FOGLIO 9

ESERCIZIO 1 Considera

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 3 & 0 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 1 & 0 & -1 \\ 2 & 2 & -1 & -1 \\ 3 & 1 & -5 & 0 \end{bmatrix}$$

ⓐ Calcola il determinante con Laplace e Gauss

ⓑ Calcola $\det(A^{-1})$ e $\det(B^{-1})$ se possibile

Con Laplace: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$

$\Rightarrow$ Per A scegliamo la seconda colonna (con più zeri)

$$\det(A) = (-1)^{2+2} \cdot 1 \cdot \det \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 3 & 1 & -2 \end{bmatrix} + (-1)^{2+3} \cdot 1 \cdot \det \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$= 1 \cdot 1 \cdot \Big((-1 \cdot 2 \cdot -2) + 0 + (1 \cdot 3 \cdot -1) - \big((3 \cdot 2 \cdot 2) + (-1 \cdot -1 \cdot 1) + 0\big)\Big) +$$

$$-1 \cdot 1 \cdot \Big((-1 \cdot 3 \cdot -2) + (2 \cdot 2 \cdot 1) + (3 \cdot 1 \cdot -1) - \big((3 \cdot 3 \cdot 2) + (-1 \cdot -1 \cdot 1) + (2 \cdot -2)\big)\Big)$$

$$= 1\big((4 + 0 - 3) - (12 + 1)\big) - 1\big((6 + 4 - 3) - (18 + 1 - 4)\big) = 1 - 13 - 7 + 15$$

$$= -12 + 8 = -4$$

Con Gauss

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 3 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[R_4 + 3R_1]{R_2 + 2R_1} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 + 4/3 R_4}$$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & -4/3 \end{bmatrix} \Rightarrow \det(A) = -1 \cdot -3 \cdot -\frac{4}{3} = -4$$

$\boxed{B}$ $$\det(B) = (-1)^{4+2} \cdot -1 \cdot \det\begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 1 & -5 \end{pmatrix} + (-1)^{4+3} \cdot (-1) \cdot \det\begin{pmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 3 & 1 & -5 \end{pmatrix} =$$

$$= -1 \left(\left(\left(1 \cdot 2 \cdot -5\right) + \left(2 \cdot 2 \cdot -1\right) + \left(-1 \cdot 3 \cdot -1\right)\right) - \left(\left(3 \cdot 2 \cdot 2\right) + \left(1 \cdot 1 \cdot -1\right) + \left(2 \cdot -5 \cdot -1\right)\right)\right) +$$

$$-1 \cdot -1 \cdot \left(\left(\left(1 \cdot 1 \cdot -5\right) + \left(-3 \cdot 2 \cdot 1\right) + \left(-1 \cdot 3 \cdot 0\right)\right) - \left(\left(3 \cdot 1 \cdot 2\right) + \left(1 \cdot 1 \cdot 0\right) + \left(-3 \cdot -5 \cdot -1\right)\right)\right)$$

$$= -\left(-10 + 4 + 3 - \left(12 - 1 + 10\right)\right) + \left(-5 - 6 + 0 - \left(6 + 0 - 15\right)\right) =$$

$$= -\left(-3 - 21\right) + \left(-11 + 9\right) = 24 - 2 = 22$$

Con Gauss:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 1 & 0 & -1 \\ 2 & 2 & -1 & -1 \\ 3 & 1 & -5 & 0 \end{bmatrix} \begin{matrix} \\ R_2+3R_1 \\ R_3-2R_1 \\ R_4-3R_1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & 6 & -1 \\ 0 & 4 & -5 & -1 \\ 0 & 4 & -11 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3+2R_2 \\ R_4+2R_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & 6 & -1 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} R_4 - \frac{1}{7}R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & 6 & -1 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 0 & -11/7 \end{bmatrix} \Rightarrow \det(B) = -2 \cdot 7 \cdot -\frac{11}{7} = 22$$

(b) $$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{-1}{4}$$

$$\det(B^{-1}) = \frac{1}{\det(B)} = \frac{1}{22}$$

ESERCIZIO 2

$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ Trova $\det(A)$, A^{-1}, AA^{-1}

$$\det(A) = (2 \cdot 1 \cdot 1) + (4 \cdot 1 \cdot 2) + (1 \cdot -2 \cdot 0) - ((-2 \cdot 1 \cdot 1) + (2 \cdot 2 \cdot 0) + (4 \cdot 1 \cdot 1))$$

$$= 2 + 8 - (-2 + 4) = 10 - 2 = 8$$

$$\det(A^{-1}) = \frac{1}{8}$$

Usiamo il metodo dell'aggiunta per calcolare A^{-1}

STEP 1: Troviamo la matrice dei cof. di A (formata dai determinanti dei minori dei vari elementi con segno $(-1)^{i+j}$)

Poi l'aggiunta di A è la trasposta della matrice dei cofattori e l'inversa è $A^{-1} = \det(A^{-1}) \cdot Adj(A)$.

$C_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$ $\quad C_{12} = -\begin{vmatrix} 4 & 0 \\ -2 & 1 \end{vmatrix} = -4$ $\quad C_{13} = \begin{vmatrix} 4 & 1 \\ -2 & 2 \end{vmatrix} = 10$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$ $\quad C_{22} = \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 4$ $\quad C_{23} = -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = -6$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$ $\quad C_{32} = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = 4$ $\quad C_{33} = \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = -2$

$Cof(A) = \begin{bmatrix} 1 & -4 & 10 \\ 1 & 4 & -6 \\ -1 & 4 & -2 \end{bmatrix}$ $\quad Adj(A) = Cof(A)^t = \begin{bmatrix} 1 & 1 & -1 \\ -4 & 4 & 4 \\ 10 & -6 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 \\ -4 & 4 & 4 \\ 10 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 1/8 & 1/8 & -1/8 \\ -1/2 & 1/2 & 1/2 \\ 5/4 & -3/4 & -1/4 \end{bmatrix}$$

Calcoliamo l'inversa con Gauss e verifichiamo coincida

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right] R_3+3R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 3 & 1 \end{array}\right] \xrightarrow[R_2-\frac{1}{2}R_3]{R_1+\frac{1}{4}R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -4 & -5 & 3 & 1 \end{array}\right] \xrightarrow{R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -4 & -5 & 3 & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ -R_2 \\ -4R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & +\frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

Verifichiamo che A^{-1} sia effettivamente l'inversa

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}-\frac{1}{2}+\frac{5}{4} & \frac{1}{4}+\frac{1}{2}-\frac{3}{4} & -\frac{1}{4}+\frac{1}{2}-\frac{1}{4} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{4}-1+\frac{5}{4} & -\frac{1}{4}+1-\frac{3}{4} & \frac{1}{4}+1-\frac{1}{4} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Il metodo di Gauss per l'inversa è più veloce, specialmente per matrici più grandi di 3×3 (con i minori ci sarebbero da calcolare 16 determinanti 3×3 per invertire una matrice 4×4...)

(3) Trova A^{-1}, B^{-1}, C^{-1}

$$A=\begin{bmatrix}1&2\\1&1\end{bmatrix}\qquad B=\begin{bmatrix}1&1&0\\1&-1&1\\2&1&2\end{bmatrix}\qquad C=\begin{bmatrix}2&-1&1\\-2&1&0\\1&0&2\end{bmatrix}$$

(A) $\det(A)=1-2=1\qquad \det(A^{-1})=-1$

$C_{11}=1\qquad C_{12}=-1$

$C_{21}=-2\qquad C_{22}=1$

$$\Rightarrow Cof(A)=\begin{bmatrix}1&-1\\-2&1\end{bmatrix}\Rightarrow Adj(A)=\begin{bmatrix}1&-2\\-1&1\end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$$

(B) $\det(B)=-2+2\cancel{4}(-2\cdot 0+1+2)=-3\qquad \det(B^{-1})=-\frac{1}{3}$

$B_{11}=\begin{pmatrix}-1&1\\1&2\end{pmatrix}=-3\qquad B_{12}=-\begin{vmatrix}1&1\\2&2\end{vmatrix}=0\qquad B_{13}=\begin{vmatrix}1&-1\\2&1\end{vmatrix}=3$

$B_{21}=-\begin{vmatrix}1&0\\1&2\end{vmatrix}=-2\qquad B_{22}=\begin{vmatrix}1&0\\2&2\end{vmatrix}=2\qquad B_{23}=-\begin{vmatrix}1&1\\2&1\end{vmatrix}=1$

$B_{31}=\begin{vmatrix}1&0\\-1&1\end{vmatrix}=1\qquad B_{32}=-\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1\qquad B_{33}=\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-2$

$$Cof(B)=\begin{bmatrix}-3&0&3\\-2&2&1\\1&-1&-2\end{bmatrix}\qquad Adj(B)=\begin{bmatrix}-3&-2&+1\\0&2&-1\\3&1&-2\end{bmatrix}$$

$$B^{-1}=\begin{bmatrix}1&2/3&-1/3\\0&-2/3&1/3\\-1&-1/3&2/3\end{bmatrix}$$

5. $\begin{bmatrix}2&-1&1\\-2&1&0\\1&0&2\end{bmatrix} \begin{matrix}R_2+R_1\\R_3-\frac{1}{2}R_1\end{matrix} \begin{bmatrix}2&-1&1\\0&0&1\\0&\frac{1}{2}&\frac{3}{2}\end{bmatrix} \to \begin{bmatrix}2&-1&1\\0&\frac{1}{2}&\frac{3}{2}\\0&0&1\end{bmatrix}$

$$\det(C) = -1\cdot\left(2\cdot\tfrac{1}{2}\cdot 1\right) = -1$$

$$\left[\begin{array}{ccc|ccc}2&-1&1&1&0&0\\-2&1&0&0&1&0\\1&0&2&0&0&1\end{array}\right] \xrightarrow{} \begin{matrix}R_2+R_1\\R_3-\frac{1}{2}R_1\end{matrix} \left[\begin{array}{ccc|ccc}2&-1&1&1&0&0\\0&0&1&1&1&0\\0&\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}2&-1&1&1&0&0\\0&\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}&0&1\\0&0&1&1&1&0\end{array}\right] \begin{matrix}R_1-R_3\\R_2-\frac{3}{2}R_3\end{matrix} \longrightarrow \left[\begin{array}{ccc|ccc}2&-1&0&0&-1&0\\0&\frac{1}{2}&0&-2&-\frac{3}{2}&1\\0&0&1&1&1&0\end{array}\right] R_1+2R_2 \longrightarrow$$

$$\left[\begin{array}{ccc|ccc}2&0&0&-4&-4&2\\0&\frac{1}{2}&0&-2&-\frac{3}{2}&1\\0&0&1&1&1&0\end{array}\right] \begin{matrix}\frac{1}{2}R_1\\2R_2\\R_3\end{matrix} \left[\begin{array}{ccc|ccc}1&0&0&-2&-2&1\\0&1&0&-4&-3&2\\0&0&1&1&1&0\end{array}\right]$$

$$\Rightarrow C^{-1} = \begin{bmatrix}-2&-2&1\\-4&-3&2\\1&1&0\end{bmatrix}$$

ESERCIZIO 4

Applica Cramer per risolvere il sistema lineare

$$\begin{cases} x+7y+3z=6\\ -x+2z=-7\\ 3x+y+z=2\end{cases} \qquad A=\begin{bmatrix}1&7&3\\-1&0&2\\3&1&1\end{bmatrix} \qquad b=\begin{bmatrix}6\\-7\\2\end{bmatrix}$$

$\det(A) = 0 - 3 + 42 + 0 - 2 + 7 = 44$

$$A_x = \begin{bmatrix} 6 & 7 & 3 \\ -7 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \Rightarrow 0 - 21 + 28 + 0 - 12 + 49 = 44$$

$$A_y = \begin{bmatrix} 1 & 6 & 3 \\ -1 & -7 & 2 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow -7 - 6 + 36 + 63 - 4 + 6 = 88$$

$$A_z = \begin{bmatrix} 1 & 7 & 6 \\ -1 & 0 & -7 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow 0 - 6 - 147 + 0 + 7 + 14 = -132$$

$$x = \frac{|A_x|}{|A|} = 1 \qquad y = \frac{88}{44} = 2 \qquad z = \frac{-132}{44} = -3$$

<u>ESERCIZIO 5</u>

Siano $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$ e $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$

Trovare (BA), $(BA)^{-1}$, (AB) e $(AB)^{-1}$ quando possibile

A è 3×2, B è $2\times3 \Rightarrow BA$ è 2×2, AB è 3×3

Le inverse esistono se il $\det \neq 0$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+4+3 & 0+2-3 \\ 2+2+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 4 & 1 \end{bmatrix}$$

$\det(BA) = 8 + 4 = 12 \quad \det((BA)^{-1}) = \frac{1}{12}$

$BA_{11} = 1 \quad BA_{12} = -4$

$BA_{21} = 1 \quad BA_{22} = 8$

$Cof(BA) = \begin{bmatrix} 1 & -4 \\ 1 & 8 \end{bmatrix} \quad Adj(BA) = \begin{bmatrix} 1 & 1 \\ -4 & 8 \end{bmatrix}$

$\Rightarrow BA^{-1} = \begin{bmatrix} 1/12 & 1/12 \\ -1/3 & 2/3 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 & 3+0 \\ 2+2 & 4+1 & 6+0 \\ 1-2 & 2-1 & 3+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$

$\det(AB) = 15 + 12 - 12 + 15 - 24 - 6 = 0$

$\Rightarrow$ Non esiste $(AB)^{-1}$

ESERCIZIO 6

Risolvere applicando Cramer se possibile

$\begin{cases} 2x + y + z = 1 \\ 4x - y + z = -5 \\ y - x + 2z = 5 \end{cases}$

Per applicare Cramer il sistema deve avere una soluzione unica

$\Rightarrow$ RANGO(A) = RANGO$(A|b) = 3$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix} \quad A|b = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & -1 & 1 & -5 \\ -1 & 1 & 2 & 5 \end{array}\right] \xrightarrow[R_3 + \frac{1}{2}R_1]{R_2 - 2R_1}$

$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -3 & -1 & -7 \\ 0 & 3/2 & 5/2 & 11/2 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -3 & -1 & -7 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

RANGO$(A) = 3$

RANGO$(A|b) = 3$

POSSO USARE CRAMER

$\det(A) = 2 \cdot -3 \cdot 2 = -12$

$$\det(A_x) = \begin{vmatrix} 1 & 1 & 1 \\ -5 & -1 & 1 \\ 5 & 1 & 2 \end{vmatrix} = -2-5+5+5-1+10 = 12$$

$$\det(A_y) = \begin{vmatrix} 2 & 1 & 1 \\ 4 & -5 & 1 \\ -1 & 5 & 2 \end{vmatrix} = -20+20-1-5-10-8 = -24$$

$$\det(A_z) = \begin{vmatrix} 2 & 1 & 1 \\ 4 & -1 & -5 \\ -1 & 1 & 5 \end{vmatrix} = -10+4+5-1+10-20 = -12$$

$x = \frac{12}{-12} = -1$ $\quad y = \frac{-24}{-12} = 2$ $\quad z = \frac{-12}{-12} = 1$

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