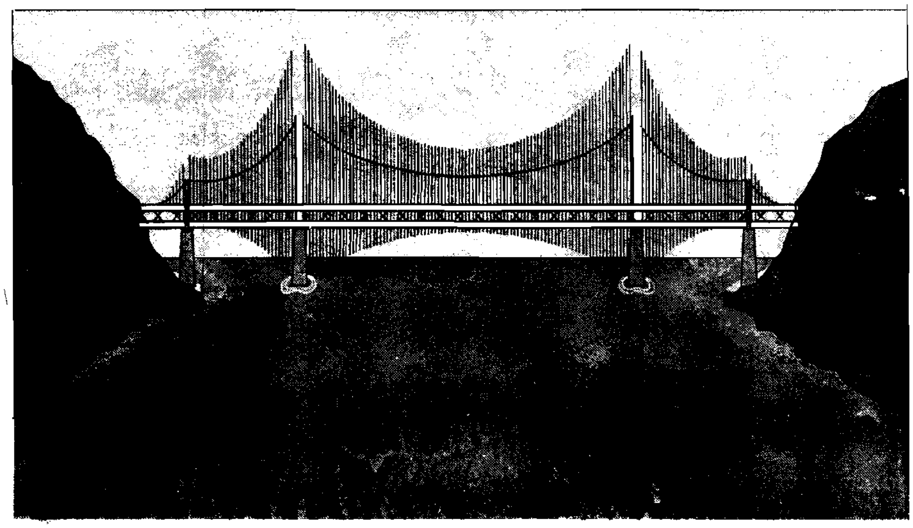


Fig. 10

In the seventeenth century Johannes Kepler discovered that planets move in elliptical orbits around the sun. Halley's comet is also known to move along a very elongated ellipse.

A comet or meteorite coming into the solar system from a great distance moves in a hyperbolic path. Hyperbolas are also used in sound ranging and navigation systems.

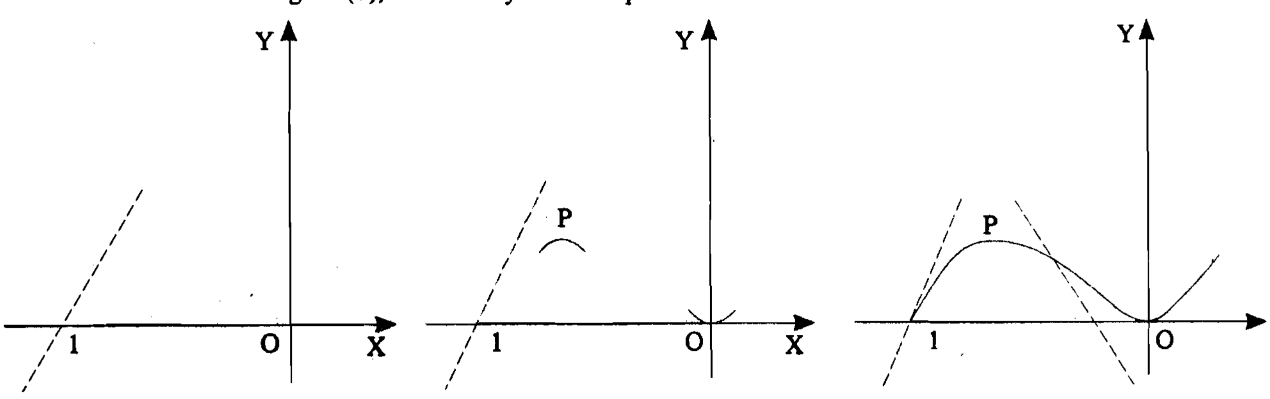
Let's look at the next example now.

Example 6 Consider the curve $y = x^3 + x^2$.

There is no symmetry and the curve meets the axes at (0, 0) and (–1, 0).

$\frac{dy}{dx} = 3x^2 + 2x$. The x-axis is the tangent at the origin as $\frac{dy}{dx} = 0$, at $x = 0$. Since $\frac{dy}{dx} = 1$ when $x = -1$, tangent at (–1, 0) makes an angle of 45° with the x-axis (Fig. 11(a)).

Further $\frac{d^2y}{dx^2} = 6x + 2$. This means (0, 0) is a minimum point as $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at $x = 0$. The point (–2/3, 4/27) is a maximum point as $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at $x = -2/3$. Thus in Fig. 11 (b), O is a valley and P is a peak.



(a) (b) Fig. 11 (c)

$\frac{d^2y}{dx^2} = 0$ at $x = -\frac{1}{3}$ and changes sign from negative to positive as x passes through $-1/3$.

Hence $(-1/3, 2/27)$ is a point of inflection.

$\frac{dy}{dx} = x(3x+2)$. Hence $\frac{dy}{dx} > 0$ when $x < \frac{-2}{3}$ or $x > 0$.

If $-\frac{2}{3} < x < 0$, then $\frac{dy}{dx} < 0$. Thus the graph rises in $]-\infty, -2/3[$ and $]0, \infty[$, but falls in $]-2/3, 0[$.

As x tends to infinity, so does y. As $x \to -\infty$, so does y. There are no asymptotes.

Hence the graph is as shown in Fig. 11(c).

So far, all our curves were graphs of functions. We shall now trace some curves which are not the graphs of functions, but have more than one branch.

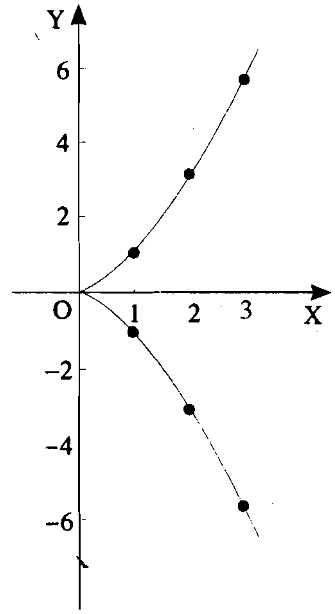


Fig. 12 : Semi cubical parabola, $y^2 = x^3$

Example 7 To trace the semi cubical parabola $y^2 = x^3$, we note that x^3 is always non-negative for points on the curve. This means x is always non-negative and no portion of the curve lies on the left on the y-axis.

There is symmetry about the x-axis (even powers of y).

The curve meets the axes only at the origin.

The tangents at the origin are given by $y^2 = 0$ so that the origin is a cusp. (see Sec. 4 in Unit 8).

In the first quadrant y increases with x, and $y \to \infty$ as $x \to \infty$.

There are no asymptotes, extreme points and points of inflection.

Taking reflection in the x-axis we get the complete graph as shown in Fig. 12.

Example 8 Suppose we want to trace the curve.

$y^2 = (x-2)(x-3)(x-4)$.

If $x < 2$, we get a negative value for y^2 which is impossible. So, no portion of the curve lies to the left of the line $x = 2$. For the same reason, no portion of the curve lies between the lines $x = 3$ and $x = 4$.

Since y occurs with even powers alone, the curve is symmetrical about the x-axis. We may thus trace it for points above the x-axis and then get a reflection in the x-axis to complete the graph.

The curve meets the axes in points A(2, 0), B(3, 0) and C(4, 0). At each of these points, the curve has a vertical tangent (see Sec. 2 of Unit 8). Combining these facts, the shape of the curve near A, B, C must be as shown in Fig. 13 (a).

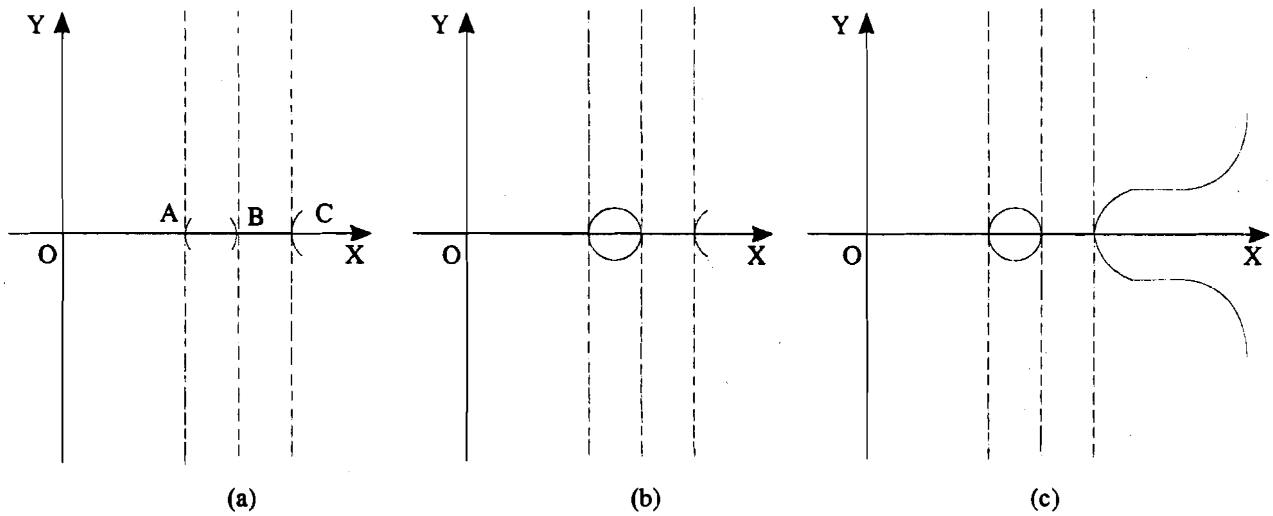


Fig. 13

Let us take $y > 0$ (i.e., consider points of the curve above the x-axis). Then

$$\frac{dy}{dx} = \frac{3x^2 - 18x + 26}{2\sqrt{(x-2)(x-3)(x-4)}}.$$ This is zero at $x = 3 \pm 1/\sqrt{3}$. If $\alpha = 3 + 1/\sqrt{3}$

and $\beta = 3 - 1/\sqrt{3}$ then α lies between 3 and 4, and can therefore be ignored. Also, $3x^2 - 18x + 26 = 3(x-\beta)(x-\alpha)$ and $2 < \beta < 3 < \alpha$. For $x \in]2, 3[$, $x - \alpha$ remains negative. Hence for $2 < x < \beta$, $\frac{dy}{dx} < 0$ since $(x - \alpha)$ and $(x - \beta)$ are both negative.

Similarly, for $\beta < x < 3$, $\frac{dy}{dx} < 0$. Hence the graph rise in $]2, \beta[$ and falls in $]\beta, 3[$. Thus the shape of the curve is oval above the x-axis, and by symmetry about the x-axis, we can complete the graph between $x = 2$ and $x = 3$ as in Fig. 13(b).

Now let us consider the portion of the graph to the right of $x = 4$. Shifting the origin to (4, 0), the equation of the curve becomes
$y^2 = x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

As x increases, so does y. As $x \to \infty$, so does y (considering points above the x-axis). When x is very small, x^3 and $3x^2$ are negligible as compared to 2x, so that near the (new) origin, the curve is approximately of the shape of $y^2 = 2x$. For large values of x, $3x^2$ and 2x are negligible as compared to x^3, so that the curve shapes like $y^2 = x^3$ for large x. Thus, at some point the curve changes its convexity.

This conclusion could also be drawn by showing the existence of a point of inflection.

There are no asymptotes or multiple points.

Considering the reflection in the x-axis, we have the complete graph as shown in Fig. 13 (c).

Example 9 Let us trace the curve $(x^2 - 1)(y^2 - 4) = 4$.

There is symmetry about both axes. We can therefore sketch the graph in the first quadrant only and then takes its reflection in the y-axis to get the graph above the x-axis. The reflection of this graph in the x-axis will give the complete graph.

Notice that the origin is a point on the graph and the tangents there, are given by $4x^2 + y^2 = 0$. These being imaginary, the origin is an isolated point on the graph. The curve does not meet the axes at any other points.

For $x > 0, y > 0$, the equation $(x^2 - 1)(y^2 - 4) = 4$ shows that x should be greater than 1 and y should be greater than 2.

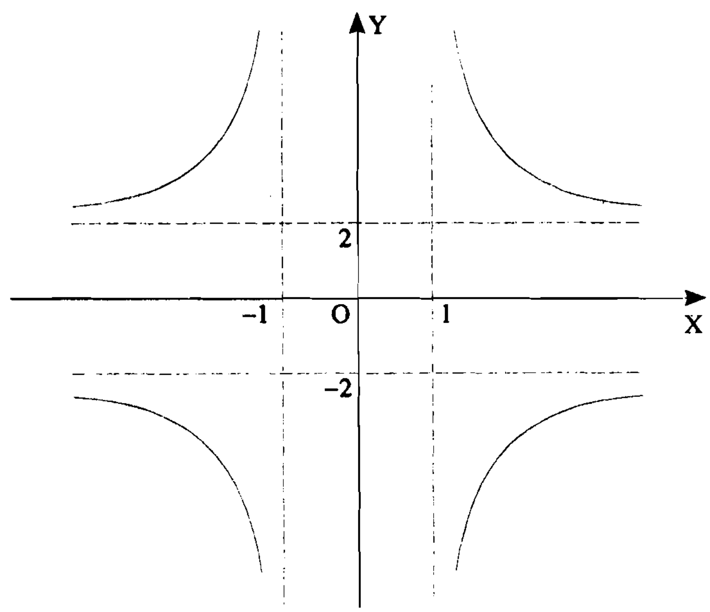


Fig. 14

Equating to zero the coefficients of the highest powers of x and y, we get $y = \pm 2$ and $x = \pm 1$ as asymptotes of the curve. Thus, the portion of the curve in the first quadrant approaches the lines $x = 1$ and $y = 2$ in the region far away from the origin.

In the first quadrant, as x increases, so does $x^2 - 1$, and since $x^2 - 1 = \frac{4}{(y^2 - 4)}$,

y decreases as x increases.

There are no extreme points, singular points or points of inflection.

As $x \to \infty$, $y \to 2$ and as $y \to \infty$, $x \to 1$. Hence the graph is as shown in Fig. 14.

Example 10 To trace the curve $y^2 = (x - 1)(x - 2)^2$ we note that there is symmetry about the x-axis.

No portion of the curve lies to the left of $x = 1$.

Points of intersection with the axes are A(1, 0) and B(2, 0) and the tangent at (1, 0) is vertical. Shifting the origin to B(2, 0), the curve transforms into $y^2 = x^2(x + 1)$. The tangents at the new origin B, are given by $y^2 = x^2$. This means that B is a node, and the tangents at B are equally inclined to the axes. Let us try to build up the graph above the x-axis between $x = 1$ and $x = 2$. Differentiating the equation of the curve with respect to x, we get.

$$2yy' = (x-2)^2 + 2(x-1)(x-2),$$
$$= (x-2)(3x-4).$$

$$\text{or } y' = \frac{(x-2)(3x-4)}{2y}$$

when $1 < x < 2$, $(x - 2) < 0$. If y is positive, then $y' > 0$ provided $3x - 4 < 0$. Thus $y' > 0$ when $x \in \,]1, 4/3[$ and $y' < 0$ which $x \in \,]4/3, 2[$. The tangent is parallel to the x-axis when $3x - 4 = 0$, that is, when $x = 4/3$ (see Fig. 15(a)). Hence, for $1 < x < 2$, the curve shapes as in Fig. 15(b).

Now for $x > 2$, As $x \to \infty$, $y \to \infty$ in the first quadrant. Note that when B(2, 0) is taken as the origin, the equation of the curve reduces to
$y^2 = x^2(x + 1) = x^3 + x^2$

This shows that when $x > 0$ and $y > 0$, the curve lies above the line $y = x$ (on which $y^2 = x^2$). Hence the final sketch (Fig. 15 (c)) shows the complete graph.

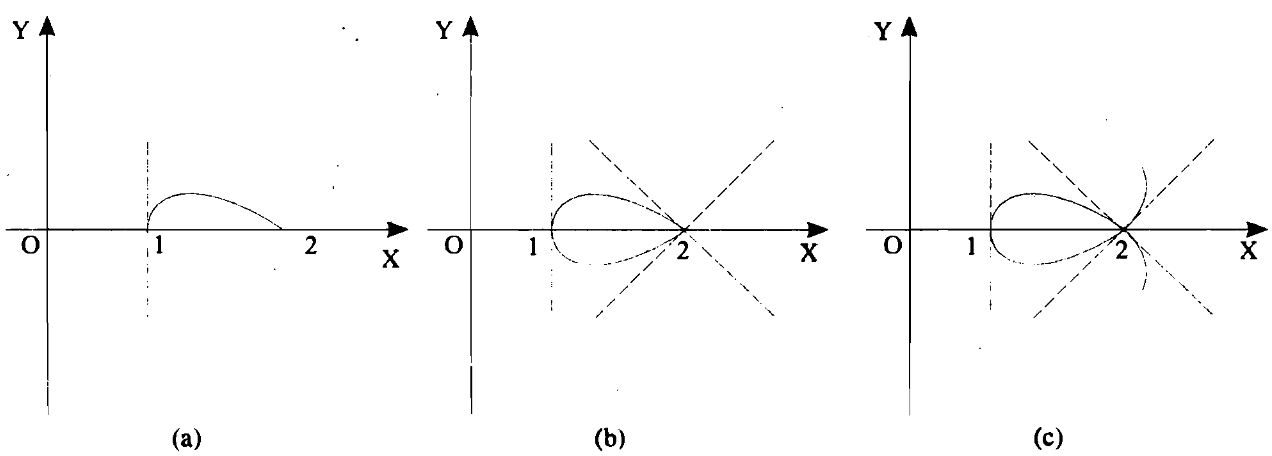


Fig. 15

If you have gone through Examples 1 — 10 carefully, you should be able to do the following exercise.

E 1) Trace the curves given by

a) $y = x^2$ b) $y^2 = (x-2)^3$

c) $y(1+x^2) = x$ d) $y^2 = x^2(1-x^2)$

(Graph paper is provided at the end of this unit.)

4.4 TRACING A CURVE : PARAMETRIC EQUATION

Sometimes a functional relationship may be defined with the help of a parameter. In such cases we are given a pair of equations which relate x and y with the parameter. You have already come across such parametric equations in Unit 4. Now we shall see how to trace a curve whose equation is in the parametric form.

We shall illustrate the process through an example.

Example 11 Let us trace the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ as t varies from $-\pi$ to π.

$\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a \sin t$, so that

$\frac{dy}{dx} = \tan(t/2)$. Since $\frac{dx}{dt} > 0$ for all $t \in]-\pi, \pi[$, x increases with t from $-a\pi$ (at $t = -\pi$) to 0 (at $t = 0$) to $a\pi$ (at $t = \pi$).

Also $\frac{dy}{dx}$ is negative when $t \in]-\pi, 0[$ and positive when $t \in]0, \pi[$. Hence y decreases from 2a to 0 in $[-\pi, 0]$ and increases from 0 to 2a in $[0, \pi]$. Let us tabulate this data.

$t \in [-\pi, 0]$	$t \in [0, \pi]$
i) x increases from $-a$ to 0	i) x increases from 0 to a
ii) y decreases from 2a to 0	ii) y increases from 0 to 2a
iii) Hence the curve falls	iii) Hence the curve rises

Also, at the terminal points $-\pi$, 0 and π of the intervals $[-\pi, 0]$ and $[0, \pi]$, we have the following.

t	(x, y)	$\frac{dy}{dx}$	$\frac{dx}{dy}$	Tangent
$-\pi$	$(-a\pi, 2a)$	not defined	0	vertical
0	(0, 0)	0	not defined	horizontal
π	$(a\pi, 2a)$	not defined	0	vertical

On the basis of the data tabulated above, the graph is drawn in Fig. 16.

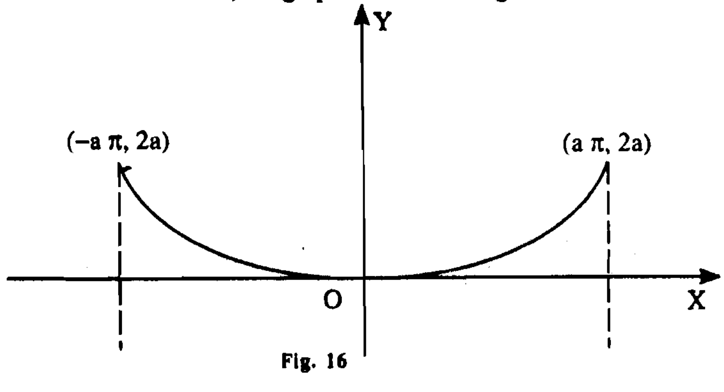


Fig. 16

Remark 1 If t is increased by 2π, x is increased by $2\pi a$ and y does not change. Thus the complete graph can be obtained in intervals....... $[-5\pi, -3\pi]$, $[-3\pi, -\pi]$, $[\pi, 3\pi]$, $[3\pi, 5\pi]$ ..... by mere translation through a proper distance.

The cycloid is known as the Helen of geometry because it was the cause of many disputes among mathematicians. It has many interesting properties. We shall describe just one of them here. Consider this question : What shape should be given to a trough connecting two points A and B, so that a ball rolls from A to B in the shortest possible time?

Now, we know that the shorter distance between A and B would be along the line AB (Fig. 17). But since we are interested in the shortest time rather than distance, we must also consider the fact that the ball will roll quicker, if the trough is steeper at A. The Swiss mathematicians Jakob and Johann Bernoulli proved by exact calculations that the trough should be made in the form of an arc of a cycloid. Because of this, a cycloid is also called the curve of the quickest descent.

The cycloid is used in clocks and in teeth for gear wheels. It can be obtained as the locus of a fixed point on a circle rolls along a straight line.

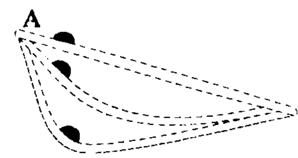


Fig. 17

See if you can do this exercise now.

E2) Trace the following curves on the graph paper given at the end of this unit.

a) $x = a(t + \sin t), y = a(1 + \cos t), -\pi \le t \le \pi$.

b) $x = a \sin 2t\,(1 + \cos 2t), y = a \cos 2t\,(1 - \cos 2t), 0 \le t \le \pi$.

c) $x = at^2, , y = 2\,at, 0 \le t \le 1$.

4.5 TRACING A CURVE : POLAR EQUATION

In this section we shall consider the problem of tracing those curves, whose equations are given in the polar form. The following considerations can be useful in this connection.

Symmetry : If the equation remains unchanged when θ is replaced by $-\theta$, then the curve is symmetric with respect to the initial line.

If the equation does not change when r is replaced by $-r$, then the curve is symmetric about the pole (or the origin).

Finally, if the equation does not change when θ is replaced by $\pi - \theta$, then the curve is symmetric with respect to the line $\theta = \pi/2$.

Extent : (i) Find the limits within which r must lie for the permissible values of θ. If $r < a$ $(r > a)$ for some $a > 0$, then the curve lies entirely within (outside) the circle $r = a$.

(ii) If r^2 is negative for some values of θ, then the curve has no portion in the corresponding region.

Angle between the line joining a point of the curve to the origin and the tangent : At suitable points, this angle can be determined easily. It helps in knowing the shape of the curve at these points. Recall that angle ϕ is given by the relation $\tan \phi = r \dfrac{d\theta}{dr}$.

We shall illustrate the procedure through some examples. Study them carefully, so that you can trace some curves on your own later.

Example 12 Suppose we want to trace the cardioid $r = a(1 + \cos \theta)$. We can make the following observations.

Since $\cos \theta = \cos(-\theta)$, the curve is symmetric with respect to the initial line.

Since $-1 \le \cos \theta \le 1$, the curve lies inside the circle $r = 2a$.

$\dfrac{dr}{d\theta} = -a \sin \theta$. Hence $\dfrac{dr}{d\theta} < 0$ when $0 < \theta < \pi$. Thus r decreases as θ increases in the interval

]0, π[. Similarly, r increases with θ in]π/2, π[. Some corresponding values of r and θ are tabulated below.

θ	0	π/2	π
r	2a	a	0

$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot(\theta/2) = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$. This shows that the angle between the line joining a pont (r, θ) on the curve to the origin and the tangent is 0 or π/2 according to θ = π or 0. Hence the line joining a point on the curve to the origin is orthogonal to the tangent when θ = 0 and coincides with it θ = π.

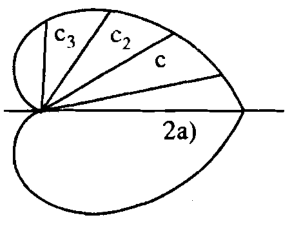


Fig. 18

Combining the above facts, we can easily draw the graph above the initial line. By reflecting this portion in the initial line we can completely draw the curve as shown in Fig. 18. Notice the decreasing radii 2a, c_1, c_2, c_3 etc.

This curve is called a cardioid since it resembles a heart.

Example 13 Let us trace the equiangular spiral $r = ae^{\theta\cot\alpha}$. We proceed as follows.

When θ = 0, r = a.

$\frac{dr}{d\theta} = r\cot\alpha$, which is positive, assuming cot α > 0. Hence as θ increases so does r.

$r\frac{d\theta}{dr} = \tan\alpha$. Thus, at every point, the angle between the line joining a point on the curve to the origin and the tangent is the same, namely α. Hence the name.

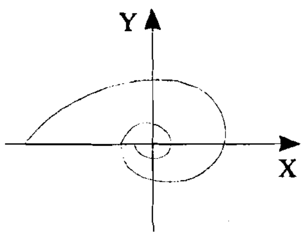


Fig. 19

Combining these facts, we get the shape of the curve as shown in Fig. 19.

The equiangular (or logarithmic) spiral $r = ae^{\theta\cot\alpha}$ is also known as the curve of pursuit. Suppose four dogs start from the four corners of a square, each pursues the dog in front with the same uniform velocity (always following the dog in front in a straight line), then each will describe an equiangular spiral. Several shells and fossils have forms which are quite close to equiangular spirals (Fig. 20). Seeds in the sunflower or blades of pine cones are also arranged in this form.

This spiral was first studied by Descartes in 1638. John Bernoulli rectified this curve and was so fascinated by it that he willed that an equiangular spiral be carved on his tomb with the words 'Though changed, I rise unchanged' inscribed below it.

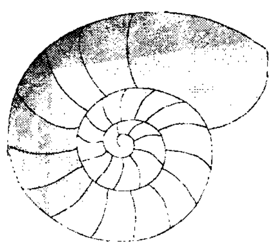


Fig. 20

The spiral r = aθ is known as the Archimedean spiral. Its study was, however, initiated by Conan. Archimedes used this spiral to square the circle, that is, to find a square of area equal to that of a given circle. This spiral is widely used as a cam to produce uniform linear motion. It is also used as casings of centrifugal pumps to allow air which increases uniformly in volume with each degree of rotation of the fan blades to be conducted to the outlet without creating back-pressure.

The spiral rθ = a, due to Varignon, is known as the reciprocal or hyperbolic (recall that xy = a is a hyperbola) spiral. It is the path of a particle under a central force which varies as the cube of the distance.

Now let's consider one last example.

Example 14 To trace the curve r = a sin 3θ, a > 0, we note that there is symmetry about the line θ = π/2, since the equation is unchanged if θ is replaced by π – θ.

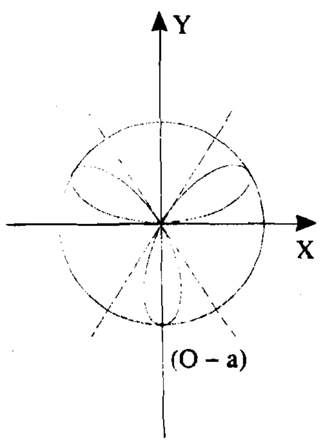


Fig. 21

The curve lies inside the circle r = a, because sin 3θ ≤ 1. The origin lies on the curve and this is the only point where the initial line meets the curve.

r = 0 ⇒ θ = nπ/3, where n is any integer. Hence the origin is a multiple point, the lines θ = 0, π/3, 2π/3, π, 4π/3, 5π/3, 2π etc. being tangents at the pole.

$\frac{dr}{d\theta} = 3\cos 3\theta$. Hence r increases in the intervals]0, π/6, [ ] π/2, 5π/6[, and]7π/6, 3π/2[, and decreases in the intervals]π/6, π/2[,]5π/6, 7π/6[and]3π/2, 5π/3[. Notice that r is negative when θ ∈]π/3, 2π/3[or θ ∈]π, 4π/3[or θ ∈]5π/3, 2π[.

Hence the curve consists of three loops as shown in Fig. 21. The function is periodic and the curve traces itself as θ increases from 2π on.

Now try to trace a few curves on your own.

E3) Trace the following curves on the graph paper provided.

a) $r = a(1 - \cos\theta), a > 0.$ b) $r = 2 + 4\cos\theta.$

c) $r = a\cos 3\theta, a > 0.$ d) $r = a\sin 2\theta, a > 0$

(Graph paper is provided at the end of this unit.)

4.6 SUMMARY

In this unit we have covered the following points.

1) Tracing a curve $y = f(x)$ or $f(x, y) = 0$ means plotting the points which satisfy this relation.

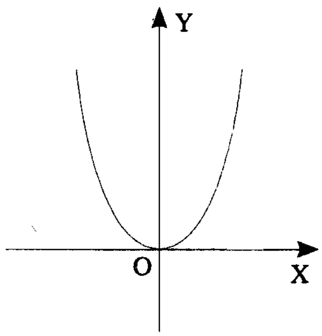
2) Criteria for symmetry and monotonicity, equations of tangents, asymptotes and points of inflection are used in curve tracing.

3) Curve tracing is illustrated by some examples when the equation of the curve is given in
   a) Cartesian form
   b) Parametric form
   c) Polar form

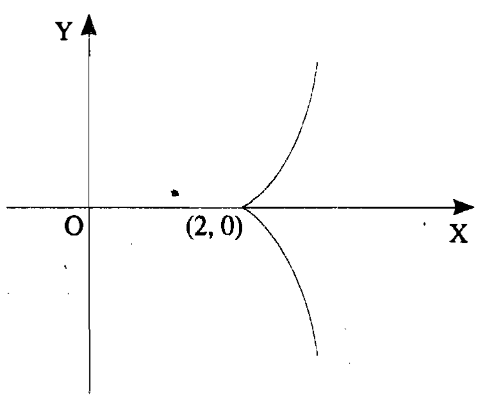
4.7 SOLUTIONS AND ANSWERS

Dotted lines represent tangents or asymptotes throughout.

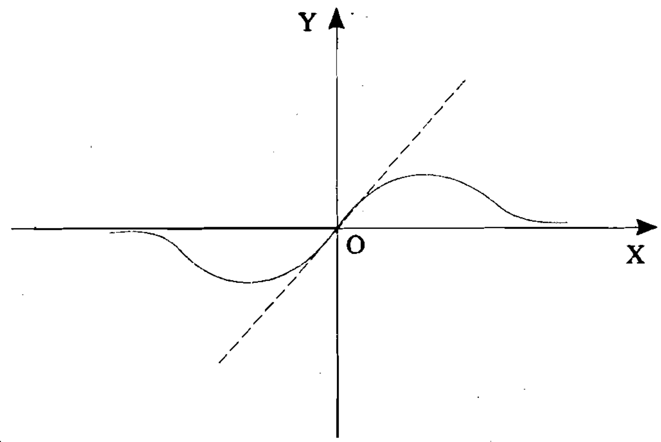
E1) a)



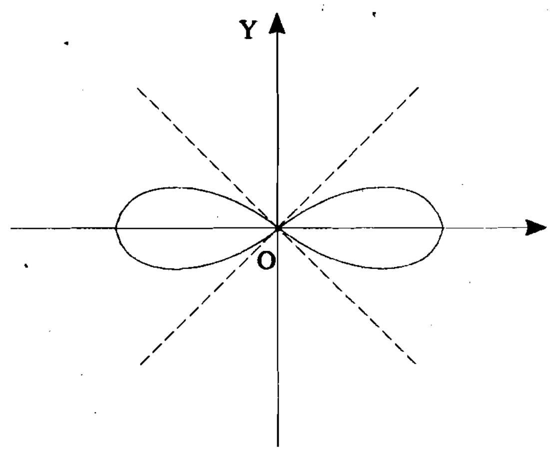
b) Shifting the origin to (2, 0) we get $y^2 = x^3$ which you know how to draw.



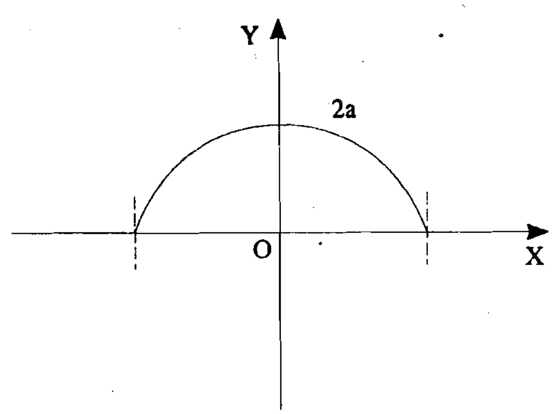
c) $y = x$ is the tangent at the origin. Origin is a point of inflection. x-axis is an asymptote. Either x, y are both positive or both negative. Function rises in $]-1, 1[$ and falls elsewhere. Graph is shown alongside.



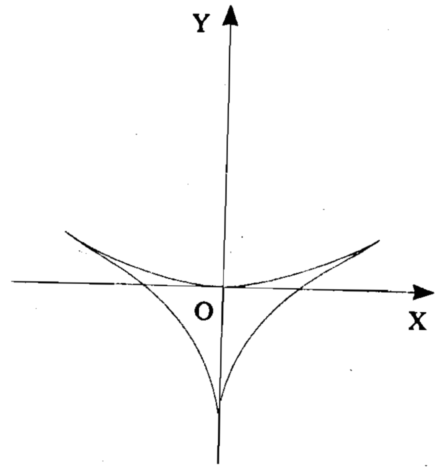
d) $\frac{y^2}{x^2} = 1 - x^2$ shows that the entire curve lies within the lines $x = \pm 1$. Tangents at the origin are $y = \pm x$. Tangents at $x = \pm 1$ are vertical. Maxima at $(\pm 1/\sqrt{2}, 1/4)$, symmetry about both axes.



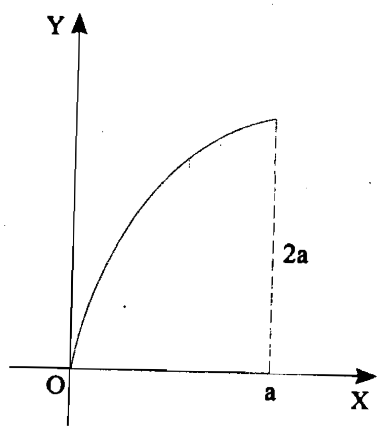
E2) a)



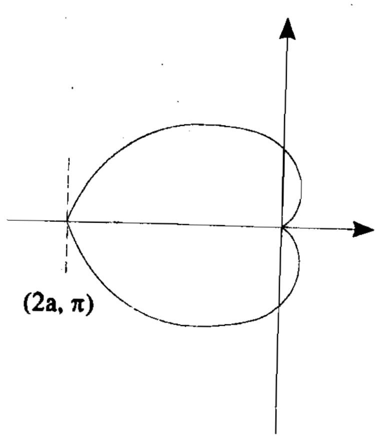
b)



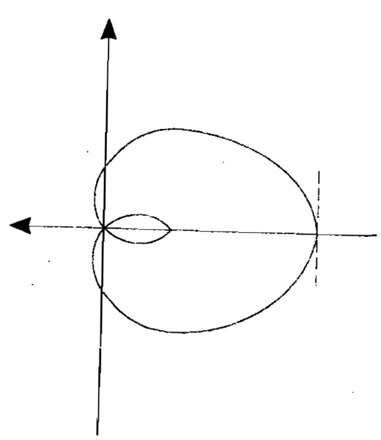
c)



E3) a)



b)



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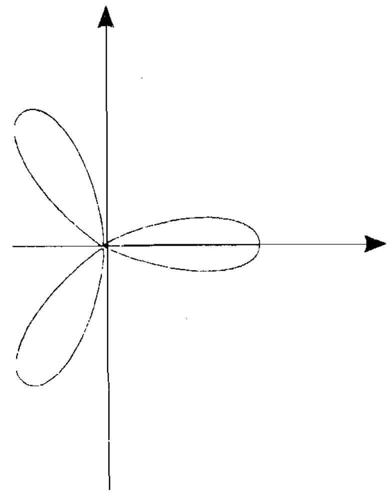

d)