

Numeri complessi

$$x_1^2 - 2x_1 + 2 = 0 \longrightarrow \underset{1=}{(1+i)^2 - 2(1+i) + 2 = 0}$$

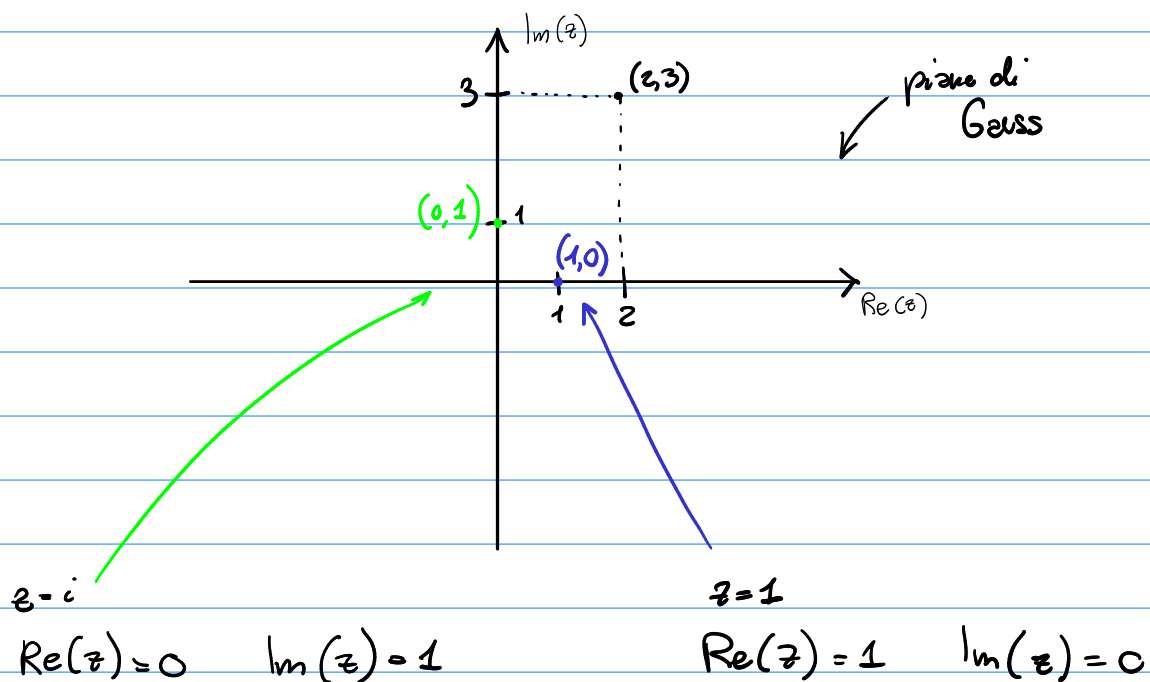
$$\underset{1=}{1 + i^2 + 2i - 2 - 2i + 2 = 0}$$

$$1 - 1 = 0 \quad \checkmark$$

Insieme numeri complessi = $\mathbb{C} : \left\{ \underbrace{a+ib}_{\substack{\text{numeri} \\ \text{complessi}}} : a, b \in \mathbb{R} \right\}$
 $i^2 = -1$

$z = \underbrace{a}_{\text{parte reale}} + \underbrace{ib}_{\text{parte immaginaria}}$ con $a, b \in \mathbb{R}$
 $\text{Re}(z) = a$
 $\text{Im}(z) = b$
 $z(2, 3) = (2, b)$

$\mathbb{R} \subseteq \mathbb{C}$



Operazioni possibili:

- SOMMA

$$\begin{aligned} z_1, z_2 &\in \mathbb{C} & z_1 + z_2 & \text{(immaginare il piano di Gauss)} \\ & & & = \\ & & & (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

props:

commutative \checkmark

associative ✓

elem. neutro : $z_1 + 0 = 0 + z_1$ (coord $(0,0) \rightarrow$ origine)

opposite: $-z_1 = -2 - ib$
 $\hookrightarrow z_1 + (-z_1) = 0$

- MOLTIPLICAZIONE

$$z_1, z_2 \in \mathcal{C}$$

$$z_1 = x_1 + iy_1 \rightarrow 2 + 3i$$

$$z_2 = x_2 + iy_2 \rightarrow 1 - i$$

$$z_1 \cdot z_2 = (2+3i)(1-i) = 2 - 2i + 3i - \underbrace{3i(i)}_3$$

$$= 5+i$$

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$

$$= \underbrace{x_1x_2 - y_1y_2}_{\text{Re}(z_1 \cdot z_2)} + i \underbrace{(x_1y_2 + y_1x_2)}_{\text{Im}(z_1 \cdot z_2)}$$

props:

props:

- commutative ✓

- associative ✓

- e./ neutro: $1 \cdot z_1 = z_1 \cdot 1 = z_1$

- reciproco $\frac{1}{z_1} \cdot z_1 = 1$

$(\mathbb{C}, +, \cdot)$ è un campo

Calcolare reciproco di un numero complesso!

$i \rightarrow \frac{1}{i} = -i$

$z_1 = 1 - 5i$

$$\frac{1}{z_1} = \frac{1}{1-5i} = \frac{1}{1-5i} \cdot \frac{1+5i}{1+5i} = \frac{1+5i}{1-(5i)^2} = \frac{1+5i}{26}$$

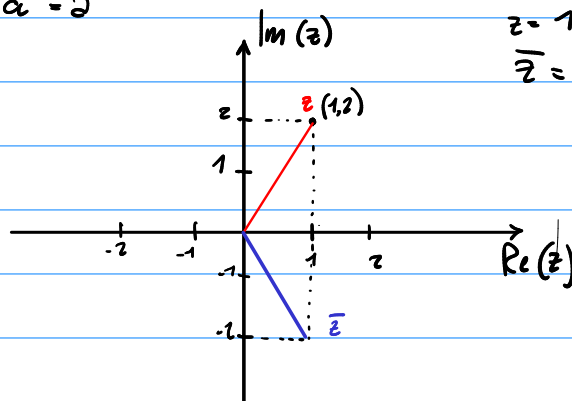
DIVISIONE

$z_1 = 1 - 5i$
 $z_2 = 1 + i \rightarrow \frac{z_1}{z_2} = \frac{1-5i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(1-5i)(1-i)}{1-(i)^2} = \frac{1-i-5i+5i^2}{2} = \frac{-4-6i}{2} = -2-3i$

CONIUGATO

$z_1 = a + ib$; $\bar{z}_1 = a - ib$
coniugato di z_1

$a \in \mathbb{R} \quad \bar{a} = a$



$z = 1 + 2i = (1, 2)$

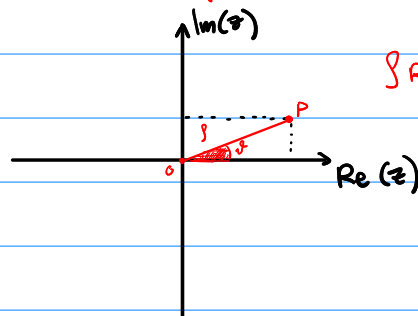
$\bar{z} = 1 - 2i = (1, -2)$

da dimostrare!

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

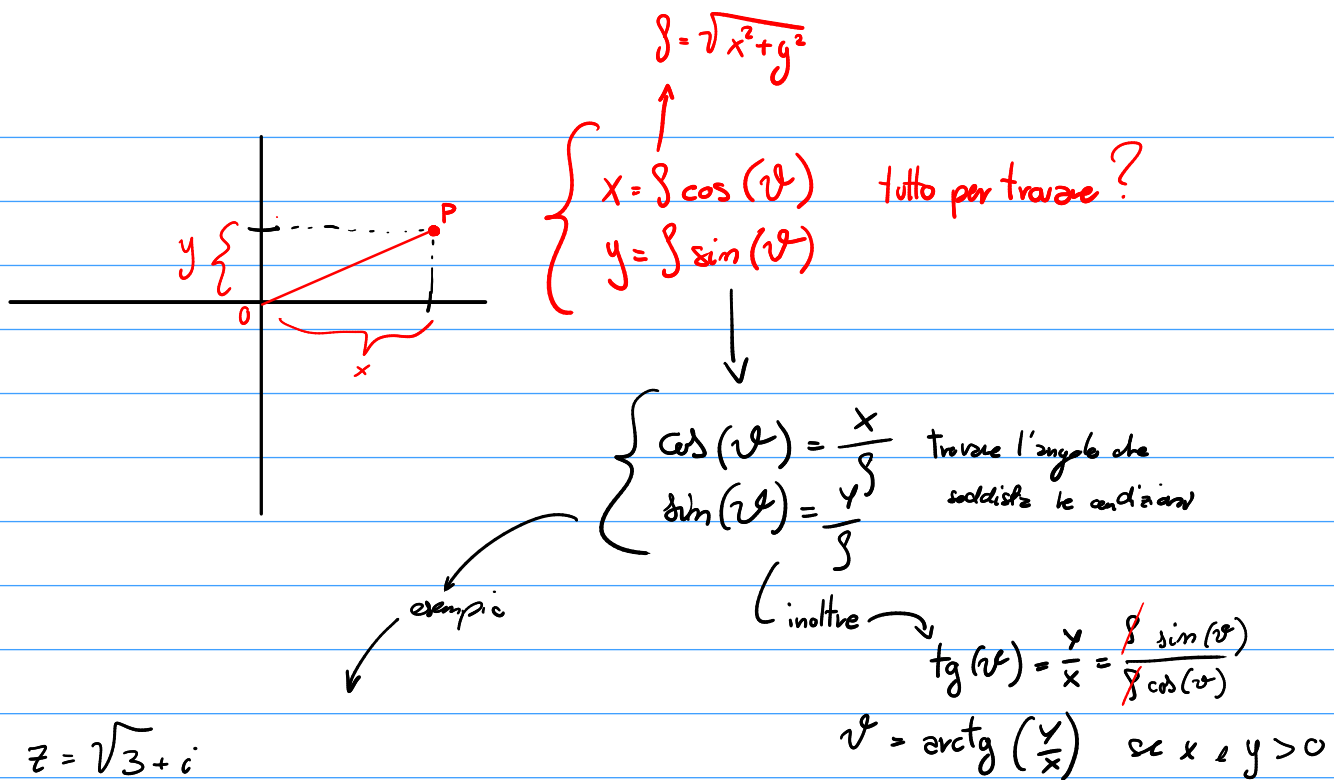
$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

FORMA TRIGONOMETRICA (FORMA POLARE)



$r = |z|$ $\rho = |z|$ - lunghezza segmento

θ theta \rightarrow argomento di z
angolo tra x e segmento OP



$$z = \sqrt{3} + i$$

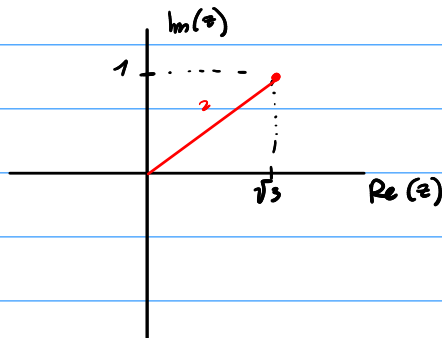
$\text{Re}(\sqrt{3}), \text{Im}(1)$

$$|z| = \rho = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$$

condizioni

$$\begin{cases} \cos(\varphi) = \frac{\sqrt{3}}{2} \\ \sin(\varphi) = \frac{1}{2} \end{cases}$$

$$\varphi = 30^\circ = \frac{\pi}{6}$$



$$0 \leq \varphi < 2\pi$$

forma trigonometrica

$$\sqrt{3} + i = 2 \cos\left(\frac{\pi}{6}\right) + i 2 \sin\left(\frac{\pi}{6}\right)$$

FORMULA DI EULERO

$$z = x + iy \rightarrow \text{rappresentazione algebrica}$$

$$= |z| (\cos(\varphi) + i \sin(\varphi))$$

$$= \rho (\cos(\varphi) + i \sin(\varphi))$$

$$0 \leq \varphi < 2\pi$$

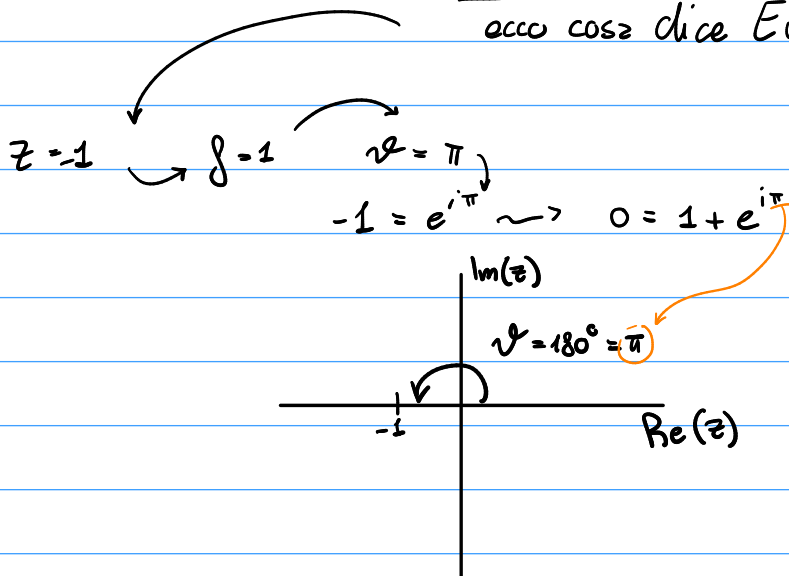
$$\rho \geq 0$$

$$\begin{cases} x = \rho \cos(\varphi) \\ y = \rho \sin(\varphi) \end{cases}$$

forma trigonometrica

$$\rho(\cos(\varphi) + i \sin(\varphi)) = \rho e^{i\varphi}$$

ecco cosa dice Eulero!



$$z = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \rightarrow \text{FORMA POLARE? (PER CASA)}$$

POTENZA

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ con $n \in \mathbb{N}$
- $|z_1^n| = |z_1|^n$
- $|z^n| = |z|^n e^{in\varphi}$

$$z_1 = \rho_1 e^{i\varphi_1} \quad \text{dove } \rho_1 = |z_1|$$

$$z_2 = \rho_2 e^{i\varphi_2} \quad \text{dove } \rho_2 = |z_2|$$

$$\begin{aligned} z_1 z_2 &= (\rho_1 e^{i\varphi_1}) (\rho_2 e^{i\varphi_2}) \\ &= \rho_1 \cdot \rho_2 e^{i\varphi_1} \cdot e^{i\varphi_2} = \underbrace{\rho_1 \cdot \rho_2}_{|z_1| \cdot |z_2|} e^{i(\varphi_1 + \varphi_2)} \end{aligned}$$

$$|z_1 z_2| = \rho_1 \cdot \rho_2 = |z_1| \cdot |z_2|$$

$$\text{Arg}(z_1 \cdot z_2) = \varphi_1 + \varphi_2 \quad (\text{angoli si sommano!})$$

$$\bullet \quad |z^n| = |z|^n$$

↑
dimostrazioni
formule
potenza

$$|z \cdot z| = |z^2| = |z|^2 \quad n \in \mathbb{N}$$

• **FORMULA DI DE MOIVRE** (importante)

$$z \in \mathbb{C} \quad \rho = |z|; \quad \varphi = \text{Arg}(z)$$

$$z^n = \rho^n \cdot e^{in\varphi}$$

forme algebrica

$$\begin{cases} x = \rho \cos(\varphi) \\ y = \rho \sin(\varphi) \end{cases}$$

$$\begin{aligned} z^n &= \rho^n \cos(n\varphi) + i \rho^n \sin(n\varphi) \\ &= \rho^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

DIMOSTRAZIONE

$$z = \rho e^{i\varphi}$$

$$\begin{aligned} z^n &= (\rho e^{i\varphi})^n \\ &= \rho^n e^{in\varphi} \end{aligned}$$

ESERCIZIO

$$z = 1 + i \quad z^7 = (1 + i)^7$$

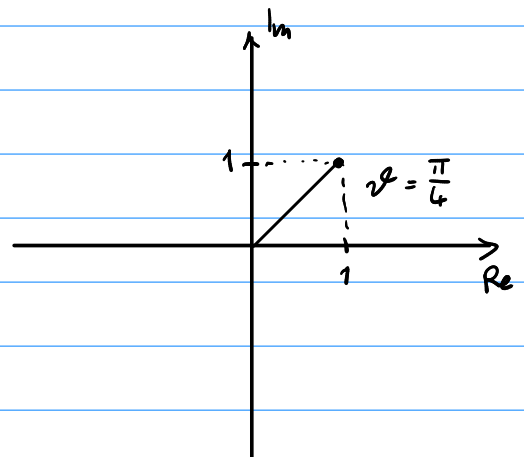
soluzione

$$|z| = \sqrt{2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{Arg}(z)$$

$$\begin{cases} x = \rho \cos(\varphi) \\ y = \rho \sin(\varphi) \end{cases}$$

$$\begin{cases} \sqrt{2} \cdot \cos\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \\ \sqrt{2} \cdot \sin\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \end{cases}$$

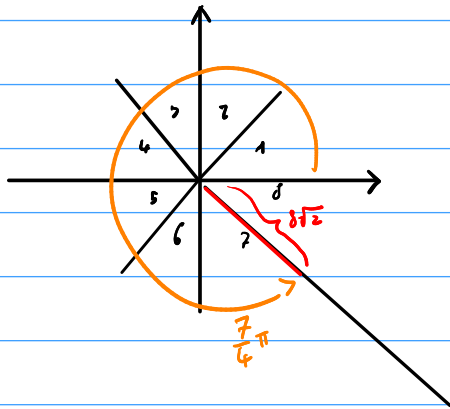


$$z = \rho e^{i\varphi} = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$z = (\sqrt{2} \cdot e^{i\frac{\pi}{4}})^2 = \sqrt{2}^2 \cdot e^{i\frac{7}{4}\pi}$$

$$= \sqrt{2 \cdot 2^6} \cdot "$$

$$= 8\sqrt{2} e^{i\frac{7}{4}\pi}$$



$$x, y? \begin{cases} x = \rho \cos(\varphi) \\ y = \rho \sin(\varphi) \end{cases}$$

$$x = 8\sqrt{2} \cdot \cos\left(\frac{7}{4}\pi\right)$$

$$= 8\sqrt{2} \cdot \frac{\sqrt{2}}{2}$$

$$= 8$$

$$y = 8\sqrt{2} \cdot \sin\left(\frac{7}{4}\pi\right)$$

$$= 8\sqrt{2} \cdot -\frac{\sqrt{2}}{2}$$

$$= -8$$

quindi

$$(1+i)^2 = 8-8i$$

Trova la soluzione a:

forma algebrica $1+i - \frac{i}{1-2i}$ e $(1-i)^{11}$

no pot. trova subito Re e Im potenza, meglio forma polare

RADICI N-ESIME DI $z \in \mathbb{C}$

dato $w \in \mathbb{C}$ diremo che $z \in \mathbb{C}$ è radice n-esima di w se $z^n = w$
e scriveremo $z = \sqrt[n]{w}$ ($n \in \mathbb{N}$)

$$w = i$$

$$z^2 = i$$

$$z = \sqrt[4]{i}$$

① Approccio geometrico

$$z = x + iy ; z^2 = i$$

$$x, y \in \mathbb{R}$$

stessa parte imm.
che vuole per
essere uguali

$$(x + iy)^2 = i$$

$$x^2 - y^2 + 2xyi = i$$

$$\begin{cases} x^2 - y^2 = 0 \\ 2xy = 1 \end{cases}$$

$$(x-y)(x+y) = 0$$

$$\textcircled{1} (x-y) = 0 \rightarrow x = y$$

$$\textcircled{2} (x+y) = 0 \rightarrow x = -y$$

$$\begin{cases} x = y \\ 2xy = 1 \end{cases}$$

$$\begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} \end{cases}$$

$$\begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \end{cases}$$

$$z_1 = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

$$z_2 = -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}$$

soluzioni!!

$$\begin{cases} x = -y \\ 2xy = 1 \end{cases}$$

$$\begin{cases} x = -y \\ y^2 = -\frac{1}{2} \end{cases} \quad \begin{matrix} \text{non esiste!!} \\ \text{soluzioni!!} \end{matrix}$$

perché

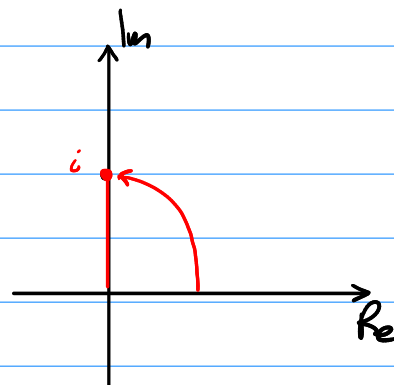
② Soluzione in forma polare

$$z = \sqrt[4]{i}$$

$$; z^2 = i$$

$$z = \rho e^{i\varphi}$$

$$i = 1 e^{i\frac{\pi}{2}}$$



Incognite

$$\rho \geq 0, \varphi \in [0, 2\pi[$$

$$(\rho e^{i\varphi})^2 = e^{i\frac{\pi}{2}} \rightarrow \rho^2 e^{i2\varphi} = 1 e^{i\frac{\pi}{2}}$$

$$\bullet \rho^2 = 1 \rightarrow \rho = \pm 1 \rightarrow \rho = 1$$

$$\bullet 2\varphi = \frac{\pi}{2} + 2k\pi \text{ (importante la periodicit\`a)} \text{ con } k \in \mathbb{N}$$

$$\varphi = \frac{\pi}{4} + k\pi$$

$$k=0 \rightarrow \varphi = \frac{\pi}{4} \quad \text{1 soluzione} \rightarrow z_1 = e^{i\frac{\pi}{4}}$$

$$k=1 \rightarrow \varphi = \frac{5\pi}{4} \quad \text{2 soluzione} \rightarrow z_2 = e^{i\frac{5\pi}{4}}$$

$$k=2 \rightarrow \varphi = \frac{\pi}{4} + 2\pi \text{ !! non va bene, perch\`e}$$

$$z_1 = e^{i\frac{\pi}{4}} \quad z_2 = e^{i\frac{5\pi}{4}}$$

$$\hookrightarrow z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_2 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

TEOREMA

data l'equazione $z^n = w$, $w \in \mathbb{C}$, $n \in \mathbb{N}$ con $n > 1$

ci sono esattamente n soluzioni distinte in \mathbb{C} , $z_0, \dots, z_{n-1} \in \mathbb{C}$

$$z_k = \sqrt[n]{\rho} e^{i\varphi_k} \text{ con } k = 0, \dots, n-1$$
$$|z_k| = \sqrt[n]{\rho} \quad \text{Arg}(z_k) = \varphi_k = \frac{\varphi}{n} + \frac{2k\pi}{n} \text{ dove } |w| = \rho \text{ Arg}(w) = \varphi$$

ESEMPI (PARTE SOPRA)

$$1) z^4 - 1 = 0 \rightarrow z^4 = 1$$

$$w = 1 \quad ; \quad \rho = |w| = 1 \quad ; \quad \varphi = \text{Arg}(w) = 0$$

$$w = 1 e^{i \cdot 0} = 1$$

$$z_k = \sqrt[n]{\rho} e^{i \varphi_k} \quad k=0, \dots, n-1$$

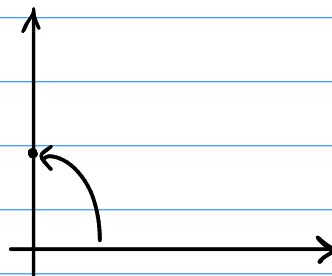
$$\varphi_k = \frac{\varphi}{n} + \frac{2k\pi}{n} = \frac{\pi}{2} \cdot k \quad k=0, \dots, 3$$

$$z_1 = 1 \cdot e^{i0} = 1$$

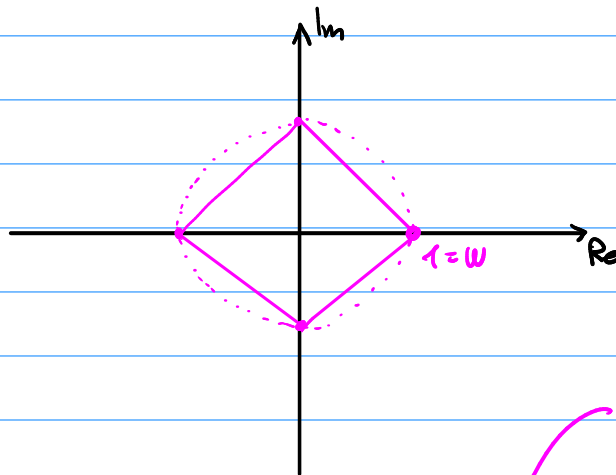
$$z_2 = 1 \cdot e^{i\frac{\pi}{2}} = i$$

$$z_3 = 1 \cdot e^{i2\frac{\pi}{2}} = e^{i\pi} = -1$$

$$z_4 = 1 \cdot e^{i3\frac{\pi}{2}} = e^{i\frac{3\pi}{2}} = -i$$



$$z^4 - 1 = 0 \xrightarrow{\text{sol}} 1, i, -1, -i \quad (-i)^4 = ((-i)^2)^2 = (-1)^2 = 1$$



$$|z_k| = \sqrt[n]{\rho}$$

$$\varphi_k = \frac{\varphi}{n} + \frac{2k\pi}{n}$$

$$\varphi_0 = \frac{\varphi}{n}$$

$$\varphi_1 = \frac{\varphi}{n} + \frac{2\pi}{n}$$

