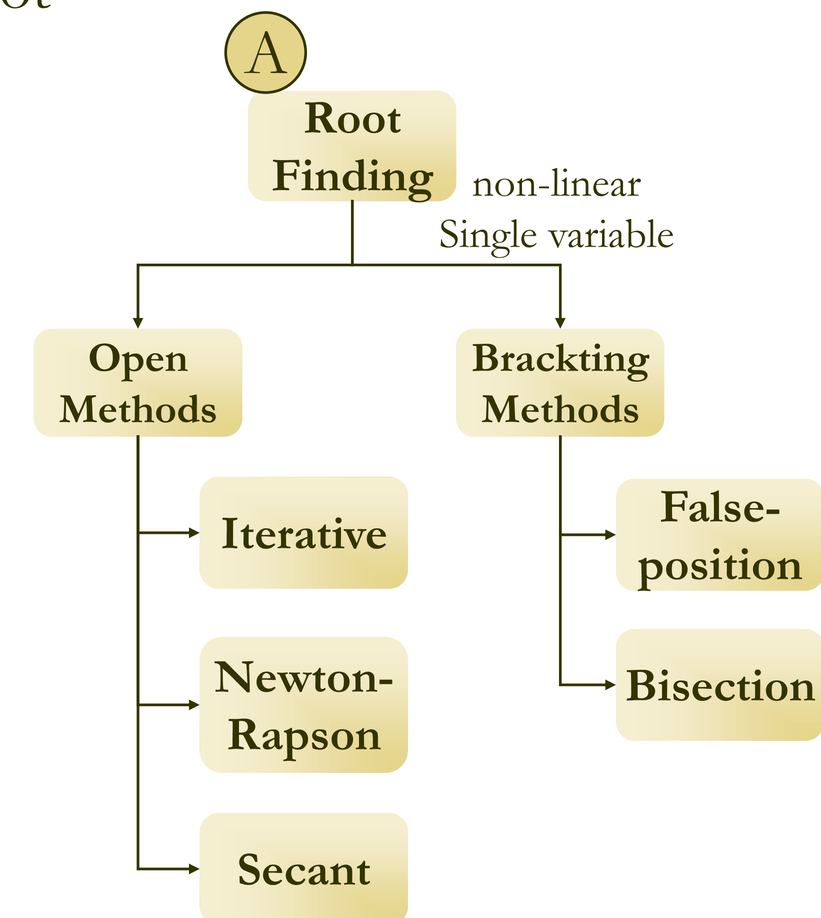
A.3

Newton-Raphson Method

Equation Solving

- Given an approximate location (initial value)
- find a single real root



Method

- We want to solve $f(x)=0$ near x_r

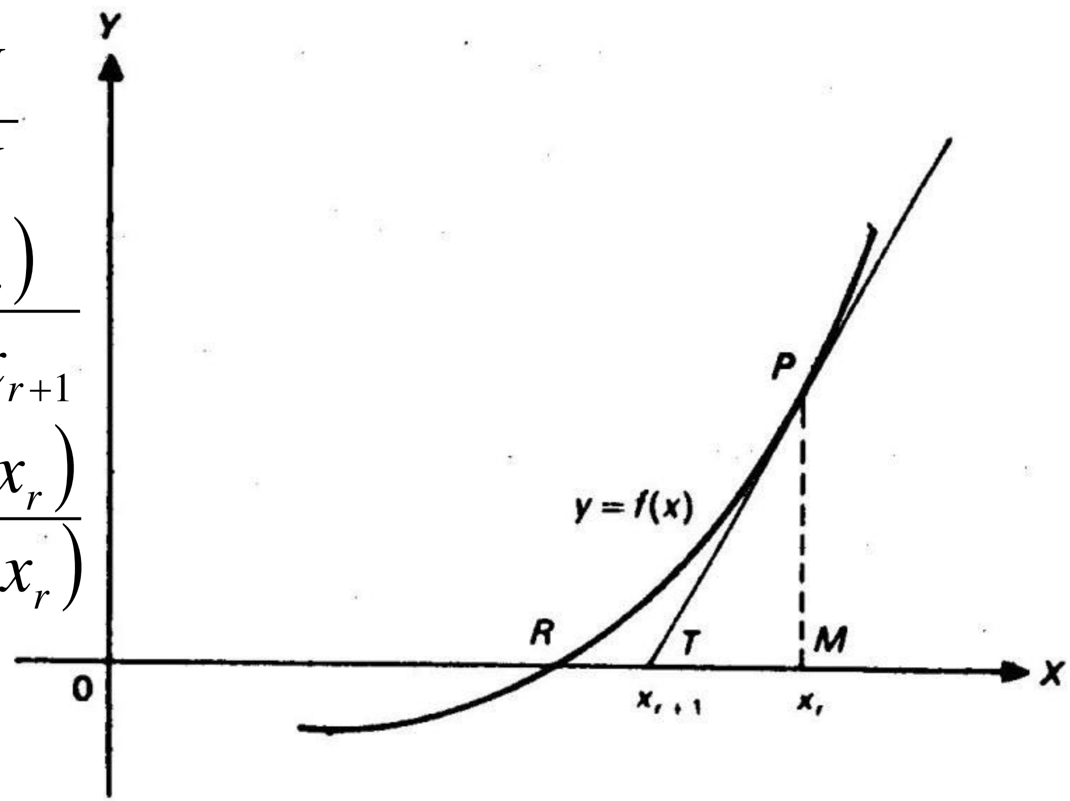
$$x_r = \text{approximate root}$$

$$x_{r+1} = \text{intersection of } f'(x_r) \text{ and } x\text{-axis}$$

$$\tan\angle PTM = \frac{PM}{TM}$$

$$\Rightarrow f'(x_r) = \frac{f(x_r)}{x_r - x_{r+1}}$$

$$\therefore \quad x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$



Algorithm

$x_{r+1} \leftarrow$ initial guess value

do {

$$x_r \leftarrow x_{r+1}$$

$$x_{r+1} \leftarrow x_r - \frac{f(x_r)}{f'(x_r)}$$

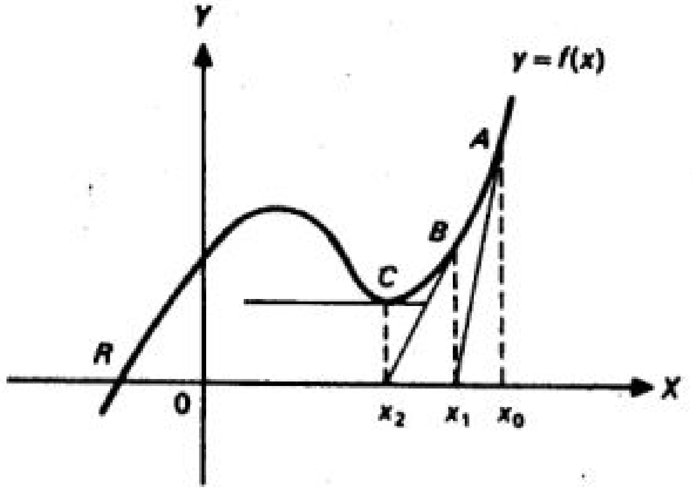
} while ($| x_{r+1} - x_r | >$ Q)

We can compute the derivative numerically as follows:

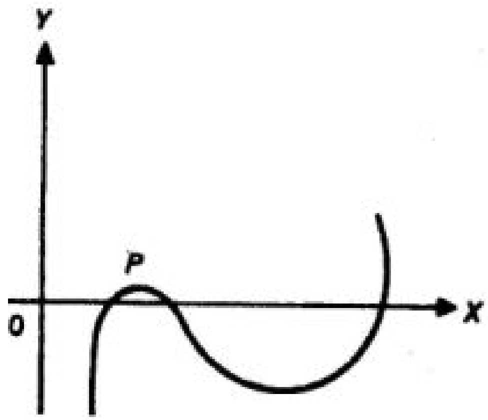
$$f'(x_r) = \frac{f(x_r + h) - f(x_r - h)}{2h}$$

Limitations

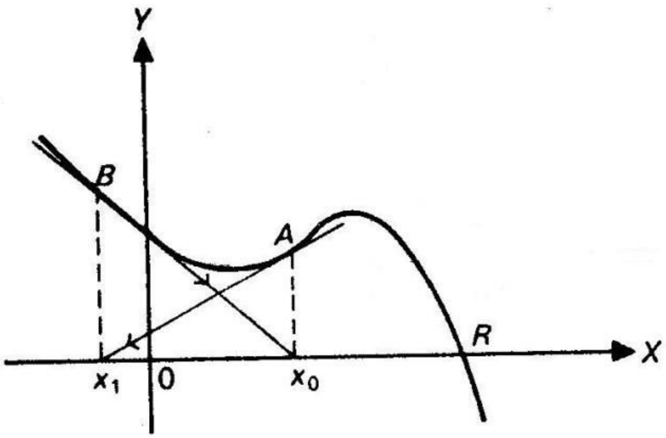
1. $f'(x_{r+1})=0$, local minima.



2. $f'(x_{r+1})\approx 0$, occurs when two roots are very close.



3. x_r and x_{r+1} recurs



Convergence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= a - \varepsilon_n - \frac{f(a-\varepsilon_n)}{f'(a-\varepsilon_n)} \qquad [\Theta\ x_n = a - \varepsilon_n]$$

$$= a - \varepsilon_n - \frac{f(a) - \varepsilon_n f'(a) + \frac{1}{2}\varepsilon_n^2 f''(a) - \Lambda}{f'(a) - \varepsilon_n f''(a) + \frac{1}{2}\varepsilon_n^2 f'''(a) - \Lambda}$$

$$\approx a - \varepsilon_n + \varepsilon_n \left[1 - \frac{1}{2}\varepsilon_n \frac{f''(a)}{f'(a)} - \Lambda\right]\left[1 - \varepsilon_n \frac{f''(a)}{f'(a)}\right]^{-1} \qquad [\Theta\ f(a) = 0]$$

$$\approx a + \frac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)}$$

$$\therefore\ \varepsilon_{n+1} = -\frac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)}$$

Second order convergence

Example

x^2-5x+4=0
0 Xr=5.000 Xr+1=4.200
1 Xr=4.200 Xr+1=4.012
2 Xr=4.012 Xr+1=4.000
3 Xr=4.000 Xr+1=4.000
3 Xr=4.000 Xr+1=4.000

e^(-x)-x=0
0 Xr=1.000 Xr+1=0.538
1 Xr=0.538 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567

xsin(x*x)-2cos(x)
0 Xr=1.000 Xr+1=0.538
1 Xr=0.538 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567

A.4

Secant Method

April 5, 2009

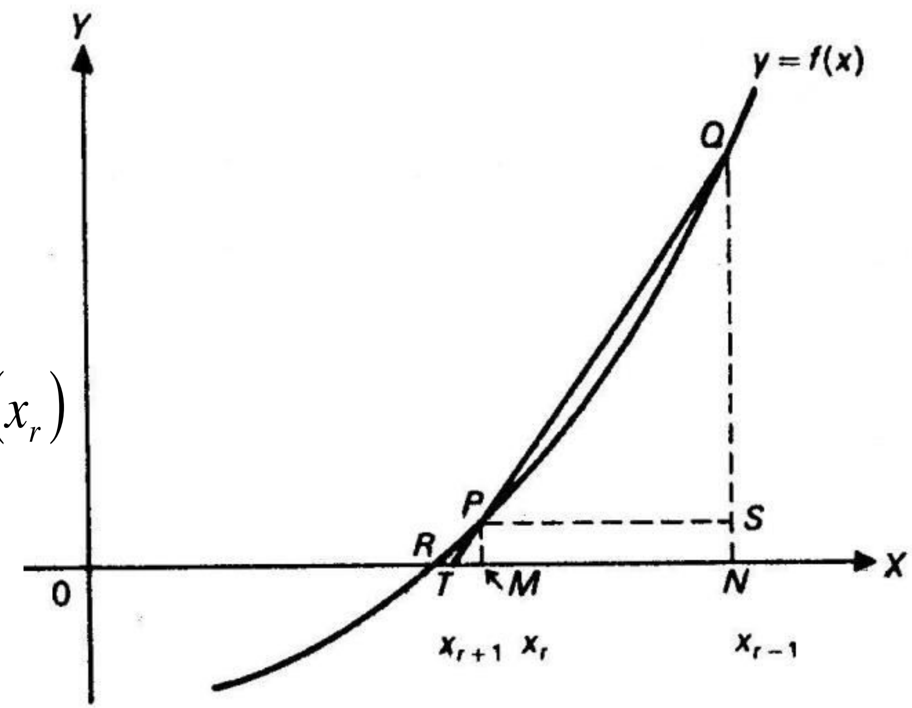
Concept

- Newton-Raphson method needs to compute $f'(x)$
  - It may be analytically complicated, or
  - Numerical evaluation may be time consuming

$$\frac{TM}{PM} = \frac{PS}{QS}$$

$$\Rightarrow \frac{x_r - x_{r+1}}{f(x_r)} = \frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}$$

$$\Rightarrow x_{r+1} = x_r - \left[\frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}\right] f(x_r)$$



- Tangent is replaced with chord $\Rightarrow$ lower convergence rate

Algorithm

$x_r \leftarrow$ initial guess value

$x_{r+1} \leftarrow$ initial guess value

do {

$$x_{r-1} \leftarrow x_r$$

$$x_r \leftarrow x_{r+1}$$

$$x_{r+1} = x_r - \left[\frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}\right] f(x_r)$$

} while ($| x_{r+1} - x_r | >$ Q)

A.4

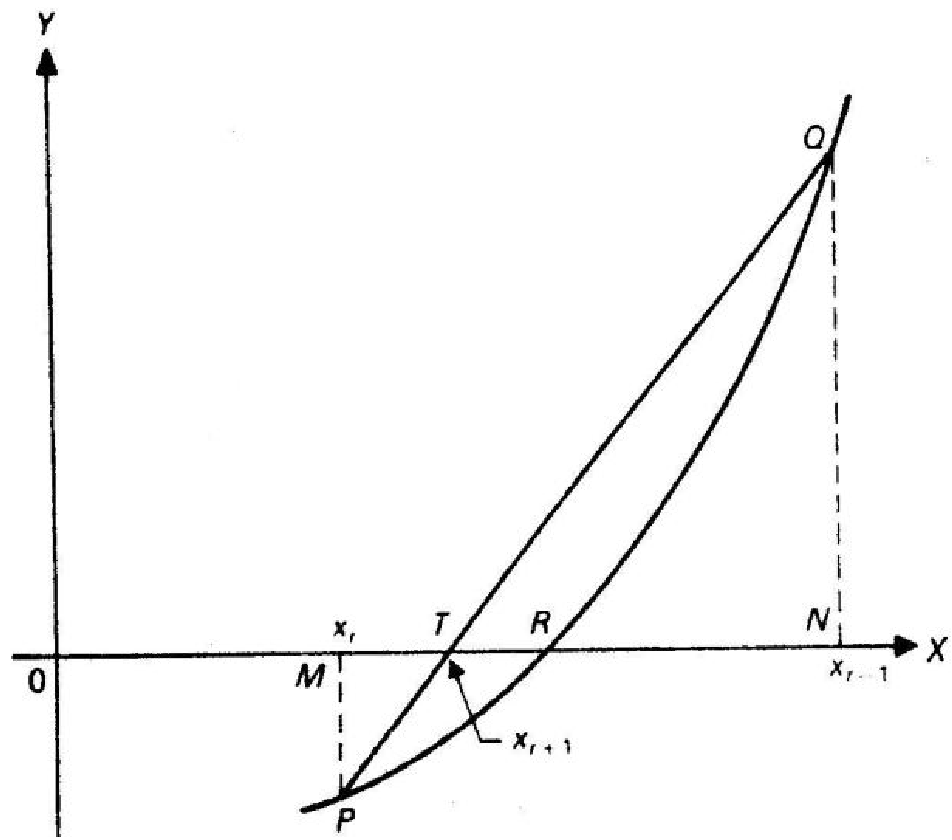
False Position Method

April 5, 2009

Concept

- Same as Secant method, except that the root is always between $\boldsymbol{x_r}$ and $\boldsymbol{x_{r-1}}$ i.e., $\boldsymbol{f(x_r)*f(x_{r-1}) < 0}$

- Convergence is lower than Newton-Raphson method
- Guarantees success



Algorithm

Choose x_r and x_{r-1} such that $f(x_r)*f(x_{r-1}) < 0$

do {

$$x_{r+1} \leftarrow x_r - \left[\frac{x_{r-1} - x_r}{f\left(x_{r-1}\right) - f\left(x_r\right)}\right] f\left(x_r\right)$$

$$\text{if} \quad f\left(x_{r+1}\right) \times f\left(x_{r-1}\right) < 0 \quad \text{then} \quad x_r \leftarrow x_{r-1}$$

$$x_{r-1} \leftarrow x_r$$

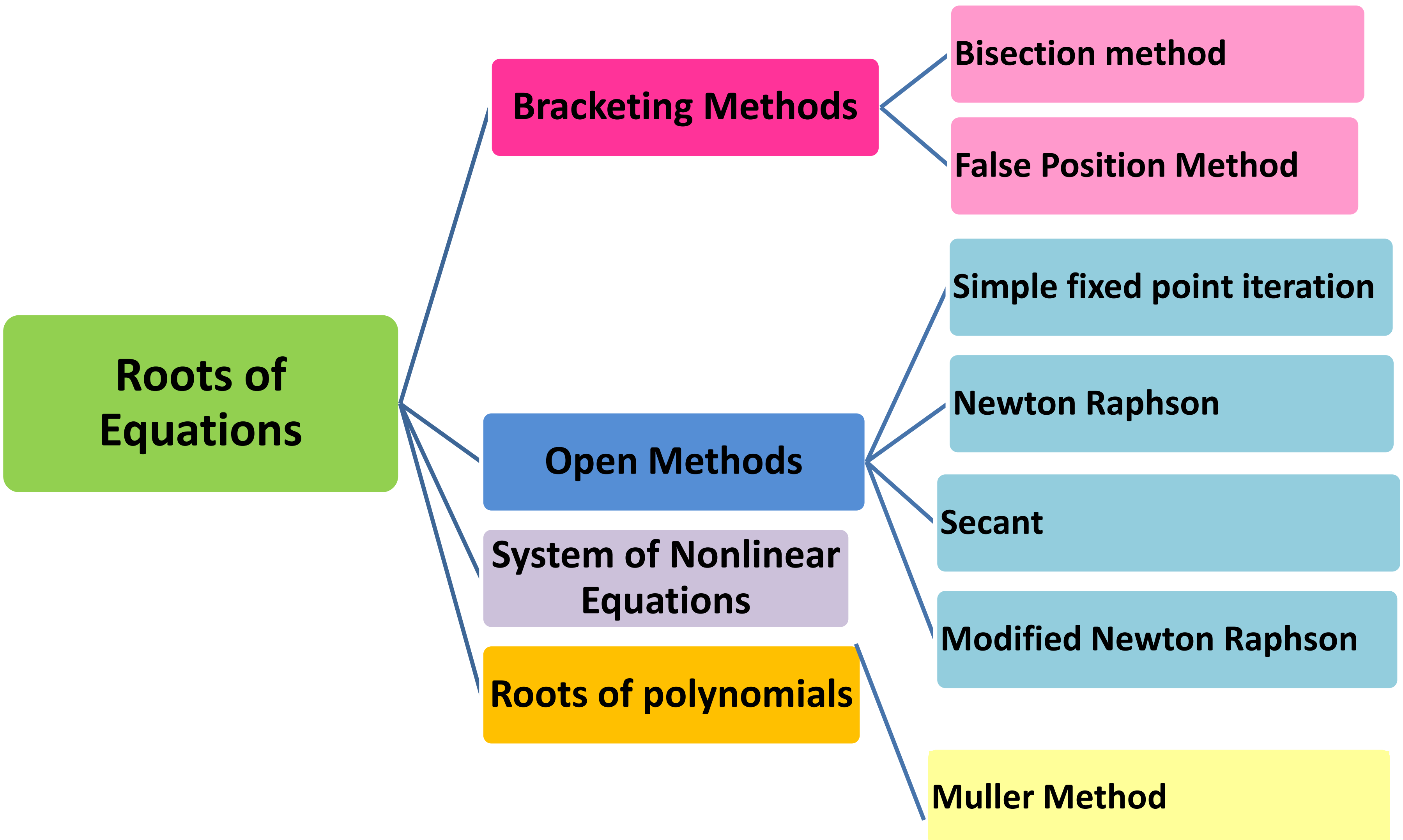
$$x_r \leftarrow x_{r+1}$$

} *while* $(|\ x_{r+1} - x_r\ | > Q)$

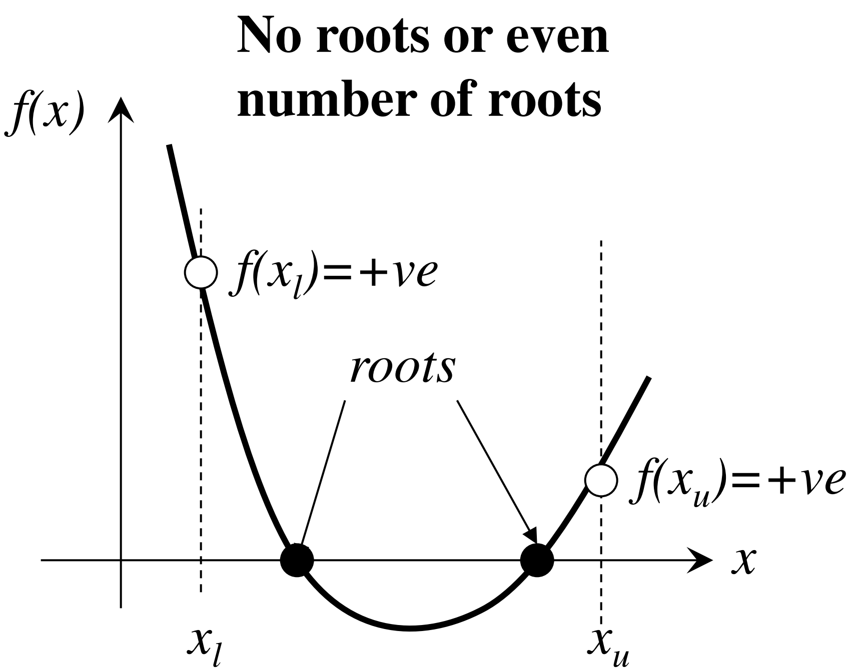
Chapter 5

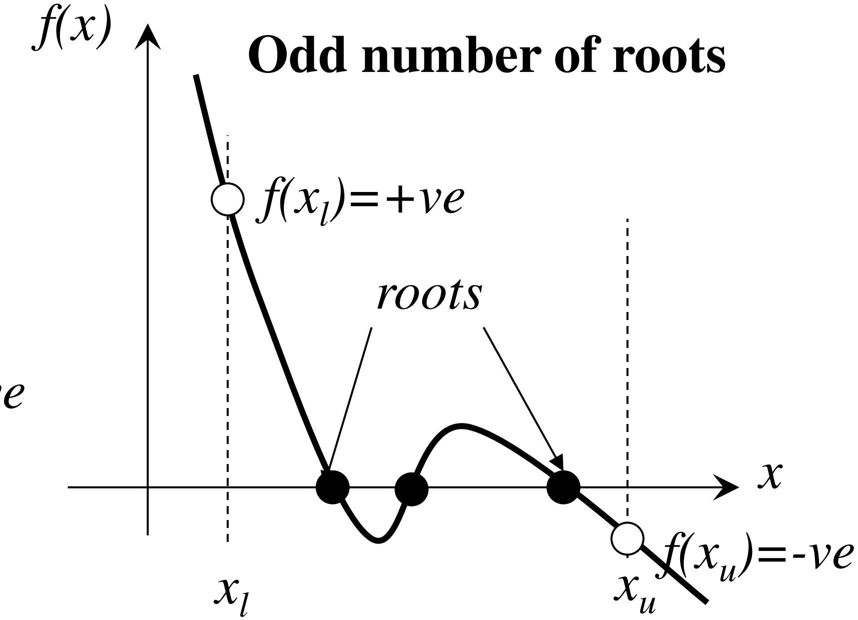
Bracketing Methods

ROOTS OF EQUATIONS



Bracketing Methods



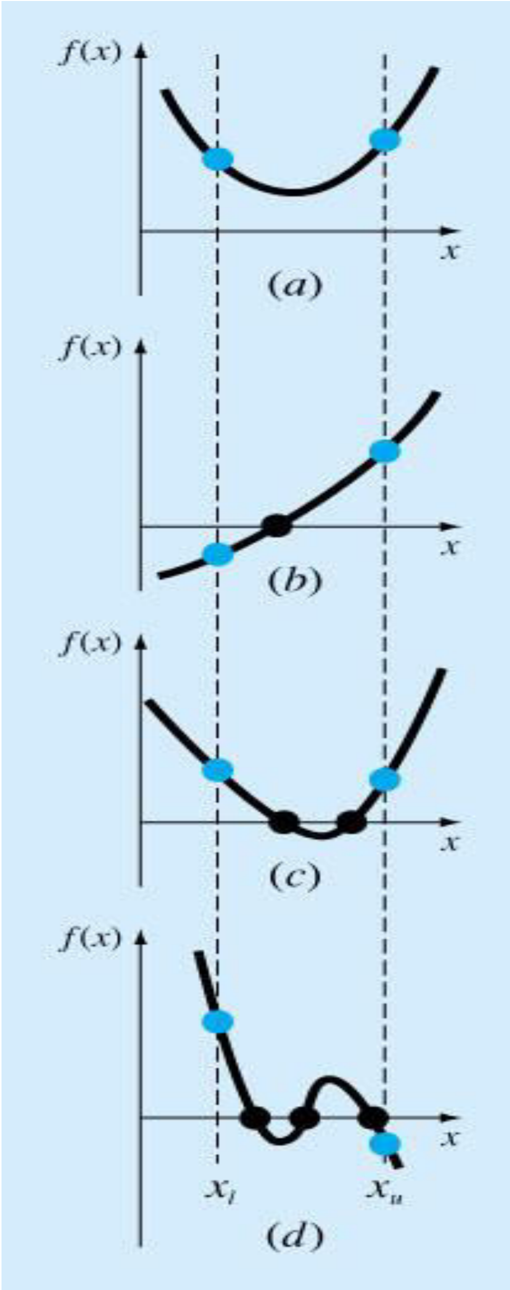


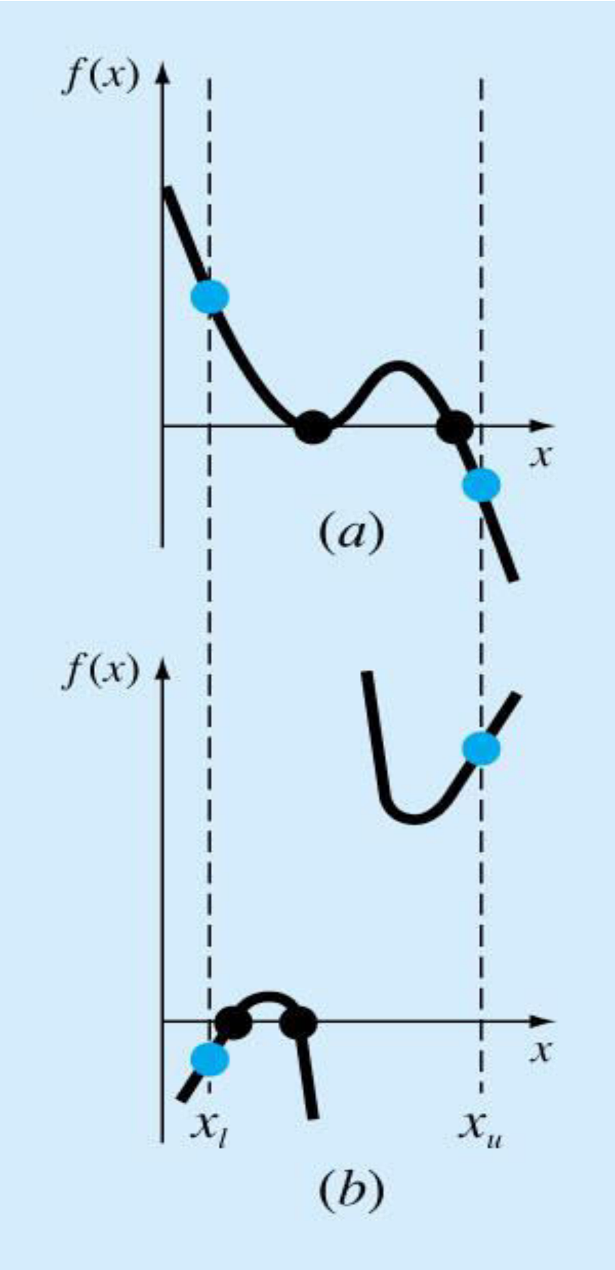
Typically changes sign in the vicinity of a root

Bracketing Methods (cont.)

- Two initial guesses (x_l and x_u) are required for the root which *bracket* the root (s).
- If one root of a **real** and **continuous** function, $f(x)=0$, is bounded by values x_l, x_u then $f(x_l).f(x_u) <0$.

(The function changes sign on opposite sides of the root)





Special Cases

Bracketing Methods
1. Bisection Method

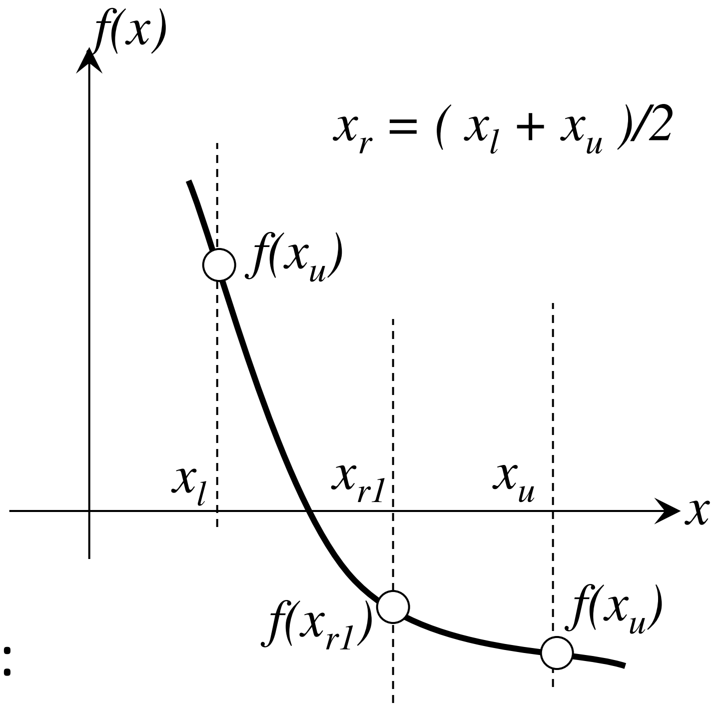
- Generally, if $f(x)$ is real and continuous in the interval x_l to x_u *and* $f(x_l).f(x_u)<0$, then there is at least one real root between x_l and x_u to this function.
- The interval at which the function changes sign is located. Then the interval is divided in half with the root lies in the midpoint of the subinterval. This process is repeated to obtained refined estimates.

Step 1: Choose lower x_l and upper x_u guesses for the root such that:

$$f(x_l).f(x_u)<0$$

Step 2: The root estimate is:

$$x_r = (x_l + x_u)/2$$

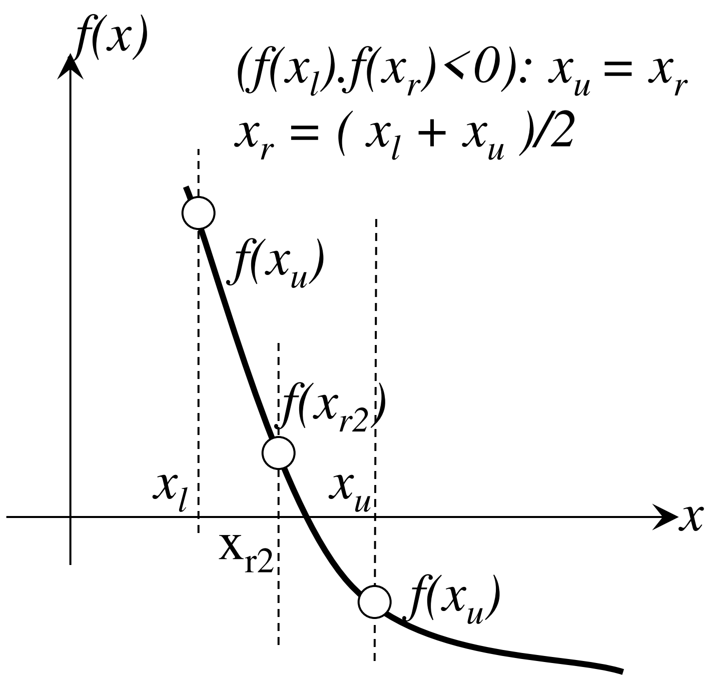


Step 3: Subdivide the interval according to:

– If $(f(x_l).f(x_r)<0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.

– If $(f(x_l).f(x_r)>0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.

– If $(f(x_l).f(x_r)=0)$ the root is x_r and stop



Bisection Method - Termination Criteria

True relaive Error :

$$\varepsilon_t = \left| \frac{X_{true} - X_{approximate}}{X_{true}} \right| \times 100\%$$

Approximate relative Error :

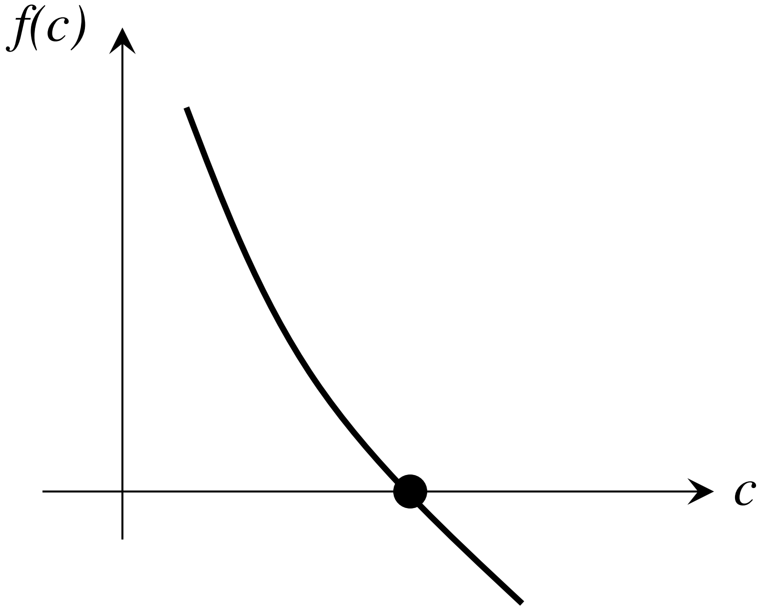
$$\varepsilon_a = \left| \frac{X_r^{n} - X_r^{n-1}}{X_r^{n}} \right| \times 100\%$$

$$\varepsilon_a = \left| \frac{X_u - X_l}{X_u + X_l} \right| \times 100\% \quad \text{(Bisection)}$$

- For the Bisection Method $\varepsilon_a > \varepsilon_t$
- The computation is terminated when ε_a becomes less than a certain criterion ($\varepsilon_a < \varepsilon_s$)

Bisection method: Example

- The parachutist velocity is $$v = \frac{mg}{c}(1 - e^{-\frac{c}{m}t})$$
- What is the drag coefficient c needed to reach a velocity of 40 m/s if m = 68.1 kg, t = 10 s, g= 9.8 m/s^2

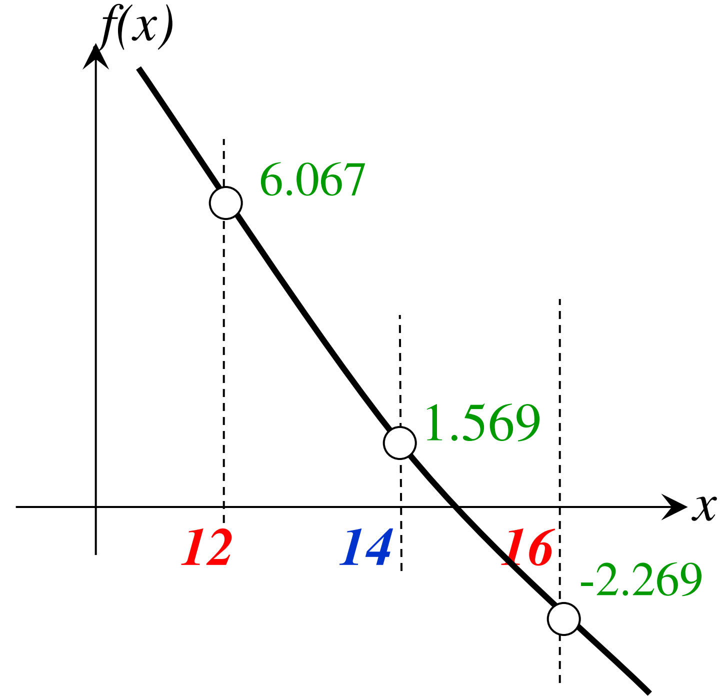


$$f(c) = \frac{mg}{c}(1 - e^{-\frac{c}{m}t}) - v$$

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$$

1. **Assume** x_l =12 and x_u=16

   $f(x_l)$=6.067 and $f(x_u)$=-2.269



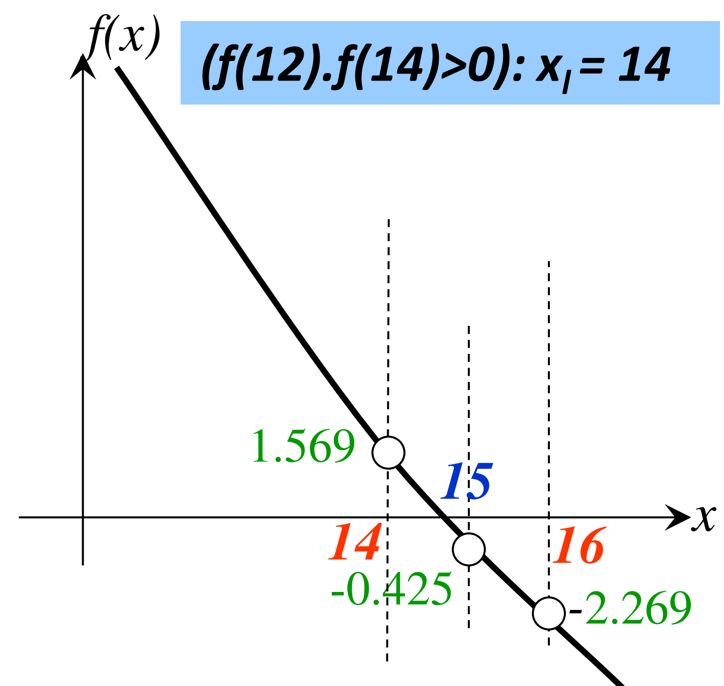
2. **The root**: $x_r=(x_l+x_u)/2= 14$

3. ***Check*** *f(12).f(14) = 6.067•1.569=9.517 >0;*

   the root lies between 14 and 16.

4. **Set** x_l = 14 and x_u=16, thus the new root

   $x_r=(14+ 16)/2= 15$



5. ***Check*** *f(14).f(15) = 1.569•-0.425= -0.666 <0;*

   the root lies bet. 14 and 15.

6. **Set** x_l = 14 and x_u=15, thus the new root

   $x_r=(14+ 15)/2= 14.5$

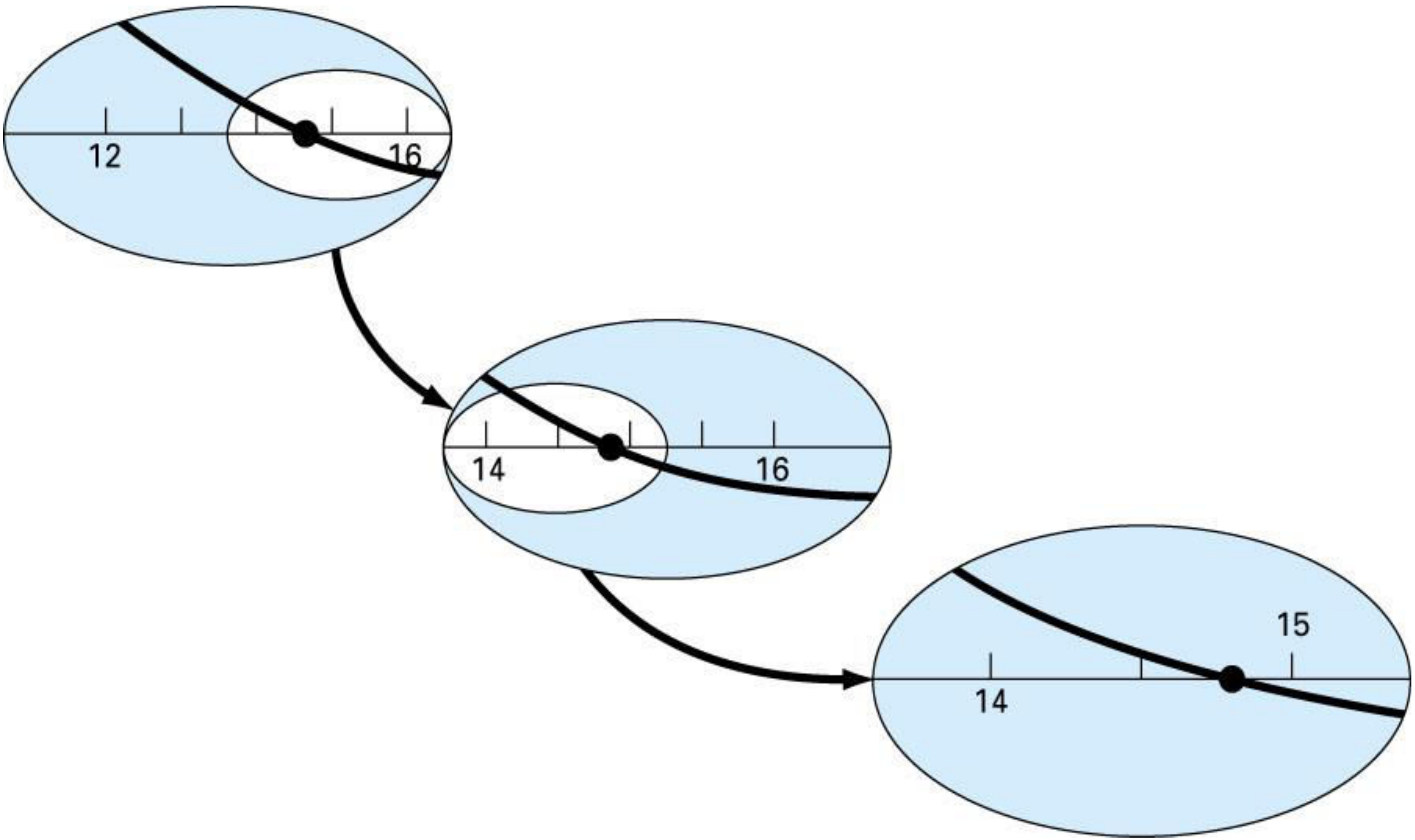
and so on……

Bisection method: Example

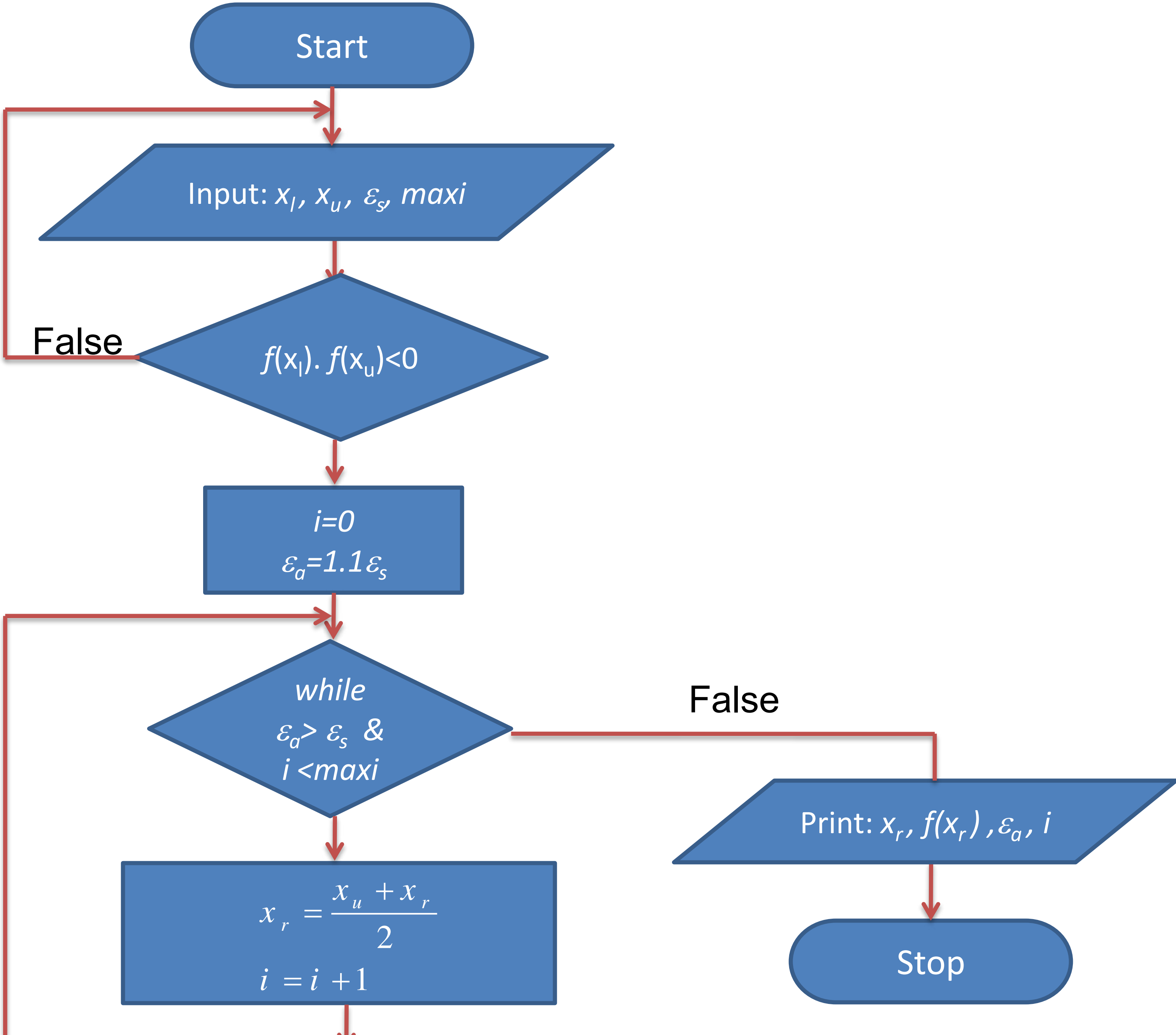
- In the previous example, if the stopping criterion is $\varepsilon_t = 0.5\%$; what is the root?

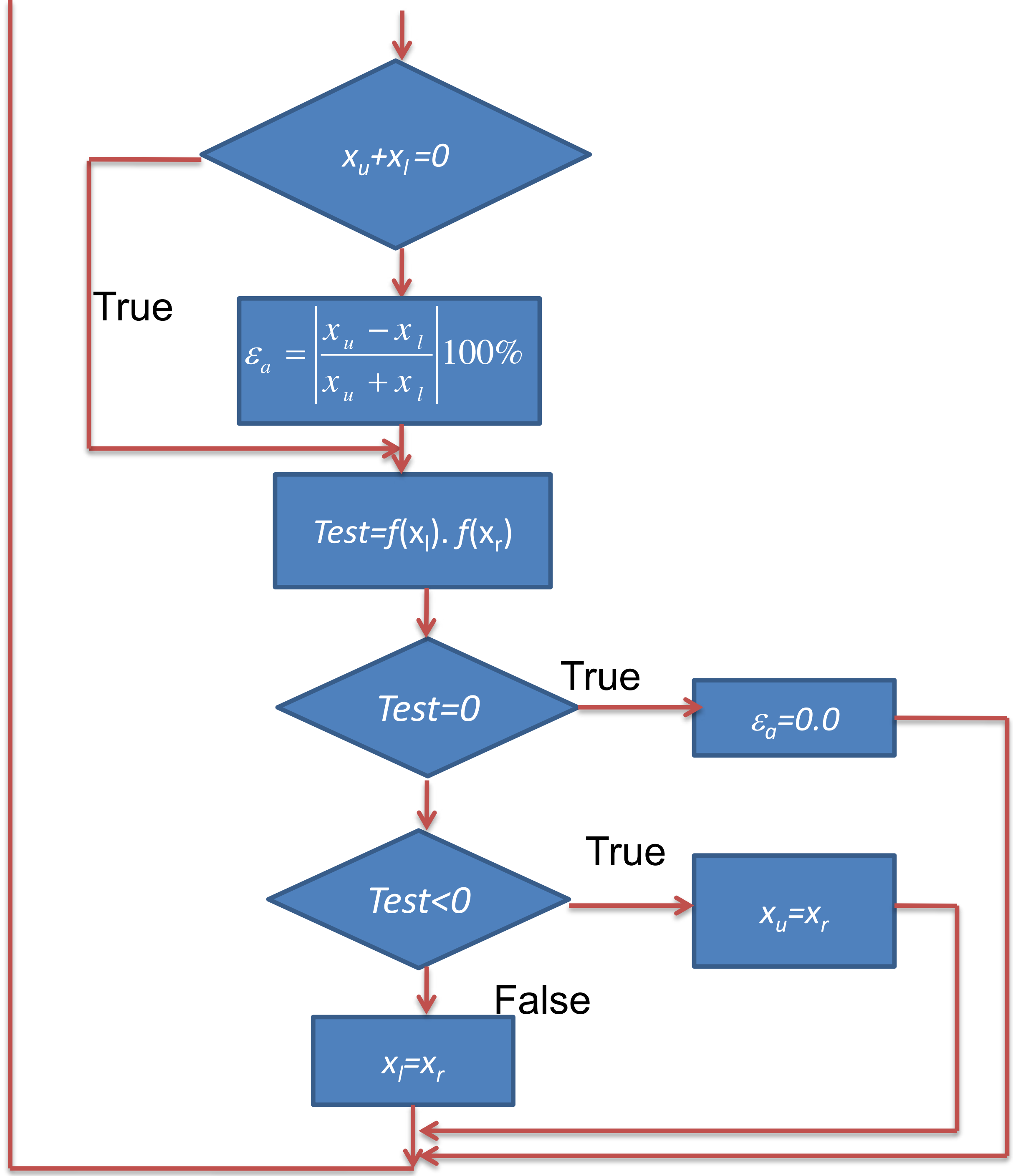
Iter.	X_l	X_u	X_r	$\varepsilon_a\%$	$\varepsilon_t\%$
1	*12*	*16*	*14*	*5.279*	--
2	*14*	*16*	*15*	*6.667*	*1.487*
3	*14*	*15*	*14.5*	*3.448*	*1.896*
4	*14.5*	*15*	*14.75*	*1.695*	*1.204*
5	*14.75*	*15*	*14.875*	*0.84*	*0.641*
6	*14.74*	*14.875*	*14.813*	*0.422*	*0.291*

Bisection method



Flow Chart –Bisection





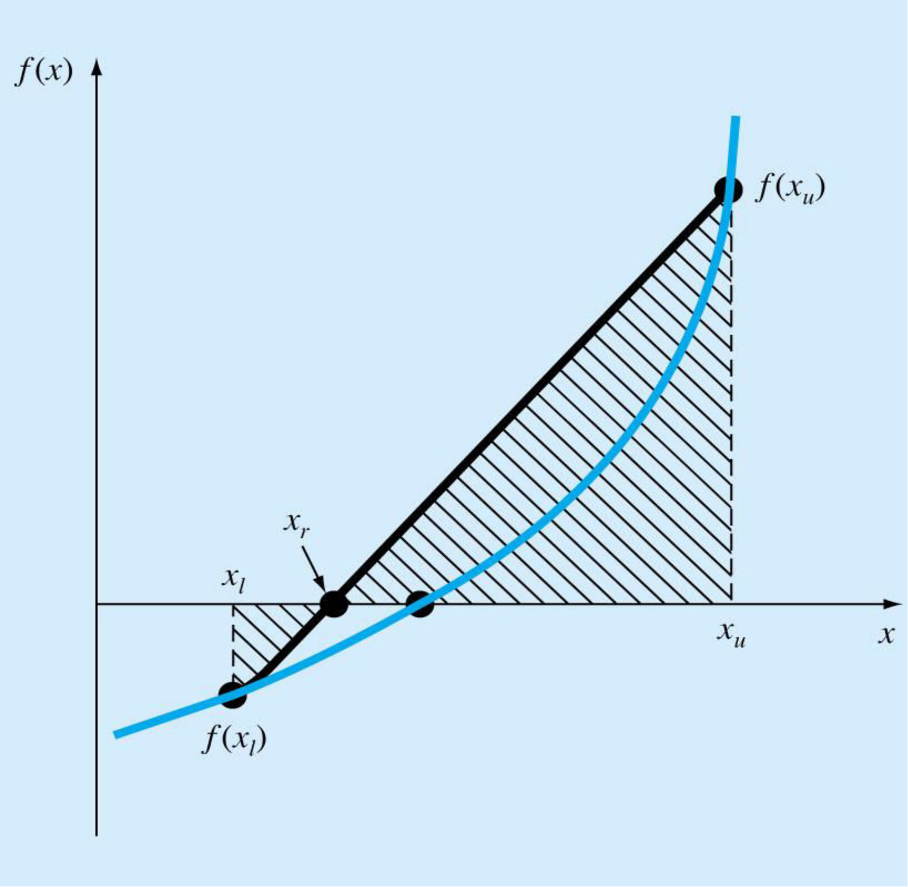
$x_u+x_l=0$
True
$\varepsilon_a = \left|\frac{x_u - x_l}{x_u + x_l}\right| 100\%$
$Test=f(x_l).\ f(x_r)$
$Test=0$
True
$\varepsilon_a=0.0$
$Test<0$
True
$x_u=x_r$
False
$x_l=x_r$

Bracketing Methods
2. False-position Method

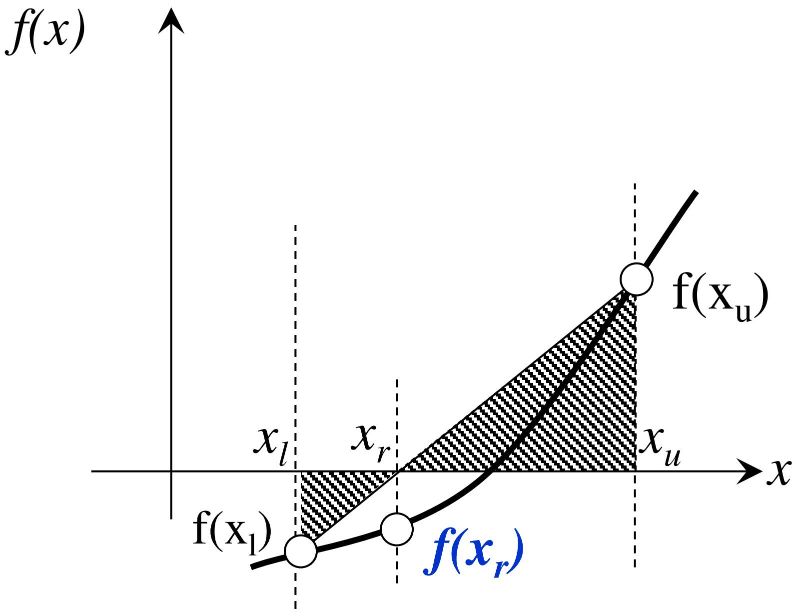
- The bisection method divides the interval x_l to x_u in half not accounting for the magnitudes of $f(x_l)$ and $f(x_u)$. For example if $f(x_l)$ is closer to zero than $f(x_u)$, then it is more likely that the root will be closer to $f(x_l)$.
- False position method is an alternative approach where $f(x_l)$ and $f(x_u)$ are joined by a straight line; the intersection of which with the x-axis represents and improved estimate of the root.

2. False-position Method

- False position method is an alternative approach where $f(x_l)$ and $f(x_u)$ are joined by a straight line; the intersection of which with the x-axis represents and improved estimate of the root.



False-position Method -Procedure



$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

False-position Method -Procedure

Step 1: Choose lower x_l and upper x_u guesses for the root such that: $f(x_l).f(x_u) < 0$

Step 2: The root estimate is:

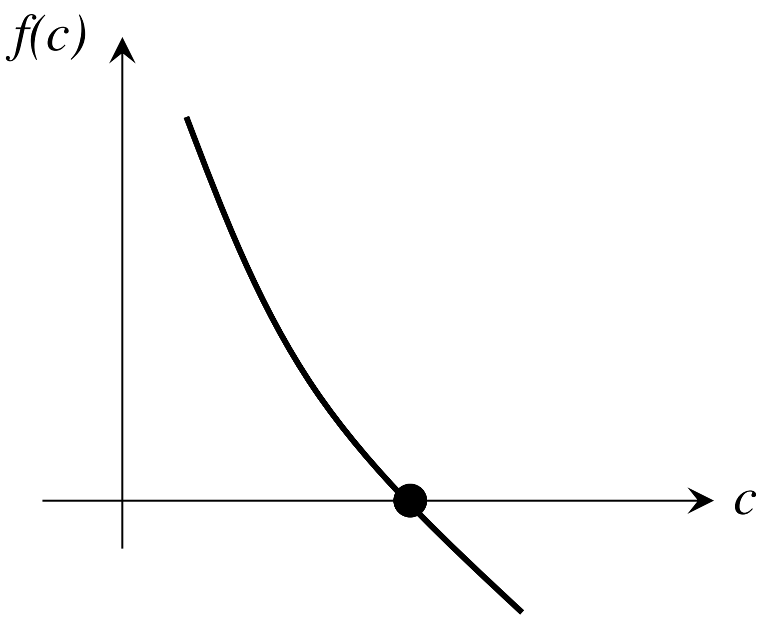
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Step 3: Subdivide the interval according to:

- If $(f(x_l).f(x_r) < 0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.
- If $(f(x_l).f(x_r) > 0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.
- If $(f(x_l).f(x_r) = 0)$ the root is x_r and stop

False position method: Example

- The parachutist velocity is $v = \dfrac{mg}{c}(1 - e^{-\frac{c}{m}t})$
- What is the drag coefficient c needed to reach a velocity of 40 m/s if m =68.1 kg, t =10 s, g= 9.8 m/s^2



$$f(c) = \frac{mg}{c}(1 - e^{-\frac{c}{m}t}) - v$$

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$$

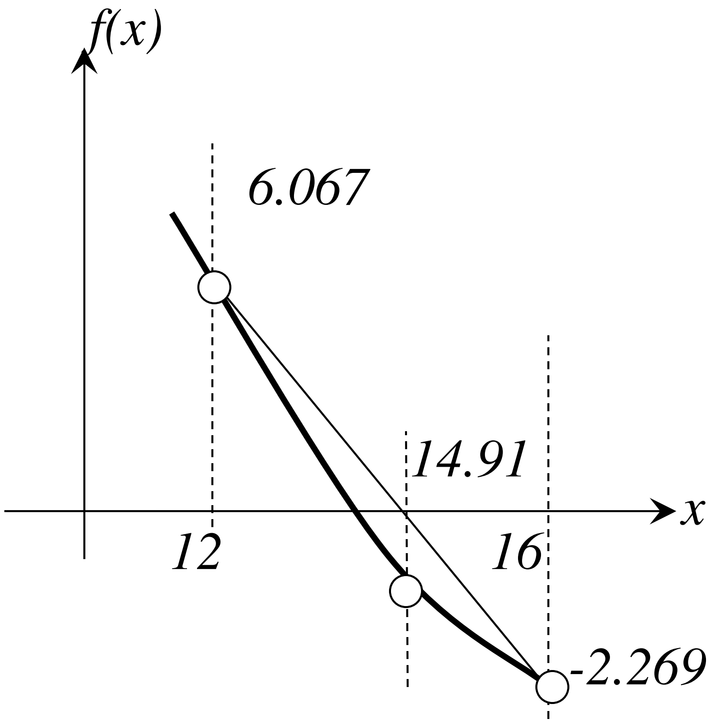
False position method: Example

1. Assume $x_l = 12$ and $x_u=16$

   $f(x_l)= 6.067$ *and* $f(x_u)= -2.269$

2. The root: $x_r=14.9113$

   $f(12) \,.\, f(14.9113) = -1.5426 < 0;$



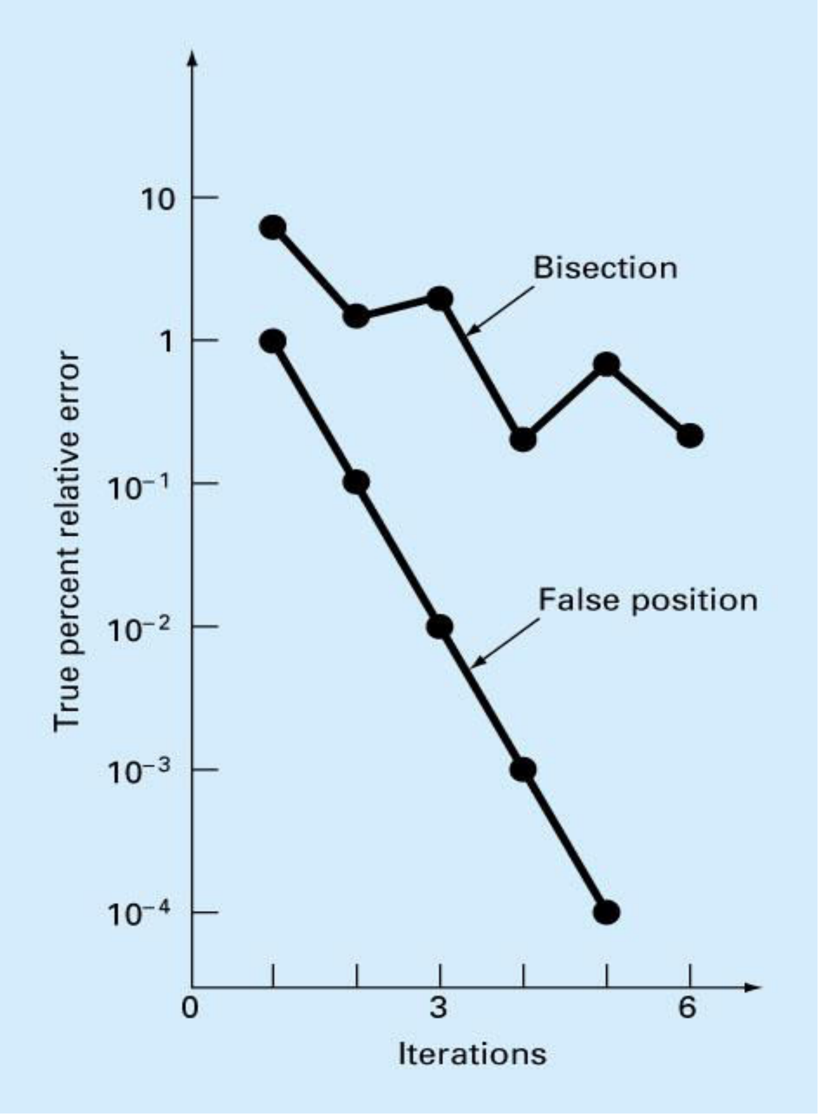
3. The root lies bet. 12 and 14.9113.

4. Assume $x_l = 12$ and $x_u=14.9113$, $f(x_l)=6.067$ *and* $f(x_u)=-0.2543$

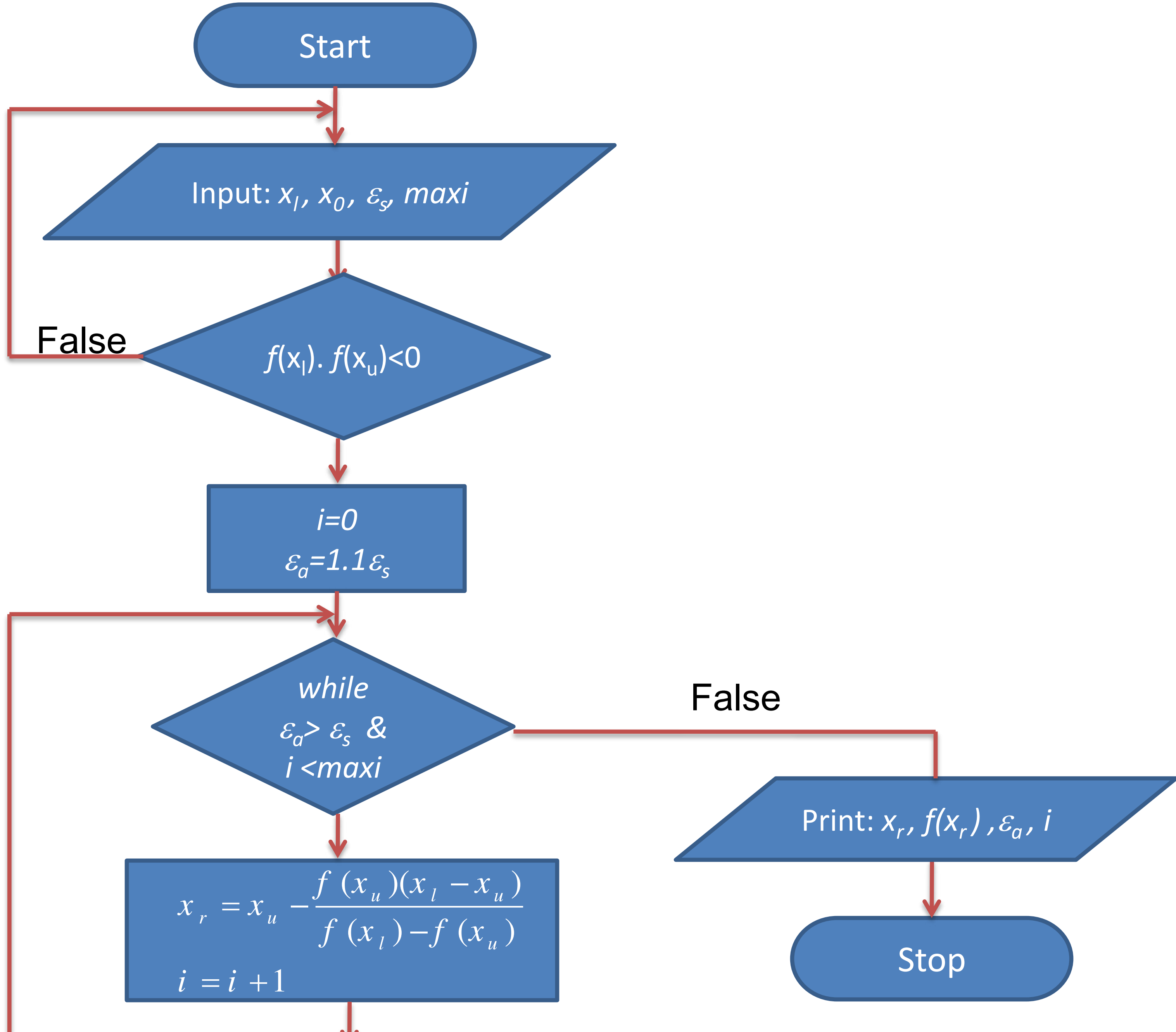
5. The new root $x_r= 14.7942$

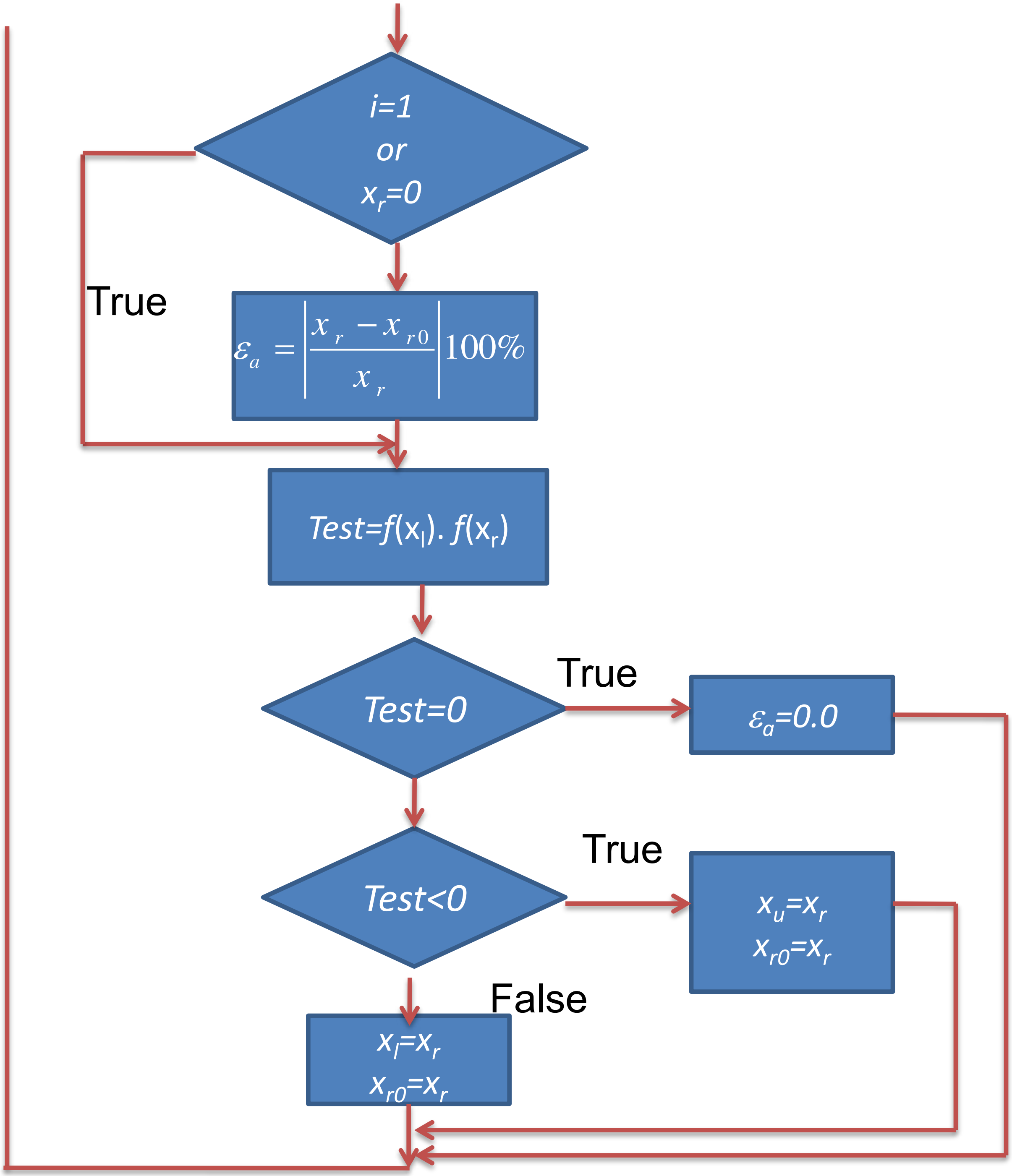
6. This has an approximate error of 0.79%

False position method: Example



Flow Chart –False Position





i=1
or
$x_r=0$
True
$\varepsilon_a = \left| \frac{x_r - x_{r0}}{x_r} \right| 100\%$
Test=f(x_l). f(x_r)
Test=0
True
ε_a=0.0
Test<0
True
$x_u=x_r$
$x_{r0}=x_r$
False
$x_l=x_r$
$x_{r0}=x_r$

False Position Method-Example 2

A Case Where Bisection Is Preferable to False Position

Problem Statement. Use bisection and false position to locate the root of

$$f(x) = x^{10} - 1$$

between $x = 0$ and 1.3.

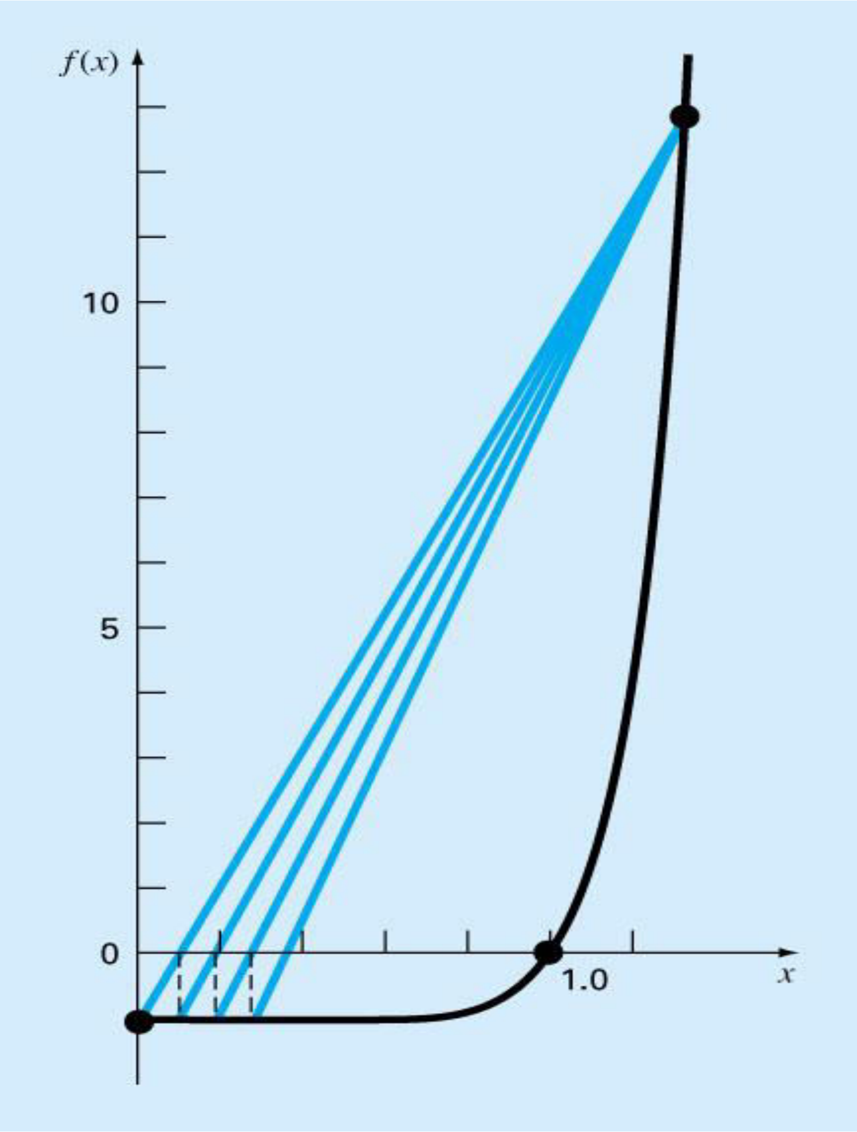
Solution. Using bisection, the results can be summarized as

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Thus, after five iterations, the true error is reduced to less than 2 percent. For false position, a very different outcome is obtained:

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.09430		90.6
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

False Position Method - Example 2



Pitfalls of the False Position Method

- Although a method such as false position is often superior to bisection, there are some cases (when function has significant curvature) that violate this general conclusion.

- In such cases, the approximate error might be misleading and the results should always be checked by substituting the root estimate into the original equation and determining whether the result is close to zero.

- Major weakness of the false-position method: its one sidedness That is, as iterations are proceeding, one of the bracketing points will tend stay fixed which lead to poor convergence.

Modified Fixed Position

- One way to mitigate the "one-sided" nature of false position is to make the algorithm detect when one of the bounds is stuck. If this occur, <u>the function value at the stagnant bound is divided in half</u>. This is thought to fasten the convergence.