Solving Linear Systems: Gaussian Elimination

Given: n equations in n variables, $x_1 x_2 \dots x_n$

Output: A unique solution $\vec{x} = (x_1 x_2 \dots x_n)$ to the system

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 1 \\ 4x_1 + 2x_2 - x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \end{aligned}$$

$$A \longrightarrow \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ n \left\{ \right. & \left[\begin{array}{ccc:c} 3 & -1 & 2 & 1 \\ 4 & 2 & -1 & 4 \\ 1 & 3 & 3 & 5 \end{array} \right] \\ & \underbrace{\qquad\qquad\qquad}_{n} \end{array}$$

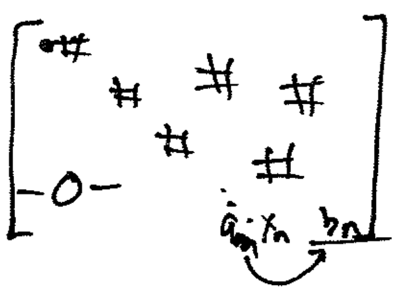
rows = equations

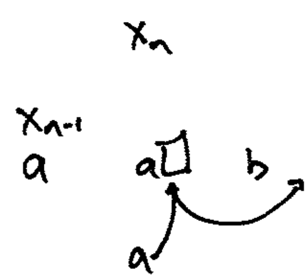
cols = variables

$n \times (n+1)$ matrix

<u>Augmented</u> • Last column: scalar value

Goal: make A into an upper triangular matrix





valid elementry operations:

- mulitply a row by a constant
- divide ~~by~~ row by a constant other than zero
- exchane rows
- exchange columns? yes as long as you keep track:
  exchanges variables
- you can add/subtract a row from another row

$$\begin{bmatrix} 3 & -1 & 2 & 1 \\ 4 & 2 & -1 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix}$$

subtract $\frac{4}{3} \times$ row 1 from row 2

subtract $\frac{1}{3} \times$ row 1 from row 3

$$\begin{bmatrix} 3 & -1 & 2 & 1 \\ 0 & 10/3 & -\frac{11}{3} & \frac{8}{3} \\ 0 & 10/3 & 7/3 & \frac{14}{3} \end{bmatrix}$$

$\longrightarrow$ subtract $\underline{1}$ x row 2 from row 3 $\longrightarrow$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \vec{b} \\ \end{array}$$
$$\left[\begin{array}{ccc:c} 3 & -1 & 2 & 1 \\ 0 & 10/3 & -11/3 & 8/3 \\ 0 & 0 & \frac{18}{3} & \frac{6}{3} \end{array}\right]$$

(6) (2)

~~line~~ row 3:

$$6x_3 = b_3$$
$$= 2$$

$$x_3 = \frac{1}{3}$$

row $\cancel{3}$ 1: $3x_1 - x_2 + 2x_3 = 1$

$$\vec{x} = \left(\frac{1}{2}, 7/6, \frac{1}{3}\right)$$

row 2:

$\frac{10}{3}$

$$\frac{\cancel{3}0}{\cancel{9}} x_2 \cancel{- \frac{11}{3} x_3} = \frac{\cancel{8}}{\cancel{3}}\left(\frac{24}{9} + \frac{11}{9}\right)\frac{9}{30} \qquad 11/9$$

$$x_2 = 7/6$$

$$x_1 = 1/2$$

outline:

$$A \quad \begin{array}{cccc} x_1 & x_2 & \cdots & \\ \end{array}$$

$$A = \left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \boxed{a_{21}} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & \boxed{a_{n2}} & \cdots & a_{nn} & b_n \end{array}\right]$$

$a_{21} \to 0$

$a_{21} - \frac{a_{21}}{a_{11}} \cdot a_{11} = 0$

Step i $\quad i = 1 \ldots n-1$

Step 1: eliminate $a_{21}\ a_{31} \ldots a_{n1}$

for $j = \overset{i+1}{\cancel{2}} \ldots n$

$\quad t \leftarrow \dfrac{a_{j1}}{\cancel{a_{11}}\ a_{ii}}$

$\quad$ for $k = j \ldots n+1$

$\quad\quad a_{jk} \leftarrow a_{jk} - t \cdot a_{\overset{i}{\cancel{1}}k}$

Step 2: eliminate $a_{32}\ a_{42}\ a_{52} \ldots a_{n2}$

for $j = \overset{i+1}{\cancel{3}} \ldots n$

$\quad t \leftarrow \dfrac{a_{j2}}{\cancel{a_{22}}\ a_{ii}}$

$\quad$ for $k = j \ldots n+1$

$\quad\quad a_{jk} \leftarrow a_{jk} - t\, a_{\overset{i}{\cancel{2}}k}$

$\vdots$

Step $n-1$

Input: An $n \times (n+1)$ augmented matrix, A

Output: An upper triangular matrix A'

```
A' ← A
for i = 1 ... n-1
|   pivotrow ← i
|   for j = (i+1) ... n
|   |   if (|A'[j,i]| > |A'[pivotrow, i]|)
|   L   L   pivotrow ← j
|   for k = i ... (n+1)
|   L   swap A'[i,k], A'[pivotrow, k]
|   for j = (i+1) ... n
|   |   t ← A'[j,i] / A'[i,i]
|   |   for k = i ... (n+1)
L   L   L   A'[j,k] ← A[j,k] − A'[i,k] · t
output A'
```

finds the "best" pivotrow

$\sum_{i=1}^{n-1} \cdots$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=i}^{n+1} 1 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} n-i+2$$

input size: $n^2 = N$

$$= \sum_{i=1}^{n-1}\left[\sum_{j=i+1}^{n} n - \sum_{j=i+1}^{n} i + \sum_{j=i+1}^{n} 2\right]$$

$$= \sum_{i=1}^{n-1}\left[n(n-i) - i(n-i) + 2(n-i)\right]$$

$(n+1)-(i-1)$

$n+2-i$

~~Gauss's Formula~~

$$\cancel{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$= \sum_{i=1}^{n-1}\left[n^2 - ni - ni + i^2 + 2n - 2i\right]$$

$$= (n-1)n^2 - 2n\sum_{i=1}^{n-1} i + \boxed{\sum_{i=1}^{n-1} i^2} + \underbrace{2(n-1)n - 2\sum_{i=1}^{n-1} i}$$

$$= n^3 - n^2 \cdots$$

$$\in \Theta(n^3) \longrightarrow \Theta(N^{3/2})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ 0 & 0 & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & 0 & & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} & b_n \end{bmatrix}$$

Back solve:

determin values for

$X_n \quad X_{n-1} \quad X_{n-2} \quad \cdots \quad X_2 \quad X_1$

$$X_n = \frac{b_n}{a_{nn}}$$

$$X_{n-1} = \frac{b_{n-1} - a_{n-1\,n}\, X_n}{a_{n-1\,n-1}}$$

$$= \frac{b_{n-1} - a_{n-1,n} X_n}{a_{n-1,n-1}}$$

x_{n-1}	x_n	RHS
$a_{n-1\,n-1}$	$a_{n-1,n}$	b_{n-1}

$$a_{n-1\,n-1}\, x_{n-1} + a_{n-1\,n}\, x_n = b_{n-1}$$

$$X_{n-2} = \frac{b_{n-2} - a_{n-2\,n} X_n - a_{n-2}\, x_{n-1}}{a_{n-2\,n-2}}$$

$\vdots$

$\cancel{X_{n-i} = b_{n-i} - a_{n-i\,n} X_n - a_{n-i,}}$

$$X_i = \frac{b_i - a_{i\,i+1}\, X_{i+1} - a_{i\,i+2}\, X_{i+2} - \cdots a_{in} X_n}{a_{ii}}$$

Back Solve

Input: An $n \times (n+1)$ upper triangular matrix n^2

Output: A solution vector $\vec{x} = (x_1\ x_2 \dots x_n)$

for $i = n \dots 1$

$t \leftarrow A[i, n+1]$ // sets t to b_i

for $j = (i+1) \dots \cancel{(n+1)}\ n$

$t \leftarrow t - x_j \cdot A[i,j]$

$x_i \leftarrow t / A[i,i]$

output $(x_1 \dots x_n)$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} 1 = ?$$

$$\in \Theta(n^2)$$

Application:

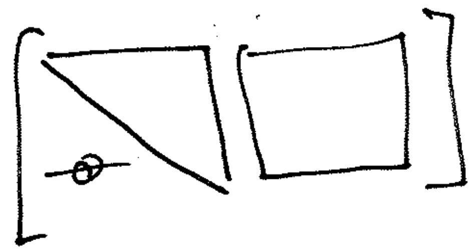
- Solving systems of linear equations
- $\cancel{A}$ $I = \begin{bmatrix} 1 & & & -0- \\ & 1 & & \\ & & 1 & \\ -0- & & & \ddots\ 1 \end{bmatrix}$

$x \to x^{-1}$

$x \cdot \frac{1}{x} = 1$

Given A, what is A^{-1} (or does it exist?)

$$n\left\{ \underbrace{[A \mid I]}_{2n} \right. \to \text{gaussian elimination}$$



$\to$ again for the upper diagonal + divide by the diagonal

$$\left[I \mid A^{-1} \right]$$

Detecting

Indeterminant Systems (infinite solutions)

A row of all zeros $\Rightarrow$ indeterminant system

Inconsistent systems (zero solutions)

A row of all zeros <u>except</u> the last column

$0 = 1$

$$\begin{matrix} 1 & 1 & 2 \\ 2 & 2 & 5 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$x = 2 - y$

$x + y = 2$

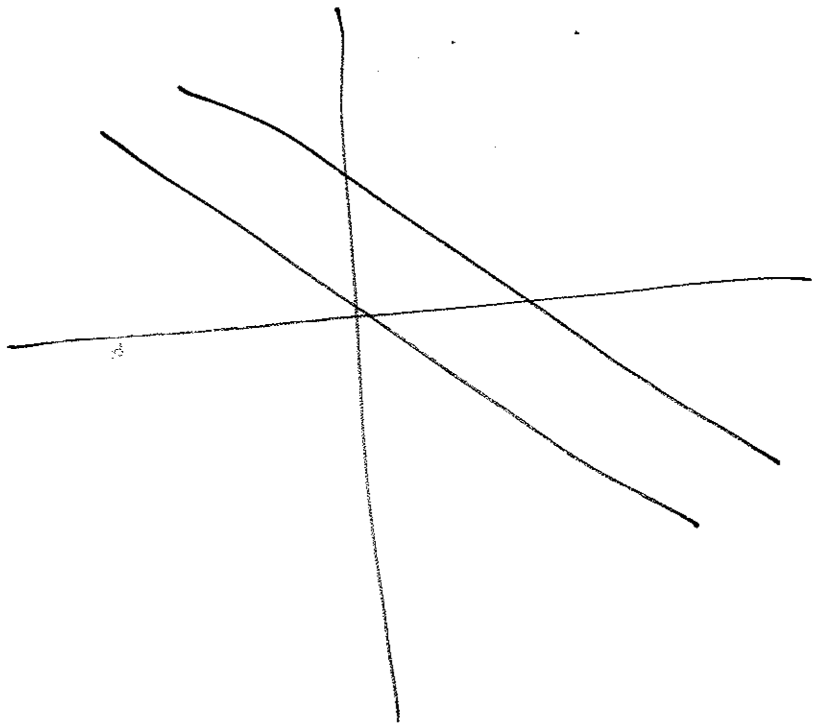
$2x + 2y = 5$

$2(2-y) + 2y = 5$

$4 - 2y + 2y = 5$

$4 = 5$

$1 =$



0 solution.

1 1 2

2 2 4

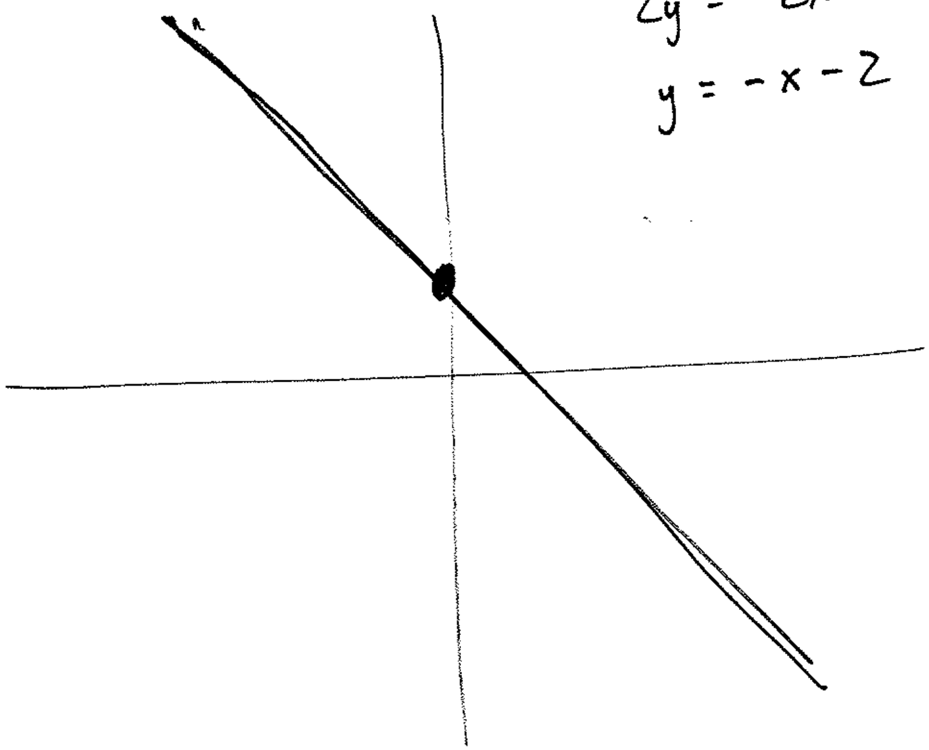
$y = -x + 2$

$x + y = 2$

$2x + 2y = 4$

$2y = -2x - 4$

$y = -x - 2$



$x = 2 - y$

$2(2 - y) + 2y = 4$

$4 - 2y + 2y = 4$

$4 = 4$

$x = ?$

$y = ?$

∞ sol.

Linear Programming

Given a collection of linear constraints

$x \geq 0$

$y \geq 0$

$x \leq 9$

$2y + x \leq 12$

$2y - x \leq 8$

linear constraints

objective function:

maximize $4x + 3y$

"payoff" function

- you can "invest" resources into 2 (or generally n) categories
- maximize profit
- minimize cost

$$\text{maximize } 4x + 3y \rightarrow y = -\frac{4}{3}x + b$$

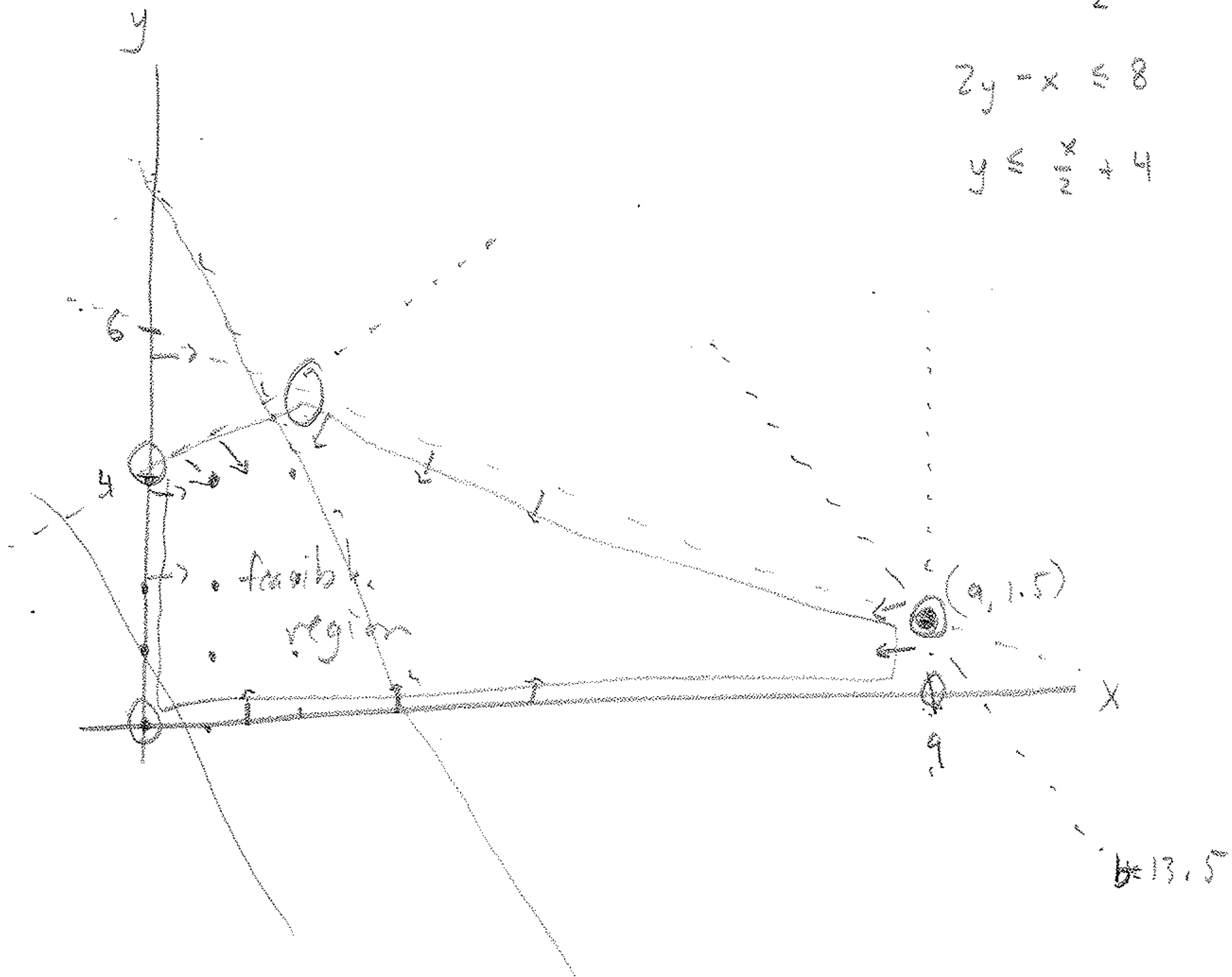
$$2y + x \le 12$$

$$y \le -\frac{x}{2} + 6$$

$$2y - x \le 8$$

$$y \le \frac{x}{2} + 4$$

point	value
(0, 0)	0
(9, 0)	36
(9, 1.5)	40.5
(2, 5)	23
(0, 4)	12



(fractional)

0-1 Knapsack Formulation

maximize you value subject to W

n items : $a_1 \cdots a_n$

n values : $v_1 \cdots v_n$

weights $w_1 \cdots w_n$

capacity W

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is taken} \\ 0 & \text{if item } i \text{ is omitted} \end{cases}$$

← indicator variables

relaxation :

$0 \leq x_i \leq 1$

$$\text{maximize} \sum_{i=1}^{n} v_i \cdot x_i$$

$$\text{subject to} \sum_{i=1}^{n} w_i x_i \leq W$$

ILP = integer linear program

$\underline{x_i \in \{0, 1\}}$, $1 \leq i \leq n$