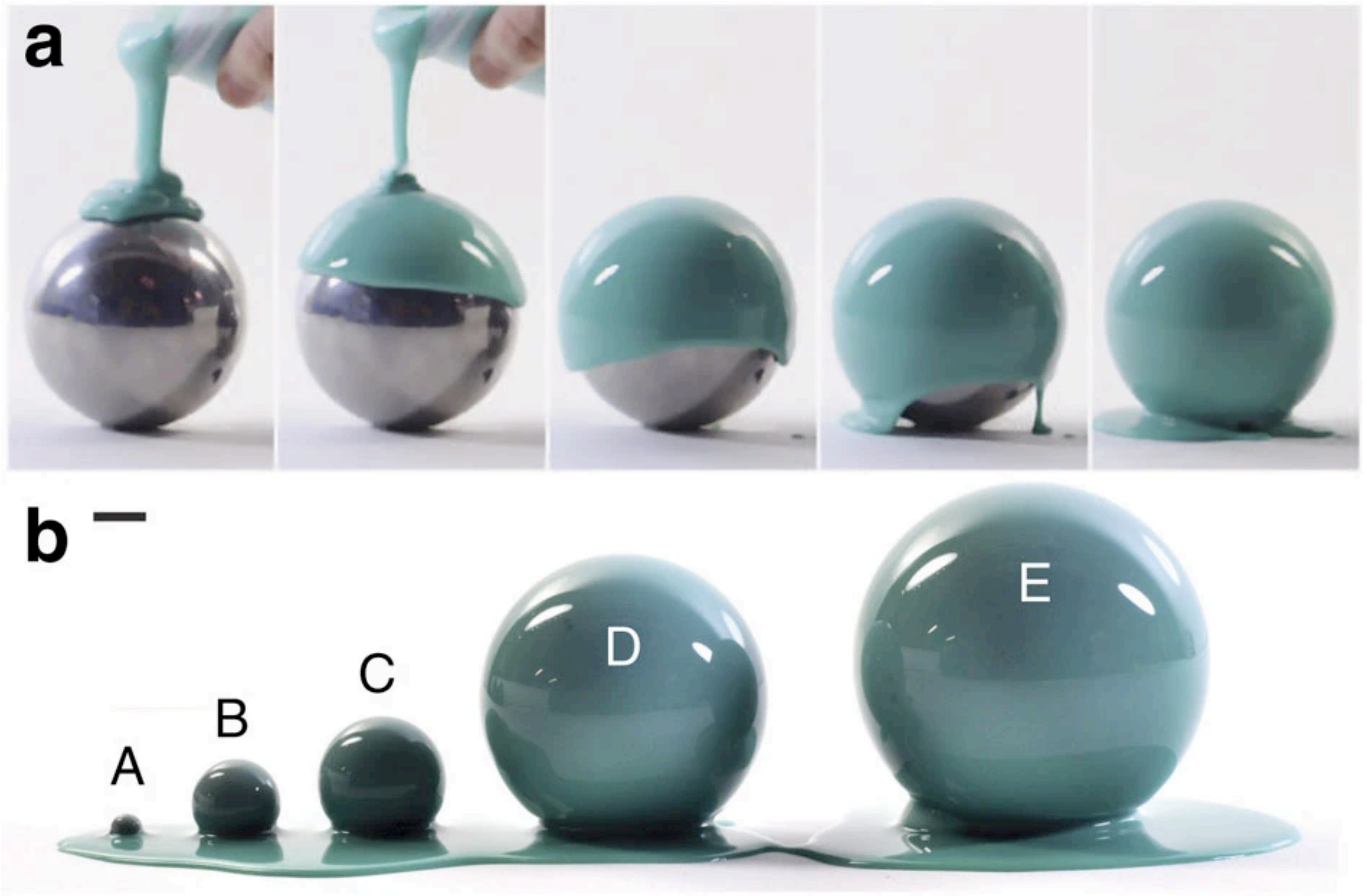
The physics of thin sheets: membranes and wrinkling

Daniel Matoz Fernandez

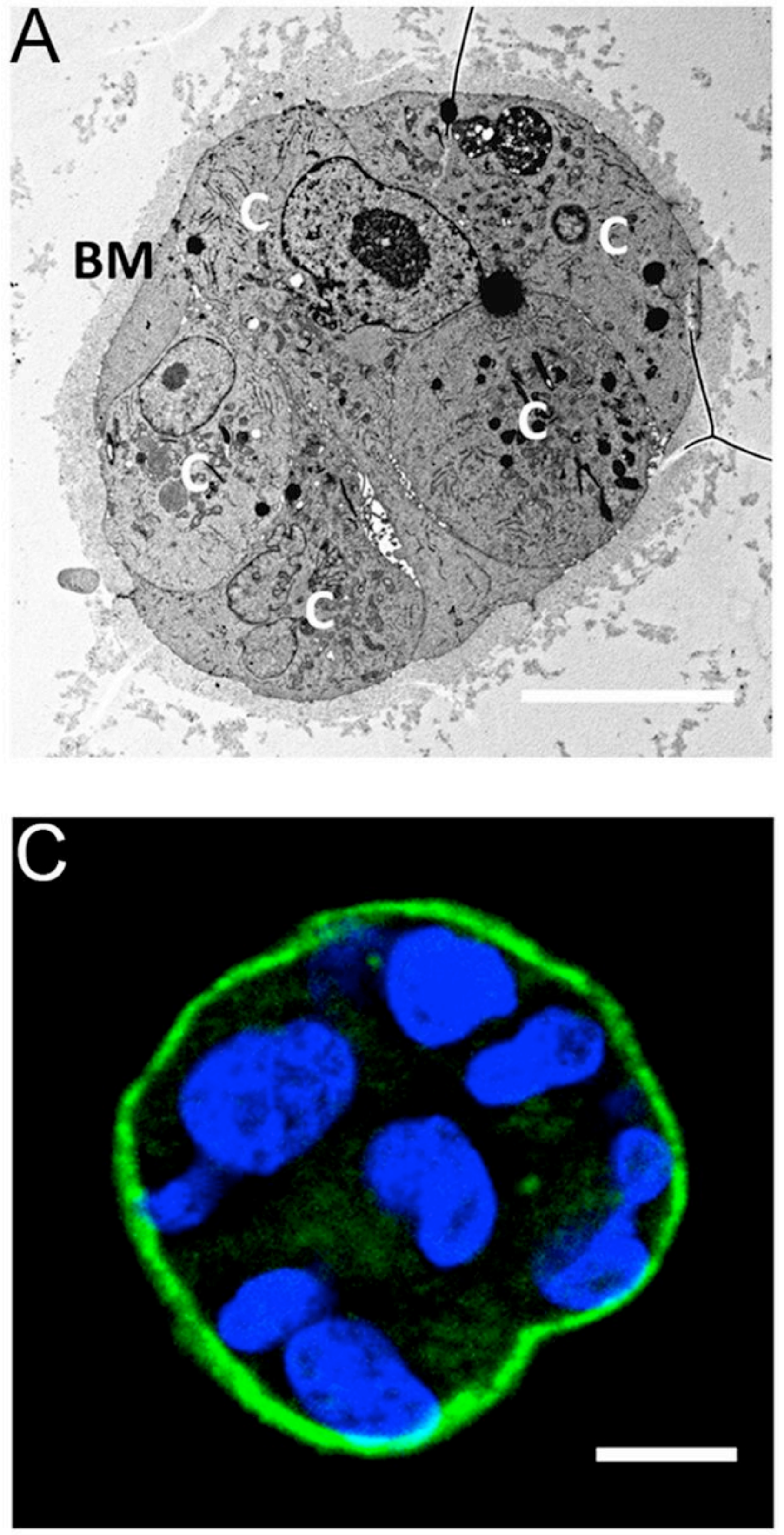
Membranes in nature

Membranes are assemblies or aggregates of molecules (atoms) which have the form of very thin sheets

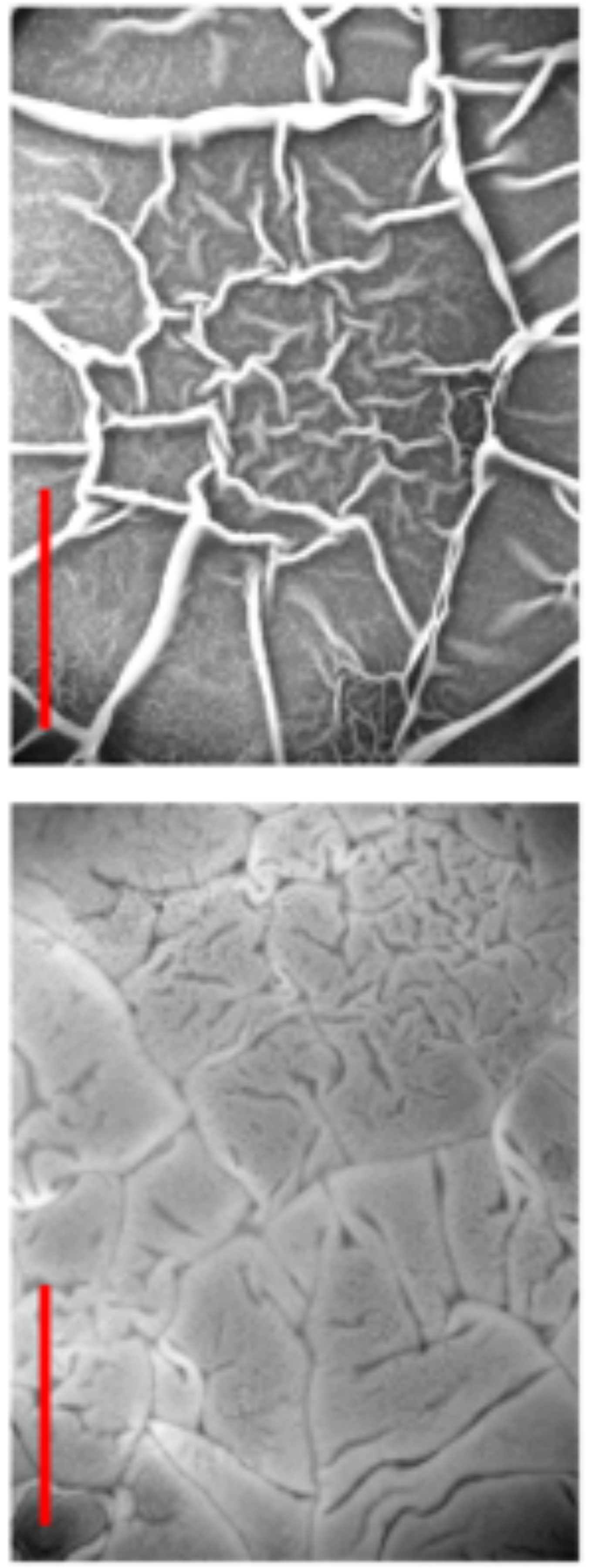
Thin coatings



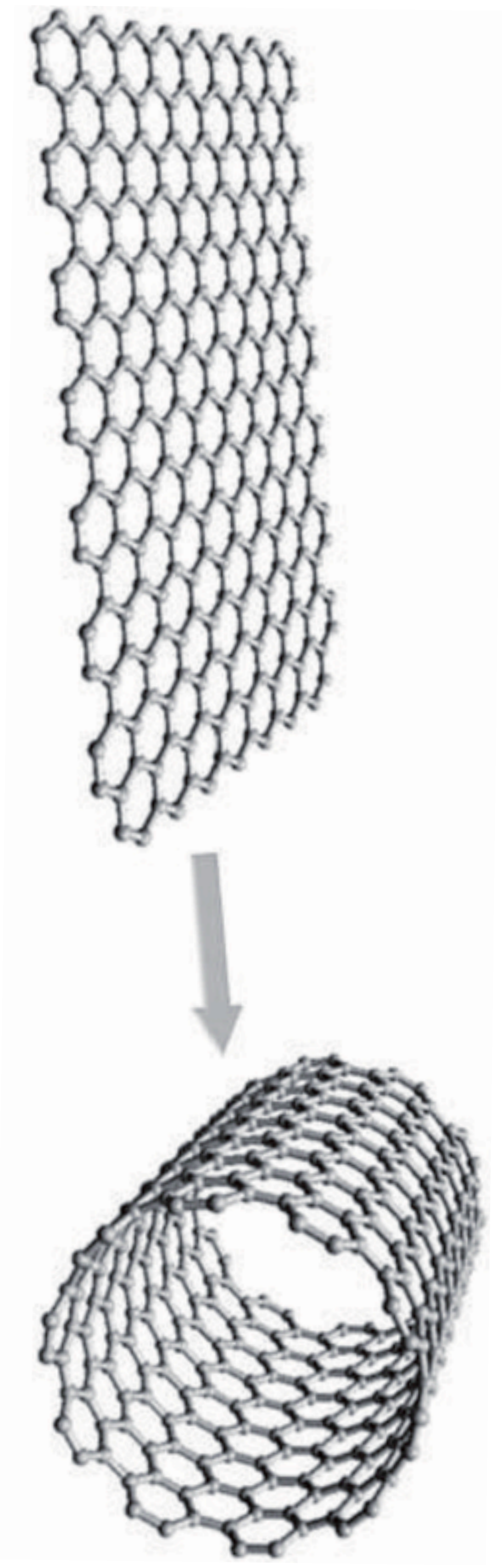
Cancer spheroids membranes



Bacterial biofilms



Carbon nanotubes



Fabrication of slender elastic shells by the coating of curved surfaces. Lee, A., Brun, PT., Marthelot, J. et al. Nat Commun 7, 11155 (2016).

Nonlinear elasticity of biological basement membrane revealed by rapid inflation and deflation. Hui Li, Yue Zheng, Yu Long Han, Shengqiang Cai, Ming Guo
Proceedings of the National Academy of Sciences Mar 2021, 118 (11) e2022422118; DOI: 10.1073/pnas.2022422118

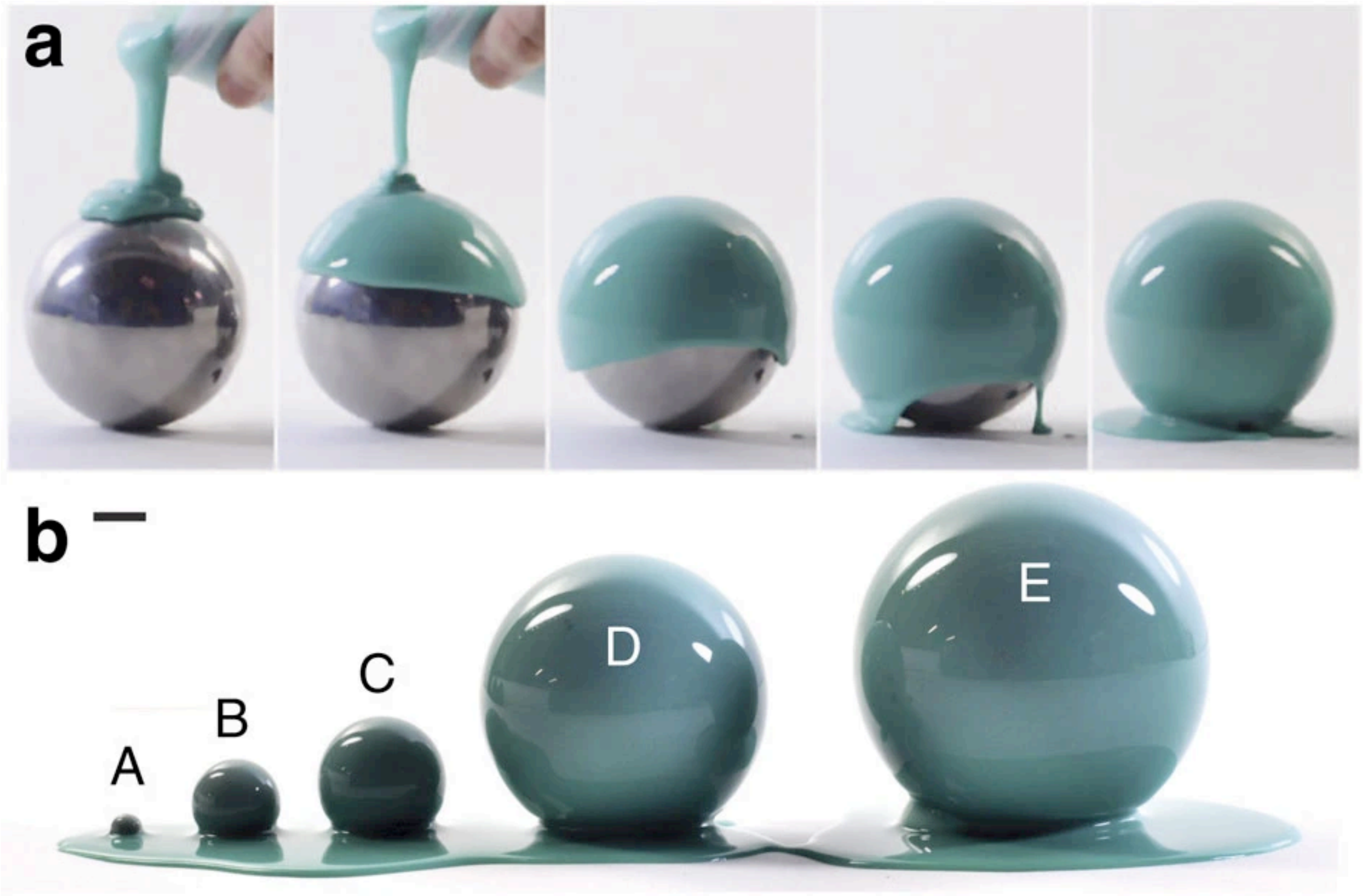
Wrinkle patterns in active viscoelastic thin sheets. D. A. Matoz-Fernandez, Fordyce A. Davidson, Nicola R. Stanley-Wall, and Rastko Sknepnek. Phys. Rev. Research 2, 013165 – Published 18 February 2020

Eleonora Secchi et al. Massive radius-dependent flow slippage in carbon nanotubes, *Nature* (2016)

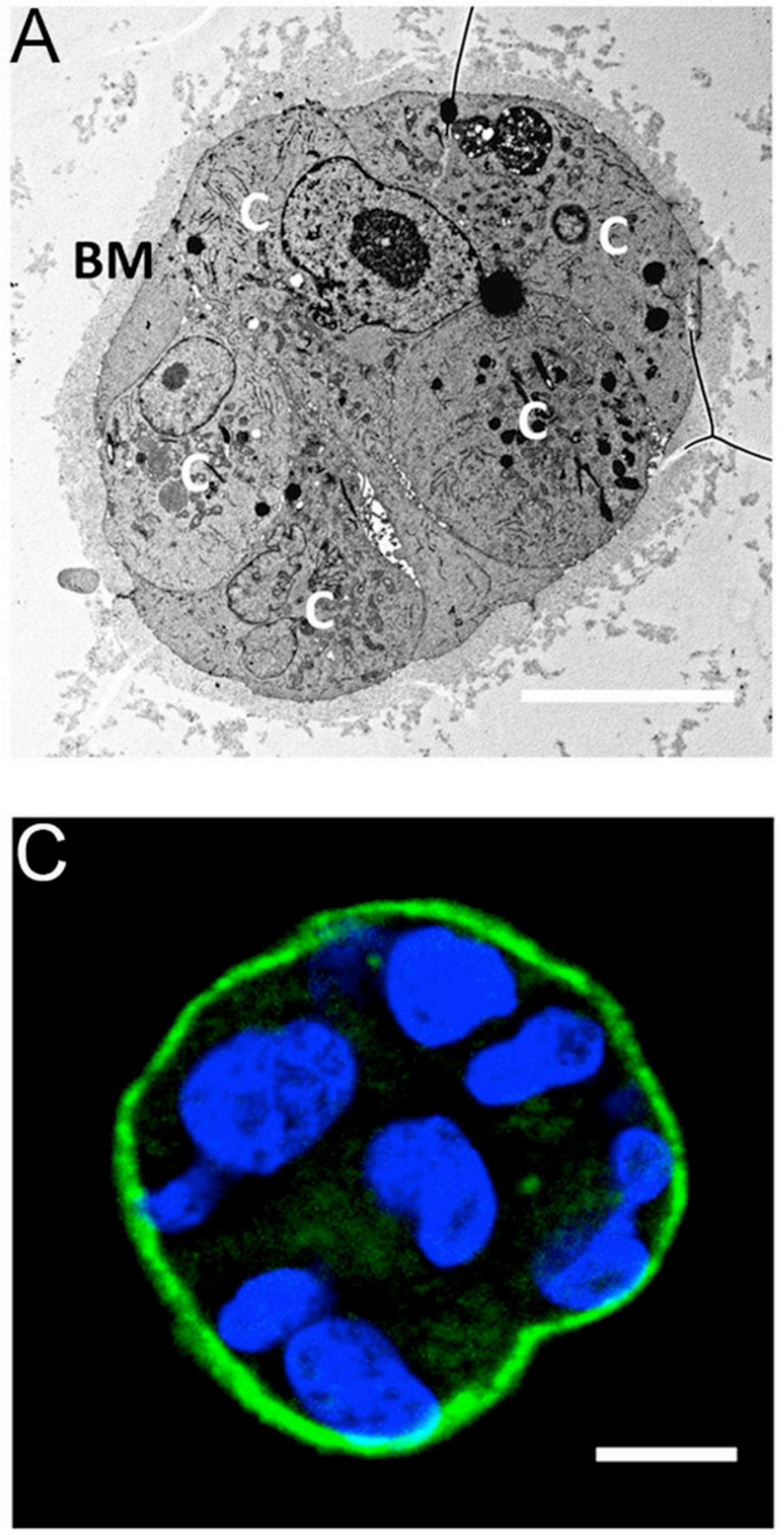
Membranes in nature

Membranes are assemblies or aggregates of molecules (atoms) which have the form of very thin sheets

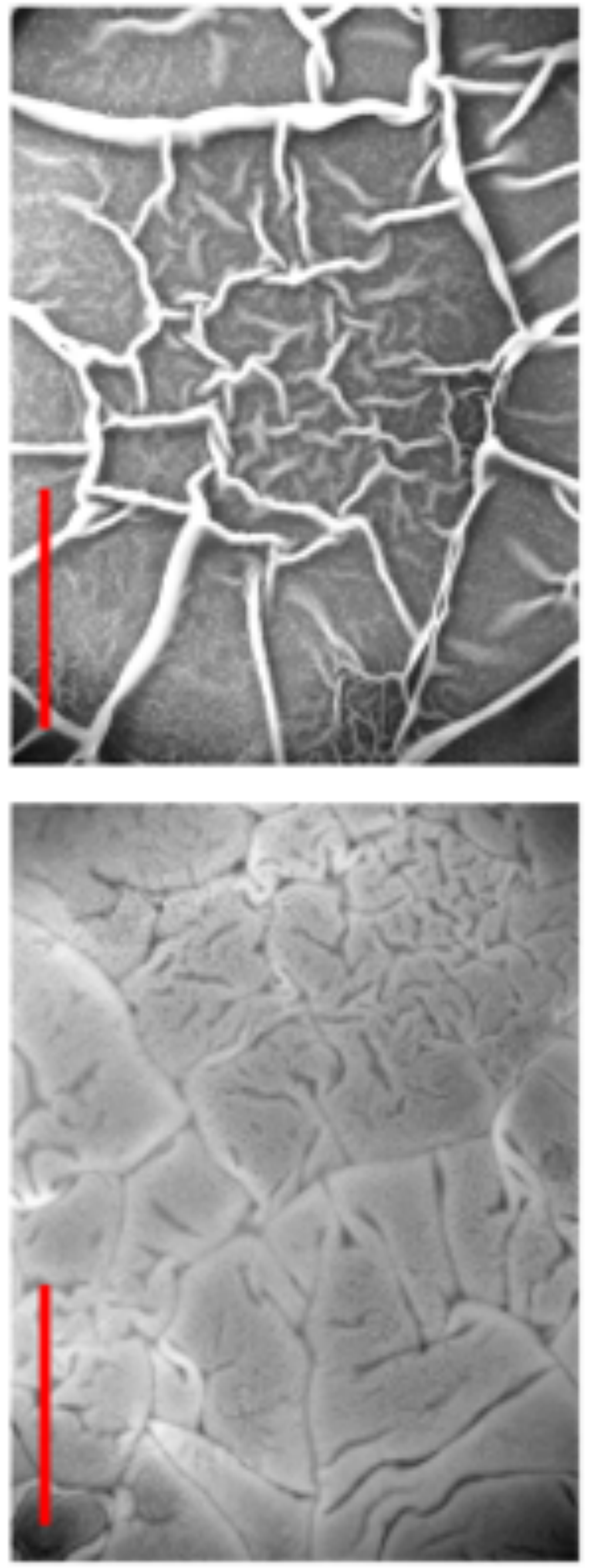
Thin coatings



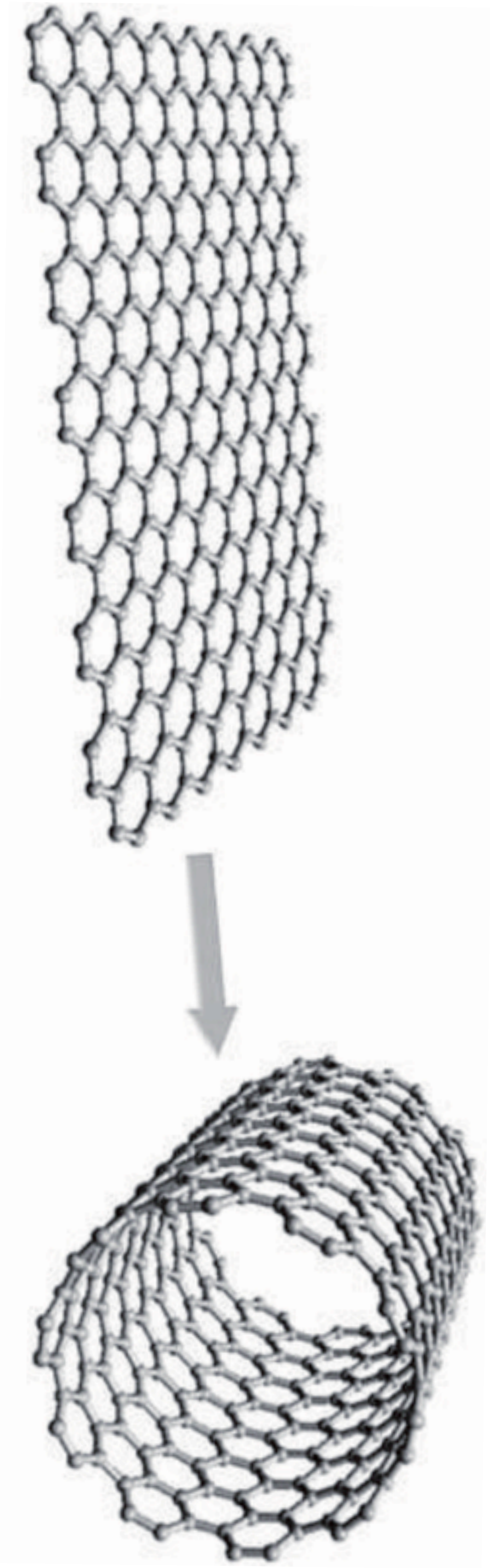
Cancer spheroids membranes



Bacterial biofilms



Carbon nanotubes



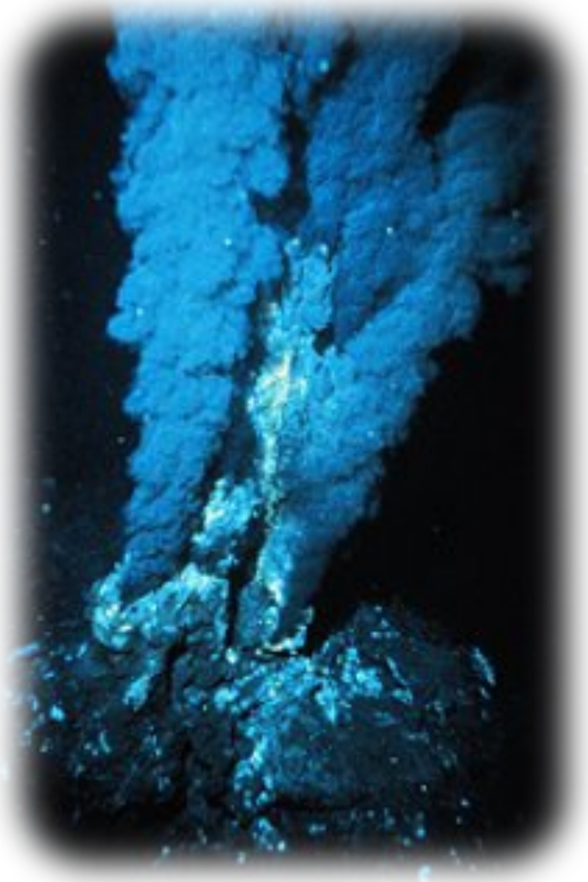
Fabrication of slender elastic shells by the coating of curved surfaces. Lee, A., Brun, PT., Marthelot, J. et al. Nat Commun 7, 11155 (2016).

Nonlinear elasticity of biological basement membrane revealed by rapid inflation and deflation. Hui Li, Yue Zheng, Yu Long Han, Shengqiang Cai, Ming Guo
Proceedings of the National Academy of Sciences Mar 2021, 118 (11) e2022422118; DOI: 10.1073/pnas.2022422118

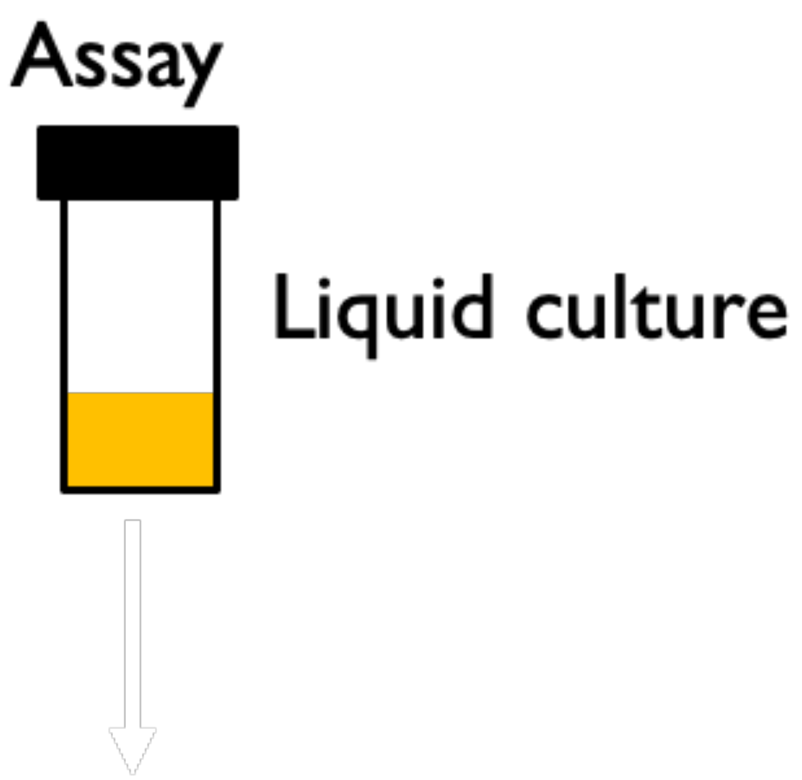
Wrinkle patterns in active viscoelastic thin sheets. D. A. Matoz-Fernandez, Fordyce A. Davidson, Nicola R. Stanley-Wall, and Rastko Sknepnek. Phys. Rev. Research 2, 013165 – Published 18 February 2020

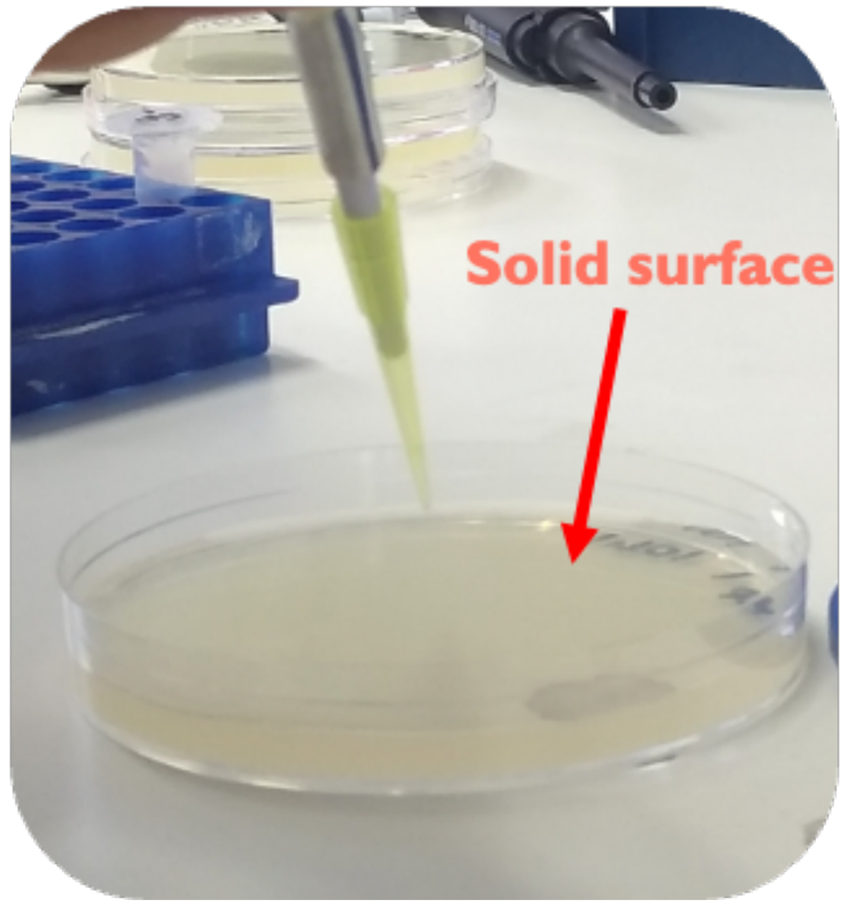
Eleonora Secchi et al. Massive radius-dependent flow slippage in carbon nanotubes, *Nature* (2016)

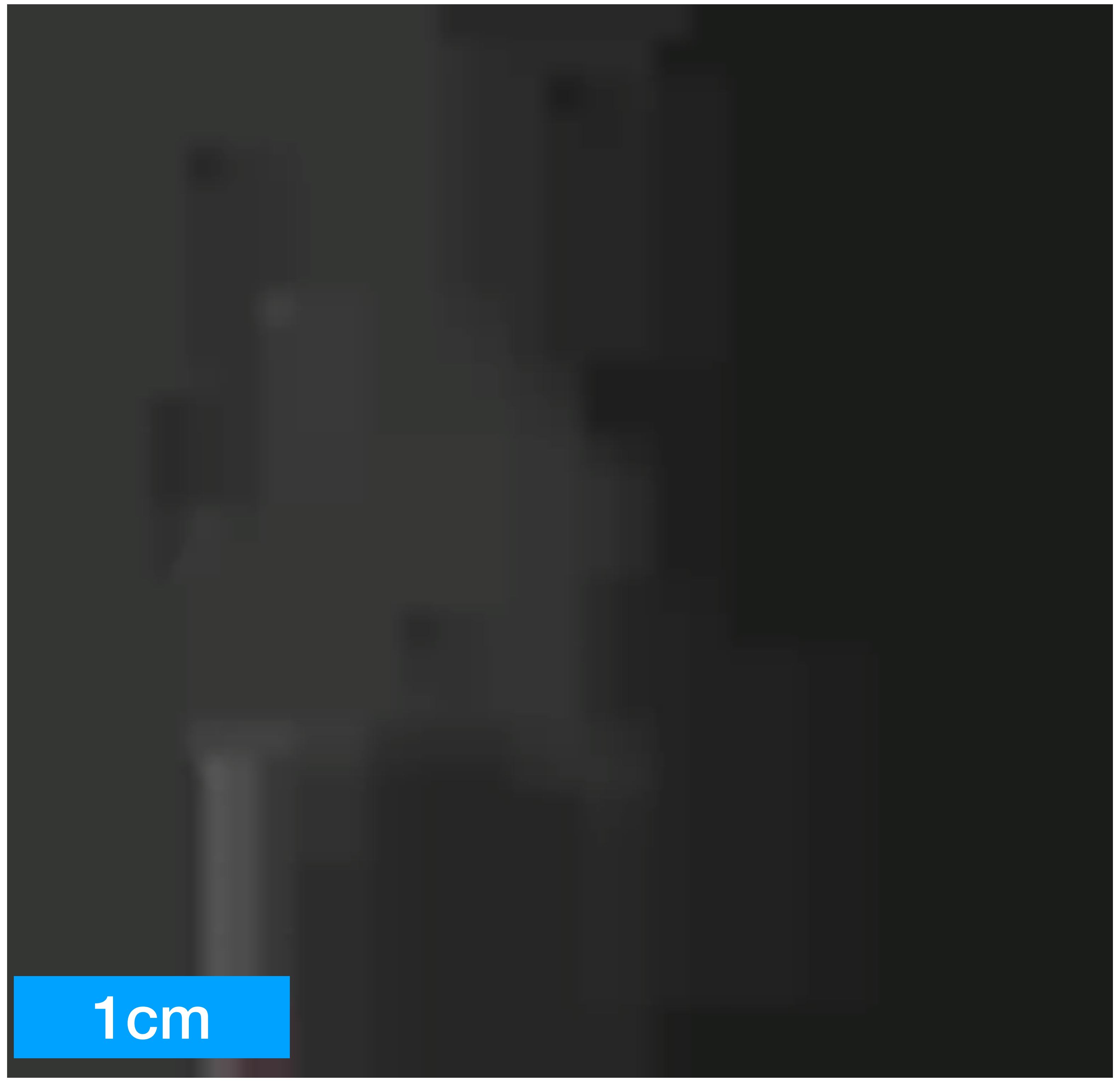
Bacterial Biofilms



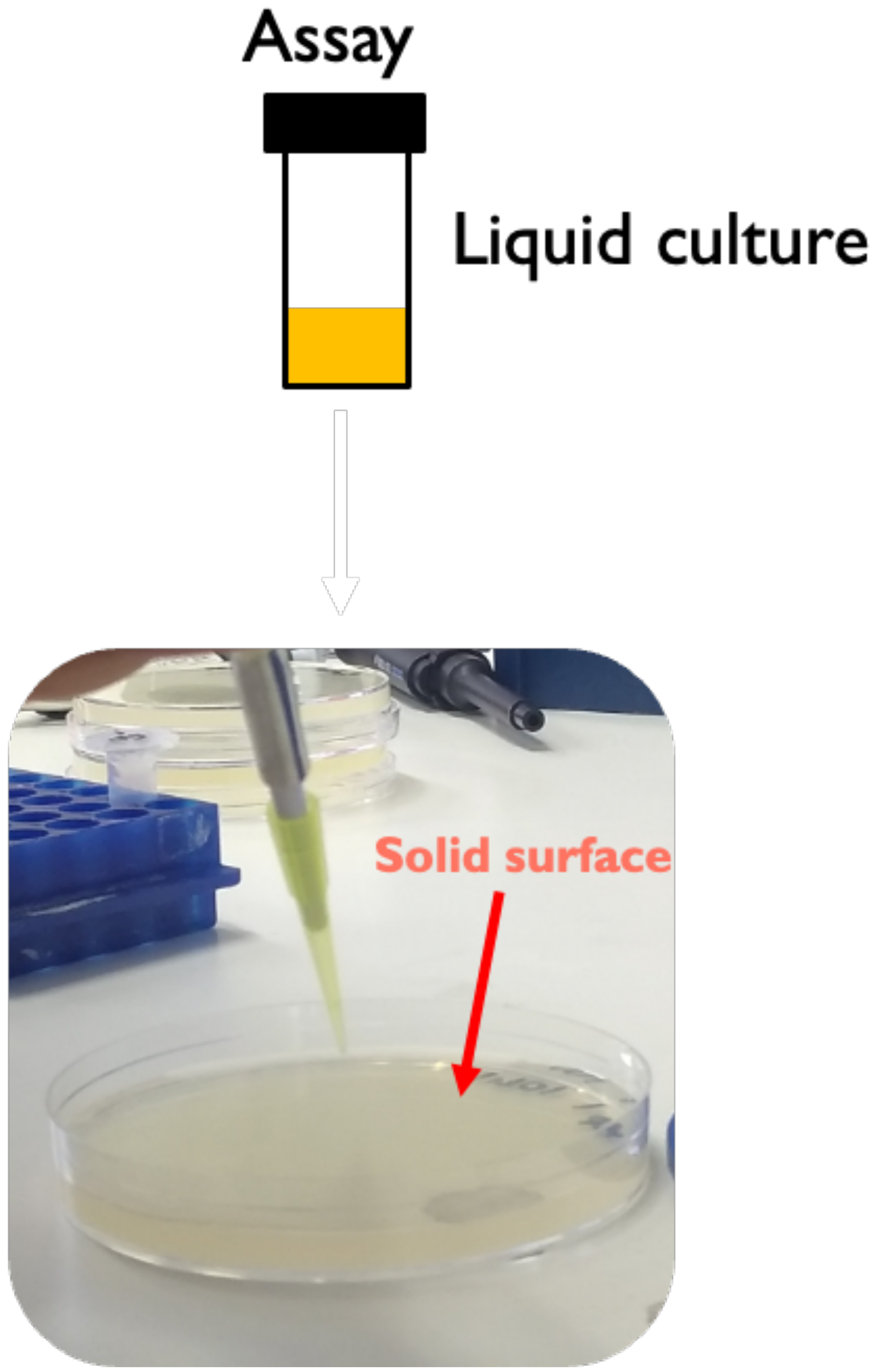
Bacterial Biofilms

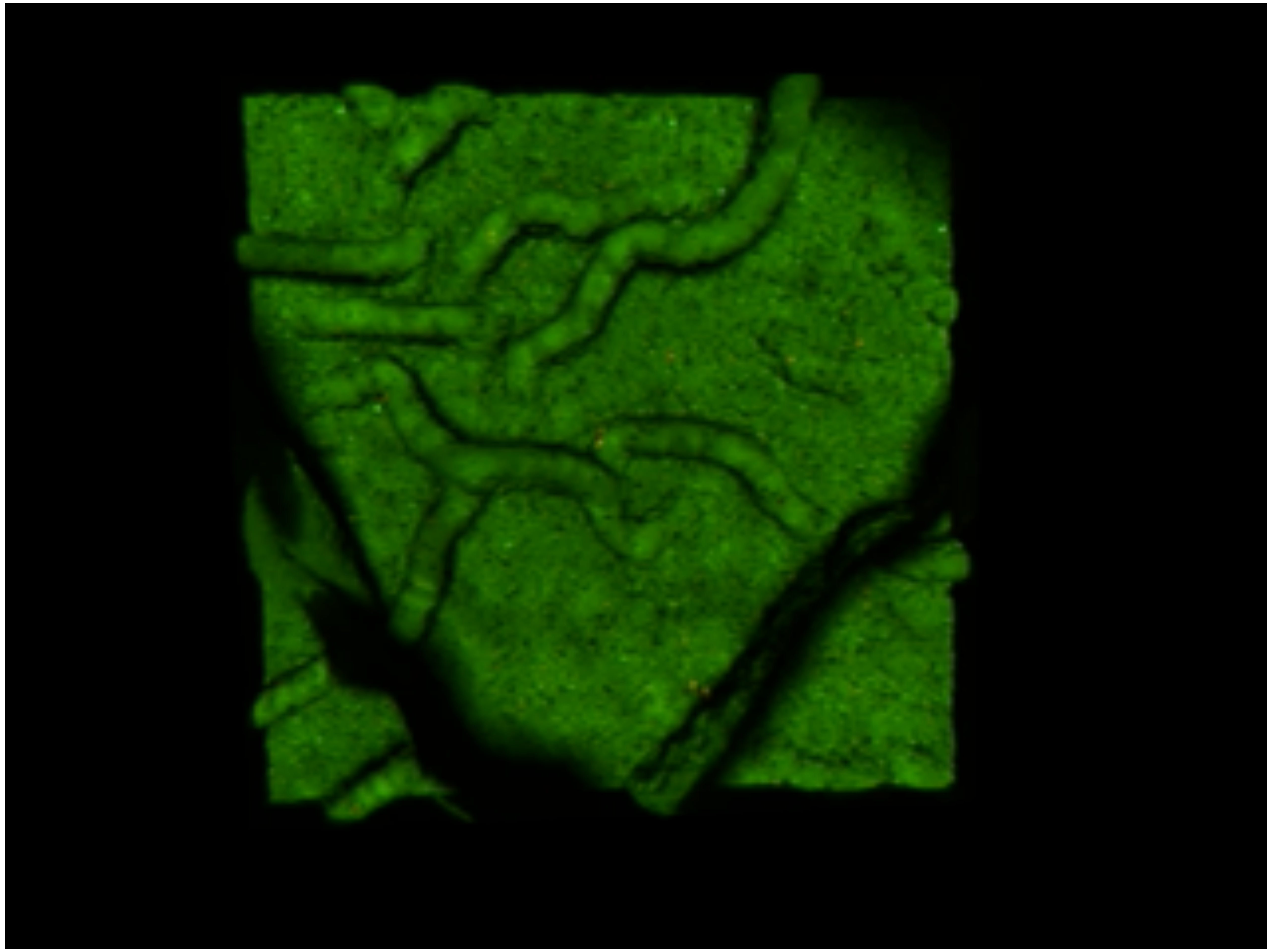






Bacterial Biofilms

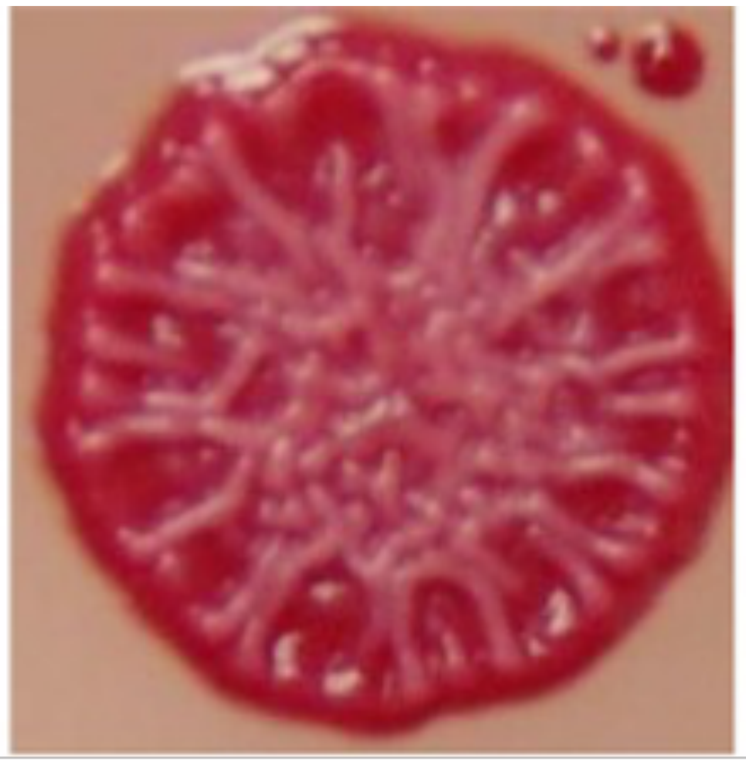




Confocal microscopy. Michael Porter, NSW Lab, Dundee

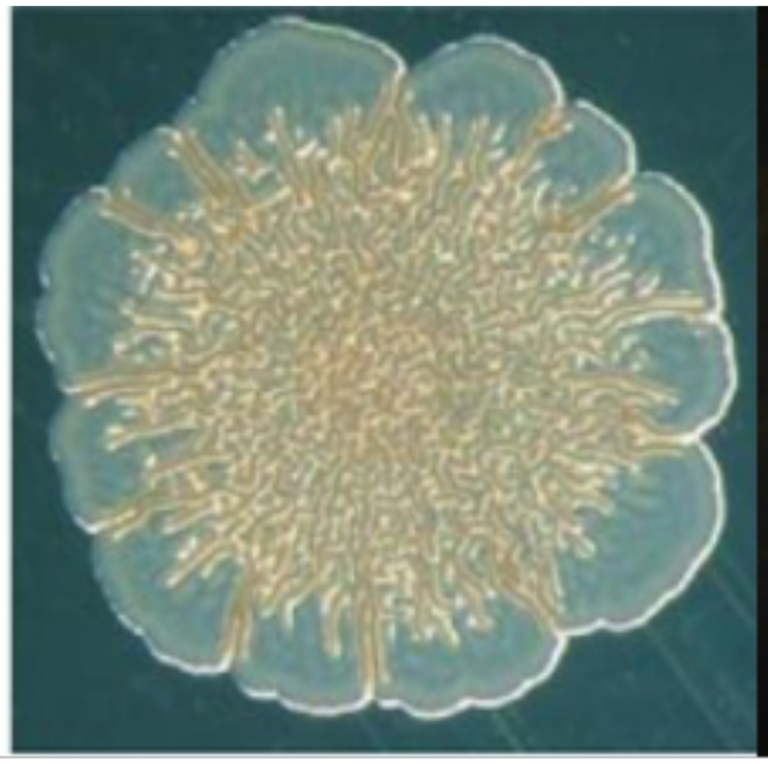
Wrinkles are a common feature

B. subtilis

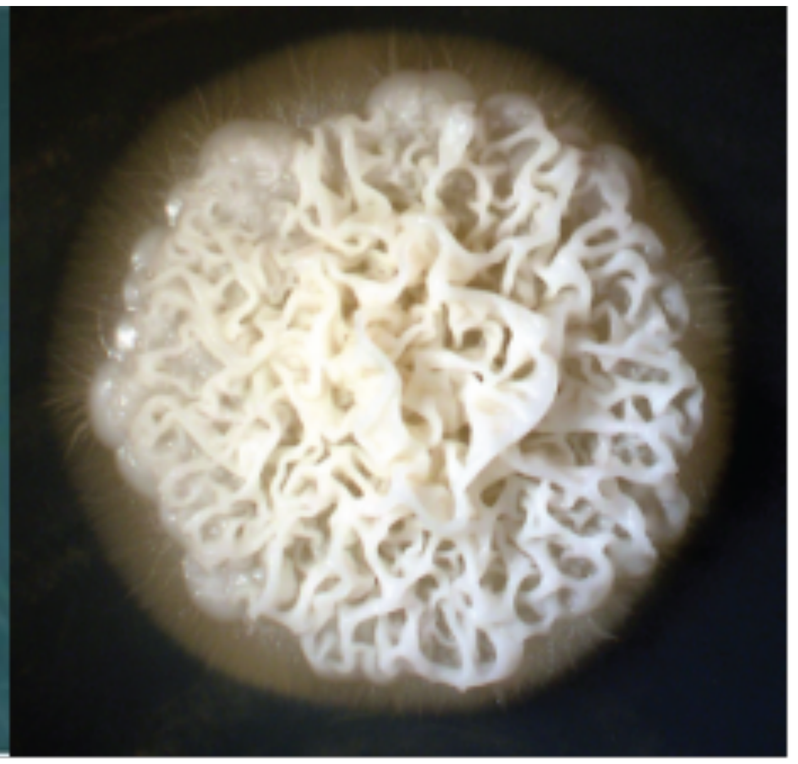


Agrobacterium tumefaciens
Congo red

Vibrio fischeri

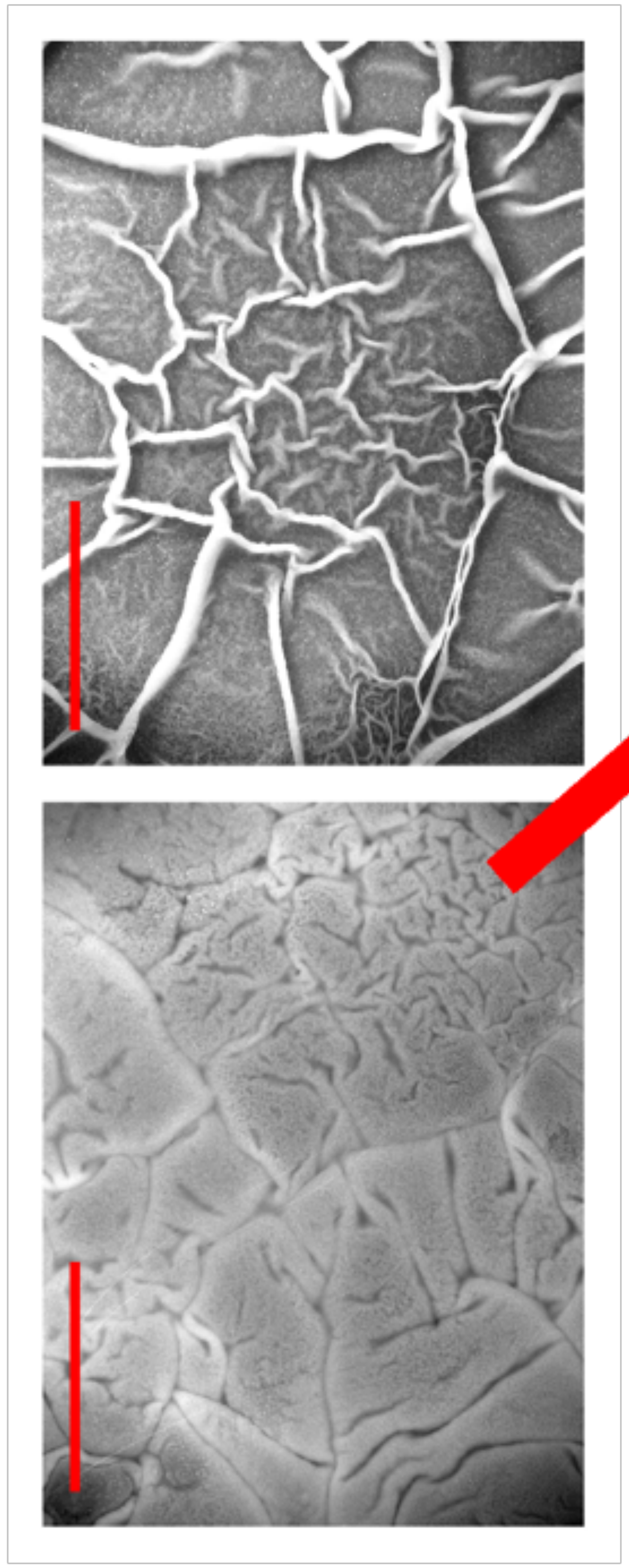


Pseudomonas fluorescens

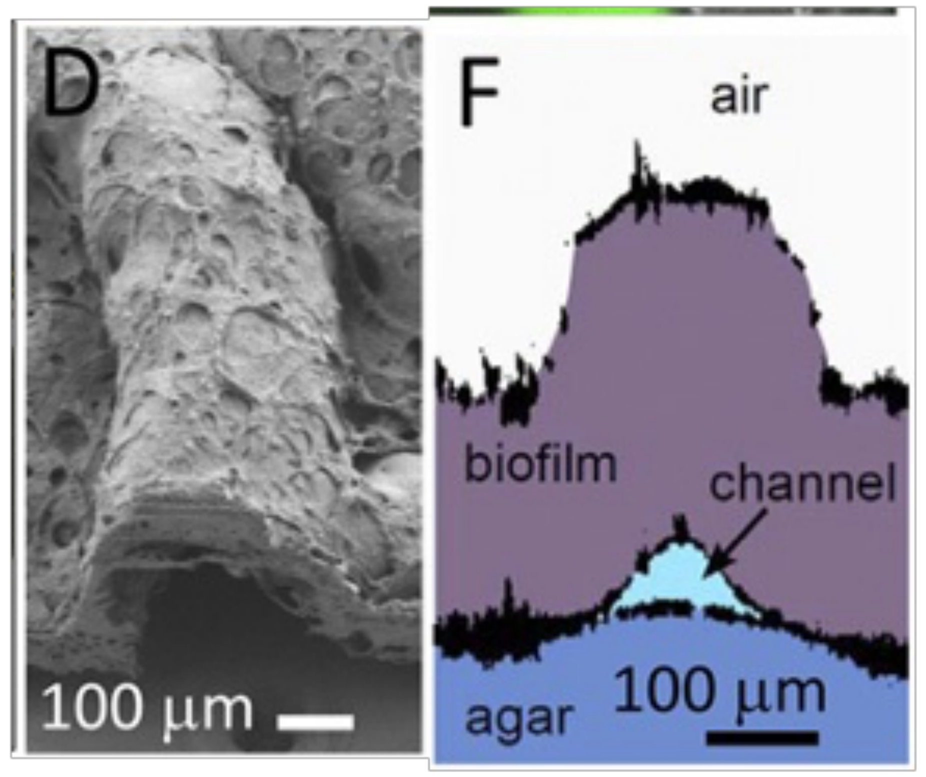


Candida albicans

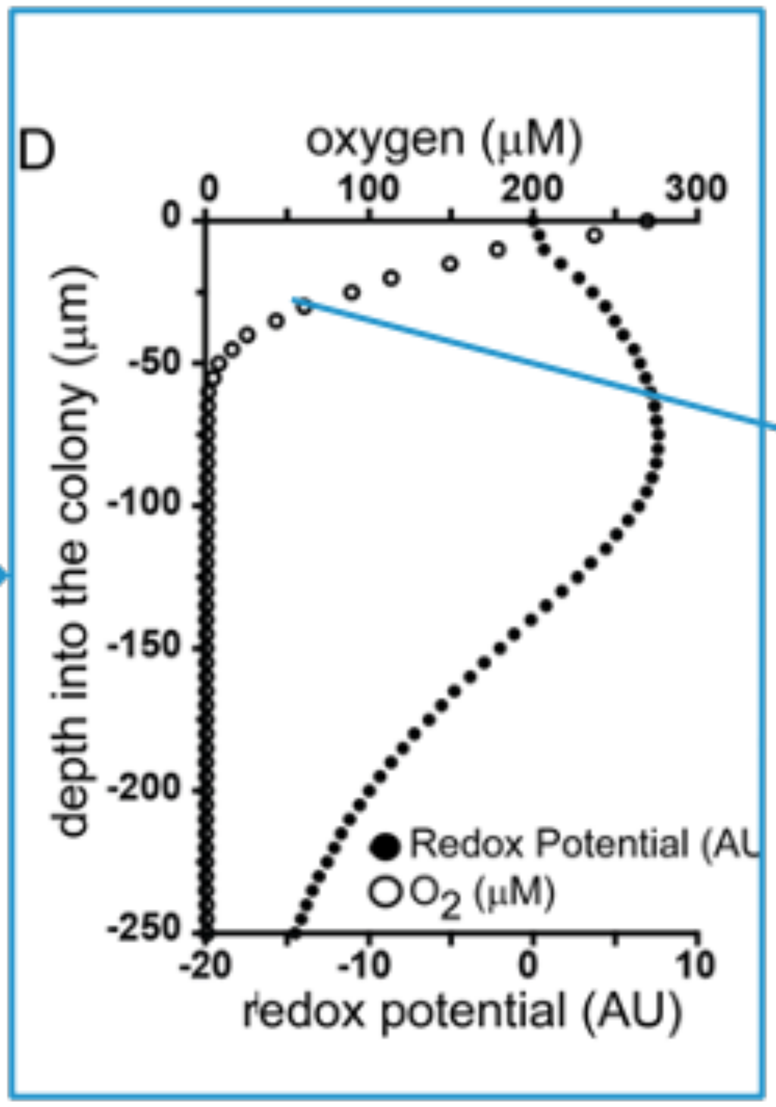
Why do wrinkles form?



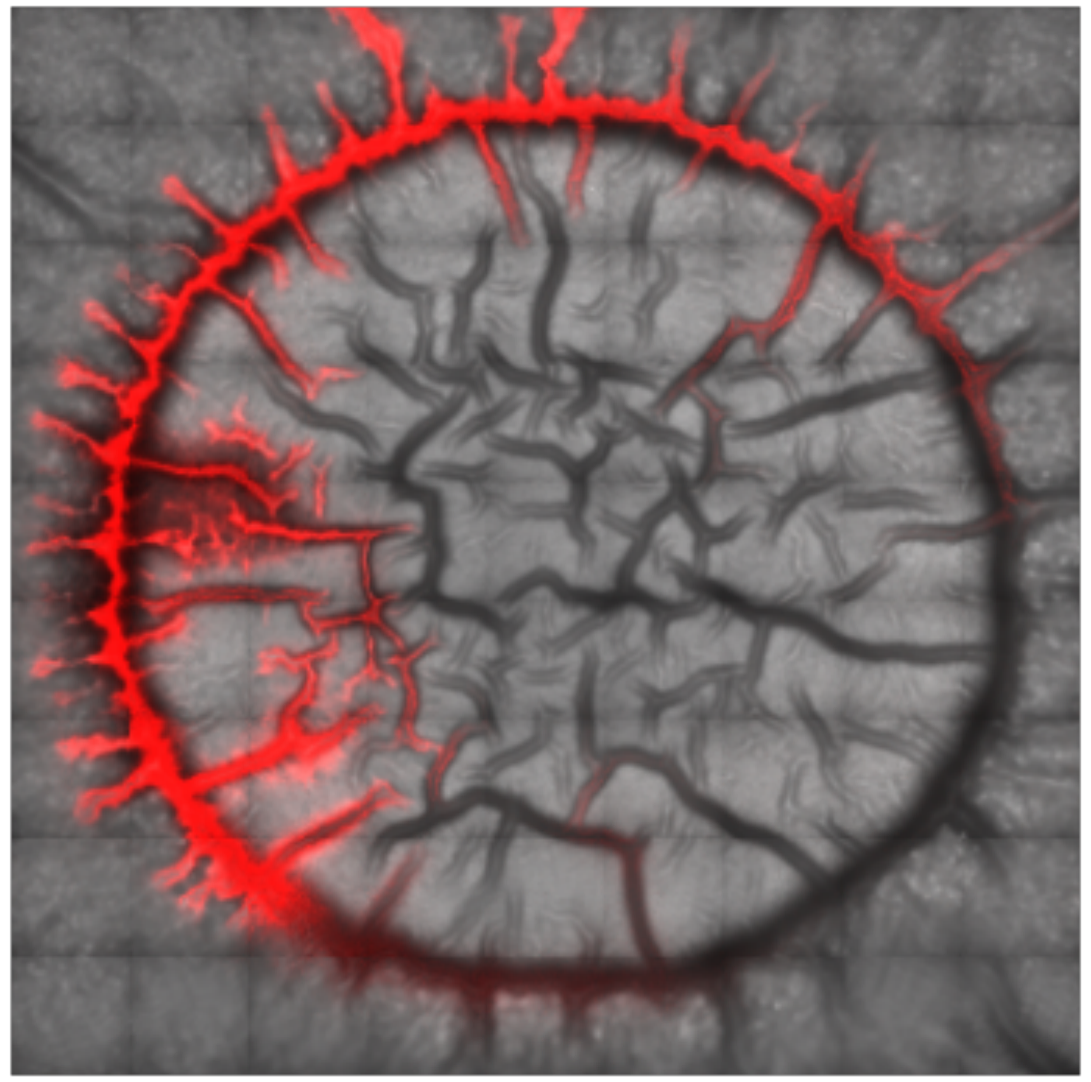
D.A Matoz-Fernandez et al. (2019)



James N. Wilking et al. PNAS 2013, 110 (3) 848-852.



Wrinkles help in creating oxygen gradient within the biofilm

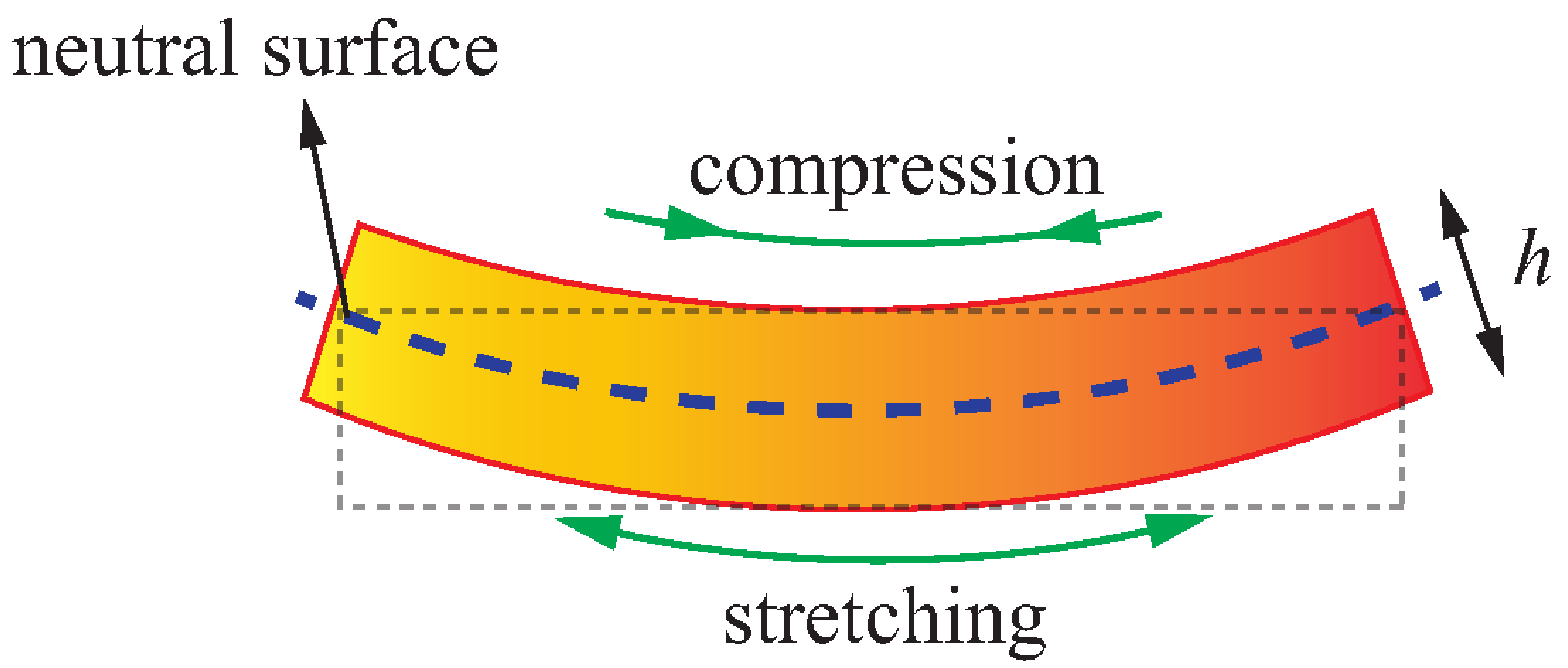


Thanks to Alan Prescott
Sofia Ferriera

Wrinkles can provide an enhanced system for nutrient and waste transport.

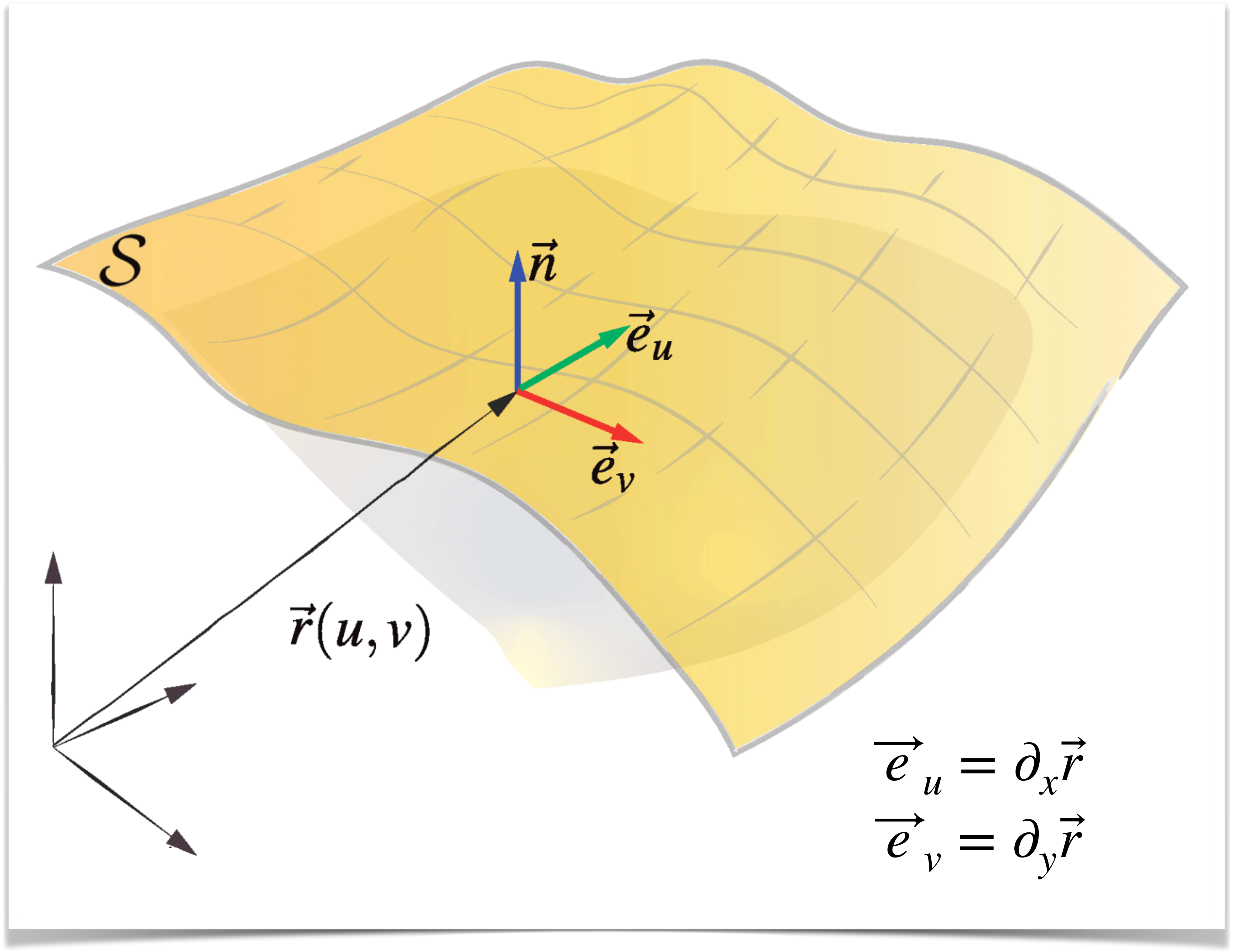
The physics of thin-membranes

Consider a piece of material



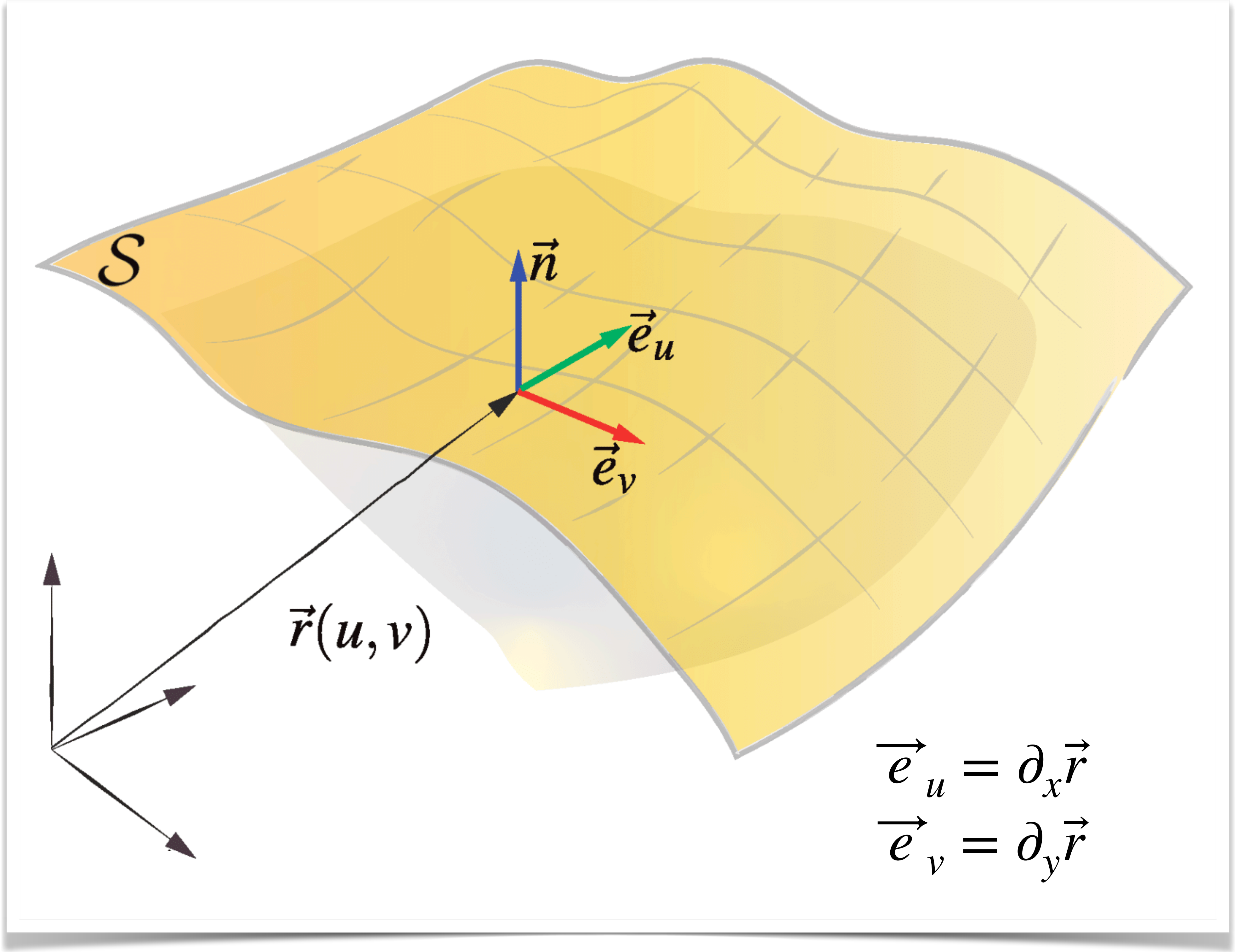
There is a surface which neither stretches nor compresses, the so-called neutral surface (blue dashed line) and is clearly exactly in the middle of the sheet.

Neutral surface representation



The physics of thin-membranes

Neutral surface representation



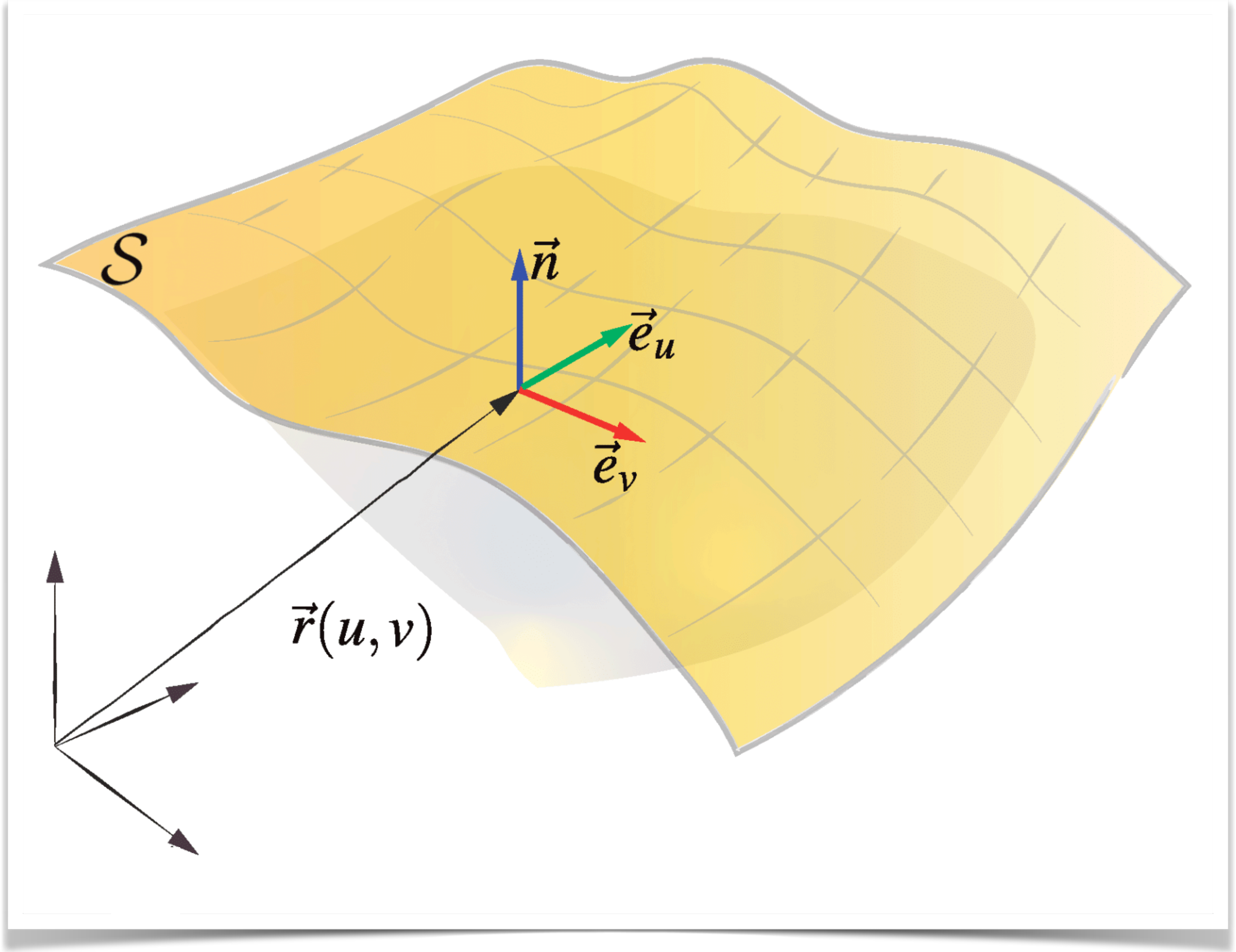
At each point of the surface, we can assign a metric of the surface

$$g_{ij} = \partial_i \vec{r} \cdot \partial_j \vec{r}$$

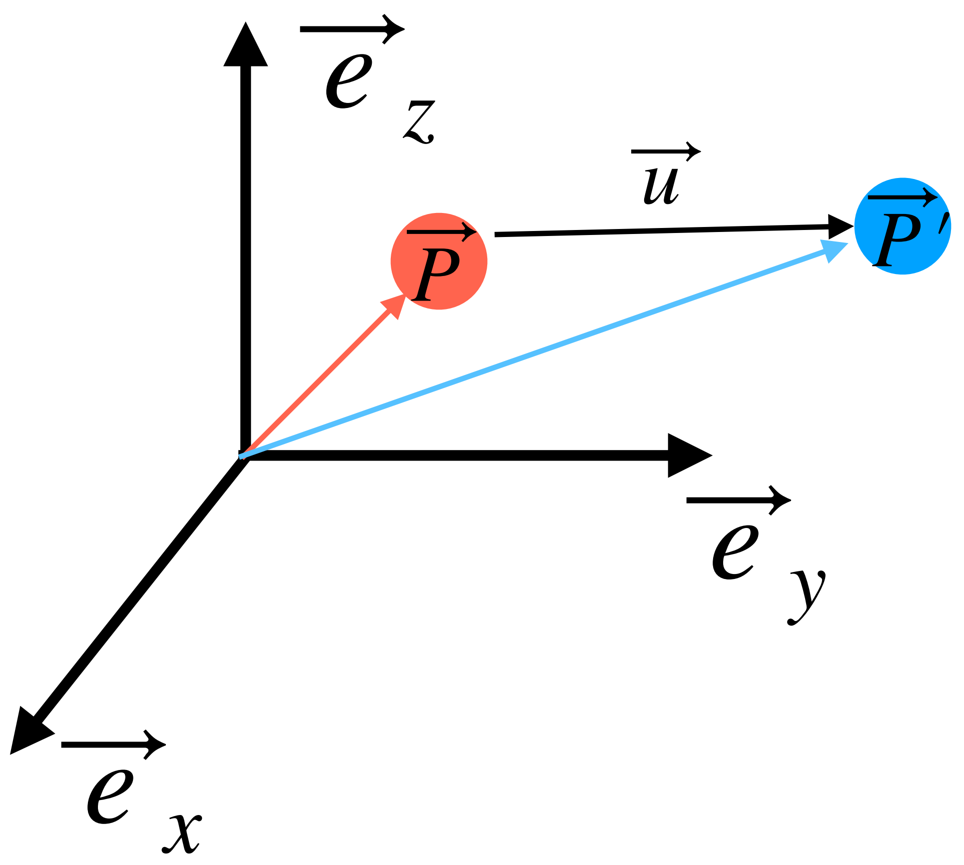
$$b_{ij} = -\,\partial_i \vec{r} \cdot \partial_j \vec{n}$$

$$i, j = \{x, y\}$$

The physics of thin-membranes



Displacement vector



$$\vec{P}' = \vec{P} + \vec{u}$$

Displacement vector and metric

Reference state

$$u_{ij} = \frac{1}{2}\left(g_{ij} - \bar{g}_{ij}\right)$$

Deformed state

The physics of thin-membranes

Elastic Energy

The elastic energy density depends on the strain tensor, i.e., on the metric, $E_{el} = E_{el}\left(g_{ij}\right)$.

If the strain is small, we can expand E_{el} *in powers of* u_{ij}

around the target configuration ($\bar{g}_{ij}$ *and* $u_{ij} = 0$*)*

$$E_{el} \approx E\left(\bar{g}_{ij}\right) + \left.\frac{\partial E}{\partial g_{ij}}\right|_{u_{ij}=0} u_{ij} + \boxed{\frac{1}{2}\left.\frac{\partial^2 E}{\partial g_{ij}\partial g_{kl}}\right|_{u_{ij}=0}} u_{ij}u_{kl} + o\left(u^3\right)$$

Vanish that only depends on the deformation gradient

Elastic Tensor

$$E_{el} - E\left(\bar{g}_{ij}\right) = \frac{1}{2}A^{ijkl}u_{ij}u_{kl} + o\left(u^3\right)$$

The physics of thin-membranes

Elastic Energy for a volumetric material

$$E_{tot} = \int_V \frac{1}{2} A^{ijkl} u_{ij} u_{kl} \, dV$$

$$A^{ijkl} = \lambda g^{ij} g^{kl} + \mu \left(g^{ik} g^{jl} + g^{il} g^{jk} \right)$$

$$E = \frac{\mu \left(3\lambda + 2\mu \right)}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2 \left(\lambda + \mu \right)}$$

Material properties

How we can go to a thin material?

The physics of thin-membranes

Elastic Energy for a volumetric material

$$E_{tot} = \int_V \frac{1}{2} A^{ijkl} u_{ij} u_{kl} \, dV$$

1. The body is in the state of plane-stress, i.e., stress normal to the surfaces parallel to the neutral surface can be neglected.

2. Points which lie on a normal to the neutral surface in the reference configuration remain on the same normal in the deformed configuration.

The physics of thin-membranes

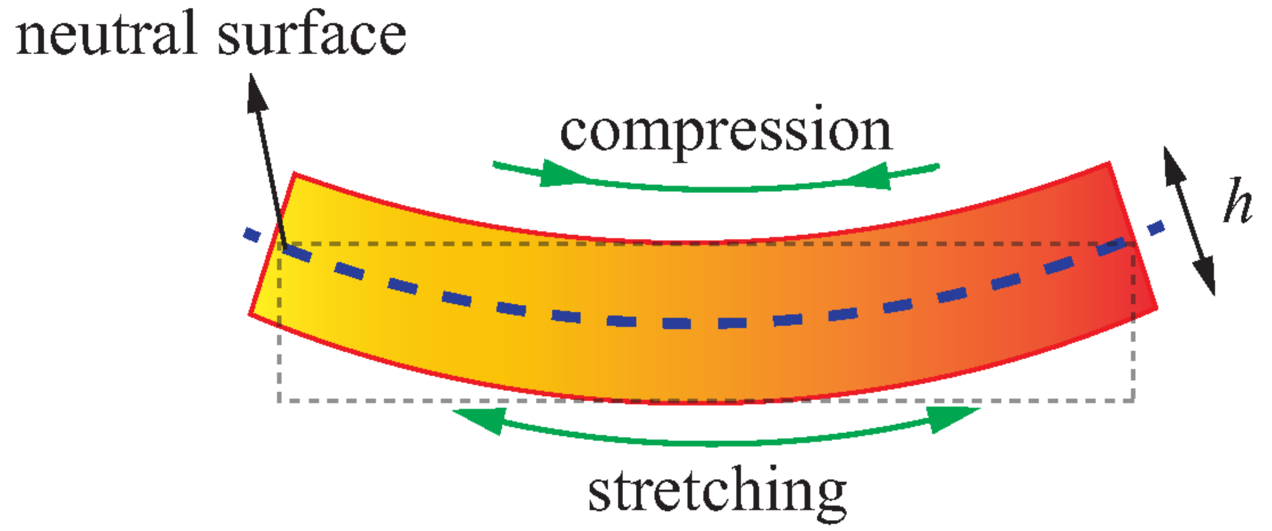
Elastic Energy for a volumetric material

$$E_{tot} = \int_V \frac{1}{2} A^{ijkl} u_{ij} u_{kl} \, dV$$

1. The body is in the state of plane-stress, i.e., stress normal to the surfaces parallel to neutral surface can be neglected.

$$\sigma^{i3} = 0 \quad i = x, y, z$$

2. Points which lie on a normal to the neutral surface in the reference configuration remain on the same normal in the deformed configuration.



$$u_3^3 = u_{33} = -\frac{\lambda}{\lambda + 2\mu} u_\alpha^\alpha$$

The physics of thin-membranes

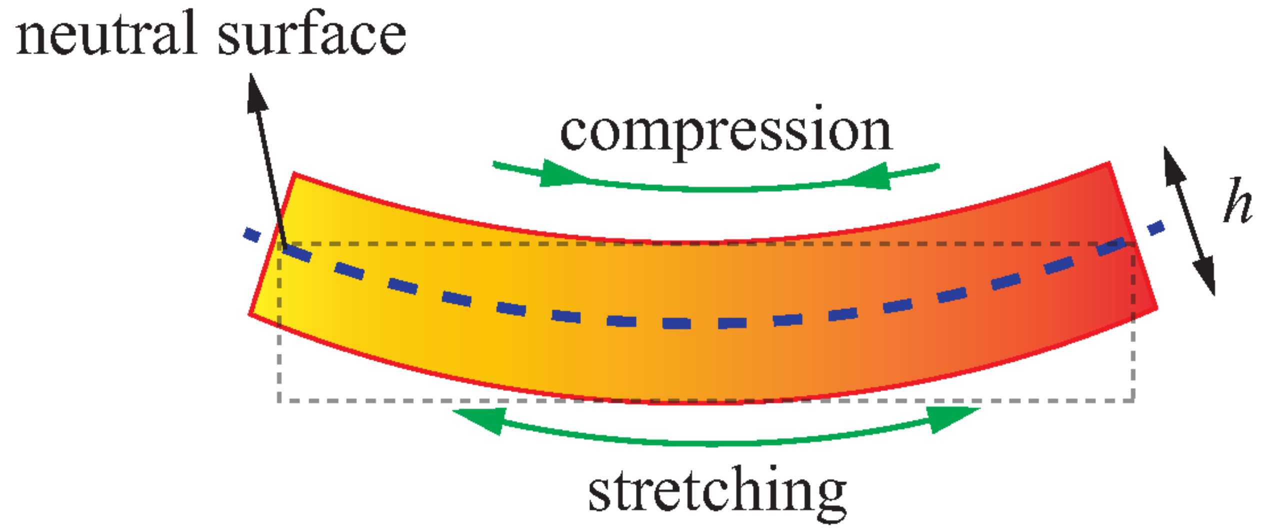
Elastic Energy for a volumetric material

$$E_{tot} = \int_V \frac{1}{2} A^{ijkl} u_{ij} u_{kl} \, dV$$

1. The body is in the state of plane-stress, i.e., stress normal to the surfaces parallel to neutral surface can be neglected.

$$\sigma^{i3} = 0 \quad i = x, y, z$$

2. Points which lie on a normal to the neutral surface in the reference configuration remain on the same normal in the deformed configuration.



$$u_3^3 = u_{33} = -\frac{\lambda}{\lambda + 2\mu} u_\alpha^\alpha$$

The physics of thin-membranes

Elastic Energy for a volumetric material

$$E_{tot} = \int_V \frac{1}{2} A^{ijkl} u_{ij} u_{kl} \, dV$$

$$E^{2D}_{tot} = \frac{1}{2} \int_S \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \sqrt{\left| g(z) \right|} A^{\alpha\beta\gamma\delta} u_{\alpha\beta}(z) \, u_{\gamma\delta}(z)$$

$$E^{2D}_{tot} = \int_S \sqrt{\left| g \right|} A^{\alpha\beta\gamma\delta} \left(\frac{h}{2} u_{\alpha\beta} u_{\gamma\delta} + \frac{h^3}{24} b_{\alpha\beta} b_{\gamma\delta} \right)$$

The physics of thin-membranes

Elastic Energy for a thin material

$$E_{tot}^{2D} = \int_S \sqrt{|g|} A^{\alpha\beta\gamma\delta} \left(\underbrace{\frac{h}{2} u_{\alpha\beta} u_{\gamma\delta}}_{\text{Streaching}} + \underbrace{\frac{h^3}{24} b_{\alpha\beta} b_{\gamma\delta}}_{\text{Bending}} \right)$$

The physics of thin-membranes

Elastic Energy for a thin material

$$E_{tot}^{2D} = \int_S \sqrt{|g|}\, A^{\alpha\beta\gamma\delta} \left(\frac{h}{2} u_{\alpha\beta} u_{\gamma\delta} + \frac{h^3}{24} b_{\alpha\beta} b_{\gamma\delta} \right)$$

Streaching

Bending

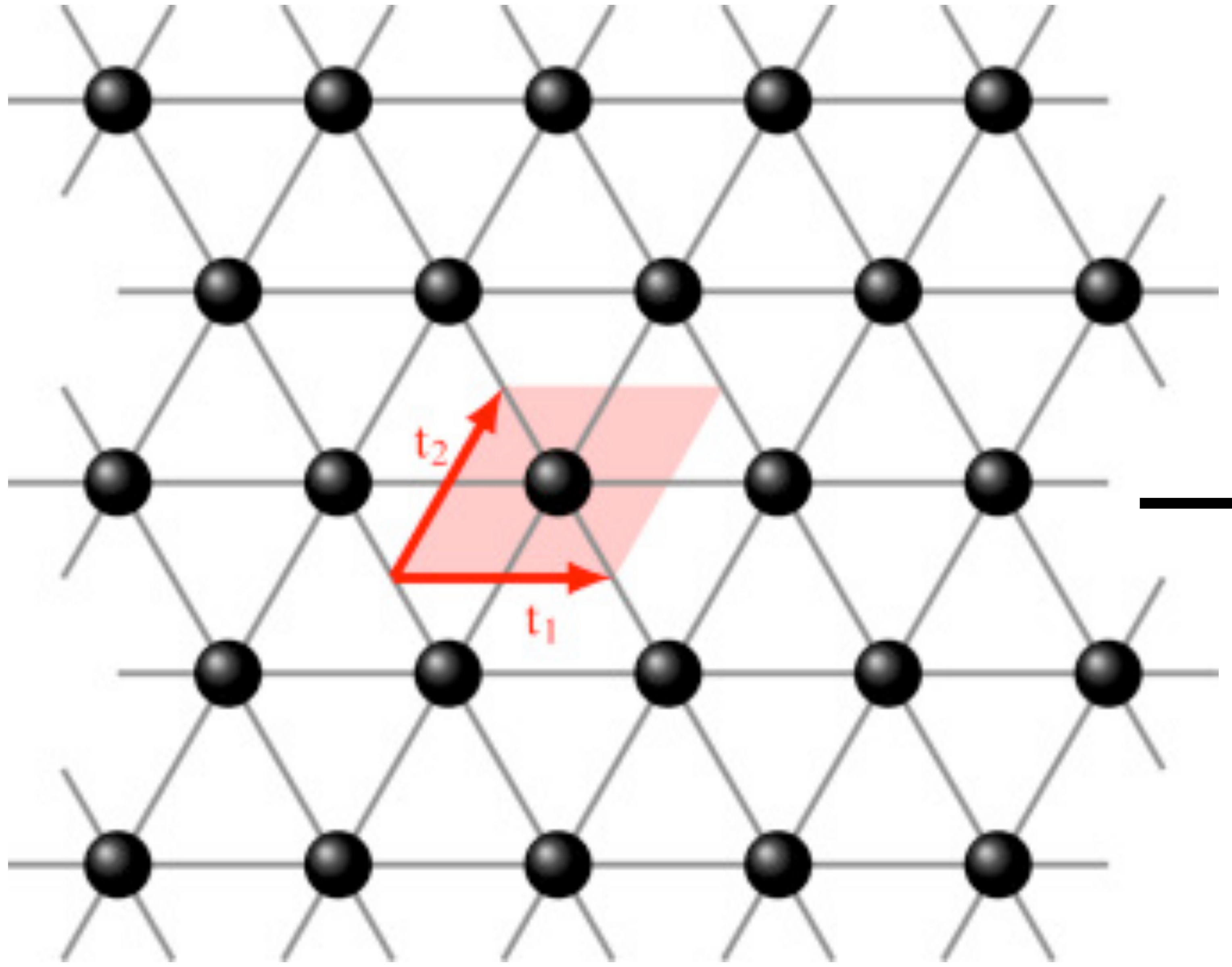
How can we solve this?

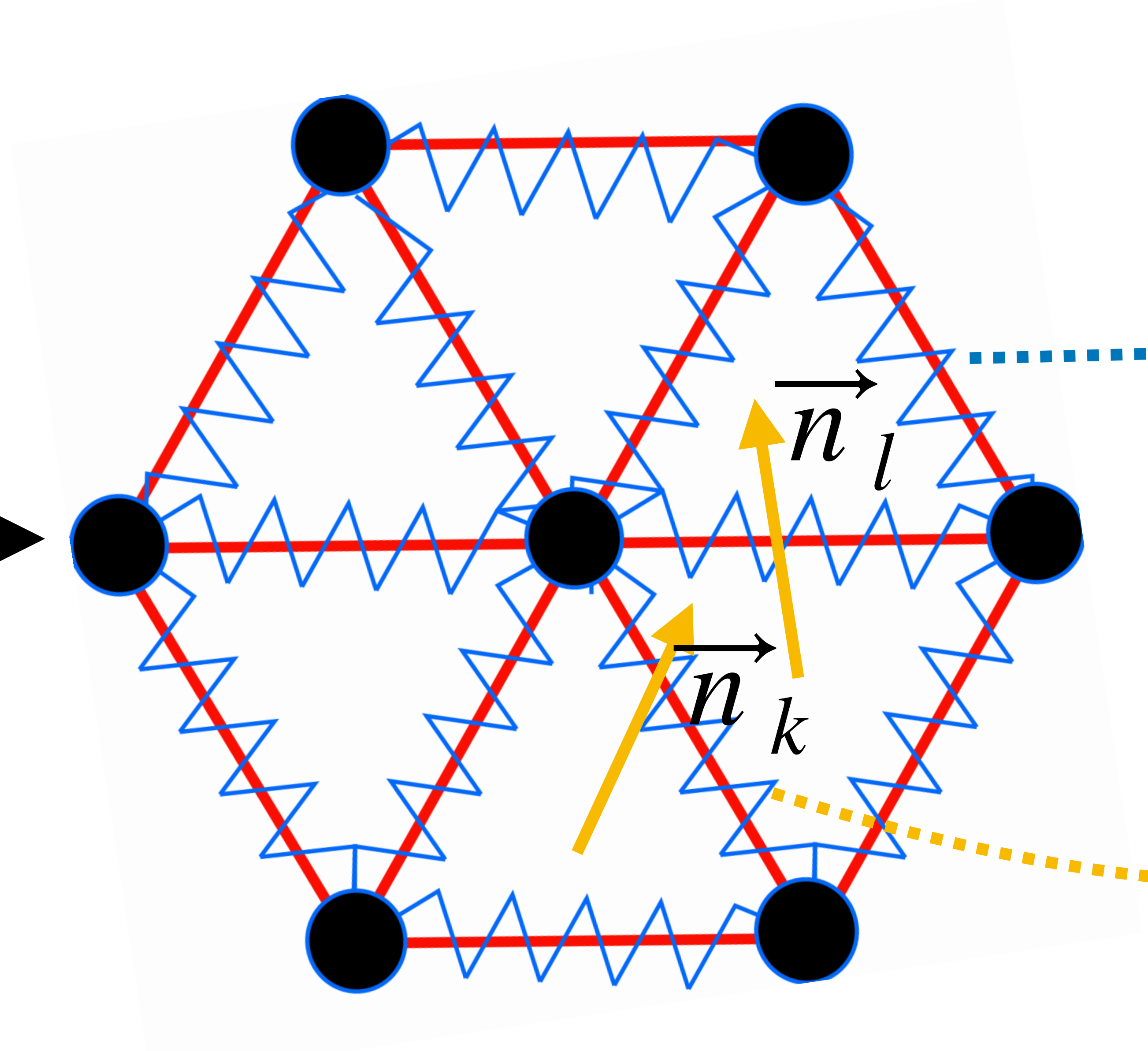
Elastic Energy for a thin material

How can we solve this?

$$E^{2D}_{tot} = \int_S \sqrt{|g|}\, A^{\alpha\beta\gamma\delta}\left(\frac{h}{2}u_{\alpha\beta}u_{\gamma\delta} + \frac{h^3}{24}b_{\alpha\beta}b_{\gamma\delta}\right)$$

Streaching **Bending**



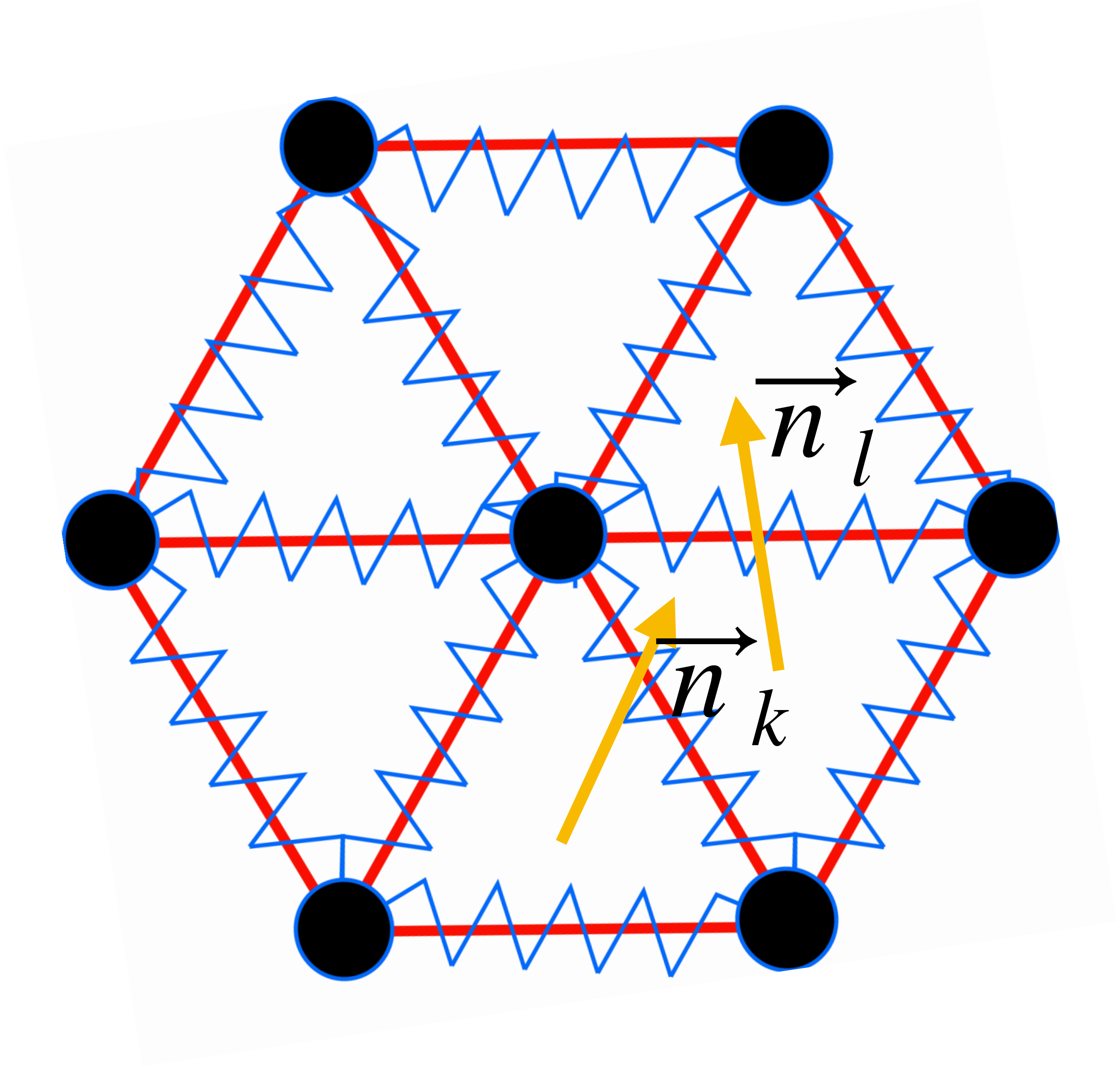


$$E_{Streatching} = \frac{1}{2}k\sum_{edges}(l - l_0)^2$$

$$E_{Bending} = k_B\sum_{edges}(1 - \vec{n}_k \cdot \vec{n}_l)$$

Elastic Energy for a thin material

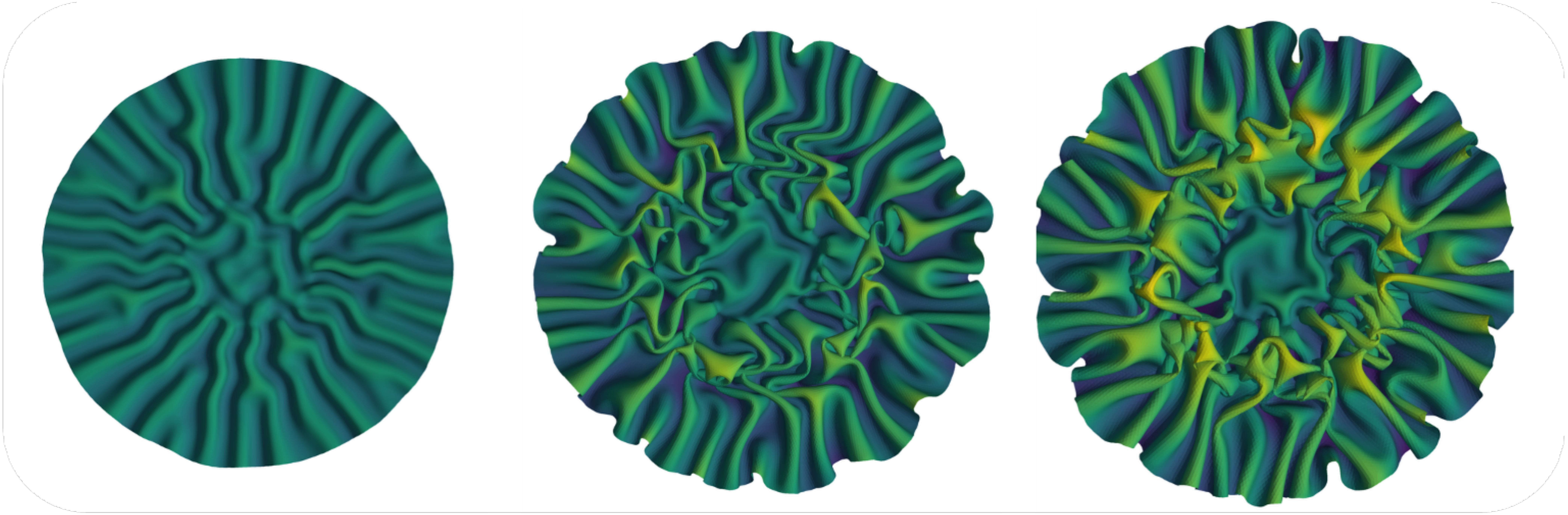
How can we solve this?



$$E_{tot}^{2D} = \sum_{edges} \frac{1}{2}k(l - l_0)^2 + k_B(1 - \vec{n}_k \cdot \vec{n}_l)$$

Streaching **Bending**

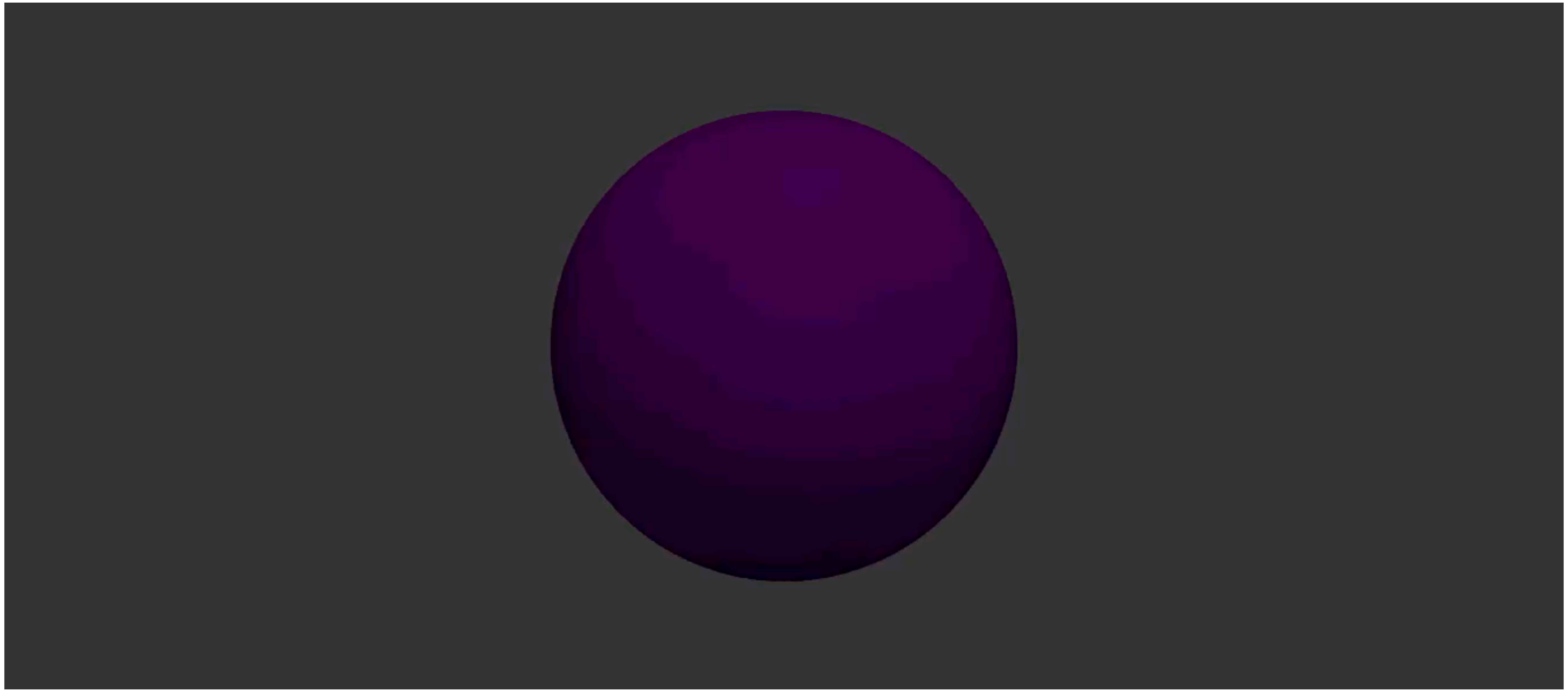
Thins sheet: Wrinkling and growth



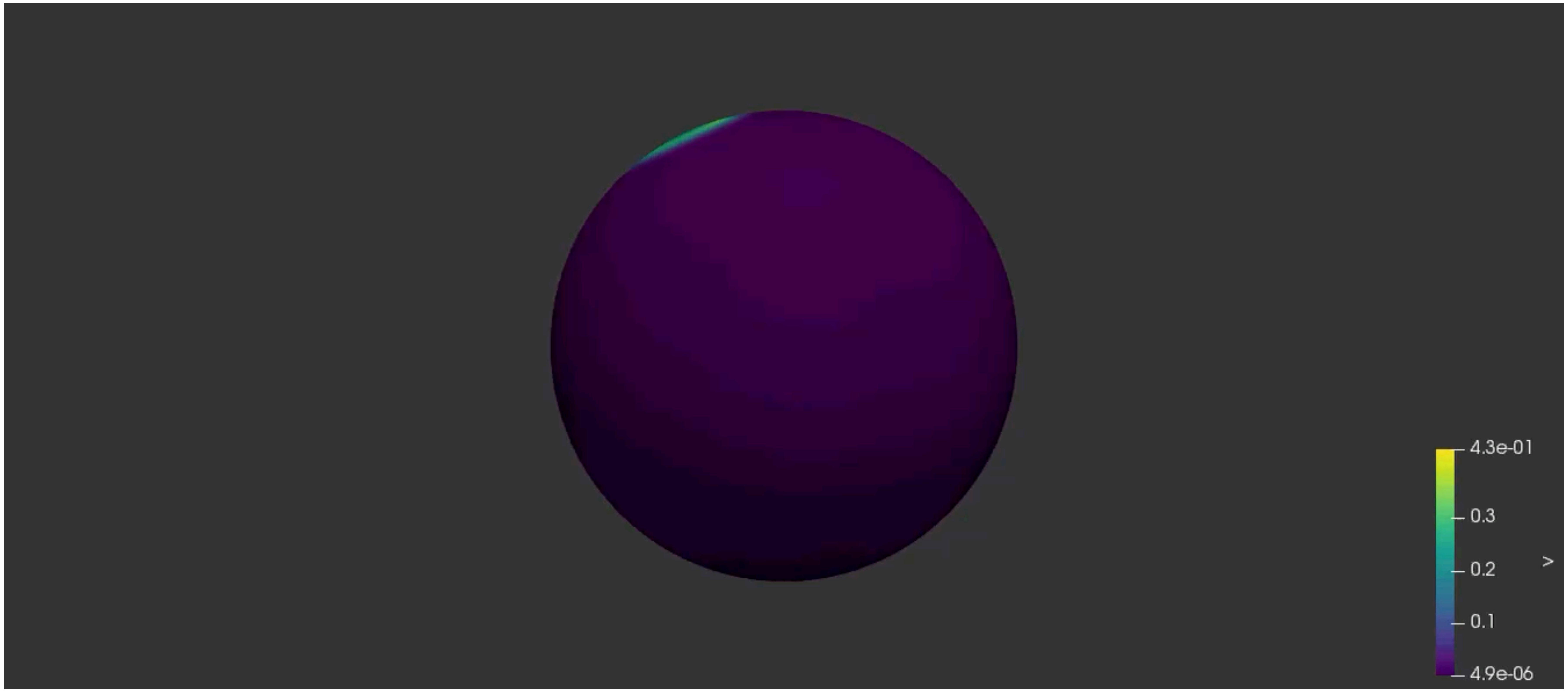
Growth rate

Daniel Matoz, et al. Physica Review Research 2020

Thins sheet: Wrinkling and growth



Thins sheet: Wrinkling and growth



Daniel Matoz, et al. 2022