A.L. LEZ. 10

TRASFORMAZIONE LINEARE

$T \in HOM(V, W)$

L S. VETT. SU $\mathbb{R}$

$T: V \to W$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) =$ CE NE SONO INFINITE DI LEGGI ESPLICITE CHE SODDISFANO QUEI 3 VINCOLI

$f(1) = 1$

$f(2) = 4$

$f(0) = 0$

→ $\dim V = n$

TEOR. DI UNIVOCITÀ

$T \in Hom(V,W) \quad \dim V = n$

$B = \{v_1, \dots, v_n\}$ BASE DI V

ALLORA FISSATI $w_1, \dots, w_n \in W$ $\exists! \, T \in Hom(V,W)$:

$T(v_j) = w_j \quad \forall j = 1, \dots, n$

DIM. $T(v) = ?\ \forall v \in V$ $\quad B = \{v_1, \dots, v_n\}$

$$v = \sum_{j=1}^{n} \alpha_j v_j$$

$$T(v) = \sum_{j=1}^{n} \alpha_j w_j = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$$

1. T È LINEARE ✓
2. $T(v_j) = w_j$ ✓
3. T È UNICA ✓

a) $T(v+v') = T(v) + T(v')$ $\quad \forall v, v' \in V$

b) $T(\lambda v) = \lambda T(v)$ → LO LASCIO A VOI

$$v = \sum_{j=1}^{n} \alpha_j v_j \qquad v' = \sum_{j=1}^{n} \alpha'_j v_j$$

$$v + v' = \sum_{j=1}^{n} \alpha_j v_j + \sum_{j=1}^{n} \alpha'_j v_j = \sum_{j=1}^{n} (\alpha_j + \alpha'_j) v_j$$

$$T(v+v') = \sum_{j=1}^{n} (\alpha_j + \alpha'_j) w_j = \sum_{j=1}^{n} \alpha_j w_j + \sum_{j=1}^{n} \alpha'_j w_j =$$

$$= T(v) + T(v')$$ → PUNTO a) DIMOSTRATO

2. $v_j = \underbrace{0}_{\alpha_1=0} v_1 + \underbrace{0}_{\alpha_2=0} \cdot v_2 + \dots + \underbrace{1}_{\boxed{\alpha_j=1}} \cdot v_j + \dots + \underbrace{0}_{\alpha_n=0} \cdot v_n$

$$T(v_j) = \sum_{j=1}^{n} \alpha_j w_j = w_j$$

$$v_j = \sum_{i=1}^{n} \alpha_i v_i = \underset{0}{\alpha_1} v_1 + \underset{0}{\alpha_2} v_2 + \dots + \underset{0}{\alpha_{j-1}} v_{j-1} + \overset{1}{\alpha_j} v_j + \underset{0}{\alpha_{j+1}} v_{j+1} + \dots + \underset{0}{\alpha_n} v_n$$

③

$$T(v_j) = \sum_{1}^{m} {}_i\, \alpha_i w_j = \underset{\underset{0}{\shortparallel}}{\alpha_1} w_1 + \underset{\underset{0}{\shortparallel}}{\alpha_2} w_2 + \dots + \underset{\underset{1}{\shortparallel}}{\alpha_j} w_j + \dots + \underset{\underset{0}{\shortparallel}}{\alpha_n} w_n$$

$$= w_j$$

P. ASS. $S \in \mathrm{Hom}(V,W)$ $\boxed{S(v_j) = w_j}$ $j = 1, \dots, n$

$\overset{?}{\Rightarrow} S = T$

$$S(v) = S\left(\sum_{1}^{n} {}_i\, \alpha_i v_i\right) \overset{\text{LINEARITÀ}}{=} \sum_{1}^{m} {}_i\, \alpha_j S(v_j) =$$

$$\boxed{\forall v = \sum_{1}^{n} {}_i\, \alpha_i v_i} \qquad = \sum_{1}^{m} {}_i\, \alpha_j w_j = T(v) \qquad \square$$

$$S(v) = S\left(\sum_{1}^{m} {}_i\, \alpha_j v_j\right) = S(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) =$$

$$\overset{\text{LINEARITÀ}}{=} S(\alpha_1 v_1) + \dots + S(\alpha_n v_n) \overset{\text{LINEARITÀ}}{=}$$

$$= \alpha_1 S(v_1) + \dots + \alpha_n S(v_n)$$

$$= \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n = T(v)$$

ESERCIZIO

$\dim V = 2$

$T \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^3)$ $\quad T(1,0) = (0,1,0) = w_1$

$V = \mathbb{R}^2 \quad W = \mathbb{R}^3$ $\quad T(1,1) = (1,1,0) = w_2$

DETERMINARE $T(-1,2) = ?$

$v_1 = (1,0) = e_1$

$v_2 = (1,1)$

$B = \{v_1, v_2\}$ È UNA BASE (FATELO VOI)

$T(V_1) = W_1$

$T(V_2) = W_2$

$\rightarrow$ IL TEOR DI UNIVOCITÀ CI DICE CHE ESISTE UNA ED UNA SOLA TR. LIN T TALE CHE

$T(V_1) = W_1$

$T(V_2) = W_2$

$\forall v \in \mathbb{R}^2 \quad V = \alpha_1 V_1 + \alpha_2 V_2 \qquad B = \{V_1, V_2\}$

$$\boxed{T(V) = \alpha_1 W_1 + \alpha_2 W_2}$$

$V = (-1,2) \qquad V = \alpha_1 V_1 + \alpha_2 V_2$

DOVETE TROVARE α_1, α_2 !!!

$(-1,2) = \alpha_1 (1,0) + \alpha_2 (1,1)$

$$\begin{cases} \alpha_1 + \alpha_2 = -1 \\ \boxed{\alpha_2 = 2} \end{cases} \qquad \alpha_1 + 2 = -1 \qquad \boxed{\alpha_1 = -3}$$

$$\boxed{V = -3V_1 + 2V_2}$$

$T(V) = T(-1,2) = -3W_1 + 2W_2 =$

$= -3(0,1,0) + 2(1,1,0)$

$= (0,-3,0) + (2,2,0)$

$= (2,-1,0)$

PROP. 1

$T \in \mathrm{Hom}(V, W)$

$U = \mathrm{span}\{v_1, \dots, v_k\}$ S.S.VETT. DI V

ALLORA $T(U) = \mathrm{span}\{T(v_1), \dots, T(v_k)\}$

DIM. $w_1 = T(v_1)$, $w_2 = T(v_2)$, $\dots$, $w_k = T(v_k)$

$\mathrm{span}\{w_1, \dots, w_k\} = T(U)$

1° PASSO: $\mathrm{span}\{w_1, \dots, w_k\} \subseteq T(U)$

$$w \in \mathrm{span}\{w_1, \dots, w_k\} \Rightarrow w = \sum_{j=1}^{k} \alpha_j w_j =$$

$$= \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_k w_k$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_k T(v_k)$$

$$\overset{(b)}{=} T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_k v_k) =$$

$$\overset{(a)}{=} T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k) \qquad \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = v$$

$$= T(v), \quad v \in U \qquad v \in U$$

$$w \in T(U)$$

2° PASSO $T(U) \subseteq \mathrm{span}\{w_1, \dots, w_k\}$ **(VOI)** $\square$

PURTROPPO, UNA TRASF. LINEARE IN GENERALE

NON CONSERVA LE BASI

$T \in Hom(V, W) \qquad \dim V = n$

$B = \{v_1, .., v_n\} \qquad T(B) \rightarrow$ **NON** È DETTO CHE SIA UNA BASE

ESEMPIO

$V = W = \mathbb{R}^3 \qquad T(x, y, z) = (x-y, 2y-2x, z)$

$B = \{e_1, e_2, e_3\}$

$e_1 = (1, 0, 0) \qquad e_2 = (0, 1, 0) \qquad e_3 = (0, 0, 1)$

$T(B) = \{T(e_1), T(e_2), T(e_3)\} = \{w_1, w_2, w_3\}$

$T(e_1) = (1, -2, 0) = w_1$

$T(e_2) = (-1, 2, 0) = w_2$

$T(e_3) = (0, 0, 1) = w_3 = e_3$

$w_1 + w_2 = \overline{0}$ C.L. CON COEFFICIENTI NON TUTTI NULLI DI w_1, w_2, w_3 CHE DÀ IL VETTORE NULLO

$\rightarrow w_1, w_2, w_3$ NON SONO L. IND.

$\Rightarrow T(B)$ **NON** È UNA BASE

PROPRIETÀ DELLE TR. LINEARI

$T \in \mathrm{Hom}(V, W)$

DEF. $\ker T = \{v \in V : T(v) = \mathbf{0}\} \subseteq V$

NUCLEO DI T

$T(\bar{0}) = \bar{0}$ → SEMPRE VERO

→ $\ker T \supseteq \{\mathbf{0}\}$

PROP. $\ker T$ È UN S.S. VETT. DI V

DIM. a) $T(v_1 + v_2) = \mathbf{0} \quad \forall v_1, v_2 \in \ker T$

$\Rightarrow v_1 + v_2 \in \ker T$

$T(v_1) + T(v_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$

b) $\forall v \in \ker T \quad \forall \lambda \in \mathbb{R} \Rightarrow \lambda v \in \ker T$

$$T(\lambda v) = \mathbf{0}$$
$$\parallel$$
$$\lambda T(v) = \lambda \cdot \mathbf{0} = \mathbf{0} \quad \square$$

$\boxed{\dim V = n} \qquad 0 \leq \dim(\ker T) \leq n$

CASI ESTREMI

$\dim(\ker T) = 0 \Rightarrow \ker T = \{\mathbf{0}\}$ **INTERESSANTE** 8

$\dim \ker T = n = \dim V \Rightarrow \ker T = V$

$\forall v \in V \quad T(v) = \mathbf{0}$ $\quad \Downarrow \quad$ $T(v) = \mathbf{0} \quad \forall v \in V$

LA TR. LIN. NULLA

NON INTERESSANTE

ESEMPIO $V = W = \mathbb{R}^2 \quad T(x,y) = (x-y, x+2y)$

$\ker T = ? \quad \forall v = (x,y) \quad \boxed{T(v) = \mathbf{0}}$

$\boxed{T(x,y) = (0,0)} = (x-y, x+2y)$

$$\begin{cases} x - y = 0 \\ x + 2y = 0 \end{cases}$$

SIST. LIN. OMOGENEO

RIDUZIONE $\rightarrow 2^a - 1^a \Rightarrow 3y = 0 \quad y = 0 \Rightarrow x = y = 0$

$\rightarrow \ker T = \{\mathbf{0}\} \quad \dim \ker T = 0$