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Covered Call on an Index
- A Comparative Study of Two Strategies

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In dedication
to the pursuit of knowledge

Abstract

This thesis undertakes a comparative analysis of two ways of performing a covered call strategy on a dual asset index. The distinguishing factor between the two approaches pertains to the writing of the call options, where one approach involves writing the call option on the entire index, while the other involves writing options on each asset within the index separately. This study is done by first, initiating an appropriate pricing method for index options using Monte Carlo simulations. Then, the two option strategies are analysed from a utility perspective by a figuration of investors with varying degrees of risk aversions. The results indicate that an investor's risk attitude has no significant relevance in their choice of strategy, but rather that the characteristics of the underlying assets within the index are important in determining the preferred approach.

Keywords: Index Options, Covered Calls, Monte Carlo, Utility

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Chapter 1

Introduction

The task of managing a portfolio of assets is to balance a risk-return profile, which is often optimised by including a risk-free asset and/or options. The field of financial engineering is constantly evolving, leading to the development of new strategies for achieving an optimal balance of risk and return. One such strategy is the use of covered calls, which has been shown by both practitioners and academics to have beneficial performance characteristics (McIntyre & Jackson 2007; Diaz & Kwon, 2019; Hill et al. 2006; Asset Consulting Group, 2012).

A covered call strategy is about simultaneously buying an asset and writing a European call option on that same underlying asset. The asset can be delivered to the counterpart if the option is exercised and the position is therefore said to be "covered". The writer of the option receives a premium, which partly offsets the downside in the case of a decrease in the underlying asset, and hence it contributes to the strategy having an attractive risk-adjusted return compared to solely being long the underlying asset. The higher the expected volatility the higher the premium received. Moreover, in the case of a price increase, the upside is capped due to the liability of having to deliver the underlying asset if the option is exercised, which at the same time means that the exercise price is lower than the price of the underlying asset at maturity. The Covered call strategy is also known as a Buy-Write strategy and the two denominations will be used interchangeably throughout this thesis.

The strategy is commonly implemented using a rolling 30-day at-the-money scheme, where options are written every 30 days with a strike price equal to the current share price and an expiration date 30 days in the future. By following this overwriting rule, the premiums are collected regularly and generate a cash flow that contributes to the overall return of the strategy. This scheme is prevalently employed on an index, where one holds all the shares according to a specific index and on a rolling basis writes at-the-money call options on the corresponding index. Many ETFs follow this rule-based strategy on different indices and some of them have faced significant growth in assets under management (AUM) in recent years.

Global X NASDAQ 100 Covered Call ETF (QYLD) performs the strategy on the Nasdaq 100 index and is the largest exchange-traded fund (ETF) of this type with 6.5 billion USD in AUM, which has quadrupled in two years (Ycharts, 2023). Another example is the Invesco S&P 500 BuyWrite ETF (PBP), which has on the contrary had a stable AUM for the last couple of years and offers exposure to a covered call strategy with the S&P 500 as the underlying index.

Cboe S&P 500 BuyWrite Index (BXM) follows the performance of a hypothetical Covered Call strategy on the S&P 500 Index and prevalently serves as a benchmark for this rule-based type of strategy. It has therefore been subject to numerous investigations of how it performs and if it can be refined to attain even better performance, and in some cases, it has shown to be so (Hill et al.,2006; Whaley, 2002; Feldman & Roy, 2005). Given that various, complex rule-based strategies have the potential to improve the performance of the BXM, this thesis replicates essentially the same setup as the BXM and compares two ways of setting up a 30-day at-the-money covered call strategy.

The objective of this thesis is to compare two ways of setting up a covered call strategy on an index. One way is the one described above and most commonly used, which is owning each share in the index while writing an index call option, meaning that one option is written with the index as the underlying asset. This approach will throughout be called the “index” approach. It will be compared to a strategy where one owns all the shares in the index but writes a call option for each share for the period which will be called the “stock-by-stock” approach. Thus, the difference lies in the writing of the options. Investors with different risk aversions may prefer different strategies, and therefore, I will analyse if one approach is better than the other measured as utility for individuals given different risk profiles. For this paper, the analysis will be done on an index constructed of two assets.

There are several reasons why the payoffs of the two approaches are expected to be different. One is that some options will expire in the money, and some will expire out of the money when writing the options individually in the stock-by-stock approach. In the index approach, the option will either be out of the money or in the money and thus have a different payoff profile. The premiums collected will also differ due to the number of options that are written where the stock-by-stock approach could potentially offer higher returns, as it grants the investor to receive premiums on each individual written option. As in the index approach, only a single option will be sold, and the investor will collect a lower premium income. Volatility is an essential factor when pricing options and the volatility of an index is lower than that of individual stocks due to diversification and will arguably affect the differences in option prices.

Formally, I aim to answer the research questions:

I Given an index consisting of two correlated stocks, each following a Geometric Brownian Motion, is there a difference in the performance of an index buy-wright strategy that employs an option written on the index, and a Stock-by-Stock buy-write strategy that utilizes options written on every stock in the index?

II Do groups of investors with diverse risk attitudes exhibit a preference for one of the two strategies over the other?

The findings are three-folded. Firstly, the expected return of the two strategies is the same, but the distribution of returns is different. Secondly, neither of the strategies is consistently favoured over the other and thus investor's attitude towards risk does not seem to matter regarding the choice of strategy. The last finding appeared without it being the objective of this study, and that is that the characteristics of the assets that is underlying the index seem to play a decisive role in the choice of strategy for the investors.

The structure of the remainder of this paper is as follows: In chapter 2, a review of previous research related to the method and objective is presented together with the theory upon which the paper is built. Chapter 3 describes the method and the technical measurements of each approach which is followed by chapter 4 which presents the result. At last, chapter 5 is an analysis chapter that is followed by the concluding chapter, 6.

Chapter 2

Literature Review & Theory

This chapter has been structured into two distinct sections in order to fulfil two primary objectives. It aims to review and synthesize existing empirical research that is closely related to the focus of this thesis, while it best requires to provide a comprehensive and coherent exposition of the relevant theory that forms the foundation of this study.

2.1 Literature Review

This section begins with a discussion of existing empirical evidence regarding whether a covered call strategy is beneficial in comparison to a simple buy-and-hold approach of the underlying stock. The purpose of this introductory discussion is to underline the significance of covered calls as a central point of research. Following this, the literature review proceeds to examine the performance of covered call strategies on indices in diverse settings versus the performance of the underlying index alone. It will be presented along with research on suggested modifications of the strategy behind the BXM and investigations on the profitability of these modified strategies, which fill similar purposes as this thesis. At last, relevant literature pertaining to the methodology of this study is presented to motivate the methods used for reaching the objective.

2.1.1 Covered Call on Individual Stocks

McIntyre and Jackson (2007) evaluate an investment strategy involving at-the-money covered calls and compare its performance to a buy-and-hold strategy consisting of 27 shares. They perform the examination on synthesised economies, using Monte Carlo simulation to mirror a real economy under different theoretical assumptions, and on historical transaction data from 1994 to 1999. The authors find that the covered call strategy outperforms the buy-and-hold strategy more frequently in the historical data, while in the simulated economies, the result indicates that the buy-and-hold strategy generates better returns on average. Nonetheless, the covered call strategy performs surprisingly well in many trials, relative to their expectations.

Diaz and Kwon (2019) construct a scheme that optimises first, the selection of the underlying shares, and then the weights of overwriting call options. They conclude that writing call options according to their framework reduces portfolio risk and that such optimization may also increase the expected return. The authors claim that their finding contrasts with the "conventional" understanding of covered calls when one asset is fully covered by one call option. This view is also shared by Rendleman (2001) who argues that a portfolio made up of a long position in a share and simultaneously writing a call option in the same share is likely to expect a lower return than owning the share alone due to leverage. His argument is based on the assumption of frequent rebalancing to maintain a 1:1 ratio, and when the stock's return exceeds the risk-free rate, and then writing call options has a negative impact on portfolio performance. While discussing myths regarding covered call strategies, Israelov and Nielsen (2014) assert that practising a covered call strategy is a way of selling volatility. The authors stress that it is a promising strategy when one thinks the implied volatility is higher than the expectations of volatility and thus the premium makes it profitable.

2.1.2 Covered Call on Index

Hill et al. (2006) analyse the risk and return characteristics for a rolling covered call strategy on the S&P 500 index, the same strategy as the BXM, along with an evaluation of other similar rolling strategies. Their analysis is built on historical data from the period 1990 to 2005 and the strategies conducted vary with respect to strike prices. The authors found that these strategies provide favourable characteristics of return at different levels of risk compared to the S&P 500 and suggest they were all promising strategies for one who owns the index and seek to decrease portfolio risk. More recently, similar results are presented in a study by Asset Consulting Group (2012) that compared the performance of the BXM, and some other option strategies, against the S&P 500 index using data from 1986 to 2012. The main finding was that the BXM and the other options strategies have lower risk-adjusted returns than the S&P 500 index. Correspondingly, Whaley (2002) also examines the BXM, studies its performance measures and compares it to the S&P 500 index. One of the findings of this paper was that over the period studied, the index obtained a monthly return almost equal to the S&P 500 index, but with approximately two-thirds of the risk. Other conventional risk-adjusted performance measures also indicate superior characteristics of the BXM over S&P 500 in this study. Similar results are found by Feldman and Roy (2005) who also examine the characteristics of a 30-day at-the-money rolling covered call strategy on the S&P 500 index. They extend the analysis of Whaley (2002) by using longer history of the index and by adding an analysis of an investable version of the index.

Furthermore, Schneeweis and Spurgin (2001) study passive and active index option-based strategies with data they collected between 1987 and 1999. Their findings suggest that for the period under study, many of these passive option-based investment strategies outperformed the underlying index.

2.1.3 Measure Preferences of Covered Calls Strategies

Existing literature has frequently utilised utility functions as a means of assessing preferences between strategies related to covered calls. Board et al. (2000) use utility functions and four other risk-return dominance criteria to evaluate the performance of covered call strategies on underlying equities to a portfolio of equities alone. These other measures of performance were used to supplement variance, which was deemed insufficient in the context of options. The authors conclude that the considered dominance criteria were not effective as performance metrics when choosing among the strategies, but that the utility functions provided an effective way of demonstrating the superiority of a covered call strategy over an equity basket.

Moreover, Rendleman (2001), use the concept of expected utility and makes a comparison between performing a covered call strategy to holding the underlying asset, a risk-free asset, or simply a long call option. The study concludes that a covered call strategy is less attractive for an investor with constant proportional risk aversion than many of the strategy advocates claim, according to the author.

Diaz and Kwon (2020) proceed with a different approach, constructing portfolios that contained positions in covered calls on equity. The authors composed positions with different expiry and strike prices on the options, and address the problem in a similar approach as this paper, with an expected utility objective, but where they used various utility functions. Their findings indicate that their progressive option strategy was more effective with a power utility function compared to when optimising the quadratic utility function.

2.1.4 Multi-Asset Option Pricing

This study is also closely related to other strands of the literature with reference to the method and theoretical aspects. One such area of research addresses the issue of index option pricing. The methodology for pricing index options is one of the cornerstones of this paper and is one major area that distinguishes it from the previous literature. Hull (2021), makes the assumption in one of the most conventional books about options, that the index follows a Geometric Brownian Motion (GBM) and thus uses the Black and Scholes formula to price the index option. Similarly, Broadie et al. (2009) move along with the same assumption that the index follows a GBM, although they state that the formula may be rejected on theoretical grounds, which they ignore for their purpose. On the other hand, Avellaneda et al.(2003) propose a mathematical formula for option pricing of an index with consideration taken to the processes of the underlying stocks and a corresponding correlation matrix.

Another strand of literature is regarding option pricing in a two-asset case. Stulz (1982) starts his analysis with the same set-up as in this paper as he derives a pricing formula for an option depending on two risky assets that follow a GBM. But unlike the objective of pricing an index option, the author derives a formula for pricing an option on the minimum or maximum of the two assets. However, the derivation in the initial part of the method relies on the same assumptions on distributions of the assets as this thesis, and that the two assets correlate. Clift and Forsyth (2008) also derive a pricing method for an option written on two assets based on correlated Brownian Motions but take jump activity into consideration when addressing the objective of developing a more complex pricing proposal than Black and Scholes that aims to mimic actual market conditions.

2.2 Theory

This section describes the concepts and assumptions underlying the method of the thesis to give the reader some technical preparation and theoretical ground. This will give some clarification to the literature review and serve as a foundation for the methodology which is outlined in the next chapter.

2.2.1 Geometric Brownian Motion

A Geometric Brownian Motion (GBM) is a common mathematical concept in economics and finance to model the behaviour of asset prices over time as it describes how the next period's price of the asset depends on the current price. It has the properties of a random walk with a drift term representing an expected trend, and a diffusion term reflecting random fluctuations. In other words, the model lets a variable follow an unpredictable path and thus returns a stochastic process where the historic particular path is irrelevant for future development. The GBM is therefore a useful model for the purpose of stock price modelling since it allows for these realistic properties and is the model the stock prices in this thesis are assumed to follow.

A Geometric Brownian Motion is defined by the equation:

$$dS(t) = S(t)\mu dt + S(t)\sigma dW(t) \tag{2.1}$$

where $dS(t)$ is the random change in the stock price during a small time period dt, with the price $S(t)$ at time t. The expected drift rate in S, is $S(t)\mu$, for each change in time, dt, plus a term for the movement of uncertainty expressed as standard deviation proportional to the stock price, $S(t)\sigma dW(t)$, where $W(t)$ is a Standard Brownian Motion i.e. an independent path with no drift and $\sim N(0, \sqrt{dt})$.

In this paper, I model the stock prices accordingly, but with an extension based on this basic formula. An important consequence of the model is that the stock has log-normal distribution property, and the predictions of the future can only be expressed in terms of expectations (Hull, 2022).

2.2.2 Black & Scholes

Estimating the value of a call option in the real world is highly subjective, however, in this thesis, some option pricing follows the framework of Black and Scholes (1973), whose formula for pricing options is widely used.

The Black and Scholes partial differential equation (PDE) is an equation that describes the evolution of the price of a European call option. It is a formula that takes relevant factors into account for pricing options and makes several important assumptions about the environment:

1. The short-term interest rate is known and constant all through the life of the option, and so is the volatility.

2. The stock price follows a Geometric Brownian Motion with continuous time.

3. There are frictionless markets and thus transaction costs when buying and writing the units are overseen.

4. The markets are efficient, meaning that all actors in the market have the same information, and there exist no arbitrage opportunities (Black & Scholes, 1973).

With these assumptions in consideration, the price of the option solely depends on the stock price, time and the parameters that are fixed. The reasoning underlying the formula is that the option price and the underlying stock are always in equilibrium, meaning that there exist no arbitrage opportunities.

The Black and Scholes formula, derived from the PDE is:

Black & Scholes formula:

$$\begin{aligned} C(S(t),t) &= SN(d_1) - Xe^{-r(T-t)}N(d_2), \\ d_1 &= \frac{ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T - t}{\sigma\sqrt{(T-t)}}, \\ d_2 &= d_1 - \sigma\sqrt{(T-t)} \end{aligned} \tag{2.2}$$

where $C(S(t),t)$ is the value of a European Call, $S(t)$ is the stock price at time t, X is the strike price, r is the short-term interest rate, N(d) is the cumulative normal density function, and t is the number of years until maturity of the option. The payoff of a European Call option is:

$$\text{Payoff European Call Option:} \qquad V(T,S(T)) = max(S(T)-K,0) \tag{2.3}$$

This tells us that the payoff is the highest value of the stock price at maturity less the strike price, or zero.

The price of the option, C, is what a buyer will pay for the option and is therefore also referred to as the premium that is paid and thus collected by the writer.

2.2.3 Ito's Lemma

If a stochastic process is dependent on time, then a function of that process is also a stochastic process. Any derivative, such as an option, is a function of the variable underlying the derivative. Ito's lemma gives us the result of the process of the derivative if we know the process of the underlying asset. So, when assuming that the stock price follows a GBM, then one can calculate the process of the price of its derivative using Ito's lemma. We also know that if we apply Ito's lemma to a GBM, the result will also be a GBM. This proves to be an important result behind the option pricing formula of Black and Scholes, and the extension of their analysis we make in this thesis when developing the pricing formula for the index option.

2.2.4 Monte Carlo Simulation

Monte Carlo simulation is a technique that uses randomly generated inputs and simulates many scenarios to generate probabilistic results, such as the expectation or variance of some problem of interest. Boyle (1977) introduced to use the method in obtaining a numerical solution to option valuation. Monte Carlo is a way to solve problems when an analytical solution is hard or not possible and is thus very handy in the case of pricing complex financial derivatives, which in this case, will be applied to calculate the risk-neutral price of the European index option.

2.2.5 Preferences & Utility Functions

A utility function is a mathematical function that represents an individual's preferences over a set of alternatives. It is used to evaluate the relative desirability of different options based on their associated utility value to an individual. It is an important tool in economics as it allows for formal analysis and representation of preferences. In this thesis, I will assume that the individual has a specific utility function called a "power utility function".

$$\text{Power utility function:} \qquad U(C) = \frac{C^{1-\gamma}}{1-\gamma} \tag{2.4}$$

where C is the level of consumption, gamma is the measure of relative risk aversion, and U, the outcome, is the level of utility. Some investors are more risk-averse than others and will choose less risky investments. This particular utility function has the property of being increasing and concave which gives diminishing marginal utility. As $\gamma \to 0, U(C) \to C$ which means that the investor is risk neutral, and $\gamma = 1, U(C) = ln(U)$.

2.3 Literature Review & Theory Summary

Overall, the literature has shown some benefits of the covered call strategy, (specifically when optimising the underlying shares or the moneyness of the option selection) and especially convincing results regarding index call option writing. Although the literature on index option-based strategies is extensive, it has largely focused on option writing on the whole holdings as one, and to my knowledge, the literature lacks evidence on writing on the separate elements of an index and is a window in the research of whether this is beneficial or not. I aim to address this gap with my own research provided in this paper.

Additionally, as the method will show, this paper relies heavily on the above-presented theoretical framework and in some aspects, it distinguishes it from prior literature where they either have used historical data on option prices or assumed that an index follows a GBM which is a common assumption when pricing index options. The distributional properties of an index consisting of stocks, which in turn is assumed to follow a GBM, do not live up to the requirements of the Black and Scholes formula and thus I undertake another method to be used when pricing an index option with inspiration from research that addresses multi-asset option pricing. That will be my second contribution to the literature.

Chapter 3

Method

The main objective of this thesis is to compare two ways of implementing the covered call strategy on an index; the stock-by-stock approach and the index approach. To accomplish this, a value-weighted index comprising of two stocks is simulated using Matlab Software and stochastic programming.

The analysis commences by simulating stochastic processes to construct the two stocks and the index. Once the dynamics of the underlying stocks are retrieved, it is followed by a valuation of a European call option on the two stocks and the index. Based on the calculated option prices, the cost and the payoffs of the two strategies are determined, assuming that a call option is written at the beginning of the period, with a strike price equal to the current market price of the underlying asset. The returns generated by the Stock-By-Stock approach are then compared to the returns of the Index approach. Finally, a preference analysis is conducted to determine the optimal strategy for different investors.

3.1 Asset Dynamics & Construction of Index

The two stocks, $S_1(t)$ and $S_2(t)$, are simulated under a Black and Scholes environment where stocks follow a GBM. To mimic the relationship between shares in a "real-world" index, some dependence between the two stocks is introduced. Letting the volatility σ be defined as:

$$\sigma_1 = \sqrt{\theta_{11}^2 + \theta_{21}^2}$$

$$\sigma_2 = \sqrt{\theta_{21}^2 + \theta_{22}^2}$$

then the dynamics of the share prices that follow a two-dimensional GBM can be set as:

$$\begin{aligned} dS_1(t) &= S_1(t)(\mu dt + \theta_{11} dB_1(t) + \theta_{12} dB_2(t)) \\ dS_2(t) &= S_2(t)(\mu dt + \theta_{21} dB_1(t) + \theta_{22} dB_2(t)) \end{aligned} \tag{3.1}$$

where $B_1(t)$ and $B_2(t)$ are two independent Standard Brownian Motions. Setting:

$$\begin{aligned} W_1(t) &= \frac{1}{\sqrt{\theta_{11}^2 + \theta_{12}^2}} (\theta_{11} dB_1(t) + \theta_{12} dB_2(t)) \\ W_2(t) &= \frac{1}{\sqrt{\theta_{21}^2 + \theta_{22}^2}} (\theta_{21} dB_1(t) + \theta_{22} dB_2(t)) \end{aligned}$$

then the processes of our two stocks can now be expressed nicely in the following way:

$$\begin{aligned} dS_1(t) &= S_1(t)(\mu dt + \underbrace{\sqrt{\theta_{11}^2 + \theta_{12}^2}}_{\sigma_1} dW_1(t)) \\ dS_2(t) &= S_2(t)(\mu dt + \underbrace{\sqrt{\theta_{21}^2 + \theta_{22}^2}}_{\sigma_2} dW_2(t)) \end{aligned} \tag{3.2}$$

Where $W_1(t)$ and $W_2(t)$ are now dependent on each other.

The analysis is done on numerical examples of 3.2, where the base case is characterized by the parameters: $S(t) = 100$, T=1/12, N=1000, dt=T/N, and $\theta = \begin{bmatrix} 0.1 & 0.282843 \\ 0.282843 & 0.1 \end{bmatrix}$

for both stocks, which means that it is a one-month perspective, a diffusion term $W_i(t)$ allowing for a correlation of 0.1 between the two stocks, and an initial stock price of 100. $S_i(0)$ will be noted as the price at time zero which is at the beginning of the period, and $S_i(1)$, is the end of the one-month period. A graph of the stock prices according to equation 3.2 and the parameters can be found in Appendix A.1, which provides a visual illustration of the processes.

Taking the compounded developments of the processes gives the value of the stocks at time T. Thus, the price at time 1, which is at the maturity of the one-month option can be written as:

$$\begin{aligned} S_1(1) &\overset{d}{=} S_1(0)exp\left(r - \frac{1}{2}(\theta_{11}^2 + \theta_{12}^2)T + \theta_{11}\sqrt{T}Z_1 + \theta_{12}\sqrt{T}Z_2\right) \\ S_2(1) &\overset{d}{=} S_2(0)exp\left(r - \frac{1}{2}(\theta_{21}^2 + \theta_{22}^2)T + \theta_{21}\sqrt{T}Z_1 + \theta_{22}\sqrt{T}Z_2\right) \end{aligned} \tag{3.3}$$

The distributional properties of the processes still hold in line with the theory despite this reconstruction, and proof of this can be found in Appendix A2.

The shares are initially held in equal proportions in the portfolio, making the index level as follows:

$$I(t) = n_1 S_1(t) + n_2 S_2(t) \tag{3.4}$$

where n_i is the number of shares of asset i, and since the number of shares are written in the same proportion as the options for the individual approach, n will also be the number of option contracts we write. Furthermore, this number will be constant since we replicate a "buy and hold" portfolio. It will later be contrasted with writing a single option on the index, I.

3.2 Stock-By-Stock Buy-Write

The stock-by-stock strategy involves:

1. Buy n_1 shares of asset one at a cost of $S_1(0)$), and sell n_1 call options on asset one, with strike price equals $S_1(0)$.

2. Buy n_2 of asset two at cost $S_2(0)$, and sell n_2 call options on asset two, with strike price equals $S_2(0)$.

$S_1(0))$ means the price of stock one at time zero.

Due to the standard format of the dynamics of the stocks, we can use the known solution to the Black and Scholes PDE, as in equation 2.2, to price the stock options, and get the single asset Black and Scholes formula. Using our two assets from equation 3.2 the pricing formulas become:

$$\begin{aligned} C_1(S_1(0)) &= S_1(0)N(d_1) - S_1(0)e^{-r}N(d_2) \\ d_1 &= \frac{ln\left(\frac{S_1(0)}{S_1(0)}\right) + \left(r + \frac{\theta_{11}^2+\theta_{12}^2}{2}\right) * T}{\sqrt{(\theta_{11}^2 + \theta_{12}^2)} * \sqrt{T}} \\ d_2 &= d_1 - \sqrt{(\theta_{11}^2 + \theta_{12}^2)} * \sqrt{T} \end{aligned} \tag{3.5}$$

where $C_1(S_1(0))$ is the value of a European call written on stock S_1 at time 0, the strike price is $S_1(0)$ since we are selling at the money call options, the starting time $t = 0$, and $T = 1$ is the maturity of the option. I let the interest rate, r, be equal to 0.05 throughout this analysis along with the assumption that market participants can lend or borrow money at the risk-free rate. The same substitution is made with regards to $S_2(0)$ is:

$$\begin{aligned} C_2(S_2(0)) &= S_2(0)N(d_1) - S_2(0)e^{-r}N(d_2) \\ d_1 &= \frac{ln\left(\frac{S_2(0)}{S_2(0)}\right) + \left(r + \frac{\theta_{21}+\theta_{22}}{2}\right) * T}{\sqrt{(\theta_{21} + \theta_{22})} * \sqrt{T}} \\ d_2 &= d_1 - \sqrt{(\theta_{21} + \theta_{22})} * \sqrt{T} \end{aligned} \tag{3.6}$$

The cost of the Stock-by-Stock strategy at time $t = 0$ is:

$$n_1S_1(0) - n_1C(S_1(0)) + (n_2S_2(0) - n_2C(S_2(0))) \tag{3.7}$$

and the payoff at time T=1 of the strategy is:

$$n_1 S_1(1) - n_1 max(S_1(1) - S_1(0),0) + n_2 S_2(1) - n_2 max(S_2(1) - S_2(0),0) \tag{3.8}$$

3.3 Index Buy-Write

In order to obtain the dynamics of the derivative on the index consisting of two stocks, we need to use a general version of Ito's lemma to include two motions. So, by applying a general two-dimensional Ito's lemma, the law of motion of the option on the index is obtained.

The calculation of the two-dimensional Ito's lemma with respect to our two GBMs i.e., the law of motion of $V(t, S_1(t), S_2(t))$ is:

$$\begin{aligned} dV = [V_t' + V_1'\mu_1 S_1(t) + V_2'\mu_2 S_2(t) + \frac{1}{2}V_1'' S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \frac{1}{2}V_2'' S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ + V_{12}'' S_1(t)S_2(t)(\theta_{11}\theta_{21} + \theta_{12}\theta_{22})]dt \\ + (V_1' S_1(t)\theta_{11} + V_2' S_2(t)\theta_{21})dB_1(t) + (V_1' S_1(t)\theta_{12} + V_2' S_2(t)\theta_{22})dB_2(t) \end{aligned} \tag{3.9}$$

(Klebaner, 2005). Where the $V_i'(i = t,1,2)$ is the partial derivatives with regards to either t, $S_1(t)$ or $S_2(t)$, and $V_i''(i = 1,2)$ is the second derivative, where the processes are as in equation 3.2.

As mentioned earlier, the Black and Scholes model assumes the stock prices follow GBM and thus have a log-normal distribution. The index, however, constructed as in the equation 3.4, does not comply with these assumptions due to algebraic reasons. So, I must derive a pricing formula for the index in order to fulfil the assumption of a log-normal distributed process. That is done by deriving a partial differential equation (PDE) for the two asset case, which will help with determining the fair price of the option on the index. This is possible since, just like Black and Scholes rest on the assumption that one can short the underlying asset, buy a call and invest the rest into an asset generating a risk-free return in such a construction that creates a risk-free portfolio. So next, looking for a mimicking portfolio of the derivative helps us derive the PDE from our result of Ito's lemma:

PDE index:

$$V_t' + \frac{1}{2}V_1''S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \frac{1}{2}V_{22}''S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) + V_{12}''S_1(t)S_2(t)(\theta_{11}\theta_{12} + \theta_{21}\theta_{22})$$
$$= Vr - V_1'S_1(t)r - V_2'S_2(t)r \quad (3.10)$$

How this pricing formula is derived in detail is shown in Appendix A3. The solution to the PDE, $V(t, S_1(t), S_1(t))$, is the price of the index option at time 0 that satisfy 3.10 and is subject to the following boundary condition:

$$V(T, S_1(1), S_2(1)) = max(n_1S_1(1) + n_2S_2(1) - K,0) \quad (3.11)$$

where $K = n_1S_1(0) + n_2S_2(0)$, which is the sum of the at-the-money strike price for each position.

Next, is solving the PDE using the Monte Carlo technique. This is done by running Monte Carlo simulations of the processes in equation 3.3 and setting $e^{-rT}V(T, S_1(1), S_2(1))$ which gives the price of the options, which is equal to the risk-neutral expectation of the discounted payoff. So, the probabilistic solution to the PDE subject to the boundary condition is the price of the option on the index:

$$C_{index}(n_1S_1(0), n_2S_2(0)) = e^{-rT}Emax(n_1S_1(1) + n_2S_2(1) - K,0) \quad (3.12)$$

The derivation of this outcome from the PDE can be found in Appendix A.4.

The cost of the index buy-write strategy is:

$$n_1S_1(0) + n_2S_2(0) - C_{index}(n_1S_1(0), n_2S_2(0)) \quad (3.13)$$

and the payoff at time 1 of the strategy is:

$$n_1S_1(1) + n_2S_2(1) - max(0, n_1S_1(1) + n_2S_2(1) - (n_1S_1(0) + n_2S_2(0))) \quad (3.14)$$

Finally, I adopt a Power utility function and compute the utility, U, associated with the anticipated return, C, obtained from 2.4 across various levels of risk aversion. It is worth noting that the presence of endowments and beliefs held by representative agents does not alter the outcome.

3.4 Limitations

This study is constrained in its scope, as it solely includes two stocks in the index. This decision stems from the escalating complexity of computations that accompanies the inclusion of additional assets in the analysis due to the distributing assumptions pertaining to the stocks. Indeed, simply going from two to three assets increases the complexity drastically.

When pricing an asset, there is an infinite number of factors that can be taken into account, yet mathematical approximations require simplification in order to derive a result, which is also true in this study. One such factor this paper overlooks is liquidity. It simply assumes that all shares and options have infinite liquidity, thus ignoring, the fact that stocks and options might lack liquidity in the trading. The real-world implication of this is significant and as the purchase of shares or options may entail costs due to the high bid-ask spread caused by low liquidity, specifically, in instruments in smaller companies.

Chapter 4

Results

The result section starts with the outcome of the calculated call option prices based on the stocks that follow the processes in equation 3.2, and the index consisting of these two stocks. This is to investigate the validity of the derived pricing formula for the index. Subsequently, the Stock-by-Stock covered call strategy is compared to the Index covered call strategy both in terms of returns and utility to investigate which strategy has superior performance.

4.1 Option pricing

The performance of the derived pricing formula for the index call option is central in properly addressing the main objective of this paper. To this end, a comparison is set up between the option price retrieved from the Black and Scholes model and from the Monte Carlo technique. The Black and Scholes option price serves as a benchmark, representing a valid option price. It is important to know that the pricing formula for index options lacks an analytical closed-form solution, and thus, the option price derived in this study is obtained through the proposed formula.

4.1.1 Call Option Prices

Table 4.1 presents the individual stock options calculated using both the Black and Scholes model and the Monte Carlo technique, which was also applied to the index option.

Table 4.1

Call Option Prices

	Index	Stock 1	Stock 2
B&S	-	3.656	3.656
Monte Carlo	6.662	3.654	3.676

Note. Analytical calculations are obtained by the Black and Scholes formula. The Monte Carlo is performed on 10^5 simulations.

The prices for both stocks 1 and 2 are 3.656 when using the Black and Scholes formula. In the case of the Black and Scholes formula, the prices of the two stock options are equal, as they should be, since the solution of that PDE is deterministic. Since the future price is uncertain, both options have the same outlook and should thus be priced identically using the same parameters. The only component that set them apart is the random path they are taking due to the randomised term, which is distributed identically and thus yields the same forecast.

In contrast, the Monte Carlo method involves computing the average of a large number of simulated discounted payoffs, which may result in different prices for the two stocks. The prices differ at the second decimal place, which is deemed reasonable given the use of 100,000 simulations. Increasing the number of simulations would supposedly decrease the difference but also increase computational demands.

The prices between the Black and Scholes formula and Monte Carlo do not exceed 0.02.

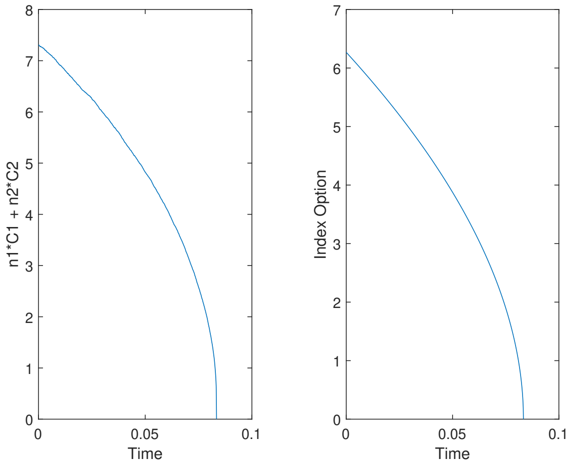
4.1.2 Characteristics of the Derived Index Options Price Formula

To further validate the pricing formula for the index call option derived in this study, a comparison is conducted between the qualitative properties of the price of the index option and the sum of the price of the two stocks calculated with the Black and Scholes formula. The so-called Greek Letters, which indicate the sensitivity of a derivative with respect to a particular parameter, are utilized to measure various dimensions of risk associated with an option position. By changing the value of one variable of interest while fixing all other parameters in the formula, the impact on the option price of that parameter can be observed.

Figure 4.1 and 4.2 depict the comparative results. The left side of the figures shows the sum of the price of the shares calculated with the Black and Scholes method times the number of shares. Meanwhile, the right side of the figures shows the price of the index option.

Figure 4.1

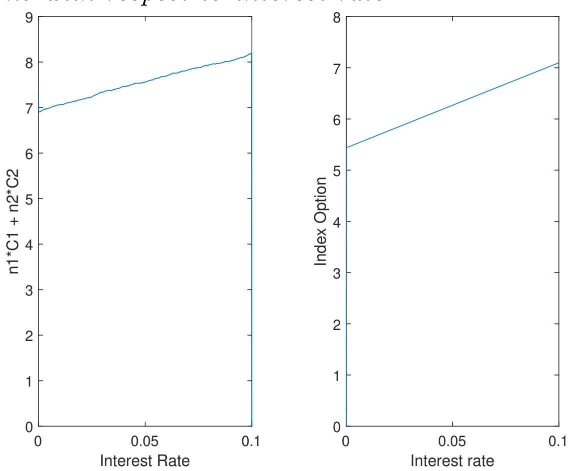
Price of the call options with respect to time



Note. Price of call options while time goes from 0 to 0.0833, which is equal to 1/12 or one month, which is the time of maturity of the call option.

Figure 4.2

Price of the call options with respect to interest rate



Note. Price of call options when the interest rate goes from 0 to 10%. In the base scenario, the interest rate was set to 5%.

Figure 4.1 shows that the pricing formula for the index option is decreasing with time, and the same relationship is true for the Black and Scholes price of the option. Figure 4.2 discloses that the price of the call option increases with interest rates in both cases.

4.2 Buy-Write Strategies - Comparative Results

The application of Monte Carlo simulations is now used to calculate all option prices in order to visualise the distribution of returns for each of the strategies and to evaluate the two strategies against each other.

4.2.1 Comparison of the Returns Between the Two Strategies

Table 4.2 discloses the result in terms of return for the two strategies.

Table 4.2

Costs and Payoffs for the strategies

	Index buy-write	Stock-by-Stock buy-write
Cost	193.337	192.67
Expected Payoff	194.202	193.532
Expected Return	0.004473	0.004474

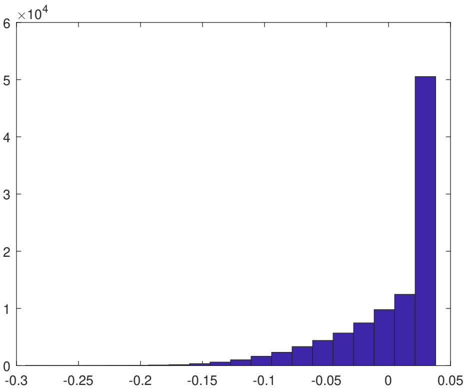
*Note.*The costs are calculated as in the equations 3.7 for the Stock-by-Stock buy-write and 3.13 for the index buy-write, and the expected payoff is the expectations of equations 3.8 and 3.14 respectively. The expected return is calculated as the expectation of payoff/cost-1 for each strategy.

The results show that the Index buy-write strategy exhibits higher costs and expected payoffs than for the Stock-by-Stock buy-write. Notably, the expected payoff exceeds the cost for both approaches, leading to a positive expected return. Furthermore, the expected return for both strategies turned out to be closely aligned, although marginal superiority of the Stock-by-Stock strategy.

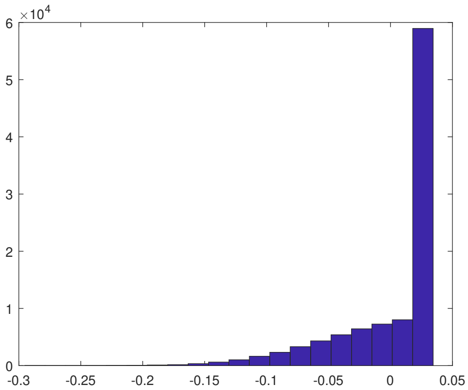
To investigate the returns further, figure 4.3 exhibits the distributions of the returns for each approach.

Figure 4.3

Distribution of returns



(a) *Stock-by-Stock strategy*



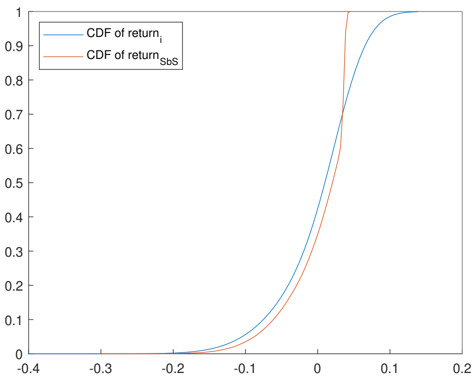
(b) *Index strategy*

Note. Distribution of returns for each of the strategies where the level of return is on the x-axis for all simulations, with returns reaching from negative to positive values.

To begin with, it is pertinent to observe that the potential returns for both strategies are constrained. Furthermore, it is noteworthy that the maximum attainable returns are equivalent for both approaches, as revealed by the pronounced bars in Figure 4.3 that correspond to instances where an option expires in-the-money, signifying that the stock price exceeds the strike price of 200, necessitating the delivery of the option to the purchaser. The figure also reveals slight divergences in the distribution of returns for the two strategies. This variation is accentuated in Figure 4.4, which plots each cumulative distribution function.

Figure 4.4

CDF



Note. The Cumulative distribution function for the Sock-by-Stock approach (red line) and for the Index approach (blue line).

4.2.2 Utility Levels & Scenario Partition

In order to discern disparities in individual preferences and establish an optimal strategy choice, the present analysis incorporates three investors characterized by distinct levels of risk aversion, as indicated by their respective gamma parameters. The expected utilities for these representative agents are subsequently computed employing the power utility function articulated in equation 2.4. The result is presented in table 4.3.

Table 4.3

Utility - Base Scenario - Low Correlation

	Index buy-write	Stock-by-stock buy-write
$\gamma = 10$	-1.5510e+20	-1.5477e+20
$\gamma = 1$	94.7340	94.7342
$\gamma = 0$	0.004473	0.004474

Note. The results of this table are based on the values $\theta_{11} = \theta_{22} = 0.28284$ and $\theta_{12} = \theta_{21} = 0.1$ which results in a volatility of 0.3 and correlation of 0.1.

Notably, higher gamma values correspond to greater risk aversion, with a gamma of ten representing high risk aversion, while a value of one denotes logarithmic risk aversion, and a gamma of zero connotes risk neutrality. The table reports the calculated utility values, with thetas set as in the base case, that is, a correlation of 0.1 between the two stocks and volatility of 0.3. It is important to note that the utility level is relevant only when comparing strategies with the same gamma. Moreover, only the relationship between the level of utility is relevant and not to what extent the utility level differs between the strategies.

Upon comparing the Index buy-write strategy with the Stock-by-Stock strategy, with reference to the parameters specified for the two underlying assets in table 4.3, it appears that the latter strategy generates higher utility levels across all three gamma values.

To explore the potential impact of changes in asset-specific variables on individual preferences, the values of the thetas are tuned to alter the volatility and interdependence of the underlying stocks. Table 4.4 demonstrates a few scenarios in addition to the original setting from the previous table, as a means of illustrating this variation. The bold values are the highest utility of the two strategies.

Table 4.4

Utility - Different Scenarios

	$\gamma = 10$		$\gamma = 1$		$\gamma = 0$	
	SbS	Index	SbS	Index	SbS	Index
IID	**-1.776e+20**	-1.784e+20	**94.7197**	94.7192	**0.00441**	0.00440
High Corr	-1.181e+20	**-1.128e+20**	94.7626	**94.767**	0.00461	**0.00463**
High Vol	**-8.467e+19**	-8.634e+19	**94.7977**	94.7956	**0.00478**	0.00477

Note. This table presents the utility for three scenarios in addition to the original setting displayed in table 4.3. SbS is short for Stock-by-Stock.
The first case with identically independent distributed, **IID**, stocks is based on the values $\theta_{11} = \theta_{22} = 0.3$ and $\theta_{12} = \theta_{21} = 0$ which results in a volatility = 0.3 and correlation = 0.
The **High Correlation** case is based on values of $\theta_{11} = \theta_{22} = 0.1$ and $\theta_{12} = \theta_{21} = 0.5$ which results in a volatility ≈ 0.51 and correlation = 0.5.
The **High Volatility** case is based on values of $\theta_{11} = \theta_{22} = 0.7$ and $\theta_{12} = \theta_{21} = 0.1$ which results in a volatility ≈ 0.71 and correlation = 0.1.

In the case of identically independent distributed stocks, the level of utility is higher for all individuals for the Stock-by-Stock strategy. Whereas in the scenario where the correlation between the stocks is high, the Index strategy generates higher utility. In the final scenario, interdependence remains the same as the base case, but the volatilities of both stocks are elevated. For this scenario, table 4.4's last column demonstrates that the Stock-by-Stock buy-write strategy continues to yield higher utility levels than the Index buy-write strategy.

Chapter 5

Analysis

The determination of index option pricing is one of the key focuses of this paper. The derivation of the partial differential equation (PDE) using two underlying assets provided two options for analysis: to obtain a deterministic formula via analytical solution, as Black and Scholes (1973) have done, or to use Monte Carlo simulations to determine the index option price. The Monte Carlo method was deemed a practical approach for this task, which is unlike the method of Stulz (1982) who derives a deterministic formula from a PDE also based on a two-asset case but on pricing options on the minimum or the maximum of two risky assets. Since I decided to take this approach when pricing the index option I found it important to investigate the validity and characteristics och the price.

The difference between the results obtained using the Black and Scholes method versus the Monte Carlo method for calculating stock options prices was negligible, lending credibility to the approach for pricing the index option. This price consistency between the stock options retrieved by the two methods forms the basis for relying on the price of the index option since the index option price was not controlled against a deterministic formula. Nonetheless, like Stulz (1982), obtaining a deterministic index option price formula through an analytical solution, which can be parameterized, holds great significance for future research.

Furthermore, the pricing formula for the index option shows the same characteristics as the analytical Black and Scholes way of calculating the option price on the single options. This is true for both time and interest rates. This means that the index option price also has a time value decay which is a known characteristic of any option. As a consequence of this result, it is shown that the derived formula for the index call option exhibits the same qualitative statistics as those of the Black and Scholes formula. Therefore, it further amplifies the credibility of the index option pricing and consequently its suitability as input for the proceedings of the analysis of the main objective, which is the strategy comparison.

Comparing the returns of the two buy-write strategies, the distributions of the returns displayed capped returns, which was just as expected in both approaches since the point with a covered call strategy makes the investor cap the upside potential and keeps the downside. The distribution, however, of the two strategies differs which opens up the question of whether the utility will be different for investors with various attitudes towards risk. The Stock-By-Stock approach showed a somewhat steeper CDF which indicates that it has a higher tendency to have less variability in its outcomes than the Index approach. And conversely, the Index approach seems to generate more widespread in outcomes, which is in line with the fact that the index option is either out- or in-the-money and thus intuitively this difference would appear, compared to the Stock-by-Stock approach where one option could be in-the-money while the other might not be. In this thesis, I limited the analysis only to considering two stocks, so the risk of the options being out of the money and thus affecting the returns is significant. If this analysis was to be made with an index consisting of more than 2 stocks, this effect could arguably diminish.

Regarding the part where the two covered call strategies on an index are evaluated in terms of utility, it was shown that for the base case, the finding suggests that the Stock-by-Stock approach is a more preferable option for all levels of risk aversion. Thus, all the representative investors prefer the Stock-by-Stock buy-write strategy over the Index buy-write strategy regardless of the risk preference. Hence, a difference could be spotted between the two strategies as expected, but this result alone is not enough to answer the question of which is preferable and that is why the different scenarios are set up.

The scenario partition indicates the same outcome as could be seen in the base case for additional two scenarios, namely, the scenario with the IID stocks, and the scenario with high-volatility stocks. However, the opposite was found for the case with high correlation, where it was found that all the representative agents preferred the Index approach to the Stock-by-Stock approach. This result of having all investor profiles choosing the same strategy in each scenario indicates that the choice of strategy does not depend on the attitude towards risk. Although it could be seen that the Stock-by-Stock approach was found to be selected more often than the index approach when setting up these four particular scenarios. This implies that the Stock-by-Stock strategy can be a beneficial choice in favour of the index approach in some settings.

Moreover, based on the scenario analysis it can be argued that, unlike the risk aversions which did not exhibit any particular importance, the characteristics of the underlying stock that the index a built on appear to have a notable impact. These findings emphasize the importance of carefully selecting the underlying stock when choosing between the two covered call strategies on an index, as it could affect the preferences of representative investors.

One could argue that the efficiency of employing an ETF to manage a Stock-by-Stock approach is considerably lower than that of an Index approach, which may result in limited concern in this matter of research. Nevertheless, advancements in trading algorithms are expected to render this argument more and more obsolete.

Chapter 6

Conclusion

The objective of this thesis is to compare two ways of implementing the covered call strategy on a two-asset index.

The two approaches compared are:

1. Hold all the shares in the index and write a call option on the index.

2. Hold all the shares in the index and write a call option on each share separately.

The analysis was performed under the assumption that each of the underlying stocks is log-normal distributed and thus it resulted in a large focus on deriving an appropriate pricing method for the index option, which exhibits valid and consistent results with the Black and Scholes option pricing formula.

Even though the expected return turned out to be higher for the Stock-by-Stock strategy in the base case, the difference in distribution does not make it obvious that all investors, regardless of their risk aversion could be assumed to prefer this strategy. However, the results were that the Stock-by-Stock approach is preferred more often than the index approach, based on the few cases that were set up in this analysis.

Conclusively, the main finding was that one strategy is not consistently preferred above the other and that all investors have the same strategy preference in all of the replicated cases. The results indicated that an investor's attitude towards risk is not significant when it comes to strategy choice but rather the characteristics of the underlying assets in the index.

Since this specific topic has not been researched to any significant extent as far as I know, I see several openings on how future research can proceed with this analysis. First of all, I propose to extend the analysis to consider an index that consists of more than two stocks. As mentioned earlier, this increases the complexity of the calculations drastically if the same assumption were to be made as in this thesis. However, an elaboration on an index consisting of 30, 100 or 500 stocks would more accurately symbolise the ETFs that are practising the covered call strategy thus such an analysis would be of high relevance. Also, an extension analysis can be done with jump-diffusion processes to mimic the real world to a larger extent than this analysis were allowed for.

Chapter 7

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Appendix A

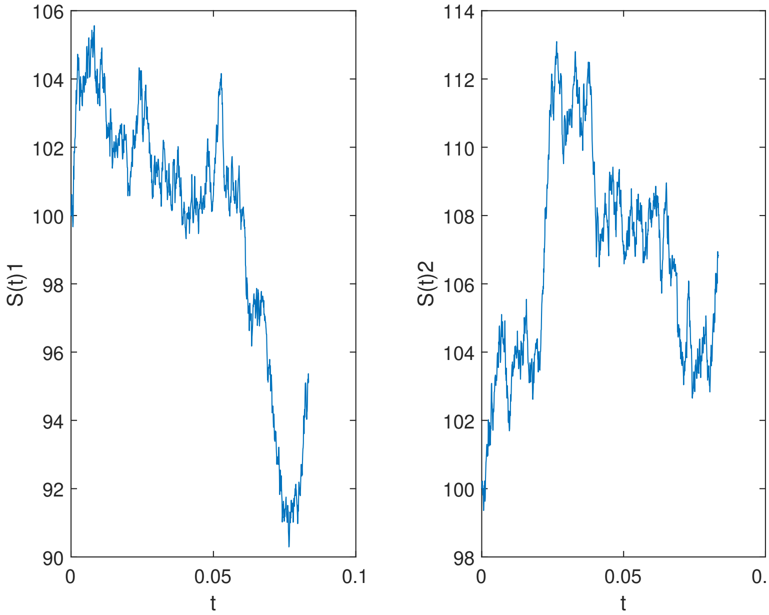
Appendix

A.1 Numerical Example on Assets Dynamics

A.1.1 Stock Dynamics

Figure A.1

Stock Price Dynamics

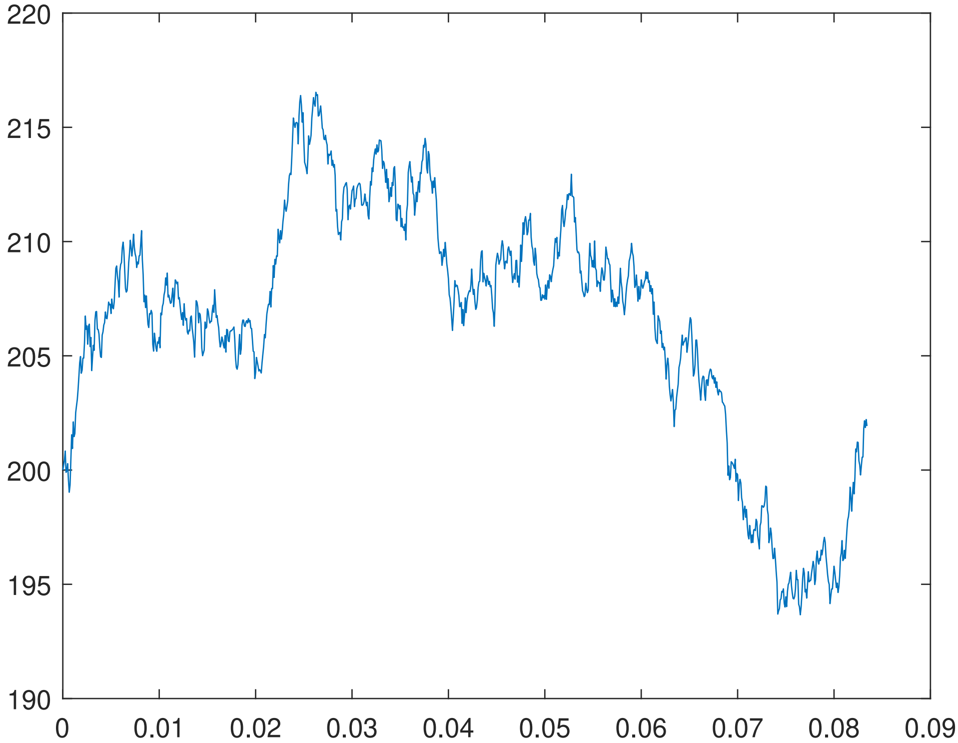


Note. Visual representation of equation 3.2 with values:
$S_i(0) = 100, T = 1/12, N = 1000, dt = T/N,\ \mu = 0.05.\ \theta = \begin{bmatrix} 0.1 & 0.282843 \\ 0.282843 & 0.1 \end{bmatrix}.$

A.1.2 Index Dynamics

Figure A.2

Index Dynamics



Note. Visual representation of the Index, constructed on the two stocks from A.1

A.2 Proof of Log-normal Distribution of Stocks

The construction of the index option formula is based on the fact that both underlying stocks have a log-normal distribution. A variable is log-normally distributed if its natural logarithm is normally distributed. So, showing that

$$S(1) \stackrel{d}{=} S(0)exp\left(r - \frac{1}{2}(\theta_{11}^2 + \theta_{12}^2)T + \theta_{11}\sqrt{T}Z_1 + \theta_{12}\sqrt{T}Z_2\right) \tag{A.1}$$

is distributed log-normal can be done by showing that ln(S) is normally distributed.

Taking the natural logarithm of S results in:

$$\ln S(1) = \ln S(0) + \left(r - \frac{1}{2}(\theta_{11}^2 + \theta_{12}^2)T + \theta_{11}\sqrt{T}Z_1 + \theta_{12}\sqrt{T}Z_2\right) \tag{A.2}$$

where:

$$\ln S(1) = \ln S(0) + \int_0^1 \text{drift}dt + \int_0^1 \text{volatility}dB_1(t) + \int_0^1 \text{volatility}dB_2(t) \tag{A.3}$$

and since the drift and volatilities are scalars it is shown that ln(S) is a normally distributed variable and thus, S is log-normal distributed.

A.3 Derivation of the PDE for the Index Call Option

This part is a description of how I derived equation 3.10

To mimic the payoff of an option that pays at maturity, one can assume a self-financing portfolio. This means that no cash is added or withdrawn from the portfolio, then we know that the value of that portfolio only depends on the changes in the elements that are included in the portfolio. The value of the replicating portfolio consists of a risk-free asset, stock 1 and stock 2 is:

$$V(t, S_1(t), S_2(t)) = M(t) + \delta_1(t)S_1(t) + \delta_2(t)S_2(t) \tag{A.4}$$

where $M(t)$ is money in the money market, r is the risk-free return and δ_i is the number of shares in asset i. The value of the replicating portfolio depends solely on time, $S_1(t)$ and $S_2(t)$.

Since we know it is self-financing its dynamics becomes:

$$dV = M(t)rdt + \delta_1(t)dS_1(t) + \delta_2(t)dS_2(t). \tag{A.5}$$

Substituting in the dynamics of the stock prices $dS_1(t)$ and $dS_2(t)$:

$$dS_1(t) = S_1(t)(\mu dt + \theta_{11} dB_1(t) + \theta_{12} dB_2(t))$$
$$dS_2(t) = S_2(t)(\mu dt + \theta_{21} dB_1(t) + \theta_{22} dB_2(t))$$

into A.5 results in:

$$dV = M(t)rdt + \delta_1(t)S_1(t)(\mu dt + \theta_{11} dB_1(t) + \theta_{12} dB_2(t)) + \delta_2(t)S_2(t)(\mu dt + \theta_{21} dB_1(t) + \theta_{22} dB_2(t)).$$

It can be rewritten as:

$$\begin{gathered} dV = (M(t)r + \delta_1\mu_1 S_1(t) + \delta_2\mu_2 S_2(t))dt + (\delta_1 S_1(t)\theta_{11} + \delta_2 S_2(t)\theta_{21})dB_1(t) \\ +(\delta_1 S_1(t)\theta_{12} + \delta_2 S_2(t)\theta_{22})dB_2(t) \end{gathered}$$

Using this result we can replicate the option together with the result from Ito's lemma, where we know that 3.9 hold. Equation 3.9 is repeated here:

$$\begin{gathered} dV = \big[V'_t + V'_1\mu_1 S_1(t) + V'_2\mu_2 S_2(t) + \tfrac{1}{2}V''_1 S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V''_2 S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ +V''_{12} S_1(t)S_2(t)(\theta_{11}\theta_{21} + \theta_{12}\theta_{22})dt \\ +[V'_1 S_1(t)\theta_{11} + V'_2 S_2(t)\theta_{21}]dB_1 + [V'_1 S_1(t)\theta_{12} + V'_2 S_2(t)\theta_{22}]dB_2 \end{gathered}$$

Matching $dV = dV$ from the two previous equations, yields:

$$\begin{gathered} (Mr + \delta_1\mu_1 S_1(t) + \delta_2\mu_2 S_2(t))dt \\ +(\delta_1 S_1(t)\theta_{11} + \delta_2 S_2(t)\theta_{21})dB_1(t) + (\delta_1 S_1(t)\theta_{12} + \delta_2 S_2(t)\theta_{22})dB_2(t) = \\ [\ V'_t + V'_1\mu_1 S_1(t) + V'_2\mu_2 S_2(t) + \tfrac{1}{2}V''_1 S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V''_2 S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ +V''_{12} S_1(t)S_2(t)\ (\theta_{11}\theta_{21} + \theta_{12}\theta_{22})]dt \\ +(V'_1 S_1(t)\theta_{11} + V'_2 S_2(t)\theta_{21})dB_1(t) + (V'_1 S_1(t)\theta_{12} + V'_2 S_2(t)\theta_{22})dB_2(t) \end{gathered}$$

Now, matching the diffusion terms, dt, from the above result

$$Mr + \delta_1\mu_1 S_1(t) + \delta_2\mu_2 S_2(t) = V_t' + V_1'\mu_1 S_1(t) + V_2'\mu_2 S_2(t)$$
$$+\tfrac{1}{2}V_1'' S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V_2'' S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) + V_{12}'' S_1(t)S_2(t)(\theta_{11}\theta_{21} + \theta_{12}\theta_{22})$$

and matching the terms ending with $dB_1(t)$ and $dB_2(t)$:

$$\delta_1 S_1(t)\theta_{11} + \delta_2 S_2(t)\theta_{21} = V_1' S_1(t)\theta_{11} + V_2' S_2(t)\theta_{21}$$
$$\delta_1 S_1(t)\theta_{12} + \delta_2 S_2(t)\theta_{22} = V_1' S_1(t)\theta_{12} + V_2' S_2(t)\theta_{22}$$

From the above, it becomes that $V_1' = \delta_1$ and $V_2' = \delta_2$. Thus,

$$M = V - \delta_1 S_1(t) - \delta_2 S_2(t) = V - V_1' S_1(t) - V_2' S_2(t)$$

. Then, matching the drift terms we get that V satisfies the PDE and we get the result as in equation 3.10:

$$V_t' + \frac{1}{2}V_1'' S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \frac{1}{2}V_2'' S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) + V_{12}'' S_1(t)S_2(t)(\theta_{11}\theta_{12} + \theta_{21}\theta_{22})$$
$$= Vr - V_1' S_1(t)r - V_2' S_2(t)r$$

A.4 Probabilistic Solution to the PDE

Here I show how I derived the price of the index call option from the PDE:

$$C_{index}(n_1 S_1(0), n_2 S_2(0)) = e^{-rT} Emax(n_1 S_1(1) + n_2 S_2(1) - K{,}0)$$

A reminder to the reader is that the beginning of the period, t, is the start of the month: t = 0, and the end of the period, T, is at the end of the month: T = 1.

The results of the multidimensional Ito's Lemma, 3.9, and the PDE 3.10 can be used to show that the value of the call option on the index can be written as $e^{-rT}EV(t, S_1(1), S_2(1))$ where $V(t, S_1(t), S_2(t))$ is the boundary condition. That is the discounted expected value of the index option.

The PDE can be rewritten as:

$$\begin{gathered} \mathrm{V}'_t + \tfrac{1}{2}V''_1 S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V''_2 S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ +V''_{12}S_1(t)S_2(t)(\theta_{11}\theta_{12} + \theta_{21}\theta_{22}) + V'_2 S_2(t)\mu_2 + V'_1 S_1(t)\mu_1 = V_r \end{gathered}$$

The dynamics of the index retrieved using multidimensional Ito's Lemma:

$$\begin{gathered} dV = [\mathrm{V}'_t + V'_1\mu_1 S_1(t) + V'_2\mu_2 S_2(t) + \tfrac{1}{2}V''_1 S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V''_2 S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ +V''_{12}S_1(t)S_2(t)(\theta_{11}\theta_{21} + \theta_{12}\theta_{22})]dt \\ +(V'_1 S_1(t)\theta_{11} + V'_2 S_2(t)\theta_{21})dB_1(t) + (V'_1 S_1(t)\theta_{12} + V'_2 S_2(t)\theta_{22})dB_2(t) \end{gathered}$$

for which the following is true:

$$\begin{gathered} V(t, S_1(t), S_2(t)) = V(0) \\ + \int_0^T [V'_t + V_1\mu_1 S_1(t) + V'_2\mu_2 S_2(t) + \tfrac{1}{2}V''_1 S_1^2(t)(\theta_{11}^2 + \theta_{12}^2) + \tfrac{1}{2}V''_2 S_2^2(t)(\theta_{21}^2 + \theta_{22}^2) \\ +V''_{12}S_1(t)S_2(t)(\theta_{11}\theta_{21} + \theta_{12}\theta_{22}]dt \\ + \int_0^T (V'_1 S_1(t)\theta_{11} + V'_2 S_2(t)\theta_{21})dB_1(t) + \int_0^T (V'_1 S_1(t)\theta_{12} + V'_2 S_2(t)\theta_{22})dB_2(t) \end{gathered}$$

Multiplying the equation above with the continuous compounding discount factor e^{-rT} and substituting in the right side of the PDE from A.4 yields:

$$e^{-rT}V(t, S_1(t), S_2(t)) = e^{-rT}V(0) + e^{-rT}\int_0^T (Vr)dt$$

$$+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{11}+V_2'S_2(t)\theta_{21})dB_1(t)+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{12}+V_2'S_2(t)\theta_{22})dB_2(t)$$

Where: $e^{-rT}\int_0^T (Vr)dt = e^{-rT}Ve^{rT} = e^{-rT+rT}V = e^0V = V$, so:

$$e^{-rT}V(t,S_1(t),S_2(t)) = e^{-rT}V(0)+V$$
$$+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{11}+V_2'S_2(t)\theta_{21})dB_1(t)+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{12}+V_2'S_2(t)\theta_{22})dB_2(t)$$

Now, taking the expectation E, of the above it can be written:

$$e^{-rT}EV(t,S_1(t),S_2(t)) = E[V(0)+\int_0^T (e^{-rT}Vr)dt$$
$$+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{11}+V_2'S_2(t)\theta_{21})dB_1(t)+\int_0^T e^{-rT}(V_1'S_1(t)\theta_{12}+V_2'S_2(t)\theta_{22})dB_2(t)]$$

Since terms of the form $\int \ldots B_i(t)$ are martingales their expected value are zero:

$$E\int_0^T e^{-rT}(V_1'S_1(t)\theta_{11}+V_2'S_2(t)\theta_{21})dB_1(t)=0$$
$$E\int_0^T e^{-rT}(V_1'S_1(t)\theta_{12}+V_2'S_2(t)\theta_{22})dB_2(t)=0$$

and V(0) is a constant so its expected value is also zero, so left is:

$$e^{-rT}EV(T,S_1(T),S_2(T)) = e^{-rT}EV(T,S_1(T),S_2(T))$$

and denoting the time period from 0 to time 1 after one month, results in:

$$V(T,S_1(1),S_2(1)) = max(n_1S_1(1)+n_2S_2(1)-K,0).$$

Which gives the price of today's index call as the discounted expected value of the future value:

$$C_{index}(n_1S_1(0),n_2S_2(0)) = e^{-rT}Emax(n_1S_1(1)+n_2S_2(1)-K,0)$$