# Numeri complessi

$$x_1^2 - 2x_1 + 2 = 0 \longrightarrow (1+i)^2 - 2(1+i) + 2 = 0$$

$$1 + i^2 + 2i - 2 - 2i + 2 = 0$$

$$1 - 1 = 0 \quad \checkmark$$

Insieme numeri complessi $= \mathbb{C} : \left\{ \underbrace{a+ib}_{\text{numeri complessi}} : \begin{array}{l} a,b \in \mathbb{R} \\ i^2 = -1 \end{array} \right\}$

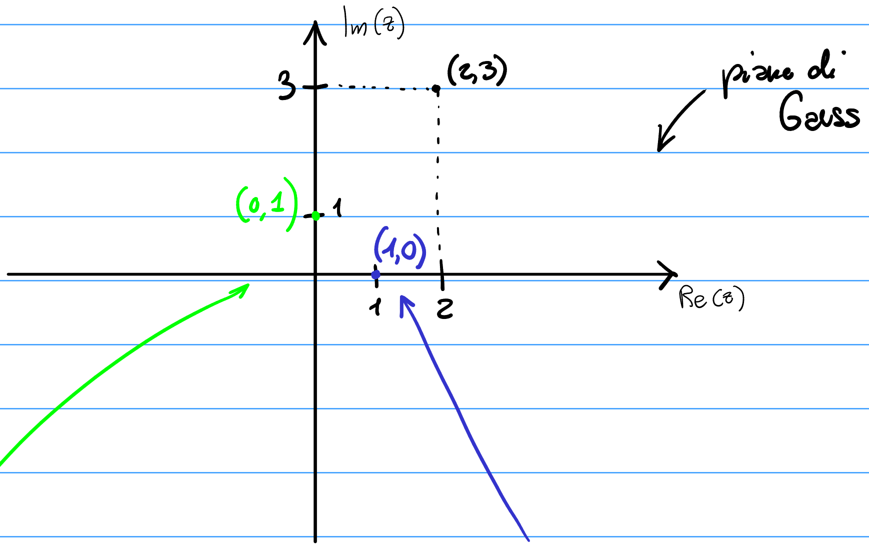
$$\mathbb{R} \subset \mathbb{C}$$

$z = a + ib$

$Re(z) = a$ parte reale   con $a,b \in \mathbb{R}$

$Im(z) = b$ parte immaginaria

$$z(2,3) = (a,b)$$




$z = i$

$Re(z) = 0 \qquad Im(z) = 1$

$z = 1$

$Re(z) = 1 \qquad Im(z) = 0$

Operazioni possibili:

- SOMMA

$z_1, z_2 \in \mathbb{C}$ $\quad z_1 + z_2$ (immagine il piano di Gauss)

$z_1 = x_1 + iy_1$
$z_2 = x_2 + i\cdot y_2$

$= (x_1+x_2)+i(y_1+y_2)$

props:

commutativa ✓
associativa ✓
elem. neutro: $z_1 + 0 = 0 + z_1$ (coord $(0,0) \rightarrow$ origine)
opposto: $-z_1 = -a - ib$
↳ $z_1 + (-z_1) = 0$

- MOLTIPLICAZIONE

$z_1, z_2 \in \mathbb{C}$

$z_1 = x_1 + iy_1 \rightarrow 2+3i$
$z_2 = x_2 + iy_2 \rightarrow 1-i$

$z_1 \cdot z_2 = (2+3i)(1-i) = 2-2i+3i\underbrace{-3i(i)}_{3}$
$= 5+i$

$z_1 \cdot z_2 = (x_1+iy_1)(x_2+iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$
$= \underbrace{x_1x_2 - y_1y_2}_{Re(z_1\cdot z_2)} + i\underbrace{(x_1y_2 + y_1x_2)}_{Im(z_1\cdot z_2)}$

props:

- commutativa ✓
- associativa ✓
- el. neutro: $1 \cdot z_1 = z_1 \cdot 1 = z_1$
- reciproco $\frac{1}{z_1} \cdot z_1 = 1$

$(\mathbb{C}, +, \cdot)$ si dice campo

Calcolare reciproco di un numero complesso!

$i \xrightarrow{rec} \frac{1}{i} = -i$ $\quad z_1 = 1-5i$

$$\frac{1}{z_1} = \frac{1}{1-5i} = \frac{1}{1-5i}\cdot\frac{1+5i}{1+5i} = \frac{1+5i}{1-(5i)^2} = \frac{1+5i}{26}$$

DIVISIONE

$z_1 = 1-5i$

$z_2 = 1+i$

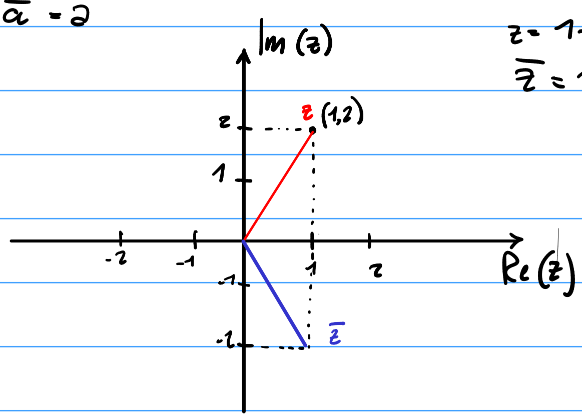
$$\longrightarrow \frac{z_1}{z_2} = \frac{1-5i}{1+i}\cdot\frac{1-i}{1-i} = \frac{(1-5i)(1-i)}{1-(i)^2} = \frac{1-i-5i+5i^2}{2} = \frac{-4-6i}{2} = -2-3i$$

CONIUGATO

$z_1 = a+ib$ ; $\quad \bar{z}_1 = a-ib$

coniugato di $z_1$

$a \in \mathbb{R} \quad \bar{a} = a$




$z = 1+2i = (1,2)$

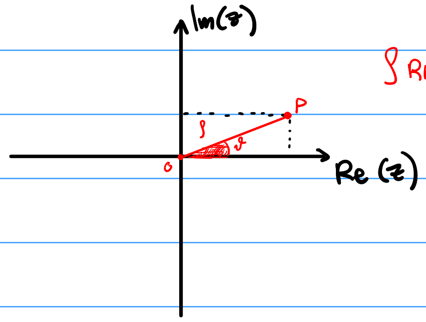
$\bar{z} = 1-2i = (1,-2)$

da dimostrare!

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

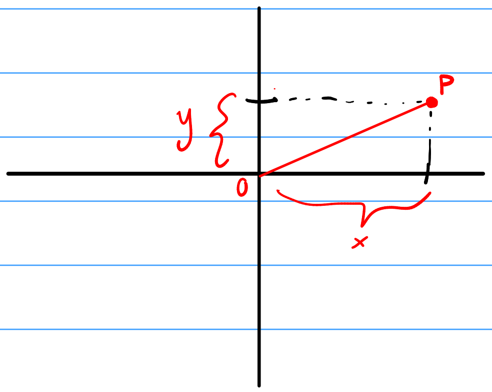
$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

FORMA TRIGONOMETRICA (FORMA POLARE)




$\rho$ RHO $\quad \rho = |z|$ - lunghezza segmento

$\vartheta$ theta → argomento di z = angolo tra x e segmento OP




$$\rho = \sqrt{x^2+y^2}$$

$$\begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$$ tutto per trovare ?

$$\begin{cases} \cos(\vartheta) = \dfrac{x}{\rho} \\ \sin(\vartheta) = \dfrac{y}{\rho} \end{cases}$$ trovare l'angolo che soddisfa le condizioni

inoltre $\rightarrow$ $tg(\vartheta) = \dfrac{y}{x} = \dfrac{\cancel{\rho}\sin(\vartheta)}{\cancel{\rho}\cos(\vartheta)}$

$\vartheta = \operatorname{arctg}\left(\dfrac{y}{x}\right)$ se $x$ e $y > 0$

esempio

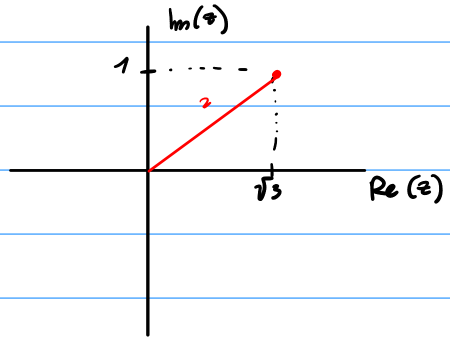
$$\underbrace{z = \sqrt{3} + i}_{}$$

$Re(\sqrt{3}), Im(1)$ $\quad |z| = \rho = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

cerchiamo

$$\begin{cases} \cos(\vartheta) = \dfrac{\sqrt{3}}{2} \\ \sin(\vartheta) = \dfrac{1}{2} \end{cases}$$

$$\vartheta = 30° = \frac{\pi}{6}$$




$0 \leq \vartheta < 2\pi$

forma trigonometrica $\sqrt{3} + i = 2\cos\left(\frac{\pi}{6}\right) + i\,2\sin\left(\frac{\pi}{6}\right)$

$\rho \qquad \rho$

## FORMULA DI EULERO

$z = x + iy \mapsto$ rappresentazione algebrica

$= |z|\,(\cos(\vartheta) + i\sin(\vartheta))$

$= \rho\,(\cos(\vartheta) + i\sin(\vartheta))$ $\downarrow$ forma trigonometrica

$0 \leq \vartheta < 2\pi$

$\rho \geq 0$

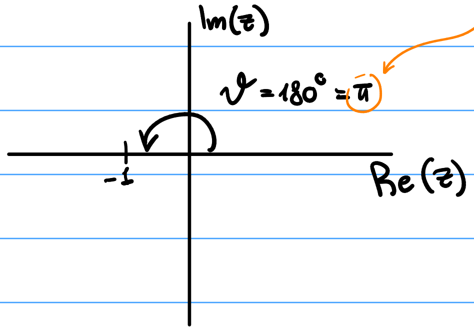
$$\begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$$

$$\rho\left(\cos(\vartheta)+i\sin(\vartheta)\right) = \underline{\underline{\rho e^{i\vartheta}}}$$

ecco cosa dice Eulero!

$z=-1 \rightarrow \rho = 1 \quad \vartheta = \pi$

$-1 = e^{i\pi} \rightsquigarrow 0 = 1 + e^{i\pi}$




$z = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ → FORMA POLARE? (PER CASA)

## POTENZA

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ con $n \in \mathbb{N}$
- $|z_1^n| = |z_1|^n$
- $|z^n| = |z|^n e^{in\vartheta}$

$z_1 = \rho_1 e^{i\vartheta_1}$ dove $\rho_1 = |z_1|$

$z_2 = \rho_2 e^{i\vartheta_2}$ dove $\rho_2 = |z_2|$

$$z_1 z_2 = \left(\rho_1 e^{i\vartheta_1}\right)\left(\rho_2 e^{i\vartheta_2}\right)$$
$$= \rho_1 \cdot \rho_2 e^{i\vartheta_1} \cdot e^{i\vartheta_2} = \underbrace{\rho_1 \cdot \rho_2 \cdot e^{i(\vartheta_1 + \vartheta_2)}}$$

$|z_1 z_2| = \rho_1 \cdot \rho_2 = |z_1| \cdot |z_2|$

$\mathrm{Arg}(z_1 \cdot z_2) = \vartheta_1 + \vartheta_2$ (angoli si sommano!)

$\bullet \quad |z^n| = |z|^n$

dimostrazioni formule potenza

$|z \cdot z| = |z^2| = |z|^2 \qquad n \in \mathbb{N}$

## • FORMULA DI DE MOIVRE (importante)

$z \in \mathbb{C} \qquad \rho = |z| \; ; \; \vartheta = Arg(z)$

$$z^n = \rho^n \cdot e^{in\vartheta}$$

forma algebrica $\begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$

$$z^n = \rho^n \cos(n\vartheta) + i\rho^n \sin(n\vartheta) = \rho^n(\cos(n\vartheta) + i\sin(n\vartheta))$$

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DIMOSTRAZIONE

$z = \rho e^{i\vartheta} \qquad z^n = (\rho e^{i\vartheta})^n = \rho^n e^{n \underbrace{i\vartheta}}$

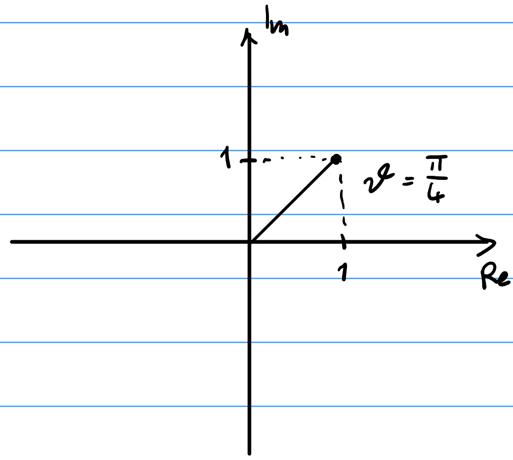
## ESERCIZIO

$z = 1 + i \qquad z^7 = (1+i)^7$

soluzione

$|z| = \sqrt{2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$Arg(z) \qquad \begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$



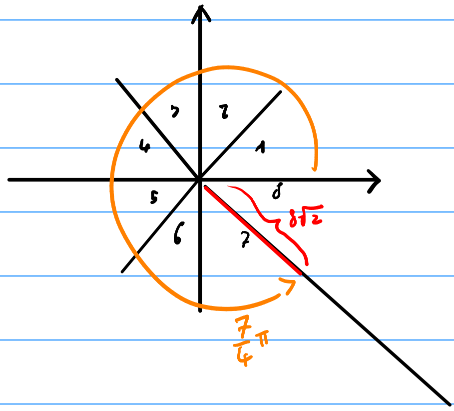

$$\begin{cases} \sqrt{2} \cdot \cos\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \\ \sqrt{2} \cdot \sin\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \end{cases}$$

$$z = \rho e^{i\vartheta} = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$z^7 = \left(\sqrt{2}\cdot e^{i\frac{\pi}{4}}\right)^7 = \sqrt{2}^{\,7} \cdot e^{i\frac{7}{4}\pi}$$

$$= \sqrt{2\cdot 2^6} \cdot \text{"}$$

$$= 8\sqrt{2}\; e^{i\frac{7}{4}\pi}$$




$x, y$? $\begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$

$x = 8\sqrt{2}\cdot\cos\left(\frac{7}{4}\pi\right) = 8\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 8$

$y = 8\sqrt{2}\cdot\sin\left(\frac{7}{4}\pi\right) = 8\sqrt{2}\cdot -\frac{\sqrt{2}}{2} = -8$

quindi $(1+i)^7 = 8-8i$

Trova la soluzione a:

forma algebrica $\underbrace{1+i-\frac{i}{1-2i}}_{\text{no pot. trova subito Re e Im}}$ e $\underbrace{(1-i)^{11}}_{\text{potenza, meglio forma polare}}$

## RADICI N-ESIME DI $z \in \mathbb{C}$

dato $w \in \mathbb{C}$ diremo che $z \in \mathbb{C}$ è radice n-esima di $w$ se $z^n = w$

e scriveremo $z = \sqrt[n]{w}$ $(n \in \mathbb{N})$

$w = i$ $\qquad z^2 = i$ $\qquad z = \sqrt{i}$

## ① Approccio geometrico

$z = x + iy \; ; \quad z^2 = i \qquad x, y \in \mathbb{R}$

$(x+iy)^2 = i$

stessa parte imm. che reale per essere uguali

$x^2 - y^2 + 2xiy = i$

$$\begin{cases} x^2 - y^2 = 0 \\ 2xy = 1 \end{cases} \qquad (x-y)(x+y) = 0$$

① $(x-y) = 0 \rightarrow x = y$

② $(x+y) = 0 \rightarrow x = -y$

$$\begin{cases} x = y \\ 2xy = 1 \end{cases} \qquad \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} \end{cases} \vee \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \end{cases}$$

$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \qquad z_2 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ soluzioni!!

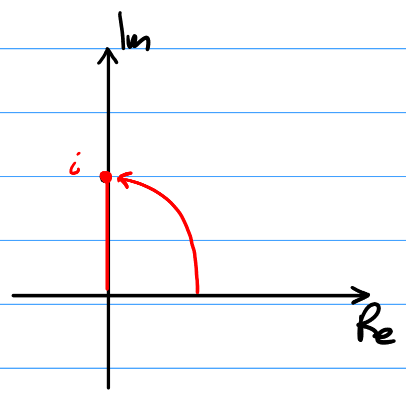
$$\begin{cases} x = -y \\ 2xy = 1 \end{cases} \qquad \begin{cases} x = -y \\ y^2 = -\frac{1}{2} \end{cases}$$ (non esiste soluzione!!)

perché

## ② Soluzione in forma polare

$z = \sqrt{i} \; ; \quad z^2 = i$

$z = \rho e^{i\theta} \qquad i = 1 e^{i\frac{\pi}{2}}$



Incognite

$$\rho \geq 0 \quad , \quad \vartheta \in [0, 2\pi[$$

$$z^2 = i$$

$$(\rho e^{i\vartheta})^2 = e^{i\frac{\pi}{2}} \longrightarrow \rho^2 e^{i2\vartheta} = 1 e^{i\frac{\pi}{2}}$$

- $\rho^2 = 1 \longrightarrow \rho = \pm 1 \longrightarrow \rho = 1$
- $2\vartheta = \frac{\pi}{2} + 2k\pi$ (importante la periodicità) con $k \in \mathbb{N}$

$$\vartheta = \frac{\pi}{4} + k\pi$$

$k = 0 \longrightarrow \vartheta = \frac{\pi}{4}$ 1 soluzione $\hookrightarrow z_1 = e^{i\frac{\pi}{4}}$

$k = 1 \longrightarrow \vartheta = \frac{5\pi}{4}$ 2 soluzione $\hookrightarrow z_2 = e^{i\frac{5}{4}\pi}$

$k = 2 \longrightarrow \vartheta = \frac{\pi}{4} + 2\pi$ !! non va bene, perché

$$z_1 = e^{i\frac{\pi}{4}} \qquad z_2 = e^{i\frac{5}{4}\pi}$$

$$\hookrightarrow z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_2 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

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**TEOREMA**

data l'equazione $z^n = w$, $w \in \mathbb{C}$, $n \in \mathbb{N}$ con $n > 1$

ci sono esattamente $n$ soluzioni distinte in $\mathbb{C}$, $z_0, \ldots, z_{n-1} \in \mathbb{C}$

$$z_n = \sqrt[n]{\rho}\, e^{i\vartheta_k} \text{ con } k = 0, \ldots, n-1$$

$$|z_k| = \sqrt[n]{\rho} \qquad \hookrightarrow \operatorname{Arg}(z_k) = \vartheta_k = \frac{\vartheta}{n} + \frac{2k\pi}{n} \quad \text{dove } |w| = \rho \quad \operatorname{Arg}(w) = \vartheta$$

## ESEMPI (PARTE SOPRA)

1) $z^4 - 1 = 0 \longrightarrow z^4 = 1$

$w = 1$ ; $\rho = |w| = 1$ ; $\vartheta = Arg(w) = 0$

$w = 1\, e^{i \cdot 0} = 1$ $\qquad z_k = \sqrt[k]{\rho}\; e^{i \vartheta_k}$ $\qquad k = 0, \dots, \underset{\uparrow}{3}^{\,n-1}$

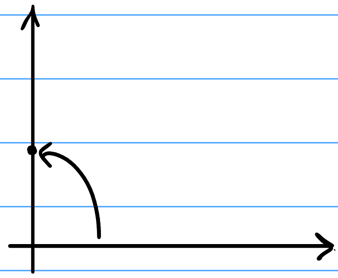
$\vartheta_k = \frac{\vartheta}{4} + \frac{2k\pi}{4} = \frac{\pi}{2} \cdot k \qquad k = 0, \dots, 3$

$z_1 = 1 \cdot e^{i0} = 1$

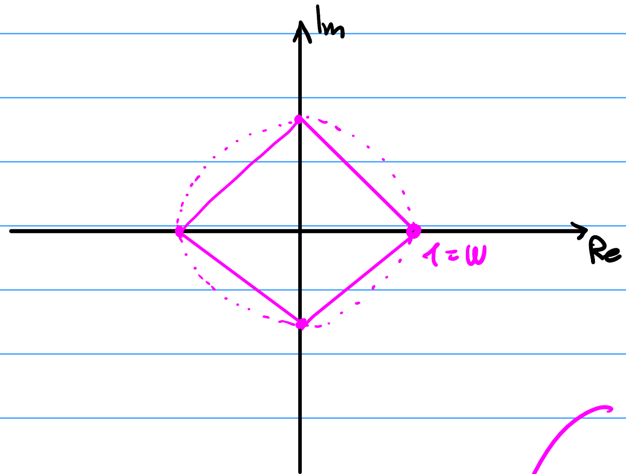
$z_2 = 1 \cdot e^{i\frac{\pi}{2}} = i$

$z_3 = 1 \cdot e^{i 2\frac{\pi}{2}} = e^{i\pi} = -1$

$z_4 = 1 \cdot e^{i 3\frac{\pi}{2}} = e^{i\frac{3}{2}\pi} = -i$



$z^4 - 1 = 0 \xrightarrow{sol.} 1, i, -1, -i \qquad (-i)^4 = \left((-i)^2\right)^2 = (-1)^2 = 1$

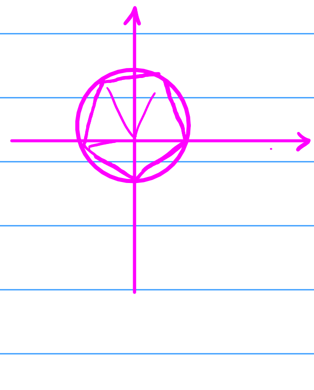



$|z_k| = \sqrt[k]{\rho}$

$\vartheta_k = \frac{\vartheta}{n} + \frac{2k\pi}{n}$

$\vartheta_0 = \frac{\vartheta}{n}$

$\vartheta_1 = \frac{\vartheta}{n} + \frac{2\pi}{n}$



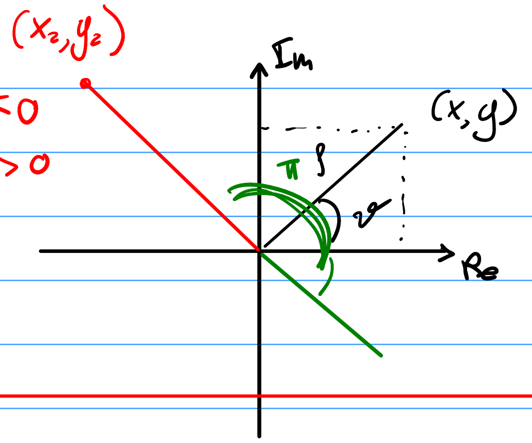
$x_2 < 0$

$y_2 > 0$

$\vartheta_2$ ?

$90 < \vartheta_2 < 180$

$\frac{\pi}{2} < \vartheta_2 < \pi$




$\rho \geq 0$

$0 \leq \vartheta < 2\pi$

$\vartheta_2 = \text{arctg}\left(\frac{y}{x}\right)$ no, non funzionerebbe

$\vartheta = \text{arctg}\left(\frac{y}{x}\right) \longrightarrow 0 < \vartheta < 90° \left(\frac{\pi}{2}\right)$

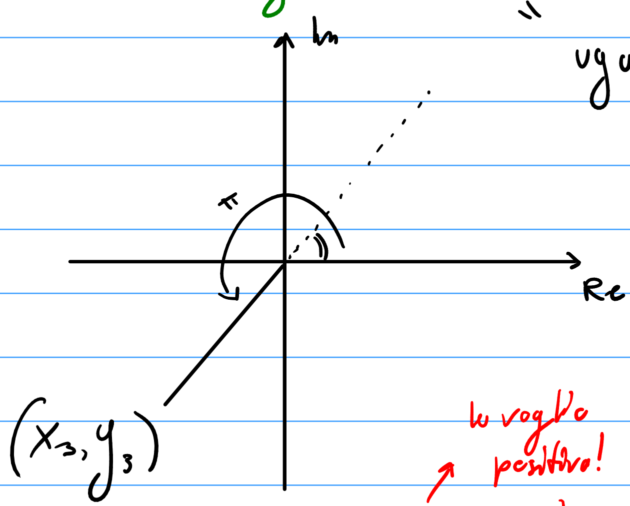
$\text{arctg} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$k \longrightarrow \vartheta = \text{arctg}(k) : \text{tg}(\vartheta) = k$

se $x_2 < 0$ aggiungo $\pi$

$\text{arctg}\left(\frac{y}{x}\right) + \pi$

" uguali "




$\vartheta_3 = \text{arctg}\left(\frac{y_3}{x_3}\right) + \pi$

$y_3 < 0$, $x_3 < 0$

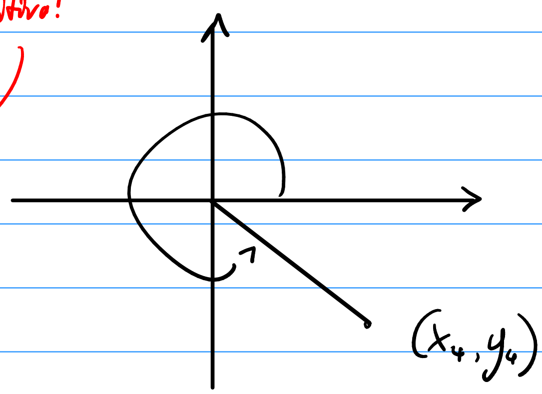
lo voglio positivo!

$\left]-\frac{\pi}{2}, 0\right[$

$< 0$

$\vartheta_4 = \text{arctg}\left(\frac{y_4}{x_4}\right) + 2\pi$

$x_4 > 0$ e $y_4 < 0$



complesso

$Im = 0 \quad Re > 0 \quad (0)$ $\qquad$ $Im > 0 \quad Re > 0 \quad \left(\frac{\pi}{2}\right)$

$Im = 0 \quad Re < 0 \quad (\pi)$ $\qquad$ $Im < 0 \quad Re = 0 \quad \left(\frac{3\pi}{2}\right)$

ESERCIZIO

$w = 1 + i$

$z \in \mathbb{C} \qquad z^3 = 1 + i \qquad z = \sqrt[3]{1+i}$

$z^n = w \longrightarrow n$ soluzioni , $z_0, z_1, \dots, z_{n-1}$

$|z_k| = \sqrt[n]{|w|}$

$\vartheta_k = \frac{Arg(w)}{n} + \frac{2\pi}{n}k \qquad k = 0, \dots, n-1$

$|w| = \sqrt{1+1} = \sqrt{2}$

$\vartheta_w : \begin{cases} x = \rho \cos(\vartheta) \\ y = \rho \sin(\vartheta) \end{cases} \quad \begin{cases} \cos(\vartheta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin(\vartheta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{cases}$

$= \vartheta_w = \frac{\pi}{4}$

$\vartheta_n = \frac{\vartheta_w}{3} + \frac{2\pi}{3}k \qquad k = 0,1,2$

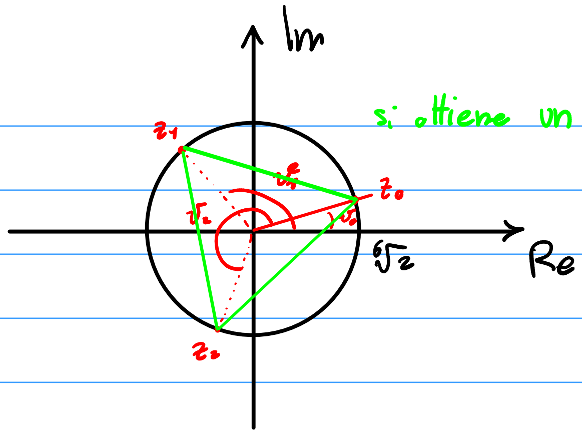
$\vartheta_0 = \frac{\pi}{12} = 15°$

$\vartheta_1 = \frac{\pi}{12} + \frac{2}{3}\pi = \frac{3}{4}\pi$

$\vartheta_2 = \frac{\pi}{12} + \frac{4}{3}\pi$

radici cubiche

$|z_i| = \sqrt[3]{|w|} = \sqrt[3]{\sqrt{2}} = \sqrt[6]{2} = 2^{\frac{1}{6}}$




si ottiene un triangolo equilatero

## Teorema fondamentale dell'algebra

un eq polinomiale di grado n in campo complesso ha esattamente n soluzioni contate con la loro molteplicità

$z^3 = 0 \quad = (3 \text{ soluzioni})$

$z \cdot 0 \cdot z \quad 0 \cdot z \cdot z \quad z \cdot z \cdot 0$

$$z^2 - (4+i)z + 4 + 2i = 0$$

1) $$z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{4+i \pm \sqrt{(4+i)^2 - (4+2i)\cdot 4}}{2}$$

2) $z = x + iy \qquad x, y \in \mathbb{R}$

$(x+iy)^2 - (4+i)(x+iy) + 4 + 2i = 0$

$x^2 - y^2 + 2xyi - 4x - 4iy - ix + y + 4 + 2i = 0$

$x^2 - y^2 - 4x + y + 4 + i(2yx - 4y - x + 2) = 0$

$$\begin{cases} x^2 - y^2 - 4x + y + 4 = 0 \\ 2yx - 4y - x + 2 = 0 \end{cases}$$

$2y(x-2) - (x-2) = (x-2)(2y-1) = 0$

1) $x - 2 = 0 \qquad x = 2$

$4 - y^2 - 8 + y + 4 = 0$

2) $y = \frac{1}{2}$

una sola dà soluzioni!!

EQ. Algebriche (polinomiali)

EQ. NON Algebriche

$$z^2 + i\,\mathrm{Im}(z) + 2\bar{z} = 0$$ ← problema

$z = x + iy \qquad x, y \in \mathbb{R}$

$$(x+iy)^2 + iy + 2(x - iy) = 0$$

$$x^2 - y^2 + 2xyi + iy + 2x - 2iy = 0$$

$$x^2 - y^2 + 2x + i(2xy + y - 2y) = 0$$

$$\begin{cases} x^2 - y^2 + 2x = 0 \\ 2xy - y = 0 \end{cases}$$

$y(2x-1) = 0$

① $y = 0$ $\vee$ (oppure) ② $x = \frac{1}{2}$

| ① | ② |
|---|---|
| $x^2 + 2x = 0$ | $\frac{1}{4} - y^2 + 1 = 0$ |
| $x(x+2) = 0$ | $\frac{5}{4} - y^2 = 0$ |
| $x = 0$ | $y = \pm \frac{\sqrt{5}}{2}$ |
| $x = -2$ | |

1.1 $z_1 = 0$

1.2 $z_1 = -2$

2.1 $= \frac{1}{2} + \frac{\sqrt{5}}{2} i$

2.2 $= \frac{1}{2} - \frac{\sqrt{5}}{2} i$