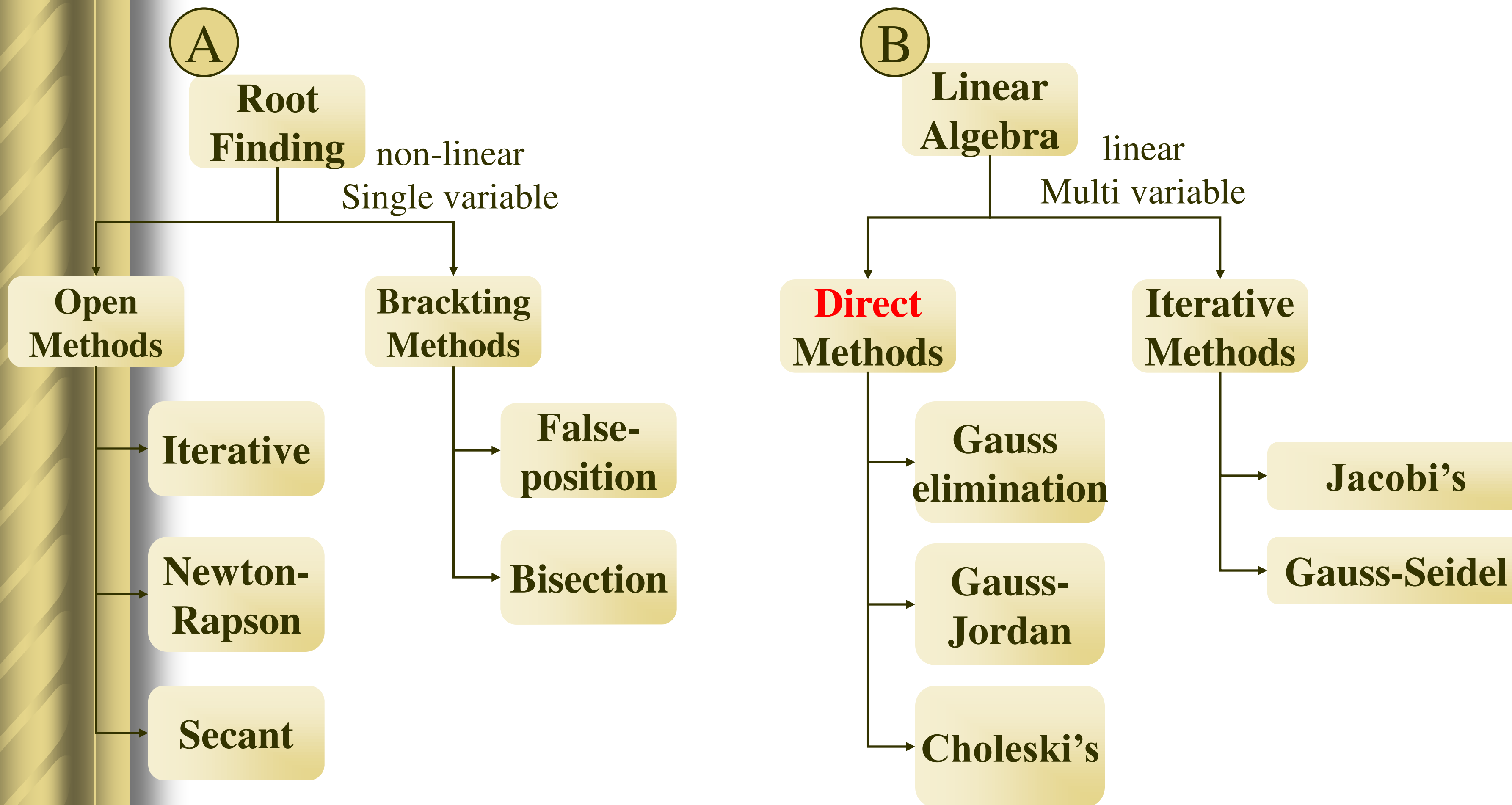
B.1

System of Linear Equations

Equation Solving



Linear Algebraic Equations

- An equation of the form $ax+by+c=0$ or equivalently $ax+by=-c$ is called a linear equation in x and y variables.
- $ax+by+cz=d$ is a linear equation in three variables, x, y, and z.
- Thus, a linear equation in n variables is $a_1x_1+a_2x_2+ \ldots +a_nx_n = b$
- A solution of such an equation consists of real numbers c_1, c_2, c_3, … , c_n
- If you need to work more than one linear equations, a system of linear equations must be solved simultaneously.

Non-computer Methods for Solving Systems of Equations

- For small number of equations ($n \le 3$) linear equations can be solved readily by simple techniques such as "**graphical method**"
- Linear algebra provides the tools to solve such a system of linear equations
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical

Solving Small Numbers of Equations

- There are many ways to solve a system of linear equations:
  - Graphical method
  - Cramer's rule
  - Method of elimination
  - Computer methods

Graphical Method

- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

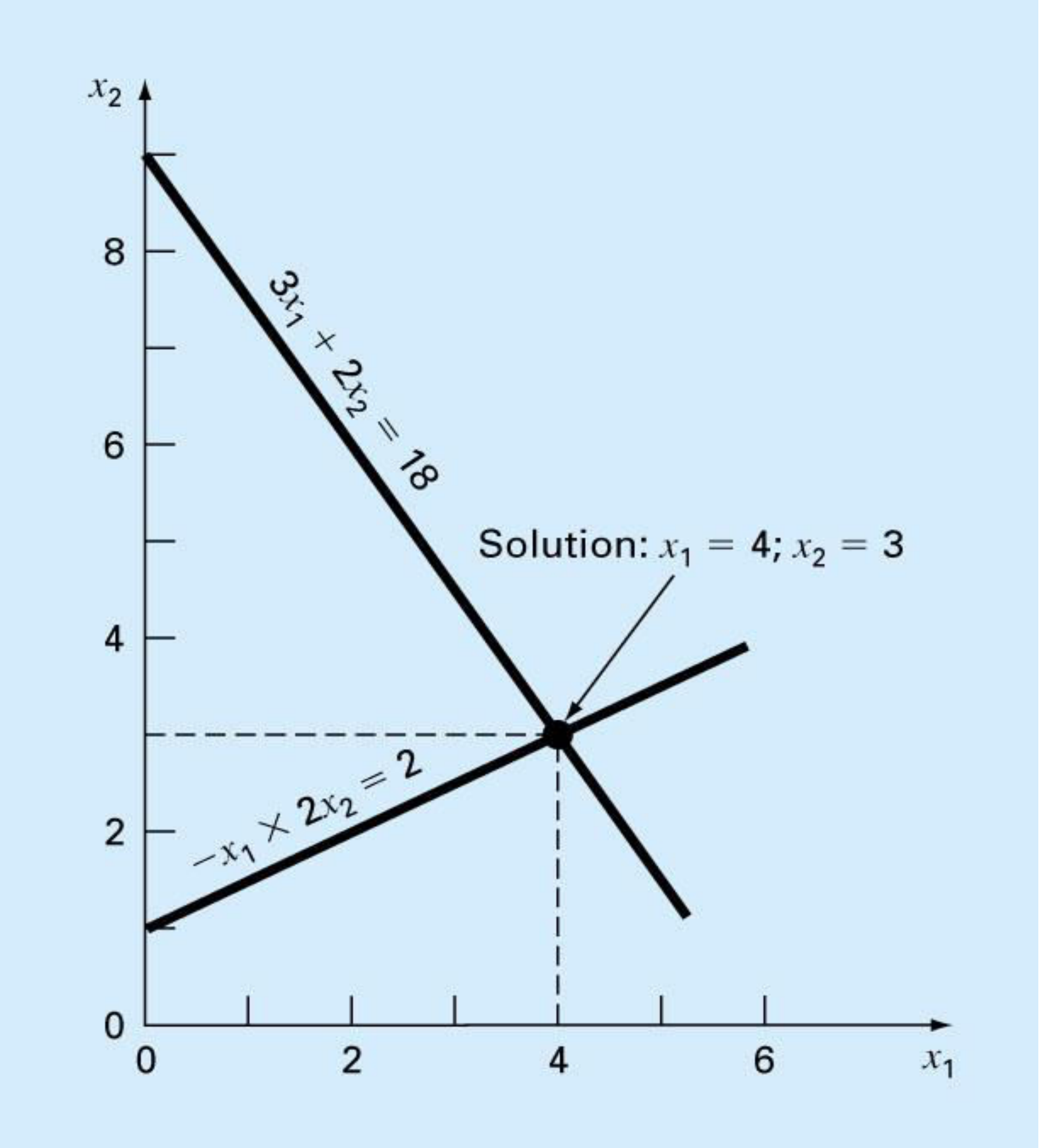
- Solve both equations for $x_{2:}$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

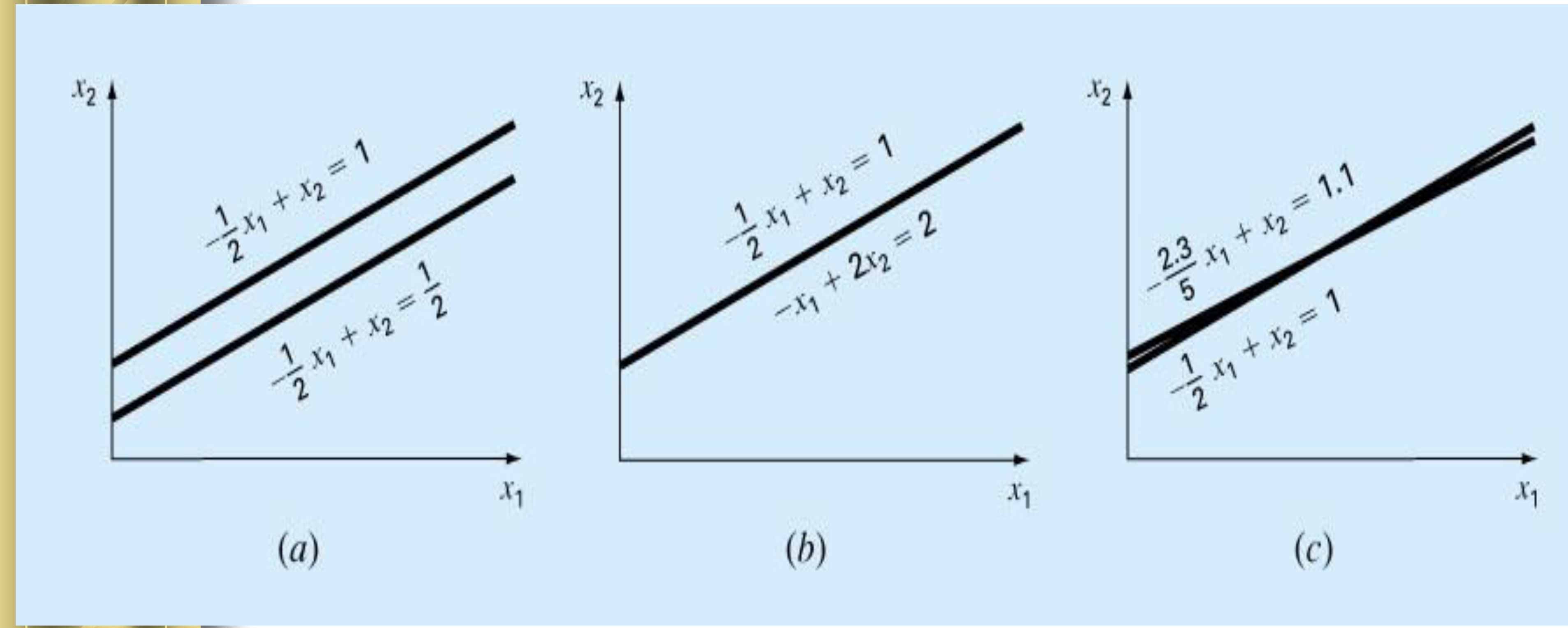
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

Graphical Method (Contd.)

Plot x_2 vs. x_1 on rectilinear paper, the intersection of the lines gives the solution



Graphical Method (Contd.)



No solution

Infinite solutions

Ill-conditioned

(Slopes are too close

Algebraic Solution

Or equate and solve for x_1

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

$$\Rightarrow \left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} - \frac{b_2}{a_{22}} = 0$$

$$\Rightarrow x_1 = -\frac{\left(\frac{b_1}{a_{12}} - \frac{b_2}{a_{22}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)} = \frac{\left(\frac{b_2}{a_{22}} - \frac{b_1}{a_{12}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)}$$

Cramer's Rule

- Gabriel Cramer was a Swiss mathematician (1704-1752)

- Cramer's rule is another solution technique that is best suited to small numbers of equations

Cramer’s Rule (Contd.)

- Each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficients of the unknown in question by the constants $b1$, $b2$,... , bn

Coefficient Matrices

- Can use determinants to solve a system of linear equations
- Use the coefficient matrix of the linear system

- Linear System

  ax+by=e

  cx+dy=f

Coefficient Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Cramer's Rule for 2x2 System

- Let A be the coefficient matrix
- Linear System

  ax+by=e

  cx+dy=f

  Coefficient Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If detA $\neq$ 0, then the system has exactly one solution

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

Example 1- Cramer's Rule 2x2

Solve the system:

- 8x+5y=2
- 2x-4y=-10

Coefficient matrix is: $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$ $\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix} = (-32)-(10) = -42$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42}$$

and

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42}$$

Example 1 (Contd.)

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \frac{-8-(-50)}{-42} = \frac{42}{-42} = -1$$

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \frac{-80-4}{-42} = \frac{-84}{-42} = 2$$

Solution: (-1,2)

Example 2- Cramer's Rule 3x3

- Solve the system:
- x+3y-z=1
- -2x-6y+z=-3
- 3x+5y-2z=4

$$z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ -2 & -6 & -3 \\ 3 & 5 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -1 \\ -2 & -6 & 1 \\ 3 & 5 & -2 \end{vmatrix}} = \frac{-4}{-4} = 1$$

Let's solve for Z

The answer is: (-2,0,1)!!!

System of Linear Equations

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Gaussian Elimination

One of the most popular techniques for solving simultaneous linear equations of the form

$$[A][X] = [B]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Consists of 2 steps

1. Forward Elimination of Unknowns.

2. Back Substitution

Forward Elimination

In general we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \Lambda & a_{1n} \\ 0 & a'_{22} & a'_{23} & \Lambda & a'_{2n} \\ 0 & 0 & a''_{33} & \Lambda & a''_{3n} \\ 0 & 0 & 0 & M & M \\ 0 & 0 & 0 & \Lambda & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ M \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ M \\ b_n^{(n-1)} \end{bmatrix}$$

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Solve the second from last equation $(n-1)^{th}$ using x_n solved for previously.

This solves for x_{n-1}.

Back Substitution

Representing Back Substitution for all equations by formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

For $i=n-1,\ n-2,\ldots,1$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

MATLAB

```
function x = GaussNaive(A,b)
% GaussNaive: naive Gauss elimination
%   x = GaussNaive(A,b): Gauss elimination without pivoting.
% input:
%   A = coefficient matrix
%   b = right hand side vector
% output:
%   x = solution vector

[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
nb = n+1;
Aug = [A b];
% forward elimination
for k = 1:n-1
  for i = k+1:n
    factor = Aug(i,k)/Aug(k,k);
    Aug(i,k:nb) = Aug(i,k:nb)-factor*Aug(k,k:nb);
  end
end
% back substitution
x = zeros(n,1);
x(n) = Aug(n,nb)/Aug(n,n);
for i = n-1:-1:1
  x(i) = (Aug(i,nb)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
end
```

Example: Rocket Velocity

The upward velocity of a rocket is given at three different times

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \, , \qquad 5 \le t \le 12.$$

Find: The Velocity at t=6,7.5,9, and 11 seconds.

Example: Rocket Velocity

$$v(t) = a_1 t^2 + a_2 t + a_3 \;, \quad 5 \le t \le 12.$$

Assume

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using date from the time / velocity table, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \qquad Row2 - \left[\frac{Row1}{25}\right] \times (64) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.81 \\ -96.21 \\ 279.2 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.81 \\ -96.21 \\ 279.2 \end{bmatrix} \quad Row3 - \left[\frac{Row1}{25}\right] \times (144) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ -336.0 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ -336.0 \end{bmatrix} \quad Row3 - \left[\frac{Row2}{-4.8}\right] \times (-16.8) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

This is now ready for Back Substitution

Example: Rocket Velocity

Back Substitution:

Solve for a_3 using the third equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Example: Rocket Velocity

Back Substitution:

Solve for a_2 using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example: Rocket Velocity

Back Substitution:

Solve for a_1 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_1 = \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$a_1 = 0.2900$$

Example: Rocket Velocity

Solution:

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$v(t) = a_1 t^2 + a_2 t + a_3$$

$$= 0.2900t^2 + 19.70t + 1.050, \; 5 \le t \le 12$$

Example: Rocket Velocity

Solution:

Substitute each value of t to find the corresponding velocity

$$v(6) = 0.2900(6)^2 + 19.70(6) + 1.050$$
$$= 129.69\ m/s.$$

$$\mathrm{v}(7.5) = 0.2900(7.5)^2 + 19.70(7.5) + 1.050$$
$$= 165.1\ \mathrm{m/s}.$$

$$\mathrm{v}(9) = 0.2900(9)^2 + 19.70(9) + 1.050$$
$$= 201.8\ \mathrm{m/s}.$$

$$\mathrm{v}(11) = 0.2900(11)^2 + 19.70(11) + 1.050$$
$$= 252.8\ \mathrm{m/s}.$$

Pitfalls

Two Potential Pitfalls

-Division by zero: May occur in the forward elimination steps. Consider the set of equations:

$$10x_2 - 7x_3 = 7$$

$$6x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

- Round-off error: Prone to round-off errors.

Pitfalls: Example

Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

Pitfalls: Example

Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$

Pitfalls: Example

Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Partial Pivoting

Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

If the maximum of the values is $|a_{pk}|$ In the p^{th} row, $k \le p \le n,$

then switch rows p and k.

Partial Pivoting

What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 3x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the **second** column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

MATLAB

```
% forward elimination
for k = 1:n-1
  % partial pivoting
  [big,i]=max(abs(Aug(k:n,k)));
  ipr=i+k-1;
  if ipr~=k
    Aug([k,ipr],:)=Aug([ipr,k],:);
  end
  for i = k+1:n
    factor=Aug(i,k)/Aug(k,k);
    Aug(i,k:nb)=Aug(i,k:nb)-factor*Aug(k,k:nb);
  end
end
```

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Summary

-Forward Elimination

-Back Substitution

-Pitfalls

-Partial Pivoting

Adapted from Prof. Autar Kaw

Gauss-Jordan Method

- Solve the system of equations using Gauss-Jordan Method

$$
\begin{aligned}
x - y + 2z &= 0 \\
x - 2y + 3z &= -1 \\
2x - 2y + z &= -3
\end{aligned}
\qquad
\left[\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
1 & -2 & 3 & -1 \\
2 & -2 & 1 & -3
\end{array}\right]
$$

$$
-R_1 + R_2 \rightarrow R_2
$$

$$
\begin{array}{cccc}
-1 & 1 & -2 & 0 \\
1 & -2 & 3 & -1 \\
0 & -1 & 1 & -1
\end{array}
$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 2 & -2 & 1 & -3 \end{array}\right]$$

$$-R_1 + R_2 \rightarrow R_2$$

$$\begin{array}{cccc} -1 & 1 & -2 & 0 \\ 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & -1 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$
\begin{aligned}
x - y + 2z &= 0 \\
x - 2y + 3z &= -1 \\
2x - 2y + z &= -3
\end{aligned}
\qquad
\left[\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
0 & -1 & 1 & -1 \\
2 & -2 & 1 & -3
\end{array}\right]
$$

$$
-2R_1 + R_3 \to R_3
$$

$$
\begin{array}{cccc}
-2 & 2 & -4 & 0 \\
2 & -2 & 1 & -3 \\
0 & 0 & -3 & -3
\end{array}
$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -3 & -3 \end{array}\right]
\begin{array}{cccc} \multicolumn{4}{c}{-2R_1 + R_3 \rightarrow R_3} \\ -2 & 2 & -4 & 0 \\ 2 & -2 & 1 & -3 \\ 0 & 0 & -3 & -3 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$-1R_2 \rightarrow R_2$$

$$\mathbf{0 \quad 1 \quad -1 \quad 1}$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$-1R_2 \rightarrow R_2$$

$$0 \quad 1 \quad -1 \quad 1$$

- Solve the system of equations using Gauss-Jordan Method

$$
\begin{aligned}
x - y + 2z &= 0 \\
x - 2y + 3z &= -1 \\
2x - 2y + z &= -3
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & -3 & -3
\end{array}\right]
$$

$$
R_2 + R_1 \rightarrow R_1
$$

$$
\begin{array}{cccc}
0 & 1 & -1 & 1 \\
1 & -1 & 2 & 0 \\
1 & 0 & 1 & 1
\end{array}
$$

- Solve the system of equations using Gauss-Jordan Method

$$x - y + 2z = 0$$
$$x - 2y + 3z = -1$$
$$2x - 2y + z = -3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$R_2 + R_1 \rightarrow R_1$$

$$\begin{array}{cccc} 0 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$x - y + 2z = 0$$
$$x - 2y + 3z = -1$$
$$2x - 2y + z = -3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$-\frac{1}{3}R_3 \rightarrow R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

- Solve the system of equations using Gauss-Jordan Method

$$
\begin{aligned}
x - y + 2z &= 0\\
x - 2y + 3z &= -1\\
2x - 2y + z &= -3
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 1 & 1\\
0 & 1 & -1 & 1\\
0 & 0 & 1 & 1
\end{array}\right]
$$

$$-\frac{1}{3}R_3 \rightarrow R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

- Solve the system of equations using Gauss-Jordan Method

$$
\begin{aligned}
x - y + 2z &= 0 \\
x - 2y + 3z &= -1 \\
2x - 2y + z &= -3
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 1 & 1 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 1
\end{array}\right]
$$

$$R_3 + R_2 \rightarrow R_2$$

$$
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 1 & -1 & 1 \\
0 & 1 & 0 & 2
\end{array}
$$

- Solve the system of equations using Gauss-Jordan Method

$$x - y + 2z = 0$$
$$x - 2y + 3z = -1$$
$$2x - 2y + z = -3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_3 + R_2 \rightarrow R_2$$

$$\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$-R_3 + R_1 \rightarrow R_1$$

$$\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$x - y + 2z = 0$$
$$x - 2y + 3z = -1$$
$$2x - 2y + z = -3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$-R_3 + R_1 \rightarrow R_1$$

$$\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}$$

- Solve the system of equations using Gauss-Jordan Method

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2y + 3z &= -1 \\ 2x - 2y + z &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$(0, 2, 1)$

B.4

Gauss-Seidel Method

Gauss-Seidel Method

An *iterative* method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Gauss-Seidel Method

Why?

The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Calculate the Absolute Relative Approximate Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns.

Gauss-Seidel Method: Example 1

The upward velocity of a rocket is given at three different times

Time, t	Velocity, v
s	m/s
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12.$$

Gauss-Seidel Method: Example 1

Using a Matrix template of the form

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

The system of equations becomes

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial Guess: Assume an initial guess of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Gauss-Seidel Method: Example 1

Rewriting each equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Gauss-Seidel Method: Example 1

Applying the initial guess and solving for a_i

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

When solving for a_2, how many of the initial guess values were used?

Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\varepsilon_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| x100 = 72.76\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| x100 = 125.47\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| x100 = 103.22\%$$

At the end of the first iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The maximum absolute relative approximate error is 125.47%

Gauss-Seidel Method: Example 1

Iteration #2

Using

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

from iteration #1

the values of a_i are found:

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{12.056-3.6720}{12.056}\right| \text{x}100 = 69.542\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882-(-7.8510)}{-54.882}\right| \text{x}100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34-(-155.36)}{-798.34}\right| \text{x}100 = 80.54\%$$

At the end of the second iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.34 \end{bmatrix}$$

The maximum absolute relative approximate error is 85.695%

Gauss-Seidel Method: Example 1

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	3.672	72.767	-7.8510	125.47	-155.36	103.22
2	12.056	67.542	-54.882	85.695	-798.34	80.540
3	47.182	74.448	-255.51	78.521	-3448.9	76.852
4	193.33	75.595	-1093.4	76.632	-14440	76.116
5	800.53	75.850	-4577.2	76.112	-60072	75.962
6	3322.6	75.907	-19049	75.971	-249580	75.931

! Notice – The relative errors are not decreasing at any significant rate

Also, the solution is not converging to the true solution of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0858 \end{bmatrix}$$

Gauss-Seidel Method: Pitfall

What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of the Gauss-Siedel method: not all systems of equations will converge.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\left|a_{\mathrm{ii}}\right| \geq \sum_{\substack{j=1 \\ j\neq \mathrm{i}}}^{n} \left|a_{ij}\right| \quad \text{for all 'i'} \qquad \text{and} \qquad \left|a_{ii}\right| \rangle \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right| \quad \text{for at least one 'i'}$$

Gauss-Seidel Method: Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Which coefficient matrix is diagonally dominant?

$$[A]=\begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix}$$

$$[B]=\begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Gauss-Seidel Method: Example 2

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Will the solution converge using the Gauss-Siedel method?

Gauss-Seidel Method: Example 2

Checking if the coefficient matrix is diagonally dominant

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge using the Gauss-Siedel Method

Gauss-Seidel Method: Example 2

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Gauss-Seidel Method: Example 2

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 67.662\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

Gauss-Seidel Method: Example 2

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Substituting the x values into the equations

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Gauss-Seidel Method: Example 2

Iteration #2 absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.62\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.887\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.876\%$$

The maximum absolute relative error after the first iteration is 240.62%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Gauss-Seidel Method: Example 2

Repeating more iterations, the following values are obtained

Iteration	a_1	$\lvert\varepsilon_a\rvert_1$	a_2	$\lvert\varepsilon_a\rvert_2$	a_3	$\lvert\varepsilon_a\rvert_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

The solution obtained

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$$

is close to the exact solution of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Gauss-Seidel Method: Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting the equations

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Gauss-Seidel Method: Example 3

Conducting six iterations

Iteration	a_1	$\left\|\in_a\right\|_1$	a_2	$\left\|\in_a\right\|_2$	a_3	$\left\|\in_a\right\|_3$
1	21.000	110.71	0.80000	100.00	5.0680	98.027
2	-196.15	109.83	14.421	94.453	-462.30	110.96
3	-1995.0	109.90	-116.02	112.43	4718.1	109.80
4	-20149	109.89	1204.6	109.63	-47636	109.90
5	2.0364×10^5	109.90	-12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	1.0990	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging.

Does this mean that the Gauss-Seidel method cannot be used?

Gauss-Seidel Method

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

But this is the same set of equations used in example #2, which did converge.

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

Gauss-Seidel Method

Not every system of equations can be rearranged to have a diagonally dominant coefficient matrix.

Observe the set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

Which equation(s) prevents this set of equation from having a diagonally dominant coefficient matrix?

MATLAB

$$x_1^{\text{new}} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{\text{old}} - \frac{a_{13}}{a_{11}} x_3^{\text{old}}$$

$$x_2^{\text{new}} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{\text{new}} - \frac{a_{23}}{a_{22}} x_3^{\text{old}}$$

$$x_3^{\text{new}} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{\text{new}} - \frac{a_{32}}{a_{33}} x_2^{\text{new}}$$

Notice that the solution can be expressed concisely in matrix form as

$$\{x\} = \{d\} - [C]\{x\}$$

where

$$\{d\} = \begin{Bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ b_3/a_{33} \end{Bmatrix}$$

and

$$[C] = \begin{bmatrix} 0 & a_{12}/a_{11} & a_{13}/a_{11} \\ a_{21}/a_{22} & 0 & a_{23}/a_{22} \\ a_{31}/a_{33} & a_{32}/a_{33} & 0 \end{bmatrix}$$

MATLAB

```
for i = 1:n
  C(i,1:n) = C(i,1:n)/A(i,i);
end
for i = 1:n
  d(i) = b(i)/A(i,i);
end
iter = 0;
while (1)
  xold = x;
  for i = 1:n
    x(i) = d(i)-C(i,:)*x;
    if x(i) ~= 0
      ea(i) = abs((x(i) - xold(i))/x(i)) * 100;
    end
  end
  iter = iter+1;
  if max(ea)<=es | iter >= maxit, break, end
end
```

Gauss-Seidel Method

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

Gauss-Seidel Method

Questions?

B.3

Choleski's Method

LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

The Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1. Decompose

$$[A] = [L][U]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21} & 1 & 0 \\ \lambda_{31} & \lambda_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2. Forward substitution:

   Given **[L]** and **[B]** find **[Y]**

$$\begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21} & 1 & 0 \\ \lambda_{31} & \lambda_{32} & 1 \end{bmatrix} \underset{}{\overset{?}{\boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$[U][X] = [Y] \qquad [L][Y] = [B]$$

3. Backward substitution

   Given **[U]** and **[Y]** find **[X]**

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \overset{?}{\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

LU Decomposition

How is this better or faster than Gauss Elimination?

Let's look at computational time.

n = number of equations

To decompose [A], time is proportional to $\frac{n^3}{3}$

To solve $[U][X]=[Y]$ and $[L][Y]=[B]$

time proportional to $\frac{n^2}{2}$

LU Decomposition

Therefore, total computational time for LU Decomposition is proportional to

$$\frac{n^3}{3} + 2\left(\frac{n^2}{2}\right) \quad \text{or} \quad \frac{n^3}{3} + n^2$$

Gauss Elimination computation time is proportional to

$$\frac{n^3}{3} + \frac{n^2}{2}$$

How is this better?

LU Decomposition

What about a situation where the **[B]** vector changes?

In LU Decomposition, LU decomposition of [A] is independent of the [B] vector, therefore it only needs to be done once.

Let m = the number of times the [B] vector changes

The computational times are proportional to

LU decomposition = $\frac{n^3}{3} + m(n^2)$ Gauss Elimination = $m(\frac{n^3}{3} + \frac{n^2}{2})$

Consider a 100 equation set with 50 right hand side vectors

LU Decomposition = 8.33×10^5 Gauss Elimination = 1.69×10^7

LU Decomposition

Method: [A] Decompose to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \lambda_{21} & 1 & 0\\ \lambda_{31} & \lambda_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[U] *is the same as the coefficient matrix at the end of the forward elimination step.*

[L] *is obtained using the multipliers that were used in the forward elimination process*

LU Decomposition

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$Row_2 - Row_1 \times \frac{64}{25}$$

$$Row_3 - Row_1 \times \frac{144}{25}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$Row_3 - Row_2 \times \frac{-16.8}{-4.8}$$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = ?$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 64/25 & 1 & 0 \\ 144/25 & -16.8/-4.8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Forward Substitution

Solve ***[L][Y]=[B]*** for ***[Y]***

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 106.8 \\ y_2 &= 177.2 - 2.56y_1 \\ &= 177.2 - 2.56(106.8) \\ &= -96.2 \\ y_3 &= 279.2 - 5.76y_1 - 3.5y_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.21) \\ &= 0.735 \end{aligned}$$

$$[Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Recall: *[Y]* is the same right hand side matrix, as obtained by Gauss elimination method

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Backward Substitution

Solve ***[U][X]=[Y]*** for ***[X]***

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

$$0.7x_3 = 0.735$$

$$x_3 = \frac{0.735}{0.7}$$

$$= 1.050$$

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$x_2 = \frac{-96.21 + 1.56x_3}{-4.8}$$

$$= \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$= 19.70$$

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$x_1 = \frac{106.8 - 5x_2 - x_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

B.4

Computing Inverse Matrix

Finding Inverse

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

$$Z = A^{-1}$$

$$AZ = I$$

$$LUZ = I$$

$$LY = I$$

$$UZ = Y$$

$$LY_1 = I_1 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{11}\\ y_{21}\\ y_{31}\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}y_{11}\\ y_{21}\\ y_{31}\end{bmatrix}=\begin{bmatrix}1\\ -2.56\\ 3.2\end{bmatrix}$$

$$LY_2 = I_2 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{12}\\ y_{22}\\ y_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}y_{12}\\ y_{22}\\ y_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ -3.5\end{bmatrix}$$

$$LY_3 = I_3 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{13}\\ y_{23}\\ y_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \Rightarrow \begin{bmatrix}y_{13}\\ y_{23}\\ y_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

Finding Inverse

$Z = A^{-1}$

$AZ = I$

$LUZ = I$

$LY = I$

$UZ = Y$

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \qquad [A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$UZ_1 = Y_1 \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix} \Rightarrow \begin{bmatrix} z_{11} \\ z_{21} \\ z_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$UZ_2 = Y_2 \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{22} \\ z_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3.5 \end{bmatrix} \Rightarrow \begin{bmatrix} z_{12} \\ z_{22} \\ z_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.4167 \\ -5 \end{bmatrix}$$

$$UZ_3 = Y_3 \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} z_{13} \\ z_{23} \\ z_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} z_{13} \\ z_{23} \\ z_{33} \end{bmatrix} = \begin{bmatrix} 0.0357 \\ -0.4643 \\ 1.4286 \end{bmatrix}$$

Finding Inverse

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$Z = A^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.0357 \\ -0.9524 & 1.4167 & -0.4643 \\ 4.571 & -5 & 1.4286 \end{bmatrix}$$

Finding Inverse by Gauss-Jordan

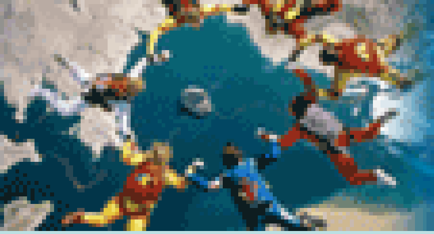
$$[A \quad I] \equiv \begin{bmatrix} a_{11} & \Lambda & a_{1n} & 1 & 0 & \Lambda & 0 \\ a_{21} & \Lambda & a_{2n} & 0 & 1 & \Lambda & 0 \\ M & O & M & M & M & O & M \\ a_{n1} & \Lambda & a_{nn} & 0 & 0 & \Lambda & 1 \end{bmatrix}$$

$$\Big\Downarrow \text{ Gauss-Jordan}$$

$$[I \quad B] \equiv \begin{bmatrix} 1 & 0 & \Lambda & 0 & b_{11} & \Lambda & b_{1n} \\ 0 & 1 & \Lambda & 0 & b_{21} & \Lambda & b_{2n} \\ M & O & M & M & M & O & M \\ 0 & \Lambda & 0 & 1 & b_{n1} & \Lambda & b_{nn} \end{bmatrix} \Rightarrow \quad B = A^{-1}$$

This method is numerically unstable unless complete pivoting is used

Thanks to Prof. Autar Kaw for his wonderful slides



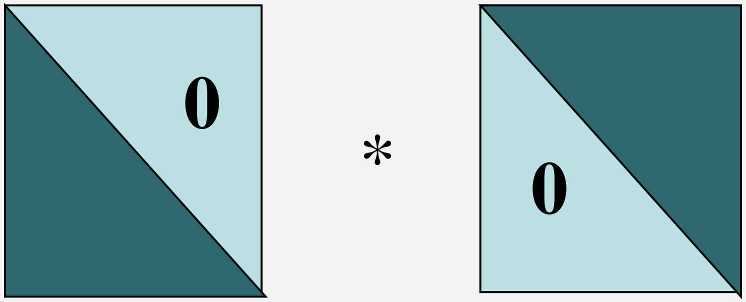
LU Decomposition and Matrix Inversion

Chapter 10

Credit: Prof. Lale Yurttas, Chemical Eng., Texas A&M University

Solve $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ *(system of linear equations)*

Decompose $\boldsymbol{A} = \boldsymbol{L} \cdot \boldsymbol{U}$



L : Lower Triangular Matrix U : Upper Triangular Matrix

To solve $[A]\{x\}=\{b\}$

$$[L][U]=[A] \quad \rightarrow \quad [L][U]\{x\}=\{b\}$$

Consider
$$[U]\{x\}=\{d\}$$
$$[L]\{d\}=\{b\}$$

1. Solve $[L]\{d\}=\{b\}$ using **forward substitution** to get $\{d\}$

2. Use **back substitution** to solve $[U]\{x\}=\{d\}$ to get $\{x\}$

Both phases, (1) and (2), take $O(n^2)$ steps.

$$[A]\{x\} = \{b\} \quad \Longrightarrow \quad [L][U]\{x\} = \{b\}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*L*] **[*U*]**

$$[A]\{x\} = \{b\} \quad \Rightarrow \quad [L][U]\{x\} = \{b\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Elimination ➔ ➔

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

$[U]$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Coefficients used during the elimination step

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a}_{31} & \boldsymbol{a}_{32} & \boldsymbol{a}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{l}_{21} & 1 & 0 \\ \boldsymbol{l}_{31} & \boldsymbol{l}_{32} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13} \\ 0 & \boldsymbol{a}_{22}^{'} & \boldsymbol{a}_{23}^{'} \\ 0 & 0 & \boldsymbol{a}_{33}^{''} \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}}$$

$$l_{31} = \frac{a_{31}}{a_{11}}$$

$$l_{32} = ?$$

$$[\ \boldsymbol{L} \cdot \boldsymbol{U}\ ]$$

Example: $A \quad = \quad L \quad \cdot \quad U$

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Gauss Elimination

Coefficients

l_{21} = -2/-1= 2

l_{31} = 4/-1= -4

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} \Rightarrow \quad \Rightarrow \begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix}$$

[*L*]

l_{32} = -12/4= -3

$$\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}}$$

[*U*]

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & ?? & 1 \end{bmatrix} \Rightarrow \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

Example: $A = L \cdot U$

Gauss Elimination with pivoting

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} \Rightarrow pivoting \quad \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ -2 & 9 & 11 \\ -1 & 2.5 & 5 \end{bmatrix}$$

Coefficients

l_{21} = -2/4= -.5

l_{31} = -1/4= -.25

$$\begin{bmatrix} 4 & -22 & -20 \\ -2 & 9 & 11 \\ -1 & 2.5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ 0 & \mathbf{-2} & \mathbf{1} \\ 0 & -3 & 0 \end{bmatrix} \Rightarrow pivoting \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

Coefficients

l_{32} = -2/-3

$$\begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-0.5} & 1 & 0 \\ -0.25 & ?? & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ \mathbf{-0.5} & ?? & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ \mathbf{-0.5} & \mathbf{0.66} & 1 \end{bmatrix}$$

[U]

LU decomposition

- *Gauss Elimination* can be used to decompose [A] into [L] and [U]. Therefore, it requires the same total FLOPs as for *Gauss elimination:* In the order of (proportional to) N^3 where N is the # of unknowns.

- l_{ij} values (the factors generated during the elimination step) can be stored in the lower part of the matrix to save storage. This can be done because these are converted to zeros anyway and unnecessary for the future operations.

- Provides efficient means to compute the matrix inverse

LU Decomposition

What about a situation where the **[B]** vector changes?

In LU Decomposition, LU decomposition of [A] is independent of the [B] vector, therefore it only needs to be done once.

Let m = the number of times the [B] vector changes

The computational times are proportional to

LU decomposition = $\frac{n^3}{3} + m(n^2)$

Gauss Elimination = $m(\frac{n^3}{3} + \frac{n^2}{2})$

Consider a 100 equation set with 50 right hand side vectors

LU Decomposition = 8.33×10^5

Gauss Elimination = 1.69×10^7

MATRIX INVERSE

$$A.\,A^{-1} = I$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve in n=3 major phases

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve each one
using A=L·U method ➔ e.g.

$$LU \cdot \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Each solution takes **O(n²)** steps.

Therefore,
The Total time = **O(n³)**

Solve Problem 10.6. Solution file is available on the web